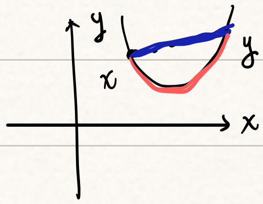
Convex Problem

Def: a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is convex.

if $\underline{f(\theta x + (1-\theta)y)} \leq \underline{\theta f(x) + (1-\theta) f(y)}$ for all $x, y \in \mathbb{R}^n$ & $\theta \in [0,1]$.

↓ line segment



Example of convex function

① $f(x) = x^2$, $\exp(x)$, $-\log(x)$ $\Rightarrow$ $f: \mathbb{R} \rightarrow \mathbb{R}$

→ how to check a function is convex?

↳ the second derivative is positive (nonnegative)

② $f(\underline{x}) = x^T x$ $\Rightarrow$ $f: \mathbb{R}^n \rightarrow \mathbb{R}$

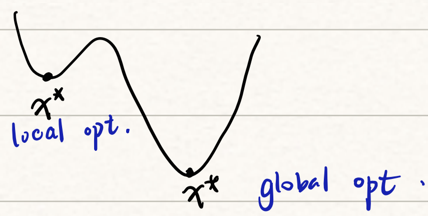
→ how to check a function is convex?

↳ check the Hessian $\left(\nabla^2 f := \frac{\partial^2}{\partial x_i \partial y_i} f\right)$

is $\underline{\text{positive semidefinite}}$ for all $\underline{x}$

Conside the following unconstrained optimization problem:

→ $\min\limits_{\underline{x} \in \mathbb{R}^n} f(x)$



i) we say that x^* is locally optimal

if $f(\underline{x}^*) \leq f(\underline{x})$ for all $\|\underline{x} - \underline{x}^*\| < \varepsilon$.

(there exists a $\varepsilon > 0$)

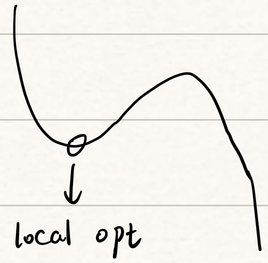
a tiny neighbourhood

↳ local

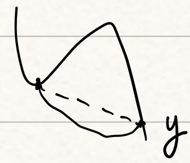
ii) we say that x^* is globally optimal

if $f(x^*) \leq f(\underline{x})$ for all $\underline{x}$

ex:



no global opt.



$$\min_{\underline{x} \in \mathbb{R}^n} f(\underline{x})$$

props: Suppose f is convex, then any locally opt. solution is globally opt. solution.

pf: Let x^* be locally optimal, we need to show:

$$f(x^*) \le f(y) \quad \text{for all } y$$

Know that $\exists\, \varepsilon > 0$ s.t $f(x^*) \le f(x)$ for all x s.t $\|x^* - x\| \le \varepsilon$.

let $y = x^* + d$

$x^* + \theta d$ where θ is sufficiently small

s.t $\|x^* - (x^* + \theta d)\| \le \varepsilon$. $\theta > 0$

From locally opt: $f(x^*) \le f(x^* + \theta d)$ $(*)$

From convexity: $f(x^* + \theta d) \le (1-\theta) f(x^*) + \theta f(x^* + d)$ $(**)$

Combining, $f(x^*) \le (1-\theta) f(x^*) + \theta f(x^* + d)$

$\Rightarrow f(x^*) \le f(x^* + d) = f(y)$ for all y #

Def: we say that a set $\mathcal{D} \in \mathbb{R}^n$ is convex if for all $x, y \in \mathcal{D}$, we have $\theta x + (1-\theta) y \in \mathcal{D}$

Qn (Zhu Dan)

Completement of Convex set is always nonconvex ?

Counterexample : half plane.

Q: is H is convex set that is not a half-plane, then is the complement non-convex ?

Q: f and $-f$ convex function $\Rightarrow$ $f(x) = a^T x + b$ (affine function)

A convex program is an optimization instance in which we minimize a convex function over a convex set.

$$\hookrightarrow \quad \min_{x \in \mathcal{D}} f(x) \begin{cases} f(x) \text{ is convex function} \\ \mathcal{D} \text{ is a convex set} \end{cases}$$

This is a constrained problem.

i) we say that x^* is locally optimal if.

$\exists \, \varepsilon > 0$, s.t. for all x s.t

$\|x - x^*\| < \varepsilon$ & $x \in \mathcal{D}$

one has $f(x^*) \leq f(x)$

ii) we say that x^* is globally optimal if.

$f(x^*) \leq f(x)$ for all $x \in \mathcal{D}$.

Props: Locally opt. solutions to convex program are globally opt.

Pf: ① Define the indicator function

$$\mathbb{I}_{\mathcal{D}}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{D} \\ +\infty, & \text{if } x \notin \mathcal{D} \end{cases}$$

② $\mathbb{I}_{\mathcal{D}}(x)$ is a convex function

$\Leftrightarrow \mathcal{D}$ is a convex set.

③ $\min_{x \in \mathcal{D}} f(x) \Leftrightarrow \min_{x} f(x) + I_{\mathcal{D}}(x)$

But defined on extended Real Line $\mathbb{R} \cup \{+\infty\}$

constrained problem $\Rightarrow$ $\boxed{\text{unconstrained function}}$

④ preview proposition still holds on the extended real line $\mathbb{R} \cup \{+\infty\}$.

Q: Are $\boxed{\text{MILPs}}$ convex programs?

A: No. e.g. $x \in \{0, 1\} \rightarrow$ feasible region

But $\frac{1}{2}$ is not in feasible region

In general, the integral constraint makes the problem non-convex.

Linear Program (LP)

In general, LP can be written as follows:

$$\min_{x \in \mathbb{R}^n} c^T x \rightsquigarrow \text{affine function}$$

$$\text{s.t. } Ax = b$$

$$x \geq 0 \rightsquigarrow \text{polyhedral}$$

Rmk: 1. x is the optimization variable

2. $A \in \mathbb{R}^{m \times n}$ as well as $b \in \mathbb{R}^m$ are the input variable

Actually LPs are subset of convex program. (special case)

LP duality. ①

Consider the LP in primal form

primal form

$\inf c^T x$

$$x$$

s.t. $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$

②

Lagrangian : $L_{\lambda,\mu}(x) = c^T x + \mu^T(\overset{b-Ax}{\cancel{Ax-b}}) + \lambda^T(-x)$

(function)

$\mu \in \mathbb{R}^m$ $\downarrow$ same as b $\lambda \in \mathbb{R}^n$ $\downarrow$ same as x $\lambda \geq 0$

③

Let's consider the problem of minimizing the Lagrangian:

$$q(\lambda,\mu) := \inf_{x \in \mathbb{R}^n} L_{\lambda,\mu}(x)$$ → unconstrained of x !

④

Let q be the minimum of the ③ problem. ⇔ $q(\lambda,\mu)$

⑤

function $q(\lambda,\mu)$ ⟷ the optimal solution to primal form

$$q(\lambda,\mu) = \inf_{x \in \mathbb{R}^n} L_{\lambda,\mu}(x)$$

$$= \inf_{x \in \mathbb{R}^n} c^T x + \mu^T(Ax-b) - \lambda^T x$$

In general, $q(\lambda,\mu)$ is a lower bound to the opt solution to primal prob.

analysis:

x_{opt} → the true solution to Primal Prob.

↳ $Ax_{opt} = b$ & $x_{opt} \geq 0$

$$c^T x_{opt} \geq c^T x_{opt} + \cancel{\mu^T(Ax_{opt}-b)} - \overset{\geq 0}{\boxed{\lambda^T}} x_{opt}$$ always true

$$= L_{\lambda,\mu}(x_{opt})$$

$$\geq \inf_{x} L_{\lambda,\mu}(x) = q(\lambda,\mu).$$

Conclusion:

In other word, $q(\lambda,\mu)$ always defines a LOWER Bound to the primal prob!

max over λ, μ to find a best possible lower bound.

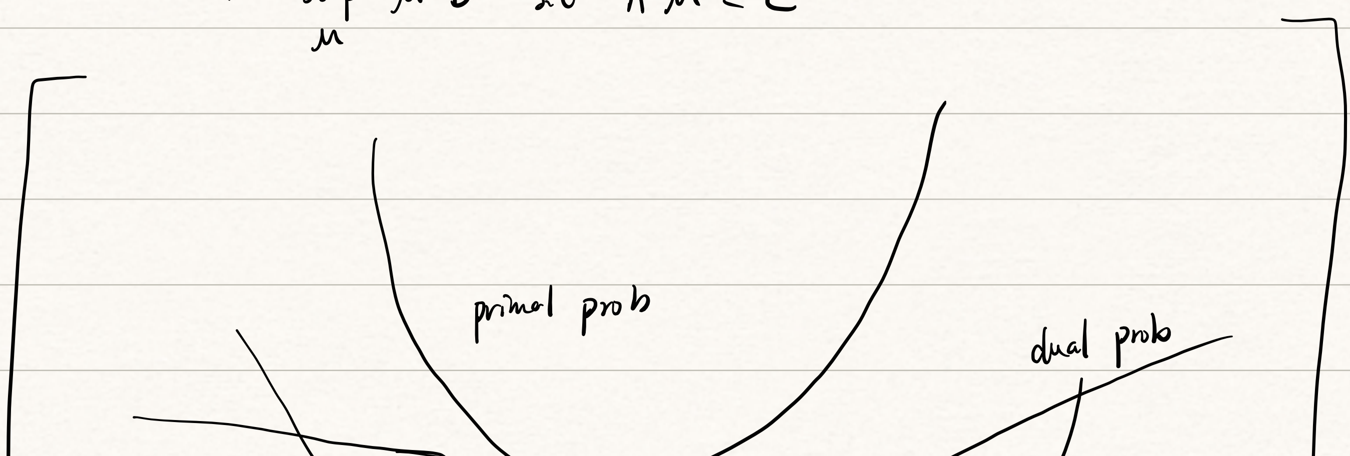
$$\boxed{\sup_{\substack{\lambda \geq 0 \\ \mu}} q(\lambda,\mu)} \rightarrow \text{dual prob}$$

$$\begin{aligned}
q(\lambda,\mu) &= \inf_{x} L_{\lambda,\mu}(x) \\
&= \inf_{x} c^T x + \mu^T(b - Ax) - \lambda^T x \\
&= \inf_{x} (c - A^T\mu - \lambda)^T x + \mu^T b \\
&= \begin{cases} -\infty, & \text{if } c - A^T\mu - \lambda \neq 0. \\ \mu^T b, & \text{else } (c - A^T\mu - \lambda = 0) \end{cases}
\end{aligned}$$

Dual prob $\Rightarrow \sup_{\substack{\lambda \geq 0 \\ \mu}} \mu^T b \quad \text{s.t} \quad c - A^T\mu - \lambda = 0 \Rightarrow \begin{cases} \lambda = c - A^T\mu \\ \lambda \geq 0 \end{cases}$

$$\Rightarrow \sup_{\mu} \mu^T b \quad \text{s.t} \quad c - A^T\mu \geq 0$$

$$\Leftrightarrow \sup_{\mu} \mu^T b \quad \text{s.t} \quad A^T\mu \leq c$$



Thm: $\boxed{\text{Weak duality.}}$

Let x be primal feasible (to Primal Prob)

μ be dual feasible (to Dual Prob)

Then $c^T x \geq b^T \mu$

pf: $c^T x = c^T x + (b - Ax)^T \mu$

$= \underbrace{(c - A^T\mu)^T}_{\geq 0} \underline{x}_{\geq 0} + b^T \mu$

$\geq b^T \mu$

holds for more general case.

$\min f(x)$

s.t $f_i(x) \leq 0$

$g_j(x) = 0$

Strong duality

the optimal value of (P) & (D) are equal.

In most case (D) $\leq$ (P)

Thm: Strong Duality for LP

Suppose that one of the Primal LP & dual LP is feasible.

Then the optimal value of primal is equal to the optimal value of the dual prob.

$$\inf_x \; c^T x \quad \text{s.t } Ax = b \quad x \geq 0$$

$$= \sup_\mu \; b^T \mu \quad \text{s.t. } A^T \mu \leq c$$

Rmk: Strong duality also holds for convex programs but one needs additional stronger qualifications.

- KKT Conditon
- Slater Condition