Lec 2 $\boxed{\text{PMP}}$

Recap: $\begin{cases} \dot{x}(t) = f(t, x(t)) . \\ x(0) = x_0 \end{cases}$ ⟶ pertube $\boxed{x_0 + \varepsilon v_0}$ $v_0 \to \dfrac{x_\varepsilon - x_0}{\varepsilon}$ $\boxed{\varepsilon \to 0}$

$\Rightarrow \dfrac{\varphi_t(x_0 + \varepsilon v_0) - \varphi_t(x_0)}{\varepsilon} \to v(t)$

$\to \begin{cases} \dot{v}(t) = \nabla_x f(t, x(t))\, v(t) \\ v(0) = v_0 \end{cases}$

Calalation Variation

Optimal Control ⟶ Extend $\boxed{\text{E-L}}$ to non-smooth case

strong minima

↓

$\boxed{\text{weak minima}}$

↓

Smooth pertubation

$\boxed{\text{Bolza Problem}}$ $J[\theta] = \underbrace{\Phi(X(T))}_{\text{terminal cost}} + \int_0^T \underbrace{L(t, x(t), \theta(t))}_{\text{running cost}}\, dt$

$$\boxed{\begin{array}{lll} \inf\limits_{\theta \in L^\infty([0,T], \Theta)} : & J[\theta] & \\ \text{s.t} & \dot{x}(t) = f(t, x(t), \theta(t)) & t \in [0,T] \\ & x(t_0) = x_0 & \end{array}}$$

- Mayer Prob: $L \equiv 0$
- Lagrange Prob: $\Phi \equiv 0$

→ E-L equation (necessary) only holds for $x^* \in C^1$

→ PMP is a necessary condition for $\theta^* \in L^\infty([0,T], \Theta)$

Goal: Derive necessary condition for Strong Optimality $\boxed{\text{More advanced}}$

non-smooth pertubation is allowed

"if $\underline{\theta}^*$ is a strong minimizer,

then $\underline{\theta}^*$ must satisfy $\underbrace{\text{something}}_{\text{we want to find}}$!"

$\boxed{\text{Rmk:}}$ Basically, this is a constrained optimization

$\downarrow$

Lagrangian Multiplier Method

Lagrangian Multiplier Method (KKT Condition)

$\downarrow$

View Bolza Prob. as a constrained Opt. problem

$$\rightarrow \begin{cases} \inf\limits_{\underline{\theta}, \underline{x}} J[\underline{\theta}, \underline{x}] = \Phi(x(T)) + \int_0^T L(t, x(t), \theta(t))\, dt \\ \\ \text{s.t.} \quad \begin{cases} \dot{x}(t) - f(t, x(t), \theta(t)) = 0 & \forall t \in [0, T] \\ x(0) - x_0 = 0 \end{cases} \end{cases}$$

$\Downarrow$

infinit constraints

$\downarrow$

need infinitely many Lagrangian Multipliers

$\rightarrow$ For each t, there is a constraint,

$\Downarrow$

introduce $\boxed{\text{a Sequence of}}$ $\boxed{\text{Lagrangian Multipliers}}$

(infinitely many)

$\downarrow$

$\{p(t) : t \in [0, T]\}$

Then $\mathcal{L}(\underline{\theta}, \underline{x}, \underline{p}) = J[\underline{\theta}, \underline{x}] + \int_0^T p(t)^T [\dot{x}(t) - f(t, x(t), \theta(t))]\, dt$

$\boxed{\text{分部积分}}$

$+ p(0)^T (x(0) - x_0)$

$\rightarrow$ Necessary Condition is:

$$0 = \nabla_{\underline{\theta}} \mathcal{L}$$
$$0 = \nabla_{\underline{x}} \mathcal{L}$$
$$0 = \nabla_{\underline{p}} \mathcal{L}$$

State

① $\nabla_{p(t)} \mathcal{L} = 0 \Rightarrow \boxed{\begin{cases} \dot{x}(t) = f(t, x(t), \theta(t)) \\ x(0) = x_0 \end{cases}} \rightarrow \boxed{\text{feasibility}}$

② $\nabla_{\underline{x}} \mathcal{L} = 0 \Rightarrow$ Rewrite $\mathcal{L}[\underline{\theta}, \underline{x}, \underline{p}]$

$$= J[\underline{\theta}, \underline{x}] - \int_0^T p(t)^T f(t, x(t), \theta(t))\, dt \quad p(0)^T (x(0) - x_0)$$

$$J[\underline{\theta}, \underline{x}] = \Phi(\underline{x}(T)) + \int_0^T L(t, x(t), \theta(t))\, dt$$

$$\boxed{+ \int_0^T p(t)\, \dot{x}(t)\, dt = [p(t)^T x(t)]\Big|_{t=0}^{T} - \int_0^T \dot{p}(t)^T x(t)\, dt}$$

$$= J[\underline{\theta}, \underline{x}] - \int_0^T \left(p(t)^T f(t, x(t), \theta(t)) + \dot{p}(t)^T x(t) \right) dt$$
$$+ p(T)^T x(T) - p(0)^T x_0$$

$\boxed{t \in (0, T)}$ $\boxed{\begin{cases} \nabla_{x(t)} \mathcal{L} = 0 \Rightarrow \dot{p}(t) = -\nabla_x [p(t)^T f(t, x(t), \theta(t)) - L(t, x(t), \theta(t))] \\ \nabla_{x(T)} \mathcal{L} = 0 \Rightarrow p(T) = -\nabla_x \Phi(x(T)) \end{cases}}$

Co-state

③ $\nabla_{\underline{\theta}(t)} \mathcal{L} = 0 \Rightarrow \boxed{\nabla_\theta \left(p(t)^T f(t, x(t), \theta(t)) - L(t, x(t), \theta(t)) \right) = 0}$

→ want to write in a Compact form

→ Hamiltonian H

$$H : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Theta \longrightarrow \mathbb{R}$$

$$H(t, \underline{x}, \underline{p}, \underline{\theta}) = p^T \underbrace{f(t, x, \theta)}_{\text{Dynamic}} - \underbrace{L(t, x, \theta)}_{\text{running cost}}$$

$\boxed{\text{Necessary Condition}}$

↑ → simplification of ① ② ③

$\boxed{\text{Attain by}}$ ① $\dot{x}(t) = \nabla_p H(t, x(t), p(t), \theta(t)) \qquad x(0) = x_0$

$\boxed{\text{L-M-M}}$

② $\dot{p}(t) = -\nabla_x H(t, x(t), p(t), \theta(t))$ $\quad p(T) = -\nabla\Phi(x(T))$

③ $\nabla_\theta H(t, x(t), p(t), \theta(t)) = 0$

$\downarrow$

if control is not good $\rightarrow$ $\boxed{\text{no } \nabla_\theta\,!}$

$\downarrow$

$\boxed{\text{PMP}}$

$\boxed{\text{PMP}}$

Define Hamiltonian $H(t, x(t), p(t), \theta(t))$

$\boxed{\text{Broader Class}}$ $= p^T f(t, x, \theta) - L(t, x, \theta)$

Theorem: $\rightarrow$ $\boxed{\text{Finance}}$ more (Application / formulation)

Let $\boxed{\theta^* \in L^\infty([0,T], \Theta)}$ be an optimal control of the Bolza Problem, and x^* is controlled (state) trajectory, then there exists a co-state process $\{p^*(t) : t \in [0,T]\}$ absolutely continuous & satisfy:

$x^*(0) = x_0$

$$\begin{cases} ① \ \dot{x}^*(t) = f(t, x^*(t), \theta^*(t)) = \nabla_p H(t, x^*(t), p^*(t), \theta^*(t)) \\ ② \ \dot{p}^*(t) = -\nabla_x H(t, x^*(t), p^*(t), \theta^*(t)) \quad p^*(T) = -\nabla\Phi(x^*(T)) \\ ③ \ \boxed{H(t, x^*(t), p^*(t), \theta^*(t)) \geq H(t, x^*(t), p^*(t), \theta)} \end{cases}$$

for $\forall t \in [0,T]$ & $\forall \theta \in \Theta$

a.e. $\rightarrow$ almost every

① $\boxed{\text{relax our condition on } H \leftrightarrow \theta}$ $\rightarrow$ no need have $\boxed{\nabla_\theta H}$

② Stronger Condition $\rightarrow$ rule out more candidates

$\boxed{\text{Fixed - End - Time}}$

$\boxed{\text{Pf}}$ (Simplest One) $\rightarrow$ terminal time fixed

- Step 1 $\rightarrow$ convertion to Mayer Problem (From Bolza Prob to $L \equiv 0$)

Method: Define an auxilary variable $\{x^0(t)\}$ satisfy an ODE

$$\underbrace{\dot{x}^0(t) = L(t, x(t), \theta(t)) \qquad x^0(0) = 0}$$

$$\Downarrow$$

$$\text{Sol}^{n}: \quad x^0(t) = \int_0^t L(\tau, x(\tau), \theta(\tau))\, d\tau$$

$$\Rightarrow \quad x^0(T) \rightarrow \text{Running cost !}$$

Therefore, define new variables

$$\begin{cases} \tilde{x} = \begin{pmatrix} x^0 \\ x \end{pmatrix} \\ \tilde{f} = \begin{pmatrix} L \\ f \end{pmatrix} \\ \tilde{\Phi}(\tilde{x}) = \Phi(x) + x^0 \\ \tilde{x}(0) = \begin{pmatrix} 0 \\ x_0 \end{pmatrix} = \tilde{x}_0 \end{cases}$$

Then Bolza Prob. becomes:

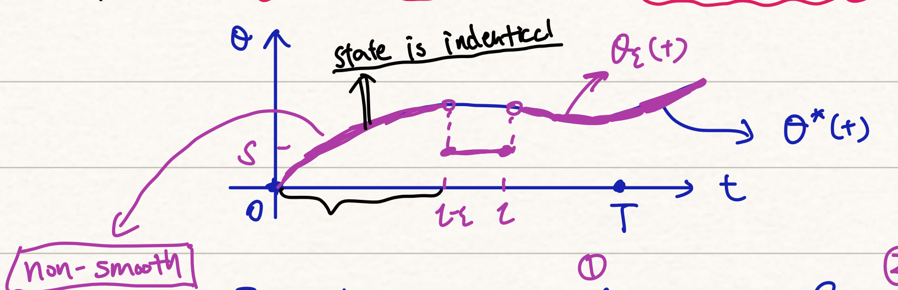
$$\begin{cases} \inf_{\theta} & \tilde{\Phi}(\tilde{x}(T)) \\ \text{s.t.} & \begin{cases} \dot{\tilde{x}} = \tilde{f} \\ \tilde{x}(0) = \tilde{x}_0 \end{cases} \end{cases} \longrightarrow \boxed{\text{Mayer Problem}}$$

$$\downarrow$$

drop the '~' and consider

Mayer Problem only W.L.O.G

- Step 2: Introduce 'Non-smooth' Pertubation $\boxed{\text{(Needle Pertubation)}}$



Let θ^* be an optimal control

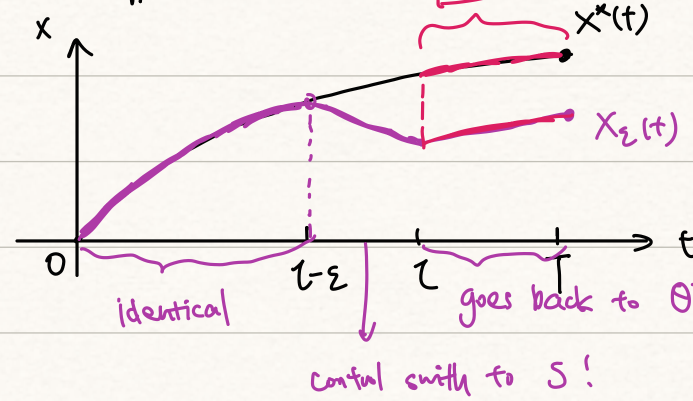
Fix arbitrary some ① $\tau \in (0, T)$ & ② $S \in \Theta$ & ③ $\varepsilon > 0$

Define a pertubed control as $\theta_\varepsilon(t) = \begin{cases} S & t \in [\tau-\varepsilon, \tau] \\ \theta^*(t) & \text{o/w} \end{cases}$

and corresponding trajectory X_ε defined by

$$\begin{cases} \dot{X}_\varepsilon(t) = f(t, x(t), \theta_\varepsilon(t)) \\ X_\varepsilon(0) = X_0 \end{cases}$$

What happens to state? $\boxed{\text{same ODE} \quad \text{different initial state}}$!!!



$\Phi(X^*(T))$

$\geq$

$\Phi(X_\varepsilon(T))$ holds for arbitrary τ, ε & S

CRUCIAL PART!!! $\longrightarrow$ Needle Pertube

- Step 3: Variation Equation $\rightarrow$ how pertabation transfer

$\boxed{\text{pertubate}}$,

Define $V(t) := \lim\limits_{\varepsilon \to 0} \dfrac{X_\varepsilon(t) - X^*(t)}{\varepsilon}$ $\quad t \in [\tau, T]$

$\hookrightarrow$ variantion Equation Reults!

Note that on $[\tau, T]$, control is the same, ODE same

$\downarrow$

X_ε & X^* satisfy the same equ.

$\dot{x}(t) = f(t, x(t), \theta(t))$

Thus, use Variational Equation Result,

$\{V(t)\}$ satisfy: (ODE)

$$\begin{cases} \dot{V}(t) = \nabla_x f(t, x^*(t), \theta^*(t)) \, V(t) & t \in [\tau, T] \\ V(\tau) = \boxed{V_0} \end{cases}$$

$\rightarrow$ haven't detemied yet!

To calculate $v(\tau)$, by defn:

$$V(\tau) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\Big[\underbrace{\Big(x_0+\int_0^{\tau} f(t,x(t),\theta_\varepsilon(t))\,dt\Big)}_{x_\varepsilon(t)} - \underbrace{\Big(x_0+\int_0^{\tau} f(t,x(t),\theta^*(t))\,dt\Big)}_{x^*(t)}\Big]$$

$\int_0^{\tau-\varepsilon} \to$ the same

$\int_{\tau-\varepsilon}^{\tau} \to$ different

$$= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\Big[\int_{\tau-\varepsilon}^{\tau} f(t,x(t),s)\,dt - \int_{\tau-\varepsilon}^{\tau} f(t,x(t),\theta^*(t))\,dt\Big]$$

$$\overset{\text{Lebesgue Differential Thm}}{=\!=\!=\!=\!=} f(\tau,x(\tau),s) - f(\tau,x(\tau),\theta^*(\tau))$$

☆ Thus, Needle Pertubation travels as:

to reach this, we make use of the arbitrariness of ε to characterize the initial pertubation $\to V(\tau)$ $\leftarrow$

$$\begin{cases} \dot V(t) = \nabla_x f(t,x^*(t),\theta^*(t))\,V(t) & t\in[\tau,T] \\ V(\tau) = f(\tau,x^*(\tau),s) - f(\tau,x^*(\tau),\theta^*(\tau)) \end{cases}$$

$\Phi: \mathbb{R}^d \to \mathbb{R}$

- Step 4. make use of $\Phi(x^*(T)) \le \Phi(x_\varepsilon(T))$

$\to$ Optimality condition at $\boxed{\text{End-Point}}$

By defn of strong local optimality

$\to \Phi(x^*(T)) \le \Phi(x_\varepsilon(T))$ for ε sufficient small

$\to \Phi(x_\varepsilon(T)) - \Phi(x^*(T)) \ge 0$ $\quad \varepsilon\to 0^+$

treat as function of ε

$\Rightarrow$ Direction Derivative ≥ 0

$\to$ optimality condition

$x_\varepsilon(T) \approx x^*(T) + \varepsilon V(T) + o(\varepsilon)$

$$\Big(\underbrace{\lim_{\varepsilon\to 0^+} \frac{x_\varepsilon(T)-x^*(T)}{\varepsilon}}_{V(T)}\Big)^T \nabla\Phi(x^*(T)) \ge 0$$

$$\Leftrightarrow \quad V(T)^T\,\nabla\Phi(x^*(T)) \ge 0 \qquad (*)$$

$\hookrightarrow$ implicitly depends on τ & s (arbitrary)

Goal: 'Push Back' $(*)$ to time τ to say 'something' about $x^*(\tau)$ & $\theta^*(\tau)$

Adjoint Equation :

Consider a linear ODE system given By

$$\dot{v}(t) = A(t)\, v(t)$$

Its adjoint equation is

$$\dot{p}(t) = -A(t)^T p(t)$$

$\varphi_t(\cdot) : \boxed{\mathbb{R}^d} \to \boxed{\mathbb{R}^d}$

$\begin{pmatrix} \varphi_{t1} \\ \vdots \\ \varphi_{td} \end{pmatrix}$ $\varphi_{ti} : \mathbb{R}^d \to \mathbb{R}$

Key property:

$$\frac{d}{dt}\left(p(t)^T v(t)\right) = p(t)^T \dot{v}(t) + \dot{p}(t)^T v(t)$$

$$= p(t)^T A(t)\, v(t) + \left(-A(t)^T p(t)\right)^T v(t))$$

$$\equiv 0 \quad !!!$$

$\Rightarrow p(t)^T v(t) \equiv$ Constant with respect t !

$\to$ use Adjoint equation on ; (ODE)

$$\begin{cases} \dot{v}(t) = \nabla_x f(t, x^*(t), \theta^*(t))\, v(t) \\ v(\tau) = f(\tau, x^*(\tau), s) - f(\tau, x^*(\tau), \theta^*(\tau)) \end{cases}$$

we define the adjoint equation (co-state) is $\{p^*(t) : t \in [\tau, T]\}$

$$\begin{cases} \dot{p}(t) = -\nabla_x f(t, x^*(t), \theta^*(t))^T p(t) \\ p^*(T) = -\nabla \Phi(x^*(T)) \end{cases}$$

$$\Rightarrow p^*(t)^T v(t) = \text{constant} = p^*(T)^T v(T) \leq 0$$

$\hookrightarrow -\nabla \Phi(x^*(T))$

holds for $\forall t \in [\tau, T]$.

$\Rightarrow p^*(\tau)^T v(\tau) \leq 0 \Leftarrow$ choose $t = \tau$

$(v(\tau) = f(\tau, x^*(\tau), s) - f(\tau, x^*(\tau), \theta^*(\tau)))$

$\Rightarrow \boxed{p^*(\tau)^T f(\tau, x^*(\tau), s) \leq p^*(\tau)^T f(\tau, x^*(\tau), \theta^*(\tau))}$

holds for a.e. $\tau \in [0, T]$

for all $s \in \Theta$

$\rightarrow$ Mayer Problem Result

$\downarrow$

Transform to the Bolza Problem Version!

- Step 5: convert Back to Bolza Problem! $(L \neq 0)$

Mayer		Bolza
$\tilde{\Phi}(\tilde{x})$	$\longrightarrow$	$\Phi(x) + x^0$
$\tilde{p}^*$	$\longrightarrow$	(p^*, p^{*0})
$\tilde{f}$	$\longrightarrow$	(f, L)

check this term

has no relation to x^0

$\boxed{\text{Jacobian}}$

$$\rightarrow \begin{cases} \dot{p}^{*0} = \nabla_{x^0} L(t, x(t), \theta(t)) \equiv 0 \\ p^*(T) = -\nabla_{x^0} \tilde{\Phi}(\tilde{x}) = -1 \end{cases} \Rightarrow p^{*0}(t) \equiv -1$$

$\boxed{\text{Mayer}} \longrightarrow \boxed{\text{Bolza}}$

$\Rightarrow \tilde{p}^* \tilde{f} \longrightarrow p^*(\tau) f(\tau, x^*(\tau), \theta^*(\tau)) - L(\tau, x^*(\tau), \theta^*(\tau))$

$\Rightarrow \tilde{p}^* \tilde{f}(s) \leq \tilde{p}^* \tilde{f}(\theta^*(\tau))$

$\Leftrightarrow \boxed{H(\tau, x^*(\tau), p^*(\tau), \theta^*(\tau)) \geq H(\tau, x^*(\tau), p^*(\tau), s)}$

$\downarrow$

3

For $\boxed{\text{Bolza-Problem}}$!!!

holds for a.e. $\forall \tau \in [0, T]$

and $\forall s \in \Theta$

#

$\boxed{\text{Rmk:}}$ PMP $\longrightarrow$ has many versions (not just 1 thm) for different problems

Bolza Problem with $\boxed{\text{fixed end point}}$ & $\boxed{\text{variable time}}$

$\Downarrow$ previously is fixed-end-time

$$\begin{cases} \inf\limits_{\theta, T} J[\theta] = \int_0^T L(t, x(t), \theta(t))\,dt + \cancel{\Phi(x(T))} \\ \text{s.t.} \quad \dot{x}(t) = f(t, x(t), \theta(t)) \\ \quad x(0) = x_0 \\ \quad x(T) = x_1 \end{cases}$$

$\longrightarrow$ $\boxed{\text{Physics More}}$! (formulation) !

$\boxed{\text{PMP:}}$ $\longrightarrow$ $\boxed{\text{Variant}}$

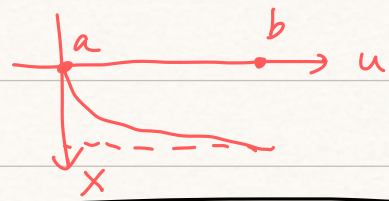
$$\begin{cases} \dot{x}^*(t) = \nabla_p H(t, x^*(t), p^*(t), \theta^*(t)) & x^*(0) = x_0 \quad \boxed{x^*(T) = x_1} \\ \dot{p}^*(t) = -\nabla_x H(t, x^*(t), p^*(t), \theta^*(t)) & \cancel{\boxed{\quad}} \\ H(t, x^*(t), p^*(t), \theta^*(t)) \geq H(t, x^*(t), p^*(t), s) & \\ \quad \text{for ae } \forall t \in [0, T] & \\ \quad \& \ \forall \theta \in U & \end{cases}$$

New

Vanish ! since $\Phi \equiv 0$

$\boxed{2小时}$

$\rightarrow$ Brachistochrone $\Rightarrow$ terminal state only fixed halfly !



$$\begin{cases} u(t_1) = b \\ x(t_1) \rightarrow \boxed{\text{not fixed}} \end{cases}$$

$\boxed{\text{Application}}$: C^1 piece-wise curve (example of PMP)

$$\left.\begin{aligned} & J[x] = \int_{-1}^{1} x(u)^2 (x'(u) - 1)^2\,du \\ & \min_{x} J[x] \\ & \begin{cases} x(-1) = 0 \\ x(1) = 1 \end{cases} \end{aligned}\right\} \rightarrow \text{Calculus Variation Prob.}$$

global minimizer (not in C^1)

$$\boxed{\underline{x}^* = \begin{cases} 0, & -1 \le u \le 0 \\ u & 0 < u \le 1 \end{cases}}$$

E-L equation is not a necessary condition

only applied when $x \in C^1$

$\Rightarrow$ $\boxed{\text{Method:}}$ Reformulate into a Controlled problem !

$$\min_{\underline{\theta} \in \Theta} \int_{-1}^{1} \underbrace{x(t)^2 (\theta(t) - 1)^2}_{} \, dt$$

$L \rightarrow$ running cost

$\Phi \equiv 0 \rightarrow$ no terminal cost

s.t. $\dot{x}(t) = \theta(t)$

$x(-1) = 0$

$x(1) = 1$

$\Theta = \mathbb{R}$

PMP Variantion

$\rightarrow$ Hamiltonian : $H(t, x, p, \theta) = p\theta - x^2 (\theta - 1)^2$

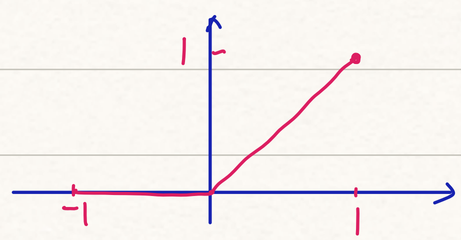
$\Rightarrow$ apply PMP :

① $\dot{x}^*(t) = \theta^*(t)$

② $\dot{p}^*(t) = 2 x^*(t) (\theta^*(t) - 1)^2$ $\quad x^*(-1) = 0 \quad x^*(1) = 1$

③ $\theta(t) \in \underset{\theta \in \Theta}{\operatorname{argmax}} \{ p^*(t) \theta - x^*(t)^2 (\theta - 1)^2 \}$

The solution we found by observation is:



$$\underbrace{x^*(t) = \begin{cases} 0, & -1 \le t \le 0 \\ t, & 0 < t \le 1 \end{cases}}$$

$$\theta^*(t) = \left.\begin{cases} 0, & -1 \le t \le 0 \\ 1, & 0 < t \le 1 \end{cases}\right\} \rightarrow \text{optimal control}$$

$\dot{p}^*(t) \equiv 0$ from $\begin{cases} x^*(t) \\ \theta^*(t) \end{cases}$

$\Downarrow$

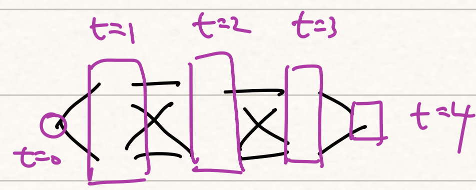
$p^*(t) \equiv \text{Constant}$

Sufficient & Necessary Condition (Dynamic Program)

$\boxed{\text{Motivation:}}$

① $\boxed{\text{Greedy}}$

② Recursive



Goal: Maximize the final score

→ Total # of paths $= 2^3 = 8$

→ More generally, if we have N circles at each step and T steps

(Brute Force)

→ total # of paths N^T (exponential in T)

We can use DP to solve the problem more efficiently.

↳ Define • $\boxed{X(t)}$ → $\boxed{\text{state}}$ of player at time (stage) t

☆ • Define $\boxed{V(t,x)}$ = max score starting at state x & time t (Value Function)

observe that $\boxed{V(0,1)}$ → Best score we can get

$\boxed{\text{our interest}}$

$\boxed{\text{DP Strategy}}$ → $\boxed{\text{extend our problems}}$

- Derive a recursion on $\{V(t,x)\}_{t,x}$

  ↳ solve for all t & x

↳ construct our optimal sol^{n}

Q1: How to determine max score? ⇒ $\boxed{\text{Backwards!}}$

$$\left[\begin{array}{l} V(3,2)=3 \\ V(3,1)=-3 \end{array}\right. \longrightarrow \left[\begin{array}{l} V(2,2) = \max\{-1+V(3,2),\ 4+V(3,1)\} = 2 \\ V(2,1) = \max\{-1+V(3,1),\ -2+V(3,2)\} = 1 \end{array}\right.$$

↓ at time 3 ↓ at time 2

$$\left[\begin{array}{l} V(1,2) = \max\{2+V(2,2),\ 1+V(2,1)\} = 4 \\ V(1,1) = \max\{3+V(2,2),\ 5+V(2,1)\} = 6 \end{array}\right.$$

↓ at time 1

$$\boxed{V(0,1) = \max\{0+V(1,1),\ 1+V(1,2)\} = 6}$$

↳ Better than GREEDY

Q2: How to attain the path? (Forward)

since we know $\{V(t,x)\}_{t,x}$

at time 0 → $x_0 = \underset{x}{\operatorname{argmax}} \{\text{score}(x) + V(1,x)\}$

at time 1 → $x_1 = \underset{x}{\operatorname{argmax}} \{\text{score}(x_0, x) + V(2,x)\}$

⋮

$\boxed{\text{Punchline}}$:

we can apply "Greedy Strategy" on $\{V(t,x)\}$ to achieve the optimal path

↓

$\boxed{\text{Complexity}}$ $O(N^2T)$

↓

$\boxed{\text{Brute Force}}$ → $\boxed{N^T}$

→ Apply to Optimal Control thm !

↳ Define Value Func $V(t,x)$

↳ Define Recursion on $V(t,x) \leftrightarrow$ PDE

↳ Solve for $\{V(t,x)\} \to$ ~~Greedily~~ Search

$\Rightarrow$ $\boxed{\text{Hamilton - Jacobi - Bellman Equation !}}$