Inventory Theory

intuitive understanding

Framework: ① Deterministic Model → 1. EOQ

- allow shortage → penalty
- without shortage → $q^* \Rightarrow t^*$

q^* → order, t^*; S^* → Stock

相似

Lagrangian M M

允许 shortage，但对 shortage 部分要支付 penalty

EOQ Model

obj: minimize TC per unit time

$$\left(\frac{q}{2} \times \frac{q}{a} \times \frac{a}{q}\right)h + K \cdot \frac{a}{q} + aC$$

→ function of q → order

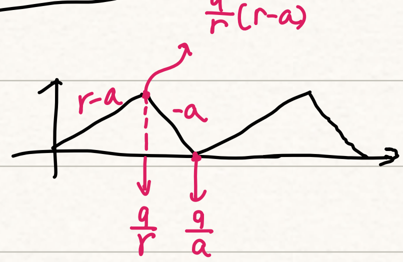
$$\Rightarrow \begin{cases} q^* = \sqrt{\frac{2aK}{h}} \\ t^* = \frac{q^*}{a} = \sqrt{\frac{2K}{ah}} \end{cases}$$

useful quantity

2. Lead Time Non-Zero Case → 提前 Order

Constant

3. Continuous-rate EOQ Model



4. quantity discounts → algorithm

holding cost per unit time = average inventory level * holding cost per item per unit time

② Non-deterministic Model → Demand is stochastic

(a)

→ (r, q) policy → solve approximately (fix q^* → EOQ) → determine r^*

- 1. Back-ordered case → Model
- 2. Lost Sales case

① $h \cdot \frac{1}{2}(r - \mathbb{E}X + q + r - \mathbb{E}X)$ — average inventory level

② $K \cdot \frac{\mathbb{E}D}{q}$ → system cost

③ $C_B \cdot \mathbb{E}[B_r] \cdot \frac{\mathbb{E}[D]}{q}$ → shortage cost

$B_r = \max\{0, X - r\}$

Cost $\begin{cases} C_{LS} \\ C_B \end{cases}$ → per item

Lost Sales case:

→ ① $h[\text{previous} + \mathbb{E}[B_r]]$

② system cost the same

③ $C_{LS}\, \mathbb{E}[B_r] \cdot \frac{\mathbb{E}[D]}{q}$

(b)

→ (s, S) policy → demand discrete

$$\begin{cases} s^* = r^* \\ S^* = r^* + q^* \end{cases}$$

5. News Vendor Problem

③ Single-Period Model $\begin{cases} \text{News Vendor Problem} \\ \\ \text{Other} \end{cases}$

① News Vendor Model

Intuitive 的理解：

$$\begin{cases} C_u(x,q) = -\overset{\alpha}{(p-c+b)}\,q + bx \\ C_o(x,q) - C_u(x,q) = \overset{\beta}{(p-s+b)}(q-x) \end{cases}$$

$\begin{cases} p \to \text{profit} \\ c \to \text{cost} \\ b \to \text{penalty of shortage} \end{cases}$

$\Rightarrow C_u \to \begin{cases} \text{sell} & q \\ \text{buy} & q \\ \text{demand} & x \end{cases}$ $\quad q < x \quad$ sell $= \min\{x, q\}$

$C_o \to \begin{cases} \text{sell} & x \\ \text{buy} & q \\ \text{demand} & x \end{cases}$ $\quad x < q$

$\Rightarrow C_o - C_u \rightsquigarrow \begin{cases} \text{少卖} \quad q-x \Longrightarrow +p \\ \text{少交} \quad x-q \quad \boxed{\text{penalty}} \Rightarrow +b \\ \text{多回收} \quad q-x \Rightarrow -s \end{cases}$

$\to$ when $\boxed{\text{demand is continuous}}$ $\Rightarrow$ $\boxed{\mathbb{E}[\cdot] \text{ is in the form of integration}}$

given

$$\Rightarrow EC(q) = \mathbb{E}_D[\,Cost(d,q)\,]$$
$$= \mathbb{E}_D[\,Cost_o(d,q)\,\mathbb{1}\{D \le q\} + cost_u(d,q)\,\mathbb{1}\{D > q\}\,]$$

① q is continuous

$\Rightarrow \boxed{\left.\dfrac{dEC}{dq}\right|_{q=q^*} = 0}$

② q is discrete

$\Rightarrow$

$\boxed{\text{optimal condition}}$

Solve by $\left.\dfrac{dEC}{dq}\right|_{q=q^*} = 0$

$\to \boxed{\text{靠近 } q^* \text{ 的两个整数}}$

$\hookrightarrow$ check!!

→ $\boxed{\text{Continuous Case}}$

$$EC(q+1) - EC(q)$$

$$= \int_0^{q+1} \beta(q+1-x) f(x)\,dx - \alpha(q+1) - \int_0^{q} \beta(q-x) f(x)\,dx + \alpha q$$

$$= \int_0^{q+1} \beta(q-x) f(x)\,dx + \int_0^{q+1} \beta f(x)\,dx - \int_0^{q} \beta(q-x) f(x)\,dx - \alpha$$

$$= \int_q^{q+1} \beta(q-x) f(x)\,dx + \beta\,\mathbb{P}(D \le q+1) - \alpha$$

$$= \boxed{\int_q^{q+1} \beta(q+1-x) f(x)\,dx} + \beta\,\mathbb{P}(D \le q) - \alpha$$

→ 0 when discrete case

→ $\boxed{\text{Discrete Case}}$ (when D is discrete)

↳ demand

$$EC(q+1) - EC(q)$$

$= \beta\,\mathbb{P}(D \le q) - \alpha$ ⇒ the optimal q^* is the smallest q

s.t $EC(q+1) - EC(q) \geq 0$

$\Leftrightarrow \beta\,\mathbb{P}(D \le q) - \alpha \geq 0$

$\Leftrightarrow \mathbb{P}(D \le q) \geq \dfrac{\alpha}{\beta}$

$\boxed{x \to \text{demand}}$ $\boxed{q \to \text{order}}$

② Other Model

→ $\pi(x, q)$

→ Basic Idea : Minimize $\boxed{\text{Cost Function}}$

(Expectation with respect to Demand)

→ $\boxed{\text{regular}}$

$\boxed{\text{④ (R,S) policy}}$ → $\boxed{\text{review policy}}$

$\boxed{\text{Punchline:}}$ $\boxed{\text{purchase cost per unit item} \Rightarrow \text{can be ignored}}$

↓

only related to demand rate $\mathbb{E}[D]$

☆☆☆

$\Rightarrow$ Back-ordered $TC(R,S)$

此处 $X \sim D_{L+R}$ $\rightarrow$ $E[\max\{0, D_{L+R}-S\}]$

$$= (K+J)/R + h\cdot\left(S - \mathbb{E}[D_{L+R}] + \tfrac{1}{2}\mathbb{E}[D_R]\right) + \frac{C_B \cdot \boxed{\int_S^\infty (x-S) f(x)\,dx}}{R}$$

$\rightarrow$ 类似于之前 $\mathbb{E}[B_r]$ $\boxed{B_r = \max\{0, X-r\}}$

求导: $\int_S^\infty x f(x)\,dx - S\int_S^\infty f(x)\,dx$

$- Sf(S) - \int_S^\infty f(x)\,dx + Sf(S)$

lost

$\Rightarrow$ Lost-sale $TC(R,S)$

$$= (K+J)/R + h\left(S - \mathbb{E}[D_{L+R}] + \tfrac{1}{2}\mathbb{E}[D_R] + \int_S^\infty (x-S) f(x)\,dx\right)$$

$$+ \frac{C_{LS}\int_S^\infty (x-S) f(x)\,dx}{R}$$

①

$\rightarrow$ fix R^*, S^* has closed form solution

① $P(D_{L+R} > S) = \dfrac{Rh}{C_B}$

② $P(D_{L+R} > S) = \dfrac{Rh}{Rh + C_{LS}}$

②

(approximator)

$\rightarrow$ the choice of R^* is from EOQ

$$q^* = \sqrt{\frac{2\mathbb{E}[D](K+J)}{h}} \Rightarrow R^* = \frac{q^*}{\mathbb{E}[D]} = \sqrt{\frac{2(K+J)}{h\,\mathbb{E}[D]}}$$

order for one cycle — time for one cycle

Br

Apply for (r, q) policy

Confusion

⑤ service level

Service Level 1 $\rightarrow$ Expected fraction of demand met on time

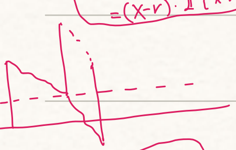
SLM1

$$1 - SLM1 = \frac{\text{Total Lost-sale}}{\text{Total Demand}} = \frac{\mathbb{E}[B_r]}{q}$$

(per cycle)

when $\begin{cases} C_B \\ C_{LS} \end{cases}$ difficult to determine

$B_r = \max\{0, X-r\}$
$= (X-r)\cdot \mathbb{1}\{X-r > 0\}$



$D = B_r + \bar{D}$

$\bar{D} = \begin{cases} D & X<r \\ D-X+r & X>r \end{cases}$ → Renewal Process

↳ choose a good r^* according to our target (SLM1)

$$\begin{cases} \text{Discrete case} \rightarrow \text{easy} \\ \text{Continuous case} \rightarrow \mathbb{E}[B_r] = \int_r^\infty (x-r) f(x)\, dx \end{cases}$$

(difficult)

Lead time Demand

$X \rightarrow D_L$

↓

Gaussian

$$= \frac{1}{\sqrt{2\pi}\sigma_x} \int_r^\infty (x-r) \exp\left\{ -\frac{(x-\mathbb{E}[X])^2}{2\sigma_x^2} \right\} dx$$

$$z = \frac{x-\mathbb{E}[X]}{\sigma_x} \qquad = \frac{\sigma_x}{\sqrt{2\pi}} \int_{\frac{r-\mathbb{E}X}{\sigma_X}}^\infty \left(z - \frac{r-\mathbb{E}X}{\sigma_x}\right) e^{-\frac{z^2}{2}} dz$$

$\Rightarrow x - r = \sigma_x z - (r - \mathbb{E}X)$

$$= \sigma_x\, NL\left(\frac{r-\mathbb{E}[X]}{\sigma_x}\right)$$

$$\boxed{NL(\alpha) = \frac{1}{\sqrt{2\pi}} \int_\alpha^\infty (z-\alpha)\, e^{-\frac{z^2}{2}} dz}$$

$\boxed{\text{SLM2}}$ → expected # of shortage cycles per unit time

$$\boxed{\text{SLM2} = \mathbb{P}(X>r) \cdot \mathbb{E}[D]/q}$$

q^* → EOQ quantity order $= \sqrt{\frac{2\,\mathbb{E}[D]\cdot K}{h}}$

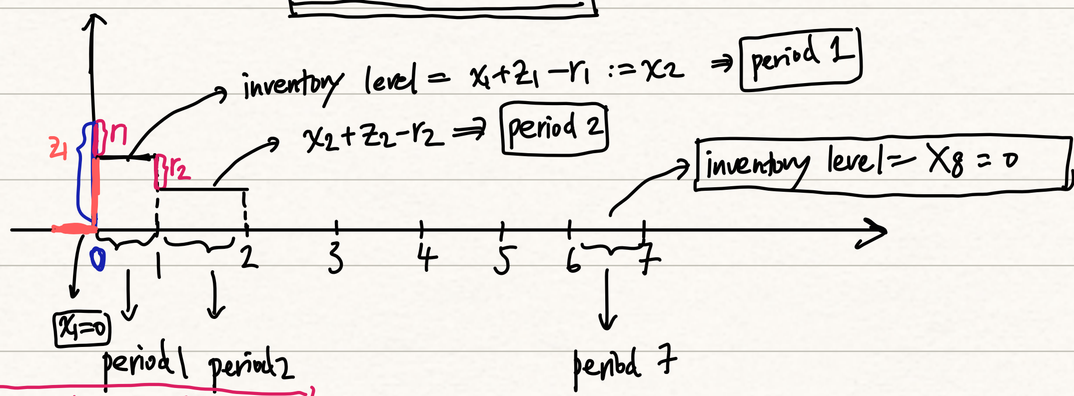
$\boxed{r - \mathbb{E}X}$ → safety stock $\Rightarrow$ $\boxed{h(r-\mathbb{E}[X])}$

$\boxed{X_1=0}$ →

1 period $\boxed{0\sim 1}$

r_1 →

⑥ Periodic Review



☆ Optimality Condition

→ a Necessary Condition for Optimality

⇒ $z_i = r_i + r_{i+1} + \cdots + r_j$ or $z_i = 0$

⇒ Moreover, $\begin{cases} x_i \neq 0, & z_i = 0 \\ x_i = 0, & z_i = r_i + \cdots + r_j \end{cases}$

⇒ Define $\boxed{f_{ij}}$ → cost from (period i) to (period $j-1$)

$$\Rightarrow f_{ij} = K + c(z_i) + \underbrace{h\left(\sum_{k=i+1}^{j-1} r_k\right)}_{\text{period } i}$$

$$+ \underbrace{h\left(\sum_{k=i+2}^{j-1} r_k\right)}_{\text{period } i+1} + \cdots + \underbrace{h(r_{j-1})}_{j-2} + h\cdot 0$$

$$\propto \underline{K + h\left(r_{i+1} + 2r_{i+2} + \cdots + (j-1-i)r_{j-1}\right)}$$

!!!

Procedure

① calculate $\{f_{ij}\}$ table

f_{ij}	$j=2$	3	4	5
$i=1$	K	$K+hr_2$	$\cdots$	
2		K	$K+hr_3$	$\cdots$
3			K	$K+hr_4$
4				K

② calculate $TC^*(i)$ Table

	$j=2$	3	4	5	TC^*	j^*
$i=4$				$\boxed{f_{45}}$	K	5
3			$\boxed{f_{34}+TC(4)}$	f_{35}		4
2		$f_{23}+TC(3)$	$f_{24}+TC(4)$	f_{25}	Backward	Forward
1	$f_{12}+TC(2)$	$\boxed{f_{13}+TC(3)}$	$f_{14}+TC(4)$	f_{15}	0	3

for example

$z_1 = r_1 + r_2$

$z_3 = r_3$

$z_4 = r_4$