Wavelet Transform (Small Part of Theory for understanding)

$\Rightarrow$ We only focus on [Haar Wavelet]

① Haar Scale Function (父小波)

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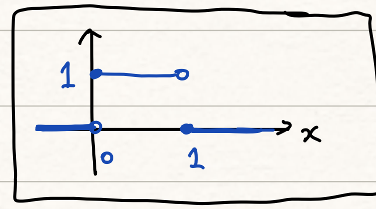

$$\phi(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{o/w} \end{cases}$$

define $\boxed{V_0 = \{ f : f(x) = \sum_{k\in\mathbb{Z}} a_k \phi(x-k), \ a_k \in \mathbb{R} \}}$

$\downarrow$

dis-continous at integer points

Generally, define $V_j = \{ f : f(x) = \sum_{k\in\mathbb{Z}} a_k \phi(2^j x - k), \ a_k \in \mathbb{R} \}$

$\downarrow$

dis-continous at $2^{-j} \cdot k$, contain the information with the resolution of 2^{-j}

orthonormal basis : $\{ 2^{j/2} \phi(2^j x - k) : k \in \mathbb{Z} \}$

Observation : $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_j$

② Haar Wavelet Function (母小波)

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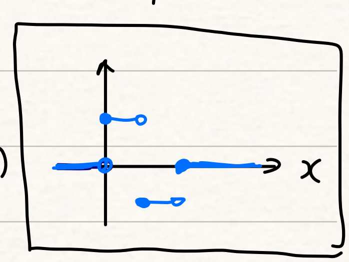

$$\psi(x) := \underbrace{\phi(2x)}_{\begin{cases} 1, & x\in[0,\frac{1}{2}) \\ 0, & \text{o/w} \end{cases}} - \underbrace{\phi(2x-1)}_{\begin{cases} 1, & x\in[\frac{1}{2},1) \\ 0, & \text{o/w} \end{cases}} = \begin{cases} 1, & x \in [0, \frac{1}{2}) \\ -1, & x \in [-\frac{1}{2}, 1) \\ 0, & \text{o/w} \end{cases}$$

observation : $\begin{cases} \psi \in V_1 \\ \psi \notin V_0 \\ \psi \in V_0^{\perp} \end{cases}$

Define $W_j := \{ g : g(x) = \sum_{k \in \mathbb{Z}} b^j_k \psi(2^j x - k),\ b^j_k \in \mathbb{R} \}$

then we all have: $\begin{cases} W_j = V_j^{\perp} \\ \boxed{V_{j+1} = W_j + V_j} \end{cases}$

$\forall f_{j+1} \in V_{j+1}$. in each small interval $[2^{-(j+1)} \cdot (k-1), 2^{-(j+1)}(k+1))$ it can be written as:

$$f_{j+1}(x) = \begin{cases} a_{k-1}, & x \in [2^{-(j+1)}(k-1), 2^{-(j+1)} \cdot k) \\ a_k, & x \in [2^{-(j+1)} \cdot k, 2^{-(j+1)} \cdot (k+1)) \end{cases}$$

$a_k - a_{k-1}$ $\quad a_{k-1} - t$ $\quad a_k - t$

then it can be written as: $f_{j+1}(x) = g_j(x) + f_j(x)$

where $\begin{cases} f_j(x) = \dfrac{a_{k-1} + a_k}{2} & x \in [2^{-(j+1)}(k-1), 2^{-(j+1)}(k+1)) \\ g_j(x) = \left(\dfrac{a_{k-1} - a_k}{2}\right) \psi(2^j x - k) \end{cases}$

$$\Rightarrow V_{j+1} = W_j \oplus V_j$$

$$= W_j \oplus W_{j-1} \oplus \cdots \oplus W_0 \oplus V_0$$

$\Rightarrow \forall f_{j+1} \in V_{j+1}$, it can be uniquely decomposed as:

$$f_{j+1} = g_j + g_{j-1} + \cdots + g_0 + f_0$$

where $g_i \in W_i$ and $f_0 \in V_0$

③ Real Application $\longrightarrow$ $\boxed{\text{Discrete Signal}}$

Assumption: with sample frequency 2^j $\longrightarrow$ j, we sample the original signal as:

$$\{ a^j_k : a^j_k = f(2^{-j} \cdot k),\ k \in \mathbb{Z} \}$$

Decomposition

Thus, the sampled signal can be expressed as:

$$f_j(x) = \sum_{k\in\mathbb{Z}} a_k^j \phi(2^j x - k) \in V_j$$

According to "Direct Sum" Decomposition $V_j = W_{j-1} \oplus V_{j-1}$

$$f_j(x) = g_{j-1}(x) + f_{j-1}(x)$$

Property : ① $\phi(2x) = \frac{1}{2}(\phi(x) + \psi(x))$

② $\phi(2x-1) = \frac{1}{2}(\phi(x) - \psi(x))$

then $f_j(x) = \sum_{k\in\mathbb{Z}} a_k^j \phi(2^j x - k)$

$$(奇偶) = \sum_{k\in\mathbb{Z}} a_{2k}^j \phi(2^j x - 2k) + \sum_{k\in\mathbb{Z}} a_{2k+1}^j \phi(2^j x - 2k - 1)$$

$$= \sum_{k\in\mathbb{Z}} a_{2k}^j \cdot \frac{1}{2}\left(\phi(2^{j-1}x - k) + \psi(2^{j-1}x - k)\right)$$

$$+ \sum_{k\in\mathbb{Z}} a_{2k+1}^j \frac{1}{2}\left(\phi(2^{j-1}x - k) - \psi(2^{j-1}x - k)\right)$$

$$= \underbrace{\boxed{\sum_{k\in\mathbb{Z}} \frac{1}{2}(a_{2k}^j + a_{2k+1}^j)\, \phi(2^{j-1}x - k)}}_{f_{j-1}(x)}$$

$$\underbrace{\boxed{+ \sum_{k\in\mathbb{Z}} \frac{1}{2}(a_{2k}^j - a_{2k+1}^j)\, \psi(2^{j-1}x - k)}}_{g_{j-1}(x)}$$

[this result corresponds to previous analysis of $V_j = V_{j-1} + W_j$]

which is very intuitive

Reconstruction

After Decomposition, we will have:

$$f_j(x) = g_{j-1}(x) + \cdots + g_0(x) + f_0(x)$$

here, $\begin{cases} f_0(x) = \sum\limits_{k\in\mathbb{Z}} a_k^0 \phi(x-k) \\ g_i(x) = \sum\limits_{k\in\mathbb{Z}} b_k^i \phi(2^i \cdot x - k) \end{cases}$ $\quad i = 0, 1, \ldots, j-1$

$\Rightarrow$ we want to Recover f_j from $\begin{cases} f_0 \\ g_0, \ldots, g_{j-1} \end{cases}$

$$\boxed{\begin{array}{ll} \text{Property: } ① & \phi(x) = \phi(2x) + \phi(2x-1) \\ ② & \psi(x) = \phi(2x) - \phi(2x-1) \end{array}}$$

consider: 1. $f_0(x) = \sum\limits_{k\in\mathbb{Z}} a_k^0 \phi(x-k)$

$$= \sum_{k\in\mathbb{Z}} \left[a_k^0 \phi(2x-2k) + a_k^0 \phi(2x-2k-1) \right]$$

$$:= \sum_{\ell\in\mathbb{Z}} \tilde{a}_\ell^1 \phi(2x-\ell)$$

$$\tilde{a}_\ell^1 := \begin{cases} a_k^0, & \ell = 2k \\ a_k^0, & \ell = 2k+1 \end{cases}$$

(Similarly) 2. $g_0(x) = \sum\limits_{k\in\mathbb{Z}} b_k^0 \psi(x-k)$

$$:= \sum_{\ell\in\mathbb{Z}} \tilde{b}_\ell^1 \phi(2x-\ell)$$

$$\tilde{b}_\ell^1 := \begin{cases} b_k^0, & \ell = 2k \\ -b_k^0, & \ell = 2k+1 \end{cases}$$

To conclude, $f_1(x) = f_0(x) + g_0(x)$

$$= \sum_{\ell \in \mathbb{Z}} \hat{a}_\ell^1 \, \phi(2x-\ell) + \sum_{\ell \in \mathbb{Z}} \tilde{b}_\ell^1 \, \phi(2x-\ell)$$

$$= \sum_{\ell \in \mathbb{Z}} a_\ell^1 \, \phi(2x-\ell)$$

$$a_\ell^1 = \hat{a}_\ell^1 + \tilde{b}_\ell^1 = \begin{cases} a_k^0 + b_k^0, & \ell = 2k \\ a_k^0 - b_k^0, & \ell = 2k+1 \end{cases}$$