LEC3 DSA5203 $\longrightarrow$ Fourier Transformation

Lecture Summary

1. Inner Product Defined in $L^2(\mathbb{R}) := \{ f : \int_{\mathbb{R}} |f(t)|^2 \, dt < +\infty \}$

$$\langle f, g \rangle := \int_{\mathbb{R}} f(t) \overline{g(t)} \, dt$$

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2. Fourier Transform $\Longrightarrow$ Generalization for Fourier Series

$\boxed{f \in L^2(\mathbb{R})} \longrightarrow$ No need for periodic

Fourier Series $\downarrow$ work for T-period function

time domain

Then $\boxed{f(t)} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) \boxed{e^{i\omega t}} \, d\omega$ $\longrightarrow \{ e_\omega(t) = e^{i\omega t} : \omega \in \mathbb{R} \}$

BASIS Function

frequency domain

Here $\boxed{\hat{f}(\omega)} = \int_{\mathbb{R}} f(t) \cdot e^{-i\omega t} \, dt$

$= \langle f, e_\omega \rangle$

can be viewed as "Contribution" for ω-frequency wave

3. Convolution

for $f, g \in L^2(\mathbb{R})$, $\underbrace{(f \otimes g)(t) := \int_{\mathbb{R}} f(x) \overline{g(t-x)} \, dx}_{\boxed{\text{Convolution}}}$

$\boxed{\text{Thm}}$

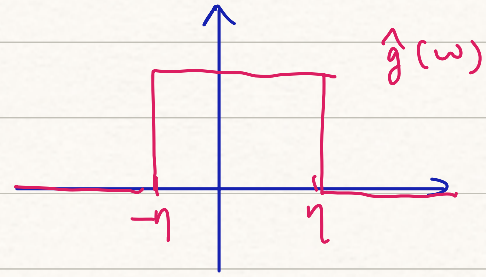
$$\widehat{f \otimes g}(\omega) = \hat{f}(\omega) \cdot \hat{g}(\omega)$$

Rmk: LHS $\rightarrow$ Design our interest in Frequency Domain

[e.g.], if we want low-pass filter $g(\cdot)$,

$\Rightarrow$ we just need:

$\hat{g}(\omega)$ behaves like:



$\Rightarrow$ Then, through "Fourier Transformation", we can recover $g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\omega) e^{i\omega t} d\omega$

$\Rightarrow$ Actually $g(t) = \frac{\sin \eta t}{\pi t} = \frac{\eta}{\pi} \cdot \frac{\sin \eta t}{\eta t}$

$= \frac{\eta}{\pi} \cdot \mathrm{sinc}(\eta t)$

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4. Discrete Fourier Transformation [DFT]

Idea: given $f \in \mathbb{C}^N$, i.e., $f = (f[1], \ldots, f[N])$

we want find $\boxed{f = \sum_{i=0}^{N-1} a_i e_i}$

wave

contribution

☆ Assume all samples constitute one whole period

DFT:

a) $f = \sum_{n=0}^{N-1} \langle f, e_n \rangle e_n := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{f}[n] \cdot e_n$

Here, $e_n[k] = \frac{1}{\sqrt{N}} e^{i \cdot 2\pi \cdot \frac{n}{N} \cdot k}$ $\quad n, k = 0, 1, 2, \ldots, N-1$

That is, $\underline{\{e_0, \ldots, e_{N-1}\}} \Rightarrow$ orthonormal basis for $\mathbb{C}^N$

b) $\hat{f}[n] = \sqrt{N} \langle f, e_n \rangle = \sum_{k=0}^{N-1} f[k] \cdot \overline{e_n[k]}$

$$= \sum_{k=0}^{N-1} f[k] \cdot e^{-i \cdot 2\pi \cdot \frac{n}{N} \cdot k}$$

Property:

① $\hat{f}[N-k] = \sum_{n=0}^{N-1} f(n) \cdot e^{-i \cdot 2\pi \cdot \frac{N-k}{N} \cdot n}$

$$= \sum_{n=0}^{N-1} f(n) \cdot e^{-i \cdot 2\pi \cdot \left(-\frac{k}{N}\right) \cdot n} \cdot e^{i \cdot (-2n\pi)}$$

$$= \sum_{n=0}^{N-1} f(n) \cdot e^{-i \cdot 2\pi \cdot \left(-\frac{k}{N}\right) \cdot n}$$

$$= \overline{\hat{f}[k]}$$

② $\hat{f}[0] = \sum_{n=0}^{N-1} f(n) \Rightarrow$ Real Number

when N is even, $\hat{f}[\frac{N}{2}]$ exists and,

$\hat{f}[\frac{N}{2}] = \hat{f}[N - \frac{N}{2}] = \overline{\hat{f}[\frac{N}{2}]} \Rightarrow$ Real Number

$$\Rightarrow \left\{ \begin{array}{l} \hat{f}[0] \\ \hat{f}[\frac{N}{2}] \end{array} \right. \longrightarrow \text{直流分量}$$

Further Exploration:

$\downarrow$

⊗ **Understanding**:

what does $\hat{f}[n]$ stands for?

suppose we sample N points (signal) within S seconds.

then sample frequency $\boxed{f_s = \frac{N}{S}}$

N samples : $f[0] \ldots f[N-1]$

First thing

$\longrightarrow$ Let's check : $f[k] = \frac{1}{N} \sum_{n=0}^{N-1} \hat{f}[n] \cdot e^{i \cdot 2\pi \cdot \frac{nk}{N}}$

$$\Leftrightarrow N \times f[k] = \hat{f}[0]\, e^{i \cdot 2\pi \cdot 0} + \hat{f}[1] \cdot e^{i \cdot 2\pi \cdot \frac{k}{N}}$$

(suppose N is even) $$+ \cdots + \hat{f}[N-1] \cdot e^{i \cdot 2\pi \cdot (N-1) \cdot \frac{k}{N}}$$

$$\left[\begin{aligned} e^{i\pi k} &= \cos(k\pi) + i\sin(k\pi) \\ &= \begin{cases} 1, & \text{if } k \text{ even} \\ -1, & \text{if } k \text{ odd} \end{cases} \end{aligned} \right]$$

$$= \underbrace{\hat{f}[0] + \hat{f}[\tfrac{N}{2}] \cdot e^{i\pi \cdot k}}_{\boxed{\text{real number}}} + \cdots\cdots$$

check ~~~ term :

$$\left.\begin{aligned} \text{suppose } \hat{f}[n] &= a_n + i \cdot b_n \\ \text{then } \hat{f}[N-n] &= a_n - i b_n \end{aligned}\right] \longrightarrow \text{property } ①$$

then $\hat{f}[n] \cdot e^{i \cdot 2\pi \cdot n \cdot \frac{k}{N}} + \hat{f}[N-n]\, e^{i 2\pi \cdot (N-n) \cdot \frac{k}{N}}$

$$= (a_n + i b_n)\left[\cos(2\pi n \tfrac{k}{N}) + i\sin(2\pi n \cdot \tfrac{k}{N})\right]$$

$$+ (a_n - i b_n)\left[\cos(2\pi n \cdot \tfrac{k}{N}) - i\sin(2\pi n \cdot \tfrac{k}{N})\right]$$

$$= 2a_n \cos\left(2\pi n \cdot \frac{k}{N}\right) - 2b_n \cdot \sin\left(2\pi n \cdot \frac{k}{N}\right) \in \underline{\underline{\mathbb{R}}}$$

$$= 2A_n \cos\left(2\pi \cdot \frac{nk}{N} + \phi_n\right)$$

where $\boxed{A_n = \sqrt{a_n^2 + b_n^2}\ , \quad \phi_n = \arctan\left(\frac{b_n}{a_n}\right)}$

Thus, $$N \times f[k] = \underbrace{\hat{f}[0] + \hat{f}[N/2] \cdot e^{i \cdot k\pi}}_{A_0 \in \mathbb{R}} + 2A_1 \cos\left(2\pi \cdot 1 \cdot \frac{k}{N} + \phi_1\right)$$

$$+ 2A_2 \cos\left(2\pi \cdot 2 \cdot \frac{k}{N} + \phi_2\right) + \cdots + 2A_{\frac{N}{2}-1} \cos\left(2\pi \cdot \left(\frac{N}{2}-1\right) \cdot \frac{k}{N} + \phi_{\frac{N}{2}-1}\right)$$

<u>Second thing</u>: $\leftarrow$ $\boxed{\text{Signal sampling}}$

<u>Recap</u>, we have $\underline{f[0], \ldots\ldots, f[N-1]}$

suppose we sample them within T seconds,

then <u>sample frequency</u> $f_s = \frac{N}{T}$

$\boxed{\text{Sample}}$	$f[0]$	$f[1]$	$\cdots\cdots$	$f[N-1]$
$\updownarrow$	$\downarrow$	$\downarrow$		$\downarrow$
$\boxed{\text{Original Signal}}$	$f(0)$	$f(1 \cdot \frac{T}{N})$, $\frac{T}{N} = \frac{1}{f_s}$		$f((N-1) \cdot \frac{T}{N})$, $\frac{T}{N} = \frac{1}{f_s}$

From <u>above discussion</u>, the conclusion is:

$$N \times f[k] = \underbrace{\hat{f}[0] + \hat{f}[N/2] \cdot e^{i \cdot k\pi}}_{2A_0 \in \mathbb{R}} + 2A_1 \cos\left(2\pi \cdot 1 \cdot \frac{k}{N} + \phi_1\right)$$

$$+ 2A_2 \cos\left(2\pi \cdot 2 \cdot \frac{k}{N} + \phi_2\right) + \cdots + 2A_{\frac{N}{2}-1} \cos\left(2\pi \cdot \left(\frac{N}{2}-1\right) \cdot \frac{k}{N} + \phi_{\frac{N}{2}-1}\right)$$

$$\Rightarrow N \times f\left(k \cdot \frac{1}{f_s}\right) = 2A_0 + 2A_1 \cos\left(2\pi \cdot 1 \cdot \frac{k}{N} + \phi_1\right)$$

$$+ \cdots + 2A_{\frac{N}{2}-1} \cos\left(2\pi\left(\frac{N}{2}-1\right)\frac{k}{N} + \phi_{\frac{N}{2}-1}\right)$$

$\boxed{k = f_s \cdot t}$

$$\Rightarrow N \times f(t) = 2A_0 + 2A_1 \cos\left(2\pi \cdot 1 \cdot \frac{f_s t}{N} + \phi_1\right)$$

$$+ \cdots + 2A_{\frac{N}{2}-1} \cos\left(2\pi\left(\frac{N}{2}-1\right)\frac{f_s t}{N} + \phi_{\frac{N}{2}-1}\right)$$

$$= 2A_0 + 2A_1 \cos\left(2\pi \cdot 1 \cdot \frac{t}{T} + \phi_1\right)$$

$$+ \cdots + 2A_{\frac{N}{2}-1} \cos\left(2\pi\left(\frac{N}{2}-1\right)\frac{t}{T} + \phi_{\frac{N}{2}-1}\right)$$

$\Rightarrow$ A_k, ϕ_k $\boxed{\text{measures}}$ the $\boxed{\text{property}}$ of $\frac{k}{T}$ - frequency wave

$\frac{k f_s}{N}$

Up to now, we can answer this question:

what does $\hat{f}[n]$ stands for ?

Answer: if we sample N points within T seconds,

& $\hat{f}[n] = a + bi$ $\quad n = 0, 1, \ldots, \frac{N}{2}-1, \frac{N}{2}$

then it explains the $\boxed{\frac{n}{T} - \text{frequency}}$ wave

(amplitude)

$\sqrt{a^2+b^2}$ explains the contribution of this wave

$\arctan\left(\frac{b}{a}\right)$ explains the phase (shift) of this wave

Convolution (Discrete) calculation

{ [1 3 4 6 8 20 11] signal
{ [1 0 -1] Filter

① circular

1 3 4 6 8 20 11

1 0 -1

→ 1 0 -1 0 0 0 0

Step ① (Location 0)

0 ... 0 -1 0 1

1 0 0 0 0 -1 0 ⇒ -19

Step ② (Location 1)

0 --- 0 -1 0 1

0 1 0 0 0 0 -1 ⇒ -8

Result: -19 -8 3 3 4 14 3

② most entries with 0-padding

1 3 4 6 8 20 11

1 0 -1

reverse → 0 0 $\boxed{1\ 3\ 4\ 6\ 8\ 20\ 11}$ 0 0

$\boxed{-1\ 0\ 1}$ Location 0 = 1

$\boxed{-1\ 0\ 1}$ Location 1 = 3

$\boxed{-1\ 0\ 1}$ Location 2 = 3

$\boxed{-1\ 0\ 1}$ Location 3 = 3

slide

same size

Result 1 $\boxed{3\ 3\ 3\ 4\ 14\ 3\ -20}$ -11