Appendix A

Complex Numbers

Complex numbers are very important in engineering and science. Engineers use complex numbers in analyzing stresses and strains on beams and in studying resonance phenomena in structures as different as tall buildings and suspension bridges. There are many other applications of complex numbers, including control theory, signal analysis, quantum mechanics, relativity, and fluid dynamics.

You have probably dealt with complex numbers before. If so this appendix will serve as a review; otherwise, there is sufficient material here for you to understand complex numbers when they arise in the book. Vectors and matrices of complex numbers are not dealt with in a formal fashion. Occasionally they will arise as eigenvectors or eigenvalues. You will encounter complex roots of polynomials when dealing with eigenvalues and a small number of proofs that involve complex numbers.

A.1 CONSTRUCTING THE COMPLEX NUMBERS

It is clear that the equation $x^2 = -1$ has no real solution, so mathematics defines $i = \sqrt{-1}$, and $i^2 = -1$. The solutions to $x^2 = -1$ are then $x = i$ and $x = -i$. The complex number i forms the basis for the set of complex numbers we call $\mathbb{C}$.

Definition A.1. A complex number has the form $z = x + iy$, where x and y are real numbers and $i = \sqrt{-1}$. We can express a real number x as a complex number $z = x + i0$.

When $z = x + iy$, we call x the *real part* of z and y the *imaginary part*..

Two complex numbers are equal if they have the same real and imaginary parts:

$$x_1+iy_1=x_2+iy_2 \Rightarrow x_1=x_2 \quad \textit{and} \quad y_1=y_2,$$

where x_1, x_2, y_1, y_2 are real numbers.

The sum of two complex numbers is a complex number:

$$(x_1+iy_1) + (x_2+iy_2) = (x_1+x_2)+i(y_1+y_2)$$

The product of two complex numbers is a complex number.

$$(x_1+iy_1)\,(x_2+iy_2) = (x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)$$

The easy way to perform this calculation is to proceed just like you are computing (a+b)(c+d) = ac + ad + bc + bd, except that i^2 = -1.

$$(x_1+iy_1)\,(x_2+iy_2) = \mathrm{x_1x_2+ix_1y_2+ix_2y_1+i^2y_1y_2=x_1x_2+ix_1y_2+ix_2y_1-y_1y_2} = \mathrm{(x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)}$$

A useful identity satisfied by complex numbers is

$$(x + iy)\,(x - iy) = x^2 + y^2.$$

This leads to a method of computing the quotient of two complex numbers.

$$\frac{x_1+iy_1}{x_2+iy_2}=\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}$$
$$=\frac{(x_1x_2+y_1y_2)+i(-x_1y_2+y_1x_2)}{x_2^2+y_2^2}.$$

The process is known as *rationalization of the denominator*.

A.2 CALCULATING WITH COMPLEX NUMBERS

We can now do all the standard linear algebra calculations with complex numbers - find the upper triangular form of a matrix whose elements are complex numbers, solve systems of linear equations, find inverses and calculate determinants.

For example, solve the system

$$(1+i)z+(2-i)w=2+7i$$
$$7z+(8-2i)w=4-9i.$$

The coefficient determinant is

$$\begin{vmatrix} 1+i & 2-i \\ 7 & 8-2i \end{vmatrix}=(1+i)(8-2i)-7(2-i)=(8-2i)+i(8-2i)-14+7i==-4+13i\neq 0.$$

Hence by Cramer's rule, there is a unique solution:

$$z=\frac{\begin{vmatrix} 2+7i & 2-i \\ 4-9i & 8-2i \end{vmatrix}}{-4+13i}=\frac{(2+7i)(8-2i)-(4-9i)(2-i)}{-4+13i}=$$
$$=\frac{2(8-2i)+(7i)(8-2i)-\{(4(2-i)-9i(2-i)\}}{-4+13i}$$
$$=\frac{16-4i+56i-14i^2-\{8-4i-18i+9i^2\}}{-4+13i}=$$
$$\frac{31+74i}{-4+13i}=\frac{(31+74i)(-4-13i)}{(-4)^2+13^2}=\frac{838-699i}{(-4)^2+13^2}=\frac{838}{185}-\frac{699}{185}i.$$

Similarly $w=\frac{-698}{185}+\frac{229}{185}i$.

A property enjoyed by complex numbers is that every complex number has a square root.

Theorem A.1. *If w is a non-zero complex number, then the equation $z^2=w$ has a solution $z\in\mathbb{C}$.*

Proof. Let $w=a+ib$, a, $b\in\mathbb{R}$.
Case 1. Suppose $b=0$. Then if $a>0$, $z=\sqrt{a}$ is a solution, while if $a<0$, $i\sqrt{-a}$ is a solution.
Case 2. Suppose $b\neq 0$. Let $z=x+iy$, x, $y\in\mathbb{R}$. Then the equation $z^2=w$ becomes

$$(x+iy)^2=x^2-y^2+2xyi=a+ib,$$

so equating real and imaginary parts gives

$$x^2-y^2=a \qquad \text{and} \qquad 2xy=b.$$

Hence $x\neq 0$ and $y=b/(2x)$. Consequently

$$x^2-\left(\frac{b}{2x}\right)^2=a,$$

so $4x^4-4ax^2-b^2=0$ and $4(x^2)^2-4a(x^2)-b^2=0$. By the quadratic equation,

$$x^2=\frac{4a\pm\sqrt{16a^2+16b^2}}{8}=\frac{a\pm\sqrt{a^2+b^2}}{2}.$$

However $x^2>0$, so we must take the + sign, since $a-\sqrt{a^2+b^2}<0$. Then $x^2=\frac{a+\sqrt{a^2+b^2}}{2}$, and the solutions are

$$x=\pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}},\quad y=b/(2x). \tag{A.1}$$

□

Example A.1. Find the solutions z = x + iy to the equation $z^2=1+i$ using equation A.1.

For our problem, $w=1+i$, so a = 1 and b = 1, and $x=\pm\sqrt{\frac{1+\sqrt{2}}{2}}$, $y=\pm\frac{1}{\sqrt{2}\sqrt{1+\sqrt{2}}}$. The solution is

$$z=\pm\left(\sqrt{\frac{1+\sqrt{2}}{2}}+\frac{i}{\sqrt{2}\sqrt{1+\sqrt{2}}}\right).$$

■

Example A.2. Find the cube roots of 1.

We have to solve the equation $z^3=1$, or $z^3-1=0$. Now
$z^3-1=(z-1)(z^2+z+1)$. So $z^3-1=0 \Rightarrow z-1=0$ or $z^2+z+1=0$.
But

$$z^2+z+1=0 \Rightarrow z=\frac{-1\pm\sqrt{1^2-4}}{2}=\frac{-1\pm\sqrt{3}i}{2}.$$

So there are 3 cube roots of 1, namely 1 and $(-1\pm\sqrt{3}i)/2$. ■

A.3 GEOMETRIC REPRESENTATION OF $\mathbb{C}$

Complex numbers can be represented as points in the plane, using the correspondence $x+iy \leftrightarrow (x, y)$. The representation is known as the *Argand diagram* or *complex plane*. The real parts lie on the x-axis, which is then called the *real axis*, while the imaginary parts lie on the y-axis, which is known as the *imaginary axis*. The complex numbers with positive imaginary part lie in the *upper half plane*, while those with negative imaginary part lie in the *lower half plane*.

Because of the equation

$$(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2),$$

complex numbers add vectorially, using the parallelogram law. Similarly, the complex number z_1-z_2 can be represented by the vector from (x_2, y_2) to (x_1, y_1), where $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ (Figure A.1).

The geometric representation of complex numbers can be very useful when complex number methods are used to investigate properties of triangles and circles. It is useful in the branch of calculus known as Complex Function theory, where geometric methods play an important role.

A.4 COMPLEX CONJUGATE

Definition A.2. **(Complex conjugate)** If $z=x+iy$, the *complex conjugate* of z is the complex number defined by $\bar{z}=x-iy$. Geometrically, the complex conjugate of z is obtained by reflecting z across the real axis (Figure A.2).

The following properties of the complex conjugate are easy to verify:

a. $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$;
b. $\overline{-z}=-\bar{z}$;
c. $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$;
d. $\overline{z_1z_2}=\overline{z_1}\ \overline{z_2}$;
e. $\overline{1/z}=1/\bar{z}$;

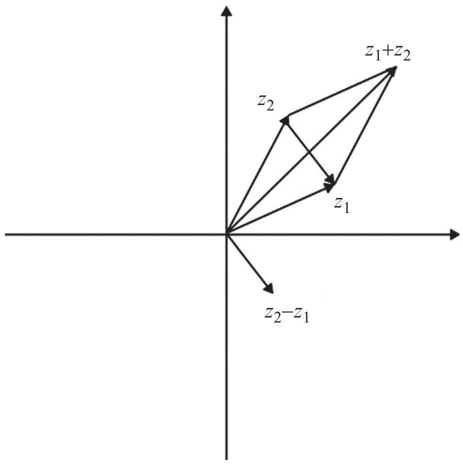

FIGURE A.1 Complex addition and subtraction.

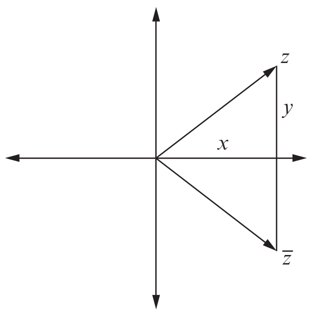

FIGURE A.2 Complex conjugate.

f. $\overline{z_1/z_2}=\overline{z_1}/\overline{z_2}$;
g. z is real if and only if $\overline{z}$=z;
h. With the standard convention that the real and imaginary parts are denoted by Re z and Im z, we have Re $z = \frac{z+\overline{z}}{2}$, Im $z = \frac{z-\overline{z}}{2}$;
i. if $z = x + iy$,then $z\overline{z}=x^2+y^2$.

The following is an interesting and useful result concerning the roots of polynomials.

Theorem A.2. *Let* $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$, *where* $a_n, \ldots, a_0$ *are real. The complex roots occur in complex-conjugate pairs, i.e.* if $f(z) = 0$, *then* $f(\overline{z}) = 0$.

Proof. If $f(z) = 0$, then $0 = \overline{0} = \overline{f(z)} = \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0}$
$= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} = a_n\overline{z}^n + a_{n-1}\overline{z}^{n-1} + \cdots + a_1\overline{z} + a_0 = f(\overline{z})$. □

The computation of the roots of a polynomial play an important role in applications. Some applications include

- Representing Geometric Figures
- Modeling of Steel Corrosion
- Electrical Circuits
- Depth of Flow in Rivers
- Numerical Integration

The computation of polynomial roots is a complex process for polynomials of degree greater than or equal to 3 and is normally done using carefully crafted computer algorithms. Roots of polynomials are also important in theory. Eigenvalues of a square matrix are defined in terms of polynomial roots, but eigenvalues are seldom computed by directly finding the roots.

A real matrix has a transpose, and a real matrix A such that $A^T = A$ is said to be symmetric. There are equivalents of these concepts for a complex matrix.

Definition A.3. The *conjugate transpose* of a complex matrix A, written A^*, is obtained from A by taking the transpose and then taking the complex conjugate of each entry. The conjugate transpose is the equivalent of the transpose of a real matrix.

Example A.3. Let

$$A = \begin{bmatrix} 1-i & \frac{1}{2}+2i & 3-5i \\ 6+i & 7+5i & 1+i \\ -1+8i & i & -i \end{bmatrix}.$$

Then,

$$A^* = \begin{bmatrix} 1+i & 6-i & -1-8i \\ \frac{1}{2}-2i & 7-5i & -i \\ 3+5i & 1-i & i \end{bmatrix}. \quad \blacksquare$$

Recall that if A and B are real matrices, then $(AB)^T = B^T A^T$. If A and B are complex matrices, then $(AB)^* = B^* A^*$.

Definition A.4. A complex matrix A is said to be *Hermitian* if $A^* = A$, or if $a_{ij} = \overline{a_{ij}}$ for $1 \le i,j \le n$. If $i = j$, then $a_{ii} = \overline{a_{ii}}$, so the diagonal entries of Hermitian matrix are real. A Hermitian matrix is the equivalent of a real symmetric matrix.

Example A.4. The matrix

$$A = \begin{bmatrix} 1 & i & 6-2i \\ -i & 2 & 4+i \\ 6+2i & 4-i & 3 \end{bmatrix}$$

is Hermitian. ■

A.5 COMPLEX NUMBERS IN MATLAB

MATLAB implements the full range of calculations with complex numbers. For instance, you can assign a complex number to a variable as follows:

```
>> z = 3 + 2i
z =
   3 + 2i
```

Alternatively, you can use the function `complex`.

```
z = complex(5,7)
z =
   5 + 7i
```

You can use any function that accepts a complex variable, and can create and solve complex systems of equations.

```
>> z1 = 4 -i;
>> z2 = 1 +i;
```

```
>> z3 = i;
>> z1^2 + 7*z2 - 8*z3
ans =
   22.0000 - 9.0000i
>> sin(i*pi)
ans =
  0 + 11.549i
>> exp(i*pi)

ans =
-1.0000 + 0.0000i

>> A = [1-i 2+3i -7;-1+i 16+4i i;3+8i -1 7+5i]

A =
   1.0000 - 1.0000i   2.0000 + 3.0000i  -7.0000
  -1.0000 + 1.0000i  16.0000 + 4.0000i        0 + 1.0000i
   3.0000 + 8.0000i  -1.0000             7.0000 + 5.0000i

>> b = [12+2i -1-9i -i]'

b =

  12.0000 - 2.0000i
  -1.0000 + 9.0000i
        0 + 1.0000i

>> z = A\b

z =

   1.6294 - 1.0376i
   0.1008 + 0.4836i
  -1.8082 + 0.0861i

>> A*z

ans =

  12.0000 - 2.0000i
  -1.0000 + 9.0000i
  -0.0000 + 1.0000i

>> E = eig(A)

E =

   0.7504 + 7.7000i
   7.6887 - 3.7542i
  15.5609 + 4.0542i
```

When the matrix is real, complex eigenvalues occur in conjugate pairs. For a complex matrix, this is not true. Recall that a real symmetric matrix has real eigenvalues. The same is true for a Hermitian matrix (Problem A.10).

```
>> A = [1 i 6-2i;-i 2 4+i;6+2i 4-i 3]

A =

            1                           0 +           1i               6 -          2i
            0 -          1i             2                              4 +          1i
            6 +          2i             4 -           1i               3

>> eig(A)

ans =

      -5.7809
       2.2035
       9.5774
```

A.6 EULER'S FORMULA

If ω is a real constant, what is $e^{i\omega}$? Let's make mathematical sense out of it. It is well known that the McLaurin series for e^x for any real number $-\infty < x < \infty$ is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \cdots = \sum_{i=0}^{\infty} \frac{x^n}{n!}$$

Now let's apply this same result to a complex number $z = ix$, where x is any real number. Note that $i^{2k} = (-1)^k, k \geq 1$ and $i^{2k+1} = i\,(-1)^k, k \geq 1$. Now,

$$e^{ix} = 1 + \frac{(ix)}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} \cdots =$$

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) i$$

Recall that the McLaurin series for $\cos x$ and $\sin x$, $-\infty < x < \infty$ are

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

We how have Euler's formula

$$e^{ix} = \cos x + i \sin x \tag{A.2}$$

Remark A.1. The equation

$$e^{ix} = \cos x + i \sin x$$

is Euler's formula. Now let $x = \pi$ to obtain the very famous *Euler's identity*.

$$e^{i\pi} = -1$$

A poll of readers that was conducted by *Physics World* magazine in 2004 chose Euler's Identity as the "greatest equation ever", in a dead heat with the four Maxwell's equations of electromagnetism.

Richard Feynman called Euler's formula "our jewel" and "one of the most remarkable, almost astounding, formulas in all of mathematics."

After proving Euler's Identity during a lecture, Benjamin Peirce, a noted American 19th-century philosopher/mathematician and a professor at Harvard University, stated that "It is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."

The Stanford University mathematics professor, Dr. Keith Devlin, said, "Like a Shakespearean sonnet that captures the very essence of love, or a painting that brings out the beauty of the human form that is far more than just skin deep, Euler's Equation reaches down into the very depths of existence."

A.7 PROBLEMS

A.1 Evaluate the following expressions.

a. $(-3+i)(14-2i)$

i. $\dfrac{2+3i}{1-4i}$

ii. $\dfrac{(1+2i)^2}{1-i}$

A.2 Find the roots of $8z^2 + 2z + 1$.

A.3 x = 2 is a real root of the polynomial $x^3 - x^2 - x - 2$. Find the remaining two roots.

A.4 Verify that $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ are roots of the polynomial $x^4 + 1$. Find the other two roots.

A.5 Express $1+(1+i)+(1+i)^2+(1+i)^3+\ldots+(1+i)^{99}$in the form $x+iy$.

A.6 Solve the system $Ax=b$, where

$$A=\begin{bmatrix} 3+i & -1+2i & 2 \\ 1+i & -1+i & 1 \\ 1+2i & -2+i & 1+i \end{bmatrix},\quad b=\begin{bmatrix} 2+3i \\ 1-i \\ 4+3i \end{bmatrix}.$$

A.7 Find the inverse of the matrix

$$A=\begin{bmatrix} 4-6i & 1+i \\ 12-7i & -i \end{bmatrix}.$$

A.8 Find the conjugate transpose of each matrix.

a. $\begin{bmatrix} 2+i & -1+2i & 2 \\ 1+i & -1+i & 1 \\ 1+2i & -2+i & 1+i \end{bmatrix}$

b. $\begin{bmatrix} -3+i & 2+i & 6-8i \\ 5+0i & 9-i & 16+3i \\ -6+12i & 14-0i & 6+5i \\ 4-i & 8+2i & 1+i \\ 18+3i & 7-i & 1+i \end{bmatrix}$

A.9 If A is a real $m\times n$ matrix, then A^TA is an $n\times n$ symmetric matrix. Prove that if A is an $m\times n$ complex matrix, then A^*A is Hermitian.

A.10 Prove that the eigenvalues of a Hermitian matrix are real. HINT: If $Av=\lambda v$, where λ is an eigenvalue with corresponding eigenvector v, then $v^*Av=\lambda v^*v$. What type of matrices are v^*Av and v^*v?

A.11

a. Prove that $x^*x\geq 0$ for any $n\times 1$ vector x.

i. Prove that A^*A is positive definite.

ii. Prove that the eigenvalues of A^*A are nonnegative.

A.7.1 MATLAB Problems

A.12 Using MATLAB, compute

a. $(2+3i)\,(-2+i)$

b. $\dfrac{6+i}{2+5i}$

c. e^{2+i}

d. $(1-2i)^{\,5i}$

A.13 A *unitary matrix* is the complex equivalent of an orthogonal matrix. A complex matrix is unitary if $A^*A=AA^*=I$. The QR decomposition applies to a complex matrix, and the matrix Q is unitary. Let

$$A=\begin{bmatrix} 1-i & -i & 3+i & 0 \\ 2+3i & -1+2i & i & 3i \\ 5-6i & 1+7i & 3 & 5+i \\ 12 & 4+9i & 1-4i & i \end{bmatrix}$$

Compute the QR decomposition of A and verify that Q is unitary.

A.14 A complex matrix has a singular value decomposition $A=U\widetilde{\Sigma}V^*$, where U and V are unitary.

a. Using the MATLAB command `[U S V] = svd`, find the SVD of the matrix in part (b) of Problem A.8. Verify that U and V are unitary.

b. Explain why a complex matrix has real, nonnegative singular values.

A.15 One of the most famous functions in all of mathematics is the Riemann zeta function

$$\zeta\,(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}.$$

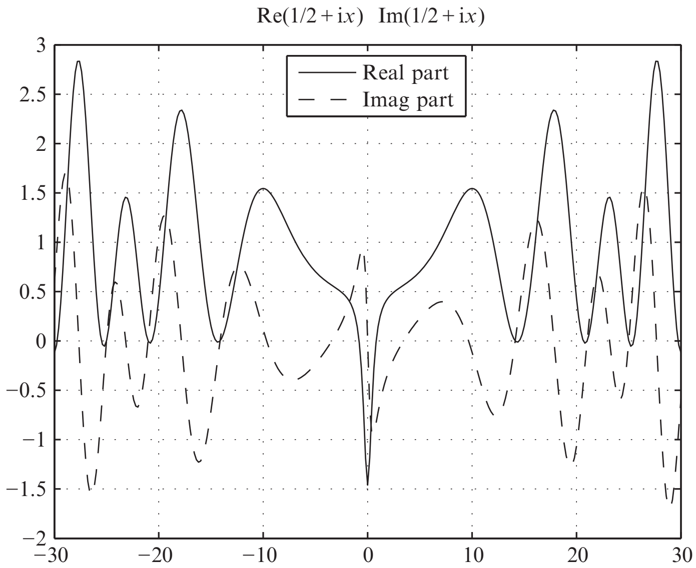

FIGURE A.3 Riemann zeta function.

The Riemann hypothesis states that all non-trivial zeros of the Riemann zeta function have real part 1/2. The trivial zeros are $-2+0i,\ -4+0i,\ \ldots$. Proving the Riemann hypothesis has been an open problem for a very long time. The MATLAB function `zeta(s)` computes the Riemann zeta function for a complex variable `s`. Figure A.3 is a MATLAB-generated graph of $Re\left(\frac{1}{2}+ix\right)$ and $Im\left(\frac{1}{2}+ix\right)$ for $\zeta\left(\frac{1}{2}+ix\right)$. Recreate the graph.

Note that the first few zeros with the imaginary part $\frac{1}{2}$ rounded to an integer are:

$$\frac{1}{2} \pm 14i,\ \frac{1}{2} \pm 21i,\ \frac{1}{2} \pm 25i$$