Chi-square Related

① $Y \sim \chi^2(p) \iff \begin{cases} Y = (X-\mu)^T \Sigma^{-1} (X-\mu) \\ X \sim N_p(\mu, \Sigma) \end{cases}$

② $Y \sim \chi^2(p, \mu^T\mu) \iff \begin{cases} Y = X^T X \\ X \sim N_p(\mu, I_p) \end{cases}$

$$X^T X = (Z+\mu)^T (Z+\mu) \qquad \boxed{Z \sim N_p(0, I_p)}$$

$$= \underbrace{Z^T Z}_{\chi^2(p)} + \underbrace{2\mu^T Z}_{\mathbb{E}[\mu^T Z]=0} + \mu^T\mu$$

且独立

③ $X \sim \chi^2(p_1, \lambda_1) \quad Y \sim \chi^2(p_2, \lambda_2) \Rightarrow X+Y \sim \chi^2(p_1+p_2, \lambda_1+\lambda_2)$

④ $\chi^2(p) = \Gamma(\frac{p}{2}, \frac{1}{2}) \Rightarrow \begin{cases} \mathbb{E}[\cdot] = p \\ Var[\cdot] = 2p \end{cases}$

⑤ [Thm] $\boxed{X \sim \text{Gaussian}}$, then $X^TAX \sim \chi^2(\cdot)$ 后，A阵必需对称幂等

$X \sim N_p(\mu, I_p)$, A symmetric

then $X^TAX \sim \chi^2(r, \mu^TA\mu) \iff \begin{cases} \text{rank}(A) = r \\ A^2 = A \end{cases}$

$\Leftarrow \quad A = P^T \begin{pmatrix} I_r & \\ & 0 \end{pmatrix} P \qquad P$ orthogonal

then denote $Y = PX \sim N_p(P\cdot\mu, I_p)$

and $X^TAX = Y^T \begin{pmatrix} I_r & \\ & 0 \end{pmatrix} Y$

$= Y_r^T Y_r$ where $Y_r \sim N_r((P\mu)_r, I_r)$

Therefore $X^TAX \sim \chi^2(r, \mu^TA\mu)$

$\Rightarrow \quad A = P^T \Sigma \cdot P \qquad P$ orthogonal (since symmetric A)

then denote $\quad y = PX \sim N_p(P\cdot\mu,\ I_p)$

$$X^TAX = y^T\Sigma y \sim \chi^2(r,\ \mu^TA\mu)$$

$\Rightarrow \ \mathrm{diag}(\Sigma) = 0$ or 1. $\Rightarrow \ \underline{\text{rank } A = r \ \ \& \ \ A^2 = A}$

$\boxed{\text{Corollary}}$: $\quad X \sim N_p(\mu, I_p)$, A symmetric

then $\quad X^TAX \sim \chi^2(r, \underset{0}{\cancel{\mu^TA\mu}}) \Leftrightarrow \begin{cases} \text{rank}(A) = r \\ A^2 = A \\ \mu^TA\mu = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} \text{rank } A = r \\ A^2 = A \\ A\mu = 0 \end{cases}$$

$\boxed{\text{Corollary}}$: $\quad X \sim N_p(\mu, \Sigma)$, A symmetric

then $\quad X^TAX \sim \chi^2(r, \mu^TA\mu) \Leftrightarrow \begin{cases} \text{rank}(A) = r \\ A\Sigma A = A \end{cases}$

IDEA: $\quad X \sim N_p(\mu, \Sigma) \Leftrightarrow \underline{\underline{\overset{Y}{\Sigma^{-\frac{1}{2}}X}}} \sim N_p(\Sigma^{-\frac{1}{2}}\mu,\ I_p)$

$$(\Sigma^{\frac{1}{2}}y)^TA(\Sigma^{\frac{1}{2}}y) \sim \chi^2(r, (\Sigma^{\frac{1}{2}}\tilde{\mu})^TA(\Sigma^{\frac{1}{2}}\tilde{\mu}))$$

$$\Leftrightarrow \begin{cases} \text{rank}(\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}) = r \\ \Sigma^{\frac{1}{2}}A\Sigma A\Sigma^{\frac{1}{2}} = \Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}} \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{rank } A = r \\ A\Sigma A = A \end{cases}$$

⑥ [Cochran Theorem] $\quad$ <u>chi-square 独立性刻画</u>

$$\begin{cases} X \sim N_p(\mu, I_p), \\ X^TAX = X^TA_1X + X^TA_2X \sim \chi^2(r, \lambda), \quad \lambda = \mu^TA\mu \end{cases}$$

$X^T A_1 X \sim \chi^2(s, \lambda_1)$ $\quad \lambda_1 = \mu^T A_1 \mu$

$\boxed{A_2 \geq 0} \to$ 不必要，只需要 $A \geq A_1$ & $A \geq A_2$

$$\Rightarrow \begin{cases} ① \ X^T A_2 X \sim \chi^2(r-s, \lambda_2) & \lambda_2 = \mu^T A_2 \mu \\ ② \ X^T A_1 X \perp\!\!\!\perp X^T A_2 X \\ ③ \ A_1 A_2 = 0 \end{cases}$$

$\boxed{\text{Proof Sketch}}$:

⓪ $A = A_1 + A_2$ since Support Set of $X = \mathbb{R}^n$

① $X^T A X \sim \chi^2(r, \lambda) \Rightarrow A^2 = A \quad r(A) = r$

$X^T A_1 X \sim \chi^2(s, \lambda_1) \Rightarrow A_1^2 = A_1 \quad r(A_1) = s$

② $A_2 \geq 0 \Rightarrow A \geq A_1$

$X^T A_1 X \sim \chi^2(s, \lambda_1) \Rightarrow A \geq A_2$

③ $A^2 = A,\ A^T = A \Rightarrow P^T A P = \begin{pmatrix} I_r & \\ & 0 \end{pmatrix}$

it must have $P^T A_1 P = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$

$P^T A_2 P = \begin{pmatrix} B_2 & 0 \\ 0 & 0 \end{pmatrix}$

from $A \geq A_1$
$A \geq A_2$

④ $A_1^2 = A_1 \Rightarrow B_1^2 = B_1 \Rightarrow Q^T P^T A_1 P Q = \begin{pmatrix} I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \boxed{Q^T P^T A_2 P Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{r-s} & 0 \\ 0 & 0 & 0 \end{pmatrix}} \Leftarrow A_2 = (Q^T P^T)^T \boxed{} (Q^T P^T)$

⊗

④ $\Rightarrow \begin{cases} A_2^2 = A_2 \\ A_2^T = A_2 \end{cases} \Rightarrow X^T A X \sim \chi^2(r-s, \mu^T A \mu)$

$r(A_2) = r-s$

④ $Y = Q^T P^T X$, $(PQ) \to$ orthogonal matrix $\Rightarrow$ preserve indept

$\Rightarrow X^T A_1 X \perp\!\!\!\perp X^T A_2 X$

④ $\Rightarrow$ $A_1 A_2 = 0$ directly holds

$\boxed{\text{Corallary}}$: $X \sim N_p(\mu, I_p)$,

then $X^T A_1 X \perp\!\!\!\perp X^T A_2 X \Leftrightarrow A_1 A_2 = 0$

$\boxed{\chi^2 \text{ test}}$ Multi-nomial $X_n = (x_{n1}, \ldots, X_{nk})$ (k-type categorical)

$\Leftrightarrow X_n \sim \text{Multi-nomial}(n; p_1, \ldots, p_k)$

χ^2 test : $\begin{cases} H_0 : p = a = (a_1, \ldots, a_k) \\ H_a : \text{o/w} \end{cases}$

$\boxed{\text{Analysis}}$ $X_n = \sum_{i=1}^{n} Y_i$ $\quad$ $Y_i \sim \text{Multi-nomial}(1 : p_1, \ldots, p_k)$

consider : ① $\mathbb{E}[Y_i] = \begin{pmatrix} p_1 \\ \vdots \\ p_k \end{pmatrix}$

covariance matrix

Σ

② $\text{Var}[Y_i] = \begin{pmatrix} p_1(1-p_1) & -p_1 p_2 & \cdots & -p_1 p_k \\ -p_1 p_2 & p_2(1-p_2) & \cdots & -p_2 p_k \\ \vdots & & \ddots & \\ -p_1 p_k & \cdots & \cdots & p_k(1-p_k) \end{pmatrix}$

$\boxed{\text{CLT}}$ $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Y_i - \mathbb{E}[Y_i]) = \frac{1}{\sqrt{n}} (X_n - np) \sim N(0, \Sigma)$

$\Rightarrow \overset{H_0}{=} \frac{1}{\sqrt{n}} (X_n - na) \sim N(0, \Sigma)$

$\boxed{\text{observation}}$ $\Rightarrow$ $\Sigma = \text{diag}(a_1, \ldots, a_k) - a a^T$ (not invertible)

$\boxed{\text{Imagination}}$ If $Y \sim N(0, \Sigma)$, then how to construct χ^2-statistic ?

Answer : Σ symmetric $\Rightarrow$ $\Sigma = Q^T \Lambda Q$

$\boxed{\text{if } \Sigma \text{ 幂等}}$, then $\Lambda = \text{diag}(1, \ldots, 1, 0, \ldots, 0)$

$\Rightarrow$ $QY \sim N(0, \Lambda)$ $\Rightarrow$ easy to construct χ^2-stats

$\Rightarrow$ $\boxed{Y^T Y \sim \chi^2(\text{rank}(\Lambda))}$ since $Q^T Q = I$

Therefore, observe that $\Sigma = \text{diag}(a_1, \ldots, a_k) - a a^T$

$D = \text{diag}(\sqrt{a_1}, \ldots, \sqrt{a_k})$

then $D \Sigma D^T = \text{diag}(1, \ldots, 1) - \sqrt{a}\sqrt{a}^T$

under the condition $\sum a_i = 1$, we have :

$D \Sigma D^T$ is 幂等 matrix and $\text{tr}(D \Sigma D^T)$

$= \text{rank}(D \Sigma D^T)$

$= k - 1$

$\boxed{\text{Recap}}$: $\frac{1}{\sqrt{n}}(X_n - na) \xrightarrow{d} N(0, \Sigma)$

$\Rightarrow$ $\frac{1}{\sqrt{n}} D (X_n - na) \xrightarrow{d} N(0, \underbrace{D \Sigma D^T}_{\text{幂等}})$

$\Rightarrow$ $\frac{1}{\sqrt{n}} (X_n - na)^T D^2 (X_n - na) \xrightarrow{d} \chi^2(k-1)$

$$\Rightarrow \sum_{i=1}^{k} \frac{[\text{observation}(i) - \text{expectation}(i)]^2}{[\text{expectation}(i)]^2} \xrightarrow{d} \chi^2(k-1)$$

$\boxed{11-14}$ #