Hypothesis Test

① H_0 & H_1

α-mistake : reject H_0 But H_0 is true $\mathbb{P}(\text{reject } H_0 \mid H_0)$

β - mistake : accept H_0 But H_0 is false $\mathbb{P}(\text{accept } H_0 \mid H_1)$

$t \geq \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}}$ or something

② aim: control α- mistake $\mathbb{P}(\text{reject } H_0 \mid H_0 \text{ is true}) \leq \alpha$

③ For example:

a) $H_0 : \mu \leq \mu_0$, $H_1 : \mu > \mu_0$, $\alpha = 0.05$

b) construct statistics $\boxed{t = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}}} \sim t(n-1)$ → r.v.

Consider $\mathbb{P}\left(t_{n-1} \geq \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}}\right)$

→ $\boxed{\text{quantity}}$

c) if $\mathbb{P}(t_{n-1} \geq *) < 0.05$, then reject H_0

Understanding of reject region for one-side hypothesis test

→ when $H_0 : \mu = \mu_0$, the accept region is (a_μ , b_μ)

→ when $H_0 : \mu \leq \mu_0$, the accept region is $\bigcup_{\mu \leq \mu_0} (a_\mu , b_\mu)$

$$\Rightarrow \text{reject region} = \bigcap_{\mu \leq \mu_0} (a_\mu , b_\mu)^c$$

$$= \bigcap_{\mu \leq \mu_0} \text{reject region}_\mu$$

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① $H_0: \theta \in \Theta_0$ $H_1: \theta \in \Theta_0^c$

② if $x \in R$, then reject H_0

$R := \{x : W(x) \leq c\}$ $\longrightarrow$ reject region

if $x \in R^c$, then accept H_0

1) Likelihood Ratio Test (LRT) $\Leftarrow$ some reasonable method to chose Reject Region

$\boxed{\text{Reject Region}}$

$R := \{x : \lambda(x) \leq c\}$ a reasonable choice intuitively

$$\lambda(x) := \frac{\sup_{\theta \in \Theta_0} L(\theta|x)}{\sup_{\theta \in \Theta} L(\theta|x)}$$

☆☆

$\boxed{\text{Property}}$!!

the reject region $\boxed{\text{only depends on}}$ $\boxed{\text{sufficient statistic}}$ of θ

$\boxed{f(x_1, \dots, x_n | T) \text{ 与 } \theta \text{ 无关}}$

[E.g.] Normal LRT $N(\theta, \sigma^2)$

$\Theta_0 = \{\theta_0\}$

$$\lambda(x) = \frac{C \cdot \exp(-\frac{1}{2\sigma^2} \Sigma (x_i - \theta_0)^2)}{C \cdot \exp(-\frac{1}{2\sigma^2} \Sigma (x_i - \bar{x})^2)}$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(\Sigma (x_i - \bar{x} + \bar{x} - \theta_0)^2 - \Sigma (x_i - \bar{x})^2\right)\right)$$

$$= \exp\left(-\frac{n}{2\sigma^2} (\bar{x} - \theta_0)^2\right)$$

$\Rightarrow$ Reject Region $R = \left\{x : \left|\frac{\bar{x} - \theta_0}{\sigma}\right| \geq \sqrt{\frac{-2\log c}{n}}\right\}$

2) union-intesection method

$H_0: \theta \in \bigcap_{r \in \Gamma} \Theta_r \Rightarrow R = \bigcup_{r \in \Gamma} R_r$

☆ If R_r is simple, then we can simplify

[E.g.] if $R_r = \{x : T_r(x) > c\}$

then $\bigcup_{r \in \Gamma} R_r = \{x : \sup_{r \in \Gamma} T_r(x) > c\}$

intersect - union method

$$H_0 : \theta \in \bigcap_{r \in \Gamma} \Theta_r \Rightarrow R = \bigcap_{r \in \Gamma} R_r$$

$(A \cup B) \cap (C \cup D)$

[Application] LRT Normal (θ, σ^2)

$H_0 : \theta \le \theta_0$ $\quad H_1 : \theta > \theta_0$

$$R_\theta = \left\{ x : \left| \frac{\bar{x}-\theta}{\sigma} \right| \ge \sqrt{\frac{-2\log c}{n}} \right\} \Rightarrow \boxed{R = \bigcap_{\theta \le \theta_0} R_\theta = \left\{ x : \frac{\bar{x}-\theta}{\sigma} \ge \sqrt{\frac{-2\log c}{n}} \right\}}$$

3) Evaluate Method for Hypothesis Testing

$H_0 : \theta \in \Theta_0$ $\quad H_1 : \theta \in \Theta_0^c$

$$\begin{cases} \text{Type I Error: } P(X \in R \mid \theta \in \Theta_0) \\ \text{Type II Error: } P(X \in R^c \mid \theta \in \Theta_0^c) = 1 - P(X \in R \mid \theta \in \Theta_0^c) \end{cases}$$

Defn: Power Function

$\Rightarrow$ define $\beta(\theta) = P_\theta(X \in R) := P(X \in R \mid \theta)$, $\theta \in \Theta$

then
$$\begin{cases} \text{Type I Error Prob.} = \beta(\theta) & \theta \in \Theta_0 \\ \text{Type II Error Prob.} = 1 - \beta(\theta) & \theta \in \Theta_0^c \end{cases}$$

$\frac{1}{\sqrt{2\pi}\sigma}$

$(2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}}$

Example: Power function for Normal LRT

$X \sim N(\theta, \sigma^2)$

$H_0 : \theta \le \theta_0$ $\quad H_1 : \theta > \theta_0$

$$\text{LRT} \Rightarrow R = \left\{ x : \frac{\bar{X}-\theta_0}{\frac{\sigma}{\sqrt{n}}} > c \right\}$$

when σ^2 unknown,

then $\lambda(x) = \dfrac{\sup L(\theta \mid x) \quad \theta \in \Theta_0}{\sup L(\theta \mid x) \quad \theta \in \Theta}$

① $\hat{\mu}_1 = \theta_0$ $\quad \hat{\sigma}_1^2 = \frac{1}{n} \sum (X_i - \theta_0)^2$

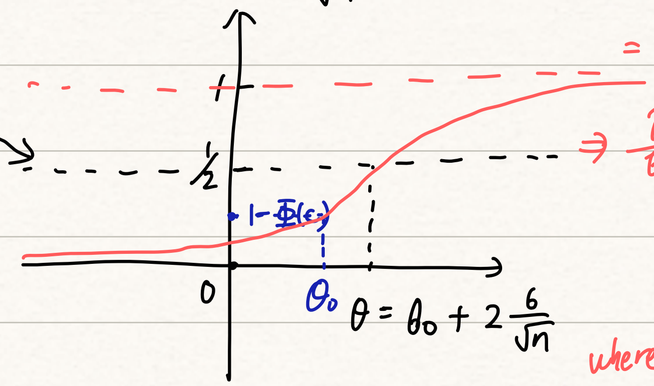
② $\hat{\mu}_2 = \bar{x}$ $\quad \hat{\sigma}_2^2 = \frac{1}{n} \sum (X_i - \bar{x})^2$

$$\beta(\theta) = \boxed{\mathbb{P}_\theta(X \in R)} = \mathbb{P}_\theta\left(\frac{\bar{X}-\theta}{\frac{\sigma}{\sqrt{n}}} > c + \frac{\theta_0-\theta}{\frac{\sigma}{\sqrt{n}}}\right)$$

$\boxed{X \text{ is r.v.}}$

$$= \mathbb{P}_\theta\left(Z > c + \frac{\theta_0-\theta}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= 1 - \Phi\left(c + \frac{\theta_0-\theta}{\frac{\sigma}{\sqrt{n}}}\right)$$



$$\lambda(x) = \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}\right)^{-\frac{n}{2}}$$

$$= \left(\frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2}\right)^{\frac{n}{2}}$$

$$\hat{\sigma}_1^2 = \frac{1}{n}\sum(x_i - \bar{x} + \bar{x} - \theta_0)^2$$

$$= \frac{1}{n}\left(\sum(x_i-\bar{x})^2 + \sum(\bar{x}-\theta_0)^2\right)$$

$$= \frac{1}{n}\sum(x_i-\bar{x})^2 + (\bar{x}-\theta_0)^2$$

$$\Rightarrow \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = 1 + \frac{n(\bar{x}-\theta_0)^2}{\sum(x_i-\bar{x})^2}$$

$$= 1 + \frac{t(x)}{n-1}$$

where $t(x) = \dfrac{\sqrt{n}(\bar{x}-\theta_0)}{\sqrt{\frac{1}{n-1}\sum(x_i-\bar{x})^2}}$

$$= \frac{\sqrt{n}(\bar{x}-\theta_0)}{\sqrt{S_{n-1}}}$$

$\boxed{\text{Hypothesis Test} \quad \text{Neyman-Pearson}}$

$\Downarrow$

$\boxed{\text{This } \alpha \text{ is called } \underline{\text{size } \alpha}} \rightarrow \boxed{\text{真实水平}}$

$\boxed{\text{Use Power Function to determine } R}$

Therefore $\{x : \lambda(x) \geq c\}$

$$= \{x : |t(x)| \geq d\} = R$$

where $\underline{t(x) \sim t_{n-1}}$

$\Rightarrow$ we can calculate $\mathbb{P}_\theta(x \in R) := \beta(\theta)$

$\Rightarrow$ let $\boxed{\alpha = \sup_{\theta \in \Theta_0} \beta(\theta)}$ to decide the reject region R

$\boxed{\text{Distinguish with significance test}}$

$\Uparrow$

$\boxed{\text{Fisher Significance Test}}$

$\downarrow$

Interpretation of such experiment:

→ If H_0 is true, then when we do N experiments (in each experiment we sample n example), the frequency of we reject H_0 converges to $N \cdot \alpha$ (at most when Θ_0 is not singleton)

→ large α ⇒ bigger reject region ⇒ easier to reject H_0

犯第一类错误概率大

Then, it is followed by Neyman-Pearson Lemma, which dives into the details of Hypothesis Testing.

↳ only works for SIMPLE vs SIMPLE test

(Uniform Most Powerful class C test)

UMP

Sufficient & Necessary Condition

Leave for future

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① $\boxed{\text{p-value}}$ definition

→ for sample $X = (X_1, \dots, X_n)$, construct statistic $p(X)$

s.t ① $0 \le p(X) \le 1$

② $p(X)$ small $\Longrightarrow$ evidence for reject H_0

② $\boxed{\text{valid p-value}}$ defn

→ for $\theta \in \Theta_0$ and $0 \le \alpha \le 1$,

we have $P_\theta (X : p(X) \le \alpha) \le \alpha$

can be viewed as $R = \{ x : p(x) \le \alpha \}$

→ then $p(x)$ is the probability of view data x

One construction for $p(X)$

[E.g.] $\boxed{\frac{\bar{X} - \mu}{\sigma} \sim N(0,1)}$ $P\left(\frac{\bar{x} - \mu}{\dots} \dots z \right) = \alpha$

$\frac{}{\sqrt{n}}$ $\quad P\left(\frac{}{\frac{6}{\sqrt{n}}} \le z_\alpha\right) = \alpha$

① $\{x : p(X) \le \alpha\} = \{x : |\bar{X}-\mu| \ge z_{\frac{\alpha}{2}} \frac{6}{\sqrt{n}}\}$

② $p : x \longmapsto \mathbb{P}\left(|Z| \ge \left|\frac{\bar{X}-\mu}{\frac{6}{\sqrt{n}}}\right|\right)$ $\Rightarrow$ X is random variable $\Rightarrow$ $p(X)$ also r.v.

sampling

$\Rightarrow$ ③ $\{x : p(x) \le \alpha\} = \{x : \mathbb{P}\left(|Z| \ge \left|\frac{\bar{X}-\mu}{\frac{6}{\sqrt{n}}}\right|\right) \le \alpha\}$

$= \{x : |\bar{X}-\mu| \ge z_{\frac{\alpha}{2}} \frac{6}{\sqrt{n}}\}$

$\Rightarrow \boxed{\mathbb{P}_\theta(p(X) \le \alpha) = \alpha}$

Consider $\boxed{CI}$ with Bootstrap:

$\boxed{\theta_0 \rightarrow \text{ground-truth}}$ ideally

$\boxed{\text{sample}}$ $\rightarrow X_i = \{X_{i1}, \ldots, X_{in}\}$ $\quad i = 1, 2, \ldots, B$

$\Rightarrow$ for each i, consider $\hat{\theta}(X_{i1}, \ldots, X_{in}) = \hat{\theta}_i$

$\Rightarrow$ $\{\hat{\theta}_1, \ldots, \hat{\theta}_B\} \longrightarrow$ approximate the sampling distribution $\hat{\theta}$

$\hat{\theta} - \theta \sim \boxed{\text{some distribution}}$

$\Rightarrow$ consider quantile in $\{\hat{\theta}_1 - \theta, \ldots, \hat{\theta}_B - \theta\}$, namely $\underline{\delta}, \bar{\delta}$

then $\mathbb{P}(\underline{\delta} \le \hat{\theta} - \theta \le \bar{\delta}) \approx 1-\alpha$

$\Updownarrow$

$\boxed{\mathbb{P}(\theta \in [\hat{\theta} - \bar{\delta}, \hat{\theta} - \underline{\delta}]) \approx 1-\alpha}$

$\boxed{\text{Real World}}$

$\begin{cases} \theta = T(F) \\ \hat{\theta} = T(\hat{F}_n) \\ X \sim F \end{cases}$

$\boxed{\text{Bootstrap World}}$

But we don't know θ_0! $\Longrightarrow$ $\hat{\theta}_{MLE} \approx \theta_0$

$$\begin{cases} \hat{\theta} = T(\hat{F}_n) \\ \theta^* = T(\hat{F}_n^*) \end{cases}$$

$X \sim \hat{F}_n \rightarrow \boxed{\text{fixed}}$

Then, the framework is:

① $\hat{\theta} \approx \theta_0 \Rightarrow$ generate $\{X_{i1}, \dots, X_{in}\}$

② compute $\theta^*(X_{i1}, \dots, X_{in}) = \theta_i^*$

③ $\mathbb{P}(\underline{\delta} \le \hat{\theta} - \theta_0 \le \bar{\delta})$

$\approx \mathbb{P}(\underline{\delta} \le \theta^* - \hat{\theta} \le \bar{\delta})$

$\approx 1-\alpha$

$\boxed{\underline{\delta} \,\&\, \bar{\delta}}$ is the quantile for $\{\theta_1^* - \hat{\theta}, \dots, \theta_B^* - \hat{\theta}\}$

④ $\mathbb{P}(\theta_0 \in [\hat{\theta} - \bar{\delta}, \hat{\theta} - \underline{\delta}]) \approx 1-\alpha$

The $\boxed{\text{duality}}$ between CI & Hypothesis Testing

$\longrightarrow$ From $\boxed{\text{Bootstrap}}$ $\boxed{\hat{\theta} = \hat{\theta}(X_1, \dots, X_n)}$

$\mathbb{P}(\theta_0 \in [\hat{\theta} - \bar{\delta}, \hat{\theta} - \underline{\delta}]) \approx 1-\alpha$ CI

$\Rightarrow$ $\boxed{\text{Construct}}$ Hypothesis Testing Thru Reversing the CI

① 从参数估计角度构造 Hypothesis Testing
- exact
- MLE asyptotic
- Bootstrap

$X_1, \dots, X_n \longrightarrow \boxed{\text{observation}}$

$\boxed{\hat{\theta}}$ $\xrightarrow{\text{generate samples and achieve}}$ $\{\theta_1^*, \dots, \theta_B^*\}$

$(\underline{\delta}, \bar{\delta}) \rightarrow \boxed{\theta_i^* - \hat{\theta}}$

$\Downarrow$

$\boxed{\text{Accept Rejion}}$

$\{X : \hat{\theta}(X) \in [\theta_0 + \underline{\delta}, \theta_0 + \bar{\delta}]\}$

② $\boxed{\dfrac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim t_{n-1}} \Rightarrow \boxed{\text{从统计量角度}}$