Convex Opt.

inequality ← equality
$\Leftrightarrow \begin{cases} g(x) \le 0 \\ -g(x) \le 0 \end{cases}$

Real Life Prob $\xrightarrow{\text{Formulation}}$ $\begin{cases} \min_{x \in \mathbb{R}^d} & f_0(x) \\ \text{s.t.} & f_i(x) \le 0 \end{cases}$ $i = 1, 2, \dots, n$

feasible space

$\rightarrow \begin{cases} f_0: & \text{objective function} \\ f_i: & \text{constraints function} \end{cases}$

Question

not just model with no aim!

What we should do ① How to model a PROBLEM (Scientific | Engineer) as an Optimization Prob?

Remember

↳ what kinds of problems can be solved

② What constitutes a good model?

③ How to solve?

④ Tractable?

use good models as guideline our modelling

↳ then model our question with compromise!

Question 2:

capture the needs of problems

Good Models: * Fidelity to the problem

(Trade-off) * Tractable / Efficiently

Model cannot be too complicated ⇓ tractability

But must capture as much as possible ⇓ Fidelity

{Convex Prog} ⊆ {Optimization Problem}

② have good theroetical guarantee

① Many problems can be translated into convex prog.

③ have solver

① convex set

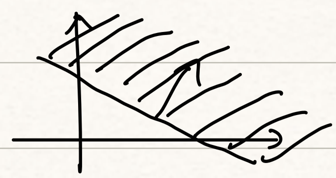
<u>Defn</u> : $S \in \mathbb{R}^d$ convex

$\Leftrightarrow$ if $x, y \in S$, then $\theta x + (1-\theta) y \in S$ $\theta \in [0,1]$

$\boxed{\text{Example}}$:

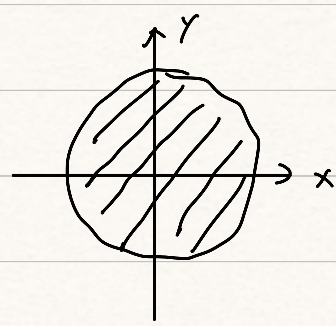
① singleton (trivial

② half-spaces $\{\underline{x}: \underline{a}^T \underline{x} \le b\}$



③ Affine subspace $\{\underline{x}: A\underline{x} = \underline{b}\}$

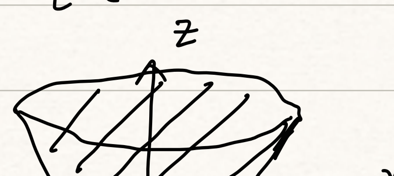
④ Euclidean - Ball $\{\underline{x}: \|\underline{x}\|_2 \le 1\}$



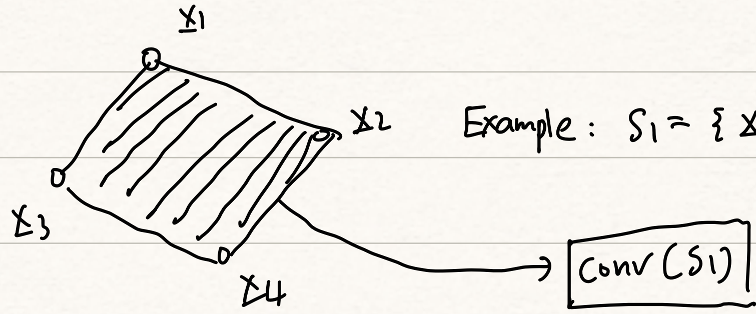
⑤ polyhedra : $\{\underline{x}: A\underline{x} \le \underline{b}\}$

④ Second Order Cone :

$\{(\underline{x}, t) \in \mathbb{R}^{d+1} : \|\underline{x}\|_2 \le t\}$



⑤ Convex Hull of S



Example : $S_1 = \{ \underline{x}_1, \dots \underline{x}_4 \}$

Defn : $S = \{ \underline{a}_i \}_{i=1}^{m}$

$$\text{conv}(S) = \left\{ \underline{x} = \sum_{i=1}^{m} \theta_i \underline{a}_i \ : \ 0 \le \theta_i \le 1 ,\ \sum_{i=1}^{m} \theta_i = 1 \right\}$$

⑥ set of all PSD matrix

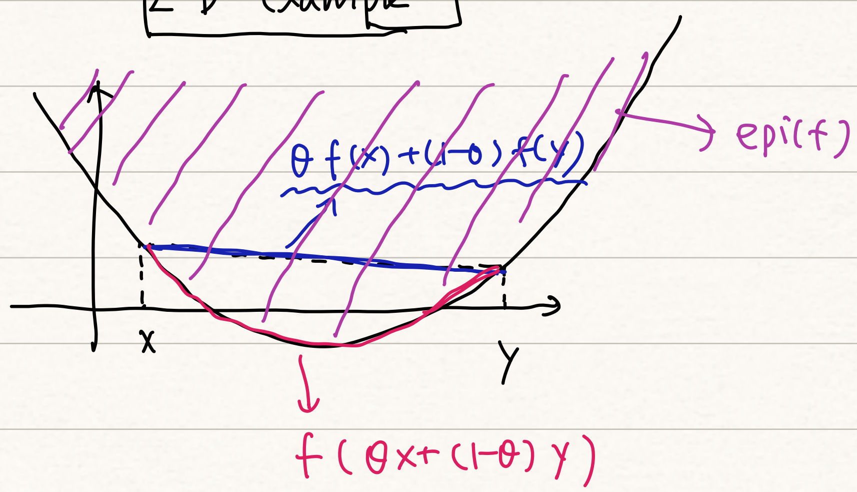
$$:= \{ X : \underline{v}^T X \underline{v} \geq 0 ,\ \forall \underline{v} \}$$

Convex function

Defn : f is convex if

$$f(\theta \underline{x} + (1-\theta) \underline{y}) \le \theta f(\underline{x}) + (1-\theta) f(\underline{y}) \qquad \forall \theta \in [0,1]$$

$\boxed{\text{2-D example}}$



Defn' (epigraph) $\rightarrow$ $\text{epi}(f) = \{(x,t) : f(x) \leq t\}$

$\Downarrow$

f is convex function $\Leftrightarrow$ $\text{epi}(f)$ is convex (set)

Example of convex $f^{=}$

① $\exp(x)$ ② $-\log x$ ③ $\|x\|_2$ ④ Entropy $x\log(x)$

⑤ $\max\{x_1, \ldots, x_n\}$ ⑥ $f(\underline{x}) = \log(\Sigma \exp(x_i))$ ⑦ $\log(\det X)$

$X \in PSD$

$\boxed{\text{Check Way}}$

if $f \in C^2$, then f convex $\Leftrightarrow$ $f^{(2)} > 0$ every where

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Convex Program

$\downarrow$

An optimization instance $\rightarrow$ $\begin{cases} \boxed{\min} \ f(x) \\ \text{s.t.} \ x \in \mathcal{D} \end{cases}$

$\boxed{\text{not max}}$!!!

where $\boxed{f(\cdot) \text{ is convex (over convex set } \mathcal{D})}$ & $\boxed{\mathcal{D} \text{ is convex}}$

Rmk:

① maximize concave $f^{=}$ $f(x)$ over a convex set $\mathcal{D}$

is convex program

② if $f(\underline{x}) \rightarrow$ convex function

then $\{\underline{x} : f(\underline{x}) \leq 0\} \Rightarrow$ convex set

③ Intersection of convex sets are also convex

$\Rightarrow$ $\{\underline{x} : f_i(\underline{x}) \le 0$ for all $1 \le i \le n\}$ convex set

if $f_i(\cdot)$ is all convex

1.

$\Rightarrow$ $\begin{cases} \min_{x \in \mathbb{R}^d} f_0(\underline{x}) \\ \text{s.t } f_i(\underline{x}) \le 0 \\ \quad i=1,\dots,n \end{cases}$ is a convex program $\boxed{\text{if}}$ $\begin{cases} f_0(\cdot) \text{ is convex} \\ f_i(\cdot) \text{ is convex } i=1,2,\dots,n \end{cases}$

2.

$\begin{cases} \min_{x \in \mathbb{R}^d} f_0(x) \\ \text{s.t } f_i(\underline{x}) \le 0 \quad i=1,2,\dots,n \\ \quad A\underline{x} = \underline{b} \end{cases}$ is a convex program $\boxed{\text{if}}$ $\begin{cases} f_0(\cdot) \text{ is convex} \\ f_i(\cdot) \text{ is convex} \end{cases}$

Q: Why we focus on Convex Opt.?

Answer: (one reason) $\boxed{\text{Local minima} \Rightarrow \text{Global minima}}$

$\underline{x}^*$ is locally minima

$\Leftrightarrow$ $\exists \varepsilon > 0$, s.t

$f(\underline{x}^*) \le f(\underline{x})$ for all $\|\underline{x} - \underline{x}^*\| < \varepsilon$

x^* is global minima

$\Leftrightarrow$ $f(\underline{x}^*) \le f(\underline{x})$ for all $\underline{x} \in \mathcal{D}$

① [Prop] suppose $\boxed{f \text{ is convex}}$, then $\boxed{\text{minimizing } f \text{ over } \underline{x} \in \mathbb{R}^n}$ unconstrainted opt

all local optimal sol$^{\underline{n}}$s are global optimal

② [Prop] Locally optimal sol$^{\underline{n}}$s of convex program $\rightarrow$ $\boxed{?}$

are globally optimal

Proof Let x^* be a locally optimal sol$^{\underline{n}}$

locally optimal $\Leftrightarrow$ $\exists \varepsilon > 0$, s.t

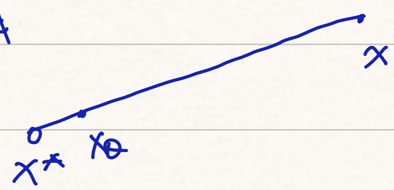
$f(x^*) \le f(x) \quad \forall \{\|x - x^*\| < \varepsilon\} \cap \mathcal{D}$

$\rightarrow$ want to show globally optimal

$\Leftrightarrow$ $f(x^*) \le f(x) \quad \forall x \in \mathcal{D}$

we can write $x = x^* + d$

consider $x_\theta = x^* + \theta d$



$\Big\downarrow$ $\underline{\theta \text{ sufficient}}$ $\underline{\text{small}}$

$\Rightarrow \|x_\theta - x^*\| = \theta \|d\| < \varepsilon$

① By convexity of $\mathcal{D}$, $x_\theta \in \mathcal{D}$

② By convexity of f, we have

$$\boxed{f(x_\theta) \le (1-\theta) f(x^*) + \theta f(x)}$$

$$\updownarrow$$

$$x_\theta = \boxed{x^* + \theta d = (1-\theta) x^* + \theta (x^* + d)}$$

③ $f(x^*) \le f(x_\theta)$

globally optimal

①+②+③ $\Rightarrow$ $\boxed{f(x^*) \le f(x) \text{ holding for } \forall x \in \mathcal{D}}$

{Conic Program} $\subseteq$ { Convex program }

↳ $\boxed{\text{nice structure}}$ → express many problems

↳ analytic result

① Defn : K is a convex[①] cone[②]

⟺ it is convex[①] & satisfy :

② $\lambda \cdot x \in K$ for all $\lambda \geq 0$ if $x \in K$

② Example :

1. $\boxed{\{ \underline{x} : \underline{x} \geq 0 \}}$ → convex cone

$\boxed{\text{Non-negative orthant}}$ NAME !

2. $\{ (\underline{x}, t) : \|\underline{x}\|_2 \leq t \}$ → $\boxed{\text{second order cone}}$

↕ { convex cone

3. set of all PSD matrices

$\{ X : v^T X v \geq 0 \text{ for all } v \}$

→ $\boxed{\text{convex cone}}$

Note : $\{ x : \|x\|_2 \leq 1 \}$ → not CONVEX CONE

↳ $\boxed{\text{Bounded Set cannot be CONE}}$

Conic Program

$\boxed{\text{Conic} = \left\{ \begin{array}{l} \text{convex} \\ \text{cone} \end{array} \right.}$

↳ $\min_x \; c^T x$

linear function

Affine constraints

where K is a convex cone

$$\text{s.t} \quad \boxed{Ax = b}$$

$$\boxed{x \in K}$$

where K is a convex cone

convex cone constraints

Rmk: ① when K is non-negative orthant

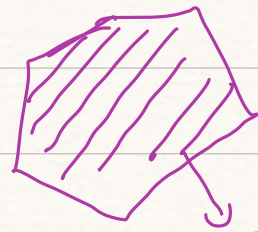
$$K = \{x : x \geq 0\}$$

Conic Prog $\Longrightarrow$ LP

$$\min_{x} \; c^T x$$

$$\text{s.t} \quad Ax = b$$

$$x \geq 0$$



Feasible set

② when K is second order cone

$$K = \{(x, t) : \|x\|_2 \leq t\}$$

Conic Prog $\Longrightarrow$ Second Order Conic Prog

$\downarrow$

$$\min_{(\underline{x}, t)} \; c_1^T \underline{x} + c_2 t$$

$$\text{s.t} \quad A \begin{pmatrix} \underline{x} \\ t \end{pmatrix} = b$$

$$\|\underline{x}\|_2 \leq t$$

Consider: 1. $\{(\underline{x}, t) : \|\underline{x}\|_2 \leq t\}$

2. $\{(\underline{x}_1, t_1, \ldots, \underline{x}_n, t_n) : \|\underline{x}_i\|_2 \leq t_i \;\; i = 1, 2, \ldots, n\}$

$\downarrow$

convex cone

③ when K is the set of PSD matrices

we get Semidefinite Prog

$\downarrow$

$$\begin{aligned} \min \quad & tr(C \cdot X) \\ s.t \quad & X \text{ is a PSD matrix} \\ & tr(A_i X) = b_i \quad i=1,2,\dots,m \end{aligned}$$

symmetric matrix ← $X \in \mathbb{R}^{n \times n}$ $\quad A_i \in \mathbb{R}^{n\times n}$ symmetric mat

$$\begin{cases} C \,\&\, A_i \quad i=1,2,\dots,m \rightarrow \boxed{\text{input matrices}} \\ X \longrightarrow \text{optimization variables} \end{cases}$$

$\boxed{\text{Duality}} \longleftrightarrow \boxed{\text{Conic Program}}$

Start From $\boxed{\text{LP Duality}}$

Consider:
$$\left.\begin{aligned} \min \quad & c^T x \\ s.t \quad & Ax = b \\ & x \geq 0 \end{aligned}\right] \rightarrow \boxed{\text{PRIMAL PROB}} \ (P)$$

write the Lagrangian function

$$\underbrace{L(x;\lambda,\mu)}_{\boxed{\text{a function}}} = \underbrace{c^T x}_{\text{obj.}} + \underbrace{\mu^T(b - Ax) - \lambda^T x}_{\text{constraints}}$$

$\boxed{\text{Given}}$ → fix (λ,μ) a specific $\lambda \geq 0$ & $\mu \in \mathbb{R}$, consider the unconstrained minimization problem

define $q(\lambda,\mu) = \min_x L(x;\lambda,\mu)$

Rmk: ① $q(\lambda,\mu)$ is a lower Bound of (P)

for given $\lambda \geq 0$ & $\mu \in \mathbb{R}$

Pf: $q(\lambda, \mu) \leq L(\hat{x}; \lambda, \mu)$

$= c^T\hat{x} - \lambda^T\hat{x} \quad (\lambda \geq 0; \hat{x} \geq 0)$

$\leq c^T\hat{x} = \begin{pmatrix} \min\limits_{x} & c^Tx \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{pmatrix}$

#

② Consider $\underbrace{\left(\max\limits_{\lambda \geq 0, \mu} q(\lambda, \mu)\right)}_{\boxed{(D)\ \text{Dual Prob}}}$ is still a lower bound to (P)

$\Rightarrow \quad (D) \leq (P)$

$\rightarrow$ weak duality

For LP (or some other questions), we can simplify the $q(\lambda, \mu)$

$\boxed{\text{Dual norm}} \longleftrightarrow \boxed{\text{Norm constraints}}$

$\downarrow$

$q(\lambda, \mu) = \inf\limits_{x} L(x; \lambda, \mu)$

$= \inf\limits_{x} c^Tx + \mu^T(b - Ax) - \lambda^Tx$

$= \inf\limits_{x} (c - A^T\mu - \lambda)^Tx + \mu^Tb$

$= \begin{cases} \mu^Tb, & c - A^T\mu - \lambda = 0 \\ -\infty, & c - A^T\mu - \lambda \neq 0 \end{cases}$

then (D) $\Rightarrow$ $\max\limits_{\lambda,\mu}$ $q(\lambda,\mu)$

s.t $\lambda \geq 0$

$$\Leftrightarrow \left[\begin{array}{ll} \max\limits_{\lambda,\mu} & \mu^T b \\ \text{s.t} & c - A^T\mu - \lambda = 0 \\ & \lambda \geq 0 \end{array}\right.$$

$\Leftrightarrow$ LP-(D) $\left[\begin{array}{ll} \max\limits_{\mu} & \mu^T b \\ \text{s.t} & c - A^T\mu \geq 0 \leftrightarrow A^T\mu \leq c \end{array}\right]$

LP-(P) $\Rightarrow$ $\left[\begin{array}{ll} \min\limits_{x} & c^T x \\ \text{s.t} & Ax = b \\ & x \geq 0 \end{array}\right]$

Duality for $\boxed{\text{conic program}}$

$\boxed{\text{Conic - Primal}}$ $\Leftrightarrow$ $\boxed{\begin{array}{ll} \min\limits_{x} & c^T x \\ \text{s.t} & Ax = b \\ & x \in K \end{array}}$

Defn : Dual cone (K^*) let $K \subseteq \mathbb{R}^d$ be a convex cone

then dual cone K^* is defined as

$K^* = \{ y : y^T x \geq 0 \text{ for all } x \in K \} \subseteq \mathbb{R}^d$

Case : $K = \{ x : x \geq 0 \}$ $\Rightarrow$ $K^* = K$

non-negative orthant

$\rightarrow$ defn of $\boxed{\text{Self-dual cone}}$ $\Leftrightarrow$ $K = K^*$

polar cone K^o

$\updownarrow$

cone K

Example: ① non-negative orthant

indicator f: $\delta_K(\cdot) = \begin{cases} 0, & x \in K \\ +\infty, & x \notin K \end{cases}$

② Second order cone

③ Set of all PSD matrices

$\delta_K^*(y) = \sup_x \{ \langle y, x \rangle - \delta_K(x) \}$

$= \sup_{x \in K} \{ \langle y, x \rangle \},$

$= \begin{cases} y \in K, & +\infty \\ y \notin K, & \begin{cases} \exists x \in K, \langle y, x \rangle > 0 \Rightarrow +\infty \\ \forall x \in K, \langle y, x \rangle \le 0 \Rightarrow 0 \end{cases} \end{cases}$

$= \begin{cases} 0, & \forall x \in K, \langle y, x \rangle \le 0 \\ +\infty, & \text{else} \end{cases}$

$= \delta_{K^o}(y)$ where $K^o = \{ y : \langle y, x \rangle \le 0, \forall x \in K \}$

Dual Form of Conic Program

Conic − P: $\min_x \; c^T x$ s.t. $Ax = b$, $x \in K$

Conic − D: $\max_y \; b^T y$ s.t. $c - A^T y \in K^*$

LP − P: $\min_x \; c^T x$ s.t. $Ax = b$, $x \ge 0$

LP − D: $\max_y \; b^T y$ s.t. $c - A^T y \ge 0$

Rmk: ① **Weak Duality**

$x \rightarrow$ primal feasible

$(\mu, \lambda) \rightarrow$ dual feasible

then $f(x) \ge q(\mu, \lambda) = \inf_x L(x; \mu, \lambda)$

$\uparrow$

$b^T\mu - x^TA^T\mu$

$\boxed{c^Tx \ge b^T\mu}$ → Pf: $c^Tx = c^Tx + (b-Ax)^T\mu$

$= \boxed{(c-A^T\mu)^Tx} + b^T\mu$

$\ge b^T\mu$

holds for { LP, $\boxed{\text{Conic Prog}}$ }

$x \in K$ $\quad c - A^T\mu \in K^*$

$\{y : \langle y, x\rangle \ge 0\}$

$\boxed{(c-A^T\mu)^Tx \ge 0}$

② Application of Weak Duality

→ Dual Prog ⇒ check the optimality of $\boxed{\text{Primal Prog}}$

↳ lower Bound of (P)

↳ if we have a lower Bound which is very close to the solution in (P)

↳ just sol$^{\underline{n}}$, not optimal

then we can say the solution is good !

③ Strong Duality → (D) = (P)

generally → (D) ≤ (P) (weak duality)

⟶ For a large class of convex prog, strong duality holds.

there exists case P → feasible, D → infeasible

Thm: (Strong Duality for LPs)

example: (P) min $-x$ s.t $x \ge 0$ → (D) max 0 s.t $-1 \ge 0$

$-\infty$ $\quad -\infty$

Suppose one of the PRIMAL and DUAL is feasible

→ e.g. $\exists x$ s.t $Ax = b$, $x \ge 0$ or $\exists y$ s.t $A^Ty \le c$

primal feasibility ... dual feasibility

then strong duality holds

$\boxed{\text{For general convex program}}$ — Slater's condition $\uparrow$ $\boxed{\text{sufficient condition}}$

$$\begin{cases} \min & f_0(x) \\ \text{s.t} & f_i(x) \quad i=1,2,\dots,m \\ & Ax=b \end{cases}$$

$\boxed{f_0, \dots, f_i \text{ convex}}$

suppose exists x^* s.t

$$\begin{cases} f_i(x^*) > 0 & \text{for all } i=1,2,\dots,m \\ Ax^* = b \end{cases}$$

(strictly feasibility)

$\Rightarrow$ strong duality holds

Is Convex Prog easy to solve?

$\downarrow$

$\boxed{\text{Not Quite}}$!

$\boxed{\text{Reason!}}$: All optimization problems can be Re-formulated into a convex program.

$\downarrow$

General problem : $\begin{cases} \min\limits_{x} & f(x) \\ \text{s.t} & x \in \mathcal{D} \end{cases} \longrightarrow \mathcal{D} \Rightarrow$ general set equivalent $\Big\downarrow$ reformulate

$$\begin{cases} \min\limits_{(x,t)} & t \\ \text{s.t} & x \in \mathcal{D} \\ & f(x) \le t \end{cases}$$

$\downarrow$ Difficulty: → C is very complicated

t^* ①

$$\min_{(x,t)} \; t \quad \text{s.t} \quad (x,t) \in C$$

$C = \{ (x,t) : f(x) \le t, \; x \in \mathcal{D} \}$

$\uparrow$

$\boxed{\text{Equivalent}}$

$\downarrow$

t^{**} ②

$$\min_{(x,t)} \; t \quad \text{s.t} \quad (x,t) \in \operatorname{conv}(C)$$

t → $\boxed{\text{linear}}$

→ ② $\le$ ① holds generally

$\boxed{\text{Reason}}$ $\forall (x,t) \in \operatorname{conv}(C)$

$\Leftrightarrow \; \exists \, (x_1,t_1) \; (x_2,t_2) \in C$

s.t $(x,t) = \theta (x_1,t_1) + (1-\theta)(x_2,t_2)$

$\Rightarrow t = \theta t_1 + (1-\theta) t_2$

$\Rightarrow t \ge t^*$ since $\begin{cases} (x_1,t_1) \in C \\ (x_2,t_2) \in C \end{cases}$

$\Longrightarrow \; t^{**} \ge t^*$

$\Longrightarrow$ ② $\ge$ ①

→ prove ① $\le$ ②