[Weak Duality] (i) → its theorem → Note : holds for every feasible solution !

not just opt. solution

(ii) → holds for the primal & dual optimal

⇓

In particular, if we take

This need to check the written notes!

P-opt → the Primal Prob. Opt. value

(can be $-\infty$)

D-opt → the Dual Prob Opt. value

(can be $+\infty$)

we have D-opt ≤ P-opt

(iii) if you have $\tilde{x}, \tilde{\mu}$ which are primal & Dual feasible solution, s.t $c^T\tilde{x} = b^T\tilde{\mu}$

⇓

the optimal solutions are found!

(both Primal & Dual)

Geometry of LP.

Recall the standard form of an MILP.

$$\begin{aligned} \min\ & c^Tx + d^Ty \\ \text{s.t}\ & Ax + By = b \\ & x \geq \underline{0},\ y \geq \underline{0} \\ & \underline{x} \in \mathbb{Z}^n \end{aligned}$$

the linear relaxation of the MILP is to ignore the integer constraints

$$\begin{aligned} \min\ & c^Tx + d^Ty \\ \text{s.t}\ & Ax + By = b \\ & x \geq \underline{0},\ y \geq \underline{0} \end{aligned}$$

Q: When are two solutions to these problems equal?

Fundamental to this question is to understand what the feasible region of a LP looks like.

$\hookrightarrow$ The feasible regions of LP's are known as $\Rightarrow$ polyhedral.

$$P = \{x : Ax \le b\}$$

$$= \{x : a_i^T x \le b_i \text{ for } 1 \le i \le m\}.$$

convex

Q: Can a polyhedral be unbounded?

Yes. Example: half-space.

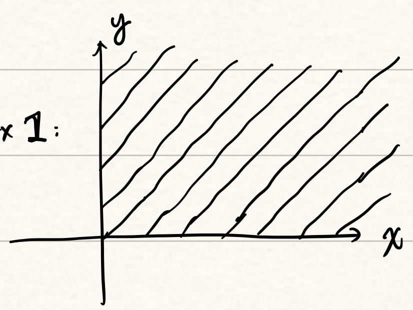
Then want to define the 'sharp points'.

o Defn. (Extreme Points) Let P be a polyhedron, we say that a point $y \in P$ is a extreme point if:

$$y = \theta z_1 + (1-\theta) z_2 \quad \text{where } 0 < \theta < 1 \text{ with } z_1, z_2 \in P.$$

$$\Rightarrow z_1 = z_2 = y.$$

Ex 1:



$\rightarrow$ only have 1 extreme point $(0,0)$

pf: ① $(0,0)$ is extreme point

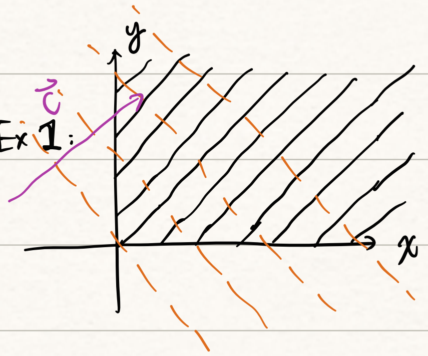
② $\{(x,y) : x \ge 0, y \ge 0\} / \{0,0\}$ all not extreme points

o Defn (Vertex) Let P be a polyhedron, we say that:

$\rightarrow$ $y \in P$ is a vertex of P if there is a derection $c \in \mathbb{R}^n$ s.t $\boxed{c^T y < c^T z}$ for all $z \in P / \{y\}$

must be strict inequality!

Ex 1:



only have 1 vertex $(0,0)$

pf: choose $c = \binom{1}{1}$

Defn: Basic Feasible Solution (BFS)

Consider a polyhedron defined as $P = \{x : Ax = b,\ x \geq 0\} \subseteq \mathbb{R}^m$

where $A \in \mathbb{R}^{m \times n}$, $m \leq n$

Assume that:

(i) P is not empty.

(ii) A has full row rank $\Leftrightarrow$ $\text{rank}(A) = m$

if not, kick out linearly dependent part!

Pick m columns of A that are linearly independent.

To simplify, let the last m columns are linearly independent

$$A = [\underset{n-m}{*} \quad \underset{m}{B}]\, m$$

Define $\hat{x} = \begin{pmatrix} 0 \\ B^{-1}b \end{pmatrix} \in \mathbb{R}^n$

Notice that $A\hat{x} = b$ → definition

$\Leftrightarrow$ $\hat{x}$ lies in the $\{x : Ax = b\}$

call this $\hat{x}$ a Basic Solution!

(all other BS are obtained by using other choices

of linearly independent columns → different B)

However, we cannot, in general, know if $\hat{x} \geq 0$.

But if indeed $\hat{x} \geq 0$, then we call this a BASIC FEASIBLE SOLUTION!

Thm: Suppose $P = \{x : Ax = b, x \geq 0\}$ is a nonempty polyhedron. Let $y \in P$

The following are equivalent:

(i) y is a vertex

(ii) y is an extreme point

(iii) y is a basic feasible solution.

$\Leftrightarrow \quad y = \begin{pmatrix} 0 \\ B^{-1}b \end{pmatrix} \in \mathbb{R}^n$

Rmk: (i) There are analogous definitions of BFS for non-standard form. (but not for today's lecture)

(ii) vertex & extreme don't rely on A & b (parameters)

$P = \{x : Ax = b, x \geq 0\} = \{x : \tilde{A}x = \tilde{b}, x \geq 0\}$

$\Rightarrow$ the BFS defined differently, but will end up being the same point because vertexes/extreme points are not defined with parametrization.

Pf: (i) $\Rightarrow$ (ii) y is vertex $\quad y \in P$ (if y is not an extreme point)

if $y = \theta z_1 + (1-\theta) z_2$ when $0 < \theta < 1$ and $z_1 \in P / \{y\}$ $z_2 \in P$

either $z_1 \neq y$ or $z_2 \neq y$.

$c^T y = \theta c^T z_1 + (1-\theta) c^T z_2 < \theta c^T y + (1-\theta) c^T z_2$ $\quad z_1 \neq y$

$\leq \theta c^T y + (1-\theta) c^T y$ $\quad z_2$ can be y

$= c^T y \quad \Rightarrow$ Contradiction!

(ii) $\Rightarrow$ (iii) if y is an extreme point

$A = [\, * \; \underbrace{B}_{k} \,] \qquad y = \begin{pmatrix} 0 \\ \tilde{y} \end{pmatrix} \}k$

$\rightarrow$ these k columns must be linearly independent

If not, there exists $d_1 \in \mathbb{R}^k$ s.t $Bd_1 = 0$

we must have $y_1 := y + \alpha \begin{pmatrix} 0 \\ d_1 \end{pmatrix}$ α is enough small.

$y_2 := y - \alpha \begin{pmatrix} 0 \\ d_1 \end{pmatrix}$

we can have $Ay_1 = b$. $Ay_2 = b$ and $y_1 \geq 0$, $y_2 \geq 0$

$\Rightarrow y_1 \in P$, $y_2 \in P$

But we have $y = \frac{1}{2}(y_1 + y_2)$

$\hookrightarrow$ contradicts to y is an extreme point

Then we have $k \leq m$.

① $k = m$ ✓

② $k < m$, then add other $m-k$ columns to make B (invertible) (linearly independent)

$\Rightarrow B\tilde{y} = b \rightarrow \tilde{y} = B^{-1}b$

then we finish $y = \begin{pmatrix} 0 \\ B^{-1}b \end{pmatrix}$.

(iii) $\Rightarrow$ (i) if y is a BFS, $y = \begin{pmatrix} 0 \\ B^{-1}b \end{pmatrix} \geq 0$.

construct $\underline{c}$ as follows:

$$\underline{c} = \begin{cases} +1, & \text{index } \{n\} \setminus B \\ 0, & \text{else.} \end{cases}$$

Consider the Optimization prob.

$$\min \ c^T x$$

$$\text{s.t} \ Ax = b, \ x \geq 0$$

(we need to show that $c^T x > c^T y \quad \forall x \in P \setminus \{y\}$

(1) the obj. is always ≥ 0

(2) it attains 0 for the point y

(3) if $c^T x = 0$ and $Ax = b$, $x \geq 0$

then since c takes $+1$ in entries not in B

$x = \begin{pmatrix} 0 \\ * \end{pmatrix}$ $\quad A = (* \ \ B)$

But $Ax = b \Rightarrow x = B^{-1}b$

(1) & (2) & (3) shows that y is the unique point that minimizes the optimization Problem

$\Rightarrow$ y is a vertex.

#

Numerical Solver for Optimization Instances.

- MILP Solvers : IBM CPLEX, Gurobi. Proprietary.
- Generic Solvers : MATLAB, Scipy
- Specialized Solvers :

Gurobi → evaluation version → solve problem instances of limited size

- knapsack problem (MILP).

$$\max \sum c_i x_i$$

$$\text{s.t. } \sum w_i x_i \le B$$

$$x_i \in \{0, 1\}.$$

- Faculty location Prob.

$$\min \sum C_j y_j + \sum_{ij} d_{ij} x_{i,j}$$

$$\text{s.t. } \sum_j x_{ij} = 1$$

$$x_{ij} \le y_j$$

$$x_{ij} \in \{0,1\}. \quad y_j \in \{0,1\}.$$

↳ relaxed to $0 \le x_{ij} \le 1$.