Chapter 6

Orthogonal Vectors and Matrices

You should be familiar with

- Distance between points in two- and three-space
- Geometric interpretation of vector addition and subtraction
- Simple geometry and trigonometry
- Rotation matrices
- Real symmetric matrices
- Computation of $\int_a^b f(t)\, g(t)\, \mathrm{d}t$ (for Section 6.5)

6.1 INTRODUCTION

We will have occasion in the book to use two- and three-dimensional vectors as examples. If we are discussing a property or operation that applies to all vectors, we can use vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ as illustrations, since we can visualize the results, whereas that is not possible for a vector in $\mathbb{R}^n$, $n \geq 4$.

In three-dimensional space, *points* are defined as ordered triples of real numbers and the *distance* between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is defined by the formula (Figure 6.1)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Directed line segments $\overrightarrow{P_1P_2}$ (Figure 6.1) are introduced as three-dimensional column vectors: If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, then

$$\overrightarrow{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}.$$

If P is a point, we let $P = \overrightarrow{OP}$ and call P the *position vector* of P, where O is the origin.

There are geometrical interpretations of equality, addition, subtraction, and scalar multiplication of vectors (Figure 6.2).

1. Equality of vectors: Suppose A, B, C, D are distinct points such that no three are collinear. Then $\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and $\overrightarrow{AC} \parallel \overrightarrow{BD}$.
2. Addition of vectors obeys the *parallelogram law*: Let A, B, C be non-collinear. Then

$$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD},$$

   where D is the point such that $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and $\overrightarrow{AC} \parallel \overrightarrow{BD}$
3. The difference of two vectors $\overrightarrow{AB} - \overrightarrow{AC}$ is a vector whose start is the tip of $\overrightarrow{AC}$ and whose tip coincides with the tip of $\overrightarrow{AB}$.
4. Scalar multiplication of vectors (Figure 6.3): Let $\overrightarrow{AP} = t\overrightarrow{AB}$, where A and B are distinct points. Then P is on the line AB, and
   a. $P = A$ *if* $t = 0$, $P = B$ *if* $t = 1$;
   b. P is between A and B if $0 < t < 1$;
   c. B is between A and P if $t > 1$;
   d. A is between P and B if $t < 0$.

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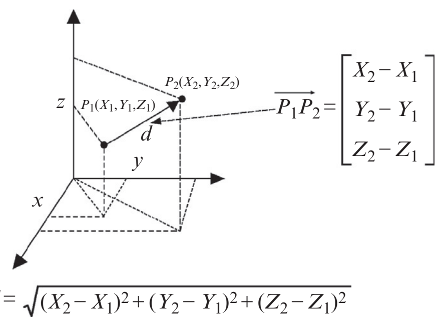

FIGURE 6.1 Distance between points.

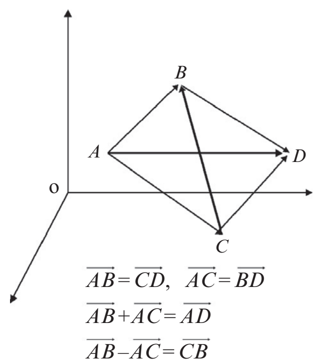

FIGURE 6.2 Equality, addition, and subtraction of vectors.

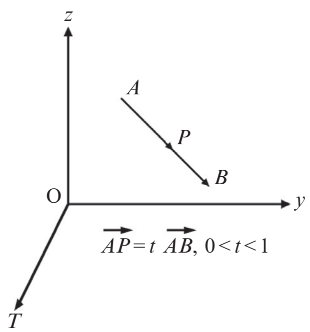

FIGURE 6.3 Scalar multiplication of vectors.

6.2 THE INNER PRODUCT

Along with matrix multiplication, the inner product is an important operator in linear algebra. It defines vector length, orthonormal bases, the L^2 matrix norm, projections, and Householder reflections. We will study these and many more constructs that use the inner product.

Definition 6.1. Given two vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbb{R}^n$, we define the *inner product* of x and y, written $\langle x, y \rangle$ to be the real number

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Note that $x^{\mathrm{T}}y = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \langle x, y \rangle$, so we can compute the inner product as the matrix product, $x^{\mathrm{T}}y$. Since $\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i$, $y^{\mathrm{T}}x$ is another way to compute the inner product.

Remark 6.1. In many books, the notation $x \cdot y$ to refers to the inner product, and it is called the *dot product*. We will seldom use this notation in the book.

Example 6.1. In $\mathbb{R}^3$, if $x = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $y = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$, then $\langle x, y \rangle = a_1 a_2 + b_1 b_2 + c_1 c_2$. For instance, if $u = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$, $v = \begin{bmatrix} -9 \\ 7 \\ -1 \end{bmatrix}$, then $\langle u, v \rangle = 4\,(-9) + 6\,(7) + 4\,(-1) = 2$ ■

There are properties of the inner product that we will use throughout the book.

Theorem 6.1. *The inner product has the following properties:*

1. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
2. $\langle cx, y \rangle = \langle x, cy \rangle = c\,\langle x, y \rangle$, *where c is a scalar*
3. $\langle x, y \rangle = \langle y, x \rangle$
4. $\langle x, 0 \rangle = 0$
5. $\langle x, x \rangle = \sum_{i=1}^{n} x_i^2$
6. *If* $\langle x, x \rangle = 0$, *then* $x = 0$

Proof. We will prove properties 5 and 6, leaving the remaining properties to the exercises.

Property 5: $\langle x, x \rangle = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2 = \sum_{i=1}^{n} x_i^2$

Property 6: If $\langle x, x \rangle = 0$, then from Property 5 $x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2 = 0$. The only way this can occur is if $x_i = 0,\ 1 \le i \le n$. □

There are many occasions where we will need to deal with the length of a vector, so we need a compact notation for vector length. A vector u in the plane has length $\sqrt{x^2 + y^2}$, and a vector u in three-dimensional space has length $\sqrt{x^2 + y^2 + z^2}$ (Figure 6.4). Using Property 5 of Theorem 6.1, we see that in either two or three dimensions length $(u) = \sqrt{\langle u, u \rangle}$. The notation $\|u\|_2$ will be used to specify the length of vector u. This notation will be fully developed in Chapter 7 when we discuss vector norms.

There is a nice geometric interpretation for the inner product of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. For simplicity, we will consider vectors in $\mathbb{R}^2$, but the same reasoning applies to three-dimensional vectors. Suppose that θ is the angle between vectors u and v such that $0 \le \theta \le \pi$ as shown in Figure 6.5. It follows that $\langle u, v \rangle = \|u\|_2\,\|v\|_2 \cos\theta$. In Figure 6.6, the three vectors form the triangle AOB. Note that the length of each side is the length of the vector forming that side. The law of cosines tells us that

$$\|u - v\|_2^2 = \|u\|_2^2 + \|v\|_2^2 - 2\,\|u\|_2\,\|v\|_2 \cos\theta. \tag{6.1}$$

Using the properties of inner products, we can write the left-hand side of Equation 6.1 as

$$\|u - v\|_2^2 = \langle u - v, u - v \rangle = \langle u, u \rangle - 2\,\langle u, v \rangle + \langle v, v \rangle = \|u\|_2^2 - 2\,\langle u, v \rangle + \|v\|_2^2\,.$$

Equating the rewritten left-hand side of Equation 6.1 with the right-hand side gives

$$\|u\|_2^2 - 2\,\langle u, v \rangle + \|v\|_2^2 = \|u\|^2 + \|v\|_2^2 - 2\,\|u\|_2\,\|v\|_2 \cos\theta. \tag{6.2}$$

After cancelation of terms in Equation 6.2, we have the result

$$\langle u, v \rangle = \|u\|_2\,\|v\|_2\, cos\theta. \tag{6.3}$$

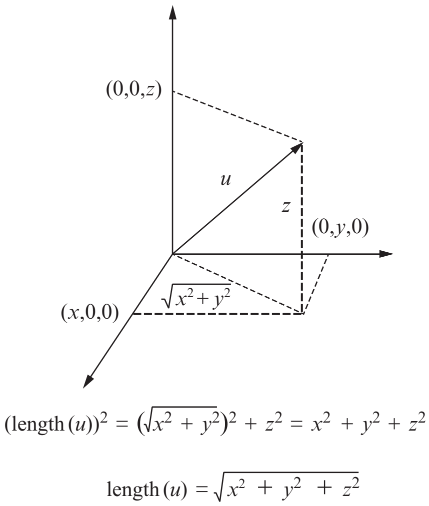

FIGURE 6.4 Vector length.

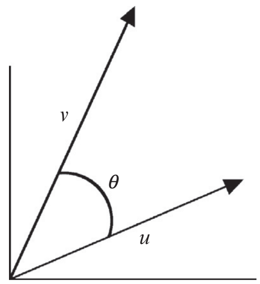

FIGURE 6.5 Geometric interpretation of the inner product.

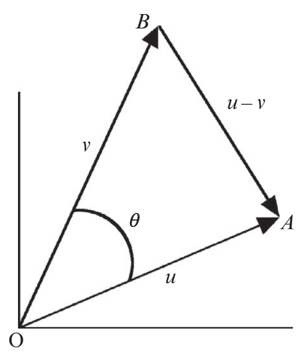

FIGURE 6.6 Law of cosines.

This formula is usually used to determine the angle between two vectors, not to compute the inner product.

Example 6.2. Determine the angle between $u = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$.

$\langle u, v \rangle = -22$, $\|u\|_2 = \sqrt{26}$, $\|v\|_2 = \sqrt{29}$. The angle between the two vectors is given by

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\|_2 \, \|v\|_2} = \frac{-22}{\sqrt{26}\sqrt{29}} = -0.8011927,$$

so

$\theta = \cos^{-1}(-0.8011927) = 2.5\,\text{rad} = 143.24°$. ■

Another application of the inner product is to determine whether two vectors are perpendicular or parallel. Vectors u and v are perpendicular, when the angle θ between them is $\pi/2$. Assume u and v are nonzero. The cosine of $\pi/2$ is 0, so by Equation 6.3, $\langle u, v \rangle = 0$, and u and v are perpendicular. Vectors u and v are *parallel* when the angle between them is either 0 radians (pointing in the same direction) or π radians (pointing in opposite directions). Since $\cos(0) = 1$ and $\cos(\pi) = -1$, it follows from Equation 6.3 that either

$$\langle u, v \rangle = \|u\|_2 \, \|v\|_2 \ \ (\theta = 0) \quad \text{or} \quad \langle u, v \rangle = -\|u\|_2 \, \|v\|_2 \ (\theta = \pi)$$

implies that u and v are parallel.

Example 6.3. Determine if the following vectors are parallel, perpendicular, or neither.

a. $u = \begin{bmatrix} 6 \\ -2 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$

$\langle u, v \rangle = 6(2) - 2(5) - 1(2) = 0$

u and v are perpendicular.

b. $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, v = \begin{bmatrix} -1/2 \\ 1/4 \end{bmatrix}$.

$\langle u, v \rangle = 2(-1/2) + (-1)(1/4) = -\frac{5}{4}$

Compute their lengths, and test to see if they are parallel.

$\|u\|_2 = \sqrt{5}$ and $\|v\|_2 = \sqrt{5/16} = \frac{\sqrt{5}}{4}$. Now, $\langle u, v \rangle = -\frac{5}{4} = -\sqrt{5}\left(\frac{\sqrt{5}}{4}\right) = -\|u\|_2 \, \|v\|_2$

The two vectors are parallel. ■

6.3 ORTHOGONAL MATRICES

Vectors u and v are called *orthogonal* if $\langle u, v \rangle = 0$. We briefly mentioned orthogonal matrices in Chapter 5, and will now provide a formal definition. Many tools in numerical linear algebra involve orthogonal matrices, such as the *QR* decomposition (introduced in Chapter 14) and the singular value decomposition (SVD) (introduced in Chapter 15). Over the course of this book, we will see that orthogonal matrices are the most beautiful of all matrices, and that they have an intimate relation with orthogonal vectors.

Definition 6.2. An $n \times n$ matrix P is orthogonal if $P^{\mathrm{T}} = P^{-1}$.

The simplest example of an orthogonal matrix is the 2×2 rotation matrix introduced in Chapter 1.

Example 6.4. A rotation matrix $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal, since

$$P^{\mathrm{T}}P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ ■

Orthogonal matrices of many sizes occur in applications, from 2×2 to 1000×1000, and larger.

Example 6.5. Let $P = \begin{bmatrix} 0.00 & -0.80 & -0.60 \\ 0.80 & -0.36 & 0.48 \\ 0.60 & 0.48 & -0.64 \end{bmatrix}$. To verify that P is an orthogonal matrix, form $P^{\mathrm{T}}P$.

$$\begin{bmatrix} 0.00 & 0.80 & 0.60 \\ -0.80 & -0.36 & 0.48 \\ -0.60 & 0.48 & -0.64 \end{bmatrix} \begin{bmatrix} 0.00 & -0.80 & -0.60 \\ 0.80 & -0.36 & 0.48 \\ 0.60 & 0.48 & -0.64 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, take a look at the columns of P.

$$\left\| \begin{matrix} 0.00 \\ 0.80 \\ 0.60 \end{matrix} \right\|_2 = \left\| \begin{matrix} -0.80 \\ -0.36 \\ 0.48 \end{matrix} \right\|_2 = \left\| \begin{matrix} -0.60 \\ 0.48 \\ -0.64 \end{matrix} \right\|_2 = 1,$$

so each column has length 1 (a *unit vector*). Take the inner product of columns 1 and 2.

$$\left\langle \begin{bmatrix} 0.00 \\ 0.80 \\ 0.60 \end{bmatrix}, \begin{bmatrix} -0.80 \\ -0.36 \\ 0.48 \end{bmatrix} \right\rangle = 0.00\,(-0.80) + 0.80\,(-0.36) + 0.60\,(0.48) = 0.$$

Verify that the two remaining inner products are also zero. In summary, for this matrix, the columns of P are orthogonal, and each column has length 1. A set of orthogonal vectors, each with unit length, are said to be *orthonormal*. It is not a coincidence that the columns are orthonormal. ■

Theorem 6.2. *Let P be an $n \times n$ real matrix. Then P is an orthogonal matrix if and only if the columns of P are orthogonal and have unit length.*

Proof. Let $P = \begin{bmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1n} \\ a_{21} & \dots & \vdots & \dots & a_{2n} \\ \vdots & & a_{ii} & & \vdots \\ \vdots & & \vdots & & \vdots \\ a_{n1} & & a_{ni} & & a_{nn} \end{bmatrix}$, $P^{\mathrm{T}}P = I$. View P as $P = \begin{bmatrix} v_1 & \dots & v_i & \dots & v_n \end{bmatrix}$, where $v_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{n-1,i} \\ a_{ni} \end{bmatrix}$, $1 \leq i \leq n$ are the columns of P. Then, $P^{\mathrm{T}} = \begin{bmatrix} v_1^{\mathrm{T}} \\ v_2^{\mathrm{T}} \\ \vdots \\ v_{n-1}^{\mathrm{T}} \\ v_n^{\mathrm{T}} \end{bmatrix}$, where v_i^{T}, $1 \leq i \leq n$ are the rows of P^{T}.

Thus,

$$P^{\mathrm{T}}P = \begin{bmatrix} v_1^{\mathrm{T}} \\ v_2^{\mathrm{T}} \\ \vdots \\ v_{n-1}^{\mathrm{T}} \\ v_n^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} v_1 & \dots & v_i & \dots & v_n \end{bmatrix} = \begin{bmatrix} v_1^{\mathrm{T}}v_1 & \dots & v_1^{\mathrm{T}}v_i & \dots & v_1^{\mathrm{T}}v_n \\ v_2^{\mathrm{T}}v_1 & \ddots & \vdots & \dots & v_2^{\mathrm{T}}v_n \\ \vdots & \vdots & v_i^{\mathrm{T}}v_i & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n^{\mathrm{T}}v_1 & \dots & v_n^{\mathrm{T}}v_i & \dots & v_n^{\mathrm{T}}v_n \end{bmatrix} \tag{6.4}$$

.

If $P^{\mathrm{T}}P = I = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix}$, then Equation 6.4 implies that $\langle v_i, v_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$, so the columns of P are orthogonal. Since $\langle v_i, v_i \rangle = 1$, $\|v_i\|_2^2 = 1$, and the columns of P have unit length.

If the columns of P are orthogonal and of unit length, Equation 6.4 implies that $P^{\mathrm{T}}P = I$, and P is an orthogonal matrix. $\square$

Orthogonal matrices have other interesting properties. Among them is the fact the their determinant is always ± 1.

Theorem 6.3. *If P is orthogonal,* $\det P = \pm 1$.

Proof. Recall that the determinant of a product is the product of the determinants, and $\det P^{\mathrm{T}} = \det P$. Then,

$$\det(I) = \det\left(P^{\mathrm{T}}P\right) = \left(\det P^{\mathrm{T}}\right)(\det P) = (\det P)(\deg P) = (\det P)^2,$$

so $(\det P)^2 = \det I = 1$, and $\det P = \pm 1$. $\square$

Remark 6.2. If the determinant of an orthogonal matrix is 1, we say it is a *proper orthogonal matrix*.

6.4 SYMMETRIC MATRICES AND ORTHOGONALITY

In this and later chapters, we will discover many interesting and useful facts about symmetric matrices; in particular, many computations can be done faster and more accurately for a symmetric matrix. A good example is the computation of the eigenvalues of a symmetric matrix. We will begin right here with Theorem 6.4 that tells us the relationship between any two distinct eigenvalues and the corresponding eigenvectors of a real symmetric matrix.

Theorem 6.4. *If A is a real symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. Let λ_1 and λ_2 be distinct eigenvalues with associated eigenvectors v_1 and v_2. Then, $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. Take the inner product of the first equation by v_2 and the inner product of the second equation by v_1:

$$v_2^{\mathrm{T}}(Av_1) = \lambda_1 \langle v_2, v_1 \rangle, \quad (Av_2)^{\mathrm{T}} v_1 = \lambda_2 \langle v_2, v_1 \rangle. \tag{6.5}$$

In Equation 6.5, $(Av_2)^{\mathrm{T}} v_1 = v_2^{\mathrm{T}} A^{\mathrm{T}} v_1$, so Equation 6.5 becomes

$$v_2^{\mathrm{T}}(Av_1) = \lambda_1 \langle v_2, v_1 \rangle, \quad v_2^{\mathrm{T}} A^{\mathrm{T}} v_1 = \lambda_2 \langle v_2, v_1 \rangle. \tag{6.6}$$

Since $A^{\mathrm{T}} = A$, in Equation 6.6, we have

$$v_2^{\mathrm{T}}(Av_1) = \lambda_1 \langle v_2, v_1 \rangle, \quad v_2^{\mathrm{T}}(Av_1) = \lambda_2 \langle v_2, v_1 \rangle,$$

and

$$\lambda_1 \langle v_2, v_1 \rangle = \lambda_2 \langle v_2, v_1 \rangle. \tag{6.7}$$

Equation 6.7 gives

$$(\lambda_1 - \lambda_2)\langle v_2, v_1 \rangle = 0.$$

.

Since $\lambda_1 \neq \lambda_2$, $\langle v_2, v_1 \rangle = 0$, and v_1, v_2 are orthogonal. $\square$

Example 6.6. Let A be the symmetric matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 6$, and eigenvectors corresponding to the eigenvalues are

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix},$$

respectively. The three eigenvectors are mutually orthogonal, and you also should note that the eigenvectors are linearly independent, so they are a basis for $\mathbb{R}^3$. As a result, the matrix $X = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ is invertible. If we form the product $X^{-1}AX$, the result is

$$X^{-1}AX = D,$$

where D is the diagonal matrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ with the eigenvalues of A on the diagonal. In other words, A is diagonalizable. Let's go one step further and build a matrix, P, whose columns are those of X converted to a unit vector. Do this by dividing each column vector by its length, and obtain $P = \begin{bmatrix} 0.7071 & 0.5774 & -0.4082 \\ -0.7071 & 0.5774 & -0.4082 \\ 0.0000 & 0.5774 & 0.8165 \end{bmatrix}$. By Theorem 6.2, P is an orthogonal matrix. Now compute $P^{\mathrm{T}}AP$ and you will again get D. Thus, A is diagonalizable using an orthogonal matrix.
■

Real symmetric matrices have wonderful properties. We can get a hint of this by taking a look at the nonsymmetric matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$. Since $\det \begin{bmatrix} 1-\lambda & 2 & 4 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda)(2-\lambda)$, the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 2$. A computation shows that the eigenvectors corresponding to the eigenvalues are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

The eigenvectors span a subspace of dimension two. This is caused by the duplicate eigenvalue 1, and this matrix cannot be diagonalized.

Example 6.7. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$. A is symmetric and has characteristic polynomial $p(\lambda) = \lambda^4(\lambda - 5)$, so A has four eigenvalues of 0. Despite this, there are five linearly independent eigenvectors. Use the MATLAB command `[V D] = eig(A)`and note there are four values of 0 on the diagonal of D. Verify the following:

- V is an orthogonal matrix.
- The rank of V is 5, so the columns of V are linearly independent and form a basis for $\mathbb{R}^5$.
- $V^{\mathrm{T}}AV = D$

Despite the fact that A has four equal eigenvalues, it can be diagonalized. ■

6.5 THE L^2 INNER PRODUCT

We have presented the inner product for the vector space $\mathbb{R}^n$, and showed that it satisfies the properties in Theorem 6.1. The general concept of an inner product extends beyond Euclidean space to any vector space for which an inner product can be defined. In particular, there are many applications for vector spaces whose elements are functions, and such vector spaces normally have infinite dimension. Chapter 12 presents Fourier series to illustrate this concept. Fourier series is one of the most useful topics in engineering and science. The applications of Fourier series include heat conduction, signal processing, analysis of sound waves, seismic imaging, and solving differential equations. The inner product used with Fourier series and many other vector spaces of functions is the L^2 inner product.

Definition 6.3. If functions $f(t)$ and $g(t)$ are continuous on the interval $a \leq t \leq b$, the L^2 *inner product* is

$$\langle f, g \rangle_{L^2} = \int_a^b f(t)\, g(t)\, \mathrm{d}t.$$

It is not difficult to show that $\langle \cdot, \cdot \rangle_{L^2}$ satisfies the requirements for an inner product. For instance,

$$\langle cf, g\rangle_{L^2} = \int_a^b (cf(t))\, g(t)\, \mathrm{d}t = \int_a^b f(t)\,(cg(t))\, \mathrm{d}t = \langle f, cg\rangle_{L^2} = c\int_a^b f(t)\, g(t)\, \mathrm{d}t = c\,\langle f, g\rangle_{L^2},$$

so

$$\langle cf, g\rangle_{L^2} = \langle f, cg\rangle_{L^2} = c\,\langle f, g\rangle_{L^2}.$$

Proving the remaining properties is left to the exercises.
The length of a vector u is $\sqrt{\langle u, u\rangle}$. We can also define the length or size of a function over the interval $a \le t \le b$ by

$$\|f\|_{L_2} = \sqrt{\langle f, f\rangle_{L^2}} = \sqrt{\int_a^b f^2(t)\, \mathrm{d}t}.$$

A function f is normalized if $\langle f, f\rangle_{L^2} = 1$, and two functions f and g are orthogonal if $\langle f, g\rangle_{L^2} = 0$. A Fourier series consists of an infinite sequence of normalized trigonometric functions that are mutually orthogonal with respect to the L^2 norm.

Example 6.8. Given the functions $f(t) = (1/\sqrt{\pi})\sin(5t)$ and $g(t) = (1/\sqrt{\pi})\cos(3t)$, compute $\langle f, g\rangle_{L^2}$ and $\|f\|_{L_2}$.

a. $\langle f, g\rangle_{L^2} = (1/\pi)\int_0^{2\pi} sin\,(5t)\, cos\,(3t)\, \mathrm{d}t = -(1/4\pi)\left(\cos 2t + \frac{1}{4}\cos 8t\right)\Big|_0^{2\pi} = 0$. ($f$ and g are orthogonal.)

b. $\|f\|_{L_2} = \sqrt{(1/\pi)\int_0^{2\pi}\sin^2 5t\, \mathrm{d}t} = \sqrt{(1/\pi)\left[(t/2) - \frac{1}{20}\sin 10t\right]\Big|_0^{2\pi}} = 1$ ■

Similarly, $\|g\|_2 = 1$, so f and g are orthogonal and have unit length using the L^2 inner product.

6.6 THE CAUCHY-SCHWARZ INEQUALITY

The *Cauchy-Schwarz inequality* is one of the most widely used inequalities in mathematics, and will have occasion to use it in proofs. We can motivate the result by assuming that vectors u and v are in $\mathbb{R}^2$ or $\mathbb{R}^3$. In either case, $\langle u, v\rangle = \|u\|_2\,\|v\|_2\cos\theta$. If $\theta = 0$ or $\theta = \pi$, $|\langle u, v\rangle| = \|u\|_2\,\|v\|_2$. This occurs when u and v are parallel, or when $v = cu$ for some scalar multiple c. For $0 < \theta < \pi$, $|\cos\theta| < 1$, so $|\langle u, v\rangle| \le ||u||_2\,||v||_2$.

Theorem 6.5 (Cauchy-Schwarz inequality). *For any n-dimensional vectors u and v,*

$$|\langle u, v\rangle| \le ||u||_2\,||v||_2,$$

and equality occurs if and only if $v = cu$.

For a proof, see (15, p. 316).

Remark 6.3. The Cauchy-Schwarz inequality applies to any vector space that has an inner product; for instance, it applies to a vector space that uses the L^2-norm.

Recall in high school geometry you were told that the sum of the lengths of two sides of a triangle is greater than the third side. This is an instance of the *triangle inequality* that follows by using the Cauchy-Schwarz inequality:

$$\|u+v\|_2^2 = \|u\|_2^2 + 2\,\langle u, v\rangle + \|v\|_2^2 \le \|u\|_2^2 + 2\,\|u\|_2\,\|v\|_2 + \|v\|_2^2 = \left(\|u\|_2 + \|v\|_2\right)^2,$$

and

$$||u+v||_2 \le ||u||_2 + ||v||_2.$$

The triangle inequality holds for any number of dimensions, but is easily visualized in $\mathbb{R}^3$. In Figure 6.7, note the progression from a normal triangle to the three sides collapsing into a line, corresponding to $v = cu$. In this case, $\|x+y\|_2 = \|x\|_2 + \|y\|_2$.

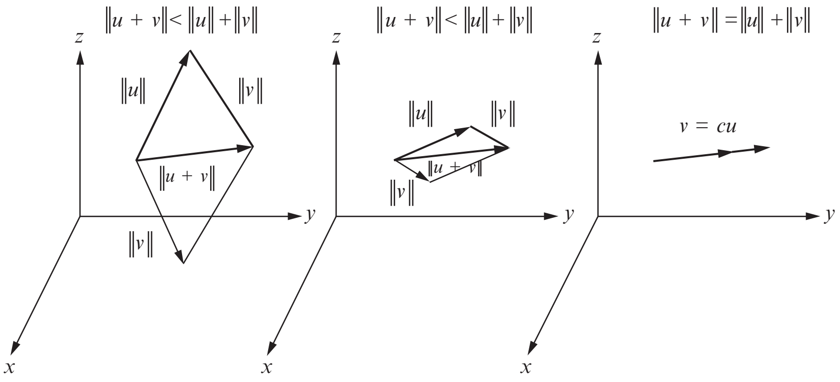

FIGURE 6.7 Triangle inequality.

6.7 SIGNAL COMPARISON

There is a particularly interesting implication of the Cauchy-Schwartz inequality [89].We ask the question "When is an expression of the form $|\langle u/\|u\|_2,\, v/\|v\|_2\rangle|$ a maximum?" Note that both $u/\|u\|_2$ and $v\|v\|_2$ are unit vectors. By the Cauchy-Schwartz inequality, we know that $|\langle u/\|u\|_2,\, v/\|v\|_2\rangle| \leq \|u/\|u\|_2\|\,\|v/\|v\|_2\| = 1$ and that $|\langle u/\|u\|_2,\, v/\|v\|_2\rangle| = 1$ if and only if $v/\|v\|_2 = cu/\|u\|_2$ for some scalar c. Hence, $|\langle u/\|u\|_2,\, v/\|v\|_2\rangle|$ attains a maximum when $v/\|v\|_2 = cu/\|u\|_2$ for some c. Now suppose we collect numerous samples of scalars of the form $|\langle u/\|u\|_2,\, v/\|v\|_2\rangle|$. The largest values will occur when $v = cu$. This result is very useful in developing *matched filter detector techniques.* When dealing with signals, we replace vectors by functions. Use the L^2 inner product to compare functions, and the Cauchy-Schwarz inequality applies to the L^2 inner product. We want to find the member signal in a set S of signals that most closely matches a target signal v. Define $f(u,v) = |\langle u/\|u\|_2,\, v/\|v\|_2\rangle|$. To find the best matching signal we need to evaluate

$$u_{\max} = \max_{u \in S} f(u, v).$$

The value $u_{\max}$ that produces the maximum value of $f(u,v)$ is not necessarily unique, so there may be more than one matching signal in S. It is possible that among the current members of S, the signal, u_{max}, giving the maximum value of $f(u,v)$ may be small and a poor match for the target signal v. A solution is to set a threshold and return no matching signals if $f(u,v)$ is below the threshold. There also may be a signal u that produces an very high value of $f(u,v)$, well above the actual match desired. This corresponds to a local maximum, and there are techniques to filter out local maxima.

Example 6.9. Here is the target signal and a set of three candidate signals (Figure 6.8). An application of the technique we have outlined will determine that among the candidate signals (c) is the best match for $f(t)$. ■

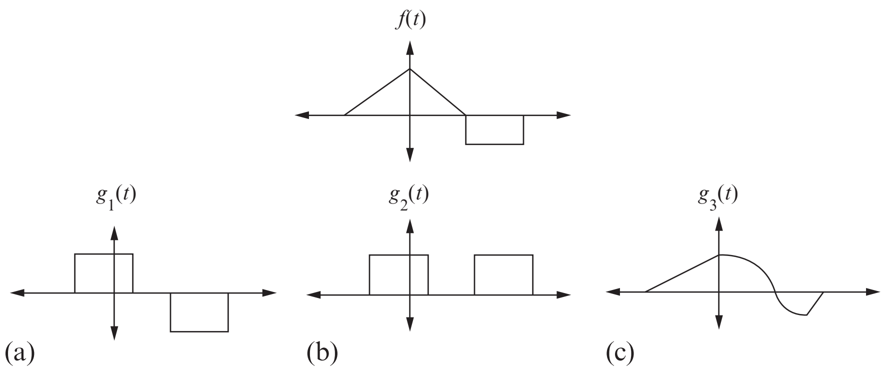

FIGURE 6.8 Signal comparison.

6.8 CHAPTER SUMMARY

The Inner Product

The inner product of two vectors is often called the dot product, although we will seldom use the term in the text. If x and y are vectors in $\mathbb{R}^n$, the inner product of x and y is $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, alternatively written $\langle x, y\rangle = x^{\mathrm{T}}y$. Since the inner product is commutative, we can also write $\langle x, y\rangle = y^{\mathrm{T}}x$. This expression for the inner product will be useful in many places throughout the book.

Among the most important properties of the inner product is that $\langle x, x\rangle = \sum_{i=1}^{n} x_i^2$, and so the length of a vector can be expressed as $\text{length}\,(x) = \sqrt{\langle x, x\rangle} = \|x\|_2$. Also, if $\langle x, x\rangle = 0$, $x = 0$.

In two and three dimensions, the inner product has the geometric interpretation

$$\langle x, y\rangle = \|x\|_2\,\|y\|_2 \cos\theta,$$

where θ is the angle between x and y.

Orthogonal Matrices

Orthogonal matrices are the most beautiful of all matrices. A matrix P is orthogonal if $P^{\mathrm{T}}P = I$, or the inverse of P is its transpose. Alternatively, a matrix is orthogonal if and only if its columns are orthonormal, meaning they are orthogonal and of unit length. An interesting property of an orthogonal matrix P is that $\det P = \pm 1$. As an example, rotation matrices are orthogonal.

Orthogonal matrices are involved in some of the most important decompositions in numerical linear algebra, the *QR* decomposition (Chapter 14), and the SVD (Chapter 15). The fact that orthogonal matrices are involved makes them invaluable tools for many applications.

Symmetric Matrices and Orthogonality

Symmetric matrices can always be diagonalized with an orthogonal matrix; in other words, there is an orthogonal matrix of eigenvectors such that $P^{\mathrm{T}}AP = D$, where D is a diagonal matrix of eigenvalues. This allows us to develop method for computing their eigenvalues more rapidly than we can find eigenvalues for nonsymmetric matrices. We begin the development of this diagonalization result by showing that any eigenvectors of a symmetric matrix corresponding to distinct eigenvalues are orthogonal.

The L^2 Inner Product

The inner product can be extended to functions by defining

$$\langle f, g\rangle_{L^2} = \int_a^b f(t)\, g(t)\, \mathrm{d}t,$$

so that

$$\|f\|_{L_2}^2 = \int_a^b f^2(t)\, \mathrm{d}t.$$

There are important sequences of functions that are orthogonal under the L^2 inner product. Chapter 12 looks at Fourier series, where the functions are trigonometric.

The Cauchy-Schwarz Inequality

The inequality,

$$|\langle x, y\rangle| \le \|x\|_2\,\|y\|_2$$

applies to any vector space with an inner product, and is called the Cauchy-Schwarz inequality. Among other things, it can be used to prove the triangle inequality

$$\|x+y\|_2 \le \|x\|_2 + \|y\|_2\,.$$

Although we will use the Cauchy-Schwarz inequality in later chapters as a theoretical tool, it has applications in matched filter detector techniques. Given function $f(t)$, it can be used to determine the best match to $f(t)$ among a set of candidate signals.

6.9 PROBLEMS

6.1 Compute the distance between the specified points.

a. $\begin{bmatrix} 1 & -6 & 7 \end{bmatrix}^T, \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^T$

b. $\begin{bmatrix} -1 & 4 & -9 & 12 & 15 \end{bmatrix}^T, \begin{bmatrix} 2 & -8 & 0 & -7 & 3 \end{bmatrix}^T$

6.2 Draw the vectors in $\mathbb{R}^2$.

a. $\begin{bmatrix} 1 \\ -9 \end{bmatrix} + 4\begin{bmatrix} -14 \\ 2 \end{bmatrix}$

b. $\begin{bmatrix} 5 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

6.3 Find the inner product of each vector pair.

a. $\begin{bmatrix} 1 & -5 & 2 \end{bmatrix}^T, \begin{bmatrix} -10 & 1 & -8 \end{bmatrix}^T$

b. $\begin{bmatrix} 17 & 0 & -4 & 12 & 3 \end{bmatrix}^T, \begin{bmatrix} 1 & -1 & 5 & 9 & 2 \end{bmatrix}$

6.4 Determine if each pair of vectors is orthogonal.

a. $\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T, \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T$

b. $\begin{bmatrix} 1 & -2 & 5 & 7 \end{bmatrix}^T, \begin{bmatrix} -1 & 2 & 1 & 1 \end{bmatrix}^T$

6.5 Normalize each vector in Problem 6.4.

6.6 If u and v are unit vectors, compute the following:

a. $\langle u+v, u-v\rangle$

b. $\langle u+v, u+v\rangle$

6.7 Find a vector parallel to the vector $\begin{bmatrix} -1 \\ 2 \\ 5 \\ 7 \end{bmatrix}$ and a vector orthogonal to it.

6.8 Find the angle between the vectors $\begin{bmatrix} -1 & 2 & 5 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & -8 & 2 \end{bmatrix}^T$.

6.9 What is the length of the 10-dimensional vector $u = \begin{bmatrix} 1 & -1 & 1 & -1 & 2 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$? Find a vector orthogonal to u and normalize each vector.

6.10 Verify the Cauchy-Schwarz and triangle inequalities for the vectors $x = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, y = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$.

6.11 Explain the result of applying the Cauchy-Schwarz inequality to the vectors $u = \begin{bmatrix} 0.88 \\ -1.55 \\ 2.68 \end{bmatrix}$, $v = \begin{bmatrix} -7.92 \\ 13.95 \\ -24.12 \end{bmatrix}$.

6.12 Prove the *parallelogram law* $2\|u\|_2^2 + 2\|v\|_2^2 = \|u+v\|_2^2 + \|u-v\|_2^2$. Explain where its name comes from by considering vectors u, v in the plane.

6.13 Determine if all possible pairings of the following vectors are parallel, orthogonal or neither.

$$u = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 1/2 \\ 1/16 \\ 1/8 \end{bmatrix}, w = \begin{bmatrix} -64 \\ -8 \\ 16 \end{bmatrix}$$

6.14 Show that the triangle formed by the points (−3, 5, 6), (−2, 7, 9), and (2, 1, 7) is a 30-60-90 triangle.

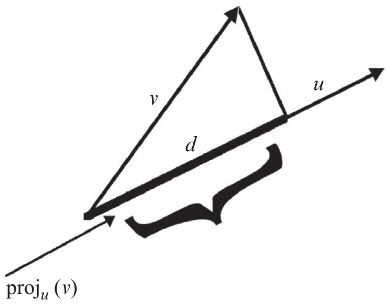

FIGURE 6.9 Projection of one vector onto another.

6.15 One of the primary applications of the inner product is the projection of one vector onto another. Looking at the Figure 6.9, develop a formula for the vector that is a projection of vector v onto vector u.
6.16 Prove parts 1, 2, 3, and 4 of Theorem 6.1.
6.17 Prove that if x and y are vectors in $\mathbb{R}^n$, then $\langle Ax, y\rangle = \left\langle x, A^{\mathrm{T}}y\right\rangle$.

Problems 6.18–6.24 deal with the *cross product*, which is defined for three dimensions as follows:

Definition 6.4. Let $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ be the standard basis for $\mathbb{R}^3$. The cross product of $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, and $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, written $u \times v$ is the vector

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{6.8}$$

Equation 6.8 is not a normal determinant. Treat vectors i, j, and k as scalars for the computation and then consider them vectors.

6.18 Show that

$$u \times v = (u_2v_3 - u_3v_2)\,i + (u_3v_1 - u_1v_3)\,j + (u_1v_2 - u_2v_1)\,k.$$

6.19 What is the relationship between $u \times v$ and $v \times u$?
6.20 Show that $u \times v$ is perpendicular to both u and v.
6.21 Show that $u \times u = 0$.
6.22 For each pair of vectors, compute the cross product.

a. $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} -7 \\ 12 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} a \\ -a \\ b \end{bmatrix}$, $\begin{bmatrix} -b \\ a \\ a \end{bmatrix}$, where a and b are scalars.

6.23 The equation of a plane is determined by three non-collinear points A, B and C. Take the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in the plane and use the cross product to find a vector n perpendicular to the plane. Pick an arbitrary point $P : (x, y, z)$ in the plane and require that $\left\langle n, \overrightarrow{PA}\right\rangle = 0$. This generates what is called the *normal equation of a plane*.

a. Draw a figure that illustrates this process.
b. Find the equation of the plane containing the points $A : (-1, 2, 3)$, $B\,(5, 1, 2)$, and $C : (-7, 1, 3)$.

6.24 Using the process of computing the normal equation to a plane in Problem 6.23, find an equation for a plane involving a determinant.
6.25 Show that if P is an orthogonal matrix, and x and y are vectors in $\mathbb{R}^n$, then $\langle Px, Py\rangle = \langle x, y\rangle$.
6.26 Show that if A and B are orthogonal matrices, then AB and BA are orthogonal matrices.

6.27 The matrix A has orthogonal columns. Convert it to an orthogonal matrix by normalizing the columns.

$$A = \begin{bmatrix} -1 & 1 & -3 \\ -1 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

6.28 Find one eigenvalue and its corresponding eigenvector of the symmetric matrix A by hand. Compute the other two using MATLAB. Using MATLAB, show that any two eigenvectors corresponding distinct eigenvalues are orthogonal, and diagonalize A with an orthogonal matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

6.29 Find the eigenvalues of matrix A. Can you diagonalize it? Explain.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

6.30 Find the eigenvalues of A. The matrix A has two equal eigenvalues, but it still has three linearly independent eigenvectors. Diagonalize A with an orthogonal matrix using MATLAB.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

6.31 Find the L^2 inner product of $f(t) = \sin(\pi t)$, $g(t) = \cos(3\pi t)$, $0 \le t \le 2\pi$. Also compute $\langle f, f\rangle^2$.

6.32 Over the interval $0 \le t \le 2\pi$, show that $\left\langle \cos it/\sqrt{\pi}, \cos jt/\sqrt{\pi}\right\rangle_{L^2} = 0$ for $i, j \ge 1$, $i \neq j$ and that $\left\langle \cos it/\sqrt{\pi}, \cos it/\sqrt{\pi}\right\rangle_{L^2} = 1$, $i \ge 1$.

6.33 A *permutation matrix* is a matrix obtained by swapping one or more rows of the identity matrix. Prove that a permutation matrix is orthogonal.

6.34 The vector *outer product* , $u \otimes v$, takes an $m \times 1$ column vector, u, an $n \times 1$ column vector, v, and returns an $m \times n$ matrix obtained by multiplying each element of u by each element of v. In particular, $(u \times v)_{ij} = u_i v_j$. The product $u \otimes v$ is also called the *tensor product.* The outer product computes the inertial tensor in rigid body dynamics, performs transform operations in digital signal processing and digital image processing, and has applications in statistics.

a. Show that $u \otimes v = u^{\mathrm{T}} v$.

b. Show that if $A = u \otimes v$, then $Av = u \|v\|_2^2$

6.9.1 MATLAB Problems

6.35 **a.** Develop `function d = veclength(v)` that computes the length of a vector v.

b. Use your function to find the length of each vector.

i. $\begin{bmatrix} -1 \\ 2 \\ 4 \\ 12 \\ -3 \end{bmatrix}$

ii. $\begin{bmatrix} -1 \\ 35 \\ 52 \\ 6 \end{bmatrix}$

6.36 The SVD discussed in Chapter 15 says that if A is any $n \times n$ matrix, there exist orthogonal matrices U and V and a diagonal matrix Σ such that $A = U\Sigma V^{\mathrm{T}}$.

a. Show that $A^{\mathrm{T}}A = V\Sigma^2 V^{\mathrm{T}}$

b. Prove that the eigenvalues of $A^{\mathrm{T}}A$ are the squares of the elements on the diagonal of Σ.

c. Let $A = \begin{bmatrix} 1 & -1 & 5 & 0 & 3 \\ 5 & -1 & 3 & 6 & 1 \\ 8 & -9 & 2 & 7 & 4 \\ 8 & 4 & -3 & 5 & 1 \\ -1 & -4 & 3 & 0 & 2 \end{bmatrix}$. Execute the MATLAB command `[U S V] = svd(A)` and verify that $A = USV^{\mathrm{T}}$ by computing `veclength(A-U*S*V')` if you did Problem 6.35; otherwise use `norm(A-U*S*V')`.

d. Find the eigenvalues of $A^{\mathrm{T}}A$ and verify the result of part (b). The elements on the diagonal of S are sorted in descending order. Sort the eigenvalues, E, of $A^{\mathrm{T}}A$ using `sort(E, 'descend')` before computing $\|E - \text{diag}(S)\|_2$.

6.37 A floating point number is a number that contains a fractional part, such as 0.3, 234.56819, and 1.56×10^{-8}. Because a computer generally cannot perform floating point calculations exactly, errors, called round-off errors, are introduced during computation. Chapter 8 discusses round-off errors and their effect on the accuracy of computer calculations. For example, if the exact value of calculation is 0.0, the computed result may be 3.0×10^{-16}. Using MATLAB, compute the inner product of u and v. Find the inner product using exact arithmetic, and comment on the results.

$$u = \begin{bmatrix} 3.2 & -1.5 & 6.3 & -2.5 \end{bmatrix}^{\mathrm{T}}, \quad v = \begin{bmatrix} 4.3 & 0 & 1.8 & 10.04 \end{bmatrix}^{\mathrm{T}}.$$

6.38 Show that each matrix is orthogonal in two different ways, using the definition and by directly showing that the columns have unit length and are orthogonal.

a. $P1 = \begin{bmatrix} -0.40825 & 0.43644 & 0.80178 \\ -0.8165 & 0.21822 & -0.53452 \\ -0.40825 & -0.87287 & 0.26726 \end{bmatrix}$

b. $P2 = \begin{bmatrix} -0.51450 & 0.48507 & 0.70711 \\ -0.68599 & -0.72761 & 0.0000 \\ 0.51450 & -0.48507 & 0.70711 \end{bmatrix}$

6.39 Are any of the two matrices orthogonal?

a. $P1 = \begin{bmatrix} -0.58835 & 0.70206 & 0.40119 \\ -0.78446 & -0.37524 & -0.49377 \\ -0.19612 & -0.60523 & 0.77152 \end{bmatrix}$

b. $P2 = \begin{bmatrix} -0.47624 & -0.4264 & 0.30151 \\ 0.087932 & 0.86603 & -0.40825 \\ -0.87491 & -0.26112 & 0.86164 \end{bmatrix}$

6.40 In Chapter 14, we will begin the study of the QR decomposition. A special case of this decomposition states that for any $n \times n$ matrix A, there exists an $n \times n$ orthogonal matrix Q and an $n \times n$ upper triangular matrix R such that

$$A = QR.$$

The MATLAB command

```
[Q R] = qr(A)
```

computes the factors Q and R. Find the QR decomposition of the matrix in Problem 6.36(c). Verify that Q is orthogonal.

6.41 For a matrix whose elements are complex numbers, there is a definition analogous to the transpose of a real matrix. A^*, called the *conjugate transpose*, is the matrix obtained by taking the complex conjugate of the entries of A and exchanging rows and columns. A matrix is said to be *Hermitian* if $A^* = A$.

a. Find the conjugate transpose of the matrix.

$$\begin{bmatrix} 1-\mathrm{i} & 3+\mathrm{i} & 7 & 8-3\mathrm{i} \\ 6+7\mathrm{i} & 4-\mathrm{i} & \mathrm{i} & 1+\mathrm{i} \\ 2-3\mathrm{i} & 6+\mathrm{i} & 3 & 9+\mathrm{i} \\ -1-\mathrm{i} & 10+\mathrm{i} & 7 & 12+2\mathrm{i} \end{bmatrix}$$

b. Using MATLAB, verify your result of part (a).

c. Investigate whether the MATLAB function `eig` applies to a complex matrix by using it with the matrix in part (a).
d. Prove that if A is Hermitian, its diagonal entries are real numbers.

6.42 **a.** Write a function `c = mycross(u,v)` that computes the cross product of vectors v and w.
b. Execute the function with two vectors. In each case, compare the result with the MATLAB function `cross`.

6.43 The inner product, or tensor product, is defined in Problem 6.34.
a. Write a function `t = tensor(u,v)` that computes the inner product of $m \times 1$ vector u and $n \times 1$ vector v.
b. Test the function for two pairs of vectors, one pair giving a 5×5 matrix and another giving a 6×4 matrix.