$\boxed{\text{Chance Constrained Prog}}$

① $\boxed{\text{Stochastic Model}}$

→ $\boxed{\text{Model 1}}$

$$\begin{cases} \min\limits_{x} & f(x) \\ \text{s.t} & g_i(x,w) \le 0 \quad i \in I \\ & g_j(x) \le 0 \quad j \in J \end{cases}$$

↓ interpret as $\boxed{\text{deterministic optimization problem}}$

$$\boxed{K(\alpha) := \{ x : \mathbb{P}(g_i(x,w) \le 0, i \in I) \geq \alpha , \; g_j(x) \le 0 , \; j \in J \}}$$

$$\Rightarrow \begin{cases} \min\limits_{x} & f(x) \\ \text{s.t} & x \in K(\alpha) \end{cases} \rightarrow \boxed{\text{deterministic model}}$$

→ $\boxed{\text{Model 2}}$

$$\begin{cases} \max\limits_{x} & \mathbb{P}(g_i(x,w) \le 0 : i \in I) \\ \text{s.t} & g_j(x) \le 0 \end{cases} \rightarrow \boxed{\text{deterministic model}}$$

$\boxed{\text{Example}}$ $b_1 \sim \exp(1/2)$ $b_2 = -1$

Find $K(\alpha) := \{ x : \mathbb{P}(3x_1 - x_2 \geq b_1 , x_1 + 2x_2 \geq b_2) \geq \alpha \}$

$\boxed{\text{Solution}}$ $x \in K(\alpha)$

$\Leftrightarrow \mathbb{P}(3x_1 - x_2 \geq b_1 , x_1 + 2x_2 \geq b_2) \geq \alpha$

$\Leftrightarrow \mathbb{P}(b_1 \le 3x_1 - x_2) \geq \alpha$ & $x_1 + 2x_2 \geq b_2$

$\Leftrightarrow 3x_1 - x_2 \geq F^{-1}(\alpha)$ $\quad x_1 + 2x_2 \geq b_2$ $\quad \boxed{F^{-1}(\alpha) := \{ x : \mathbb{P}(b_1 \le x) = \alpha \}}$

$\Rightarrow K(\alpha) = \{ x : 3x_1 - x_2 \geq F^{-1}(\alpha) , \; x_1 + 2x_2 \geq b_2 \}$

$= \{ x : 3x_1 - x_2 \geq \ln 100 , \; x_1 + 2x_2 \geq -1 \}$

③ **Lemma** $K(\alpha) = \{x : \mathbb{P}(Tx \geq \xi) \geq \alpha\}$

① $\xi \rightarrow \boxed{\xi^i \leftrightarrow p_i}$

② $X_i = \{x : Tx \geq \xi^i\}$

③ $J_1, \ldots, J_L \longrightarrow$ index set such that $\sum_{i \in J_l} p_i \geq \alpha$

$$\Rightarrow K(\alpha) = \bigcup_{l=1}^{L} \Big[\underbrace{\bigcap_{i \in J_l} X_i}_{\text{guarantee } \boxed{\mathbb{P}(Tx \geq \xi) \geq \alpha}} \Big]$$

Rmk: when application, we just need to consider the $\boxed{\text{NON-OVERLAP}}$

Example: $K(\alpha) = \{x : \mathbb{P}(x \geq \xi) \geq \alpha\}$

$$\xi = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\quad\ \ 1/4 \qquad 1/4 \qquad 1/4 \qquad 1/4$$

Solution ① $\alpha = 0$

$K(\alpha) = \mathbb{R}^2$

$$\boxed{\begin{aligned} X_1 &= \{x : x \geq 0\} \\ X_2 &= \{x : x_1 \geq 1,\ x_2 \geq 0\} \\ X_3 &= \{x : x_1 \geq 0,\ x_2 \geq 1\} \\ X_4 &= \{x : x \geq 1\} \end{aligned}}$$

$X_1 \supset X_2 \supset X_4$

$X_1 \supset X_3 \supset X_4$

② $0 < \alpha \leq \frac{1}{4}$

$K(\alpha) = X_1 \cup X_2 \cup X_3 \cup X_4 = \mathbb{R}^2_+$ $\quad J_1 = \{1\}$ $\ J_2 = \{2\}$ $\ J_3 = \{3\}$ $\ J_4 = \{4\}$

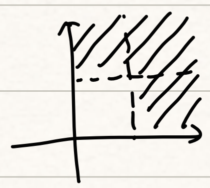
③ $\frac{1}{4} < \alpha \leq \frac{1}{2}$

$J_1 = \{1,2\}$ $\ J_2 = \{1,3\}$ $\ J_3 = \{1,4\}$ $\ J_4 = \{2,3\}$ $\ J_5 = \{2,4\}$ $\ J_6 = \{3,4\}$

$X_1 \cap X_2 = X_2$ $\qquad X_2 \cap X_3 = X_4$ $\qquad X_3 \cap X_4 = X_4$

$X_1 \cap X_3 = X_3$ $\qquad X_2 \cap X_4 = X_4$

$X_1 \cap X_4 = X_4$



$\Rightarrow K(\alpha) = X_2 \cup X_3 \cup X_4$

④ $\frac{1}{2} < \alpha \leq \frac{3}{4}$

$J_1 = \{1,2,3\}$ $\ J_2 = \{1,3,4\}$ $\ J_3 = \{1,2,4\}$ $\ J_4 = \{2,3,4\}$

$X_1 \cap X_2 \cap X_3 = X_1 \cap X_3 \cap X_4 = \cdots = X_4$

$\Rightarrow K(\alpha) = X_4$

⑤ $\frac{3}{4} < \alpha \le 1$

$J_1 = \{1,2,3,4\} \qquad X_1 \wedge X_2 \wedge X_3 \wedge X_4 = X_4$

$\Rightarrow K(\alpha) = X_4$

Convexity - Based Chance Constrained Prog

$\boxed{\text{Model}}$ $\quad K(\alpha) := \{x : \mathbb{P}(T(w)x \geqslant h(w)) \geqslant \alpha\}$

① One Constraint, LHS FIXED, RHS RANDOM

$$\begin{aligned} K(\alpha) &:= \{x : \mathbb{P}(T_1x_1 + \cdots + T_nx_n \geqslant w) \geqslant \alpha\} \\ &= \{x : T_1x_1 + \cdots + T_nx_n \geqslant \bar{w}(\alpha)\} := X(\bar{w}(\alpha)) \\ \bar{w}(\alpha) &= \inf\{w \in \Omega ;\ F(w) \geqslant \alpha\} \rightarrow \boxed{F^{\leftarrow}(\alpha)} \end{aligned}$$

$$\begin{aligned} K(\alpha) &= \{x : \mathbb{P}(T_1x_1 + \cdots + T_nx_n \le w) \geqslant \alpha\} \\ &= \{x : T_1x_1 + \cdots + T_nx_n \le \vec{w}(\alpha)\} \\ \bar{w}(\alpha) &= \sup\{w \in \Omega : F(w) \le 1-\alpha\} \end{aligned}$$

② One Constraint, LHS RANDOM (Gaussian), RHS FIXED

$$K(\alpha) = \{x : \mathbb{P}(w_1x_1 + \cdots + w_nx_n \geqslant h) \geqslant \alpha\}$$

$$w \sim N(\mu, V) \Rightarrow \underset{\substack{\shortparallel \\ z}}{x^Tw} \sim N(x^T\mu, x^TVx)$$

$\boxed{\Phi(x) = \mathbb{P}(z \le x)}$

$$\left[\begin{aligned} &\mathbb{P}(z \geqslant h) \geqslant \alpha \\ \Leftrightarrow\ &\mathbb{P}\left(\tilde{z} \geqslant \frac{h - x^T\mu}{\sqrt{x^TVx}}\right) \geqslant \alpha \\ \Leftrightarrow\ &\frac{h - x^T\mu}{\sqrt{x^TVx}} \le \Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha) \rightarrow \boxed{\text{Non-trivial}} \end{aligned}\right.$$

$$\Rightarrow K(\alpha) = \{x : x^T\mu \geqslant h + \Phi^{-1}(\alpha)\sqrt{x^TVx}\}$$

$y = V^{\frac{1}{2}}x$ $\qquad \begin{pmatrix} V^{\frac{1}{2}} \end{pmatrix} x + \begin{pmatrix} 0 \end{pmatrix}$

$\boxed{f(A) \subseteq B}$ Mapping $t = \frac{\mu^T x - h}{\Phi^{-1}(\alpha)}$ $\longleftrightarrow$ $f(x) = \begin{pmatrix} \\ \mu^T \end{pmatrix} \begin{pmatrix} \\ -\frac{h}{\Phi^{-1}(\alpha)} \end{pmatrix}$

$B \swarrow \{ (t, y) : \Phi^{-1}(\alpha) \cdot t \geq \Phi^{-1}(\alpha) \cdot \|y\|_2 \}$

$\boxed{x \in K(\alpha) \Leftrightarrow f(x) \in B}$

$\boxed{K(\alpha) \text{ is convex}}$ (Proof) $\boxed{\text{Suppose } \Phi^{-1}(\alpha) > 0}$ $\boxed{f \text{ is affine}}$ $\boxed{B \text{ convex}}$

$\rightarrow x_1, x_2 \in K(\alpha)$ $\exists \begin{pmatrix} y_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ t_2 \end{pmatrix}$ s.t $f(x_1) = \begin{pmatrix} y_1 \\ t_1 \end{pmatrix} \in B$

$\boxed{\text{要证 } x_\lambda \in K(\alpha)}$ $f(x_2) = \begin{pmatrix} y_2 \\ t_2 \end{pmatrix} \in B$

B is convex $\Rightarrow \begin{pmatrix} y_\lambda \\ t_\lambda \end{pmatrix} \in B$ $\boxed{\begin{pmatrix} y_\lambda \\ t_\lambda \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ t_1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} y_2 \\ t_2 \end{pmatrix}}$

$\Leftrightarrow \lambda f(x_1) + (1-\lambda) f(x_2) = f(x_\lambda) \in B$ (affine f)

$\Leftrightarrow x_\lambda \in A$

③ Multi-constraints
$\left\{ \begin{array}{l} \boxed{\text{Form 1} \quad \max_x \mathbb{P}(Tx \geq h)} \\ \boxed{\text{Form 2} \quad \min f(x) \quad \text{s.t } x \in K(\alpha) = \{ x : \mathbb{P}(Tx \geq h) \geq \alpha \}} \end{array} \right.$

$\boxed{h \rightarrow \text{multi-variate random variable}} \rightarrow \boxed{F(\cdot)}$

$\boxed{\text{log-concave}}$ & $\boxed{\text{quasi-concave}}$

① $\boxed{\text{log-concave}}$: f log-concave $\Leftrightarrow$ log f concave

$\not\Leftrightarrow$ f concave

f log-concave $\Leftrightarrow$ log f concave

$\Rightarrow$ log f quasi-concave

$\Rightarrow$ f quasi-concave

$\rightarrow \boxed{\text{Defn}}$: $\underline{f(\lambda z_1 + (1-\lambda) z_2) \geq [f(z_1)]^\lambda [f(z_2)]^{1-\lambda}}$

$\rightarrow \boxed{\text{保运算}}$

(a) f, g log-concave

$\Rightarrow f * g \rightarrow$ log concave

(b) $a * f$ log-concave

(c) 积分 $f \to$ log-concave

$\Rightarrow F(x) = \int_{-\infty}^{x} f(y)\, dy \to$ $\boxed{\text{log-concave}}$

(d) $f(Tx+b) \to$ log-concave

② quasi-concave: $\boxed{\text{Defn}}$ ① $S(\alpha) = \{x : f(x) \geq \alpha\} \to$ $\boxed{\text{convex}}$

② $\boxed{f(\lambda x + (1-\lambda) y) \geq \min\{f(x), f(y)\}}$

$\boxed{\text{保运算}}$ ① f: quasi-concave $\boxed{g \text{ 增}}$

$\Rightarrow g \circ f$: quasi-concave

② $f(Ax+b)$

① $\max_{x} \mathbb{P}(Tx \geq h)$

if $F(\cdot)$ log-concave $\Rightarrow F(T\cdot)$ log-concave

$\Leftrightarrow \max_{x} F(Tx)$

$\Rightarrow$ $\boxed{\log F(Tx) \text{ concave}}$

$\Leftrightarrow$ $\boxed{\max_{x} \log F(Tx)}$

② $K(\alpha) := \{x : \mathbb{P}(Tx \geq h) \geq \alpha\}$ if $F(\cdot)$ quasi-concave,

$= \{x : F(Tx) \geq \alpha\}$ then $K(\alpha)$ convex