Chapter 7

Vector and Matrix Norms

You should be familiar with

- Basic two- and three-dimensional geometry
- Vector and matrix properties
- The inner product
- Partial derivatives (for Section 7.1.2)
- Eigenvalues
- Symmetric matrices
- Orthogonal matrices

Chapters 1–6 provide sufficient background for the remainder of this book. This chapter actually begins our study of numerical linear algebra that, as we have stated, is very different from theoretical linear algebra. This is because we use a computer and are concerned with numerical accuracy and execution time. Since we are dealing with a numerical subject, it is natural to assume there must be a means of measuring the length of a vector and the "size" of a matrix. In each case, we define what is termed a *norm*. Vector norms have applications in many areas, including signal processing, quantum information theory, measuring deflections, and determining convergence of sequences of vectors. We studied the solution of square linear algebraic systems in Chapter 2. In some cases, the coefficient matrix is sensitive to changes in data; for instance, if there are small changes to the vector b in the system $Ax = b$ due to experimental error, the solution may differ widely, leading to incorrect results. In such a case, the matrix is said to be *ill-conditioned*. The matrix norm plays a critical role in determining if a matrix is ill-conditioned. In addition, there are many applications of matrix norms to specific disciplines such as structural analysis and input-output response in electrical engineering problems.

We begin with a definition of a vector norm and develop some examples of vector norms. These norms are important in their own right, and we will see that some frequently used matrix norms are derived from a vector norm.

7.1 VECTOR NORMS

A vector norm gives us a way of measuring vector length. You are already familiar with the most-used vector norm, the formula for the length of a vector u in $\mathbb{R}^n$. In Chapter 6, we used the notation $\|u\|_2$ for the length of a vector, where

$$\|u\|_2 = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}. \tag{7.1}$$

Figure 6.4 graphically shows why this function computes the length of a three-dimensional vector. Let's examine some properties of this length function.

- Since $u_i^2 \geq 0$, $1 \leq i \leq n$, $\|u_2\| = 0$ if and only if $u = 0$.
- If α is a scalar,

$$\begin{aligned}\|\alpha u\|_2 &= \sqrt{(\alpha u_1)^2 + (\alpha u_2)^2 + \cdots + (\alpha u_n)^2} \\ &= \sqrt{\alpha^2 \left[u_1^2 + u_2^2 + \cdots + u_n^2\right]} = |\alpha| \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}\end{aligned}$$

- In Chapter 6, we developed the triangle inequality for vectors x, y in $\mathbb{R}^n$, which states that $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$.

Any function that takes a vector argument, computes a real number, and satisfies these three conditions is called a vector norm, so $\|u\|_2 = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$ is our first vector norm.

Numerical Linear Algebra with Applications. http://dx.doi.org/10.1016/B978-0-12-394435-1.00007-7

Definition 7.1. A function $\|\cdot\|:\mathbb{R}^n \to \mathbb{R}$ is a *norm* provided:

1. $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$; $\|x\| = 0$ if and only if $x = 0$ (*positivity*);
2. $\|\alpha x\| = |\alpha| \, \|x\|$ for all $\alpha \in \mathbb{R}$ (*scaling*);
3. $\|x + y\| \leq \| x \| + \|y\|$ for all $x,\ y \in \mathbb{R}^n$ (*triangle inequality*).

In this book, the only vector norms we will use are the *p-norms* defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p},$$

for $p = 1, 2, \ldots$ The values $p = 1, 2$, and ∞ are the most commonly used norms. Equation 7.1 corresponds to $p = 2$ and we will refer to it as the *Euclidean norm* or the *2-norm*, and indicate this using the notation

$$\|u\|_2 = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = \sqrt{u^{\mathrm{T}} u}.$$

For $p = 1$ and ∞, the norms are

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$
$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$$

Problem 7.9 justifies the formula for $\|x\|_\infty$.

Example 7.1. Let $u = \begin{bmatrix} -1 \\ -9 \\ 2 \end{bmatrix}$.

$$\|u\|_1 = |-1| + |-9| + |2| = 12,$$
$$\|u\|_2 = \sqrt{(-1)^2 + (-9)^2 + (2)^2} = \sqrt{86} = 9.2736,$$
$$\|u\|_\infty = \max\{|-1|,\ |-9|,\ |2|\} = 9,$$
$$\|u\|_5 = \left((|-1|)^5 + (|-9|)^5 + (|2|)^5\right)^{1/5} = 9.0010.$$ ■

We have already shown that $\|.\|_2$ is a norm, but we should not take for granted that the 1- and ∞-norms satisfy the three requirements for a norm. Theorem 7.1 shows that $\|\cdot\|_\infty$ satisfies the required properties. Showing that $\|.\|_1$ is a norm is left to the exercises.

Theorem 7.1. *$\|.\|_\infty$ is a norm.*

Proof. *Positivity*: Clearly, $\|x\|_\infty \geq 0$ for all $x \neq 0$. If

$$\|x\|_\infty = \max_{1 \leq i \leq} |x_i| = 0,$$

then $x = 0$. If $x = 0$, then all its components are 0.

Scaling:

$$\|\alpha x\|_\infty = \max_{1\leq i\leq n} |\alpha x_i| = |\alpha| \max_{1\leq i\leq n} |x_i| = |\alpha| \, \|x\|_\infty .$$

Triangle inequality:

$$\|x + y\|_\infty = \max_{1\leq i\leq n} |x_i + y_i| \leq \max_{1\leq i\leq n} (|x_i| + |y_i|) = \max_{1\leq i\leq n} |x_i| + \max_{1\leq i\leq n} |y_i| = \|x\|_\infty + \|y\|_\infty .$$ □

Which norm to use can depend on the application. Also, all three norms are equivalent, which means that each norm is bounded below and above by a multiple of one of the other norms.

Lemma 7.1.

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty\,,$$
$$\|x\|_\infty \le \|x\|_1 \le n\,\|x\|_\infty\,,$$
$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2\,.$$

Proof. We will prove the first of the three inequalities. Proofs of the remaining ones are left to the exercises.

Assume that the maximum absolute value of the vector components occurs at index i, so $|x_i| = \|x\|_\infty$. Now, $\|x\|_\infty = \sqrt{x_i^2} \le \sqrt{x_1^2 + x_2^2 + \cdots + x_i^2 + \cdots + x_n^2} = \|x\|_2$.

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_i^2 + \cdots + x_n^2} \le \sqrt{n\,(x_i)^2} = \sqrt{n}\,|x_i| = \sqrt{n}\,\|x\|_\infty$$ □

Example 7.2. Let $x = \begin{bmatrix} 1 \\ 4 \\ -9 \end{bmatrix}$. Then,

$\|x\|_\infty = 9$, $\|x\|_1 = 14$, $\|x\|_2 = \sqrt{86}$.

Now test each inequality in Lemma 7.1.

$9 \le \sqrt{86} \le 9\sqrt{3}$, $\sqrt{86} \le 14 \le \sqrt{258}$, $9 \le 14 \le 27$

The 2-norm is more computationally expensive than the ∞- or the 1-norm. If an application requires the computation of a norm many times, it could be advantageous to use the ∞- or the 1-norm. ■

Example 7.3. The MATLAB `norm` command will compute norms of a vector. For instance, if $v = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}$, the following MATLAB statements compute the ∞-norm, the 1-norm, and the 2-norm of v. ■

```
>> norm(v, 1)
ans =
10.0000
>> norm(v, 'inf')
ans =
7
>> norm(v, 2)
ans =
7.3485
```
■

7.1.1 Properties of the 2-Norm

The 2-norm is the norm most frequently used in applications, and there are good reasons why this is true. There are many relationships satisfied by the 2-norm, and one of the most frequently used is the *Cauchy-Schwarz inequality*:

$$|\langle x, y\rangle| = \left|x^{\mathrm{T}} y\right| \le \|x\|_2\,\|y\|_2\,,$$

with equality holding when x and y are collinear (Theorem 6.5). Another relationship involving the Euclidean norm is the *Pythagorean Theorem* for orthogonal x and y,

$$\|x + y\|_2^2 = \|x\|_2^2 + \|y\|_2^2\,.$$

The vector 2-norm enjoys yet another property: it is *orthogonally invariant*. This means that for any $n \times n$ orthogonal matrix P

$$\|Px\|_2 = \|x\|_2$$

for all x in $\mathbb{R}^n$, since

$$\|Px\|_2^2 = (Px)^{\mathrm{T}}\,Px = x^{\mathrm{T}} P^{\mathrm{T}} Px = x^{\mathrm{T}} Ix = x^{\mathrm{T}} x = \|x\|_2^2\,.$$

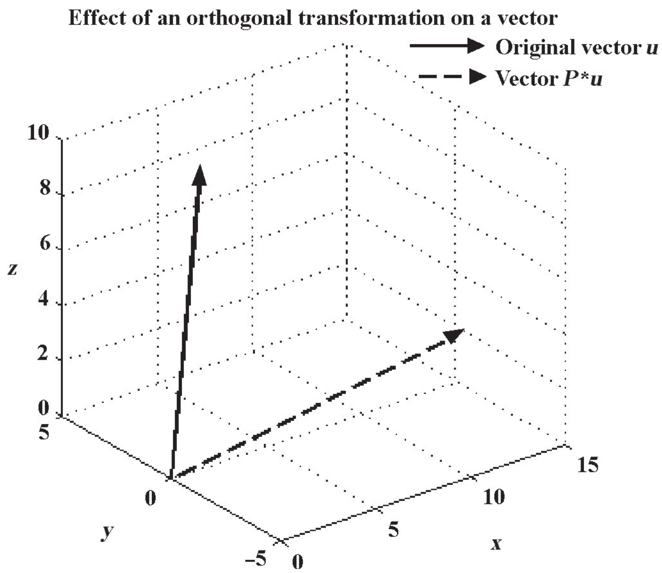

FIGURE 7.1 Effect of an orthogonal transformation on a vector.

In other words, multiplication of a vector, x, by an orthogonal matrix will likely rotate x, but the resulting vector Px has the same length. This is one of the reasons that orthogonal matrices are useful in computer graphics. Figure 7.1 shows the result of applying the orthogonal transformation $P = \begin{bmatrix} 0.4082 & 0.5774 & 0.7071 \\ 0.4082 & 0.5774 & -0.7071 \\ 0.8165 & -0.5774 & 0 \end{bmatrix}$ to $u = \begin{bmatrix} 8 \\ 5 \\ 12 \end{bmatrix}$ producing the vector $v = Pu = \begin{bmatrix} 14.6380 \\ -2.3325 \\ 3.6452 \end{bmatrix}$. Note that $\|u\|_2 = \sqrt{8^2+5^2+12^2} = 15.2643$, and $\|Pu\|_2 = \sqrt{(14.6380)^2+(-2.3325)^2+(3.6452)^2} = 15.2643$.

Theorem 7.2 gives us another important property. A set of orthogonal vectors is a basis for the subspace spanned by those vectors.

Theorem 7.2. *If the nonzero vectors $u_1, u_2, \ldots, u_k$ in $\mathbb{R}^n$ are orthogonal, they form a basis for a k-dimensional subspace of $\mathbb{R}^n$.*

Proof. Let

$$c_1u_1 + c_2u_2 + \cdots + c_{i-1}u_{i-1} + c_iu_i + c_{i+1}u_{i+1} + \cdots + c_ku_k = 0.$$

Choose any i, $1 \le i \le k$. Then,

$$\begin{aligned} &c_1\langle u_1, u_i\rangle + c_2\langle u_2, u_i\rangle + \ldots + c_{i-1}\langle u_{i-1}, u_i\rangle + c_i\langle u_i, u_i\rangle + c_{i+1}\langle u_{i+1}, u_i\rangle + \ldots + c_k\langle u_k, u_i\rangle \\ &= c_1(0) + c_2(0) + \cdots + c_{i-1}(0) + c_i(1) + c_{i+1}(0) + \cdots + c_k(0) = c_i = 0. \end{aligned}$$

Since $c_i = 0$, $1 \le i \le k$, the u_i are linearly independent, and thus are a basis. □

If the vectors in a basis $u_1, u_2, \ldots, u_n$ are mutually orthogonal and each vector has unit length $\left(\langle u_i, u_j\rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}\right)$, we say the basis is *orthonormal*. We proved in Theorem 6.2 that if P is an $n \times n$ real matrix, then P is an orthogonal matrix if and only if the columns of P are orthogonal and have unit length. It follows from Theorem 7.2 that the columns of an $n \times n$ orthogonal matrix P are an orthonormal basis for $\mathbb{R}^n$. This fact has a natural physical interpretation. The vector

$$Ux = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i u_i$$

is a representation of the vector x in the coordinate system whose axes are given by $u_1, u_2, \dots, u_n$. The statement $\|Ux\|_2 = \|x\|_2$ simply means "the length of x does not change when we convert from the standard orthonormal basis $e_1 = [\ 1\ \ 0\ \ \dots\ \ 0\ \ 0\ ]^{\mathrm{T}}$, $e_2 = [\ 0\ \ 1\ \ \dots\ \ 0\ \ 0\ ]^{\mathrm{T}}, \dots, e_n = [\ 0\ \ 0\ \ \dots\ \ 0\ \ 1\ ]^{\mathrm{T}}$ to the new orthonormal basis $u_1,\ u_2, \dots, u_n$."

7.1.2 Spherical Coordinates

A very good example of this change of basis is the spherical coordinate system used in geography, astronomy, three-dimensional computer games, vibration problems, and many other areas. The representation for a point in space is given by three coordinates $(r,\ \theta,\ \phi)$. Fix a point O in space, called the origin, and construct the usual standard basis $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ centered at O. The r coordinate of a point P is the length of the line segment from O to P, θ is the angle between the direction of vector k and P, and ϕ is the angle between the i direction and the projection of $\overrightarrow{OP}$ onto the ij plane. In order for coordinates to be unique we require, $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$ (Figure 7.2). The name spherical coordinates comes from the fact that the equation of a sphere in this coordinate system is simply $r = a$, where a is the radius of the sphere. An application of trigonometry shows that rectangular coordinates are obtained from spherical coordinates $(r,\ \theta,\ \phi)$ as follows:

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta. \end{aligned}$$

The position vector for a point, P, in space is $\overrightarrow{P} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Now write the vector with x, y, and z replaced by their equivalents in spherical coordinates.

$$\overrightarrow{P} = r \sin\theta \cos\phi \mathbf{i} + r \sin\theta \sin\phi \mathbf{j} + r \cos\theta \mathbf{k}.$$

Our aim is to develop a basis in spherical coordinates. Such a basis must have a unit vector $\mathbf{e_r}$ in the direction of r, $\mathbf{e}_\theta$ in the direction of θ, and $\mathbf{e}_\phi$ in the direction of ϕ (Figure 7.3) such that the position vector

$$\overrightarrow{P} = r\mathbf{e_r} + \theta\mathbf{e}_\theta + \phi\mathbf{e}_\phi.$$

The vectors $\mathbf{e_r}$, $\mathbf{e}_\theta, \mathbf{e}_\phi$ change direction as the point P moves. If θ and ϕ are fixed and we increase r, $\mathbf{e_r}$ is a unit vector in the direction of change in r. This means we take the partial derivative.

$$\frac{\partial \overrightarrow{P}}{\partial r} = \frac{\partial \left(r \sin\theta \cos\phi \mathbf{i} + r \sin\theta \sin\phi \mathbf{j} + r \cos\theta \mathbf{k} \right)}{\partial r}$$

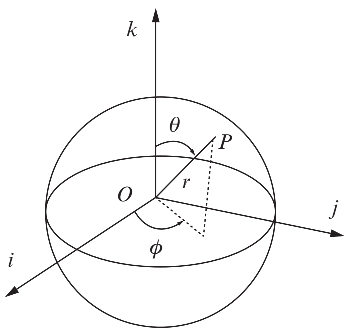

FIGURE 7.2 Spherical coordinates.

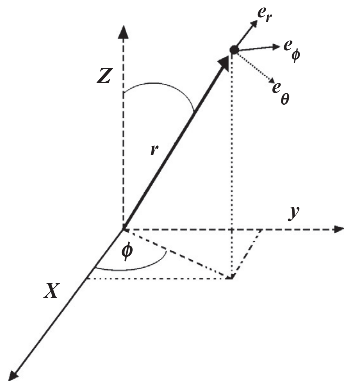

FIGURE 7.3 Orthonormal basis for spherical coordinates.

$$
\begin{aligned}
&= \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k} \\
&= \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix},
\end{aligned}
$$

and

$$\mathbf{e_r} = \frac{\frac{\partial \overrightarrow{\mathbf{P}}}{\partial \mathbf{r}}}{\left\| \frac{\partial \overrightarrow{P}}{\partial \mathbf{r}} \right\|}.$$

Similarly,

$$\mathbf{e}_\theta = \frac{\frac{\partial \overrightarrow{\mathbf{P}}}{\partial \theta}}{\left\| \frac{\partial \overrightarrow{\mathbf{P}}}{\partial \theta} \right\|}, \quad \mathbf{e}_\phi = \frac{\frac{\partial \overrightarrow{\mathbf{P}}}{\partial \phi}}{\left\| \frac{\partial \overrightarrow{\mathbf{P}}}{\partial \phi} \right\|}.$$

After performing the differentiation and division, the result is

$$e_r = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}, \quad e_\theta = \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix}, \quad e_\phi = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}.$$

Now,

$$\overrightarrow{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = r\mathbf{e_r} + \theta\mathbf{e}_\theta + \phi\mathbf{e}_\phi = r\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} + \theta\begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix} + \phi\begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix},$$

and

$$
\begin{aligned}
x &= (\sin\theta\cos\phi)\, r + (\cos\theta\cos\phi)\, \theta - (\sin\phi)\, \phi, \\
y &= (\sin\theta\sin\phi)\, r + (\cos\theta\sin\phi)\, \theta + (\cos\phi)\, \phi, \\
z &= (\cos\theta)\, r - (\sin\theta)\, \theta.
\end{aligned}
$$

In matrix form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix} = P\begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}. \tag{7.2}$$

The matrix P is orthogonal, as we can see by applying simple trigonometry, so $P^{-1} = P^{\mathrm{T}}$ and

$$\begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

Furthermore, $\{\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ is an orthonormal basis. It must be noted that the basis is a local basis, since the basis vectors change. Applications include the analysis of vibrating membranes, rotational motion, and the Schrodinger equation for the hydrogen atom. In Equation 7.2, the xyz-coordinate system is fixed, but the $r\theta\phi$-coordinate system moves. If we choose $r = 1,\ \theta = \pi/4, \phi = \pi/4$, the orthogonal matrix in Equation 7.2 is

$$P = \begin{bmatrix} 0.5000 & 0.5000 & -0.7071 \\ 0.5000 & 0.5000 & 0.7071 \\ 0.7071 & -0.7071 & 0 \end{bmatrix},$$

and

$$P\begin{bmatrix} 1 \\ \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} 0.33734 \\ 1.4481 \\ 0.15175 \end{bmatrix}.$$

This says that in the coordinate system defined by the spherical basis with $r = 1,\ \theta = \pi/4, \phi = \pi/4$, the vector $\begin{bmatrix} 1 \\ \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix}$ corresponds to the vector $\begin{bmatrix} 0.33734 \\ 1.4481 \\ 0.15175 \end{bmatrix}$ in the Cartesian coordinate system (Figure 7.4).

```
P =

         0.5           0.5      -0.70711
         0.5           0.5       0.70711
     0.70711      -0.70711             0

>> P*[1 theta phi]'

ans =

     0.33734
      1.4481
     0.15175
```

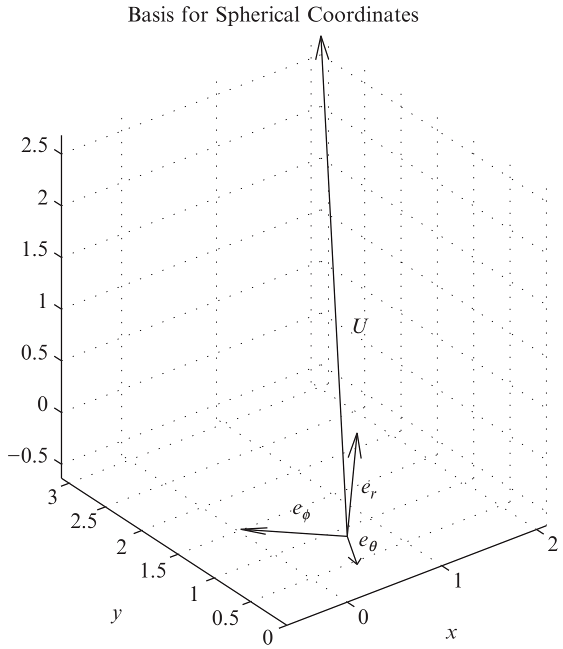

FIGURE 7.4 Point in spherical coordinate basis and Cartesian coordinates.

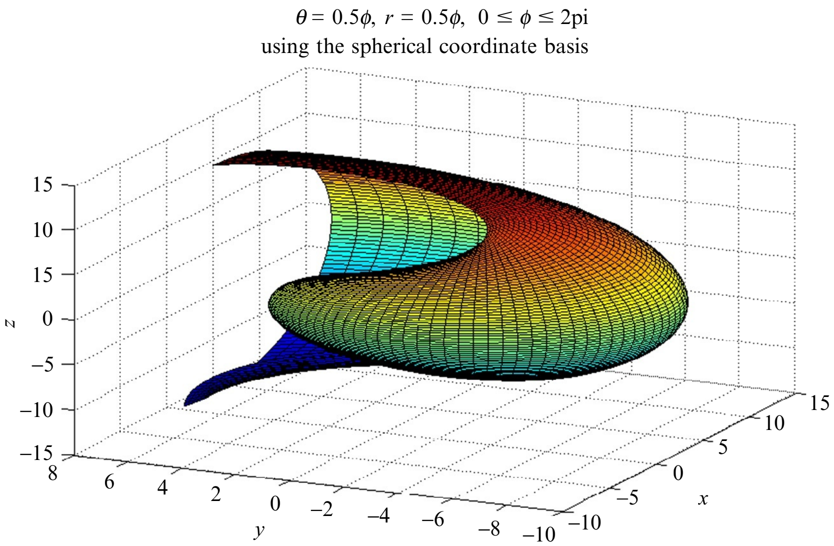

FIGURE 7.5 Function specified in spherical coordinates.

Example 7.4. Apply the change of basis from spherical to Cartesian coordinates to graph the surface formed by the equations $\theta = \frac{1}{2}\phi$, $r = 2\phi$, $0 \le \phi \le 2\pi$. Unfortunately, there is no fixed standard for spherical coordinates. We have defined spherical coordinates as commonly used in physics. MATLAB switches the roles of θ and ϕ; furthermore, ϕ is the angle between the projection of $\overrightarrow{OP}$ onto the xy-plane and OP. As a result, $-(\pi/2) \le \phi \le \pi/2$. The book software distribution contains a MATLAB function

```
[x y z] = sph2rect(r, theta, phi)
```

that uses our definition of spherical coordinates. It takes `r`, `theta`, `phi` in the local basis and returns Cartesian coordinates suitable for graphing a function using the MATLAB `surf` or `mesh` functions. The resulting graph is shown in Figure 7.5.

```
>> phi = linspace(0,2*pi);
>> theta = 0.5*phi;
>> [theta phi] = meshgrid(theta, phi);
>> r = 2*phi;
>> [x y z] = sph2rect(r, theta, phi);
>> surf(x,y,z);
```

■

7.2 MATRIX NORMS

We used vector norms to measure the length of a vector, and we will develop matrix norms to measure the size of a matrix. The size of a matrix is used in determining whether the solution, x, of a linear system $Ax = b$ can be trusted, and determining the convergence rate of a vector sequence, among other things. We define a matrix norm in the same way we defined a vector norm.

Definition 7.2. A function $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$ is a *matrix norm* provided:

1. $\|A\| \ge 0$ for all $A \in \mathbb{R}^{m\times n}$; $\|A\| = 0$ if and only if $A = 0$ (*positivity*);
2. $\|\alpha A\| = |\alpha|\,\|A\|$ for all $\alpha \in \mathbb{R}$ (*scaling*);
3. $\|A + B\| \le \| A \| + \|B\|$ for all $A, B \in \mathbb{R}^{m\times n}$ (*triangle inequality*)

7.2.1 The Frobenius Matrix Norm

One of the oldest and simplest matrix norms is the *Frobenius norm*, sometimes called the *Hilbert-Schmidt norm*. It is defined as the square root of the sum of the squares of all the matrix entries, or

$$\|A\|_{\mathrm{F}} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}.$$

Clearly, it measure the "size" of matrix A. A matrix with small (large) entries will have a small (large) Frobenius norm, but we need to prove it is actually a matrix norm.

Theorem 7.3. *$\|\cdot\|_{\mathrm{F}}$ is a matrix norm.*

Proof. *Positivity*: Clearly, $\|A\|_{\mathrm{F}} \geq 0$, and $\|A\|_{\mathrm{F}} = 0$ if and only if $A = 0$.

Scaling: $\|\alpha A\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left(\alpha a_{ij}\right)^2 \right)^{1/2} = \left(\alpha^2 \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} = |\alpha| \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} = |\alpha| \, \|A\|_{\mathrm{F}}$.

Triangle inequality: Consider the $n \times n$ matrix A to be a vector in $\mathbb{R}^{n^2}$ by forming the column vector $v_A = \left[\begin{array}{cccccccccc} a_{11} & \ldots & a_{m1} & a_{12} & \ldots & a_{m2} & \ldots & a_{1n} & \ldots & a_{mn} \end{array} \right]^{\mathrm{T}}$. Similarly, form the vector v_B from matrix B. Then,

$$\|A + B\|_{\mathrm{F}} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} + b_{ij}\right)^2 \right)^{1/2} = \|v_A + v_B\|_2 \leq \|v_A\| + \|v_B\| = \|A\|_{\mathrm{F}} + \|B\|_{\mathrm{F}}$$

by applying the triangle inequality to the vectors v_A and v_B. □

Example 7.5. If $A = \begin{bmatrix} -1 & 2 & 5 \\ -1 & 2 & 7 \\ 23 & 4 & 12 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -6 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, then

$\|A\|_{\mathrm{F}} = \sqrt{(-1)^2 + 2^2 + 5^2 + \cdots + 12^2} = 27.8029$, $\|B\|_{\mathrm{F}} = 7.5498$, and

$\|A + B\|_{\mathrm{F}} = \left\| \begin{bmatrix} 0 & 3 & 5 \\ 1 & -4 & 10 \\ 24 & 5 & 14 \end{bmatrix} \right\|_{\mathrm{F}} = 30.7896$. Note that $30.7896 < 27.8029 + 7.5498 = 35.3527$ as expected by the triangle inequality. ■

7.2.2 Induced Matrix Norms

The most frequently used class of norms are the *induced matrix norms*, that are defined in terms of a vector norm.

Definition 7.3. Assume $\|\cdot\|$ is a vector norm, A is an $m \times n$ matrix, and x an $n \times 1$ vector. Then the matrix norm of A induced by $\|\cdot\|$ is

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}. \tag{7.3}$$

The definition measures the norm of a matrix by finding the largest size of Ax relative to x. If $\|Ax\|$ becomes large for a particular range of vectors, x, that do not have large norms, then $\|A\|$ will be large. An induced matrix norm measures the maximum amount the matrix product Ax can stretch (or shrink) a vector relative to the vector's original length. Figure 7.6 illustrates the effect of a matrix on two vectors in $\mathbb{R}^2$.

Remark 7.1. We use the notation $\|A\|_p$ to denote that the norm of A is derived from the p-norm in Equation 7.3.
An induced matrix norm is often called a subordinate matrix norm.

We have commented that orthogonal matrices are beautiful things, and when it comes to their norms, they do not disappoint.

Lemma 7.2. *If P is an orthogonal matrix, then $\|P\|_2 = 1$.*

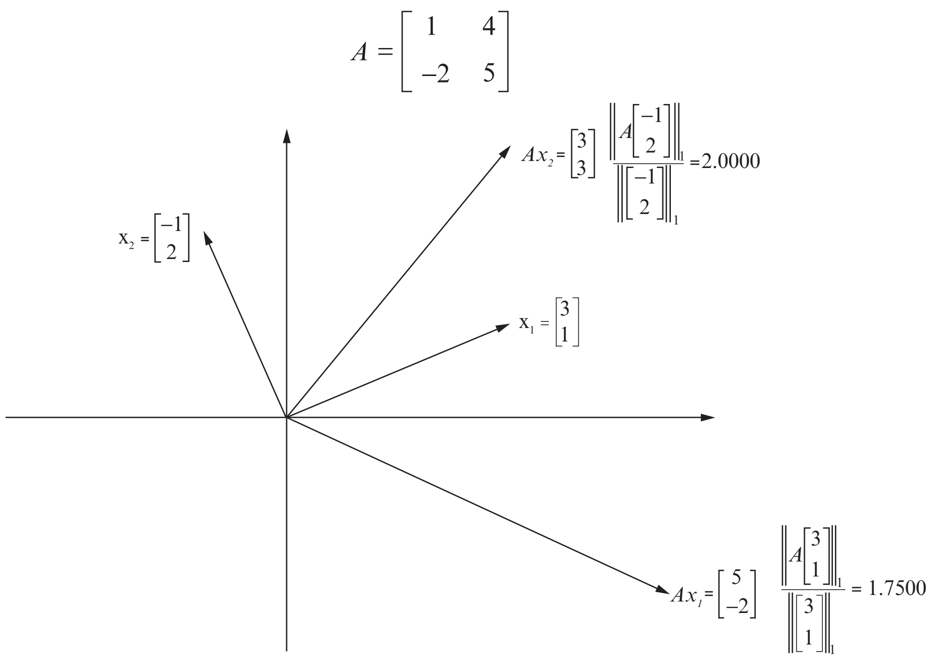

FIGURE 7.6 Effect of a matrix on vectors.

Proof. An orthogonal matrix maintains the 2-norm of x when forming Px, so $\|Px\|_2 = \|x\|_2$. It follows that

$$\|P\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \frac{\|x\|_2}{\|x\|_2} = 1. \qquad \square$$

Not all matrix norms are induced. The *Frobenius norm* is not induced by any vector norm (Problem 7.10).

Another, perhaps easier, way to understand the concept of an induced matrix norm is to use the scaling property of a vector norm as follows:

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{x \neq 0} \left\| \left(\frac{1}{\|x\|} \right) Ax \right\| = \max_{x \neq 0} \left\| A \left(\frac{x}{\|x\|} \right) \right\| = \max_{\|x\|=1} \|Ax\| . \tag{7.4}$$

Equation 7.4 says that to compute induced norm $\|A\|$, find the maximum value of $\|Ax\|$, where x ranges over the unit sphere $\|x\| = 1$. It is helpful to view the *unit* sphere *of a norm*, which is possible for $\mathbb{R}^2$ and $\mathbb{R}^3$. For vectors in $\mathbb{R}^2$, the unit spheres for the ∞-, 1-, and 2-norm have equations

$$\begin{aligned} -1 \leq x \leq 1,\ |y| = 1, \quad -1 \leq y \leq 1,\ |x| = 1, \\ \|(x, y)\|_1 = |x| + |y| = 1, \\ \|(x, y)\|_2 = \sqrt{x^2 + y^2} = 1. \end{aligned}$$

Figure 7.7 shows a graph of all three unit spheres.

You may not have seen a definitions like Equations 7.3 and 7.4. For instance, if $A = \begin{bmatrix} 1 & -8 \\ -1 & 3 \end{bmatrix}$, then

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \left[\max \left(|x_1 - 8x_2| , |-x_1 + 3x_2| \right) \right].$$

Computing this value seems complex. Let's run a numerical experiment so we can make an educated guess for the value of the induced infinity matrix norm. We need x to vary over the unit sphere, which is the set of points (x, y) in the plane such that $\max\{|x| , |y|\} = 1$. This is a square (Figure 7.7). Example 7.6 estimates$\|Ax\|_\infty$ using the function `approxinfnorm(A)`. The function generates 2500 random vectors on each side of the square and finds the maximum value of $\|Ax\|_\infty$ among the 10,000 values. Following the estimation of $\|A\|_\infty$, the example applies the function to the matrix $B = \begin{bmatrix} 0.25 & -0.75 \\ -0.75 & 0.30 \end{bmatrix}$.

Example 7.6. Experimentally estimate $\|\cdot\|_\infty$ for two matrices.

```
function max = approxinfnorm(A)
%APPROXINFNORM Generate 10,000 random values on the unit circle for the
%infinity norm in the plane. Return the maximum value of norm(A*x, 'inf').
%input  : Matrix A
%output : real value max
   max = 0.0;
   for i = 1:10000
      r = 1 - 2*rand;
      if i <= 2500
         x = [1.0 r]';
      elseif i <= 5000
         x = [r, 1.0]';
      elseif i <= 7500
         x = [-1.0 r]';
      else
         x = [r -1.0]';
      end

      bvalue = norm(A*x,'inf');
      if bvalue > max
         max = bvalue;
      end
   end
```

```
>> approxinfnorm(A)

ans =
    9.0000

>> approxinfnorm(A)

ans =
    8.9998

>> approxinfnorm(B)

ans =
    1.0500

>> approxinfnorm(B)

ans =
    1.0497
```

■

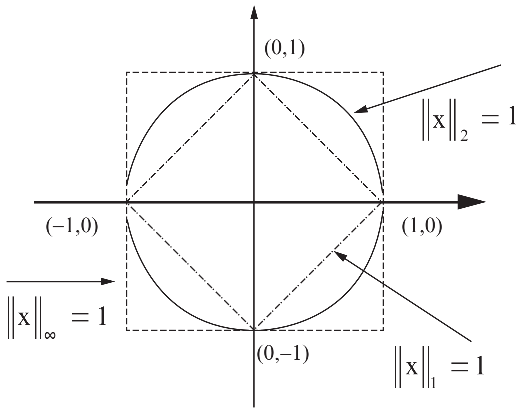

FIGURE 7.7 Unit spheres in three norms.

Notice that in Example 7.6, the experimental results using matrix A indicate an actual value of 9.0, which is the sum of the absolute value of the entries in row 1. For matrix B, the experiment indicates that the norm is the sum of the absolute values of the entries in row 2. In fact, here is the way to compute$\|A\|_\infty$ for any $m \times n$ matrix. Find the sum of the absolute values of the elements in each row of A and take $\|A\|_\infty$ be the maximum of these sums; in other words,

$$\|A\|_\infty = \max_{1 \le k \le n} \sum_{j=1}^{n} \left| a_{kj} \right|.$$

How do we get from the definition of the infinity-induced matrix norm

$$\max_{\|x\|=1} \|Ax\|_\infty$$

to this simple expression? Theorem 7.4 answers that question. The proof is somewhat technical, and is here for the interested reader.

Theorem 7.4. *If A is an $m \times n$ matrix,*

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} \left| a_{ij} \right|.$$

Proof. We can assume $A \neq 0$, since the result is certainly true for $A = 0$.

$$Ax = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m-1,1}x_1 + \cdots + a_{n-1,n}x_n \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix},$$

and

$$\|Ax\|_\infty = \max_{1 \le i \le m} \left| \sum_{j=1}^{n} a_{ij}x_j \right|.$$

Using the fact that $|x_i| \le \|x\|_\infty$, $1 \le i \le n$ for vector x, it follows that

$$\|Ax\|_\infty \le \max_{1 \le i \le m} \sum_{j=1}^{n} \left| a_{ij} \right| \left| x_j \right| \le \|x\|_\infty \max_{1 \le i \le m} \sum_{j=1}^{n} \left| a_{ij} \right|. \tag{7.5}$$

By the definition of an induced matrix norm and Equation 7.5,

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty \le \max_{1 \le i \le m} \sum_{j=1}^{n} \left| a_{ij} \right|. \tag{7.6}$$

Let $i_{\max}$ be the row index that gives the maximum sum in Equation 7.6 so that

$$\max_{1 \le i \le m} \sum_{j=1}^{n} \left| a_{ij} \right| \le \sum_{j=1}^{n} \left| a_{i_{\max},j} \right|. \tag{7.7}$$

Let vector $x^{\max} \in \mathbb{R}^n$ be defined by $x_j^{\max} = 0$ if $a_{i_{\max},j} = 0$ and $x_j^{\max} = a_{i_{\max},j} / \left| a_{i_{\max},j} \right|$ if $a_{i_{\max},j} \neq 0$. Since $A \neq 0$, $x^{\max} \neq 0$, and $\|x^{\max}\|_\infty = 1$. Furthermore, using Equation 7.7 we have

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty \ge \left| \sum_{j=1}^{n} a_{i_{\max},j} x_j^{\max} \right| = \sum_{j=1}^{n} \frac{\left(a_{i_{\max},j}\right)^2}{\left| a_{i_{max},j} \right|} = \sum_{j=1}^{n} \left| a_{i_{\max},j} \right| \ge \max_{1 \le i \le m} \sum_{j=1}^{n} \left| a_{ij} \right|. \tag{7.8}$$

Now, Equation 7.6 says

$$\|A\|_\infty \le \max_{1 \le i \le m} \sum_{j=1}^{n} \left| a_{ij} \right|,$$

and Equation 7.8 says

$$\|A\|_\infty \geq \max_{1\leq i\leq m} \sum_{j=1}^{n} \left|a_{ij}\right|,$$

and so

$$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} \left|a_{ij}\right|.$$

The same type of argument (Problem 7.5) shows that the 1-norm is the maximum absolute column sum, or

$$\|A\|_1 = \max_{1\leq k\leq n} \sum_{i=1}^{m} |a_{ik}|. \qquad \square$$

Example 7.7. If $A = \begin{bmatrix} -2 & 1 & -8 & 1 \\ 0 & -4 & -21 & 18 \\ -33 & 16 & -6 & 20 \\ 14 & -20 & -18 & 5 \\ 8 & -1 & 12 & 16 \end{bmatrix}$, then

$$\|A\|_\infty = 33 + 16 + 6 + 20 = 75, \quad \|A\|_1 = 8 + 21 + 6 + 18 + 12 = 65. \qquad \blacksquare$$

MATLAB computes matrix norms using the same command, `norm`, that it uses for a vector. For a matrix, in addition to the ∞-, 1-, and 2-norms, the Frobenius norm is available. We apply those norms to the matrix of Example 7.7.

```
>> norm(A,'inf')

ans =
     75

>> norm(A,1)

ans =
     65

>> norm(A,'fro')

ans =
   63.5767
```

Remark 7.2. From the definition of an induced norm, $(\|Ax\|/\|x\|) \leq \|A\|$, and so

$$\|Ax\| \leq \|A\| \, \|x\| \tag{7.9}$$

We will have occasion to use Equation 7.9 numerous times throughout the remainder of the book.

7.3 SUBMULTIPLICATIVE MATRIX NORMS

Example 7.8. Let $A = \begin{bmatrix} -1 & 2 & 5 \\ -1 & 2 & 7 \\ 23 & 4 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -6 & 3 \\ 1 & 1 & 2 \end{bmatrix}$. The product $AB = \begin{bmatrix} 8 & -8 & 16 \\ 10 & -6 & 20 \\ 43 & 11 & 36 \end{bmatrix}$, and its Frobenius norm is $\|AB\|_F = 64.6993$. The product of the Frobenius norms is $\|A\|_F \, \|B\|_F = 209.9071$, and so

$$\|AB\|_F \leq \|A\|_F \, \|B\|_F. \qquad \blacksquare$$

The inequality $\|AB\|_F \leq \|A\|_F \, \|B\|_F$ is not a coincidence. Our first matrix norm, the Frobenius norm is sub-multiplicative.

Definition 7.4. If the matrix norm $\|.\|$ satisfies $\|AB\| \leq \|A\| \, \|B\|$ for all matrices A and B in $\mathbb{R}^{n\times n}$, it said to be *sub-multiplicative*.

Theorem 7.5. *The Frobenius matrix norm is sub-multiplicative.*

Proof. Let $C = AB$. Then, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, and

$$\|AB\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{i,n-1} \\ a_{in} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{n-1,j} \\ b_{nj} \end{bmatrix}\right)^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left\| \begin{matrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{i,n-1} \\ a_{in} \end{matrix} \right\|_2 \left\| \begin{matrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{n-1,j} \\ b_{nj} \end{matrix} \right\|_2\right)^2 \tag{7.10}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\sum_{k=1}^{n} a_{ik}^2\right)\left(\sum_{k=1}^{n} b_{kj}^2\right)\right] = \|A\|_F^2 \, \|B\|_F^2 . \tag{7.11}$$

We used the Cauchy-Schwarz inequality in Equation 7.10. Verifying Equation 7.11 is left to the exercises. □

The induced matrix norms are sub-multiplicative:

$$\|AB\| = \max_{x \ne 0} \frac{\|ABx\|}{\|x\|} = \max_{x \ne 0} \frac{\|A\,(Bx)\|}{\|Bx\|}\frac{\|Bx\|}{\|x\|} \le \left(\max_{x \ne 0} \frac{\|Ax\|}{\|x\|}\right)\left(\max_{x \ne 0} \frac{\|Bx\|}{\|x\|}\right) = \|A\|\,\|B\|$$

There exist norms that satisfy the three basic matrix norm axioms, but are not submultiplicative; for instance,

$$\|A\| = \max_{1 \le i,j \le n} \left|a_{ij}\right|$$

satisfies the positivity, scaling, and triangle inequality properties, but is not sub-multiplicative (see Exercise 7.6).

7.4 COMPUTING THE MATRIX 2-NORM

The most useful norm for many applications is the induced matrix 2-norm (often called the *spectral norm*):

$$\|A\|_2 = \max_{x \ne 0} \frac{\|Ax\|_2}{\|x\|_2}$$

or

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\| .$$

It would seem reasonable that this norm would be more difficult to find, since

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

is a more complex calculation than $\|x\|_\infty$ or $\|x\|_1$. In fact, it is a nonlinear optimization problem with constraints. In $\mathbb{R}^2$, $\|A\|_2$ is the maximum value of $\|Ax\|_2$ for x on the unit circle. As you can see in Figure 7.8, the image of the matrix $A = \begin{bmatrix} 1 & -8 \\ -1 & 3 \end{bmatrix}$ as x varies over the unit circle is an ellipse. The problem is to find the largest value of $\|Ax\|_2$ on this ellipse.

Let's examine Figure 7.8 in more detail. A semi-major axis of the ellipse is the longest line from the center to a point on the ellipse, and the length of the semi-major axis for our ellipse is 8.6409. Create `A` with MATLAB and use the `norm` command to compute its 2-norm.

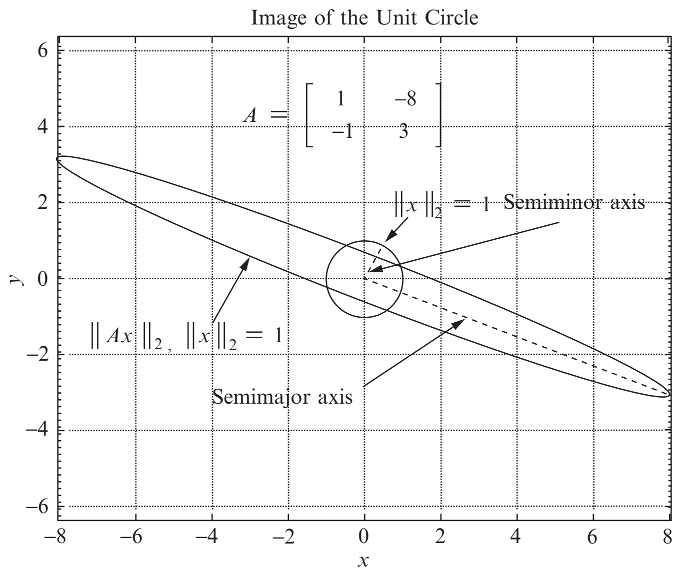

FIGURE 7.8 Image of the unit circle.

```
>> A = [1 -8;-1 3];
>> norm(A)

ans =
    8.6409
```

This is not a coincidence. We will show in Chapter 15 that the norm of a matrix is the length of a semi-major axis of an ellipsoid formed from the image of the unit sphere in k-dimensional space, $k \leq m$.

If A is an $m \times n$ matrix, $A^{\mathrm{T}}A$ is of size $n \times n$ and since $\left(A^{\mathrm{T}}A\right)^{\mathrm{T}} = A^{\mathrm{T}}A$, it is also symmetric. The method for computing $\|A\|_2$ exploits properties of $A^{\mathrm{T}}A$. The following is a summary of the process, followed by an example.

- The eigenvalues of a symmetric matrix are real.
- $A^{\mathrm{T}}A$ is symmetric, so it has real eigenvalues. Furthermore, it can be shown that the eigenvalues of $A^{\mathrm{T}}A$ are nonnegative (≥ 0).
- The square roots of the eigenvalues of $A^{\mathrm{T}}A$ are termed *singular values* of A. The norm of an $m \times n$ matrix, A, is the largest singular value.

Example 7.9. Find the 2-norm of $A = \begin{bmatrix} 1 & 13 & 5 & -9 \\ 12 & 55 & 5 & -6 \\ 18 & 90 & 1 & -1 \\ 3 & 0 & 2 & 3 \end{bmatrix}$ using the MATLAB commands `eig` and `norm`.

```
>> E = eig(A'*A)

E =
  1.0e+004 *

    0.0000
    0.0021
    0.0131
    1.1802

>> sqrt(max(E))

ans =

  108.6373

>> norm(A,2)

ans =
```

```
  108.6373
>> norm(A)  % default without second argument is the 2-norm
ans =
  108.6373
```

■

Remark 7.3. The fact that

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \max\left(\sqrt{s_i}\right),$$

where s_i are the eigenvalues of $A^{\mathrm{T}}A$, is really a theoretical result that will lead to methods for efficiently computing $\|A\|_2$. Computing the norm like we did in Example 7.9 is too slow and prone to errors. Chapter 8 provides justification for this statement.

It is useful to note that

$$\langle Bx,\, y\rangle = (Bx)^{\mathrm{T}}\, y = x^{\mathrm{T}}B^{\mathrm{T}}y = \left\langle x,\, B^{\mathrm{T}}y\right\rangle.$$

This says that you can move matrix B from one side of an inner product to the other by replacing B by B^{T}. Now, if B is symmetric $B = B^{\mathrm{T}}$, and we have

$$\langle Bx,\, y\rangle = \langle x,\, By\rangle\,. \tag{7.12}$$

The remainder of this section mathematically derives the computation of $\|A\|_2$ from the eigenvalues of $A^{\mathrm{T}}A$, and can be skipped if the reader does not need to see the details.

The proof of Lemma 7.3 uses the concept of the conjugate of a complex number and the conjugate transpose of a complex matrix (Definition A.3).

Lemma 7.3. *The eigenvalues of a symmetric matrix are real, and the corresponding eigenvectors can always be assumed to be real.*

Proof. Suppose λ is an eigenvalue of the symmetric matrix A, and u is a corresponding eigenvector. We know that the eigenvalues of an $n \times n$ matrix with real coefficients can be complex and, if so, occur in complex conjugate pairs $a{+}ib$ and a-ib. Since λ might be complex, the vector u may also be a complex vector. Because u is an eigenvector with eigenvalue λ,

$$Au = \lambda u$$

Now take the conjugate transpose of both sides of the latter equation and we have

$$u^*A = \overline{\lambda}u^* \tag{7.13}$$

Multiply 7.13 by u on the right, and

$$\begin{aligned}
\left(u^*A\right)u &= \overline{\lambda}u^*u \\
u^*\left(Au\right) &= \overline{\lambda}u^*u \\
u^*\left(\lambda u\right) &= \overline{\lambda}u^*u \\
\lambda u^*u &= \overline{\lambda}u^*u
\end{aligned} \tag{7.14}$$

From 7.14, there results

$$\left(\lambda - \overline{\lambda}\right)u^*u = 0.$$

Now, $u^*u > 0$, since u is an eigenvector and cannot be 0. It follows that

$$\lambda - \overline{\lambda} = 0,$$

and $\lambda = \overline{\lambda}$ means that λ is real.

This finishes the first portion of the proof. We now need to show that for any eigenvalue λ, there is a corresponding real eigenvector. Assume that u is an eigenvector of A, so

$$Au = \lambda u.$$

If we take the complex conjugate of both sides, we obtain

$$A\overline{u} = \lambda \overline{u}.$$

By adding the two equations, we have

$$A\left(u + \overline{u}\right) = \lambda\left(u + \overline{u}\right).$$

Thus, $u + \overline{u}$ is an eigenvector of A. If $u = x + \mathrm{i}y$, then $u + \overline{u} = (x + \mathrm{i}y) + (x - \mathrm{i}y) = 2x$, which is real. □

Remark 7.4. If A is a symmetric square matrix, all its eigenvalues are real, and A has real eigenvectors; however, this does not mean that A has no complex eigenvectors. For instance, consider the symmetric matrix

$$A = \begin{bmatrix} 1 & -8 \\ -8 & 1 \end{bmatrix}.$$

A has eigenvalues $\lambda = -7$ and $\lambda = 9$. For $\lambda = 9$, $u = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$ is an eigenvector, but so is $(1 - 9\mathrm{i})\begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$. Lemma 7.3 says we can always find a real eigenvector for each real eigenvalue. If A is not symmetric, it may have complex eigenvalue λ, in which case a corresponding eigenvector will be complex.

We are getting closer to deriving the formula for $\|A\|_2$. Because $A^{\mathrm{T}}A$ is an $n \times n$ symmetric matrix, Lemma 7.3 says it has real eigenvalues. The following lemma shows that the eigenvalues of $A^{\mathrm{T}}A$ are in fact always greater than or equal to 0.

Lemma 7.4. *If A is an $m \times n$ real matrix, then the eigenvalues of the $n \times n$ matrix $A^{\mathrm{T}}A$ are nonnegative.*

Proof. The eigenvalues of $A^{\mathrm{T}}A$ are real from Lemma 7.3. Let λ be an eigenvalue of $A^{\mathrm{T}}A$ and $u \neq 0$ be a corresponding eigenvector, so that

$$\left(A^{\mathrm{T}}A\right)u = \lambda u.$$

Take the inner product of this equality with u to obtain

$$\left\langle \left(A^{\mathrm{T}}A\right)u,\, u\right\rangle = \lambda\, \|u\|_2^2$$

from which we arrive at

$$\lambda = \frac{\left\langle \left(A^{\mathrm{T}}A\right)u,\, u\right\rangle}{\|u\|_2^2}.$$

Note that $\left\langle \left(A^{\mathrm{T}}A\right)u,\, u\right\rangle = \left(A^{\mathrm{T}}(Au)\right)^{\mathrm{T}} u = (Au)^{\mathrm{T}}Au = \langle Au,\, Au\rangle$, and so

$$\lambda = \frac{\|Au\|_2^2}{\|u\|_2^2} \geq 0.$$ □

We are almost in a position to compute the 2-norm of a matrix, but first we need to define the singular values of A.

Definition 7.5. The *singular values* $\{\sigma_i\}$ of an $m \times n$ matrix A are the square roots of the eigenvalues of $A^{\mathrm{T}}A$.

Remark 7.5. We can always compute $\sqrt{\lambda}$, where λ is an eigenvalue of $A^{\mathrm{T}}A$ because Lemmas 7.3 and 7.4 guarantee λ is real and nonnegative.

Example 7.10. Let A be the 4×2 matrix $\begin{bmatrix} 2 & 5 \\ 1 & 4 \\ -1 & 6 \\ 7 & 8 \end{bmatrix}$. $A^{\mathrm{T}}A = \begin{bmatrix} 55 & 64 \\ 64 & 141 \end{bmatrix}$, and the eigenvalues of AA^{T} are $\lambda_1 = 175.1$, and $\lambda_2 = 20.896$, so the singular values are $\sigma_1 = 13.233$, $\sigma_2 = 4.571$. ■

Before proving how to compute $\|A\|_2$ in Theorem 7.7, we state a result concerning symmetric matrices called the *spectral theorem* that we will prove in Chapter 19. It says that any real symmetric matrix A can be diagonalized with a real orthogonal matrix.

Theorem 7.6 (Spectral theorem). If A is a real symmetric matrix, there exists an orthogonal matrix P such that

$$D = P^{\mathrm{T}}AP,$$

where D is a diagonal matrix containing the eigenvalues of A, and the columns of P are an orthonormal set of eigenvalues that form a basis for $\mathbb{R}^n$.

We are now in a position to prove how to compute $\|A\|_2$.

Theorem 7.7. *If A is an $m \times n$ matrix, $\|A\|_2$ is the square root of the largest eigenvalue of $A^{\mathrm{T}}A$.*

Proof. The symmetric matrix $A^{\mathrm{T}}A$ is diagonalizable ($D = P^{\mathrm{T}}A^{\mathrm{T}}AP$), and its eigenvalues λ_i are nonnegative real numbers. Let $y = P^{\mathrm{T}}x$, and

$$\begin{aligned}
\|A\|_2^2 &= \max_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \max_{x\neq 0} \frac{(Ax)^{\mathrm{T}}Ax}{\|x\|_2^2} \\
&= \max_{x\neq 0} \frac{x^{\mathrm{T}}A^{\mathrm{T}}Ax}{\|x\|_2^2} = \max_{x\neq 0} \frac{x^{\mathrm{T}}PDP^{\mathrm{T}}x}{\|x\|_2^2} \\
&= \max_{x\neq 0} \frac{\left(P^{\mathrm{T}}x\right)^{\mathrm{T}} D \left(P^{\mathrm{T}}x\right)}{\left\|P^{\mathrm{T}}x\right\|_2^2} = \max_{y\neq 0} \frac{y^{\mathrm{T}}Dy}{\|y\|_2^2} \\
&= \max_{y\neq 0} \frac{\sum_{i=1}^{n} \lambda_i y_i^2}{\sum_{i=1}^{n} y_i^2} \leq \lambda_{\max} \frac{\sum_{i=1}^{n} y_i^2}{\sum_{i=1}^{n} y_i^2} = \lambda_{\max}.
\end{aligned} \tag{7.15}$$

Assume that $\lambda_{\max}$ occurs at index (k,k). We have $Pe_k = x_k$, where x_k is column k of P and e_k is the kth standard basis vector, so $e_k = P^{\mathrm{T}}x_k$. By choosing $y = e_k$, the inequality in Equation 7.15 is an equality, and $\|A\|_2 = \sqrt{\lambda_{\max}}$. □

Remark 7.6. For values of p other than 1, 2, and ∞, there is no simple formula for the induced matrix p-norm.

Example 7.11. In Example 7.10, we computed the largest singular value of $A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ -1 & 6 \\ 7 & 8 \end{bmatrix}$, $\sigma_1 = 13.233$. Using the MATLAB command `norm`, compute the 2-norm of A.

```
>> norm(A,2)
ans =
        13.233
```

■

7.5 PROPERTIES OF THE MATRIX 2-NORM

There are a number of important properties of the matrix 2-norm. Since we have the tools to develop the properties, we will do so now and then refer to them as needed.

The matrix 2-norm inherits *orthogonal invariance* from the vector 2-norm. This means that multiplying matrix A on the left and right by orthogonal matrices does not change its 2-norm.

Theorem 7.8. *For any orthogonal matrices U and V, $\|UAV\|_2 = \|A\|_2$.*

Proof. We first consider the case of multiplication on the left by a single orthogonal matrix P, and then use this to prove that multiplication on the right by a single orthogonal matrix also preserves the 2-norm. The combination of these two results will allow us to prove the more general result.

$$\|PA\|_2^2 = \max_{x\neq 0} \frac{(PAx)^{\mathrm{T}}(PAx)}{\|x\|_2^2} = \max_{x\neq 0} \frac{\left(x^{\mathrm{T}}A^{\mathrm{T}}P^{\mathrm{T}}\right)(PAx)}{\|x\|_2^2}$$
$$= \max_{x\neq 0} \frac{x^{\mathrm{T}}A^{\mathrm{T}}IAx}{\|x\|_2^2} = \max_{x\neq 0} \frac{(Ax)^{\mathrm{T}}(Ax)}{\|x\|_2^2} = \max_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \|A\|_2^2 .$$

Now, noting that $\|x\|_2 = \|Px\|_2$ for any vector x,

$$\|AP\|_2^2 = \max_{x\neq 0} \frac{\|APx\|_2^2}{\|x\|_2^2} = \max_{x\neq 0} \frac{\|A(Px)\|_2^2}{\|Px\|_2^2} = \max_{y\neq 0} \frac{\|Ay\|_2^2}{\|y\|_2^2} = \|A\|_2^2 .$$

Now consider multiplication by orthogonal matrices U on the left and V on the right.

$$\|UAV\|_2 = \|U(AV)\| = \|AV\|_2 = \|A\|_2 .$$

This section concludes with the development of four properties of the matrix 2-norm. One of the properties involves the spectral radius of a matrix. □

Definition 7.6. Let A be an $n \times n$ real matrix with eigenvalues, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then the *spectral radius,* $\rho(A)$, of A is

$$\rho(A) = \max_{1\leq i\leq n} |\lambda_i| .$$

Before considering the five properties in Theorem 7.9, there is a fact we have not needed until now.

Lemma 7.5. *If A is invertible, the eigenvalues of A^{-1} are inverses of the eigenvalues of A; in other words, the eigenvalues of $A^{-1}$are $\{1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n\}$, where the λ_i are the eigenvalues of A. Note that there is no problem with division by 0, since an invertible matrix cannot have a 0 eigenvalue (Proposition 5.1). Furthermore, A and A^{-1} have the same eigenvectors. Also, the maximum eigenvalue of A^{-1} in magnitude is $1/\lambda_{\min}$, where $\lambda_{\min}$ is the smallest eigenvalue of A in magnitude.*

Proof. Let λ be an eigenvalue A and x be a corresponding eigenvector. Then $Ax = \lambda x$, and $A^{-1}x = (1/\lambda)x$. Thus, $1/\lambda$ is an eigenvalue of A^{-1} and x is a corresponding eigenvector. Clearly, if $\lambda_{\min}$ is the smallest eigenvalue of A in magnitude, then $1/\lambda_{\min}$ is the largest eigenvalue of A^{-1}. □

Theorem 7.9. *The matrix 2-norm has the following properties:*

1. *If A is a symmetric matrix, $\|A\|_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A.*
2. *If A is a symmetric matrix, its singular values are the absolute value of its eigenvalues.*
3. $\|A\|_2 = \left\|A^{\mathrm{T}}\right\|_2$.
4. $\left\|A^{\mathrm{T}}A\right\|_2 = \left\|AA^{\mathrm{T}}\right\|_2 = \|A\|_2^2$.
5. $\left\|A^{-1}\right\|_2 = \frac{1}{\sigma_{\min}}$, *where $\sigma_{\min}$ is the minimum singular value of A.*

Proof. The proof of parts 1 and 5 are left to the exercises.

For part (2), assume that λ is an eigenvalue of A with associated eigenvector, so $Av = \lambda v$. To obtain singular values, we need to find the eigenvalues of $A^{\mathrm{T}}A = A^2$ since A is symmetric. Multiply both sides of $Av = \lambda v$ by A, and we have

$$A^2 v = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v.$$

Assume the eigenvalues are sorted by decreasing absolute value. The singular values are nonnegative and are square roots of λ^2, so $\sigma_1 = |\lambda_1|$, $\sigma_2 = |\lambda_2|, \ldots, \sigma_n = |\lambda_n|$.

Begin the proof of part (3) with

$$\|Ax\|_2^2 = (Ax)^{\mathrm{T}}(Ax) = x^{\mathrm{T}}A^{\mathrm{T}}Ax = \left\langle x, A^{\mathrm{T}}Ax\right\rangle \leq \|x\|_2 \left\|A^{\mathrm{T}}Ax\right\|_2 \tag{7.16}$$

by the Cauchy-Schwarz inequality. From Equation 7.9

$$\|x\|_2 \left\|A^{\mathrm{T}}Ax\right\|_2 \leq \|x\|_2^2 \left\|A^{\mathrm{T}}A\right\|_2 , \tag{7.17}$$

and by putting Equations 7.16 and 7.17 together, we have

$$\|Ax\|_2 \leq \sqrt{\left\|A^{\mathrm{T}}A\right\|_2}\,\|x\|_2\,. \tag{7.18}$$

By Equation 7.18,

$$\|A\|_2 = \max_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} \leq \max_{x\neq 0}\frac{\sqrt{\left\|A^{\mathrm{T}}A\right\|_2}\,\|x\|_2}{\|x\|_2} = \sqrt{\left\|A^{\mathrm{T}}A\right\|_2}. \tag{7.19}$$

The matrix 2-norm is submultiplicative so by squaring both sides of Equation 7.19,

$$\|A\|_2^2 \leq \left\|A^{\mathrm{T}}A\right\|_2 \leq \left\|A^{\mathrm{T}}\right\|_2 \|A\|_2\,.$$

Divide by $\|A\|_2$ when $A \neq 0$ to obtain

$$\|A\|_2 \leq \left\|A^{\mathrm{T}}\right\|_2.$$

Clearly, this is also true if $A = 0$. In the last inequality, replace A by A^{T} and we now have the two inequalities, $\|A\|_2 \leq \left\|A^{\mathrm{T}}\right\|_2$ and $\left\|A^{\mathrm{T}}\right\| \leq \left\|\left(A^{\mathrm{T}}\right)^{\mathrm{T}}\right\|_2 = \|A\|_2$,
so

$$\|A\|_2 = \left\|A^{\mathrm{T}}\right\|_2.$$

For part (4), from Theorem 7.7, $\|A\|_2 = \sqrt{\lambda_{\max}}$, where $\lambda_{\max} \geq 0$ is the maximum eigenvalue of $A^{\mathrm{T}}A$, so $\|A\|_2^2 = \lambda_{\max}$. $A^{\mathrm{T}}A$ is a symmetric matrix, its eigenvalues are positive real numbers, and from (1) its 2-norm is its spectral radius, $\lambda_{\max}$, so

$$\left\|A^{\mathrm{T}}A\right\|_2 = \lambda_{\max} = \|A\|_2^2\,.$$

Now replace A by A^{T} to obtain

$$\left\|\left(A^{\mathrm{T}}\right)^{\mathrm{T}}A^{\mathrm{T}}\right\|_2 = \left\|AA^{\mathrm{T}}\right\|_2 = \left\|A^{\mathrm{T}}\right\|_2^2 = \|A\|_2^2$$

using part (3). □

7.6 CHAPTER SUMMARY

Vector Norms

A vector norm measures length. The most commonly used norm is the 2-norm, or the Euclidean norm, where

$$\|u\|_2 = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = \sqrt{\langle u, u\rangle}.$$

Motivated by the properties of the 2-norm, we define a general vector norm as a function mapping $\mathbb{R}^n$ into the real numbers with the following properties:

a. $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$; $\|x\| = 0$ if and only if $x = 0$ (*positivity*);
b. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{R}$ (*scaling*);
c. $\|x + y\| \leq \|\ x\ \| + \|y\|$ for all $x,\ y \in \mathbb{R}^n$ (*triangle inequality*).

Aside from the 2-norm, other useful norms are the 1-norm,

$$\|u\|_1 = \sum_{i=1}^{n} |u_i|\,,$$

and the ∞-norm,

$$\|u\|_\infty = \max_{1\leq i\leq n} |u_i|\,.$$

Properties of the 2-Norm

The 2-norm has unique properties that will be useful throughout the book. It satisfies

- the Cauchy-Schwarz inequality, $|\langle u, v\rangle| = \left|u^{\mathrm{T}}v\right| \le \|u\|_2 \|v\|_2$,
- the Pythagorean Theorem, $\|u+v\|_2 = \|u\|_2^2 + \|v\|_2^2$, when u and v are orthogonal,
- orthogonal invariance, $\|Px\|_2 = \|x\|_2$ when P is an orthogonal matrix.

Orthogonal invariance is of particular importance in computer graphics. If a rotation is applied to the elements in an object, the object's size and shape do not change.

A set of k orthogonal vectors $u_1, u_2, \ldots, u_k$ is linearly independent and forms a basis for a $k-$dimensional subspace of $\mathbb{R}^n$. If we normalize each vector to form, $v_i = u_i/\|u_i\|_2$, then the matrix $[v_1 v_2 \ldots v_k]$ is orthogonal.

Spherical Coordinates

Spherical coordinates are useful in computer graphics, vibrating membranes, and the Schrodinger equation for the hydrogen atom. With some work, we can build an orthogonal matrix that implements a change of coordinates from rectangular to spherical and spherical to rectangular.

Matrix Norms

The definition of a matrix norm is the same as that for a vector norm. It has the properties of positivity, scaling, and the triangle inequality. Essentially, it measure the size of a matrix. The oldest matrix norm is the Frobenius norm defined by

$$\|A\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2},$$

which can be viewed as the norm of a vector in $\mathbb{R}^{mn}$.

Induced Matrix Norm

A vector norm, $\|.\|$, can be used to define a corresponding matrix norm as follows:

$$\|A\| = \max_{x\neq 0}\frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1}\|Ax\|.$$

From the definition, there results a frequently used inequality, $\|Ax\| \le \|A\|\,\|x\|$.

The three most used induced matrix norms are the 2-norm, the 1-norm, and the ∞-norm. The definition can be used to develop simple formulas for the matrix ∞- and 1-norms:

$$\|A\|_\infty = \max_{1\le i\le m}\sum_{j=1}^{n}\left|a_{ij}\right|,$$

$$\|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^{m}\left|a_{ij}\right|.$$

Submultiplicative Matrix Norms

The induced norms are submultiplicative, which means that

$$\|AB\| \le \|A\|\,\|B\|.$$

The Frobenius norm is submultiplicative, but is not an induced matrix norm.

Computation of $\|A\|_2$

The computation of $\|A\|_2$ is more complex. First, note that for any $m \times n$ matrix $A^{\mathrm{T}}A$ is a symmetric $n \times n$ matrix. As such, its eigenvalues are all real, but in addition all its eigenvalues are greater than or equal to zero. The singular values of A are the square root of the eigenvalues of $A^{\mathrm{T}}A$, and the 2-norm is the largest singular value. In addition, the 2-norm of A^{-1} is the reciprocal of the smallest singular value of A.

Properties of $\|A\|_2$

The matrix 2-norm has the following properties:

1. For any orthogonal matrices U and V, $\|UAV\|_2 = \|A\|_2$.
2. If A is a symmetric matrix, $\|A\|_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A, the magnitude of its largest eigenvalue.
3. If A is a symmetric matrix, its singular values are the absolute value of its eigenvalues.
4. $\|A\|_2 = \left\|A^{\mathrm{T}}\right\|_2$.
5. $\left\|A^{\mathrm{T}}A\right\|_2 = \left\|AA^{\mathrm{T}}\right\|_2 = \|A\|_2^2$.

7.7 PROBLEMS

7.1 Compute the 1-norm, the ∞-norm, and the 2-norm for the following vectors. Do the calculations with paper, pencil, and a calculator.

a. $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

b. $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

c. $\begin{bmatrix} -1 \\ -9 \\ -6 \end{bmatrix}$

d. $\begin{bmatrix} 1 \\ 4 \\ -1 \\ 2 \end{bmatrix}$

7.2 Compute the 1-norm, the ∞-norm, the 2-norm, and the Frobenius norm for the following vectors. Except for the 2-norm, do the calculations with paper and pencil. To compute the 2-norm, you may use MATLAB to find the required eigenvalues.

a. $\begin{bmatrix} 1 & 9 \\ -1 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 5 & 3 \\ 0 & 4 & 1 \end{bmatrix}$

7.3 Show that $\|x\|_1$ is a vector norm by verifying Definition 7.1.

7.4 Prove that

a. $\|I\|_2 = 1$

b. $\|I\|_{\mathrm{F}} = \sqrt{n}$.

7.5 Prove that

$$\|A\|_1 = \max_{1 \le k \le n} \sum |a_{ik}|$$

using the proof of Theorem 7.4 as a guide.

7.6 Prove the following:

a.

$$\|A\|_{\max} = \max_{1 \le i,j \le n} \left|a_{ij}\right|$$

is a matrix norm.

b.

$$\|A\|_{\max} = \max_{1 \le i,j \le n} \left|a_{ij}\right|$$

is not sub-multiplicative. Hint: Let A be a matrix whose entries are all equal to 1.

7.7

a. In Lemma 7.1, we proved that $\|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$. Prove that $\|x\|_1 \leq \sqrt{n}\,\|x\|_2$ Hint: Consider using the Cauchy-Schwarz inequality involving vectors x and $\begin{bmatrix} 1 & \dots 1 \end{bmatrix}^{\mathrm{T}}$.

b. Find a vector $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ for which the inequalities in (a) are an equality.

7.8 Show that if $\|.\|$ is an induced matrix norm, then $|\lambda| \leq \|A\|$, where λ is an eigenvalue of A.

7.9 This problem demonstrates why $\|\cdot\|_\infty$ is a p-norm by showing that

$$\lim_{p \longrightarrow \infty} \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} = \max_{1 \leq i \leq n} |x_i| = \|x\|_\infty .$$

Reorder the elements in $\sum_{i=1}^{n} |x_i|^p$ so that $|x_1| = |x_2| = \cdots = |x_k|$ are the largest elements in magnitude, and let $|x_{k+1}|, |x_{k+2}|, \ldots, |x|_n$ be the remaining elements. Show that

$$\left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} = |x_1| \left(k + \left| \frac{x_{k+1}}{x_1} \right|^p + \cdots + \left| \frac{x_n}{x_1} \right|^p \right)^{1/p},$$

and complete the argument.

7.10

a. If $\|.\|$ is an induced matrix norm, show that $\|I\| = 1$.

b. Show that the Frobenius norm is not an induced norm.

7.11 Let A be an $n \times n$ matrix, $\rho(A)$ its spectral radius, and $\|\cdot\|$ be an induced matrix norm. Prove that for every $k \geq 1$

$$\rho(A) \leq \left\| A^k \right\|^{1/k} .$$

Hint: Begin by showing that $A^k v = \lambda^k v$ if λ is an eigenvalue of A with corresponding eigenvector v.

7.12 Show that the Frobenius norm can be computed as $\|A\|_{\mathrm{F}} = \left(\mathrm{trace}\left(A^{\mathrm{T}}A\right)\right)^{1/2}$ or $\|A\|_{\mathrm{F}} = \left(\mathrm{trace}\left(AA^{\mathrm{T}}\right)\right)^{1/2}$.

7.13 Verify that $\sum_{i=1}^{n} \sum_{j=1}^{n} \left[\left(\sum_{k=1}^{n} a_{ik}^2 \right) \left(\sum_{k=1}^{n} b_{kj}^2 \right) \right] = \|A\|_{\mathrm{F}}^2 \|B\|_{\mathrm{F}}^2$.

7.14 If A is an $m \times n$ matrix, prove that

$$\max_{1 \leq i \leq m,\, 1 \leq j \leq n} \left| a_{ij} \right| \leq \|A\|_{\mathrm{F}} .$$

7.15 Prove the *Pythagorean Theorem* for orthogonal x and y,

$$\|x + y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 .$$

7.16 Prove that if A is a symmetric matrix and $\rho(A) < 1$, then

$$\lim_{k \to \infty} A^k = 0.$$

Use the spectral theorem.

7.17 Prove that if A is an $n \times n$ matrix and

$$\lim_{n \to \infty} A^n = 0,$$

then $\rho(A) < 1$. Consult the hint for Problem 7.11.

7.18 A matrix norm and a vector norm are compatible if it is true for all vectors x and matrices A that $\|Ax\| \leq \|A\|\,\|x\|$. Show that the Euclidean vector norm is compatible with the Frobenius matrix norm; in other words, show that

$$\|Ax\|_2 \leq \|A\|_{\mathrm{F}} \|x\|_2 .$$

7.19 A Schatten p-norm is the p-norm of the vector of singular values of a matrix. If the singular values are denoted by σ_i, then the Schatten p-norm is defined by

$$\|A\|_p = \left(\sum_{i=1}^{n} \sigma_i^p \right)^{1/p} .$$

a. Show that the Schatten ∞-norm is the spectral norm, $\|A\|_2$.
b. In Chapter 15, we will be able to use the SVD to show that trace $\left(A^{\mathrm{T}}A\right) = \sum_{i=1}^{n} \sigma_i^2$. Using this result, show that the Schatten 2-norm is the Frobenius norm.

7.20 Prove
a. If A is a symmetric matrix, $\|A\|_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A.
b. Assume that $A^{\mathrm{T}}A$ and AA^{T} have the same eigenvalues, a fact we will deal with in Chapter 15. Prove that $\left\|A^{-1}\right\|_2 = 1/\sigma_{\min}$, where $\sigma_{\min}$ is the minimum singular value of A.

7.21 Complete the proof of Lemma 7.1 by showing that
a. $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$
b. $\|x\|_\infty \le \|x\|_1 \le n\,\|x\|_\infty$

7.22 Prove that for any induced matrix norm, $\rho(A) \le \|A\|$.

7.23 If A is an $n \times n$ matrix, $\rho(A) \ge 0$. Show that the spectral radius is not a matrix norm by using the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix}.$$

7.7.1 MATLAB Problems

7.24 It can be shown that if A is an $m \times n$ matrix, then

$$\frac{1}{\sqrt{n}}\,\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\,\|A\|_\infty$$

and

$$\frac{1}{\sqrt{m}}\,\|A\|_1 \le \|A\|_2 \le \sqrt{n}\,\|A\|_1 .$$

Using MATLAB, verify these relationships for the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

7.25 Run an experiment like that in Example 7.6 to formulate a good guess for the value of the matrix 1-norm.

7.26 The software distribution contains a function `P =sphereorthog(theta, phi)` that returns the orthogonal matrix in Equation 7.2. Convert $\begin{bmatrix} 2 \\ \frac{3}{8}\pi \\ \pi/6 \end{bmatrix}$ from the spherical coordinate basis to Cartesian coordinates. Using the MATLAB function `quiver3`, draw the spherical basis vectors $\mathbf{e_r}$, $\mathbf{e}_\phi$, $\mathbf{e}_\theta$, and the vector

$$u = 2\mathbf{e_r} + \frac{3}{8}\pi\,\mathbf{e}_\theta + \frac{\pi}{6}\mathbf{e}_\phi.$$

7.27 Let A be the 2×2 matrix $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$. The following function plots the unit circle and the range of Ax as x varies over the circle. Add code that computes vectors x_1 and x_2 such that $\|Ax_1\|_2$ and $\|Ax_2\|_2$ are largest and smallest, respectively, among the points generated on the unit circle. Using the MATLAB function `quiver`, draw the two vectors Ax_1, Ax_2. Also output $\|Ax_1\|_2$ and $1/\|Ax_2\|$. Compute $\|A\|_2$ and $\left\|A^{-1}\right\|_2$. What conclusion can you make?

```
function matimage(A)

% build the unit circle
t=0:0.01:2*pi;
x=cos(t)';y=sin(t)';
npts = length(t);
Ax = zeros(npts,1);
%Image of the unit circle under A
for i = 1:npts
```

```
    v=[x(i);y(i)];
    w=A*v;
    Ax(i)=w(1);
    Ay(i)=w(2);
end
% Plot of the circle and its image
plot(x,y,Ax,Ay,'.','MarkerSize',10,'LineWidth',3);
grid on; axis equal;
title('Action of a linear transformation on the unit circle');
xlabel('x'); ylabel('y');
```

7.28 Plot the surface $r = \theta^2$, $\theta = \phi$, $0 \le \theta \le \pi$.

7.29 A magic square is an $n \times n$ matrix whose rows, columns, and both diagonals add to the same number. For instance, M is a 4×4 magic square, whose sums are 34.

$$M = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}$$

a. What is the sum for an $n \times n$ magic square?

b. The time required to execute one or more MATLAB commands can be timed by entering a single line_using `tic/toc` as follows:

`tic;command1;command2;...;last command;toc;`

It is faster to compute the Frobenius norm, the 1-norm, and the ∞-norm of a matrix than to compute the 2-norm. You are to perform an experiment. Generate a 1000×1000 magic square using the MATLAB command "`M = magic(1000);`", and do not omit the ";" or 1,000,000 integers will begin spewing onto your screen. Using `tic/toc`, time the execution of `norm(A)` and then the execution of the other norms. Comment on the results.

7.30 As we have noted, a computer does not perform exact floating point arithmetic, and errors occur. Norms play a role in determining if one can depend on the solution to a linear system obtained using Gaussian elimination. Assume that the entries of matrix A are precise. Let x be the true solution to the system $Ax = b$ and that x_a is the solution obtained using Gaussian elimination. If the product Ax_a is not exact, then $Ax_a = b + \Delta b$, $\Delta b \neq 0$.

a. Using $Ax = b$, show that $x_a - x = A^{-1}\Delta b$.

b. Noting that $\|b\|_2 = \|Ax\|_2$, show that $(\|x_a - x\|_2/\|x\|_2) \le \left\|A^{-1}\right\|_2 \|A\|_2 \|\Delta b\|_2/\|b\|_2$.

c. The product $\left\|A^{-1}\right\|_2 \|A\|_2$ is called the *condition number* of A. If it is large, errors relative to the correct values can be large. For each matrix, find the condition number.

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 5 & -1 & 2 \\ 1 & 7 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} -4.0000 & 0.5000 & 0.3333 & 0.2500 \\ -120.0000 & 20.0000 & 15.0000 & 12.0000 \\ 240.0000 & -45.0000 & -36.0000 & -30.0000 \\ -140.0000 & 28.0000 & 23.3333 & 20.0000 \end{bmatrix}.$$

d. For each matrix, let b be a vector consisting of all ones. Find the MATLAB solution x. Then multiply b by 0.999 and solve the system again to obtain xp. Compute $\|x - xp\|_2$. What are your conclusions?

7.31 Let $A = \begin{bmatrix} 0.6 & 1 & 6 & -1 & 5 \\ 0 & 0.6 & 1 & 1 & 0 \\ 0 & 0 & 0.6 & 1 & 3 \\ 0 & 0 & 0 & 0.6 & 1 \\ 0 & 0 & 0 & 0 & -0.7 \end{bmatrix}$.

a. Without using MATLAB, find the eigenvalues of A.

b. Is there a basis of eigenvectors for $\mathbb{R}^5$?

c. What is the spectral radius of A?

d. Plot $\|A^n\|_2$ for $n = 0, 1, \ldots, 50$.

e. Find the maximum value of $\|A^n\|_2$, $0 \le n \le 50$.

f. Build another nonsymmetric matrix with $\rho(A) < 1$. Do parts (d) and (e) for it.

g. Perform the same actions with the symmetric matrix SYMMAT in the software distribution.

h. Attempt to explain your results. For a symmetric matrix, use the spectral theorem. We will develop Schur's triangularization in Chapter 19. It states that very $n \times n$ real matrix A with real eigenvalues can be factored into $A = PTP^{\mathrm{T}}$, where P is an orthogonal matrix and T is an upper-triangular matrix. Apply this result to the nonsymmetric case. In particular, is there a relationship between the eigenvalues of A and T that can explain what happens to $\|A^n\|_2$ as n increases?

7.32 The inner product of two $n \times 1$ vectors u, v is the real number $\langle u, v\rangle = u^{\mathrm{T}}v$. Now let's investigate the $n \times n$ matrix $A = uv^{\mathrm{T}}$. For $n = 5,\ 15,\ 25$, generate vectors $u = \mathit{rand}\,(n, 1)$ and $v = \mathrm{rand}\,(n, 1)$. In each case, compute rank $\left(uv^{\mathrm{T}}\right)$, $\|u\|_2\,\|v\|_2$, and $\left\|uv^{\mathrm{T}}\right\|_2$. What do you conclude from the experiment? Prove each assertion. It will help to recall that *rank* + *nullity* = *n* (Theorem 3.4).

7.33 This problem investigates how some norm values compare with the maximum of the absolute values of all matrix entries,

$$\max_{1\le i,j\le n} \left|a_{ij}\right|,$$

that can be computed using the MATLAB command `max(max(abs(A)))`. For $n = 5,\ 15,\ 25$, build matrices

$A_n =$`randi([-100 100],n,n)`

and compute

$$\max_{1\le i,j\le n} \left|a_{ij}^{(n)}\right|,\ \|A_n\|_\infty,\ \|A_n\|_1,\ \|A\|_2 \text{ and } \|A_n\|_{\mathrm{F}}.$$

What is the apparent relationship between

$$\max_{1\le i,j\le n} \left|a_{ij}^{(n)}\right|$$

and the matrix norms? Prove your assertion. Hint: For the induced norms, assume

$$m = \max_{1\le i,j\le n} \left|a_{ij}^{(n)}\right|$$

occurs at indices $(i_{\max}, j_{\max})$. By definition

$$\|A\| = \max_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}.$$

If e_k is a standard basis vector, then $(\|Ae_k\|/\|e_k\|) = \|Ae_k\| \le \|A\|$.