Mid Term ⇒ 9.29

Modification Q1 fgh → CE

Application of TU ⇒ Different to check directly ⟷ Equitable Bicolouring (Another equivalent Defn.)

↳ IP relaxation (Graph) Network

TU ⇔ every submatrix (column) is Equitable Bicolouring (column).

Application in Graph

① → Bipartite Graph incidence matrix ⇒ TU (row equitable bicolouring)

② • Matching is a subset of independent edge (NO intersaction)

• $\delta(V) \subseteq E$ → the edges of vertex V.

Max cardinality Bipartite matching. A → TU

Let $G=(V,E)$ be a bi-partite graph. The max cardinality bipartite matching seeks the largest matching of a graph, where size is measured by cardinality. This CAN BE formulated as an ILP:

$$\boxed{\begin{aligned} \max \quad & \sum_{e\in E} x_e \\ \text{s.t} \quad & \sum_{e\in \delta(V)} x_e \le 1 \\ & x_e \in \{0,1\} \end{aligned}}$$

$\Downarrow$

Note: This is valid for any graph. $\Rightarrow$ Useful when considering the bipartite graph

$\Downarrow$

Relaxation

And we can rewrite the above as:

$$\boxed{\begin{aligned} \max \quad & \sum_{e \in E} x_e \\ \text{s.t} \quad & Ax \leq 1. \\ & x_e \in \{0,1\} \end{aligned}} \qquad (1)$$

we have $A \to TU$

we still need to show:

$\begin{bmatrix} A \\ I \end{bmatrix} \to TU$

A is an incidence matrix of bipartite graph

$\Downarrow$

linear relaxation is tight! $\Leftrightarrow$ (1) & (2) share equal opt. val.

$$\boxed{\begin{aligned} \max \quad & \sum_{e \in E} x_e \\ \text{s.t} \quad & Ax \leq 1. \\ & 0 \leq x_e \leq 1 \quad e \in E \end{aligned}} \qquad (2)$$

To conclude this, rewrite (1) as follows:

$$\begin{aligned} \max \quad & \sum_{e \in E} x_e \\ \text{s.t} \quad & \begin{bmatrix} A \\ I \end{bmatrix} x \leq \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix} \\ & x \geq 0 \end{aligned}$$

Side note:

Q: What will happen if obj. is non-linear?

A: Things break down.

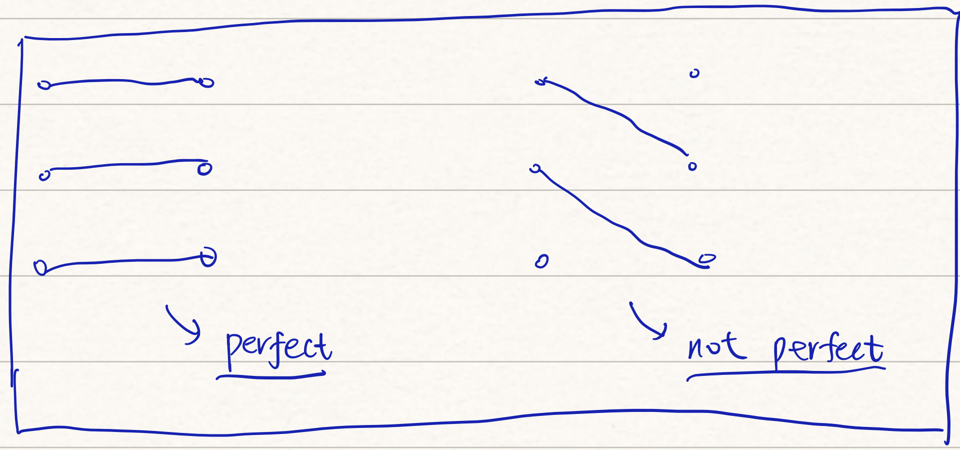
The opt. soln. to such problems do not need to lie on AN EXTREME POINT

And our conclusion now is related to EXTREME POINT

Ex: Minimize weight Perfect Matching.

$G=(V,E) \rightsquigarrow$ Graph.

We say that a matching is PERFECT if all vertices are selected.



This prob. can be formulated as follows:

$$\boxed{\begin{aligned} \min \ & \sum_{e\in E} c_e x_e \\ \text{s.t } & \sum_{e\in\delta(v)} x_e = 1 \quad v\in V \\ & x_e \in \{0,1\} \end{aligned}}$$

$\rightsquigarrow$ c_e: weight of vertex e

perfect (matching)

If a graph is bi-partite, then $A \rightsquigarrow TU \Rightarrow \begin{bmatrix} A \\ I \end{bmatrix} \rightsquigarrow TU$

$\Rightarrow$ Linear relaxation is tight

Specifically, (3) can be written as:

$$\left.\begin{aligned} \min \ & \sum_e c_e x_e \\ \text{s.t } & Ax = 1 \\ & x \in \{0,1\} \end{aligned}\right] \ (3)$$

what if $x > 1$?

the relaxation is tight

The Linear Relaxation is:

$$\min \sum c_e x_e \quad \text{s.t.} \quad \boxed{\begin{array}{l} Ax = \mathbb{1} \\ 0 \le x \le \mathbb{1} \end{array}}$$

(4) ⟶ LR form

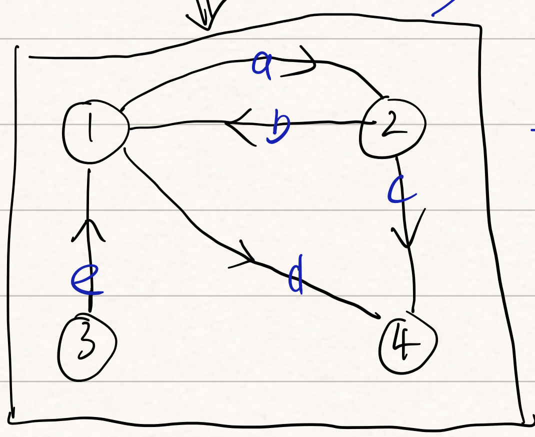
can remove $x \le 1$

⇓

use uni-modularity ✓

Directed Graph

Given a directed graph $G = (V, E)$



	a	b	c	d	e
1	+1	−1		+1	−1
2	−1	+1	+1		
3					+1
4			−1	−1	

E is the set of arcs (directed edges), the node-arc incidence matrix is the matrix whose rows are vertices, and columns are the arcs, and whose entries are.

$$A_{\text{vertex } i,\ \text{edge } j} = \begin{cases} +1, & \text{start from } i \\ -1, & \text{end at } i \\ 0, & \text{otherwise.} \end{cases}$$

[Props]. Directed Graph incidence matrix is TU

⇓

Arbitrary

Pf: (use equitable bicolouring)

Let J be any subsets of rows of A.

Now, note that every cloumn of A corresponds to an arc, and therefore was exactly one $+1$ & -1

Define $J_1 = J$ and $J_2 = \phi$. $\rightarrow$ summation 0

J_1: summation $\begin{cases} +1 \\ 0 \\ -1 \end{cases}$ $\Rightarrow$ TU

In the directed Graph $G=(V,E)$.

Given a vertex v, we let

$\delta^-(v)$: all arcs ending at v

$\delta^+(v)$: all arcs starting from v

Example: (Shortest Path) $G=(V,E)$ a directed Graph.

Let l_e, $e \in E$, be the length of arc e.

Given 2 vertices $s, t \in V$, the goal is to find a path of the form

$$s \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k \rightarrow t$$

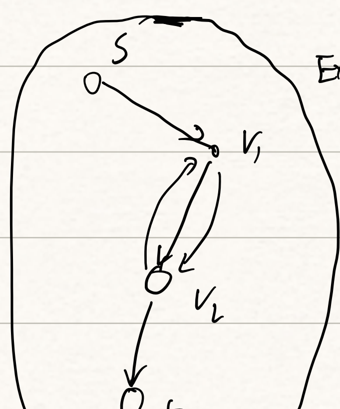
$\Downarrow$ formulate as ILP

$$\min \sum_{e \in E} l_e x_e$$

$$\text{s.t.} \begin{cases} \sum_{e \in \delta^-(v)} x_e - \sum_{e \in \delta^+(v)} x_e = 0 \quad \text{for all } v \in V \setminus \{s,t\} \\ \sum_{e \in \delta^-(s)} x_e - \sum_{e \in \delta^+(s)} x_e = -1 \\ \sum_{e \in \delta^-(t)} x_e - \sum_{e \in \delta^+(t)} x_e = +1 \end{cases}$$

可以某些点不走

Example



$x_e \in \{0,1\}$

$\Downarrow$ Rewritten

$$\boxed{\begin{aligned} \min \ & \sum_e l_e x_e \\ \text{s.t} \ & Ax = b \\ & x \in \{0,1\}^E \end{aligned}}$$

$\rightarrow$ $A \rightarrow TU$

Travelling Salesman Problem $\rightsquigarrow$ $\boxed{\text{subtour estimation}}$ (TSP)

Let $D = (V, E)$ $\rightsquigarrow$ directed Graph

C_e $\rightsquigarrow$ cost of each edge

$\downarrow$ The third constraint.

A tour $\Rightarrow$ $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k \rightarrow v_1$

$\Downarrow$

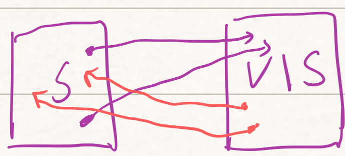
$$\text{Cost} = C_{v_1, v_2} + \cdots + C_{v_{k-1}, v_k} + C_{v_k, v_1}$$

Goal: Find the sequence of vertices to minimize the cost.

Notation: Let $S \subset V$, define:

$$\begin{cases} \delta^-(S): & S \longrightarrow V \setminus S \\ \delta^+(S): & V \setminus S \longrightarrow S \end{cases}$$

TSP Formulation



要求经过每个点

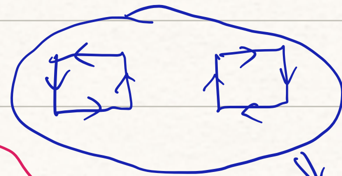
$$\min \sum c_e x_e$$

s.t. $\sum_{e \in \delta^+(v)} x_e = 1$

$\sum_{e \in \delta^-(v)} x_e = 1$

☆ $\sum_{e \in \delta^+(S)} x_e \geq 1$ for $\boxed{\text{all}}$ $\emptyset \subset S \subset V$

if we just have the TWO constraints, it is not sufficient!



subtour

subtour elimination

difficult to check!

$\Downarrow$

exponential constraints!

The proper model for

↳ TSP – subtour elimination

Difficult! – Why?

Note: The subtour elimination impose an exponential number of constraints $\Rightarrow$ In principle, this LP formulation appears to be intractable

(special method) Actually, there are methods for solving the above LP (derived from relaxation) by adding constraints one at a time.

$\boxed{\text{TU}}$

A is TU $\Leftrightarrow$ $[A, I]$ is U

$\Leftrightarrow$ A^T is TU

$\Leftrightarrow$ $[A, -A, I, -I]$ is TU $\Rightarrow$ useful when formulation

Application:

when we formulate, we may come across.

$$\left.\begin{array}{l} Ax \le b \\ x \le \mathbb{1} \\ \text{etc.} \end{array}\right\} \to \underbrace{\begin{bmatrix} A \\ I \end{bmatrix}}_{TU} x \le \begin{bmatrix} b \\ 1 \end{bmatrix}$$

TU.

Characterization of TU (equitable bicolouring)

[Thm]: A is TU.

$\Leftrightarrow$ every subset of columns of A admits an equitable column bicolouring.

construct equitable bicolouring.

Pf: $\Rightarrow$ A is TU. Pick any subset of columns $\rightsquigarrow J$.

define $d_j = \begin{cases} 1, & j \in J \\ 0, & \text{otherwise} \end{cases}$

Consider: $P := \{x : 0 \le x \le d, \underbrace{\lfloor \frac{1}{2} Ad \rfloor}_{\text{floor}} \le Ax \le \underbrace{\lceil \frac{1}{2} Ad \rceil}_{\text{ceiling}} \}$

$\Downarrow$

is bounded and nonempty

$\frac{d}{2} \in P$

$\Downarrow$

has at least ONE Extreme point.

$\Downarrow$

Let $x_{ext} \rightsquigarrow$ an ext. point

Since A is TU, $\Rightarrow$ P is integral $\Rightarrow x_{ext} \in \mathbb{Z}^n$

[props 5.2]

Now, $d \in \{0,1\}^n$, Since $0 \le x_{ext} \le d \Rightarrow x_{ext} \in \{0,1\}^n$

Define $y = d - 2x_{ext}$

$\Downarrow$ check

$$y_j = \begin{cases} \pm 1, & j \in J \\ 0, & \text{otherwise} \end{cases}$$

This implies that:

$Ay \in \{0, +1, -1\}^m$ → use $y = d - 2x_{ext}$.

$\Downarrow$

Use this to construct the equitable bicolouring partition

$J_1 = \{j \in J : y_j = +1\}$

$J_2 = \{j \in J : y_j = -1\}$.

equitable bicolouring

Exams:

① Unimodularity → IP → LR $\{x : Ax = b, x \geq 0\}$

A is U $\Rightarrow$ $\{Ax = b, x \geq 0, x \in \mathbb{Z}\} \xrightarrow{LR} \{Ax = b, x \geq 0\}$

② TU $\Rightarrow$ $\{Ax \leq b, x \geq 0, x \in \mathbb{Z}\} \xrightarrow{LR} \{Ax \leq b, x \geq 0\}$

③ Connection between TU $\Leftrightarrow$ equitable column bicolouring

→ Important.

when this thm can be used?

④ $Ax \le b,\ x \ge 0,\ x \in \mathbb{Z}$ → 不容易直接得到 ⇒ if $x \in \{0,1\}^n$

要从 $A \rightsquigarrow TU$

↓

$[A, I] \cdots$ 等形式也是 TU