Appendix C

Chebyshev Polynomials

Chebyshev polynomials have important uses in developing convergence results for algorithms; for instance, the analysis for the convergence of the conjugate gradient method in Chapter 21 can be done using these polynomials (see Ref. [1, pp. 312-316]). They also play a large role in convergence results for methods that compute eigenvalues of large sparse matrices (see Ref. [3, pp. 151-159]). There are additional applications to least-squares and interpolation. The book does not actually use Chebyshev polynomials in proofs, but the statement of certain theorems involve Chebyshev polynomials, and so we will give a brief overview of their definition and properties.

C.1 DEFINITION

The definition begins with the form of these polynomials on the interval $-1 \leq t \leq 1$, and then their definition is extended to all real numbers.

Definition C.1. On the interval $-1 \leq x \leq 1$, the Chebyshev polynomial of degree $n \geq 1$ is defined as $T_n(x) = \cos(n\theta)$, where $\theta \in [\ 0,\ \pi\ ]$ and $\cos(\theta) = x$. More compactly, $T_n(x) = \cos\left(n \cos^{-1}(x)\right)$.

Looking at the definition, it is not clear that $T_n(x)$ is a polynomial. From the definition, $T_0(x) = 1$ and $T_1(x) = x$, and we can use some trigonometry to find $T_n(x)$, $n \geq 2$.

Recall that

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Now,

$$\begin{aligned} T_{n+1}(x) &= \cos[(n+1)\theta] = \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) \\ T_{n-1}(x) &= \cos[(n-1)\theta] = \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta) \end{aligned}$$

Add the two equations and obtain

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x),$$

and after rearrangement

$$\begin{aligned} T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x),\ n \geq 1, \\ T_0(x) &= 1,\ T_1(x) = x \end{aligned} \tag{C.1}$$

Using Equation C.1, we can extend $T_n(x)$ for all $x \in \mathbb{R}$, and by application of the recurrence relation we can determine a Chebyshev polynomial of any degree. For instance,

$$\begin{aligned} T_2(x) &= 2xT_1(x) - T_0(x) = 2x^2 - 1 \\ T_3(x) &= 2x\left(2x^2 - 1\right) - x = 4x^3 - 3x \\ T_4(x) &= 2x\left(4x^3 - 3x\right) - \left(2x^2 - 1\right) = 8x^4 - 8x^2 + 1 \end{aligned} \tag{C.2}$$

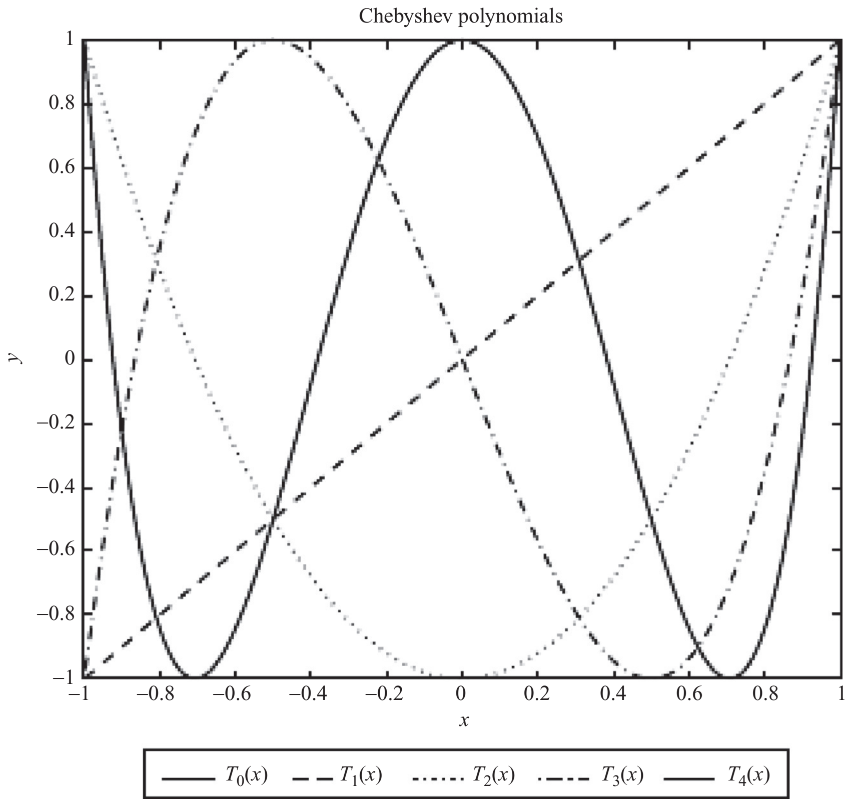

FIGURE C.1 The first five Chebyshev polynomials.

C.2 PROPERTIES

There are many interesting and useful properties of the Chebyshev polynomials, and we mention just a few.

- Since $T_n(x) = \cos(n\theta)$, it follows that

$$|T_n(x)| \leq 1, \ -1 \leq x \leq 1, \ n \geq 0.$$

- We can discern a pattern by looking at Equations C.2. For $n = 2, 3, 4$, the Chebyshev polynomials have the form

$$T_n(x) = 2^{n-1}x^n + O\left(x^{n-1}\right) \tag{C.3}$$

  This is actually true for all $n \geq 1$ (Problem C.2). As a result of Equation C.3, the Chebyshev polynomials grow very quickly with increasing n.
- The roots of $T_n(t)$ are

$$x_i = \cos\left((2i-1)\pi/(2n)\right), \ for \ i = 1, 2, \ldots, n.$$

A polynomial whose highest degree term has a coefficient of 1 is called a *monic polynomial.* If we define $\hat{T}_n(x) = \left(\frac{1}{2^{n-1}}\right) T_n(x)$, then $\hat{T}_n(x)$ is a monic polynomial. By application of these and other properties, we have following theorem (see [88], Chapter 3, Theorem 3.3)

Theorem C.1. *Consider all possible monic polynomials of degree n. Of all these, the monic polynomial $\hat{T}_n(x)$ has the smallest maximum over $-1 \leq x \leq 1$, and its maximum is $\frac{1}{2^{n-1}}$.*

C.3 PROBLEMS

C.1 Using the recurrence relation C.1,find the Chebyshev polynomials $T_5(x)$ and $T_6(x)$.

C.2 Using the recurrence relation C.1 and mathematical induction, prove relationship C.3.

C.3.1 MATLAB Problems

C.3

a. Write a function, chebyshev, that takes the degree n as an argument and returns the MATLAB form of the polynomial.

b. Use your function to graph $T_5(x)$ and $T_6(x)$.

c. We have said the Chebyshev polynomials grow quickly as n increases. Let $x = 1 + \epsilon$, $\epsilon = \left\{\ 10^{-6},\ \ 10^{-5},\ \ 10^{-4},\ \right\}$. For each ϵ, evaluate $T_n(x)$, $n = \left\{\ 5\ \ 10\ \ 50\ \ 75\ \right\}$.

C.4

a. Let $p(x) = x^3 + 0.00001x^2 - 0.00001x$. Graph $p(x)$ and $T_3(x)/4$ over $\left[\ -1,\ \ 1\ \right]$.

b. Compute the maximum of $p(x)$ and $T_3(x)/4$.