MA5232: Project 2

- For Question 3, find a partner to work with you if necessary. Only one report is required for a teamwork.
- Due date: 6pm, 25 March 2021.
- The report of the project should be submitted to the folder "Submission to Project 2" on LumiNUS.

Question 1

Given two distribution functions $f(\boldsymbol{v})$ and $g(\boldsymbol{v})$, define

$$\tilde{f}(\boldsymbol{v}) = f(\mathbf{R}\boldsymbol{v} + \boldsymbol{w}), \quad \tilde{g}(\boldsymbol{v}) = g(\mathbf{R}\boldsymbol{v} + \boldsymbol{w}),$$

where $\mathbf{R}$ is an orthogonal matrix and $\boldsymbol{w}$ is an arbitrary vector. Show that

$$Q[\tilde{f}, \tilde{g}](\boldsymbol{v}) = Q[f, g](\mathbf{R}\boldsymbol{v} + \boldsymbol{w}),$$

which indicates the Galilean invariance of the collision operator.

Question 2

For the linearized collision operator $\mathcal{L}[\cdot]$, show that

1. For any function $\psi(\boldsymbol{v})$, it holds that

$$\begin{aligned}\int_{\mathbb{R}^3} \psi(\boldsymbol{v})\mathcal{L}[f]\,\mathrm{d}\boldsymbol{v} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^{\pi}\int_{\boldsymbol{n}\perp\boldsymbol{g}} |\boldsymbol{g}|B(|\boldsymbol{g}|,\chi) \\ \times [\psi(\boldsymbol{v}_1') + \psi(\boldsymbol{v}') - \psi(\boldsymbol{v}_1) - \psi(\boldsymbol{v})] f(\boldsymbol{v}) f_{\mathrm{eq}}(\boldsymbol{v}_1)\,\mathrm{d}\boldsymbol{n}\,\mathrm{d}\chi\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{v}.\end{aligned}$$

2. The collision operator satisfies the conservation of mass, momentum and energy:

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \boldsymbol{v} \\ |\boldsymbol{v}|^2 \end{pmatrix} \mathcal{L}[f]\,\mathrm{d}\boldsymbol{v} = 0.$$

3. For the spatially homogeneous equation

$$\frac{\partial f}{\partial t} = \mathcal{L}[f],$$

the solution $f(\boldsymbol{v}, t)$ satisfies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f \log f\,\mathrm{d}\boldsymbol{v} \leqslant 0.$$

Question 3

Consider the following spatially homogeneous BGK equations for a two-species gas mixture:

$$\frac{\partial f_1}{\partial t} = \nu_{11}(f_{\mathrm{eq},11} - f_1) + \nu_{12}(f_{\mathrm{eq},12} - f_2),$$
$$\frac{\partial f_2}{\partial t} = \nu_{21}(f_{\mathrm{eq},21} - f_1) + \nu_{22}(f_{\mathrm{eq},22} - f_2),$$

where for $i, j = 1, 2$,

$$f_{\mathrm{eq},ij}(\boldsymbol{v}, t) = n_i \left(\frac{m_i}{2\pi k_B T_{ij}(t)}\right)^{3/2} \exp\left(-\frac{m_i|\boldsymbol{v} - \boldsymbol{u}_{ij}(t)|^2}{2k_B T_{ij}(t)}\right),$$
$$\boldsymbol{u}_{ij} = \frac{\rho_i \nu_{ij} \boldsymbol{u}_i + \rho_j \nu_{ji} \boldsymbol{u}_j}{\rho_i \nu_{ij} + \rho_j \nu_{ji}},$$
$$T_{ij} = \frac{n_i \nu_{ij} T_i + n_j \nu_{ji} T_j}{n_i \nu_{ij} + n_j \nu_{ji}} + \frac{\rho_i \nu_{ij}(|\boldsymbol{u}_i|^2 - |\boldsymbol{u}_{ij}|^2) + \rho_j \nu_{ji}(|\boldsymbol{u}_j|^2 - |\boldsymbol{u}_{ij}|^2)}{3(n_i \nu_{ij} + n_j \nu_{ji})}.$$

The notations appearing in the equations are given as follows:

- f_i, $i = 1, 2$: The distribution function for the ith species.
- m_i, $i = 1, 2$: The mass of a single gas molecule of the ith species.
- n_i, $i = 1, 2$: The number density of gas molecules of the ith species (the integral of f_i with respect to $\boldsymbol{v}$).
- $\rho_i = m_i n_i$, $i = 1, 2$: The density of the ith species.
- $\boldsymbol{u}_i$, $i = 1, 2$: The velocity of the ith species.
- T_i, $i = 1, 2$: The temperature of the ith species.

Consider the dimensionless parameters

$$m_1 = 1, \quad m_2 = 2, \quad k_B = 1,$$
$$\nu_{11} = 1, \quad \nu_{22} = 2, \quad \nu_{12} = \nu_{21} = 3/2$$

and the initial conditions

$$\boxed{f_i(\boldsymbol{v}, 0) = n_i \left(\frac{m_i}{2\pi k_B T_i(0)}\right)^{3/2} \exp\left(-\frac{m_i|\boldsymbol{v} - \boldsymbol{u}_i(0)|^2}{2k_B T_i(0)}\right),} \quad i = 1, 2$$

with

$$n_1 = 2, \quad n_2 = 1,$$
$$\boldsymbol{u}_1(0) = (-1, 0, 0)^T, \quad \boldsymbol{u}_2(0) = (1, 0, 0)^T,$$
$$T_1(0) = 1, \quad T_2(0) = 2.$$

Solve the equations numerically and write a report including a brief introduction to the numerical method and the numerical solutions at various times to show the evolution of the distribution functions.