① Cauchy Point.

$$m_k(p) = f(x_k) + \nabla f_k^T p + \frac{1}{2} p^T B_k p \Rightarrow \text{Model function}$$

$$\min_{\|p\|\le\Delta_k} m_k(p) \quad \xrightarrow{\text{Cauchy point}} \quad \min_{\|t p_k^s\| \le \Delta_k} m_k(t p_k^s) \qquad p_k^s = \frac{-\nabla f_k}{\|\nabla f_k\|}\cdot\Delta_k$$

$\downarrow$

$$\min_{0\le t\le 1} f(x_k) - \|\nabla f_k\|\cdot t\,\Delta_k + \frac{1}{2}t^2\left(\frac{\Delta_k}{\|\nabla f_k\|}\right)^2\cdot \nabla f_k^T B_k \nabla f_k$$

$$h'(t) = -\|\nabla f_k\|\Delta_k + t\left(\frac{\Delta_k}{\|\nabla f_k\|}\right)^2\cdot\nabla f_k^T B_k \nabla f = 0$$

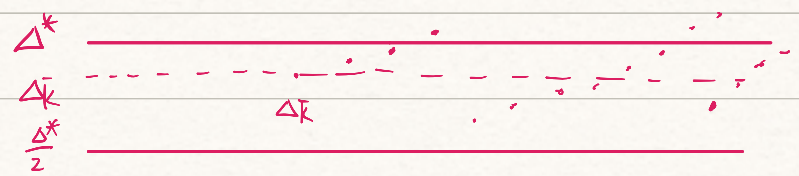
$$\Rightarrow t = \frac{\|\nabla f_k\|\,\Delta_k}{\left(\frac{\Delta_k}{\|\nabla f_k\|}\right)^2 \nabla f_k^T B_k \nabla f}$$

$$= \begin{cases} 1, & \nabla f_k^T B_k \nabla f < 0 \\ t^*, & \nabla f_k^T B_k \nabla f \ge 0 \end{cases}$$

$$= \min\left\{1, \frac{\|\nabla f_k\|^3}{\Delta_k\cdot\nabla f_k^T B_k \nabla f}\right\}$$

The Diagram of Convergence for Cauchy Point (Global)

satisfy some inequality



重要结论

$|\rho_k - 1| < C\cdot\Delta_k$

$\Downarrow$

Δ_k 可以Bound住 0 $\longrightarrow$

$\Downarrow$ 永远都会有个下界 $\tilde{\Delta}$

$|\rho_k - 1|$ $\Downarrow$

choose Δ_k 小 可以让 ρ_k 表现好

$\Downarrow$ Δ_k 更新策略保证 Δ_k 不能太小 $\rightarrow$ $\Delta_k \ge \Delta^*$

$\boxed{\text{结论: Convergence in the sense of } \liminf_{n\to\infty} \|\nabla f_n\| = 0}$

① Δ_k 小 $\Rightarrow$ $\rho_k \to 1$ $\Rightarrow$ Δ_k 变大 $\Rightarrow$ Δ_k bounded from below

i.e., $\Delta_k \ge \tilde{\Delta}$

② 不能有只有finite项不是 $\rho_k < \frac{1}{4}$ $\Rightarrow$ $\Delta_k \to 0$

不能存在'无限项 $\rho_k \ge \frac{1}{4}$

$\Rightarrow$ 无限项 p_k 为本 $\Rightarrow$ 无穷下降 $\Rightarrow$ $f \to -\infty$

Sufficient & Necessary Condition for TR subprob.

$p^* = \arg\min m(p) = f + g^T p + \frac{1}{2} p^T B p \quad \|p\| \le \Delta.$

$\Leftrightarrow$ p^* is feasible & $\exists \lambda$ $\begin{cases} \left.\begin{aligned} (B+\lambda I)\, p^* &= -g \\ \lambda(\|p^*\| - \Delta) &= 0 \end{aligned}\right\} \Rightarrow \text{trivial from KKT condition} \\ B + \lambda I \succeq 0. \Rightarrow \text{Non-trivial} \end{cases}$

$\longrightarrow$ 理论寻找 p^*:

① $B \to \begin{matrix} \lambda_1, \lambda_2, \dots, \lambda_n \\ \downarrow \quad\quad \downarrow \\ q_1 \quad \dots \quad q_n \end{matrix}$

② i) $\lambda > -\lambda_1 \Rightarrow B+\lambda I \succ 0 \Rightarrow p^* = -(B+\lambda I)^{-1} g$

$$= \sum_{i=1}^{n} -(B+\lambda I)^{-1} \alpha_i q_i \qquad \alpha_i = q_i^T g$$

$$= \sum_{i=1}^{n} -q_i^T g\, (B+\lambda I)^{-1} q_i$$

$$= \sum_{i=1}^{n} -q_i^T g \frac{1}{\lambda_i + \lambda} q_i$$

$$= \sum_{i=1}^{n} -\frac{q_i^T g}{\lambda_i + \lambda} q_i$$

如果 $q_1^T g \ne 0 \Rightarrow$ 寻找 λ s.t $\|p^*\| = \Delta \Rightarrow$ solvable!

如果 $q_1^T g = 0$ 失效

ii) choose $\lambda = -\lambda_1 \Rightarrow (B+\lambda I) \succeq 0$

$\Rightarrow$ 考虑 $(B+\lambda I)\, p^* = -g$

$p^* = \sum \alpha_i q_i \Rightarrow (\lambda_i - \lambda_1) \cdot \alpha_i = -q_i^T g.$

$\Rightarrow \alpha_i = \dfrac{-q_i^T g}{\lambda_i - \lambda_1}$ for $i \ge 2$

when $i=1$, $\|p^*\| = \|\sum_{i=2}^{n} \alpha_i q_i + \alpha_1 q_1\| = \Delta \Rightarrow \alpha_1$