Chapter 3

Subspaces

You should be familiar with

- Vector operations
- Row reducing a matrix
- Solving linear systems using row elimination

3.1 INTRODUCTION

Throughout this chapter, we will be studying $\mathbb{R}^n$, the set of n-dimensional column vectors with real-valued components. We continue our study of matrices by considering a class of subsets of $\mathbb{R}^n$ called *subspaces*. These arise naturally, for example, when we solve a system of n linear homogeneous equations in n unknowns. The *row space, column space*, and *null space* of the coefficient matrix play a role in many applications. We also study the concept of linear independence of a set of vectors, which gives rise to the concept of subspace dimension.

Remark 3.1. We will use the mathematical symbol $\in$ that means "contained in." For instance, $u \in \mathbb{R}^2$ means that u is a vector in the plane, so $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, where u_1, u_2 are real numbers.

3.2 SUBSPACES OF $\mathbb{R}^n$

Definition 3.1. A subset S of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if

1. The zero vector belongs to S (i.e., $0 \in S$);
2. If $u \in S$ and $v \in S$, then $u + v \in S$ (S is said to be closed under vector addition);
3. If $u \in S$ and $t \in \mathbb{R}$, then $tu \in S$ (S is said to be closed under scalar multiplication).

$\mathbb{R}^n$ is a subspace of itself, and we call $\mathbb{R}^n$ a *vector space*. The complete definition of a vector space is very general, and we will not provide it here. In Chapter 12, we will introduce Fourier series as an example of a vector space whose elements are functions.

Example 3.1. Let A be an $n \times n$ matrix. Then the set of vectors $x \in \mathbb{R}^n$ satisfying $Ax = 0$ is a subspace of $\mathbb{R}^n$ called the *null space* of A and is denoted by $N(A)$.
Verify each property of a subspace.

1. $A \times 0 = 0$, so $0 \in N(A)$.
2. If $x, y \in N(A)$, then $Ax = 0$ and $Ay = 0$, so $A(x + y) = Ax + Ay = 0 + 0 = 0$ and $x + y \in N(A)$.
3. If $x \in N(A)$ and $t \in \mathbb{R}$, then $A(tx) = t(Ax) = t0 = 0$, so $tx \in N(A)$. ∎

Example 3.2. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the only solution to $Ax = 0$ is $x = 0$, so $N(A) = \{0\}$, the set consisting of just the zero vector.

The matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, so $x_1 + 2x_2 = 0$, and $x_1 = -2x_2$. $N(A)$ is the set of all scalar multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, a line in the plane. ∎
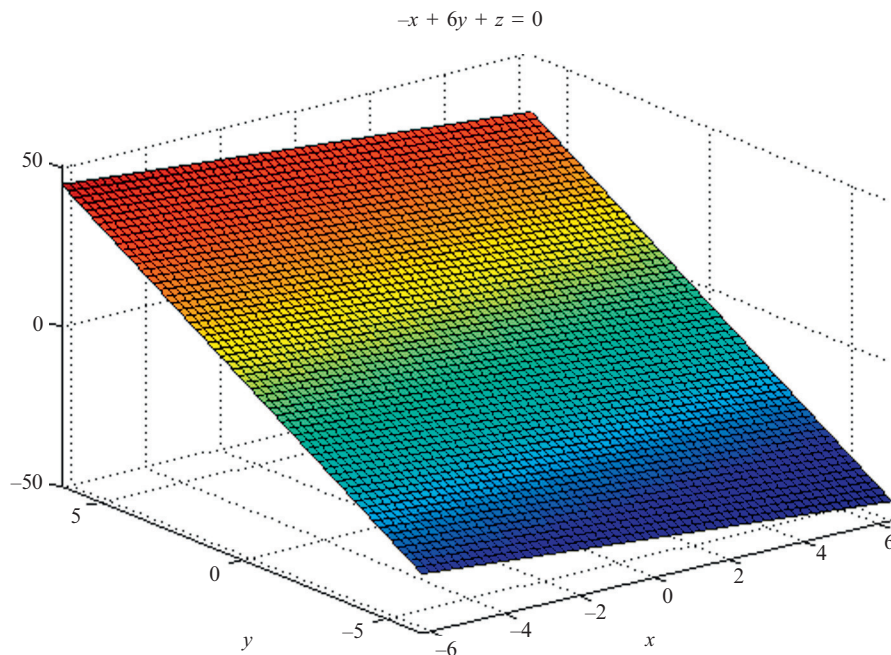
FIGURE 3.1 Subspace spanned by two vectors.

In the vector space $\mathbb{R}^3$, take the vectors $\begin{bmatrix} 1 & 1 & 5 \end{bmatrix}^T$ and $\begin{bmatrix} 2 & -1 & -8 \end{bmatrix}^T$ and form all possible linear combinations $c_1 \begin{bmatrix} 1 & 1 & 5 \end{bmatrix}^T + c_2 \begin{bmatrix} 2 & -1 & -8 \end{bmatrix}^T$. This set of vectors is a plane in $\mathbb{R}^3$ (Figure 3.1). ∎

Definition 3.2. If $x_1, \ldots, x_m$ is a set of vectors in $\mathbb{R}^n$, then an expression of the form $c_1 x_1 + \cdots + c_m x_m$ is said to be a *linear combination* of $x_1, \ldots, x_m$.

Theorem 3.1. *Let $x_1, \ldots, x_m \in \mathbb{R}^n$. Then the set consisting of all linear combinations*

$$c_1 x_1 + \cdots + c_m x_m,$$

where $c_1, \ldots, c_m \in \mathbb{R}$ is a subspace of $\mathbb{R}^n$. This subspace is called the subspace spanned by $x_1, \ldots, x_m$ and is denoted by

$$\text{span}\{x_1, \ldots, x_m\}.$$

Proof. To show that the set of all linear combinations of $x_1, x_2, \ldots, x_m$ is a subspace, we must verify properties 1, 2, and 3 of Definition 3.1.

Property 1: $0 = 0x_1 + \cdots + 0x_m$, so $0 \in \text{span}\{x_1, \ldots, x_m\}$.

Property 2: If $x, y \in \text{span}\{x_1, \ldots, x_m\}$, then $x = c_1 x_1 + \cdots + c_m x_m$ and $y = d_1 x_1 + \cdots + d_m x_m$, so

$$\begin{aligned} x + y &= (c_1 x_1 + \cdots + c_m x) + (d_1 x_1 + \cdots + d_m x_m) \\ &= (c_1 + d_1) x_1 + \cdots + (c_m + d_m) x_n \end{aligned}$$

and $x + y \in \text{span}\{x_1, \ldots, x_m\}$.

Property 3: If $x \in \text{span}\{x_1, \ldots, x_m\}$ and $t \in \mathbb{R}^n$, then $x = c_1 x_1 + \cdots + c_m x_m$, $tx = t(c_1 x_1 + \cdots + c_m x_m) = (tc_1) x_1 + \cdots + (tc_m) x_m \in \text{span}\{x_1, \ldots, x_m\}$. □

Definition 3.3. If A is an $n \times n$ matrix, the subspace spanned by the columns of A is a subspace of $\mathbb{R}^n$, called the *column space* of A. Also, the subspace spanned by the rows of A is a subspace of $\mathbb{R}^n$ called the *row space* of A.

Example 3.3. The $n \times n$ identity matrix has columns

$$e_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T, e_2 = \begin{bmatrix} 0 & 1 & \ldots & 0 \end{bmatrix}^T, \ldots, e_{n-1} = \begin{bmatrix} 0 & \ldots & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^T.$$

Since $\begin{bmatrix} x_1 & x_2 & \ldots & x_{n-1} & x_n \end{bmatrix}^{\mathrm{T}} = x_1 e_1 + x_2 e_2 + \cdots + x_{n-1} e_{n-1} + x_n e_n$, the column space of I is $\mathbb{R}^n$. The e_i are called the *standard basis vectors*. Any vector in $\mathbb{R}^n$ can be written as a linear combination of the standard basis vectors. In a similar fashion, the rows of I span $\mathbb{R}^n$. ∎

Example 3.4.   The equation $2x - 3y + 5z = 0$ defines a relationship between the components of a vector $\begin{bmatrix} x & y & z \end{bmatrix}^{\mathrm{T}}$. Find the subspace S of $\mathbb{R}^3$ spanned by all such vectors. If $[x, y, z]^{\mathrm{T}} \in S$, then $x = \frac{3}{2}y - \frac{5}{2}z$, so

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2}y - \frac{5}{2}z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}.
$$

Thus, S is the subspace spanned by $\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}$. This subspace is not $\mathbb{R}^3$. Consider the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and determine

if it can be written as a linear combination of $\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}$. There must be scalars c_1 and c_2 such that

$$
c_1 \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.
$$

This requires that

$$
\frac{3}{2}c_1 - \frac{5}{2}c_2 = 1.
$$

We must have $c_1 = 2$ and $c_2 = 3$, but

$$
\frac{3}{2}(2) - \frac{5}{2}(3) = -4\frac{1}{2} \neq 1.
$$

The two vectors do not span $\mathbb{R}^3$. In general, it takes n vectors to span $\mathbb{R}^n$. ∎

3.3   LINEAR INDEPENDENCE

Definition 3.4.   The concept of linear independence of a set of vectors in $\mathbb{R}^n$ is extremely important in linear algebra and its applications.

Vectors $x_1, \ldots, x_m$ in $\mathbb{R}^n$ are said to be *linearly dependent* if there exist scalars $c_1, \ldots, c_m$, *not all zero*, such that

$$
c_1 x_1 + \cdots + c_m x_m = 0. \tag{3.1}
$$

Suppose $c_i \neq 0$. Then, $x_i = -(c_1 x_1 + c_2 x_2 + \cdots + c_{i-1} x_{i-1} + c_{i+1} x_{i+1} + \cdots + c_m x_m)/c_i$. The vector x_i can be written as a linear combination of the remaining vectors; in other words, it is dependent on them. The vectors $x_1, \ldots, x_m$ are called *linearly independent* if they are not linearly dependent. To test for linear independence, Equation 3.1 is a linear homogeneous equation with unknowns $\begin{bmatrix} c_1 & c_2 & \ldots & c_{m-1} & c_m \end{bmatrix}^{\mathrm{T}}$. The vectors are linearly independent if the system has only the trivial solution $c_1 = 0, \ldots, c_m = 0$. Conversely, if $x_1, x_2, \ldots, x_m$ are linearly independent, then the homogeneous system has only the trivial solution.

Example 3.5.   Are the following three vectors in $\mathbb{R}^3$ linearly independent or dependent?

$$
x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -1 \\ 7 \\ 12 \end{bmatrix}.
$$

Form

$$
c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

This corresponds to the homogeneous system.

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 7 \\ 3 & 2 & 12 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 7 \\ 3 & 2 & 12 \end{bmatrix} \xrightarrow[\substack{R2 = R2 - 2R1 \\ R3 = R3 - 3R1}]{} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 9 \\ 0 & 5 & 15 \end{bmatrix} \xrightarrow[\substack{R3 = R3 - (5/3)\,R2}]{} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{bmatrix}.$$

The row of zeros in the row-reduced matrix indicates that there are infinitely many solutions to the homogeneous system, so x_1, x_2, x_3 are linearly dependent. ∎

Example 3.6. Are the vectors $u = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ linearly independent? Let $c_1 u + c_2 v + c_3 w = 0$.

This corresponds to the linear homogeneous system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ -1 & 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0.$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ -1 & 3 & 3 \end{bmatrix} \xrightarrow[\substack{R2 = R2 - 2R1 \\ R3 = R3 - (-1)\,R1}]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow[\substack{R3 = R3 - (-4/3)\,R2}]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The final system in the row-elimination process has the unique solution $c_1 = c_2 = c_3 = 0$, so u, v, and w are linearly independent. ∎

3.4   BASIS OF A SUBSPACE

We now come to the fundamental concept of a *basis* for a subspace.

Definition 3.5.   Vectors $x_1, \ldots, x_m$ belonging to a subspace S are said to form a *basis* for S if

1. $x_1, \ldots, x_m$ span S.
2. $x_1, \ldots, x_m$ are linearly independent.

Example 3.7.   The standard basis vectors $e_1, \ldots, e_n$ form a basis for $\mathbb{R}^n$. This is the reason for the term "standard basis."

If $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$, so e_1, e_2, ..., e_n span $\mathbb{R}^n$. They are linearly independent, since if

$$c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0, \text{ then } c_1 = c_2 = \cdots = c_n = 0.$$ ∎

A subspace normally has more than one basis; for instance, let u, v, and w be the linearly independent vectors of Example 3.6. To show that the vectors are a basis for $\mathbb{R}^3$, it is necessary to show that the vectors span $\mathbb{R}^3$. Let x be any vector in $\mathbb{R}^3$. There must be a linear combination of u, v, and w that equals x; in other words, there must be scalars c_1, c_2, c_3, such that $c_1 u + c_2 v + c_3 w = x$. This is a system of linear equations

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ -1 & 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Form the augmented matrix $\begin{bmatrix} 1 & 1 & 1 & x_1 \\ 2 & -1 & 2 & x_2 \\ -1 & 3 & 3 & x_3 \end{bmatrix}$. The row-reduction operations performed in Example 3.6 show that there

is a unique solution for $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, and so u, v, and w form a basis for $\mathbb{R}^3$.

There are some important properties of a basis that are stated in Theorem 3.2.

Theorem 3.2. *If S is a subspace of $\mathbb{R}^n$, then*

1. *Each vector in a basis for S must be nonzero.*
2. *If u is a vector in S, there is one and only one way to write u as a linear combination of basis vectors for S.*
3. *A subspace span $\{x_1,\ldots,x_m\}$, where at least one of $x_1,\ldots,x_m$ is nonzero, has a basis $v_1,\ldots,v_p$, where $p \leq m$.*

Proof. To prove 1, assume that $x_1,\ldots,x_m$ is a basis for S, and that $x_1 = 0$. Then we have the nontrivial linear combination $1(x_1) + 0(x_2) + \cdots + 0(x_m) = 0$, and $x_1,\ldots,x_m$ are linearly dependent.

Let $x_1,\ldots,x_m$ be a basis for S. Assume that $u = c_1 x_1 + c_2 x_2 + \cdots + c_m x_m$ and that $u = d_1 x_1 + d_2 x_2 + \cdots + d_m x_m$. Subtract the two equations to obtain

$$(c_1 - d_1)x_1 + (c_2 - d_2)x_2 + \cdots + (c_m - d_m)x_m = 0.$$

Since $x_1,\ldots,x_m$ are linearly independent, $(c_1 - d_1) = 0$, $(c_2 - d_2) = 0$, $\ldots$, $(c_m - d_m) = 0$, and $c_i = d_i$, $1 \leq i \leq m$. We have proved 2.

Statement 3 tells us that the span of any set of vectors has a basis as long as all the vectors are not zero. Scan $x_1,\ldots,x_m$ from left to right and let x_{j_1} be the first nonzero vector. If $j_1 = m$ or all the vectors x_k, $j_1 + 1 \leq k \leq m$ are multiples of x_{j_1}, then $p = 1$. Otherwise, let x_{j_2} be the next nonzero vector following x_{j_1} such that x_{j_2} is not a multiple of x_{j_1}. If $j_2 = m$ or if all the vectors x_k, $j_2 + 1 \leq k \leq m$ are linear combinations of x_{j_1} and x_{j_2}, then $p = 2$. This argument will eventually terminate with a set of vectors $x_{j_1}, x_{j_2}, \ldots, x_{j_p}$ that must be linearly independent. Any vectors among $x_1,\ldots,x_m$ that were combinations of other vectors have been eliminated, so $x_{j_1}, x_{j_2}, \ldots, x_{j_p}$ spans S, and is a basis for S.   □

Example 3.8.   Let x and y be linearly independent vectors in $\mathbb{R}^n$. Consider the subspace span $\{0, 2x, x, -y, x+y\}$. Apply the technique used in proving part 3 of Theorem 3.1. Skip over 0 and record $2x$ as a nonzero vector. Move right and discard x because it is a multiple of $2\times$. The next vector, $-y$, is not a multiple of x because x and y are linearly independent. The final vector $x+y$ is a linear combination of $2x$ and $-y$. The subspace $\{0, 2x, x, -y, x+y\}$ has a basis $2x$, $-y$.   ∎

By using arguments similar to those in Theorem 3.2, the following claims can be proved. The proofs will not be provided, but the interested reader can consult Strang [8, pp. 175-175] for very readable arguments.

Claim 3.1.   If S is a subspace of $\mathbb{R}^n$, then

1. Any two bases for a subspace S must contain the same number of elements. This number is called the *dimension* of S and is written dim S.
2. If a subspace has dimension m, then any set of m linearly independent vectors is a basis for S.

3.5   THE RANK OF A MATRIX

In this section, we will determine how to find a basis for the row space of a matrix and discuss the relationship between the row space and the column space of a matrix.

Definition 3.6.   The number of elements in a basis for the row space is called the *rank* of the matrix.

The rank is defined for any $m \times n$ matrix, but we will deal only with square matrices for now. The process we develop to find the rank of a matrix will involve row reductions, but we will go beyond just getting to upper-triangular form and will also "zero out" as many elements in the upper triangle as we can. The process is illustrated with examples and is based upon the following lemma and a theorem that follows from it.

Lemma 3.1. *Subspaces* span $\{x_1, x_2, \ldots, x_r\}$ *and* span $\{y_1, y_2, \ldots, y_s\}$ *are equal if each of* $x_1, x_2, \ldots, x_r$ *is a linear combination of* $y_1, y_2, \ldots, y_s$ *and each of* $y_1, y_2, \ldots, y_s$ *is a linear combination of* $x_1, x_2, \ldots, x_r$.

Proof. Let $x = c_1 x_1 + \ldots + c_r$. Since each of $x_1, x_2, \ldots, x_r$ is a linear combination of $y_1, y_2, \ldots, y_s$, it follows that x is a linear combination of $y_1, y_2, \ldots, y_s$. Similarly, if $y = d_1 y_1 + \cdots + d_s y_s$, then y is a linear combination of $x_1, x_2, \ldots, x_r$. This shows the two subspaces are equal.   □

Theorem 3.3. *The row space of a matrix is the same as the row space of any matrix derived from it using row reduction.*

Proof. Suppose that matrix B is obtained from matrix A by a sequence of elementary row operations. Then each row of B is a linear combination of the rows of A. But A can be obtained from B by a sequence of elementary row operations, so each row of A is a linear combination of the rows of B. By Lemma 3.1, the two row spaces are equal.   □

Example 3.9. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. The upper-triangular form for A is $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, and we cannot eliminate any more elements. The row space of A and B is the same and consists of all multiples of the vector $\begin{bmatrix} 1 & 2 \end{bmatrix}$. Hence, $\begin{bmatrix} 1 & 2 \end{bmatrix}$ is a basis for the row space of A and the rank of A is 1.

Find the row space and rank of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. We begin row reduction with

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R2 = R2 - 3R1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$

Continue on and use the -2 in row 2, column 2 to eliminate the element above it in row 1, column 2. Also, remember we can multiply any row by a constant during row reduction.

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow{R1 = R1 - (-1)R2} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \xrightarrow{R2 = R2/(-2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The row space consists of all linear combinations of the vectors $\begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}$, and so the row space is $\mathbb{R}^2$, and the rank of A is 2.   ■

Example 3.10. Consider $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 4 & 5 \end{bmatrix}$. Perform row reductions to determine a basis for the row space and the rank of A.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 4 & 5 \end{bmatrix} \xrightarrow{R3 = R3 - (1)R1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix} \xrightarrow{R3 = R3 - 4R2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -9 \end{bmatrix}$$

This is upper-triangular form, but continue eliminating as many elements as we can. To make things easier, divide row 3 by -9.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -9 \end{bmatrix} \xrightarrow{R3 = R3/(-9)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R1 = R1 - 2R3]{R2 = R2 - 3R3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row space of A is $\mathbb{R}^3$, and the rank of A is 3. Note that this matrix has no null space; in other words, the null space of A is the empty set.   ■

Example 3.11.   Let $A = \begin{bmatrix} 2 & 5 & -4 & 1 \\ 3 & 8 & -9 & 2 \\ 1 & 1 & 7 & -1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$. Perform row reductions to determine a basis for the row space and the rank

of A.

$$\begin{bmatrix} 2 & 5 & -4 & 1 \\ 3 & 8 & -9 & 2 \\ 1 & 1 & 7 & -1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \xrightarrow[\substack{R2 = R2 - (3/2)\,R1 \\ R3 = R3 - (1/2)\,R1 \\ R4 = R4 - (1/2)\,R1}]{} \begin{bmatrix} 2 & 5 & -4 & 1 \\ 0 & \frac{1}{2} & -3 & \frac{1}{2} \\ 0 & -\frac{3}{2} & 9 & -\frac{3}{2} \\ 0 & -\frac{1}{2} & 3 & -\frac{1}{2} \end{bmatrix} \xrightarrow[\substack{R3 = R3 - (-3)\,R2 \\ R4 = R4 - (-1)\,R2}]{} \begin{bmatrix} 2 & 5 & -4 & 1 \\ 0 & \frac{1}{2} & -3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now eliminate the 5 in row 1, column 2.

$$\begin{bmatrix} 2 & 5 & -4 & 1 \\ 0 & \frac{1}{2} & -3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R1 = R1 - (10)\,R2]{} \begin{bmatrix} 2 & 0 & 26 & -4 \\ 0 & \frac{1}{2} & -3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We cannot go any further without removing the zero we just produced; however, let's multiply row 1 by $\frac{1}{2}$ and row 2 by 2 to make the leading element of each nonzero row 1.

$$\begin{bmatrix} 2 & 0 & 26 & -4 \\ 0 & \frac{1}{2} & -3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{R1 = (1/2)\,R1 \\ R2 = (2)\,R2}]{} \begin{bmatrix} 1 & 0 & 13 & -2 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of A is 2, and $\begin{bmatrix} 1 & 0 & 13 & -2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & -6 & 1 \end{bmatrix}$ are a basis for the row space.   ∎

This process of finding the rank of a matrix seems somewhat disorganized. In reality, the rank of a matrix is found by computing the singular value decomposition (SVD). We will discuss the SVD in Chapter 15, and will show how to compute it accurately and efficiently in Chapter 23.

Example 3.12.   We have computed the row space for some matrices, and now we will find the null space and nullity of a matrix. Let A be the matrix of Example 3.11. The null space of A is the subspace of vectors x such that $Ax = 0$. Using the computations in Exercise 3.11, we have

$$A = \begin{bmatrix} 2 & 5 & -4 & 1 \\ 3 & 8 & -9 & 2 \\ 1 & 1 & 7 & -1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 13 & -2 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This says that

$$x_1 + 13x_3 - 2x_4 = 0,$$
$$x_2 - 6x_3 + x_4 = 0.$$

From these equations,
$x_1 = -13x_3 + 2x_4$, $x_2 = 6x_3 - x_4$. The variables x_4 and x_3 are arbitrary, so a basis for the null space of A is

$$z_1 = \begin{bmatrix} -13 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Note the presence of two zero rows and a nullity of two.   ∎

Find the rank of a matrix using the MATLAB command `rank(A)`. The command `Z = null(A)` is used to find a basis for the null space of a matrix as a set of column vectors that form the matrix Z. If you only want the nullity of the matrix, obtain the number of columns of the matrix returned by `null` using `size(null(A),2)`.

Example 3.13. Note that in the MATLAB output, $(A*Z(:,1))'$ should be a row vector of zeros, but very small nonzero vector components are the result. In Chapter 8, we will discuss why this occurs.

```
>> A = [2 5 -4 1;3 8 -9 2;1 1 7 -1;1 2 1 0];
>> rank(A)

ans =
     2

>> Z = null(A);
>> (A*Z(:,1))'

ans =
  1.0e-014 *

   -0.0860   -0.1277   -0.0583   -0.0472

>> size(null(A),2)

ans =
     2
```
∎

Notice that in Example 3.13, the rank of the matrix is 2 and the nullity is 2, so rank + nullity = 4, the matrix size. This is an example of the relation between the dimension of the row space of a matrix and its null space.

Theorem 3.4. *If A is an n×n matrix, the rank of A plus the nullity of A equals n.*

We will prove this result in Chapter 15 when we present the singular value decomposition of a matrix.

Remark 3.2. An $n \times n$ matrix A is said to have *full rank* if the rank of A is n. It follows from Theorem 3.4 that if a matrix has full rank, then there is no null space.

Example 3.14. Let A be the matrix in Examples 3.12 and 3.13. As stated in Definition 3.1, the column space of a matrix A is the subspace spanned by the columns of A. We will find a basis for the column space of A, the subspace spanned by $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$. Form A^T, the matrix whose rows are the columns of A and perform row operations on A^T.

$$
\begin{bmatrix} 2 & 3 & 1 & 1 \\ 5 & 8 & 1 & 2 \\ -4 & -9 & 7 & 1 \\ 1 & 2 & -1 & 0 \end{bmatrix}
\xrightarrow[\substack{R2 = R2 - (5/2)\,R1 \\ R3 = R3 - (-2)\,R1 \\ R4 = R4 - (1/2)\,R1}]{}
\begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & 1/2 & -3/2 & -1/2 \\ 0 & -3 & 9 & 3 \\ 0 & 1/2 & -3/2 & -1/2 \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & 1/2 & -3/2 & -1/2 \\ 0 & -3 & 9 & 3 \\ 0 & 1/2 & -3/2 & -1/2 \end{bmatrix}
\xrightarrow[\substack{R3 = R3 - (-6)\,R2 \\ R4 = R4 - (1)\,R2}]{}
\begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & 1/2 & -3/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & 1/2 & -3/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\xrightarrow[R1 = R1 - (6)\,R2]{}
\begin{bmatrix} 2 & 0 & 10 & 4 \\ 0 & 1/2 & -3/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

This is as far as we can go, so a basis for the column space of A is

$$\begin{bmatrix} 2 \\ 0 \\ 10 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ -3/2 \\ -1/2 \end{bmatrix}.$$ ∎

Notice that the dimension of the column space of A in Example 3.14 is the same as the dimension of the row space of A. This is true for any $m \times n$ matrix, and we will prove this in Chapter 15.

3.6   CHAPTER SUMMARY

Subspaces of $\mathbb{R}^n$

$\mathbb{R}^n$ is a vector space, and subsets of $\mathbb{R}^n$, including $\mathbb{R}^n$ itself, are called subspaces. A subspace satisfies three properties:

- The zero vector is in the subspace.
- If x is a vector in the subspace, then cx is in the subspace, where c is a number.
- If x and y are in the subspace, then so is $x + y$.

If $x_1, x_2, \ldots, x_k$ are in $\mathbb{R}^n, k \le n$, then the set of all linear combinations $c_1x_1 + c_2x_2 + \cdots + c_kx_k$ is a subspace, and is said to be the subspace, S, of $\mathbb{R}^n$ spanned by $x_1, \ldots, x_k$, and we write

$$S = \text{span}\,\{x_1, \ldots, x_k\}.$$

As an example, let $x_1 = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 3 \\ 1 \\ -8 \end{bmatrix}$. The subspace spanned by these vectors is a plane in three-space.

Important examples of subspaces include the column space, row space, and the null space of a matrix.

Linear Independence

A collection of vectors is linearly independent if no vector can be written as a linear combination of the others. For instance, if $u = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then these vectors are linearly independent. But, if we let $v = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$, then $v = u + 3w$, and the vectors are linearly dependent.

In general, a set of vectors $v_1, \ldots, v_k$ is linearly independent if

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

only when $c_1 = c_2 = \cdots = c_k = 0$. This means that the homogeneous system

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_{k-1} & v_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k-1} \\ c_k \end{bmatrix} = 0$$

has only the zero solution.

Basis of a Subspace

The basis for a subspace is a set of vectors that span the subspace and are linearly independent. For instance, the standard basis for $\mathbb{R}^n$ is

$$e_1 = \begin{bmatrix} 1 & \cdots & 0 & 0 \end{bmatrix}^{\mathrm{T}}, e_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}, \ldots, e_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$

A subspace can have more than one basis. For instance, the vectors u, v, w in Section "Linear Independence" are a basis for $\mathbb{R}^3$, as is e_1, e_2, e_3. However, all bases for a subspace have the same number of vectors, and this number is the dimension of the subspace.

Matrix Rank

The rank of a matrix is the dimension of the subspace spanned by its rows. As we will prove in Chapter 15, the dimension of the column space is equal to the rank. This has important consequences; for instance, if A is an $m \times n$ matrix and $m \geq n$, then rank $(A) \leq n$, but if $m < n$, then rank $(A) \leq m$. It follows that if a matrix is not square, either its columns or its rows must be linearly dependent.

For small square matrices, perform row elimination in order to obtain an upper-triangular matrix. If a row of zeros occurs, the rank of the matrix is less than n, and it is singular. As we will see in Chapters 7, 15 and 23, finding the rank of an arbitrary matrix is somewhat complex and relies on the computation of what are termed its singular values.

For any $m \times n$ matrix, rank (A) + nullity $(A) = n$. Thus, if A is $n \times n$, then for A to be nonsingular, nullity (A) must be zero.

3.7   PROBLEMS

3.1 Which of the following subsets of $\mathbb{R}^2$ are subspaces?

a. $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $x = 2y$;

b. $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $x = 2y$ and $2x = y$;

c. $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $x = 2y+1$;

d. $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $xy = 0$;

e. $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $x \geq 0$ and $y \geq 0$.

3.2 Determine if $x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$, and $x_4 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 5 \end{bmatrix}$ are linearly independent in $\mathbb{R}^4$.

3.3 For which real numbers λ are the following vectors linearly independent in $\mathbb{R}^3$?

$$x_1 = \begin{bmatrix} \lambda \\ -1 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ \lambda \\ -1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -1 \\ -1 \\ \lambda \end{bmatrix}.$$

3.4 Find a basis for the row space of the following matrix. What is the rank of A?

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 11 & 19 & 0 \end{bmatrix}.$$

3.5 Find a basis for the row space of the following matrix. What is the rank of A? Find a basis for the column space of A.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

3.6 Determine the rank of A, and find a basis for the column space of A.

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 3 & -5 \end{bmatrix}.$$

3.7 Find a basis for the subspace S of $\mathbb{R}^3$ defined by the equation

$$x + 2y + 3z = 0.$$

Verify that $y_1 = [\, -1, \; -1, \; 1]^T \in S$, and find a basis for S that includes y_1.

3.8 Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 7 & 2 \\ -1 & 23 & 8 \\ 3 & 6 & 1 \end{bmatrix}.$$

3.9 If A is a 4×4 nonsingular matrix, what can you say about the columns of A?

3.10 This problem deals with the very important concept of *matrix range*.

    a. The *range* of an $n \times n$ matrix A is set of all vectors Ax as x varies through all vectors in $\mathbb{R}^n$. Describe the range in terms of the rows or columns of A. How would you find a basis for the range of A?

    b. Find a basis for the range of the matrix $A = \begin{bmatrix} 1 & 6 & 2 \\ -1 & 3 & 0 \\ -2 & 15 & 2 \end{bmatrix}$.

3.11 The *left nullspace* of a matrix A is the set of all vectors x such that $x^T A = 0$.

    a. Find the left nullspace of the matrix A in Problem 3.10.

    b. Find the nullspace of A^T.

    c. Prove that if A is an $n \times n$ matrix, the left nullspace of A is equal to the nullspace of A^T.

3.12 If $u_1, u_2, \ldots, u_n$ is a basis for $\mathbb{R}^n$ and A is an invertible $n \times n$ matrix, show that $Au_1, Au_2, \ldots, Au_n$ is also a basis for $\mathbb{R}^n$.

3.13 If v_1, v_2, v_3 are linearly independent, for what values of c are the vectors $v_2 - v_1$, $cv_3 - v_2$ and $v_1 - v_3$ linearly independent?

3.14 Prove that if A is an $n \times n$ matrix, rank $(A) = $ rank $\left(A^T\right)$.

3.15 Given $n \times n$ matrices A and B, prove that rank $(AB) \leq$ min $\{$rank (A), rank $(B)\}$.

    Hint: Does $ABx = A(Bx)$ give you a portion of the required result? The result of Problem 3.14 says that rank $(AB) = $ rank $\left([AB]^T\right)$

3.16   **a.** If a matrix A has dimension $m \times n$, with $m > n$, what is the maximum rank of A?

    b. If a matrix A has dimension $m \times n$ with $m < n$, what is the maximum rank of A?

    c. Assuming that the MATLAB function `rand` returns a matrix of random numbers, if the computation

$$\text{rank } (\text{rand } (k, 5))$$

is executed for $k = 1, 2, \ldots, 8$, what output do you expect?

3.7.1 MATLAB Problems

3.17 An $n \times n$ matrix is *rank deficient* if its rank is less than n. The 8×8 rosser matrix is often used for testing, and we will have occasion to use it later in this book. Assign A to be the rosser matrix using the MATLAB command `A = rosser`.

    a. Is A rank deficient?

    b. Using your answer to (a), is A invertible? Try to find the inverse using the MATLAB command `inv(A)`.

    c. Is A symmetric?

3.18 Let $D = \begin{bmatrix} 1 & 3 & 7 & 1 & 1 & 1 & 5 & 0 \\ 2 & 2 & 6 & 0 & 1 & 7 & 3 & 1 \\ 1 & 1 & 3 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$.

The MATLAB functions `rank` and `null` apply to any matrix. Obtain the rank of a matrix D using `rank(D)`. The function `null` has two variations, and for this problem use `null(A,'r')`. The parameter "r" specifies that MATLAB is to use row reduction to determine a basis for the null space. Use MATLAB to find the following:

a. the nullity of D.

b. a basis for the null space of D

c. the rank of D

3.19 MATLAB has a suite of matrices used for testing software and is accessed using the `gallery` command. The first argument is a string naming the matrix.

a. Assign A to be the clement matrix using the command A = `gallery('clement', 5)`. What is the rank of A? Is A rank deficient?

b. Is A singular?

c. Execute the commands B = `A; B(1,1) = 1.0e-14;B(2,2) = 1.0e-14;B(3,3) = 1.0e-14`. What is the rank of B?

d. Using the MATLAB command `inv`, find the inverse of B.

e. Is it reasonable to say that B is nearly rank deficient?

3.20 Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$.

a. What is the rank of A?

b. Find the rank of $U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

c. What is the rank of UA and AU? Does this support the result of Problem 3.15?

3.21 Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 & 2 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$

a. Compute the rank of A and the rank of B.

b. Compute the rank of the square matrices $A^{\mathrm{T}}A$ and $B^{\mathrm{T}}B$.

c. Is there a general relationship you can postulate by considering the specific results in parts (a) and (b)?