$\boxed{\text{SQP.}}$

Notation:

$$A(x)^T = [\nabla C_1(x), \dots, \nabla C_m(x)]$$

$A(x) \to$ Jacobbi Matrix of $c(x) = \begin{pmatrix} C_1(x) \\ \vdots \\ C_m(x) \end{pmatrix}$

$A(x) \leftrightarrow C'(x)$

$$\Rightarrow \begin{cases} C(x) \approx C_k + A_k p \\ f(x) \approx f_k + \nabla f_k^T p + p^T \nabla^2 f_k p \end{cases} \Rightarrow \text{Basic Model of SQP}$$

$\boxed{\text{Before}}$

Origin Problem: $\boxed{\begin{array}{ll} \min & f(x) \\ \text{s.t} & C(x) = 0 \end{array}}$ → Penalty Method; → Augmented Lagrangian Method

$$L_A(x) = f(x) + \lambda^T C(x) + \frac{1}{2\mu} \sum C_i^2(x)$$

↓

Reformulate (at x_k)

① $\min \quad f_k + \nabla f_k^T p + p^T \nabla^2 f_k p$

$\quad \text{s.t} \quad C_k + A_k p = 0$

② $\min \quad f_k + \nabla f_k^T p + p^T \nabla_{xx}^2 L_k(x_k, \lambda_k) p$

$\quad \text{s.t} \quad C_k + A_k p = 0$

Motivation: use Newton Method to solve KKT of [O] → origin

1. KKT of [O] $\begin{cases} \nabla f(x) - A(x)^T \lambda = 0 \\ C(x) = 0 \end{cases}$

$$\Rightarrow \begin{bmatrix} \nabla f(x) - A(x)^T \lambda \\ C(x) \end{bmatrix} = 0 := F(x, \lambda)$$

$[\nabla C_1 \ \cdots \ \nabla C_m]$

2. Newton Method

$F(x,\lambda)=0$

$$\Rightarrow F(x_{k+1},\lambda_{k+1}) \approx F(x_k,\lambda_k) + F'(x,\lambda)\begin{pmatrix} p_x \\ p_\lambda \end{pmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \nabla^2_{xx}L(x_k,\lambda_k) & -A_k^T \\ A_k & 0 \end{bmatrix}\begin{bmatrix} p_x \\ p_\lambda \end{bmatrix} = \begin{bmatrix} -\nabla f_k + A_k^T\lambda_k \\ -c_k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \nabla^2_{xx}L(x_k,\lambda_k) & -A_k^T \\ A_k & 0 \end{bmatrix}\begin{bmatrix} p_x \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} -\nabla f_k \\ -c_k \end{bmatrix} \quad (**)$$

$$\Rightarrow \boxed{\begin{aligned} &\min \quad p^T\nabla^2_{xx}L(x_k,\lambda_k)p + \nabla f_k^T p \\ &\text{s.t} \quad A_k p + c_k = 0 \end{aligned}} \quad (*)$$

KKT Condition of $(*)$

以上, work locally.

Globally, it doesn't work $\Longleftrightarrow$ Newton Method converges locally

$\downarrow$

take step length α_k into consideration

Unconstraint problem $\rightarrow$ $\alpha_k \approx \arg\min_\alpha f(x_k+\alpha p_k)$

$\updownarrow$

需要p_k对于$\phi(\cdot)$是descent direction → 定义 by $(**)$

↓

不然没意义

Constraint Problem $\rightarrow$ $\boxed{\alpha_k \approx \arg\min_\alpha \phi(x_k+\alpha p_k)}$

$\boxed{\phi(\cdot) \rightarrow \text{merit function}}$

2 types $\left\{ \begin{array}{l} l_1\text{-merit function} \\ \text{Fletcher's Augmented Lagrangian Merit fun}^{\underline{c}}. \end{array} \right.$

Note: For a merit $f^{\underline{n}}$ $\phi(\cdot)$ for problem : $\boxed{\begin{array}{l} \min f(x) \\ \text{s.t } c(x) = 0 \end{array}}$

it needs to satisfy:

① decrease $\phi \Rightarrow \left\{ \begin{array}{l} \text{less objective value} \\ \text{more feasibility} \end{array} \right.$

② minimizer of $\phi(\cdot) \Rightarrow$ KKT of $f(\cdot)$ (or optimal)

③ step p of SQP $\Rightarrow$ descent direction for $\phi(\cdot)$

$c(x)^T c(x)$ $\quad \nabla c_1$

why $\phi_2(x,\mu) = f(x) + \frac{1}{2\mu}\|c(x)\|_2^2$ not a merit $f^{\underline{n}}$? $\quad \nabla c_2$

$\left(c_1^2(x) + \cdots + c_m^2(x)\right) = 2c_1(x)\, c_{1}'(x)$

$\Downarrow$

$$\nabla_x \phi_2(x,\mu) = \nabla f(x) + \frac{1}{\mu} \nabla c(x)\, c(x)$$

$\Rightarrow$ p is a descent direction

$$\nabla_x \phi_2(x,\mu)^T p = p^T \nabla_x \phi_2$$

$$= p^T \nabla f(x) + \frac{1}{\mu} p^T \nabla c(x)\, c(x)$$

$A(x)\, p = -c(x)$

$$= -p^T W(x) p + \lambda^T A(x) p + \frac{1}{\mu} p^T \nabla c(x)\, c(x)$$

$$= -p^T W(x) p + \lambda^T A(x) p - \frac{1}{\mu} c(x)^T c(x)$$

$$= -p^T W(x) p - \lambda^T c(x) - \frac{1}{\mu} c(x)^T c(x)$$

$$\text{maximum} = \text{const} + \underbrace{\frac{\mu}{2}\lambda^T\lambda - \frac{\mu}{4}\lambda^T\lambda}_{>0\,!}$$

$$-\lambda^T a - \frac{1}{\mu} a^T a$$

$$\nabla h = -\lambda - \frac{2}{\mu} a \Rightarrow a = -\frac{\mu}{2} \lambda$$