Appendix B

Mathematical Induction

This appendix is a brief disussion of the topic, and is intended to be sufficient for the times in the book that a proof uses mathematical induction.

Suppose you are given a statement, S, that depends on a variable n; for instance,

$$1 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}, n \geq 1$$

Let n_0 be the first value of n for which S applies, and prove the statement true. This is called the *basis step*. For our example, $n_0 = 1$. Now, assume S is true for any $n \geq n_0$ and prove that this implies S is true for $n + 1$. This is called the *inductive step*. Then,

- S is true for n_0, so S is true for $n_1 = n_0 + 1$.
- S is true for n_1, so S is true for $n_2 = n_1 + 1$.
- S is true for n_2, so S is true for $n_2 = n_2 + 1$.
- $\ldots$

This sequence can be continued indefinitely, so S is true for all $n \geq n_0$.

Example B.1. Prove that for $n \geq 1$, $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
Basis step: For $n_0 = 1$, $\frac{1(2)(3)}{6} = 1^2$.
Inductive step: Assume that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. We need to show that

$$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}. \tag{B.1}$$

Now,

$$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \left[1^2 + 2^2 + \ldots + n^2\right] + (n+1)^2 = \left[\frac{n(n+1)(2n+1)}{6}\right] + (n+1)^2$$

by the induction assumption. Then,

$$\left[\frac{n(n+1)(2n+1)}{6}\right] + (n+1)^2 = \frac{n+1}{6}\left(2n^2 + n + 6(n+1)\right) = \frac{(n+1)(n+2)(2n+3)}{6},$$

and the proof is complete. ■

Suppose you have an eigenvalue/eigenvector pair, λ/u, so that $Au = \lambda u$, and you need a way to compute powers $A^n u$. Do some experimenting:

$$\begin{aligned} A^2u &= A(Au) = A(\lambda u) = \lambda Au = \lambda(\lambda u) = \lambda^2 u, \\ A^3u &= A\left(A^2u\right) = A\left(\lambda^2 u\right) = \lambda^2 Au = \lambda^3 u \end{aligned}$$

There is a clear pattern:

$$A^n u = \lambda^n u.$$

When some experimentation yields a pattern, mathematical induction is often the easiest way to prove a result.

Example B.2. Prove that if A is an $n \times n$ matrix, and λ is an eigenvalue with corresponding eigenvector u, then

$$A^n u = \lambda^n u,\ n \geq 1.$$

Basis step: $A^1 u = Au = \lambda u = \lambda^1 u$.

Inductive step: Assume that $A^n u = \lambda^n u$. Then,

$$A^{n+1} u = A\left(A^n u\right) = A\left(\lambda^n u\right) = \lambda^n A u = \lambda^n\left(\lambda u\right) = \lambda^{n+1} u,$$

and the statement is true for $n + 1$. ■

A *geometric series* is a series with a constant ratio between successive terms. Since geometric series have important applications in science and engineering, the formula for the sum of a geometric series is a very useful result.

Example B.3. If a and r are numbers, $r \neq 1$, then

$$a + ar + ar^2 + ar^{n-1} = \frac{a - ar^n}{1 - r}.$$

Basis step: $\dfrac{a - ar^1}{1 - r} = a$, so the statement if true for $n = 1$.

Inductive step: Assume that

$$a + ar + ar^2 + ar^{n-1} = \frac{a - ar^n}{1 - r}.$$

Thus,

$$\begin{aligned} a + ar + ar^2 + ar^{n-1} + ar^n = \left[\frac{a - ar^n}{1 - r}\right] + ar^n &= \\ \frac{a - ar^n + (1 - r)\, ar^n}{1 - r} &= \\ \frac{a - ar^n}{1 - r}&, \end{aligned}$$

and the proof is complete. ■

Strong Induction

It is sometimes necessary to use a variant of mathematical induction called *strong induction*. The basis case is as before

Let n_0 be the first value of n for which S applies, and prove the statement true.

The inductive step is

Assume that S is true for all $n_0 \leq k \leq n$. Prove it is true for $n + 1$.

Use this form of induction when the assumed truth for n is not enough. This occurs when several instances of the inductive hypothesis are required to prove the statment true for $n + 1$.

Example B.4. Prove that any positive integer $n \geq 2$ is either prime or a product of primes.

Basis: $n = 2$ is prime.

Inductive step: Assume that for all $2 \leq k \leq n$, k is either prime or a product of primes. Consider $n + 1$. If it is prime, we are done; otherwise, it must be a composite number $n + 1 = ab$, where both a and b are in the range $2 \leq k \leq n$. By the inductive hypothesis, a and b are either prime or a product of primes, and the proof is complete. ■

B.1 PROBLEMS

B.1 Prove that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2\,(n+1)^2}{4}.$$

B.2 Assume A is an $n \times n$ matrix, X is an invertible matrix, and D is a diagonal matrix such that

$$X^{-1}AX = D.$$

Prove that

$$A^n = XD^nX^{-1},\ n \geq 1.$$

B.3 Assume that any $n \times n$ matrix M can be factored into the product of an $n \times n$ orthogonal matrix Q, and an $n \times n$ upper triangular matrix R so that $M = QR$. Let A be an $n \times n$ matrix. Prove that there exist orthogonal matrices Q_i, $1 \leq i \leq k$ and an upper triangular matrix R_k such that

$$(Q_0Q_1 \ldots Q_k)^T A\,(Q_0Q_1 \ldots Q_k) = R_kQ_k,$$

for any $k \geq 0$.

B.4 Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps. HINT: First show that 12, 13, 14, and 15 cents can be formed using 4-cent and 5-cent stamps.