Ch 1 Convexity

① Convex Hull → $\text{conv}(S) = \{ \sum \lambda_i x_i : \sum \lambda_i = 1 \ \ \lambda_i \geq 0 \ \ x_i \in S \}$

$$\Downarrow$$

[Caratheodory's Thm]

给出一个上界

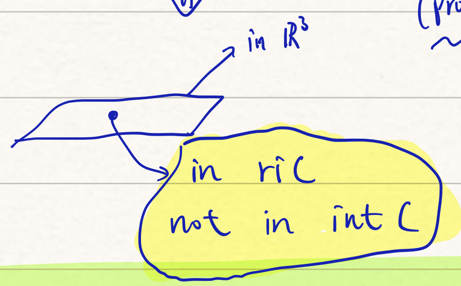
$S \subset \mathbb{R}^n \Rightarrow \text{conv}(S) = \{ \sum_{i=1}^{n+1} \lambda_i x_i : \sum \lambda_i = 1 \ \ \lambda_i \geq 0 \ \ x_i \in S \}$.

(Some defn)

② Affine Hull → $\text{aff}\, S = \{ \sum \lambda_i x_i : \sum \lambda_i = 1 \ \ x_i \in S \}$

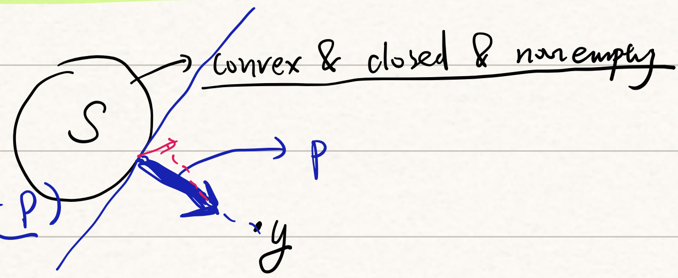
relative int → $\text{ri}\, C = \{ x \in C : \exists \varepsilon > 0,\ (x + \varepsilon B') \cap \text{aff}\, C \subset C \}$

(promotion of interior)



③ $\boxed{\text{Separation Thm}}$!!!

(guarantee the existence of p)



Pf Part

⇒ After 'simple' trick, we can get unique $\bar{x}$.

Such that $\|\bar{x} - y\| \leq \|x - y\|$ for all $x \in S$

point ⇔ set

This part is relatively difficult!

Q: How to use this to achieve conclusion?

(*) $\begin{cases} p^T y > \alpha & ① \\ p^T x \leq \alpha . & ② \end{cases}$

obviously, we can GUESS that

$p = y - \bar{x}$, $\alpha = p^T \bar{x}$.

↳ How to prove the (*)?

// ↳②

Contradiction! ① obviously true

if for some $x \in S$, $(y-\bar{x})^T(x-\bar{x}) > 0$

$\Downarrow$

$\|\bar{x}-y\|$ not the minimal

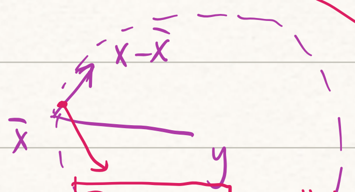
$$\underbrace{y-\bar{x}}_{A} \qquad \underbrace{x-\bar{x}}_{B}$$

$$\|y-x\| = \langle A-B, A-B\rangle$$

$$= \|A\|^2 + \|B\|^2 - \underbrace{2\langle A, B\rangle}_{<0}$$

$\Longrightarrow$ Contradiction

Motivation



$\boxed{\bar{x}+\lambda(x-\bar{x})}$ will be smaller!

small

$\rightarrow$ Key Point of the Pf.

Application Part

if we have ① a closed & convex set S.

and ② a point $y \notin S$

$\Downarrow$

there exists one p and α

such that $\begin{cases} p^T y > \alpha \\ p^T x \le \alpha \quad \text{for all } x \in S. \end{cases}$

used in Farka's Lemma etc.

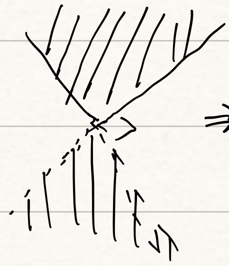
① Farka's Lemma:

$\{ \; Ax \le 0 \quad c^T x > 0$, for some x. ①

$A^T y = c \quad y \geq 0$ for some y ②

Geometric

① $\Rightarrow$ $A = \begin{pmatrix} \bigcirc \\ \bigcirc \\ \bigcirc \\ \bigcirc \end{pmatrix} = \begin{pmatrix} a_1^T \\ \vdots \\ a_m^T \end{pmatrix}$



$\Rightarrow \boxed{c \notin \text{cone}(A)}$

② $\Rightarrow$ $c \to$ linear combination of $\{a_1, \dots, a_m\}$. $\Rightarrow$ $c \in \text{cone}(A)$

Formally (pf)

key point: use separation Thm through ② ✗ $\Rightarrow$ ① ✓

(is apparent) $\leftarrow$ $\boxed{② ✓ \Rightarrow ① ✗}$

$S = \{z : z = A^T y, \ y \geq 0\}$

Outline:

$\boxed{\text{closed}} \to$ DIFFICULT

① minimal rep..

② A_K^T linearly independ.

③ $z = \boxed{A_K^T} y_K$

row (column) full rank.

$\Rightarrow$ $\{z\}$ Bounded

(y_k Bounded) $\Leftrightarrow$ $\{y\}$ Bounded

④ $z_k \to z \Rightarrow z \in S$

$z_k = A^T y_k$

$\boxed{y_{k_t} \to y}$

$\Rightarrow \boxed{z = A^T y \quad y \geq 0}$ ✓

② Farka's Lemma $\Leftrightarrow$ Gordan's Theorem $\Rightarrow$ Use the equivalence to prove this

$\{$ S1: $Ax \leq 0, \quad c^T x > 0$ for some x

S_2: $A^T y = c \quad y \geq 0 \quad$ for some y

S_1': $Ax < 0$ for some x

S_2': $A^T y = 0$, $y \geq 0$ for some $y \neq 0$

ky is also soln.

Analysis

$S_1' \Leftrightarrow Ax + s \cdot e \leq 0 \quad s > 0$

$\Leftrightarrow [A \; e] \begin{bmatrix} x \\ s \end{bmatrix} \leq 0 \quad [0, \cdots 0, 1] \begin{bmatrix} x \\ s \end{bmatrix} > 0$

Key: Construction

Farka's Lemma!

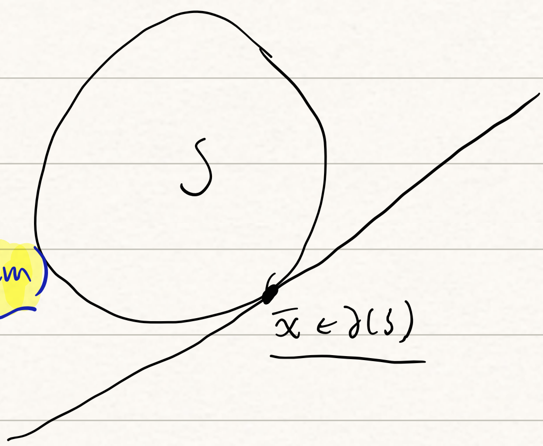
$S_2' \Leftrightarrow A^T y = 0$, $y \geq 0 \quad e^T y = 1$

$\Leftrightarrow \begin{bmatrix} A^T \\ e^T \end{bmatrix} y = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad y \geq 0$

$\boxed{\text{Supporting Hyperplane}} \Rightarrow \boxed{S \to \text{convex}} \Rightarrow$ existence of Separating Hyperplane

relatively trivial

(use separating thm)



subgradient!

Polyhedron, extreme points & direction

characterization

$\boxed{\text{Form}}$

Polyhedron $\rightarrow$ $S = \{ x : p_i^T x \le \alpha_i \ , i \in N \}$
$= \{ x : p^T x \le \alpha \}$

$\Updownarrow$ 没什么意义？

$\{ x : Ax = b , x \geq 0 \}$

Useful Conclusion

Defn of Extreme Point $\rightarrow$ $x \in E.P. \Leftrightarrow$ $\boxed{x = \lambda x_1 + (1-\lambda) x_2 \quad 0 < \lambda < 1 \Leftrightarrow x_1 = x_2 = x}$

Defn of Extreme Direction $\rightarrow$
① Recession Direction d
$x \in S \Rightarrow x + \lambda d \in S$ all $\lambda \geq 0$
② Extreme Direction d
$d = \lambda_1 d_1 + \lambda_2 d_2 \quad \lambda_1, \lambda_2 > 0$
$\Rightarrow d_1 = \alpha d_2$ for some α

① Characterization of E.P.

$\underbrace{S = \{ x : Ax = b, x \geq 0 \} \Leftrightarrow \text{BFS}}$ E.P.

$A = (a_1 \cdots a_n)$

pf: "$\Leftarrow$" obvious

"$\Rightarrow$" $x \in S$. E.P. $\quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ 0 \end{bmatrix}$

Claim 1: $\{ a_1 , \cdots , a_k \}$ linearly independ.

$\sum \alpha_i a_i = 0$ (if not)

$x + \lambda \alpha \quad x - \lambda \alpha \Rightarrow$ Contradiction

# of E.P. $\le \boxed{C_n^m}$

$\Rightarrow k \le m \Rightarrow \begin{cases} k = m & \checkmark \\ k < m & \Rightarrow \text{basis } \cancel{\text{add}} \text{ expanded} \end{cases}$

② Characterization of Extreme Direction.

$S=\{x: Ax=b,\ x\geq 0\}$ E.D $\Leftrightarrow$ $d=\begin{pmatrix}-A_B^{-1}a_j\\ e_j\end{pmatrix}$

"$\Leftarrow$"

Pf: $\begin{pmatrix}-A_B^{-1}a_j\\ e_j\end{pmatrix}=\lambda_1\begin{pmatrix}d_{11}\\ d_{12}\end{pmatrix}+\lambda_2\begin{pmatrix}d_{21}\\ d_{22}\end{pmatrix}$

$\downarrow \boxed{\begin{pmatrix}d_{11}\\ ke_j\end{pmatrix}}$ $\quad \downarrow \boxed{\begin{pmatrix}d_{21}\\ te_j\end{pmatrix}}$ $\rightarrow$ find the Analysis Soln of $\begin{pmatrix}d_{11}\\ *\end{pmatrix}$ & $\begin{pmatrix}d_{21}\\ *\end{pmatrix}$

$\Rightarrow \begin{pmatrix}d_{11}\\ *\end{pmatrix}=\alpha\begin{pmatrix}d_{21}\\ *\end{pmatrix}$

$\Downarrow$

$\boxed{\text{E.D.}}$

$\rightarrow$ recession dire.

d recession direc. $\Rightarrow$ $Ax=b,\ x\geq 0$, $x+\lambda d\in S$

$A(x+\lambda d)=b$, $x+\lambda d\geq 0$

$\Downarrow$ $\underline{Ad=0}$ $\qquad$ $\searrow$ $\underline{d\geq 0}$

"$\Rightarrow$" d is $\boxed{\text{ED}}$ $\Rightarrow$ ① $Ad=0$ $\quad d\geq 0$

② $\underbrace{d=(d_1,\dots,d_k,\dots,d_j,\dots,0)^T}$

$\{a_1,\dots,a_k\}$

$\Downarrow$

linearly independent

① $k=m$ $\quad A_B d_m+d_j a_j=0$

$\Rightarrow d_m=-d_j A_B^{-1}a_j$

$\Rightarrow d=\begin{pmatrix}d_m\\ d_j\end{pmatrix}=d_j\begin{pmatrix}-A_B^{-1}a_j\\ e_j\end{pmatrix}$ ✓

② $k<m$. claim: a_j cannot lie in the linearly independent part

if so, $A_B d_m=0\Rightarrow \boxed{d_m=0}$

$$\{a_1, \dots, a_k\} \xrightarrow{\text{expanded}} \{a_1, \dots, a_m\}$$

$$Ad = (A_B \ \ A_K) \begin{pmatrix} d_1 \\ d_m \\ 0 \\ d_j \\ 0 \end{pmatrix}$$

$$= A_B d + A_K d_j = 0 \Rightarrow \text{SAME!}$$

$\Downarrow$

$d = 0$

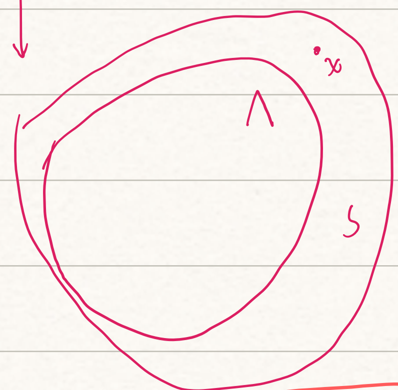
$\Downarrow$

contradicts to d is recession direc. (extreme direc.)

③ Representation of Polyhedron (With EP and ED)

$$S = \{x : Ax = b,\ x \geq 0\} \Leftrightarrow \boxed{x = \sum \lambda_i x_i + \sum \mu_j d_j \quad \boxed{\sum \lambda_i = 1 \quad \mu_j \geq 0}}$$

$\Rightarrow$ finite



Pf: $\Leftarrow$ obviously True $(\supseteq)$

$\Rightarrow$ $(\subseteq)$ $\quad L \subseteq R$

if $\exists x \in S.\ x \notin R$

$\Downarrow$

① [Separating Thm]

$\Updownarrow$

② rewrite

$$\begin{cases} p^T z > p^T x_i \\ p^T d_j \leq 0 \end{cases}$$

③ Find contradiction

choose $\bar{x} = \operatorname{argmax}\{p^T x_i : x_i \to ED\}$

$\Uparrow$

use 2 to construct $\tilde{x}$.

closed

$\Lambda = \{\sum \lambda_i x_i + \sum \mu_i d_i : \sum \lambda_i = 1,\ \mu_i \geq 0\}$

$C_k = \sum \lambda_i^k x_i + \sum \mu_i^k d_i$

$C_k \to C \Rightarrow C \in \Lambda$

$C_k' = \sum \bar{\lambda}_i^k x_i + \sum \bar{\mu}_i^k d_i$

s.t. $\lambda_i^k \to \lambda_i^*$

$\Rightarrow C_k' - \sum \bar{\lambda}_i^k x_i = d\, \bar{\mu}^k = \boxed{(d_1, \dots, d_s)\, \bar{\mu}^k}$

最小表出

$\Rightarrow \boxed{\bar{\mu}^k} = \underline{\quad\quad} (\quad)$

Bounded

$\Downarrow$

subsequence convergent

$\Rightarrow C_k^2 = \sum \bar{\bar{\lambda}}_i^k x_i + \sum \bar{\bar{\mu}}_i^k d_i$

$\downarrow \qquad \downarrow$

$$\text{s.t} \begin{cases} \tilde{x} \to ED \\ p^T\tilde{x} > p^T\bar{x} \quad (\text{use } p^Tz > p^T\bar{x}) \end{cases}$$

λ_i^* μ_i^*

$\Rightarrow \bar{C}_k^2 \to C^*$

$\Rightarrow C_k \to C^*$!

④ Method: $\bar{x} = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix}$ $\quad z = \begin{pmatrix} z_B \\ z_N \end{pmatrix}$

$$Az = b \Rightarrow A_B z_B + A_N z_N = b$$

$$p^T z - p^T \bar{x} > 0$$

$$\Rightarrow [p_N^T - p_B^T A_B^{-1} A_N] z_N > 0$$

$z_N \to > 0$

$\Downarrow$

$p_j^T - p_B^T A_B^{-1} a_j > 0$ → find a direction

$\Leftrightarrow p^T \begin{pmatrix} -A_B^{-1} a_j \\ e_j \end{pmatrix}$

$\Downarrow$

not $\boxed{E.D}$!

$\tilde{x} = \bar{x} + \lambda d$ → 保证 $p^T\tilde{x} > p^T\bar{x}$

check $\tilde{x}$ $\boxed{E.P}$

$$A_{B \cup \{j\} / \{r\}} = A_{\bar{B}}$$

$$A_{\bar{B} \cup \{r\}} = A_{B \cup \{j\}} = (A_B \ a_j)$$

$$A_B^{-1} A_{\bar{B} \cup \{r\}} = (I \ \ A_B^{-1} a_j)$$

r-th $\neq 0$

$\Rightarrow A_B^{-1} A_{\bar{B}}$ invertible

$\Rightarrow A_{\bar{B}}$ invertible

$\Rightarrow \tilde{x} \to E.P$

$\boxed{\text{Convex Function}}$ — $\boxed{\text{Subgradient}}$

$f: \mathbb{R}^n \to [-\infty, +\infty]$

dom f.

epi $f = \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R},\ \mu \geq f(x)\}$

proper

closed $\boxed{\text{epi}(f)}$

$\text{epi}(\text{cl} f) = \text{cl}(\text{epi}(f))$

$\boxed{\text{the closure of } f}$

① Defn of $\boxed{\text{convex}}$

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

② property :

1. φ convex

$$\frac{\varphi(x+\lambda_1 d) - \varphi(x)}{\lambda_1} \leq \frac{\varphi(x+\lambda_2 d) - \varphi(x)}{\lambda_2}$$

2. $\varphi : S \to \mathbb{R}$ convex. (S convex)

Liptchiz

3. φ convex in $\binom{\text{open}}{\text{convex}}$ set S

$\Leftrightarrow \forall \bar{x} \in S,\ \exists \xi$ s.t $\varphi(x) \geq \varphi(\bar{x}) + \xi^T (x - \bar{x})$ for $\forall x$

any $\bar{x} \in S$

φ strictly convex $\Rightarrow$ $\varphi(x) > \varphi(\bar{x}) + \xi^T (x - \bar{x})$

holds for $\forall x \in S$, $\xi \in \partial f(\bar{x})$

$\forall \bar{x} \in S,\ \exists \xi$. s.t $\varphi(x) > \varphi(\bar{x}) + \xi^T (x - \bar{x})$ for $\forall x$

$\Rightarrow$ strictly convex

4. f differentiable (twice) convex in <u>open convex</u> S

f convex $\Leftrightarrow$ $\nabla^2 f(x) \succeq 0$ for all $x \in S$