$\boxed{\text{Continuous}}$

① Gamma

$$\Gamma(\alpha, \lambda) \Longrightarrow f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

$\boxed{\text{Mean}}$

$$\mathbb{E}[X] = \int \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot x^\alpha e^{-\lambda x} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}$$

$$= \frac{\alpha}{\lambda}$$

$\boxed{\text{2-nd moment}}$

$$\mathbb{E}[X^2] = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+2)}{\lambda^{\alpha+2}}$$

$$= \frac{\alpha(\alpha+1)}{\lambda^2}$$

$$\Rightarrow Var[X] = \frac{\alpha}{\lambda^2}$$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx$$

$$= \int_0^\infty \frac{1}{\sqrt{x}} e^{-x} dx$$

$x = u^2$

$$= \int_0^\infty e^{-u^2} du \rightarrow N(0, \tfrac{1}{2})$$

$$= \sqrt{2\pi} \cdot \frac{\sqrt{2}}{2}$$

$$= \sqrt{\pi}$$

a) $Gamma(1, \lambda) \Rightarrow exp(\lambda)$

b) $Gamma(\frac{n}{2}, \frac{1}{2}) \Rightarrow \chi^2(n)$

② Beta distribution

$$Beta(a, b) \longrightarrow f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1} \qquad \mathcal{X} = [0,1]$$

$$\mathbb{E}[X] = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)}$$

$$= \frac{a}{a+b}$$

$$\mathbb{E}[X^2] = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(a+2)\cdot\Gamma(b)}{\Gamma(a+b+2)}$$

$$= \frac{a\cdot(a+1)}{(a+b)(a+b+1)}$$

a) Beta(1,1) $\Rightarrow$ Uniform(0,1)

b) $\begin{cases} a \longleftrightarrow 1 \\ b \longleftrightarrow 1 \end{cases} \Rightarrow$ 不同单调性 $f(x)$

③ Cauchy dist

$$\boxed{\frac{N(0,1)}{N(0,1)} \rightsquigarrow \text{Cauchy} \to \boxed{\text{ill-posed}}}$$

!! critical problem

④ Double exponential distribution

$$\boxed{f(x) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}}$$

$E[X] = \mu$

$Var[X] = 2\sigma^2$

Location- Scale family

$\begin{cases} \mu \to \text{location parameter} \\ \sigma \to \text{scale parameter} \end{cases}$

① if $f(x)$ is a p.d.f.

then $\frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$ is also a valid pdf

② $Z \sim f(x)$, $X \sim \frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$

$\Leftrightarrow$ $X = \sigma Z + \mu$

use Z to compute $\boxed{X}$ → have some 'general property' like $N(\mu, \sigma^2)$

convenience

have some 'individual property' like $N(0,1)$

$\boxed{11/21}$ More details about Beta(a, b) distribution!

① Expression:

$X \sim \text{Beta}(a,b)$

$$\Rightarrow f_X(x) = \frac{1}{B(a,b)} \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

Here, $B(a,b)$ is the normalization term

$$\Leftrightarrow B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

② $\boxed{\text{Interpretation}}$

a) check the expectation & variance

$$\mathbb{E}[X] = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \quad \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)}$$

$$= \frac{a}{a+b}$$

$$\mathbb{E}[X^2] = \frac{a(a+1)}{(a+b)(a+b+1)}$$

$a^2(a+b) - a^2(a+b+1)$

$+ a^2 + ab$

$$Var[X] = \frac{a(a+1)}{(a+b)(a+b+1)} - \frac{a \cdot a}{(a+b)(a+b)}$$

$$= \frac{a(a+1)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)} = \frac{ab}{(a+b)^2(a+b+1)}$$

To conclude: $\begin{cases} \mathbb{E}[X] = \dfrac{a}{a+b} \\ Var[X] = \dfrac{ab}{(a+b)^2(a+b+1)} \end{cases}$ and Support Set of X is $[0,1]$

☆

b) $X \to$ view as the random probability

whose $\begin{cases} \dfrac{a}{a+b} \text{ can be modeled as Success Probability} \\ \text{if } a, b \text{ becomes bigger, then uncertainty shrinks!} \end{cases}$

(Variance becomes smaller)

[e.g.] Beta(2, 2) versus Beta(4, 4)

$\downarrow$

more skewed $\leftrightarrow$ $\boxed{\text{large variance}}$

③ Advantage : Conjugate Prior w.r.t many likelihood

$\downarrow$

Binomial / NegBin

[e.g.] $\begin{cases} X \mid \Lambda = \lambda \sim \text{Binomial}(n, \lambda) \\ \Lambda \sim \text{Beta}(a, b) \end{cases}$

$$P_{\Lambda|X}(\lambda \mid k) \propto P_{X|\Lambda}(k \mid \lambda) \cdot P_\Lambda(\lambda)$$

$$\propto \binom{n}{k} \lambda^k (1-\lambda)^{n-k} \cdot \lambda^{a-1} (1-\lambda)^{b-1}$$

$$\propto \lambda^{(a+k-1)} (1-\lambda)^{(n-k+b-1)}$$

$$\Rightarrow \Lambda \mid X = k \sim \text{Beta}(a+k,\ b+n-k)$$

④ Order-Statistic $\longleftrightarrow$ Beta Dist

a) $X_1, \dots, X_n \sim F$

$\longrightarrow$ what is the distribution of $X_{(k)}$?

$$P(X_{(k)} \in [x, x+\Delta x])$$

$$= \binom{n}{1\ \ k\ \ n-k-1} \cdot (F(x))^k \cdot (F(x+\Delta x) - F(x)) \cdot (1 - F(x+\Delta x))^{n-k-1}$$

$$= \frac{n!}{1 \cdot k! \cdot (n-k-1)!} F(x)^k (F(x+\Delta x) - F(x)) (1 - F(x+\Delta x))^{n-k+1}$$

$$= P_{X_{(k)}}(x) \Delta x + O(\Delta x)$$

$$\Rightarrow P_{X_{(k)}}(x) = \frac{n!}{k!(n-k-1)!} F(x)^k f(x) (1 - F(x))^{n-k-1}$$

b) when $X_1, \dots, X_n \sim \text{Uniform}[0,1]$

then $$P_{U_{(k)}}(x) = \frac{n!}{k!(n-k-1)!} x^{k+1-1} (1-x)^{n-k-1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k)} x^{k+1-1} (1-x)^{n-k-1}.$$

that is $U_{(k)} \sim \text{Beta}(k+1,\ n-k)$

$\rightarrow$ intuition about the Random Variable Beta

⑤ Lastly, determine $\underline{B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx}$

$\rightarrow$ $\boxed{\text{normalization constant}}$

$\boxed{\text{Technique}}$ $\rightarrow$ <u>integration by parts</u>

$\boxed{\text{Lemma 1}}$

$$\rightarrow B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

$$= -\int_0^1 x^{a-1}\,d\left(\frac{1}{b}(1-x)^b\right)$$

$$= -x^{a-1}\frac{1}{b}(1-x)^b\Big|_0^1 + \int_0^1 \frac{a-1}{b}x^{a-2}(1-x)^b\,dx$$

$$= \frac{a-1}{b}B(a-1, b+1)$$

$\boxed{\text{Lemma 2}}$

$$B(1,k) = \int_0^1 (1-x)^{k-1}\,dx$$

$$= -\frac{1}{k}(1-x)^k\Big|_0^1 = \frac{1}{k}$$

Therefore, $B(a,b) = \dfrac{a-1}{b}B(a-1, b+1)$

$$= \frac{a-1}{b}\cdots\frac{1}{a+b-2}B(1, a+b-1)$$

$$= \frac{(a-1)\cdots 1}{(a+b-1)\cdots b} = \frac{\Gamma(a)}{\frac{\Gamma(a+b)}{\Gamma(b)}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$