5232 part 4 lecture3 Optimal Control

↳ Dynamic Prog. ⇔ $\boxed{\text{HJB equation}}$

$\boxed{\text{Recall}}$ (PMP)

① $\boxed{\text{Bolza Problem}}$

$$\begin{cases} \inf\limits_{\underline{\theta}} J[\underline{\theta}] := \int_0^T L(t, x, \theta)\, dt + \Phi(x(T)) \\ \text{s.t.} \begin{cases} \dot{x}(t) = f(t, x(t), \theta(t)) & t \in [0,T] \\ x(0) = x_0 \end{cases} \end{cases}$$

→ $\boxed{\text{necessary condition}}$ for $\boxed{\text{strong minimizer}}$ $\underline{\theta}^* \in \mathcal{U}$ (PMP)

↓

$\boxed{\text{Punchline: Needle Pertubation}}$

② DP perspective → give a necessary & sufficient condition!

↓

$\boxed{\text{HJB equation}}$

$\boxed{\text{DP}}$ → Punchline ⇒ aim: solve $V(0, x)$

$\boxed{\text{Solution}}$ → extend to $V(t,x)$! / develop recursive method for $V(t,x)$!

↓

$\boxed{\text{Bolza Prob}}$!

Lecture 3

→ value function for $\boxed{\text{Bolza Problem}}$

$$\inf_{\theta \in L^\infty([0,T],\Theta)} J[\theta] = \int_0^T L(t, x(t), \theta(t))\,dt + \Phi(x(T))$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = f(t, x(t), \theta(t)) \\ x(0) = x_0 \end{cases} \quad t \in [0,T]$$

→ Define value function $V: [0,T] \times \mathbb{R}^d \to \mathbb{R}$

$$V(s,z) := \inf_{\theta \in L^\infty([s,T],\Theta)} J_s[\theta] := \int_s^T L\,dt + \Phi(x(T))$$

✱ $\begin{cases} s \to \text{initial time} \\ z \to \text{initial state} \end{cases}$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = f(t, x(t), \theta(t)) \\ x(s) = z \end{cases} \quad t \in [s, T]$$

$\boxed{V(s,z) \to \text{a BIG class of problems!}}$

Note (Rmk): In particular, $V(0, x_0)$ is the optimal cost of the original Bolza Problem!

→ **DP Principle** (DPP)

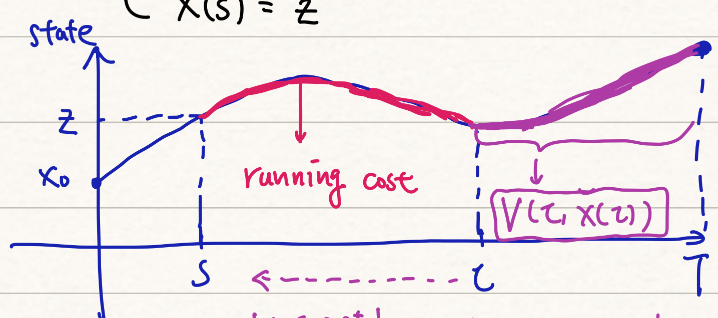
Theorem: for every $\tau, s \in [0,T]$, & $(s \le \tau)$, $z \in \mathbb{R}^d$, we have :

$$V(s,z) = \inf_{\theta} \int_s^\tau L(t,x,\theta)\,dt \quad \int_\tau^T L(t,x,\theta)\,dt + \Phi(X(T))$$

$$V(s,z) := \inf_{\theta \in L^\infty([s,\tau],\theta)} \left\{ \int_s^\tau L(t, x(t), \theta(t))\,dt + V(\tau, x(\tau)) \right\}$$

$$\text{s.t} \quad \begin{cases} \dot{x}(t) = f(t, x(t), \theta(t)) \\ x(s) = z \end{cases} \quad t \in [s, \tau]$$

Explanation



let $\tau \to s^+$! ⇒ achieve infinitesimal equation of $V(s,z)$!

Proof : Define $J^{\tau} := \inf_{\theta \in L^{\infty}([s,\tau],\Theta)} \int_s^{\tau} L(x,t,\theta)\,dt + V(\tau, x(\tau))$

we need to show $\begin{cases} J^{\tau} \leq V(s,z) \\ J^{\tau} \geq V(s,z) \end{cases}$

① $J^{\tau} \leq V(s,z)$

Fix $\varepsilon > 0$, pick $\theta^{*}: [s,T] \to \Theta$ s.t

$$J_s[\theta^{*}] \leq V(s,z) + \varepsilon$$

infimum (starting from time s state z)

under this control θ^{*}, we have:

$$V(\tau, x(\tau)) \leq \int_{\tau}^{T} L(t,x,\theta^{*})\,dt + \Phi(x(T))$$

infimum (starting from time τ state $x(\tau)$)

Then,
$$\begin{aligned} J^{\tau} &\leq \int_s^{\tau} L(t, x(t), \theta^{*}(t))\,dt + V(\tau, x(\tau)) \\ &\leq \int_s^{T} L(t,x,\theta^{*})\,dt + \Phi(x(T)) \\ &= J_s[\theta^{*}] \\ &\leq V(s,z) + \varepsilon \qquad ! \qquad \varepsilon \to 0 \end{aligned}$$

$$\Rightarrow J^{\tau} \leq V(s,z)$$

② $J^{\tau} \geq V(s,z)$

Fix $\varepsilon > 0$, there exists $\theta^{*}: [s,\tau] \to \Theta$ s.t

$$\int_s^{\tau} L(t,x,\theta^{*})\,dt + V(\tau, x(\tau)) \leq \boxed{J^{\tau}} + \varepsilon$$

inf!

& there exists $\theta^{**}: [\tau, T] \to \Theta$ st

$$\int_{\tau}^{T} L(t, x(t), \theta^{**}(t))\,dt + \Phi(x(T)) \leq \boxed{V(\tau, x(\tau))} + \varepsilon$$

inf

$\Rightarrow$ construct combined control (common technique)

$$\hat{\theta} : [s, T] \longrightarrow \Theta \quad \text{s.t}$$

$$\hat{\theta}(t) = \begin{cases} \theta^*(t), & t \in [s, \tau] \\ \theta^{**}(t), & t \in [\tau, T] \end{cases}$$

Therefore, $\rightarrow$ consider $\hat{\theta}$

$$\underbrace{\int_s^T L(t, x(t), \hat{\theta}(t))\,dt + \Phi(x(T))}_{J[\hat{\theta}]} \leqslant J^\tau + 2\varepsilon$$

Thus, $V(s, z) \leq J^\tau \quad (\varepsilon \to 0)$

$$\Rightarrow \boxed{V(s, z) = J^\tau}$$

HJB Equation: (Hamilton-Jacobi-Bellman)

Recap DPP:

$$V(s, z) = \inf_{\theta} \left\{ \int_s^\tau L(t, x(t), \theta(t))\,dt + V(\tau, x(\tau)) \right\}$$

$$\text{s.t} \quad \begin{cases} \dot{x}(t) = f(t, x(t), \theta(t))\,dt & t \in [s, \tau] \\ x(s) = z \end{cases}$$

$\rightarrow$ trick: set $\tau = s + \Delta s$ $\quad \Delta s \to 0$! $\longrightarrow$ $\boxed{\text{infintesimal behavior}}$

$\Downarrow$

$$\rightarrow V(s, z) = \inf_{\theta} \left\{ \int_s^{s+\Delta s} L(t, x, \theta)\,dt + V(s+\Delta s, x(s+\Delta s)) \right\}$$

$\Downarrow$

$\rightarrow$ perform Taylor Expansion around s $(\Delta s \to 0)$

① $X(s+\Delta s) = X(s) + \int_s^{s+\Delta s} \dot{X}(t)\,dt$

$$= X(s) + \int_s^{s+\Delta s} f(t, X(t), \theta(t))\,dt$$

$$= X(s) + \Delta s \cdot f(s, X(s), \theta(s)) + o(\Delta s)$$

$$= x(s) + \Delta s\, f(s, X(s), \theta(s)) + o(\Delta s)$$ (initial cond)

② $V(s+\Delta s, X(s+\Delta s)) \longleftrightarrow \boxed{\text{Chain Rule}}$

$$= V(s, X(s)) + \partial_s V(s, X(s)) \cdot \Delta s$$
$$+ [\nabla_x V(s, X(s))]^T f(s, X(s), \theta(s))\,\Delta s + o(\Delta s)$$

$\boxed{\text{Chain rule:}}$ $\frac{\partial V}{\partial s} = \frac{\partial V}{\partial s} + \frac{\partial V}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial s}}_{f(s, X(s), \theta(s)) + o(1)}$

& $\int_s^{s+\Delta s} L(t, x, \theta)\,dt = \Delta s \cdot L(s, X(s), \theta(s)) + o(\Delta s)$

Therefore, DPP becomes:

$$\cancel{V(s,z)} = \inf_{\theta} \Big\{ \cancel{V(s,z)} + \underbrace{\Delta s \big[\partial_s V(s,z) + [\nabla_x V(s,z)]^T f(s, z, \theta(s)) + L(s, z, \theta(s)) \big] + o(\Delta s)}_{\text{this part only depends on } \theta(s)} \Big\}$$

☆☆☆ $\xrightarrow{\Delta s \to 0}$ $\inf_{\theta(s) \in \Theta} \{ \partial_s V(s,z) + \nabla_x V(s,z)^T f(s, z, \theta(s)) + L(s, z, \theta(s)) \}$

$\boxed{\text{PDE} \to \text{very complicated}}$

Hamilton
Jacobi
Bellman
Equation
(HJB equation)

$$\begin{cases} \partial_s V(s,z) + \inf_{\theta(s) \in \Theta} \{ \nabla_x V(s,z)^T f(s, z, \theta(s)) + L(s, z, \theta(s)) \} = 0 \\ z \in \mathbb{R}^d, \quad s \in [0, T] \\ V(T, z) = \Phi(z) \end{cases}$$

$\boxed{\text{Backward recursion}}$

$V(T,z) = \Phi(z)$ → at end, Best we can do!

⇒ $\boxed{\text{Rmk:}}$ Taylor Expansion requires value function to be sufficiently regular! ⇒ our derivation is not purely formal!

$\boxed{\text{Implication}}$: (of HJB Equation)

① → First, by defn of value function

$$\boxed{V(0, x_0) = \inf_{\theta \in L^\infty([0,T], \Theta)} J[\theta]} \to \boxed{\text{optimal cost of Bolza Problem}}$$

② $\boxed{\text{What's More?}}$

→ Fix some $s \in [0,T]$, $z \in \mathbb{R}^d$

① assume there exists an optimal control $\theta^* : [s,T] \to \Theta$

From DDP, we know that, for any $\tau \geq s$

$$V(s,z) = \inf_{\theta} \left\{ \int_s^\tau L(t, x, \theta) dt + V(\tau, x(\tau)) \right\}$$

$\Downarrow$ attained at $\theta^* \in L^\infty([s,T], \Theta)$

$$\int_s^\tau L(t, x^*(t), \theta^*(t)) dt + V(\tau, x^*(\tau))$$

As before, set $\tau = s + \Delta s$ and let $\Delta s \to 0$,

then we derive:

$$-\partial_s V(s, x^*(s)) = \inf_{\theta \in \Theta} \left\{ L(s, x^*(s), \theta) + \nabla_x V(s, x^*(s))^T f(s, x^*(s), \theta) \right\}$$

(**)

$$\overset{!}{=} L(s, x^*(s), \theta^*(s)) + \nabla_x V(s, x^*(s))^T f(s, x^*(s), \theta^*(s))$$

$$\Rightarrow [-\nabla_x V(s, x^*(s))]^T f(s, x^*(s), \theta^*(s)) - L(s, x^*(s), \theta^*(s))$$

$$\geq [-\nabla_x V(s, x^*(s))]^T f(s, x(s), \theta) - L(s, x^*(s), \theta)$$

for all $\theta \in \Theta$ and all $s \in [0,T]$

If we set $p^*(s) = -\nabla_x V(s, x^*(s))$, this becomes:

$$H(s, x^*(s), p^*(s), \theta^*(s)) \geq H(s, x^*(s), p^*(s), \theta)$$

for all $\theta \in \Theta$ and all $s \in [0, T]$

(necessary condition) $\leftarrow$ induced by HJB Equ (similar to PMP)

$\Rightarrow$ In other words, if $\underline{\theta^*}$ is optimal control,

then $\theta^*(s) \in \underset{\theta \in \Theta}{\arg\max} \left\{ \underbrace{[-\nabla_x V(s, x^*(s))^T f(s, x^*(s), \theta) - L(s, x^*(s), \theta)]}_{\text{Hamiltonian}} \right\}$

(*)

Hamiltonian where

$p^*(s) = -\nabla_x V(s, x^*(s))^T$

$\Rightarrow$ p^* is something fixed

not every p^* (satisfy PMP)

缩小3 candidate

② $\boxed{\text{Sufficient Part}}$

consider $\boxed{\hat{\theta} : [0, T] \to \Theta \text{ satisfy } (*) \text{ and } (**)}$

we need to show it is the optimal control! $J[\hat{\theta}] = V(0, \hat{x}(0))$

$\hat{x} \to$ the corresponding trajectory

(**)

$\boxed{\text{Pf}}$: $\partial_t V(t, \hat{x}(t)) + [\nabla_x V(t, \hat{x}(t))]^T f(t, \hat{x}(t), \hat{\theta}(t)) + L(t, \hat{x}(t), \hat{\theta}(t)) = 0$

$\frac{d}{dt} V(t, \hat{x}(t))$

holds for all t

$$\Rightarrow \frac{d}{dt} V(t, \hat{x}(t)) + L(t, \hat{x}(t), \hat{\theta}(t)) = 0$$

$$\frac{d}{dt} V(t, \hat{x}(t)) = -L(t, \hat{x}(t), \hat{\theta}(t))$$

integration $\Rightarrow$

$$\underbrace{V(T, \hat{x}(T))}_{\Phi(\hat{x}(T))} - \underbrace{V(0, \hat{x}(0))}_{V(0, x_0)} = \underbrace{-\int_0^T L(t, \hat{x}(t), \hat{\theta}(t))\, dt}_{-\text{running cost}} \quad (\triangle)$$

$$= \inf_{\theta} J[\theta]$$

definition

$$\Rightarrow \boxed{J[\hat{\theta}] = \Phi(\hat{x}(T)) + \int_0^T L(t, \hat{x}(t), \hat{\theta}(t))\, dt}$$

$$= \inf_{\theta} J[\theta]$$

But $\hat{\theta}^*$

$\boxed{\frac{\partial V}{\partial t}} \rightarrow$ use to determine $\hat{\theta}^*$

DPP

To summarize:

$\boxed{\text{HJB}} \xrightarrow[\text{solve}]{\text{(PDE)}} \boxed{V}$ (value function)

Then $\hookrightarrow$ generate optimal control θ^*

necessary

① any optimal control θ^* must satisfy (necessary condition)

$$\theta^*(t) \in \arg\max_{\theta \in \Theta} \{ [-\nabla_x V(t, x^*(t))]^T f(t, x^*(t), \theta) - L(t, x^*(t), \theta) \}$$

sufficient

② any θ^* that satisfy is optimal !

(sufficient condition)

Remarks on HJB

- HJB equation is of the form

$$\partial_t V(t,x) + \underbrace{\inf_{\theta \in \Theta} \{ -H(t, x, -\nabla_x V(t,x)), \theta \}}_{} = 0$$

define

$\mathcal{H}(t, x, \nabla_x V(t,x)) \rightarrow$ effective Hamiltonian

shrink the scope of $p^*(\cdot)$

$$\Rightarrow \partial_t V(t,x) + \mathcal{H}(t, x, \nabla_x V(t,x)) = 0 \quad \text{(Hamilton-Jacobi Equ)}$$

$\hookrightarrow$ effective Hamiltonian

- Remark $\rightarrow$ derivation of HJB equation

$\boxed{\text{use Taylor Expansion}}$ → need $V(\cdot,\cdot)$ be regular

- In general, no smooth solutions to the HJB — seems $\boxed{\text{荒唐的}}$!!!

⇒ we need the concept of viscosity solutions ← $\boxed{\text{多亏他们}}$

$\boxed{\text{to explain}}$ (Crandall and Lions)

- Connection between HJB & PMP

$$\boxed{\text{HJB} \xrightarrow[\text{method of characteristic}]{} \text{PMP}}$$

$\boxed{\text{Example part:}}$ (Driving car dynamics)

$$\dot{x}(t) = \theta(t)\, x(t) \qquad t \in [0,T] \qquad x(0) = x_0 \in \mathbb{R}$$

$$\theta(t) \in \Theta = [-1, 1]$$

Consider $\boxed{\text{Bolza Problem}}$

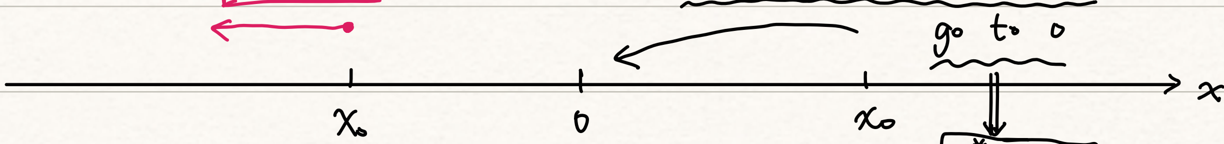
$$\inf_{\theta} \; x(T) \quad \longleftarrow \quad (\Phi(x) = x \qquad L(t,x,\theta) \equiv 0)$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = \theta(t)\, x(t) & t \in [0,T] \\ x(0) = x_0 \end{cases}$$

$\boxed{\text{case 1}}$ if $x_0 > 0$

best we can do is to go to 0

$\boxed{\theta^*(t) \equiv 1}$



$\boxed{\theta^*(t) \equiv -1}$

$\boxed{\text{case 2}}$: if $x_0 < 0$

By observation, an optimal control is

$$\theta^*(t) = \begin{cases} -\operatorname{sign}(x_0) & x_0 \neq 0 \\ \text{arbitrary} & x_0 = 0 \end{cases}$$

Question: what is the value-function $V(t,x)$?

① $x=0 \longrightarrow V(t,x) \equiv 0$

② $x>0 \longrightarrow \begin{cases} \dot{x}^*(s) = -x^*(s) & s \in [t,T] \\ x^*(t) = x \end{cases}$

$$\Rightarrow x^*(s) = e^{-(s-t)} x$$

$$\Rightarrow V(t,x) = \Phi(X^*(T)) = X^*(T) = e^{-(T-t)} x$$

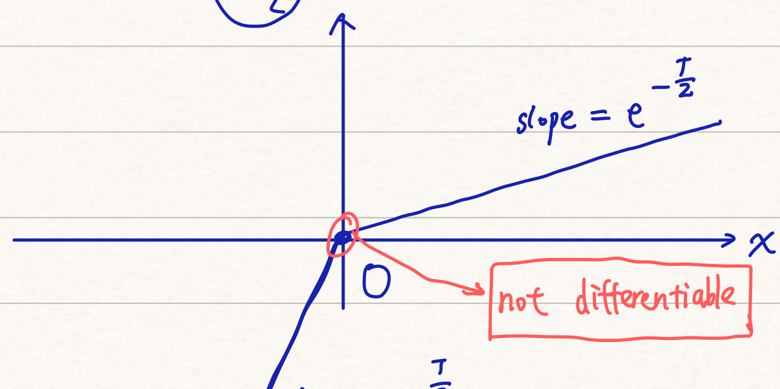
③ $x<0 \longrightarrow \begin{cases} \dot{x}^*(s) = x^*(s) & s \in [t,T] \\ x^*(t) = x \end{cases}$

$$\Rightarrow x^*(s) = e^{s-t} \cdot x$$

$$\Rightarrow V(t,x) = \Phi(X^*(T)) = X^*(T) = e^{T-t} x$$

To conclude, $V(t,x) = \begin{cases} e^{-(T-t)} x & , x>0 \\ 0 & , x=0 \\ e^{(T-t)} x & , x<0 \end{cases}$

$t=\frac{T}{2}$



slope $= e^2$

check: HJB Equation:

$$\partial_t V(t,x) + \inf_{\theta\in\Theta} \{ \partial_x V(t,x)^* \underbrace{f(t,x,\theta)}_{\theta x} + \underbrace{L(t,x,\theta)}_{\to 0} \} = 0$$

$$\Rightarrow \begin{cases} \text{if } x\,\partial_x V(t,x) > 0 \Rightarrow \theta^* = -1 \\ \text{if } x\,\partial_x V(t,x) \leq 0 \Rightarrow \theta^* = +1 \end{cases}$$

$$\Rightarrow \begin{cases} \partial_t V(t,x) = |\partial_x V(t,x)\cdot x| \\ V(T,x) = x \end{cases}$$

$\boxed{\text{Stochastic Version}}$ → Non-determinstic system

1. Stochastic DE

→ ODE: $\begin{cases} \dot{x}(t) = f(t, X(t)) \\ X(0) = x_0 \end{cases} \Rightarrow dX(t) = f(t, X(t))\, dt$

→ Stochastic DE: (SDE)

$$\begin{cases} dX(t) = \underbrace{f(t, X(t))\,dt}_{\boxed{\text{drift}}} + \underbrace{\sigma(t, X(t))\, dW_t}_{\text{diffusion / volatility}} \\ X(0) = x_0 \end{cases}$$

where $\begin{cases} f: [0,T]\times\mathbb{R}^d \to \mathbb{R}^d \\ \sigma: [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times p} \end{cases}$

$\{W(t)\}_{t\geq 0}$ → p-dim Wienner Process

Γ Heuristic Intepretation:

For small time interval Δt,

$$X(t+\Delta t) - X(t) \approx f(t, X(t))\,\Delta t + \overbrace{\sigma(t, X(t))\cdot\sqrt{\Delta t}\cdot Z(t)}^{\boxed{\text{noise term}}}$$

$\boxed{\text{diffusion matrix } \mathbb{R}^{d\times p}}$

$\boxed{\text{Std Normal}}$

$\mathcal{N}(0, I)$

Bolza Problem for SDE → Randomness

$$\inf_{\underline{\theta}\in\mathcal{A}_{0,T}} J[\underline{\theta}] = \mathbb{E}\left[\int_0^T L(t, X(t), \theta(t))\,dt + \Phi(X(T))\right]$$

$$\text{s.t.} \begin{cases} dX(t) = f(t, X(t), \theta(t))\,dt + \sigma(t, X(t), \theta(t))\,dW_t \\ X(0) = x_0 \qquad\qquad t\in[0,T] \end{cases}$$

$\mathcal{A}_{0,T}$ → $\boxed{\text{admissable set}}$ → If $\underline{\theta}\in\mathcal{A}_{0,T}$, then we need to satisfy

$\boxed{\text{technique condition}}$

- $\theta(t)\in\Theta$ $\forall t$ a.s.
- $\boxed{\underline{\theta} := \{\theta(t) : t\in[0,T]\}}$ is $\{W(t): t\in[0,T]\}$ adapted

$\boxed{\text{Random Control}}$

Control cannot look into the future

- Often, need a moment condition, e.g.

$$\mathbb{E}\left[\int_0^T \|\theta(t)\|^2\,dt\right] < \infty$$

HJB Equation for $\boxed{\text{SDE}}$: (Stochastic Control)

→ Define Value Function:

$$V(s,z) = \inf_{\underline{\theta}\in\mathcal{A}_{s,T}} J[\underline{\theta}] \quad \text{s.t.} \begin{cases} dX(t) = f\,dt + \sigma\,dW_t \\ X(s) = z \end{cases} \quad t\in[s,T]$$

Then, value function $\{V(s,z)\}_{s,z}$ is the unique sol$^{\text{n}}$ to the following HJB equ. :

$$\begin{cases} \partial_s V(s,z) + \inf\limits_{\theta \in \Theta} \Big\{ \nabla_x V(s,z)^T f(s,z,\theta) + L(s,z,\theta) \\ \qquad + \frac{1}{2} \mathrm{Tr}\left[\sigma(s,z,\theta)^T \nabla_x^2 V(s,z)\, \sigma(s,z,\theta) \right] \Big\} = 0 \\ V(T,z) = \Phi(z) \end{cases}$$

Numerical Algorithms : (for Optimal Control)

Roughly divide into 3 classes :

→ Aim for optimality condition (necessary)

① PMP - based → (ODE) ↱ high - dim system

② HJB - based → (PDE)

} → $\boxed{\text{indirect method}}$

③ optimization - based (Non-linear Programming) → $\boxed{\text{direct method}}$

→ aim for discretize the optimal control problem & solve

① Method of Succesive Approximation / Sweeping Method

Recall PMP equations

$$\begin{cases} ① \quad \dot{x}^*(t) = \nabla_p H(t, x^*, p^*, \theta^*) & x^*(0) = x_0 \\ ② \quad \dot{p}^*(t) = -\nabla_x H(t, x^*, p^*, \theta^*) & p^*(T) = -\nabla \Phi(x^*(T)) \\ ③ \quad \theta^*(t) \in \operatorname{argmax} H(t, x^*, p^*, \theta) & \text{for a.s } t \in [0,T] \end{cases}$$

$\theta \in \Theta$

$$H(t, x, p, \theta) = p^T f(t, x, \theta) - L(t, x, \theta)$$

Notice that :

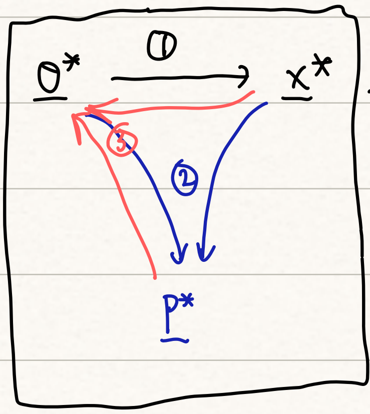
ODE

- If we know θ^*, then we can know x^* via ①
- If we know θ^*, x^*, then we can compute p^* via ②

over ① optimization ←

- If we know x^*, p^*, then we can compute θ^* via ③

↳ point-wise for t

⇒ 

→ Algorithm →

MSA

↓

name

- initialize $\theta \in L^\infty([0,T], \Theta)$
- Stopping Criteria not reached
  - $x \leftarrow \text{ODE}(\theta)$ ①
  - $p \leftarrow \text{ODE}(\theta, x)$ ②
  - $\theta \leftarrow \text{optimization}(x, p)$ ③

    ↓

    new control
- return θ

If the MSA algorithm converges, then θ is a solution to PMP !

↳ However, PMP is only a necessary condition

↳ still need to check if it is optimal

Rmk: we must have a good initialization θ !

(for Basic MSA)

↳ (Generally speaking) the iteration is unstable!

↳ fail to converge for

arbitrary initialization!

→ Extension of the MSA

$$\boxed{\theta^*(t) \in \underset{\theta \in \Theta}{\operatorname{argmax}} \; H(t, x^*(t), p^*(t), \theta)}$$

↳ idea: don't want θ change too much!

↳ stability

① • replace the maximization by steepest ascent

$$\theta(t) \leftarrow \underset{\theta \in \Theta}{\operatorname{argmax}} \; H(t, x(t), p(t), \theta)$$

(we need Θ to be regular)

$$\theta(t) \leftarrow \theta(t) + \eta \nabla_\theta H(t, x(t), p(t), \theta(t))$$

⇒ $\boxed{\text{Convergence guarantee}}$ when $\eta \ll 1$

② • consider a regularized problem

→ replace the maximization by regularized problem

$$\theta(t) \leftarrow \underset{\theta \in \Theta}{\operatorname{argmax}} \; H(t, x(t), p(t), \theta) - \lambda \| \theta - \theta(t) \|^2$$

force new θ to be close to $\theta(t)$

⇒ convergence guarantee when $\lambda \gg 1$

Two-Point Boundary Value Problem (2PBVP)

→ $\boxed{\text{starting point:}}$ for ③ → $\theta^*(t) \in \underset{\theta \in Ⓤ}{\operatorname{argmax}} H(t, x^*(t), p^*(t), \theta)$

punchline

if we can attain $\theta^*(t)$ is
some function of (x^*, p^*)
then back to solve for $\begin{cases} ① \\ ② \end{cases}$

Assume $\exists$ a function $\xi : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow Ⓤ$

s.t

$$H(t, x, p, \xi(t, x, p)) = \max_{\theta \in Ⓤ} H(t, x, p, \theta) \quad \forall t, x, p$$

$$\Rightarrow \xi(t, x, p) \in \underset{\theta \in Ⓤ}{\operatorname{argmax}} H(t, x, p, \theta)$$

⟶ then the PMP equations become:

$$\begin{cases} \dot{x}^*(t) = f(t, x^*(t), \xi(t, x^*(t), p^*(t))) & x^*(0) = x_0 \\ \dot{p}^*(t) = -\nabla_x H(t, x^*(t), p^*(t), \xi(t, x^*(t), p^*(t))) & \\ & p^*(T) = -\nabla_x \Phi(x^*(T)) \end{cases}$$

Two-Points Boundary Value Problem (2PBVP)

↳ $\boxed{\text{shooting method}}$ (for 2PBVP)

For $a \in \mathbb{R}^d$, define $(\underline{x}^a, \underline{p}^a)$ as the sol^n of the initial value problem:

$$\begin{cases} \dot{x}^a(t) = f(t, x^a(t), \xi(t, x^a(t), p^a(t))) & x^a(0) = x_0 \\ \dot{p}^a(t) = -\nabla_x H(t, x^a(t), p^a(t), \xi(t, x(t), p(t))) & \end{cases}$$

$\boxed{p^a(0) = a}$ $\Downarrow$ $\boxed{\text{SHOOTING}}$!!!

want to find a^* s.t.

$$p^{a^*}(T) = -\nabla \Phi(X^{a^*}(T))$$

$a^* \longleftarrow$ Root Finding $\{ p^a(T) + \nabla \Phi(X^a(T)) = 0 \}$

$\boxed{\text{Newton Method}}$

$$\Rightarrow \boxed{\theta^*(t) = \xi(t, x^{a^*}(t), p^{a^*}(t))} \longrightarrow \boxed{\text{optimal control}}$$