Review:

Geometry of Polyhedron!

$P = \{x : Ax = b,\ x \geq 0\}$ ⇝ non-empty polyhedron

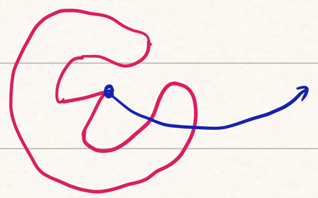
let $y \in P$. then we have:

i) y is an extrem point

ii) y is a vertex

iii) y is BFS of P, i.e. $y = \begin{pmatrix} 0 \\ B^{-1}b \end{pmatrix}$ $\begin{cases} ① \text{ BASIC} \\ ② \text{ FEASIBLE.} \end{cases}$

Example



extreme point

but not vertex!

Consider the following IP

$$\left.\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t} \quad & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned}\right\} \longrightarrow (IP\ 1)$$

(A Unimodularity)

什么条件可以保证？

如果我们求解 Relaxation 后的 (LP 1) 的解 $X_{opt} \in \mathbb{Z}^n$

⇒ 我们完成了对(IP 1)的求解

(tight relaxation!)

A Natural linear relaxation → drop $x \in \mathbb{Z}^n$

$$\left.\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t} \quad & Ax = b \\ & x \geq 0 \end{aligned}\right\} \longrightarrow (LP\ 1)$$

Q: Say, we solve (LP-1), and we obtain $X_{opt} \in \mathbb{Z}^n$, what can we say?

A: The Feasible Region of LP1 CONTAINS Feasible Region of IP1.

$\Rightarrow$ opt. value of IP1 $\geq$ opt value of LP1

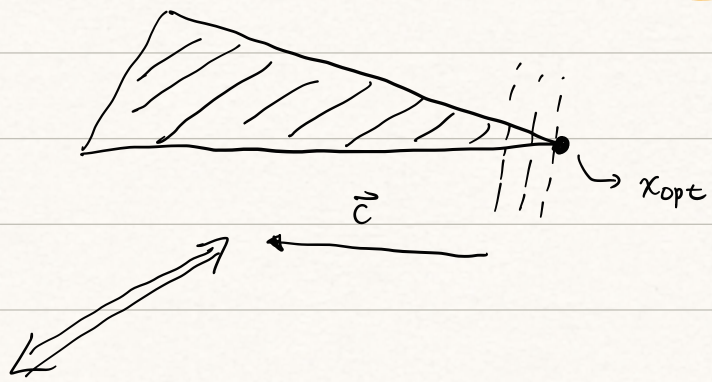
Since $x_{opt} \in \mathbb{Z}^n \Rightarrow$ feasible point in IP1.

① $\Rightarrow x_{opt}$ attains OPT val of IP1.

② opt. value of IP1 $=$ opt value of LP1

Intuition:

Solutions of LPs are attained at extreme points!

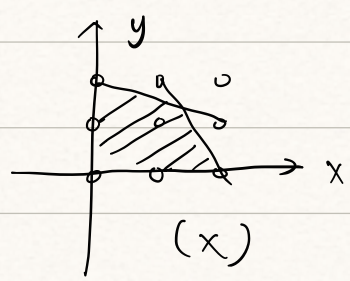


$\min_x \; c^T x$

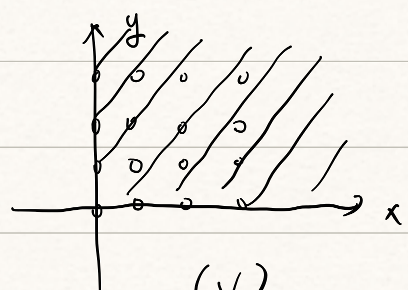
[Integral Polyhedron]

Defn. We say that a polyhedron $P \subseteq \mathbb{R}^n$ is INTEGRAL if all extreme points are integer vectors. (i.e. in $\mathbb{Z}^n$)

(V)



(X)



(V)

Defn: [Unimodularity] $\Rightarrow$ the condition that makes Linear relaxation works. (reasonable)

We say that a Matrix $A \in \mathbb{Z}^{m \times m}$ is unimodular,

if its determinant is ± 1 (i.e. $|A| = \pm 1$)

$$\begin{array}{cc} & n \\ m & \boxed{\text{Unimodular}} \end{array} \quad n > m$$

We say that a matrix $A \in \mathbb{Z}^{m \times n}$ is full row-rank is unimodular if: the sub-matrix obtained by ANY SUBSET of m columns is either SINGULAR or Unimodular.

(i.e. its determinant $\in \{-1, 0, 1\}$)

Example:

① $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ det $= -1$ ✓

② $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ det $= -2$ X

③ $\begin{pmatrix} n+1 & n \\ n & n-1 \end{pmatrix}$ det $= -1$ ✓

④ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rightsquigarrow$ det $= -2$.

X

[THM]. Let $A \in \mathbb{Z}^{m \times n}$ be a matrix is Full Row-RANK.

Then A is Unimodular $\Leftrightarrow$ the polyhedron

$P(b) := \{x : Ax = b, x \geq 0\}$

is integral for all $b \in \mathbb{Z}^m$ for which b is non-empty.

Application: for a IP1 Problem,

If A is unimodular,

$b \in \mathbb{Z}^m$.

then we just need to solve LP1 Problem! (linear relaxation) $\Rightarrow$

$$\left[\begin{array}{l} \min \ c^T x \\ \text{s.t.} \ \boxed{\begin{array}{l} Ax = \boxed{b} \\ x \geq 0 \end{array}} \\ \quad\ x \in \mathbb{Z} \end{array}\right.$$

polyhedron $\Rightarrow$ integral

$b \in \mathbb{Z}^m$

Rmk: Result has a qualification, which is that $P(b)$ is non-empty.

The case where $P(b)$ is troublesome.

Let's clear away those matrix, in interesting cases, the Bad cases never show up!

Q: Does this polyhedron have E.P.?

Without Proof

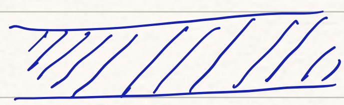
Thm: Every non-empty Bounded Polyhedron has ≥ 1 extreme points.

Thm: The polyhedron $\{x: Ax=b,\ x\geq 0\}$ has ≥ 1 extreme points.

polyhedrons can have no E.P.

☆ Note: it is possible for polyhedrons to have NO Extreme points!

e.g. $\{(x,y) : 0\leq y\leq 1\}$.



(But they are rather troublesome!)

Polyhedron $\begin{cases} \text{Bounded} \\ \{Ax=b,\ x\geq 0\} \end{cases}$ $\Rightarrow$ have at least 1 E.P.

Thm: Consider LP

$$\min_x c^T x$$
$$\text{s.t. } x\in P$$

where P is a polyhedron with ≥ 1 extreme point

Then: $\begin{cases} \text{(i) opt solution is } -\infty. \\ \text{(ii) opt solution is attained at an extreme point.} \end{cases}$

ONLY TWO POSSIBLE Case!

In summary, empty polyhedron and those are no extreme points are pathologies to avoid.

In any case, a Polyhedron in standard form $\{x: Ax=b,\ x\geq 0\}$ has ≥ 1 extreme point as long as it is non-empty.

Consequence of Unimodularity

IP1: $\min_x c^T x$

s.t $Ax = b$

$x \geq 0$

$x \in \mathbb{Z}^n$

Suppose $A \in U$ (unimodular), if: (i) $b \in \mathbb{Z}^m$

(ii) $P(b) = \{x : Ax = b,\ x \geq 0\}$ is non-empty.

Then. consider the linear relaxation:

$$\min_x \ C^T x$$
$$\text{s.t} \quad Ax = b$$
$$x \geq 0$$

Or Unbounded Below $(-\infty)$

$\Rightarrow$ solution is attained at the extreme point

- From previous theorem. $P(b)$ is integral.
- All extreme points are integral
- Solution (at least one) is attained at an integral point.
  - ↳ not necessary that all solutions are integral, but at least one!
- opt. val. of LR = opt. val. of IP

---

proof of Unimodularity $\Leftrightarrow$ $P(b) = \{x : Ax = b,\ x \geq 0\}$ is integral

$A^{m \times n}$

Pf: "$\Rightarrow$" If A is Uni-modular, then show that $P(b)$ is

Integral for $\forall b \in \mathbb{Z}^m$.

$\hookrightarrow$ all extreme points of $P(b)$ is integral.

Assume that $A \in U$.

Given a $b \in \mathbb{Z}^m$, Suppose that $P(b)$ is not empty.

Let x be an extreme point of $P(b)$.

from theorem (equivalence of Extreme Points)

x is a BFS ; i.e., $x = \begin{pmatrix} 0 \\ B^{-1}b \end{pmatrix}$

where B is a Basis of A!

$\hookrightarrow$ m. Linearly independent columns.

up to re-arrangement of the coordinates.

By Cramer's Rule. $(B^{-1}b) = \dfrac{\det(B_i)}{\det(B)}$ $\det(B_i) \to \in \mathbb{Z}$ ($B_i \in \mathbb{Z}^{m \times m}$) ; $\det(B) \to \pm 1$ (A unimodular)

$\Downarrow$

$\in \mathbb{Z}^m$

$\Rightarrow x \in \mathbb{Z}^n$

"$\Leftarrow$" Suppose $P(b)$ is integral for $\forall b \in \mathbb{Z}$, show that $A \in U$

Let B to be any basis of A

Let $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \to i\text{-th}$

Construct: $z \in \mathbb{Z}^m$

$z + B^{-1}e_i \geq 0$ $\to$ aims to pick every entry (column) of B^{-1} (to show that B^{-1} integer)

to guarantee nonnegative (to do this, pick the vector z to to big enough) positive

Define $b := Bz + e_i \in \mathbb{Z}^m$

By construction, the point $x = \begin{pmatrix} 0 \\ \ldots \end{pmatrix}$

$\left(z + B^{-1}e_i\right)$ $B^{-1}b$

satisfies ① $x \geq 0$

② $Ax = [\,*\ \ B\,]\begin{pmatrix} 0 \\ z + B^{-1}e_i \end{pmatrix} = \underbrace{Bz + e_i := b}_{\text{as we define}}$

Hence x is a BFS! $\Rightarrow$ x is an extreme point. (of $P(b)$)

$\Rightarrow$ x is an integral

($P(b)$ is INTEGRAl and

x is an extrem point of $P(b)$)

$\Rightarrow$ $z + B^{-1}e_i \in \mathbb{Z}^m$

$\Rightarrow$ $B^{-1}e_i \in \mathbb{Z}^m$

$\Leftrightarrow$ the i-th column of B^{-1} is

integral.

$\Rightarrow$ $B^{-1} \in \mathbb{Z}^{m\times m}$

We also have: $\det(B) * \det(B^{-1}) = 1$.

$\underbrace{B \in \mathbb{Z}^{m\times m}}_{\text{Assumption}}$. $\underbrace{B^{-1} \in \mathbb{Z}^{m\times m}}_{\text{Our construction!}}$ $\Rightarrow$ $\det(B) \in \{+1, -1\}$.

This holds for all B $\Rightarrow$ A is uni-modular.

Total Unimodularity.

Motivation:

IP2.

consider $\min_x \ c^T x$

s.t $Ax \leq b$. $\longrightarrow$ more useful

$$x \in \mathbb{Z}^n, \quad x \geq 0 \Big\}$$

(previously, it was $Ax = b$, $x \geq 0$, $x \in \mathbb{Z}^n$)

$\downarrow$ linear relaxation

$$\left.\begin{array}{ll} \min_x & c^T x \\ \text{s.t} & Ax \leq b \\ & x \geq 0 \end{array}\right\} \longrightarrow LP2$$

we need what condition to transform IP2 into LP2?

Q: when is opt. VAL of LP2 = opt. VAL of IP2.

Answer: modify the Problem into UNIMODULARITY setting!

Defn: (Total Unimodularity) We say that a matrix (full row-rank.) $A \in \mathbb{Z}^{m \times n}$ is totally unimodular $\Leftrightarrow$ the determinant of every squared sub-matrix (of any size.) is $\{-1, 0, +1\}$.

[Thm]. $A \in \mathbb{Z}^{m \times n}$ be a matrix.

Then A is TU $\Leftrightarrow$ $P(b) = \{x : Ax \leq b, x \geq 0\}$ is integral for all $b \in \mathbb{Z}^m$ for which $P(b)$ is non-empty.