Self-Understanding

→ disrete r.v. everything is good

① pmf $f_X(x) := \mathbb{P}(X=x)$

cdf $F_X(x) := \mathbb{P}(X\le x)$

$$f_X(x) = F_X(x) - F_X(x-1)$$

② Conditional distribution

pmf $f_{X|Y}(x|y) := \mathbb{P}(X=x|Y=y) = \mathbb{P}(\tilde{X}_y = x)$

cdf $F_{X|Y}(x|y) := \mathbb{P}(X\le x|Y=y) = \mathbb{P}(\tilde{X}_y \le x)$

where $\tilde{X}_y = X|Y=y$

$f_X(x)\Delta x = \mathbb{P}(x<X\le x+\Delta x)$

→ continuous r.v. something brakes down

① pdf $f_X(x) :\neq \mathbb{P}(X=x)$

$$f_X(x) := \lim_{\Delta x\to 0}\frac{\mathbb{P}(x<X\le x+\Delta x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{F(x+\Delta x)-F(x)}{\Delta x} = F'(x)$$

cdf $F_X(x) = \mathbb{P}(X\le x)$

② Conditional distribution

pdf $f_{X|Y}(x|y)$

derivative

cdf $F_{X|Y}(x|y) :\neq \mathbb{P}(X\le x|Y=y)$

$$F_{X|Y}(x|y) := \lim_{\Delta y\to 0}\mathbb{P}(X\le x|\, y\le Y\le y+\Delta y) = \frac{\int_{-\infty}^{x} f(u,y)\,dy}{f_Y(y)}$$

not well defined

since $\frac{\mathbb{P}(X\le x, Y=y)}{\mathbb{P}(Y=y)} = \frac{0}{0}$

since $f_{X|Y}(x|y)$ is the derivative of $F_{X|Y}(x|y)$

then
$$f_{X|Y}(x|y) := \lim_{\Delta x\to 0}\frac{F(x+\Delta x|y) - F(x|y)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\lim_{\Delta y\to 0}\frac{\mathbb{P}(x<X\le x+\Delta x|\, y\le Y\le y+\Delta y)}{\Delta x}$$

$$= \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$\boxed{X \mid Y = y_0}$

a) cdf $\boxed{\mathbb{P}(X \le x_0 \mid Y = y_0)} := \lim_{\Delta y \to 0} \mathbb{P}(X \le x_0 \mid Y \in [y_0, y_0 + \Delta y]) = \dfrac{\int_{-\infty}^{x_0} f(u, y_0)\, du}{f_Y(y_0)}$

note that $P_X(x) := \lim_{\Delta x \to 0} \dfrac{\mathbb{P}(X \in [x, x + \Delta x])}{\Delta x}$

$$= \frac{d\mathbb{P}(X \le x)}{dx}$$

b) pdf $\boxed{P_{X|Y}(x_0 \mid y_0)} = \left.\dfrac{d\,\mathbb{P}(X \le x \mid Y = y_0)}{dx}\right|_{x = x_0} = \dfrac{f(x_0, y_0)}{f_Y(y_0)}$

$\boxed{\text{pf:}}$ $P_{X|Y}(x_0 \mid y_0) = \dfrac{P_{X,Y}(x_0, y_0)}{P_Y(y_0)}$

$$\longrightarrow \mathbb{P}(X \le x \mid Y = y_0) = \lim_{\Delta y \to 0} \frac{\mathbb{P}(X \le x, Y \in [y_0, y_0 + \Delta y])}{\mathbb{P}(Y \in [y_0, y_0 + \Delta y])}$$

$$= \frac{\lim_{\Delta y \to 0} \frac{F(x, y_0 + \Delta y) - F(x, y)}{\Delta y}}{\lim_{\Delta y \to 0} \frac{F_Y(y_0 + \Delta y) - F_Y(y_0)}{\Delta y}}$$

$$= \frac{\frac{\partial F(x, y)}{\partial y}}{\frac{d F_Y(y)}{dy}}$$

$$= \frac{\int_{-\infty}^{x} f(u, y)\, du}{f_Y(y)}$$

c) from b), we have $\boxed{P_{X|Y}(x_0 \mid y_0) = \dfrac{P_{X,Y}(x_0, y_0)}{P_Y(y_0)}}$ pdf

$$\boxed{P_{X,Y}(x_0, y_0) = P_{X|Y}(x_0 \mid y_0)\, P_Y(y_0)}$$

$$\Updownarrow$$

$$\mathbb{P}(X = x_0, Y = y_0) = \mathbb{P}(X = x_0 \mid Y = y_0)\, \mathbb{P}(Y = y_0)$$

d) $\boxed{P_X(x) = \int_{-\infty}^{\infty} P_{\dots}(x, y)\, du}$ & $\mathbb{P}(X = x) = \sum \mathbb{P}(X = x, Y = y)$

d) $\boxed{p_X(x) = \int_{-\infty}^{+\infty} p_{X,Y}(x,y)\,dy}$ & $\mathbb{P}(X=x) = \sum_{y} \mathbb{P}(X=x, Y=y)$

$\boxed{\text{Pf:}}$ ① $\mathbb{P}(X \le x) = \sum_{\text{partition}} \mathbb{P}(X \le x, Y \in \text{Partition}_i)$

② $\mathbb{P}(X \le x)$

$= \mathbb{P}(X \le x, Y \le +\infty)$

$= \int_{-\infty}^{x} \int_{-\infty}^{+\infty} p_{X,Y}(u,v)\,dv\,du$

$\Rightarrow \underline{p_X(x) = \int_{-\infty}^{+\infty} p_{X,Y}(x,v)\,dv}$

$= \sum_{\text{part}} \underline{\mathbb{P}(X \le x \mid Y \in \text{Partition})} \cdot \boxed{\mathbb{P}(Y \in \text{Partition})}$

$\boxed{\mathbb{P}(Y \in [y, y+\Delta y])}$

from defn

$\overset{\Delta y \to 0}{=\!=} \int_{\mathbb{R}} \boxed{\mathbb{P}(X \le x \mid Y = y)}\, \boxed{d\,\mathbb{P}(Y \le y)}$

defn

$p_X(x) = \dfrac{d\mathbb{P}(X \le x)}{dx} = \int_{-\infty}^{+\infty} \boxed{p_{X|Y}(x|y)}\, \boxed{p_Y(y)}\, dy$

$= \underline{\int_{-\infty}^{+\infty} p_{X,Y}(x,y)\,dy}$

e) $\mathbb{E}[X] = \mathbb{E}\left[\sum_i X \cdot \mathbb{1}_{\{X \in P_i\}}\right]$

$= \sum_i \mathbb{E}\left[X \cdot \mathbb{1}_{\{X \in P_i\}}\right]$

$\overset{\Delta x \to 0}{=\!=} \int x\, \mathbb{P}(X \in P_i)$

$= \int_{\mathbb{R}} x\, d\,\mathbb{P}(X \le x)$

Change Of Variable Summary

① 1 - variate case

$\boxed{\begin{array}{ll} \to X \sim p_X & \underline{\underline{Y = f(X)}} \\ \to p_Y\ ? & \end{array}}$

(stronger condition)

Answer: Condition (Assump.) $\longrightarrow$ $f(\cdot)$ is monotone

then $\mathbb{P}(Y \le y)$

we need $f'(x) \neq 0$

$= \mathbb{P}(f(X) \le y) = \mathbb{P}(X \le f^{-1}(y))$

$y = f(x)$

$= \int_{\mathcal{X}} p(x)\,dx \qquad \mathcal{X} := \{x : f(x) \le y\}$

$\rightarrow \boxed{y - f(x) = 0}$

$\overset{y=f(x)}{=\!=} \int_{\mathcal{Y}} p(f^{-1}(y)) \left| \frac{d f^{-1}(y)}{dy} \right| dy \qquad \mathcal{Y} := \{ y : y \le y \}$

$\rightarrow \begin{cases} ① \ x = f^{-1}(y) \\ ② \ f^{-1}(y)' = -\dfrac{F_y}{F_x} \end{cases}$

$= \dfrac{1}{f'(x)}$

$\boxed{\text{Implicit Function}}$

② Multi-variate case

$\longrightarrow (X, Y) \sim P_{X,Y} \qquad \begin{cases} u = f_1(X, Y) \\ v = f_2(X, Y) \end{cases}$

$\longrightarrow P_{u,v}$?

Answer: Similarly, $\mathbb{P}(U \le u, V \le v)$

$= \mathbb{P}(f_1(X,Y) \le u, f_2(X,Y) \le v)$

change of variable:

$= \int_S P_{X,Y}(x, y)\,dx\,dy \qquad S = \left\{ (x,y) : \boxed{\begin{array}{l} f_1(x,y) \le u \\ f_2(x,y) \le v \end{array}} \right\}$

$\begin{cases} f_1(x,y) = a \\ f_2(x,y) = b \end{cases}$

$\begin{cases} x = g_1(a,b) \\ y = g_2(a,b) \end{cases}$

$\overset{\text{change of variable}}{=\!=} \int_S P_{X,Y}(g_1(a,b), g_2(a,b)) \begin{vmatrix} g_{1,a} & g_{1,b} \\ g_{2,a} & g_{2,b} \end{vmatrix} da\,db$

$\boxed{\text{implicit function theorem}}$

$S = \left\{ (a,b) : \boxed{\begin{array}{l} a \le u \\ b \le v \end{array}} \right\}$

$\begin{cases} u - f_1(x,y) = 0 \\ v - f_2(x,y) = 0 \end{cases}$

$$\Rightarrow \quad \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{F(x,y;u,v) := \begin{pmatrix} u - f_1(x,y) \\ v - f_2(x,y) \end{pmatrix}}$$