$\boxed{\text{Dual}}$ & $\boxed{\text{KKT}}$

PRIMAL: $\min\limits_{x} \ f(x)$

s.t $\quad g_i(x) \le 0$

$\qquad h_i(x) = 0$

Dual $\quad \max\limits_{u,v} \ J(u,v)$ $\qquad J(u,v) = \min\limits_{x} L(x; u,v)$

s.t $\quad u \geq 0$ $\qquad = \min\limits_{x} \ f(x) + \sum u_i g_i + \sum v_i h_i$

1. Weak duality: $\forall u \geq 0, v$,

we have $J(u,v) \le f(x) \quad \forall x$ feasible

推论 $\Rightarrow$ $\underset{\boxed{\text{取 max}}}{J^*} \le \underset{\boxed{\text{取 min}}}{f^*}$

2. strong duality: sufficient condition

like $\boxed{\text{slater's condition}}$

$\Rightarrow J^* = f^*$

完整刻画:

if $\begin{cases} f \text{ convex} \\ g_i \text{ convex} \rightarrow \text{inequality} \\ h_i: \text{affine} \rightarrow \text{equality} \end{cases}$

and slater's condition ✓

then strong duality holds

sufficient condition

$\boxed{\text{slater}}$

More results:

① Suppose $\boxed{J^* = f^*}$ $\qquad$ $\boxed{\text{(holds generally)}}$

$(x^*; u^*, v^*)$ is sol$^{\underline{n}}$ to (PRIMAL; Dual) Problem

$\Rightarrow (x^*; u^*, v^*)$ satisfy KKT condition that

$\begin{cases} ① \text{ Dual Feasibility} \quad u^* \geq 0 \\ ② \text{ Primal Feasibility} \quad g_i(x^*) \le 0 \quad h_i(x^*) = 0 \\ ③ \text{ stationarity}: \quad \partial_x L(x; u,v) \Big|_{x^*, u^*, v^*} = 0 \\ ④ \text{ slackness}: \quad u_i^* g_i(x^*) = 0 \end{cases}$

④ slackness :

Reason:

$$f^* = \min_{x \in F} f(x) = f(x^*) \geq f(x^*) + \underbrace{\sum u_i^* g_i}_{\leq 0} + \underbrace{\sum v_i^* h_i}_{=0}$$

$$\max_{\lambda \geq 0, v} L(x^*; \lambda, v) = f(x^*) \qquad \geq \min_{x} L(x; u^*, v^*)$$

$$= J(u^*, v^*)$$

$$= J^*$$

$\Rightarrow \begin{cases} \text{Stationarity (one necessary cond)} \\ \text{slackness} \end{cases}$

② Suppose $\boxed{J^* = f^*}$, $\boxed{\begin{array}{l} \min_x f(x) \\ \text{s.t. } g_i \leq 0 \\ \quad\;\; h_i = 0 \end{array}} \rightarrow \boxed{\text{Convex programe}}$

Then $(x^*; u^*, v^*)$ satisfy KKT condition

$\Rightarrow (x^*; u^*, v^*)$ is the sol^{n} to $\boxed{P / D}$

Reason: $\boxed{\text{Convex Prog}}$ $\Rightarrow \begin{cases} f \text{ convex} \\ g_i \text{ convex} \\ h_i \text{ affine} \end{cases}$

$$\Rightarrow L(x; u^*, v^*) = \underline{f(x) + \sum u_i^* g_i(x) + \sum v_i^* h_i(x)}$$

$\downarrow$

convex as for x

we have : $0 \in \nabla_x L(x; \lambda, \mu) \Big|_{(x^*, \lambda^*, \mu^*)}$

$$\Rightarrow x^* = \arg\min L(x; u^*, v^*) \Rightarrow J(u^*, v^*)$$

$$= \min_x L(x; u^*, v^*)$$

Analysis: $J^* = f^*$

$$= L(x^*; u^*, v^*)$$

① $L(x^*; u^*, v^*) = J(u^*, v^*)$ (u^*, v^*) is dual feasible

② $L(x^*; u^*, v^*) = f(x^*)$ x^* is PRIMAL Feasible

$$\Rightarrow \quad J^* = \max_{u \geq 0, v} J(u,v) \geq J(u^*, v^*) = f(x^*) \geq \min_{x \in F} f(x) = f^*$$

$$\Rightarrow \quad J(u^*, v^*) = \max_{u \geq 0, v} J(u,v)$$

$$f(x^*) = \max_{x} f(x)$$

$$x \in \{x : g_i(x) \leq 0, h_i(x) = 0\}$$

Conclude :

SUPPOSE STRONG Duality holds

$\boxed{\text{weak slater's condition}}$

inequality constraints are affine function, then we do not need to check strictly feasibility.

if $\boxed{\text{Slater's condition}}$ holds

then { strong duality ✓
convex prog ✓

then :

① if PRIMAL Problem is Convex Prog.

then $(x^*; u^*, v^*)$ is solution to $(P; D)$

$\Leftrightarrow$ $(x^*; u^*, v^*)$ satisfy KKT cond

② if Primal Program is NOT Convex Prog.

then $(x^*; u^*, v^*)$ is solution to $(P; D)$

$\Longrightarrow$ $(x^*; u^*, v^*)$ satisfy KKT

$\boxed{\text{(necessary condition)}}$

Moreover, suppose strong duality holds, i.e., $\boxed{J^* = f^*}$

then $(x^*; \lambda^*, \mu^*) \rightarrow$ sol^n to $(P; D)$

$\Rightarrow$ satisfy KKT condition (previous result)

also. $f(x^*) = f^* \geq f(x^*) + \sum \lambda_i^* g(x^*) + \sum \mu_i^* h(x^*)$

$$= L(x^*; \lambda^*, \mu^*)$$

$$\geq \min_x L(x; \lambda^*, \mu^*)$$

$$= q(\lambda^*, \mu^*)$$

$$= J^*$$

$\Rightarrow$ $\boxed{f^* = L(x^*; \lambda^*, \mu^*) = J^*}$

$\Leftrightarrow$ $\inf_x L(x; \lambda^*, \mu^*) = q(\lambda^*, \mu^*) = J^* = L(x^*; \lambda^*, \mu^*)$

$\sup_{\lambda, \mu \geq 0} L(x^*; \lambda, \mu) = f(x^*) = f^* = L(x^*; \lambda^*, \mu^*)$

$\Rightarrow$ $\boxed{(x^*; \lambda^*, \mu^*) \text{ is saddle point of } L(\cdot\,;\cdot,\cdot)}$

反之, if $(x^*; \lambda^*, \mu^*)$ is an saddle point of L

i.e., $L(x^*; \lambda^*, \mu^*) = \inf_x L(x; \lambda^*, \mu^*) = q(\lambda^*, \mu^*)$

$L(x^*; \lambda^*, \mu^*) = \sup_{\lambda \geq 0, \mu} L(x^*; \lambda, \mu) = f(x^*)$

generally, $q(\lambda, \mu) \leq f(x)$ for $\begin{cases} x \text{ feasible} \\ \lambda, \mu \text{ feasible} \end{cases}$

$\downarrow$

$\exists (x^*; \lambda^*, \mu^*)$ s.t $q(\lambda^*, \mu^*) = f(x^*)$

$\Rightarrow$ strong duality holds

check $(x^*; \lambda^*, \mu^*)$ is the solution that $(\lambda^*, \mu^*) = \arg\max_{\lambda \geq 0} q(\lambda, \mu)$

$x^* = \arg\min_{x \in \mathcal{F}} f(x)$

$\rightarrow$ $\boxed{\text{directly from Strong Duality \& } q(\lambda^*, \mu^*) = f(x^*)}$

$\boxed{\text{To conclude,}}$ $(x^*; \lambda^*, \mu^*)$ feasible & saddle point of $L(x; \lambda, \mu)$

$\Leftrightarrow$ Strong Duality holds & $(x^*; \lambda^*, \mu^*)$ is the optimal solution of (P) & (D)

$\boxed{\text{推论:}}$

Slater's condition + $\begin{cases} f \text{ convex} \\ g_i \text{ convex} \\ h_i \text{ affine} \end{cases}$

$\Rightarrow$ strong duality holds

$\Rightarrow$ $L(x; \lambda, \mu)$ has saddle point $(x^*; \lambda^*, \mu^*)$,

which is the optimal to (P) & (D) Problem.