Chapter 5

Eigenvalues and Eigenvectors

You should be familiar with

- Solution to homogeneous systems of equations
- Polynomials and their roots
- Nonsingular and singular matrices
- Diagonal matrices

Let A be an $n \times n$ matrix. For a large number of problems in engineering and science, it is necessary to find vector v such that Av is a multiple, λ, of v; in other words, $Av = \lambda v$. Av is parallel to v, and λ either stretches or shrinks v. The value λ is an eigenvalue and v is an eigenvector associated with λ. Computing eigenvalues and eigenvectors is one of the most important problems in numerical linear algebra. Eigenvalues are critical in such fields as structural mechanics, nuclear physics, biology, the solution of differential equations, computer science, and so on. Eigenvalues play a critical role in the study of vibrations, where they represent the natural frequencies of a system. When vibrating structures begin to have larger and larger amplitudes of vibration, they can have serious problems. Some examples include the wobbling of the Millennium Bridge over the River Thames in London and the collapse of the Tacoma Narrows Bridge in the state of Washington. These are examples of a phenomenon known as *resonance*. A mathematical analysis of a general model for vibrating structures shows that when the system is excited by a harmonic force that depends on time, the system approaches a resonance state when the forces approach or reach a particular eigenvalue. We will discuss specific applications of eigenvalues at various places in the remainder of this book.

Note that some applications involve matrices with complex entries and vectors spaces of complex numbers. We do not deal with matrices of this type in the book; however, many of the techniques we discuss can be adapted for use with complex vectors and matrices. The reader can consult books such as in Refs. [1, 2, 9] for details.

5.1 DEFINITIONS AND EXAMPLES

If A is an $n \times n$ matrix, in order to find eigenvalue λ and an associated eigenvector v, it must be the case that $Av = \lambda v$, and this is equivalent to the homogeneous system

$$(A - \lambda I)\, v = 0. \tag{5.1}$$

We know from Theorem 4.5 that

$$\det (A - \lambda I) = 0 \tag{5.2}$$

in order that the system 5.1 have a nonzero solution. The determinant of $A - \lambda I$ is a polynomial of degree n, so Equation 5.2 is a problem of finding roots of the polynomial

$$p(\lambda) = \det (A - \lambda I)\,.$$

Remark 5.1. A polynomial $p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$ of degree n has exactly n roots, and any complex roots occur in conjugate pairs. (If you are unfamiliar with complex numbers, see Appendix A.) The polynomial may have one or more roots of multiplicity two or more. This means that the polynomial has a factor $(x - r)^k$, $2 \le k \le n$. The root r is counted k times.

Definition 5.1 officially defines the eigenvalue problem and introduces some terms.

Definition 5.1. If A is an $n \times n$ matrix, the polynomial $p(\lambda) = \det (A - \lambda I)$ is called the *characteristic polynomial* of A, and the equation $p(\lambda) = 0$ is termed the *characteristic equation*. If λ is a root of p, it is termed an *eigenvalue* of A, and if v

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is a nonzero column vector satisfying $Av = \lambda v$, it is an *eigenvector* of A. We say that v is an eigenvector corresponding to the eigenvalue λ.

When multiplied by a matrix A, an arbitrary vector v normally changes direction and length. Eigenvectors are special. The product Av may expand, shrink, or leave the length of v unchanged, but Av will always point in the same direction as v or in the opposite direction.

Example 5.1. Let $A = \begin{bmatrix} -0.4707 & 0.7481 \\ 1.7481 & 1.4707 \end{bmatrix}$, $w = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$, $v = \begin{bmatrix} 0.2898 & 0.9571 \end{bmatrix}^{\mathrm{T}}$.

$$Aw = \begin{bmatrix} -1.2188 & 0.2774 \end{bmatrix}^{\mathrm{T}},$$
$$Av = \begin{bmatrix} 0.5796 & 1.9142 \end{bmatrix}^{\mathrm{T}}.$$

Note that $Av = \begin{bmatrix} 0.5796 & 1.9142 \end{bmatrix}^{\mathrm{T}} = 2.0v$, and v is an eigenvector of A corresponding to eigenvalue 2.0. Figure 5.1 is a graph of w, Aw, and v, Av. Note how Aw has a different direction than w, but Av points in the same direction as v. ■

The following steps show how to use Definition 5.1 to find the eigenvalues and eigenvectors of an $n \times n$ matrix. It must be noted that in the case of a root, λ, with multiplicity of two or more, there may be only one eigenvector associated with λ. In practice, the accurate computation of eigenvalues and associated eigenvectors is a complex task and is not generally done this way for reasons we will explain in later chapters.

Given an $n \times n$ matrix A:

1. Find the polynomial $p(\lambda) = \det(A - \lambda I)$.
2. Compute the n roots of $p(\lambda) = 0$. These are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A.
3. For each distinct λ_i, find an eigenvector x_i such that

$$Ax_i = \lambda_i x$$

by solving

$$(A - \lambda I)\, v = 0$$

If λ_i is a multiple root, there may be only one associated eigenvector. If not, compute the distinct eigenvectors.

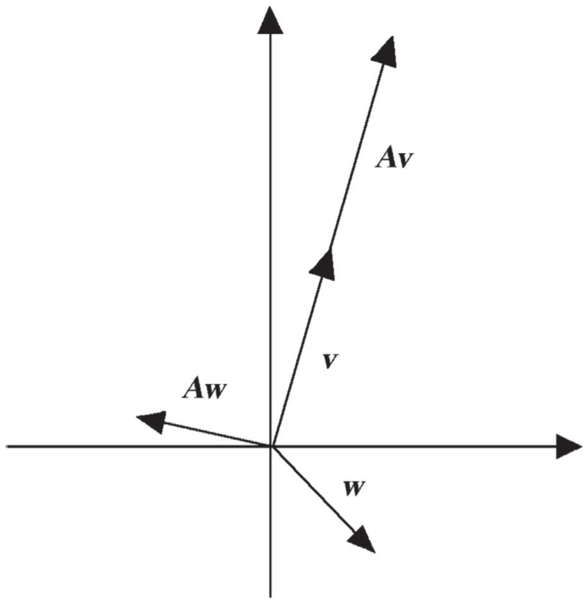

FIGURE 5.1 Direction of eigenvectors.

Example 5.2. Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and all the eigenvectors. The characteristic polynomial is $\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$, so the eigenvalues are the roots of the characteristic equation $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$. Hence, $\lambda = 1$ and $\lambda = 3$ are the eigenvalues of A. To find the eigenvector corresponding to $\lambda = 1$, solve the homogeneous system $\begin{bmatrix} 2-(1) & 1 \\ 1 & 2-(1) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which corresponds to the equations

$$\begin{aligned} x + y &= 0, \\ x + y &= 0. \end{aligned}$$

These two equations simply say that the sum of x and y must be 0, so $x = -y$. Consider y to be a parameter that varies through all real numbers not equal to 0. Consequently, the eigenvectors corresponding to $\lambda = 1$ are the vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, with $y \neq 0$. Choose $y = 1$ to obtain a specific eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Taking $\lambda = 3$ gives the two equations

$$\begin{aligned} -x + y &= 0, \\ x - y &= 0. \end{aligned}$$

These equations require that $x = y$. Again, considering y to be a parameter, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $y = 1$ to obtain the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. ■

Summary: The final result is

λ	Eigenvectors
1	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
3	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

■

Although more work is involved, the same procedure can be performed to compute the eigenvalues and corresponding eigenvectors of a 3×3 matrix.

Example 5.3. Let $A = \begin{bmatrix} 4 & 8 & 3 \\ 0 & -1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$. Determine the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det\begin{bmatrix} 4-\lambda & 8 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{bmatrix} \\ &= (4-\lambda)\det\left(\begin{bmatrix} -1-\lambda & 0 \\ -2 & 2-\lambda \end{bmatrix}\right) = (4-\lambda)(-1-\lambda)(2-\lambda). \end{aligned}$$

The roots of the characteristic polynomial are $\lambda_1 = 4$, $\lambda_2 = -1$, $\lambda_3 = 2$. We will find three eigenvectors

$x_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}$, $x_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}$, $x_3 = \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix}$ by finding nonzero solutions to

$$\begin{bmatrix} 4-\lambda & 8 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{bmatrix} x = 0. \tag{5.3}$$

for each value of λ.

$\lambda_1 = 4$ in Equation 5.3:

Solve the homogeneous system $\begin{bmatrix} 0 & 8 & 3 \\ 0 & -5 & 0 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = 0$. Using Gaussian elimination, we have $\begin{bmatrix} 0 & 8 & 3 \\ 0 & -5 & 0 \\ 0 & -2 & -2 \end{bmatrix} \overrightarrow{R3 = R3 - 2/5\,R2} = \begin{bmatrix} 0 & 8 & 3 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Thus, $x_{31} = x_{21} = 0$. The first row specifies that $(0)\,x_1 + 8\,(0) + 3\,(0) = 0$. The component x_{11} is not constrained. Any value of x_{11} will work. Choose $x_{11} = 1$ to obtain the eigenvector $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$\lambda_2 = -1$ in Equation 5.3:

The homogeneous system we need to solve is $\begin{bmatrix} 5 & 8 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = 0$. Exchange rows 2 and 3 to obtain the system $\begin{bmatrix} 5 & 8 & 3 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = 0$. The second row of the system specifies that $-2x_{22} + 3x_{32} = 0$, so $x_{22} = 3/2x_{32}$. The first row requires that $5x_{12} + 8\,(3/2x_{32}) + 3x_{32} = 0$, and $x_{12} = -3x_{32}$. This gives a general eigenvector of $x_{32} \begin{bmatrix} -3 \\ 3/2 \\ 1 \end{bmatrix}$. If we choose $x_{32} = 1$, the eigenvector is $x_2 = \begin{bmatrix} -3 \\ 3/2 \\ 1 \end{bmatrix}$.

$\lambda_3 = 2$ in Equation 5.3:

Solve the homogeneous system $\begin{bmatrix} 2 & 8 & 3 \\ 0 & -3 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = 0$. Gaussian elimination gives $\begin{bmatrix} 2 & 8 & 3 \\ 0 & -3 & 0 \\ 0 & -2 & 0 \end{bmatrix} \overrightarrow{R3 = R3 - (-2/3)\,R2} = \begin{bmatrix} 2 & 8 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The second row requires that $x_{23} = 0$. Row 1 specifies that $2x_{13} + 8\,(0) + 3x_{33} = 0$, and $x_{13} = -3/2x_{33}$. This gives the general eigenvector $x_3 = x_{33} \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}$. By choosing $x_{33} = 1$, the eigenvector is $x_3 = \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}$. ■

Example 5.4. Let $A = \begin{bmatrix} 6 & 12 & 19 \\ -9 & -20 & -33 \\ 4 & 9 & 15 \end{bmatrix}$. The characteristic polynomial of A is $(\lambda + 1)\,(\lambda - 1)^2$, so $\lambda = 1$ is a multiple root. To find an eigenvector(s) associated with $\lambda = 1$, we need to solve

$$\begin{bmatrix} 5 & 12 & 19 \\ -9 & -21 & -33 \\ 4 & 9 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After Gaussian elimination, we obtain the upper-triangular matrix $\begin{bmatrix} 5 & 12 & 19 \\ 0 & 3/5 & 6/5 \\ 0 & 0 & 0 \end{bmatrix}$, and a solution to the upper-triangular system is $x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. There is only one linearly independent eigenvector associated with $\lambda = 1$. ■

There are cases where an eigenvalue of multiplicity k does produce k linearly independent eigenvectors.

Example 5.5. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, the characteristic equation is $\lambda^2 (\lambda - 3)$, and $\lambda = 0$ is an eigenvalue of multiplicity 2. After performing Gaussian elimination, the homogeneous equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and its solution is $x_1 = -x_2 - x_3$, where x_2 and x_3 are arbitrary. Thus, any solution of the homogeneous system is of the form

$$x = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent eigenvectors. ■

Remark 5.2. Note that the matrix in Example 5.5 is symmetric. Whenever an $n \times n$ real matrix is symmetric, it has n linearly independent eigenvectors, even if its characteristic equation has roots of multiplicity 2 or more. This will be proved in Chapter 19.

5.2 SELECTED PROPERTIES OF EIGENVALUES AND EIGENVECTORS

There are some properties of eigenvalues and eigenvectors you should know, and developing them will support your understanding of the eigenvalue problem. First, there is a relation between the eigenvalues of a matrix and whether the matrix is invertible.

Proposition 5.1. *An $n \times n$ matrix A is singular if and only if it has a 0 eigenvalue.*

Proof. If A is singular, by Theorem 4.3, $Ax = 0$ has a solution $x \neq 0$. Thus, $Ax = (0)\,x = 0$, and $\lambda = 0$ is an eigenvalue.

If A has a eigenvalue $\lambda = 0$, then there exists a vector $x \neq 0$ such that $Ax = \lambda x = 0$, and the homogeneous system $Ax = 0$ has a nontrivial solution. If A is nonsingular, then $x = A^{-1}0 = 0$ is the unique solution. Thus, A is singular. □

Note that if v is an eigenvector of A corresponding to eigenvalue λ and α is a constant, then $A(\alpha v) = \alpha A v = \alpha (\lambda v) = \lambda (\alpha v)$, and αv is an eigenvector of A. This causes us to suspect that the set of eigenvectors corresponding to λ is a subspace.

Lemma 5.1. *Together with 0, the eigenvectors corresponding to λ form a subspace called an eigenspace.*

Proof. To show that a set of vectors form a subspace S, we must show that 0 is in S, that αv is in S for any constant α and any v in S, and that if v_1, v_2 are in S, then so is $v_1 + v_2$. Let S be the set containing the zero vector and all eigenvectors of A corresponding to eigenvalue λ. By hypothesis, 0 is in S. We already showed that if v is an eigenvector corresponding to λ, then so is αv for any constant α. If v_1 and v_2 are eigenvectors of A, then $A(v_1 + v_2) = Av_1 + Av_2 = \lambda (v_1 + v_2)$, and $v_1 + v_2$ is in S, so S is a subspace. □

The next result will support your understanding of the relationship between eigenvalues and the roots of the characteristic equation.

Proposition 5.2. *If A is an $n \times n$ matrix, then* $\det A = \Pi_{i=1}^{n} \lambda_i$.

Proof. First assume that all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are distinct; in other words $\lambda_k \neq \lambda_j$, $k \neq j$. The characteristic polynomial $p(\lambda)$ is of degree n, and is the determinant of

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & a_{n-1,n-1}-\lambda & a_{n-1,n} \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} & a_{nn}-\lambda \end{bmatrix}.$$

Expansion by minors shows us that the leading term of $p(\lambda)$ is $(-1)^n \lambda^n$. By Remark 5.1,

$$\det (A-\lambda I) = p(\lambda) = (-1)^n (\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_n).$$

Let $\lambda = 0$, and we have $\det A = (-1^n)(-1)^n \lambda_1\lambda_2\dots\lambda_n = \Pi_{i=1}^n \lambda_i$.

Now assume that one or more eigenvalues are repeated. In this case, $p(\lambda)$ has one or more factors of the form $(\lambda-\lambda_i)^k$, where $k \geq 2$. Think of such a factor as

$$(\lambda-\lambda_{i1})(\lambda-\lambda_{i2})\dots(\lambda-\lambda_{ik}),$$

where $\lambda_{i1} = \lambda_{i2} = \dots = \lambda_{ik}$. The same argument we just gave shows that $\det A = \Pi_{i=1}^n \lambda_i$. □

5.3 DIAGONALIZATION

In this section, we will show that, under the right conditions, we can use the eigenvectors of a matrix to transform it into a diagonal matrix of eigenvalues. The process is termed diagonalization, and is an important concept in matrix algebra; in fact, it is critical to developing results such as the singular value decomposition (Chapter 15), and computing eigenvalues of symmetric matrices (Chapter 19).

Definition 5.2. Matrix B is *similar* to matrix A if there exists a nonsingular matrix X such that

$$B = X^{-1}AX.$$

Example 5.6. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 4 & 7 & 6 \end{bmatrix}$. Now, $X^{-1} = \begin{bmatrix} 3 & 5 & -3 \\ 0 & -2 & 1 \\ -2 & -1 & 1 \end{bmatrix}$, so

$$B = X^{-1}AX = \begin{bmatrix} 14 & 27 & 23 \\ 1 & 2 & 0 \\ -9 & -18 & -13 \end{bmatrix}$$

is similar to A. ■

Definition 5.3. The $n \times n$ matrix A is *diagonalizable* if it is similar to a diagonal matrix. We also say that A can be *diagonalized*.

Example 5.7. Let $A = \begin{bmatrix} -3 & 6 & -2 \\ -12 & 7 & 0 \\ -24 & 16 & -1 \end{bmatrix}$, and X be the matrix in Example 5.6. Then,

$$X^{-1}AX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

and A is diagonalizable. ■

Remark 5.3. Determining whether a matrix can be diagonalized and performing the diagonalization requires that we prove some results.

The fact that $\det(AB) = \det(A)\det(B)$ can be used to show the relation between the determinant of A and that of its inverse.

Lemma 5.2. *If A is invertible, then* $\det(A)\det(A^{-1}) = 1$.

Proof. Since A is invertible, $AA^{-1} = I$. The determinant of a product is the product of the determinants, so

$$\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(I) = 1.$$

We can use Lemma 5.2 to prove the following useful result. □

Theorem 5.1. *Two similar matrices have the same eigenvalues.*

Proof. Assume A and B are similar matrices. Then, $B = X^{-1}AX$ for some nonsingular matrix X. It follows that

$$\begin{aligned}\det(B - \lambda I) &= \det\left(X^{-1}AX - \lambda X^{-1}X\right) = \det\left(X^{-1}[A - \lambda I]X\right)\\ &= \det\left(X^{-1}\right)\det(A - \lambda I)\det(X) = \det\left(X^{-1}\right)\det(X)\det(A - \lambda I) = \det(A - \lambda I).\end{aligned}$$

A and B have the same characteristic polynomial and thus the same eigenvalues. □

If A is diagonalizable, then there exist matrices X and D such that $D = X^{-1}AX$. A is similar to D and thus has the same eigenvalues as D. Now, D has the form

$$D = \begin{bmatrix} d_{11} & & & & \\ & d_{22} & & & \\ & & \ddots & & \\ & & & d_{n-1,n-1} & \\ & & & & d_{nn} \end{bmatrix},$$

so the eigenvalues of D are the roots of

$$\det\left(\begin{bmatrix} d_{11}-\lambda & & & & \\ & d_{22}-\lambda & & & \\ & & \ddots & & \\ & & & d_{n-1,n-1}-\lambda & \\ & & & & d_{nn}-\lambda \end{bmatrix}\right) = (d_{11}-\lambda)(d_{22}-\lambda)\dots(d_{nn}-\lambda).$$

The eigenvalues of D are $\{d_{11}, d_{22}, \dots, d_{nn}\}$.

The next result is useful in its own right, and we will have occasion to apply it a number of times in this book.

Theorem 5.2. *Eigenvectors $v_1, v_2, \dots, v_i$ that correspond to distinct eigenvalues are linearly independent.*

Overview:

The proof is algebraic. If $c_1v_1 + c_2v_2 + \dots + c_iv_i = 0$ and we show that $c_k = 0$, $1 \le k \le i$, then $v_1, v_2, \dots, v_i$ are linearly independent. Pairs of equations are created and subtracted, and in the process v_i is eliminated. Continue the process and eliminate v_{i-1} and so forth until arriving at an equation $Kc_1v_1 = 0$, $K \ne 0$. Since $v_1 \ne 0$, $c_1 = 0$. The same process can be used to show that $c_2 = 0, \dots, c_i = 0$.

Proof. Suppose that

$$c_1v_1 + c_2v_2 + \dots + c_iv_i = 0. \tag{5.4}$$

Multiply by A, noting that $Av_i = \lambda_iv_i$, to obtain

$$c_1\lambda_1v_1 + c_2\lambda_2v_2 + \dots + c_i\lambda_iv_i = 0. \tag{5.5}$$

Multiply Equation 5.4 by λ_i and subtract from Equation 5.5 to obtain Equation 5.6 that does not involve v_i.

$$\begin{aligned} &c_1(\lambda_1 - \lambda_i)v_1 + c_2(\lambda_2 - \lambda_i)v_2 + c_3(\lambda_3 - \lambda_i)v_3\\ &+ \dots + c_{i-1}(\lambda_{i-1} - \lambda_i)v_{i-1} = 0. \end{aligned} \tag{5.6}$$

Multiply Equation 5.6 by A to obtain

$$\begin{aligned} &c_1(\lambda_1-\lambda_i)\lambda_1 v_1 + c_2(\lambda_2-\lambda_i)\lambda_2 v_2 + c_3(\lambda_3-\lambda_i)\lambda_3 v_3 \\ &+\cdots + c_{i-1}(\lambda_{i-1}-\lambda_i)\lambda_{i-1}v_{i-1} = 0. \end{aligned} \tag{5.7}$$

Multiply Equation 5.6 by λ_{i-1} and subtract from Equation 5.7 to get an equation not involving v_{i-1}.

$$\begin{aligned} &c_1(\lambda_1-\lambda_i)(\lambda_1-\lambda_{i-1})v_1 + c_2(\lambda_2-\lambda_i)(\lambda_2-\lambda_{i-1})v_2 \\ &+\cdots + c_{i-2}(\lambda_{i-2}-\lambda_i)(\lambda_{i-2}-\lambda_{i-1})v_{i-2} = 0. \end{aligned}$$

If we continue by eliminating v_{i-2}, v_{i-3}, and so forth until eliminating v_2, we are left with

$$(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_i)c_1v_1 = 0.$$

Now $(\lambda_1-\lambda_2)\neq 0$, $(\lambda_1-\lambda_3)\neq 0, \ldots, (\lambda_1-\lambda_i)\neq 0$ by hypothesis, and so $c_1 = 0$. In a similar fashion, we can show that $c_2 = c_3 = \cdots = c_i = 0$, and thus $v_1, v_2, \ldots, v_i$ are linearly independent. □

If A has n linearly independent eigenvectors, we are now in a position to develop a method for diagonalizing A.

Theorem 5.3. *Suppose the $n\times n$ matrix A has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$. Place the eigenvectors as columns of the* eigenvector matrix $X = [v_1, v_2, \ldots v_n]$. *Then*

$$X^{-1}AX = D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

and A can be diagonalized.

Proof.

$$\begin{aligned} AX &= A\begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} = \begin{bmatrix} Av_1 & Av_2 & \ldots & Av_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \ldots & \lambda_n v_n \end{bmatrix} \\ &= \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = XD. \end{aligned} \tag{5.8}$$

Since $v_1, v_2, \ldots, v_n$ are linearly independent, X is invertible. From Equation 5.8, we have

$$D = X^{-1}AX.$$ □

To diagonalize matrix A we need to know that it has n linearly independent eigenvalues. Having distinct eigenvalues does the trick.

Theorem 5.4. *If an $n \times n$ matrix A has distinct eigenvalues, it can be diagonalized.*

Proof. Let $v_1, v_2, \ldots, v_n$ be eigenvectors of A corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. By Theorem 5.2, $v_1, v_2, \ldots, v_n$ are linearly independent and, by Theorem 5.3, A can be diagonalized. □

Remark 5.4. If the matrix does not have n linearly independent eigenvectors, it cannot be diagonalized.

Example 5.8. Let $A = \begin{bmatrix} 4 & 8 & 3 \\ 0 & -1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$ be the matrix of Example 5.3. We found that the eigenvalues are $\lambda_1 = 4$, $\lambda_2 = -1$, and $\lambda_3 = 2$. By Theorem 5.4, A can be diagonalized. In Example 5.3, we found eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 3/2 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}$ corresponding to λ_1, λ_2, and λ_3, respectively. To diagonalize A, form the eigenvector matrix

$$X = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -3/2 \\ 0 & 3/2 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is straightforward to verify that $X^{-1}AX = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. ■

Remark 5.2 states that a real symmetric matrix A always has n linearly independent eigenvectors and so can be diagonalized. If the characteristic equation of a nonsymmetric matrix A has a factor $(\lambda - \lambda_i)^k$, $k \geq 2$ there must be k linearly independent eigenvectors associated with eigenvalue λ_i for A to be diagonalizable.

Example 5.9. Let $A = \begin{bmatrix} 6 & 12 & 19 \\ -9 & -20 & -33 \\ 4 & 9 & 15 \end{bmatrix}$ be the matrix of Example 5.4. The characteristic polynomial of A is $(\lambda + 1)(\lambda - 1)^2$, so $\lambda = 1$ is a multiple eigenvalue with multiplicity $k = 2$. We found that there is only one linearly independent eigenvector associated with $\lambda = 1$, so A cannot be diagonalized. ■

In summary, the procedure for diagonalizing a matrix A can be done in series of steps:

1. Form the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of A.
2. Find the roots of p. If there are complex roots, the matrix cannot be diagonalized in $\mathbb{R}^{n\times n}$.
3. For each eigenvalue λ_i of multiplicity k_i, find k_i linearly independent eigenvectors. If this is not possible, A cannot be diagonalized.
4. Form the matrix $X = \begin{bmatrix} v_1 & v_2 & \dots & v_{n-1} & v_n \end{bmatrix}$ whose columns are eigenvectors of A corresponding to eigenvalues $\lambda_1, \lambda_2, \dots \lambda_{n-1}, \lambda_n$. Then, $D = X^{-1}AX$, where D is the diagonal matrix with $\lambda_1, \lambda_2, \dots \lambda_{n-1}, \lambda_n$ on its diagonal.

Example 5.10. Let $A = \begin{bmatrix} -5 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$. Its eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -5$. For A to be diagonalizable, there must be two linearly independent eigenvectors corresponding to $\lambda = 1$. We must solve the homogeneous system

$$\begin{bmatrix} -6 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix} x = 0.$$

Its solution space is all multiples of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This subspace has dimension 1, so A is not diagonalizable. ■

Example 5.11. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 5 & -1 & 0 & 0 \end{bmatrix}$. By performing expansion by minors down column 4, we see that the determinant of A is 0. Thus, A is not invertible and has an eigenvalue of 0. In fact, the eigenvalues of A are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 3$, $\lambda_4 = 1$, so the multiplicity of the 0 eigenvalue is 2. The homogeneous system

$$\begin{bmatrix} 1-0 & 0 & 0 & 0 \\ 4 & 3-0 & 0 & 0 \\ -2 & 2 & 0-0 & 0 \\ 5 & -1 & 0 & 0-0 \end{bmatrix} x = 0$$

can be row-reduced to the problem

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = 0,$$

which has the two solutions $x_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$, $x_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$. Now, $\begin{bmatrix} 0 & -3 & -2 & 1 \end{bmatrix}^{\mathrm{T}}$ is an eigenvector corresponding to $\lambda_3 = 3$, and $\begin{bmatrix} -1/6 & 1/3 & 1 & -7/6 \end{bmatrix}^{\mathrm{T}}$ corresponds to $\lambda_4 = 1$. A can be diagonalized by

$$X = \begin{bmatrix} 0 & 0 & 0 & -1/6 \\ 0 & 0 & -3 & 1/3 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -7/6 \end{bmatrix}. \qquad \blacksquare$$

Remark 5.5. The procedure we have presented for diagonalizing a matrix is based upon finding the roots of the characteristic equation. As we will see in the beginning of Chapter 10, the procedure is never done this way in practice because the roots of a polynomial are generally difficult to compute accurately.

5.3.1 Powers of Matrices

If a matrix A can be diagonalized, computing A^n is greatly simplified. Since $D = X^{-1}AX$, $A = XDX^{-1}$, and

$$A^2 = \left(XDX^{-1}\right)\left(XDX^{-1}\right) = (XD)\, I \left(DX^{-1}\right) = XD^2X^{-1}.$$

Continuing, we have

$$A^3 = A^2A = \left(XD^2X^{-1}\right)\left(XDX^{-1}\right) = XD^3X^{-1},$$

and in general by mathematical induction (Appendix B)

$$A^n = XD^nX^{-1}.$$

Example 5.12. The matrix $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is called the Fibonacci matrix because its powers can be used to compute the Fibonacci numbers

$$\begin{aligned} &f_0 = 0, \quad f_1 = 1, \\ &f_n = f_{n-1} + f_{n-2}, \quad n \geq 2. \end{aligned}$$

The first few numbers in the sequence are 0, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89. F is symmetric, so it can be diagonalized. The eigenvalues of F are (verify)

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2},$$

and the corresponding eigenvectors are (verify)

$$v_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

The eigenvalue $(1+\sqrt{5})/2 = 1.61803\ldots$ is called the *Golden ratio* and was known to the ancient Greeks. Some artists and architects believe the Golden ratio makes the most pleasing and beautiful shape. Using the eigenvalues and eigenvectors of F, we have

$$\begin{aligned} F &= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2\sqrt{5}} \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix}. \end{aligned}$$

Now $F^n = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix}$. Using MATLAB to compute F^{50} gives

$$\begin{bmatrix} 20365011074 & 12586269025 \\ 12586269025 & 7778742049 \end{bmatrix}.$$

It can be shown that

$$F^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

and

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right].$$

The latter formula is quite remarkable since each term in the formula involves $\sqrt{5}$. From F^{50}, we deduce that $f_{51} = 20365011074, f_{50} = 12586269025$, and $f_{49} = 7778742049$. ■

5.4 APPLICATIONS

In the introduction of this chapter, we noted that eigenvalues and eigenvectors have very significant applications to engineering and science. In this section, two applications are outlined. Later in this book, when we have more understanding of eigenvalues and eigenvectors, other applications will be presented in more detail.

5.4.1 Electric Circuit

In Section 2.8, we solved for the currents in a circuit that involved three batteries and four resistors (Figure 2.4). The resistors obey Ohm's Law, $V = RI$, where V is the voltage, R is the resistance, and I is the current. To determine the currents, we had to solve a system of linear algebraic equations. We will now add two inductors to the circuit (Figure 5.2). The relationship between the voltage $v(t)$ across an inductor with inductance L and the current $x(t)$ passing through it is described by the relation $v(t) = L\mathrm{d}x/\mathrm{d}t$. In other words, the voltage across an inductor is proportional to the rate of change of the current. As a result, the problem of determining the currents becomes a system of differential equations

$$\begin{aligned} x_1 - x_3 + x_2 &= 0, \\ (R_1 + R_4)\, x_1 + R_2 x_3 + L_2 \frac{\mathrm{d}x_1}{\mathrm{d}t} &= V_1 - V_2, \\ R_2 x_3 + R_3 x_2 + L_1 \frac{\mathrm{d}x_2}{\mathrm{d}t} &= V_3 - V_2. \end{aligned}$$

Choose the following values for the batteries, the resistors, and the inductors.

Components	V_1	V_2	V_3	R_1	R_2	R_3	R_4	L_1	L_2
	2 V	3 V	5 V	1 Ω	2 Ω	5 Ω	3 Ω	1 H	1 H

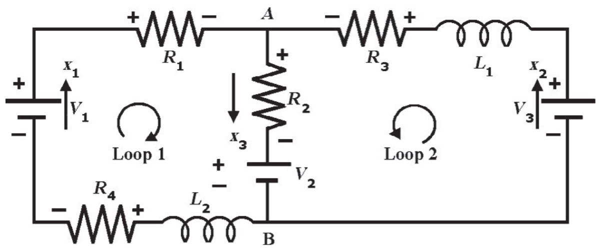

FIGURE 5.2 Circuit with an inductor.

The equations for current flow then become

$$x_1 - x_3 + x_2 = 0,$$
$$4x_1 + 2x_2 + \frac{dx_1}{dt} = -1,$$
$$2x_2 + 5x_3 + \frac{dx_2}{dt} = 2.$$

Solving for x_3 in terms of x_1 and x_2 results in the following system of differential equations:

$$\frac{dx_1}{dt} = -4x_1 - 2x_2 - 1,$$
$$\frac{dx_2}{dt} = -5x_1 - 7x_2 + 2,$$

which after conversion to matrix form is

$$\frac{dx}{dt} = Ax + b, \tag{5.9}$$

where $A = \begin{bmatrix} -4 & -2 \\ -5 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

To find the general solution to a 2×2 system of first-order differential equations with constant coefficients, first find a general solution to the homogeneous system

$$\frac{dx_h}{dt} = Ax_h, \tag{5.10}$$

and then determine a particular solution, $x_p(t)$ to Equation 5.9. The function $x(t) = x_h(t) + x_p(t)$ is a solution to Equation 5.9, since

$$\frac{dx}{dt} = \frac{dx_h}{dt} + \frac{dx_p}{dt} = Ax_h(t) + \left(Ax_p(t) + b\right) = A\left(x_h(t) + x_p(t)\right) + b = Ax + b.$$

For a proof that $x(t)$ is a general solution to Equation 5.9, see Ref. [10] or any book on elementary differential equations.

To determine a general solution to Equation 5.9, let $x_h(t) = vf(t)$,where v is a vector and f varies with time. Substituting $x_h(t)$ into Equation 5.10 results in

$$vf'(t) = f(t)\,Av,$$

so

$$Av = \left(\frac{f'(t)}{f(t)}\right)v.$$

For a fixed t, this is an eigenvalue problem, where $\lambda = f'(t)/f(t)$. If the eigenvalues of A are distinct, there are two eigenvalues λ_1 and λ_2 corresponding to linearly independent eigenvectors v_1 and v_2. For $i = 1, 2$, let

$$\lambda_i = \frac{f'(t)}{f(t)},$$

so

$$f'(t) = \lambda_i f(t),$$

for which a solution is

$$f(t) = c_i e^{\lambda_i t}.$$

Thus, the general solution to the homogeneous equation (5.10) is

$$x_h(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}.$$

It remains to determine a particular solution $x_p(t)$. The right-hand side of Equation 5.9 contains the constant vector b, so we will try a solution of the form $x_p(t) = w$, where w is a constant vector. Substituting this into Equation 5.9 gives $0 = Aw + b$ and, assuming A is nonsingular, w is the unique solution to

$$Aw = -b.$$

We now have the general solution

$$x(t) = c_1 v_1 \mathrm{e}^{\lambda_1 t} + c_2 v_2 \mathrm{e}^{\lambda_2 t} + w,$$

and are able to solve our problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} -4 & -2 \\ -5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = -9$, with corresponding eigenvectors $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}$, so

$$x_{\mathrm{h}}(t) = c_1 v_1 \mathrm{e}^{-2t} + c_2 v_2 \mathrm{e}^{-9t}.$$

To find $x_{\mathrm{p}}(t)$, solve the system

$$Aw = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The unique solution is $w = \begin{bmatrix} -0.61111 \\ 0.72222 \end{bmatrix}$, and so the general solution is

$$x(t) = c_1 \mathrm{e}^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-9t} \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix} + \begin{bmatrix} -0.61111 \\ 0.72222 \end{bmatrix}.$$

Assume that at $t = 0$, $x_1(0) = x_2(0) = 0$, so

$$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix} + \begin{bmatrix} -0.61111 \\ 0.72222 \end{bmatrix} = 0,$$

that results in the system

$$\begin{bmatrix} -1 & \frac{2}{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.61111 \\ -0.72222 \end{bmatrix},$$

whose solution is

$$c_1 = -0.64286, \quad c_2 = -0.079365.$$

The solution to Equation 5.9 is

$$x(t) = -0.64286\,\mathrm{e}^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 0.079365\,\mathrm{e}^{-9t} \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix} + \begin{bmatrix} -0.61111 \\ 0.72222 \end{bmatrix},$$

and the solution to the circuit problem is

$$\begin{aligned} x_1(t) &= 0.64286\,\mathrm{e}^{-2t} - 0.031746\,\mathrm{e}^{-9t} - 0.61111, \\ x_2(t) &= -0.64286\,\mathrm{e}^{-2t} - 0.079365\,\mathrm{e}^{-9t} + 0.72222, \\ x_3(t) &= -0.11111\mathrm{e}^{-9t} + 0.11111, \end{aligned}$$

whose graph is shown in Figure 5.3.

Note that the solution to the homogeneous equation (the transient solution) dies out quickly (Figure 5.3), leaving the particular solution (the steady state).

5.4.2 Irreducible Matrices

Our aim in Section 5.4.2 is to show how the eigenvalue problem can be used to create a ranking method. In our case, we will develop a simple method for ranking sports teams, but more advanced methods of ranking using eigenvalues are in use. For instance, sophisticated methods are used to rank NFL teams. Many of these ranking methods require that the matrix be irreducible.

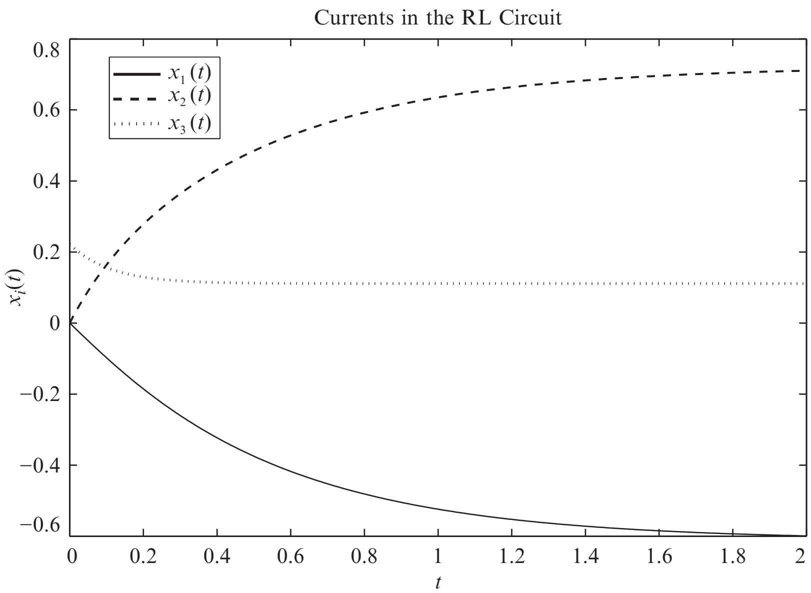

FIGURE 5.3 Currents in the *RL* circuit.

Definition 5.4. An $n \times n$ matrix A is reducible if its indices 1 to n can be divided into two disjoint nonempty sets $S = \{i_1, i_2, \ldots, i_\alpha\}$, $T = \left\{j_1, j_2, \ldots, j_\beta\right\}$, $\alpha + \beta = n$, such that

$$a_{i_p, j_q} = 0,$$

for $1 \le p \le \alpha$ and $1 \le q \le \beta$.

Example 5.13. Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 3 \\ 8 & 0 & 0 & 0 & 9 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 \end{bmatrix}$. Let $S = \{1, 3, 4, 5\}$ and $T = \{2\}$. Now, $a_{12} = a_{32} = a_{42} = a_{52} = 0$, so A is reducible. ■

For a matrix A to be *irreducible*, it must not be possible to perform such a partitioning. We will discuss two ways other using the definition to show that a matrix is irreducible, one involving simple graph theory, and the other an algebraic approach. For the graph approach, let the set $V = \{1, 2, \ldots, n\}$ and create vertices labeled "1," "2," . . ., "n." Connect vertex i to vertex j by a directed arc when $a_{ij} \neq 0$. Such a structure is called a *digraph*. For instance, consider the matrix T

$$T = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{5.11}$$

It is convenient to label the rows and columns:

$$\text{T} = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 5.4 is the digraph for T.

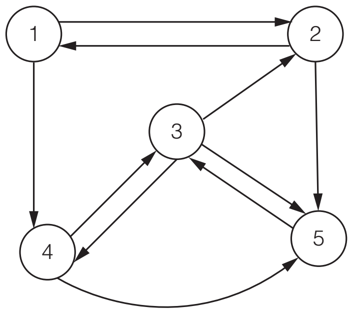

FIGURE 5.4 Digraph of an irreducible matrix.

A matrix is irreducible if beginning at any vertex, arcs can be followed to any other vertex, so that the partitioning in Definition 5.4 is not possible. This type of structure is called a *strongly connected digraph*. Our matrix T is irreducible.

The matrices we deal with for ranking purposes will consist entirely of nonnegative elements (all entries ≥ 0). Such a matrix A is said to be *nonnegative*, and we write $A \geq 0$. If all the elements of A are positive, then $A > 0$. An algebraic approach to verify that a nonnegative matrix is irreducible is specified by the following theorem. For a proof, see Ref. [11].

Theorem 5.5. *A is a nonnegative irreducible $n \times n$ matrix if and only if*

$$(I + A)^{n-1} > 0.$$

Example 5.14. Perform this computation for the matrix T (5.11) and for the matrix A of Example 5.13.

$$(I+T)^4 = \begin{bmatrix} 9 & 14 & 17 & 14 & 22 \\ 9 & 13 & 14 & 12 & 19 \\ 9 & 18 & 26 & 18 & 32 \\ 5 & 13 & 22 & 14 & 26 \\ 4 & 9 & 14 & 9 & 17 \end{bmatrix}, \quad (I+A)^4 = \begin{bmatrix} 9 & 0 & 8 & 4 & 7 \\ 19 & 1 & 12 & 11 & 9 \\ 12 & 0 & 9 & 7 & 8 \\ 8 & 0 & 7 & 2 & 4 \\ 7 & 0 & 4 & 4 & 2 \end{bmatrix}$$

T is irreducible and A is not. ■

Now we need to see how eigenvalues/eigenvectors are connected with irreducible matrices. The key is the *Perron-Frobenius theorem* for irreducible matrices (for a proof, see Ref. [12]). In the theorem, an eigenvector of matrix A is said to be *simple* if its corresponding eigenvalue is not a multiple root of the characteristic equation for A.

Theorem 5.6. *If the $n \times n$ matrix A has nonnegative entries, then there exists an eigenvector r with nonnegative entries, corresponding to a positive eigenvalue λ. Furthermore, if the matrix A is irreducible, the eigenvector r has strictly positive entries, is unique and simple, and the corresponding eigenvalue is the largest eigenvalue of A in absolute value.*

You know how to compute the distance between two vectors. If v is a vector in $\mathbb{R}^n$, then the length of v is written as $\text{length}\,(v) = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. The eigenvector r in Theorem 5.6 is computed by the formula

$$r = \lim_{n \longrightarrow \infty} \frac{A^n r_0}{\text{length}\,(A^n r_0)},$$

for any nonnegative vector r_0. For the purposes of computing the ranking vector, the book software distribution contains a MATLAB function, `perronfro`, that takes the matrix as an argument and returns an approximation to r and the corresponding eigenvalue λ. This process of computing an eigenvector is called the power method and will be discussed in Chapter 18.

Example 5.15. The matrix T (5.11) is irreducible, and $T \geq 0$, so the Perron-Frobenius theorem applies. The following MATLAB statements find the unique eigenvector and the corresponding largest eigenvalue in magnitude.

```
>> [r lambda] = perronfro(T)

r =
    0.4366
    0.3829
    0.5945
    0.4643
    0.3064

lambda =
    1.9404
```

■

5.4.3 Ranking of Teams Using Eigenvectors

Have you ever wondered how the Google search engine orders the results of a search? It uses a very large matrix and applies the PageRank process, which involves computing eigenvectors. We will not attempt to explain the process (see Refs. [13, 14]) but rather will present a much simpler procedure that is related to the *PageRank process*.

This discussion derives from Ref. [12], and the paper presents other ranking schemes. The problem is to rank things in order of importance based on some measure of the influence that they have over each other. Suppose that a set of n football teams represented by variables x_i, $1 \leq i \leq n$, are to be ranked. We assume that each team played every other team, and that elements $\{r_{ij}\}$ are weights used in ranking, where i refers to team i, j to team j and $r_{ii} = 0$. The ranking of team i is proportional to the sum of the rankings of the remaining teams weighted by r_{ij}, so

$$x_i = k \sum_{j=1}^{n} r_{ij} x_j, \quad 1 \leq i \leq n, \tag{5.12}$$

where k is the constant of proportionality. We can write Equations 5.12 in the matrix form $kRx = x$, where $R = [r_{ij}]$. This is an eigenvalue/eigenvector problem!

$$Rx = \frac{1}{k} x. \tag{5.13}$$

Theorem 5.6 applies to our eigenvalue problem 5.13 if the matrix R is irreducible. We have the problem of defining the r_{ij} so this is the case. There are many ways to do this, the simplest of which is to let $r_{ij} = 1$ if team i defeats team j or $r_{ij} = 0$ if team i loses to team j. The problem with this assignment is that the losing team gets no credit at all if the score is close, and the winning team gets no extra benefit if it scores many more points than the losing team. Also, this assignment will result in a row of 0s if a team loses all of its games, and such a matrix is not irreducible (convince yourself of this). A better approach is to base the value of r_{ij} on the score of the game. Let S_{ij} be the number of points scored by team i when it played team j, and define $r_{ij} = S_{ij}/(S_{ij} + S_{ji})$. This is an improvement but has the problem that if a game ends in a score like 6-0, the losing team gets no credit at all even though the score was close. We will settle on the following definition of r_{ij}:

$$r_{ij} = \begin{cases} \dfrac{S_{ij} + 1}{S_{ij} + S_{ji} + 2}, & i \neq j, \\ 0, & i = j. \end{cases} \tag{5.14}$$

The losing team gets some credit, there cannot be a zero row, and so R will be irreducible. For an example, assume that eight teams played each other and Table 5.1 contains the scores. For instance, when teams 1 and 2 played, team 1 scored 14 points and team 2 scored 7 points.

Applying Equation 5.14 to the data in Table 5.1 gives the matrix

$$R = \begin{bmatrix} 0.0000 & 0.6522 & 0.3333 & 0.5806 & 0.4717 & 0.2000 & 0.4800 & 0.4286 \\ 0.3478 & 0.0000 & 0.3191 & 0.7442 & 0.6133 & 0.1071 & 0.5556 & 0.6304 \\ 0.6667 & 0.6809 & 0.0000 & 0.5513 & 0.1818 & 0.3000 & 0.2222 & 0.5303 \\ 0.4194 & 0.2558 & 0.4487 & 0.0000 & 0.4286 & 0.8000 & 0.4815 & 0.5000 \\ 0.5283 & 0.3867 & 0.8182 & 0.5714 & 0.0000 & 0.2632 & 0.3333 & 0.6000 \\ 0.8000 & 0.8929 & 0.7000 & 0.2000 & 0.7368 & 0.0000 & 0.4375 & 0.8000 \\ 0.5200 & 0.4444 & 0.7778 & 0.5185 & 0.6667 & 0.5625 & 0.0000 & 0.5814 \\ 0.5714 & 0.3696 & 0.4697 & 0.5000 & 0.4000 & 0.2000 & 0.4186 & 0.0000 \end{bmatrix}.$$

The book software distribution contains a function `rankmatrix` that takes the matrix S of scores and returns the ranking matrix R obtained by applying Equation 5.14. Then apply the function `perronfro` to R to obtain the ranking vector.

TABLE 5.1 Ranking Teams

Team	1	2	3	4	5	6	7	8
1	0	14	3	17	24	0	35	2
2	7	0	14	31	45	2	29	28
3	7	31	0	42	7	17	7	34
4	12	10	34	0	20	31	12	14
5	27	28	35	27	0	14	15	20
6	3	24	41	7	41	0	13	35
7	38	23	27	13	31	17	0	49
8	3	16	30	14	13	8	35	0

Example 5.16. Assuming the scores in Table 5.1, the following command sequence finds the ranking vector. The largest component of the ranking vector is the top-rated team, the second largest the second rated team, and so forth.

```
>> R = rankmatrix(S);
>> [r lambda] = perronfro(R)

r =
    0.3198
    0.3330
    0.3134
    0.3506
    0.3448
    0.4404
    0.4050
    0.2981

lambda =

    3.9342
```

By looking at the vector r, we see that the teams are ranked from first to last as follows:

$$6 \quad 7 \quad 4 \quad 5 \quad 2 \quad 1 \quad 3 \quad 8$$

■

5.5 COMPUTING EIGENVALUES AND EIGENVECTORS USING MATLAB

The computation of eigenvalues and eigenvectors in MATLAB is done by the function `eig(A)`. To obtain the eigenvalues and associated eigenvectors, call it using the format

```
>> [V,D] = eig(A);
```

D is a diagonal matrix of eigenvalues and V is a matrix whose columns are the corresponding eigenvectors; for instance, if the eigenvalue/eigenvector pairs are

$$\lambda_1 = -12.2014, \quad v_1 = \begin{bmatrix} 0.0278 \\ 0.4670 \\ -0.8838 \end{bmatrix},$$

$$\lambda_2 = 1.3430, \begin{bmatrix} 0.9925 \\ 0.0357 \\ 0.1167 \end{bmatrix},$$

$$\lambda_3 = 5.8584, \begin{bmatrix} 0.5911 \\ 0.7333 \\ 0.3359 \end{bmatrix},$$

then

$$V = \begin{bmatrix} 0.0278 & 0.9925 & 0.5911 \\ 0.4670 & 0.0357 & 0.7333 \\ -0.8838 & 0.1167 & 0.3359 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -12.2014 & 0 & 0 \\ 0 & 1.3430 & 0 \\ 0 & 0 & 5.8584 \end{bmatrix}.$$

If you want only the eigenvalues, call `eig` as follows:

```
>> E = eig(A);
```

Example 5.17. Compute the eigenvalues and eigenvectors for the matrix $B = \begin{bmatrix} 1 & 6 & 3 \\ -1 & 4 & 9 \\ 12 & 35 & 1 \end{bmatrix}$. ■

```
>> [V E] = eig(B)

V =
    0.0118    0.9119    0.2500
    0.4211   -0.3220    0.4278
   -0.9069    0.2545    0.8686

E =
  -15.4092         0         0
         0   -0.2812         0
         0         0   21.6905
>> eig(B)

ans =
  -15.4092
   -0.2812
   21.6905
```

■

5.6 CHAPTER SUMMARY

Defining Eigenvalues and Their Associated Eigenvectors

λ is an eigenvalue of $n \times n$ matrix A, and $v \neq 0$ is an eigenvector if $Av = \lambda v$; in other words, Av is parallel to v and either shrinks or contracts it. The relationship $Av = \lambda v$ is equivalent to $(A - \lambda I)\,v = 0$, and in order for there to be a nontrivial solution, we must have

$$\det(A - \lambda I) = 0.$$

This is called the characteristic equation, and the polynomial

$$p(\lambda) = \det(A - \lambda I)$$

is the characteristic polynomial. The eigenvalues are the roots of the characteristic polynomial, and an eigenvector associated with an eigenvalue λ is a solution to the homogeneous system

$$(A - \lambda I)\,v = 0.$$

The process of finding the eigenvalues and associated eigenvectors would seem to be

Locate the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ *of* p *and find a nonzero solution to* $(A - \lambda_i I)\,v_i = 0$ *for each* λ_i.

There is a serious problem with this approach. If p has degree five or more, the eigenvalues must be approximated using numerical techniques, since there is no analytical formula for roots of such polynomials. We will see in Chapter 10 that polynomial root finding can be difficult. A small change in a polynomial coefficient can cause large changes in its roots.

Selected Properties of Eigenvalues and Eigenvectors

A matrix with a 0 eigenvalue is singular, and every singular matrix has a 0 eigenvalue. If we can find the eigenvalues of A accurately, then $\det A = \prod_{i=1}^{n} \lambda_i$. If we happen to need the determinant, this result can be useful.

Matrix Diagonalization

Square matrices A and B are similar if there exists an invertible matrix X such that $B = X^{-1}AX$, and similar matrices have the same eigenvalues. The eigenvalues of A are the diagonal elements of B, and we are said to have diagonalized A. As we will see in later chapters, diagonalization is a primary tool for developing many results.

To diagonalize a matrix requires that we find n linearly independent eigenvectors. If the matrix has n distinct eigenvalues, then it has a basis of n eigenvectors. Form X by making its columns the eigenvectors, keeping the eigenvalues in the same order in the diagonal matrix. If a matrix is symmetric, it has n linearly independent eigenvectors, even in the presence of eigenvalues of multiplicity two or more. Furthermore, the matrix X is orthogonal. If a matrix does not have n linearly independent eigenvectors, it cannot be diagonalized.

If a matrix A is diagonalizable, then it is simple to compute powers of A, since

$$A^k = XD^kX^{-1} = X \begin{bmatrix} \sigma_1^k & & & \\ & \sigma_2^k & & \\ & & \ddots & \\ & & & \sigma_n^k \end{bmatrix} X^{-1}.$$

Applications

The applications of eigenvalues are vast, including such areas as the solution of differential equations, structural mechanics, and the study of vibrations, where they represent the natural frequencies of a system.

In electrical engineering, when a circuit contains resistors, inductors, and batteries, there results a system of first-order differential equations of the form $\mathrm{d}x/\mathrm{d}t = Ax + b$, and the eigenvalues of A are required for the solution.

A very interesting application of eigenvalues and eigenvectors is in the theory of ranking. The text provides a simple example of ranking teams in a tournament.

Using MATLAB to Compute Eigenvalues and Eigenvectors

The computation of eigenvalues or both eigenvalues and eigenvectors using MATLAB is straightforward. To compute just the eigenvalues, use the format

```
>> E = eig(A);
```

and to find the eigenvectors and a diagonal matrix of eigenvalues, use

```
>> [V,D] = eig(A);
```

If A has distinct eigenvalues, then $V^{-1}AV = D$. If A has n linearly independent eigenvectors, this is also true. If A is symmetric, then things are even nicer, since $P^{\mathrm{T}}AP = D$, where P is orthogonal.

5.7 PROBLEMS

5.1 Find the eigenvalues and associated eigenvectors for the matrix $\begin{bmatrix} 1 & 3 \\ 0 & 9 \end{bmatrix}$.

5.2 Find the eigenvalues and associated eigenvectors for the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$.

5.3 Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix}$. Verify that $\left(A^{-1}\right)^{\mathrm{T}} = \left(A^{\mathrm{T}}\right)^{-1}$.

5.4 Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 7 \\ 1 & -4 \end{bmatrix}$. Verify that AB and BA have the same eigenvalues.

5.5 Show that the following properties hold for similar matrices A and B.

a. A is similar to A.
b. If B is similar to A, then A is similar to B.
c. If A is similar to B and B is similar to C, then A is similar to C; in other words, similarity is transitive.

5.6 Let $A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$.

a. Verify that $\det(A - \lambda I)$, the characteristic polynomial of A, is given by$(\lambda - 1)\lambda(\lambda - \frac{1}{4})$.
b. Diagonalize A.

5.7 Assume A can be diagonalized. Under what conditions will

$$\lim_{k \longrightarrow \infty} A^k = 0?$$

5.8 Solve the first-order system of differential equations with initial conditions:

$$\frac{dx_1}{dt} = -3x_1 + x_2,$$

$$\frac{dx_2}{dt} = 2x_1 - 4x_2 + 1,$$

$$x_1(0) = 1, \quad x_2(0) = 0.$$

5.9 Draw the digraph for each matrix and determine which matrices are irreducible.

a. $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

5.10

a. Show that $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ has only one eigenvalue $\lambda = a$ of multiplicity two and that all eigenvectors are multiples of $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

b. Consider a general version of part (a), $A^{n \times n} = \begin{bmatrix} a & 1 & & & 0 \\ & a & 1 & & \\ & & a & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & a \end{bmatrix}$. Show that A has one eigenvalue of multiplicity n and that all eigenvectors are multiples of $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

5.11 Using matrix of Problem 5.2, verify that the eigenvalues of A^{T} and A are equal. Can you say the same about the eigenvectors?

5.12

a. If A and B are $n \times n$ upper-triangular matrices, are the eigenvalues of $A + B$ the sum of the eigenvalues and A and B?

b. Is part (a) true for two arbitrary $n \times n$ matrices A and B?

5.13 If A is an $n \times n$ nonsingular matrix, prove that $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$.

5.14 If A and B are $n \times n$ matrices, prove that AB and BA have the same eigenvalues. Hint: You must show that every eigenvalue of AB is an eigenvalue of BA, and every eigenvalue of BA is an eigenvalue of AB. Suppose λ is an eigenvalue of AB. Then $ABx = \lambda x$, so $(BA)\,Bx = \lambda\,(Bx)$.

5.15 Show that the trace of a 3×3 matrix is equal to the sum of its eigenvalues using the following steps:

a. Find the characteristic polynomial $p\,(\lambda)$ and show that the coefficient of λ^2 is the trace of A.

b. Explain why $p\,(\lambda) = (-1)\,(\lambda - \lambda_1)\,(\lambda - \lambda_2)\,(\lambda - \lambda_3)$, where the λ_i are the eigenvalues of A.

c. Show that the coefficient of λ^2 is $\lambda_1 + \lambda_2 + \lambda_3$, and argue that this completes the proof.

5.16 Assume A is a real $n \times n$ matrix with a complex eigenvalue λ, and v is an associated eigenvector. If $\overline{v}$ is the complex conjugate of v, show that $\overline{v}$ is an eigenvector of A associated with eigenvalue $\overline{\lambda}$.

5.17 Prove that A and A^{T} have the same eigenvalues. Hint: By Problem 4.19, $\det A = \det A^{\mathrm{T}}$. Apply this result to the characteristic equation of A.

5.7.1 MATLAB Problems

5.18 The MATLAB function `eigshow` is a graphical demonstration of eigenvalues and eigenvectors. When invoked by `eigshow(A)`, where A is a 2×2 matrix, a graphical dialog appears. Do not press the button labeled **eig/(svd)**. You will see two vectors, a unit vector x, and the vector Ax. We will show in Chapter 15 that as the tip of x traces out the unit circle $x_1^2 + x_2^2 = 1$ the tip of the vector Ax traces out an ellipse whose center is the center of the circle. Move x with the mouse until x and Ax are parallel, if you can. If you are successful, Ax is a multiple of x, so $Ax = kx$. Since x has length 1, length $(Ax) = |k|$ and k is an eigenvalue of A corresponding to the eigenvector x, and its magnitude is $|k|$. Run `eigshow` for each of the following three matrices and estimate the eigenvalues, if you can make Ax parallel to x. There are three possibilities: there are two distinct eigenvalues, a double eigenvalue, and two complex conjugate eigenvalues.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}$$

5.19 Diagonalize $A = \begin{bmatrix} 26 & 48 & 8 \\ 35 & 28 & 13 \\ 45 & 7 & 43 \end{bmatrix}$.

5.20 Use MATLAB to find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 6 & 0 & -1 & 5 \\ 5 & -9 & 22 & 2 & 1 \\ 0 & 1 & 3 & 5 & 7 \\ 9 & 0 & -4 & -7 & -1 \\ 3 & 5 & 2 & 15 & 35 \end{bmatrix}.$$

Note that two of its eigenvalues and its corresponding eigenvectors are complex.

5.21 In MATLAB, `W = wilkinson(n)` returns one of Wilkinson's $n \times n$ eigenvalue test matrices. The matrix is symmetric and tridiagonal, with pairs of nearly, but not exactly, equal eigenvalues. The most frequently used case is `wilkinson(21)`. Its two largest eigenvalues are both about 10.746; they agree to 14, but not to 15, decimal places. Find the eigenvalues of the matrices `wilkinson(11)` and `wilkinson(21)`.

5.22

Team	1	2	3	4	5	6
1	0	17	25	25	10	30
2	38	0	24	48	21	29
3	20	31	0	14	24	17
4	36	3	25	0	24	45
5	24	30	13	14	0	0
6	28	24	20	10	23	0

a. Rank the teams whose scores are given in the table using Equation 5.14.
b. Use the simple scheme $a_{ij} = 1$ if team i defeats team j and $a_{ij} = 0$ if team i loses to team j. Compare the results of part (a).
c. Use the formula $a_{ij} = S_{ij}/(S_{ij} + S_{ji})$, and compare the results with those of parts (a) and (b).

5.23 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -7 & -4 \\ -1 & -3 & 1 \end{bmatrix}$.

a. Perform the computation

```
>> EA = eig(A);
>> EAINV = eig(inv(A));
```

b. Does part (a) motivate a general result concerning the eigenvalues of A and A^{-1}? If your answer is yes, prove it.

5.24 The Cayley-Hamilton theorem is an interesting result in theoretical linear algebra. It says that any $n \times n$ matrix satisfies its own characteristic equation. For instance, if the characteristic polynomial for a matrix A is $\lambda^3 + 3\lambda^2 - \lambda + 1$, then $A^3 + 3A^2 - A + I = 0$. Verify the Cayley-Hamilton theorem for each matrix. *Note*: The MATLAB function `poly(A)` returns a vector containing the coefficients of A's characteristic polynomial from highest to lowest power of λ.

a. $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & -6 \\ 0 & 2 & 3 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & -5 & 2 & 55 & 12 \\ 0 & 4 & 13 & 6 & -8 \\ 0 & 0 & 18 & 1 & -56 \\ 0 & 0 & 0 & -7 & 88 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

5.25 The MATLAB statement

```
>> F = gallery('frank',n,1);
```

returns an $n \times n$ matrix. Create the matrix `F = gallery('frank',15,1)` and perform the following computations:
a. The determinant of any Frank matrix is 1. Verify this for F.
b. If n is odd, 1 is an eigenvalue. Verify this for F.
c. Some of the eigenvalues of F are sensitive to changes in the entries of F. Perturb the entries of F by executing the statement

```
>> F = F + 1.0e-8*ones(15,15);
```

and compute the eigenvalues. Comment on the change in eigenvalues between the original F and the perturbed F.

5.26 If a matrix has eigenvalues of multiplicity greater than 1, generally those eigenvalues are more sensitive to small changes in the matrix. This means that small changes in the matrix might cause significant changes in its eigenvalues.

a. Build the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$ and execute

```
>> [V D] = eig(A);
```

b. How many linearly independent eigenvectors does A have?

c. Build the matrix $B = \begin{bmatrix} 0.9999 & 0 & 0 \\ 0.9999 & 0.9998 & 0 \\ 4.9999 & -3.0001 & 1.0001 \end{bmatrix}$ and show it has three distinct eigenvalues.

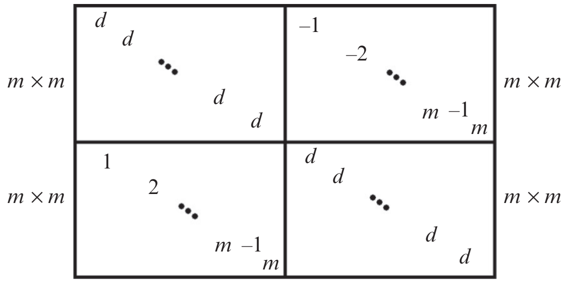

FIGURE 5.5 Hanowa matrix.

d. Let $\delta = \begin{bmatrix} 0 & 1.0 \times 10^{-6} & 1.0 \times 10^{-6} \\ 0 & 0 & 1.0 \times 10^{-6} \\ 0 & 0 & 0 \end{bmatrix}$ and compute the eigenvalues of $B + \delta$. Comment on the results and propose a relationship between A and $B + \delta B$ that might account for what you see.

5.27 A *block structured matrix* is built by putting together submatrices, where each submatrix is a block. An example is a Hanowa matrix. If m is an integer and d is a real number, then a $(2m) \times (2m)$ Hanowa matrix has block structure (Figure 5.5). We will discuss block matrices in Section 9.1.4.

a. Using the MATLAB functions `eye` and `diag`, construct a 6×6 Hanowa matrix H with $d = 3$.

b. Find the eigenvalues of H.

c. Using $d = 3$, build Hanowa matrices of dimensions 10×10 and 20×20 and compute their eigenvalues.

d. From your results in parts (b) and (c), propose a formula for the eigenvalues of an $n \times n$ Hanowa matrix.