MA5232
Modeling and Numerical Simulation

Convex Program

Wang Jiangyi, A0236307J, Mathematics Department
National University of Singapore
E0732857@U.NUS.EDU
March 29, 2022

1 Convexity

1.1 Question 1

Suppose $\{C_i\}_{i=1}^K$ are convex sets. Show that $\cap_{i=1}^K C_i$ is also convex.

Proof: To prove this, cosidering that for $\forall x_1, x_2 \in \cap_{i=1}^K C_i$, we have $x_1, x_2 \in C_i,\ i = 1, ..., K$. Since C_i is convex for $i = 1, 2, .., K$, then we have $\lambda x_1 + (1-\lambda)x_2 \in C_i$ holds for all $\lambda \in [0,1]$ and $i = 1, ..., K$. This is equivalent to the statement that, $\lambda x_1 + (1-\lambda)x_2 \in \cap_{i=1}^K C_i$ for all $\lambda \in [0,1]$, which shows the convexity of Set $\cap_{i=1}^K C_i$.

1.2 Question 2

Suppose $\{C_i\}_{i\in I}$ are convex sets. Show that $\cap_{i\in I} C_i$ is also convex.

Proof: To prove this, cosidering that for $\forall x_1, x_2 \in \cap_{i\in I} C_i$, we have $x_1, x_2 \in C_i,\ \forall i \in I$. Since C_i is convex for $i \in I$, then we have $\lambda x_1 + (1-\lambda)x_2 \in C_i$ holds for all $\lambda \in [0,1]$ and $\forall i \in I$. This is equivalent to the statement that, $\lambda x_1 + (1-\lambda)x_2 \in \cap_{i\in I} C_i$ for all $\lambda \in [0,1]$, which shows the convexity of Set $\cap_{i\in I} C_i$.

1.3 Question 3

Show that Polyhedra $P = \{Ax \le b\}$ are convex sets.

Proof: To prove this, cosidering that $\forall x, y \in P$, they must satisfy that $Ax \le b, Ay \le b\}$. Therefore, $A(\lambda x + (1-\lambda)y) \le \lambda b + (1-\lambda)b = b$ holds for all $\lambda \in [0,1]$. That is to say, $\lambda x + (1-\lambda)y \in P$ holds for all $\lambda \in [0,1]$, which shows the convexity of Polyhedra Sets P.

1.4 Question 4

$\{f_i\}_{i\in I}$ is a collection of convex fucntions. Show that $f(x) = sup\{f_i(x) : i \in I\}$ is also a convex function.

Proof: Since $f_i(x)$ is convex function for every $i \in I$, then for $\forall x, y \in I$ we have:

$$\begin{aligned}
&f_i(\lambda x + (1-\lambda)y) \le \lambda f_i(x) + (1-\lambda)f_i(y) \quad \forall i \in I \\
\Rightarrow &f_i(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \\
\Rightarrow &f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)
\end{aligned}$$

This result holds for $\forall \lambda \in [0,1]$, which shows the convexity of $f(x)$.

Notice that the **second line** is from the definition $f(x) = sup\{f_i(x) : i \in I\} \ge f_i(x), \forall i \in I$ and the **third line** is attained by picking supreme with respect to $i \in I$ for both sides.

1.5 Question 5

Briefly justify $\lambda(X) = sup_{||u||_2=1} u^T X u$ and show that $\lambda(X)$ is convex function with respect to X.

Proof: $\lambda(X) = sup_{||u||_2=1} u^T X u$ is because:

$$\begin{aligned}
&k = \lambda(X) \\
\Leftrightarrow\ &k = max\{\tilde{k} : \exists v\ s.t.\ Xv = \tilde{k}v\} \\
\Leftrightarrow\ &k = max\{\tilde{k} : \exists v\ s.t.\ ||v||_2 = 1\ Xv = \tilde{k}v\} \\
\Leftrightarrow\ &k = max\{\tilde{k} : \exists v\ s.t.\ ||v||_2 = 1\ v^T X v = \tilde{k}\} \\
\Leftrightarrow\ &k = sup\{v^T X v : ||v||_2 = 1\}
\end{aligned}$$

The convexity of $\lambda(X)$ is a corollary from question 4. Denote $f_u(X) = u^T X u$, then $f_u(X)$ is definitely a collection of convex functions with respect to X (because it is linear function in X).

Therefore, $\lambda(X) = sup\{f_u(X) : ||u||_2 = 1\}$ must be a convex function in X from the conclusion of question 4.

2 GP

2.1 Question 1

Show that GP can be transformed to a Convex Program.

Solution: Considering the transformation: $y_i = logx_i$. We just need to show that, after transformation, each posynomial can be turned to a convex function.

Without Loss of Generality, consider a general posynomial function $f(x_1, ..., x_n) = \sum_{j=1}^{N} c_j x_1^{a_{1j}} ... x_n^{a_{nj}}$. After transformation, the function becomes $\tilde{f}(y_1, ..., y_n) = \sum_{j=1}^{N} c_j e^{a_{1j}y_1} ... e^{a_{nj}y_n}) = \sum_{j=1}^{N} c_j e^{a_{1j}y_1 + ... + a_{nj}y_n}$.

Since $c_j > 0$, we can make such notation to give function $\tilde{f}$ a compact expression: $a_j = (a_{1j}, ..., a_{nj})^T, y = (y_1, ..., y_n)^T, b_j = log(c_j)$. Then, $\tilde{f}(y) = \sum_{j=1}^{N} exp(a_j^T y + b_j)$

Similiarly, as for a general monomial $h(x_1, ..., x_n) = cx_1^{a_1} ... x_n^{a_n}$, after transformation, it has the compact form $\tilde{h}(y) = exp(a^T y + b)$.

Lemma: When $f(x)$ is convex, then $g(x) := f(w^T x + b)$ is also convex.

For $\forall x, y \in \mathbb{R}^n$, we want to show $g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda)g(y)$ for $\forall \lambda \in [0, 1]$. This holds because:

$$
\begin{aligned}
&g(\lambda x + (1-\lambda)y) \\
=&f(w^T(\lambda x + (1-\lambda)y) + b) \\
=&f(\lambda(w^T x + b) + (1-\lambda)(w^T y + b)) \\
\le&\lambda f(w^T x + b) + (1-\lambda)f(w^T y + b) \\
=&\lambda g(x) + (1-\lambda)g(y)
\end{aligned}
$$

Since $f(x) = exp(x)$ is a convex function, then $exp(a_j^T y + b_j)$ is a convex function with respect to y. Therefore, $\tilde{f}(y)$ is the summation of finite convex functions, which is still a convex function.

Claim: When $f(x)$ is a monotone mapping of x and $1 \in f(X)$, then $g(x) := f(w^T x + b) = 1$ if and only if $w^T x + b = f^{-1}(1)$.

Since $f(x) = exp(x)$ is a monotone mapping and $f(0) = 1$. Therefore, $\tilde{h}(y) = 1$ if and only if $a^T y + b = 0$.

Therefore, the GP formulation after transformation becomes:

$$
\begin{aligned}
\min_{y \in \mathbb{R}^n} \ & \tilde{f}_0(y) \\
s.t. \ & \tilde{f}_i(y) - 1 \le 0 \quad 1 \le i \le m \\
& a_j^T y + b_j = 0 \quad 1 \le j \le p
\end{aligned}
$$

where $\tilde{f}_0(y), \tilde{f}_i(y) - 1$ are all convex functions with respect to y (and equality constraints are affine). Therefore, this is a Convex Program.

2.2 Question 2

Express the Max Volume Problem into GP.

Solution: We set the Decision Variables as, **Height: H, Length: L, Width: W**, then GP formulation is:

$$
\begin{aligned}
\min_{W,L,H} \ & HLW \\
s.t. \ & 2S^{-1}(LH + HW + LW) \le 1 \\
& C^{-1}LW \le 1 \\
& R_{u1}^{-1}HW^{-1} \le 1 \quad R_{l1}WH^{-1} \le 1 \\
& R_{u2}^{-1}HL^{-1} \le 1 \quad R_{l2}LH^{-1} \le 1 \\
& W, L, H > 0
\end{aligned}
$$

where the upper limit of surface area is S, the upper limit of ceiling is C and the upper and lower bound for the ratio is R_{ui} and R_{li} respectively ($i = 1, 2$).

3 Compressed Sensing

3.1 Question 1

Express $\{min\ ||x||_1\ s.t.\ y = Ax\}$ as a LP in the standard form.

Solution: The equivalent LP standard-form formulation is given as follows:

$$\begin{aligned} \min_{\tilde{x}} \quad & \mathbb{I}_{2n}^T \tilde{x} \\ s.t. \quad & \tilde{A}\tilde{x} = y \\ & \tilde{x} \geq 0 \end{aligned}$$

Here, $\mathbb{I}_{2n}$ is $2n \times 1$ all-one vector, $\tilde{x} = \begin{pmatrix} x^+ \\ x^- \end{pmatrix} \in \mathbb{R}^{2n}$ and $\tilde{A} = \begin{pmatrix} A & -A \end{pmatrix} \in \mathbb{R}^{m \times 2n}$.

To prove the equivalence, we want to show that these 2 formulations achieve the same optimal objective value. The process is given as follows (we denote the optimal objective value as **(PR)** and **(LP)** respectively):

On one hand, for arbitrary feasible solution x in Primal Problem (W.L.O.G, assume $x \geq 0$), we can construct the corresponding $\tilde{x} = \begin{pmatrix} x \\ 0 \end{pmatrix}$, which is feasible in LP Formulation. These 2 solutions achieve the same objective value in their own problems. **This direction shows that, (LP)≤(PR)**.

On the other hand, for arbitrary feasible solution $\tilde{x} = \begin{pmatrix} x^+ \\ x^- \end{pmatrix}$ in LP Formulation, we can construct the corresponding $x = x^+ - x^-$, which is a feasible solution in Primal Problem. Notice that $|x| \leq x^+ - x^-$, which implying that **(PR)≤(LP)**.

These 2 directions rigorously show that these 2 formulations are equivalent.

3.2 Question 2

Display the matrix recording the number of successes as a grayscale image.

Solution: The result of visualization is given as follows:

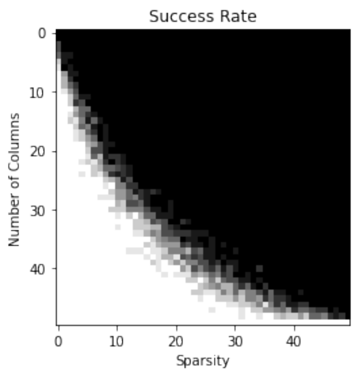

Figure 1: Sparsity vs Number of Constraints

From Figure 1, it can be easily observed there exists one curve that distinguish this figure into 2 parts. Above the curve, it is very difficult to recover the true signal, while below the curve, we can almost recover the true signal with 100% probability.

Moreover, if the true signal is very sparse, although its length is very big, we just need very few number of measurements to recover it. However, as the number of non-zero entries increase, the number of measurements we need to recover the true signal also increases very quickly. I think **an accpetable sparsity is around 20**. If the sparsity is larger than that, then we need almost 50 measurements to recover the true signal with high probability.

4 Matrix Completion

4.1 Question 1

Express Matrix Completion Problem as SDP.

Solution: Firstly, notice that we can achieve $\mathbf{X}_{ij} = \mathbf{M}_{ij}$ in the form of trace computation as follows:

$$tr(\mathbf{I}_{ji}\mathbf{X}) = \mathbf{M}_{ij}$$

where $\mathbf{I}_{ji} \in \mathbb{R}^{m\times m}$ is all-zero matrix except that (j,i)-th element is 1. Here, $\mathbf{X}, \mathbf{M} \in \mathbb{R}^{m\times n}$. This can be easily checked by matrix computation.

Secondly, since we have the equivalent expression for nuclear norm, then we have the following formulation:

$$\begin{aligned} &\min_{\mathbf{X}} \ \frac{1}{2}\min_{\mathbf{W_1},\mathbf{W_2}} \{tr(\mathbf{W_1}) + tr(\mathbf{W_2}) : \begin{pmatrix} \mathbf{W_1} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{W_2} \end{pmatrix} \in PSD\} \\ &s.t. \quad tr(\mathbf{I}_{ji}\mathbf{X}) = \mathbf{M}_{ij} \quad \forall (i,j) \in \Omega \end{aligned}$$

For simplicity, we can attain a compact expression of the optimization instance by the following denotation: $\tilde{\mathbf{X}} = \begin{pmatrix} \mathbf{W_1} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{W_2} \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}$, $\tilde{\mathbf{I}}_{ji} \in \mathbb{R}^{(m+n)\times(m+n)}$ ia all-zero matrix except that (j,i)-th element is 1. Then, this optimization instance is equivalent to:

$$\begin{aligned} &\min_{\tilde{\mathbf{X}}} \ \frac{1}{2}tr(\tilde{\mathbf{X}}) \\ &s.t. \quad tr(\tilde{\mathbf{I}}_{(m+j),i}\tilde{\mathbf{X}}) = \mathbf{M}_{ij} \quad \forall (i,j) \in \Omega \\ &\qquad\ tr(\tilde{\mathbf{I}}_{i,(m+j)}\tilde{\mathbf{X}}) = \mathbf{M}_{ij} \quad \forall (i,j) \in \Omega \\ &\qquad\ \tilde{\mathbf{X}} \in PSD \end{aligned}$$

which is a SDP instance.

4.2 Question 2

Plot a graph of the MSE over the different choices of k $\in \{100, 200, ..., 3000\}$.

Solution: The visualization result (MSE over different choices of k) is given as follows:

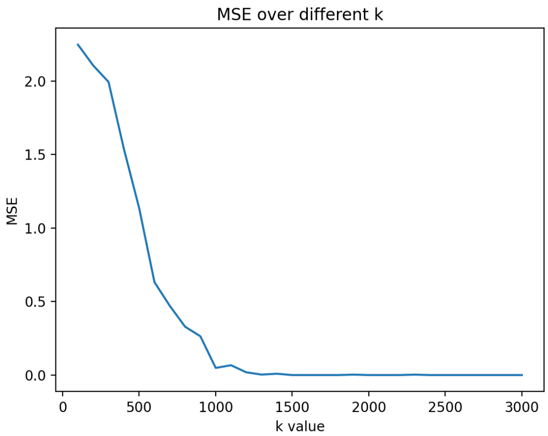

Figure 2: MSE for unobserved entries

From Figure 2, it can be observed that, when our ground-truth matrix is low-rank (here the rank of matrix is 2), we only need very few entries to recover the matrix (**only about 15% entries**). And MSE for unobserved entries is almost 0 when $k \geq 1500$.

5 Movie Lens

Instead of the SDP formulation, here we use heuristic algorithm to solve this large-scale problem.

5.1 Question 1: Baseline

The Squared Error Loss of Baseline on testing set.

Solution: After coding, we have that **SE Loss of Baseline is 10454.81** on testing set.

5.2 Question 2: Convergence Curve

Plot the errors over these iterations to see if the matrix $\mathbf{X}$ converges.

Solution: Take $r = 5$ as an example. The convergence curve is given as follows:

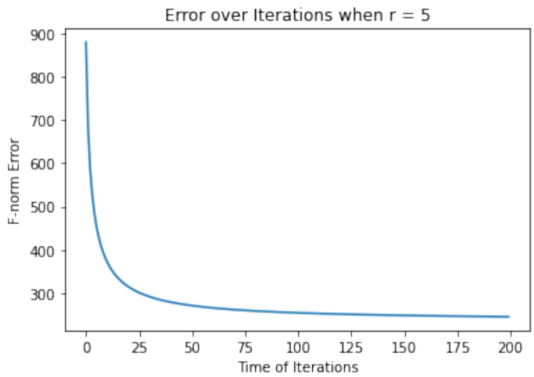

Figure 3: Convergence Curve when r=5

From Figure 3, we can be sure that matrix $\mathbf{X}$ converges because the errors over iterations change slightly after 200 iterations.

5.3 Question 3: Comparison

Squared error loss on the testing dataset for $r \in \{1, 2, ..., 20\}$.

Solution: The result of SE loss on testing set for different r is given as follows:

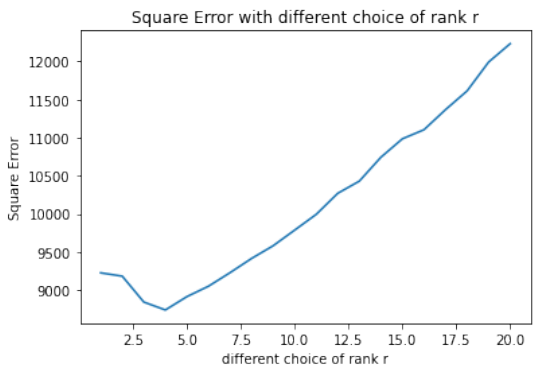

Figure 4: Convergence Curve when r=5

1. When $r \leq 10$, Square Error Loss on testing set for Heuristic algorithm is smaller than Baseline, which is 10454.81. And the minima is attained at $r = 4$. I think this is because, the ground-truth matrix $\mathbf{X}$ may have the low-rank property. Therefore, for small value of r, we are likely to recover the testing set value, which leads to the small Square Error Loss on testing set.

2. When $r \geq 10$, it seems that the SE loss on testing set is monotonously increasing, which behaves even worse than Baseline. That may comes from that, our choce of r (large r) violates the property of ground-truth matrix $\mathbf{X}$ (low-rank). Therefore, the larger value of r we choose, the worse performance we will attain.

6 Appendix

All python codes (*.py & *.ipynb) will be attached in .zip.