Lecture 07. Submodularity.

⇒ ① Greedy Alg.

⇓

A method or Heuristic that can be applied to many discrete problems

⇒ ② Submodularity and matroids

⇓

Discrete Settings ⇒ greedy alg. computes the opt. soln.

⇒ The greedy alg. remains a valid strategy that one can try in many settings.

Defn. [Submodularity]

Let N be a discrete set.

element is subset of N (set)

We say that a non-negative function $f: 2^N \to \mathbb{R}$ is Submodular if:

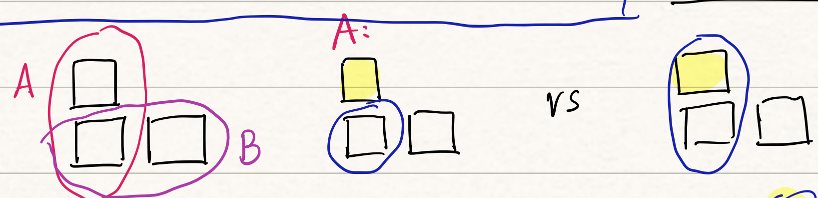
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (1)$$

holds for all subsets $A, B \subseteq N$

one can Rewrite (1) as follows:

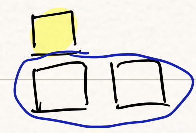
$$\boxed{f(A) - f(A \cap B) \geq f(A \cup B) - f(B)}$$ for all $A, B \subseteq N$

Note: 增量部分均是 $A - A \cap B$!

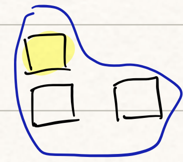


what is the increment

B:



IV



$\Downarrow$

Iuition :

集合越大，增量越小

Submodularity captures the Notion of :

dimensioning returns

Example :

$N = [n]$.

Let $f(S) = \#$ of elements in S

$\Rightarrow$ f is submodular

$f(A) + f(B) = f(A \cup B) + f(A \cap B)$

Example :

$N = [n]$

Let $f(S) = \sum_{i \in S} w_i \quad w_i \geq 0$

$\Rightarrow$ f is submodular

Example :

$N = [n]$

Let $f(S) = \min \{ T, \sum_{i \in S} w_i \}$

check : $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$

$\rightarrow$ case 1 : $f(A) = \sum_{i \in A} w_i \quad f(B) = \sum_{i \in B} w_i$

$\rightarrow$ case 2 : $f(A) = T \quad f(B) = \sum_{i \in B} w_i$

$$\begin{cases} f(A \cap B) = \sum_{i \in A \cap B} w_i & (A \cap B \subseteq B) \\ f(A \cup B) = T & (A \cup B \supseteq A) \end{cases}$$

→ case 3: $f(A) = f(B) = T$

$\Rightarrow f(A \cup B) = f(A \cup B) = T$

Another Way of Check!

Submodularity can be characterized in terms of **INCREMENT** means.

Proposition: A non-neg function fn is sub-modular

$\Leftrightarrow \quad f(S \cup \{j\}) - f(S) \geq f(S \cup \{j,k\}) - f(S \cup \{k\})$ (2)

for all subsets $S \subseteq N$ & $\{j,k\} \in N \setminus S$

Note: $\Rightarrow$ obvious

$\Leftarrow$ Non-trivial

Pf of "$\Leftarrow$": Assume (2) holds for all S and $\{j,k\} \in N \setminus S$

Take $S = A \cap B \qquad A \setminus B = \{j_1, \dots, j_r\}$

$B \setminus A = \{k_1, \dots, k_s\}$

We need to show:

$$f(B) - f(B \cap A) \geq f(A \cup B) - f(A)$$

Check: $f(B) - f(B \cap A)$

$= f(S \cup \{k_1, \dots, k_s\}) - f(S)$

$= \sum_{i=1}^{s} f(S \cup \{k_1, \dots, k_i\}) - f(S \cup \{k_1, \dots, k_{i-1}\})$

$\geq \sum_{i=1}^{s} f(S \cup \{k_1, \dots, k_i, j_1\}) - f(S \cup \{k_1, \dots, k_{i-1}, j_1\})$

$\geq \cdots \geq \sum_{i=1}^{s} f(S \cup \{k_1, \dots, k_i\} \cup (A \setminus B)) - f(S \cup \{k_1, \dots, k_{i-1}\} \cup (A \setminus B))$

$$= \sum_{i=1}^{s} f(A \cup \{k_1, \dots, k_i\}) - f(A \cup \{k_1, \dots, k_{i-1}\})$$

$$= f(A \cup (B \setminus A)) - f(A)$$

$$= f(A \cup B) - f(A)$$

Defn : (Non - decreasing)

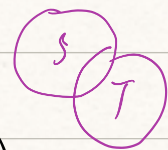
$N \to$ discrete set

$f : 2^N \longrightarrow \mathbb{R}$ is non-decreasing $\Leftrightarrow$ $f(A) \le f(B)$ for all $A \subseteq B$

Ex: ① Counting function $f(S) =$ # of element in S $\Rightarrow$ Non-decreasing

② $f(S) = \sum_{i \in S} w_i$

Proposition: f is submodular & non-decreasing



$$\Leftrightarrow f(T) \le f(S) + \sum_{j \in T \setminus S} [f(S \cup \{j\}) - f(S))$$

Pf : $\Rightarrow$ f $\begin{cases} \text{submodular} \\ \text{non-decreasing} \end{cases}$

Non-decreasing : $\cancel{f(S) \le f(S \cup \{i\})}$

$$f(T) \le f(S \cup T)$$

$$= [f(S \cup T) - f(S)] + f(S)$$

$$= \sum_{i} f(S \cup T_i) - f(S \cup T_{i-1}) + f(S)$$

$$\leq \sum_i \left[f(S \cup (T_i \setminus T_{i-1})) - f(S) \right] + f(S)$$

$$= \sum_{j \in T \setminus S} \left[f(S \cup \{j\}) - f(S) \right] + f(S)$$

$\Leftarrow$ $T = S \cup \{j, k\}$ in (3)

$$f(S \cup \{j,k\}) \leq f(S) + \left[f(S \cup \{j\}) - f(S) \right] + \left[f(S \cup \{k\}) - f(S) \right]$$

$$\Leftrightarrow f(S \cup \{j,k\}) \leq f(S \cup \{j\}) + f(S \cup \{k\}) - f(S)$$

$\Leftrightarrow$ <u>f is submodular</u>

Next $B \to$ Arbitrary $T = B \setminus \{k\}$. $S = B$

$\Rightarrow f(B \setminus \{k\}) \leq f(B)$

$\Downarrow$ Multiple times

$f(A) \leq f(B)$ for all $A \subseteq B$

$\Rightarrow$ Non-decreasing

#

$N \to$ discrete set

$f: 2^N \to \mathbb{R}$

we are interested in solving:
$$\begin{cases} \min & \sum c_j x_j \\ \text{s.t} & \sum_{j \in S} x_j \leq f(S) \quad \forall S \subseteq N \\ & x_j \geq 0, \ j \in N \end{cases}$$