Ch2 Optimality Condition

① Unconstrained Prob.

1. A direction d → $\nabla f(x)^T d < 0$ ⇓ $f(x+\lambda d) < f(x)$ ⇒ d 下降方向

$\nabla f(x)^T d < 0$ ⇓

2. Optimality Condition for Un. Pb.

1) Sufficient:

$\nabla f(\bar{x}) = 0$ & $\nabla^2 f(\bar{x}) > 0$ ⇒ $\boxed{\text{strictly local minimum}}$

2) Necessary

$\nabla f(\bar{x}) = 0$ & $\nabla^2 f(\bar{x}) \geq 0$

② Constrained Pb.

→ suff. ⇒ Relatively Easy

→ necc. ⇒ Need SOME GEOMETRIC LEMMA

Model: $\min f(x)$

s.t $x \in \boxed{S}$ → Feasible Region

1. Define $F^<$ $F^<$ $F^\leq$ $F_0^\leq$

- easy to check
- essential DESCENT

$F_0^< \subseteq F^< \subseteq F^\leq \subseteq F_0^\leq$

improving direction

$d \in F^\leq$

$f(x+\lambda d) \leq f(x)$ for some λ

$f(x+\lambda d) = f(x) + \lambda \nabla f(x) d + o(\lambda)$

→ important conclusion

f convex $\Rightarrow F_0^< = F^<$

f strictly convex $\Rightarrow F_0^{\le} = F^{\le}$

2. Feasible diretion D

$d \in D \Leftrightarrow \bar{x} \in S$, $\bar{x} + \lambda d \in S$ for all $\lambda \in (0, \delta)$

3. General Optimality Condition about Constrained problems

→ Later will be applied to some certain case

→ Geometric

i) Necessary. Part

$F_0^< \subseteq F^< \subseteq F^{\le} \subseteq F_0^{\le}$ $\quad$ $\bar{x}$ local minimum

$\Rightarrow F^< \cap D = \phi$.

$\Rightarrow$ easily $(F_0^< \cap D = \phi)$

ii) Sufficient Part

Pf

$f(\hat{x}) < f(\bar{x})$

$\hat{x} - \bar{x} = d \in D$.

$\hat{x} = \bar{x} + d$,

$f(\bar{x}+d) < f(\bar{x})$

$d \in F^< = F_0^<$

$\Rightarrow d \in F_0^< \cap D$

① $F_0^< \cap D = \phi$

② f convex at $\bar{x}$

③ S is locally convex → $\bar{x} \to N_\varepsilon(\bar{x})$ ↓ $x - \bar{x} \in D$

$\Rightarrow \bar{x}$ local minimum

3. Consider A Subset Of Constrained Prob.

$$\begin{cases} \min\ f(x) \\ \text{s.t.}\ \boxed{\begin{matrix} g_i(x) \le 0 \\ x \in X \end{matrix}} \quad i = 1, \dots, m \end{cases}$$

→ care about the $\boxed{\partial}$ point

as for these points.

$\boxed{D = G^{\le}}$

Application of General Conclusion.

→ Optimality Condition

1) Necessary Part

$\bar{x}$ local minimum $\Rightarrow$ $F^< \cap G^{\le} = \phi$ D

$\Rightarrow$ $F_0^< \cap G_0^< = \phi$ → Smaller

$F_0^< \cap G^{\le} = \phi$

$\Rightarrow$ algrebraically

$\Downarrow$

System no soln.

ii) Sufficient Part.

$\downarrow$

Two version:

a) ① f is convex at $\bar{x}$.

② $\boxed{F_0^< \cap G_0^< = \phi}$ FJ

③ g_i strictly convex

$\Rightarrow \begin{cases} ① \ S \text{ locally convex} \\ ② \ G_0^< = G^{\le} \end{cases}$

b) ① f convex

② $\boxed{F_0^< \cap G_0^{\le} = \phi.}$ KKT

③ g_i convex $\Rightarrow$ S convex locally.

Only Trick is: use g_i convex $\Rightarrow$ S locally convex at $\bar{x}$

pf: we need to show, $\exists N_\varepsilon(\bar{x})$. $\forall x \in N_\varepsilon(\bar{x})$,

$\bar{x} - x \in d$ $\quad g_i(\bar{x}) = 0 \quad i \in I$

$g_i(x) \le 0$

$\Rightarrow g_i(\bar{x} + \lambda d) = g_i(\cdots) \le g_i \cdots + g_i \cdots \le 0$

$\Rightarrow \bar{x} + \lambda d \in S \Rightarrow d \in D = G^{\le}$

Then FJ OPTIMALITY Condition

$\nabla f(\bar{x})^T d < 0$

$\nabla g_i(\bar{x})^T d < 0$

$A = \begin{pmatrix} \nabla f(\bar{x})^T \\ \nabla g_i(\bar{x})^T \end{pmatrix}$

① Nec. $\Rightarrow$ $\boxed{F_0^< \cap G_0^< = \phi}$ $\Rightarrow$ $Ad<0$ 无解 $\Leftrightarrow$ $A^T p = 0 \quad p \ge 0 \quad p \ne 0$ 有解

$A^T = (\nabla f(\bar{x}) \quad \nabla g_i(\bar{x}))$

$\Leftrightarrow$ $\boxed{\mu_0 \nabla f(\bar{x}) + \sum_{i \in I} \mu_i \nabla g_i(\bar{x}) = 0 \quad \mu \ge 0 \quad \mu \ne 0}$

② Suf → a) ① FJ

② g_i strictly convex

③ f convex

important

$$\Leftrightarrow \begin{cases} \mu_0 \nabla f(\bar{x}) + \sum_{k=1}^{m} \mu_k \nabla g_k(\bar{x}) = 0 \\ \mu_i g_i(\bar{x}) = 0 \\ \mu \geq 0, \mu \neq 0 \end{cases}$$

KKT Optimality Condition.

① Necessary ⇒ Modification of FJ → Actually Beyond: $F^{<} \cap G^{\leq} = \phi$

⇑ need more requirement

$\{\nabla g_i(\bar{x}) : i \in I\}$ L. I.

② Sufficient

Second-order Sufficient & Necessary Condition

↓

① Lagrangian f^{n}

$$\phi(x, u, v) = f(x) + \sum u_i g_i(x) + \sum v_i h_i(x)$$

↓ for KKT Point $\bar{x}$. $\exists \bar{u}, v$. s.t $\nabla\phi(\bar{x}, \bar{u}, \bar{v}) = 0$

$$L(x) = \phi(x, \bar{u}, \bar{v})$$

② Sufficient Condition (second-order) $\boxed{\bar{x} \to \text{KKT Point}}$

$d^T \nabla^2 L(\bar{x}) d > 0$ for all $d \in C$ ⇒ strict local optimum

$$C = \{d \neq 0 : \nabla g_i(\bar{x})^T d = 0, \ i \in I^+$$
$$\nabla g_i(\bar{x})^T d \leq 0, \ i \in I_0,$$
$$\nabla h_i(\bar{x})^T d = 0\}$$

③ Necessary Condition (Second-order)

$\bar{x}$ is local minimum & qualification constraint (LICQ)

⇓

$$d^T \nabla^2 L(\bar{x}) d \geq 0 \quad \text{for all} \quad d \in C$$