Implicit Function Related

① Explicit Function

$$y = f(x)$$

[e.g.] $y = x^2$

② Implicit Function

$$F(x,y) = 0$$

[e.g.] 1. $y - x^2 = 0 \longrightarrow y = x^2$

2. $x - y^2 = 0 \longrightarrow$ Restrict $x \in [0,+\infty)$ $y \in [0,+\infty)$

$\Rightarrow y = \sqrt{x}$

3. $\sin x + x^3 \ln(y+1) - y^2 = 0$

③ Existence of Implicit Function (Thm)

Suppose $F: D \subseteq \mathbb{R}^2 \longrightarrow G \subseteq \mathbb{R}$

and satisfy
1. F_x, F_y continuous
2. $F(x_0, y_0) = 0$
3. $F_y(x_0, y_0) \neq 0$

$\Rightarrow$
1. $\exists\ U \times V$ and $(x_0, y_0) \in U \times V \subseteq D$ s.t. $F(x,y) = 0$ determines unique $f(x) \in V$, $\forall x \in U$
2. $y_0 = f(x_0)$
3. $f \in C^1$ and $f'(x) = -\frac{F_x}{F_y}$

④ Can be extended to $F: \mathbb{R}^n \to \mathbb{R}^k$ $\quad F(x, y) = 0$

$\Downarrow$ $\quad y \in \mathbb{R}^k$

$y = f(x)$ $\quad \begin{cases} x \in \mathbb{R}^{n-k} \\ y \in \mathbb{R}^k \end{cases}$

Condition: $\frac{\partial F}{\partial y} := \left(\frac{\partial F_i(x, y)}{\partial y_j} \right)_{i,j}$ → full-rank