Duality

① $\rightarrow$ conjugate fun$^{=}$

$$f^*(y) = \sup_x \{ \langle x, y \rangle - f(x) : x \in \mathbb{R}^n \}$$

$$= \sup \{ \langle x, y \rangle - \mu : (x, \mu) \in epi(f) \}$$

f^* is closed and convex

since $f(y) = \sup \{ \underbrace{h_t(y)}_{\text{convex, closed}} : t \}$ $\Rightarrow f(y)$ is $\begin{cases} \text{closed} \Leftrightarrow epi(f) \text{ is closed} \\ \text{convex} \end{cases}$

② a special conjugate $f^{=}$ $\rightarrow$ $\delta(x|C) \Leftrightarrow \delta^*(x|C)$

$$\delta(x|C) = \begin{cases} 0, & x \in C \\ +\infty, & x \notin C \end{cases}$$

is conjugate if C is closed and convex

$\Downarrow$

separating them

$$\Rightarrow \delta^*(x|C) = \sup \{ \langle x, y \rangle : y \in C \}$$

$f(x) = \sup \{ h(x) : h \le f \}$

③ for closed, proper, convex $f^{=}$

1. $(f^*)^* = f \rightarrow$ f 是比它小的 Affine $f^{=}$ 的上界.

$\downarrow$

$f(x) = \sup \{ \langle x, y \rangle - \mu : h(x) \le f(x) \text{ for } \forall x \}$

$\Updownarrow$

$\langle x, y \rangle - \mu \le f(x) \quad \forall x$

$\Updownarrow$

$\mu \ge \langle x, y \rangle - f(x) \quad \forall x$

$\Rightarrow \mu \ge \sup \{ \langle x, y \rangle - f(x) : \forall x \}$

$= f^*(y)$

$\Rightarrow \mu \ge f^*(y)$

$\Rightarrow (y, \mu) \in epi(f^*)$

$= \sup \{ \langle y, x \rangle - \mu : (y, \mu) \in epi(f^*) \}$

$= (f^*)^*(x)$

2. $f \rightarrow$ positive homogeneous $\quad C = \partial f(0)$

$\Leftrightarrow f^* \rightarrow$ non-empty closed & convex set C (indicator function)

$\hookrightarrow f^*$ proper & closed & convex

④ $f \to$ closed & proper & convex

一般来说，

$$f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x) : x \in \mathbb{R}^n\}$$

$$\geq \langle x, x^*\rangle - f(x)$$

$$\Rightarrow f^*(x^*) + f(x) \geq \langle x, x^*\rangle$$

$\Downarrow$

$$f^*(x^*) + f(x) = \langle x, x^*\rangle$$

$$\Leftrightarrow x^* \in \partial f(x)$$

$$\Leftrightarrow x \in \partial f^*(x^*)$$

$$\Leftrightarrow \langle x, x^*\rangle - f^*(x^*) = \sup\{\langle z^*, x\rangle - f^*(z^*) : z^*\}$$

$$\Leftrightarrow \langle x, x^*\rangle - f(x) = \sup\{\langle z, x^*\rangle - f(z) : z\}$$

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② Fenchel Duality

$\begin{cases} f \to \text{convex} \\ g \to \text{concave} \end{cases}$

$$\inf\{f(x) - g(x)\}$$

$$\sup\{g^*(x^*) - f^*(x^*)\}$$

$\downarrow$

Lagrangian Duality $\to$ Strong Duality. $\to$ Slaters condition

$$\boxed{\inf\{p(x) : q_i(x) \leq 0,\ i = 1, 2, \dots, m\}}$$

$\|$ $\updownarrow$ equal

$$\sup_{p \geq 0}\{g(p)\} \qquad g(p) = p(x) + \langle p, q(x)\rangle$$

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③ Strong Duality

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④ $\boxed{\text{Dual Fun}^c}$

$$\begin{cases} P(x) = (g(x), h(x)) \\ w = (u, v) \end{cases}$$

$$\min f(x)$$

$$\text{s.t. } g(x) \leq 0$$

$$\ell(w) = \inf\{f(x) + w^T P(x) : x \in X\}$$

$h(x) = 0$ $\longrightarrow$ $\theta(w) = \inf\{ f(x) + w^T \beta(x) : x \in X \}$

$x \in X$

$$X(w) = \text{argmin} \{ \underbrace{f(x) + w^T\beta(x)}_{\boxed{\text{continuous } f(\cdot), \beta(\cdot)}} : \underbrace{x \in X}_{\text{compact}} \}$$

$\Downarrow$

$X(w)$ nonempty

① $\underline{\theta(\cdot) \rightarrow \text{concave}}$

② subgradient of $\underline{\theta}(\cdot)$

1. $\bar{x} \in X(\bar{w})$, $\beta(\bar{x}) \in \partial\theta(\bar{w})$

2. $X(\bar{w}) \rightarrow$ singleton

$\Downarrow$

$\theta(\cdot)$ is differentiable at $\bar{w}$ $\Rightarrow$ $\nabla\theta(\bar{w}) = \beta(\bar{x})$

③ 方向导数 $\Rightarrow$ $\theta'(\bar{w}, d) = \inf\{ d^T\xi , \ \xi \in \partial\theta(\bar{w}) \}$

④ subgradient $\Rightarrow$ $\partial\theta(\bar{w}) = \text{conv}\{ \beta(y) : y \in X(\bar{w}) \}$

(上4)

⑤ 最速下降方向 $\xi \in \partial\theta(\bar{w}) \longrightarrow$ has the smallest Euclidean norm

$$\underline{d = \frac{\xi}{\|\xi\|}}$$