$\boxed{\text{Optimal Control}} \to \boxed{\text{Basic}}$

$\boxed{\text{Calculus Variations}}$ → classical mechenics

→ deep learning

→ Finance

Calculus Variantion Example

↳ $\boxed{\text{最速下降 Curve}}$ ← Bernoulli → $\boxed{\text{Newton}}$

→ optimization over curves → characterized by $\boxed{\text{ode}}$

→ Some Basic Defn:

① ODE

- An ODE (initial value problem)

$$\dot{x}(t) = f(x(t)) \quad , \quad x(0) = x_0 \in \mathbb{R}^d$$

$$(f : \mathbb{R}^d \to \mathbb{R}^d)$$

- $\dot{x} := \dfrac{dx}{dt}$
- $f : \mathbb{R}^d \to \mathbb{R}^d$ vector field
- x_0 : initial condition
- Here, $f(\cdot)$ is independent of t

$\Updownarrow$

time-homogeneous

(e.g.) $f(y) = y$

Time-inhomogeneous ODE : $\boxed{f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d}$

e.g. $f(u,t) = t\cdot u$

"equivalent" in some way

④ regularity is different

$\Rightarrow$ Actually, we can change time-homogeneous to time-inhomogeneous ODE.

Pf: Transform a time-inhomogeneous ODE to a time-homogenous one

important technique in application (TRICK)

☆ $\Rightarrow$ use auxilary variable $x^0(t)$: $\underbrace{\dot{x}^0(t) \equiv 1, \quad x^0(0)=0}_{\Downarrow}$

$$\boxed{x^0(t) \equiv t}$$

$$\frac{d}{dt}\begin{pmatrix} x^0(t) \\ x(t) \end{pmatrix} = \begin{pmatrix} 1 \\ f(x^0(t), x(t)) \end{pmatrix}$$

initial condition $\Rightarrow \begin{pmatrix} x^0(0) \\ x(0) \end{pmatrix} = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}$

$\Rightarrow$ define $\tilde{x}(t) := \begin{pmatrix} x^0(t) \\ x(t) \end{pmatrix}$

Time-homogeneous ODE

then
$$\boxed{\begin{aligned} &\dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t)) \qquad \tilde{f}: \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \\ &\tilde{x}(0) = \begin{pmatrix} 0 \\ x_0 \end{pmatrix} := \tilde{x}_0 \in \mathbb{R}^{d+1} \end{aligned}}$$

Examples of ODEs and solutions

① $f(t,x) \equiv f(x) = ax$ (scaler $a, x \to a, x \in \mathbb{R}$)

time homogeneous

$$\begin{cases} \dot{x} = ax \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = x_0 e^{at}$$

② $f(t,x) = Ax \quad A \in \mathbb{R}^{d\times d}$ time-homogeneous

$x \in \mathbb{R}^d$

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = e^{At} x_0 \quad \text{where } \underline{e^{At} := \sum_{n=0}^{\infty} \frac{(At)^n}{n!}}$$

$$\underline{= \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot A^n}$$

$\boxed{\text{time-inhomogeneous}}$

③ $f(t,x) = a(t)\, x \qquad x \in \mathbb{R} \qquad a : [0,T] \longrightarrow \mathbb{R}$

$$\begin{cases} \dot{x}(t) = a(t)\, x(t) \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = e^{\int_0^t a(s)\, ds} x_0$$

$\boxed{\text{Recall:}}$ <u>Integral</u> vs <u>Differential Forms</u>

DF: $\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0 \end{cases}$ $\rightarrow$ $\boxed{\text{differentiable on } x \text{ over } t}$ (stronger)

$\Big\downarrow \int_0^t$ $\qquad \boxed{\text{Not exactly equivalent}}$

IF: (integral form) $\boxed{x(t) = x_0 + \int_0^t f(\tau, x(\tau))\, d\tau}$ $\rightarrow$ <u>does not require</u> <u>differentiable over time</u>

$\downarrow$

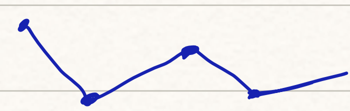
$\boxed{\text{Benefits}}$

$\downarrow$

weaker conditions

$\downarrow$

$x(t)$ is just $\boxed{\text{absolutely continuous}}$



$\downarrow$

almost every where differentiable

Thm: (Existence & Uniqueness) $\rightarrow$ <u>Picard-lindelöf</u>

Let f be continuous in t & uniformly Lipschitz in x, Then, there exists one unique (absolutely continuous) solution to your ODE

$$x(t) = x_0 + \int_0^t f(\tau, x(\tau))\, d\tau$$

on $[0,T]$

$\Downarrow$

exist $C>0$, s.t

$$\| f(t,x) - f(t,x') \| \le C \|x - x'\|$$

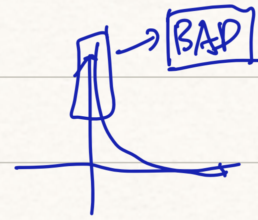
holds for $\boxed{\forall\, x, x' \in \mathbb{R}^d \ \& \ t \in [0,T]} \to \boxed{\text{uniformly}}$

Counter-example: ① $\begin{cases} \dot{x}(t) = x(t)^2 \\ x(0) = 1 \end{cases}$ $\quad (f(t,x) = x^2)$ $\to$ $\boxed{\text{not Lipschitz}}$

$\downarrow$

$x(t) = \dfrac{1}{1-t}$ only exists for $t \ne 1$

② $\begin{cases} \dot{x}(t) = x(t)^{\frac{2}{3}} \\ x(0) = 0 \end{cases}$ $\quad (f(t,x) = x^{2/3})$ $\to$ $\boxed{\text{not Lipschitz}}$



$\Rightarrow$ 2 sol^{n} $\begin{cases} x(t) \equiv 0 \\ x(t) = (t/3)^3 \end{cases}$ $\Rightarrow \boxed{\text{no uniqueness}}$

$\boxed{\text{Flow Map}}$ φ_t

consider time-homogeneous ode

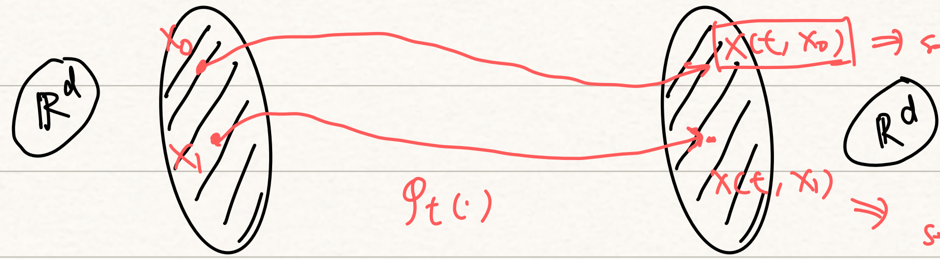
$$\begin{cases} \dot{x}(t) = f(x(t)) \\ \boxed{x(0) = x_0} \end{cases} \quad t \in [0,T]$$

Define Flow-map: φ_t associated with the ODE initial condition

$$\varphi_t : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

$$x_0 \longmapsto x(t)$$

for each t, we diffine a flow-map φ_t



$x(t, x_0) \Rightarrow$ sol^n of $\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0 \end{cases}$

$x(t, x_1) \Rightarrow$ sol^n of $\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_1 \end{cases}$

→ Time-inhomogeneous case : $\begin{cases} \varphi_{t_0, t_1}(x) = X(t_1) \\ X(t_0) = X \end{cases}$

⇓

$\boxed{\text{use auxilary trick}}$

↓

function f will relate to start time t

For a Time-inhomogeneous ODE

$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$

$\Leftrightarrow \begin{cases} \dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t)) \\ \tilde{x}(t_0) = \boxed{\begin{pmatrix} t_0 \\ x_0 \end{pmatrix}} \end{cases}$ ⤳ $\tilde{x}_0$ is related to (t_0, x_0)

$\Rightarrow \boxed{\varphi_t(x_0, t_0)}$ for Time-inhomogeneous Case

$\boxed{\text{Examples}}$:

① $\begin{cases} \dot{x}(t) = Ax \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = e^{At} x_0$

↓

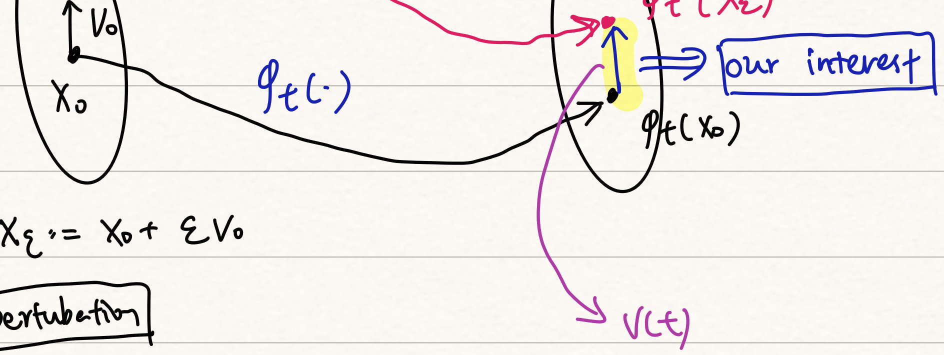
$$\boxed{\varphi_t(x) = e^{At} x}$$

② $\begin{cases} \dot{x}(t) = x(t)^2 \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = \dfrac{x_0}{1 - x_0 t}$

↓

$$\boxed{\varphi_t(x) = \frac{x}{1 - xt}}$$

Dependence On initial condition



$\boxed{\text{perturbe}}$: $x_\varepsilon := x_0 + \varepsilon V_0$

$\boxed{\text{perturbation}}$

$\boxed{\text{Result:}}$

Let $f(\cdot)$ be C^1 and Lipschitz in x uniformly in t

consider $\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0 \end{cases}$ Variantional Equation

方向导数

↑

$\dfrac{\varphi_t(x+\varepsilon v_0) - \varphi_t(x)}{\varepsilon}$

$\nabla_x \varphi_t(x)^T \cdot V_0$

Let $\{V(t)\}$ solve $\boxed{\begin{cases} \dot{V}(t) = \nabla_x f(t, x(t))\, V(t) \\ V(0) = V_0 \end{cases}}$

Then we have $\boxed{\lim_{\varepsilon \to 0^+} \left\| \dfrac{\varphi_t(x_\varepsilon) - \varphi_t(x_0)}{\varepsilon} - V(t) \right\| = 0}$

uniformly in $t \in [0, T]$

& $\|V_0\| \le 1$

φ_t ☆ 刻画了 How perturbation will evolve in time

$\boxed{\text{1-hour}}$

$f(x+tv) = \dfrac{\partial f}{\partial u}\Big|_{u=} \dfrac{\partial u}{\partial t}$

V_0

$\varepsilon \to 0$ $\boxed{\dfrac{\varphi_t(x_\varepsilon) - \varphi_t(x_0)}{\varepsilon}} := \nabla_x \varphi_t(x_0)^T V_0 = J(t) \cdot V_0$

$\mathbb{R}^{d \times d}$ $\boxed{\text{fix } x_0}$

$\varphi_t : \mathbb{R}^d \to \mathbb{R}^d$

Corallary: Define $\boxed{J(t) := \nabla_x \varphi_t(x_0)^T}$

$\boxed{\text{important}}$ $\boxed{\text{Then } \begin{cases} \dot{J}(t) := \nabla_x f(t, x(t))\, J(t) \\ J(0) = I \end{cases}}$

$\varphi_0(x) = x$ solves

$\Rightarrow J(0) = I_d$ $\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0 \end{cases}$

"$\boxed{\text{Proof}}$": $\boxed{\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0}$

$\Leftrightarrow x(t) = \varphi_t(x_0) \to \boxed{\text{fix } x_0}$

since $\dot{x}(\tilde{t}) = f(\tilde{t}, x(\tilde{t}))$ for $\tilde{t} \in [0,T]$

$\frac{\partial}{\partial x_0}$

$\Rightarrow \frac{d}{dt}(\varphi_{\tilde{t}}(x_0)) = f(\tilde{t}, \varphi_{\tilde{t}}(x_0))$ for $\tilde{t} \in [0,T]$ and $\forall$ fixed x_0

$\Rightarrow \frac{\partial}{\partial x}\left(\frac{d}{dt}\varphi_{\tilde{t}}(x)\right)\Big|_{x=x_0} = \frac{\partial f}{\partial y} f(\tilde{t}, y)\Big|_{y=\varphi_{\tilde{t}}(x_0)} \cdot \frac{\partial \varphi_{\tilde{t}}}{\partial x}\Big|_{x=x_0}$

$\Rightarrow \frac{d}{dt} J(\tilde{t}) = \nabla_y f(\tilde{t}, y)^T\Big|_{y=\varphi_{\tilde{t}}(x_0) = x(\tilde{t})} \cdot J(\tilde{t})$

$\nabla_x f(\tilde{t}, x(\tilde{t}))\, J(\tilde{t})$

$\boxed{\text{Some Examples}}$:

① $\begin{cases} \dot{x}(t) = A x(t) \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = e^{At} \cdot x_0$

$\Downarrow$

$\varphi_t(x) = e^{At} x$

$\Rightarrow \nabla_x \varphi_t(x) = e^{At} := J(t)$

we know that

$\begin{cases} \dot{J}(t) = \underbrace{\nabla_x f(t, x(t))}_{A} J(t) \\ J(0) = \mathbb{I} \end{cases} \Rightarrow J(t) = e^{At}.$

Motivation for $\boxed{\text{Calculus of Variation}}$

$\updownarrow$

$\boxed{\text{Optimization}}$

Calculus of Variation

→ Finite-dimensional opt. problem:

$\min_{x \in \mathbb{R}^d} \Phi(x) \qquad \Phi(\cdot): \mathbb{R}^d \to \mathbb{R}$

→ Infinite-dimensional opt. problem:

$$\min_{x \in X} J[x] : \quad J : X \to \mathbb{R}$$

X → $\boxed{\text{function}}$ $\quad$ J → $\boxed{\text{functional}}$

e.g. $\boxed{x := \{x(u) : a \le u \le b\}}$ just $x : [a,b] \to \mathbb{R}$

$u \mapsto x(u)$

$$X = C^1([a,b], \mathbb{R})$$

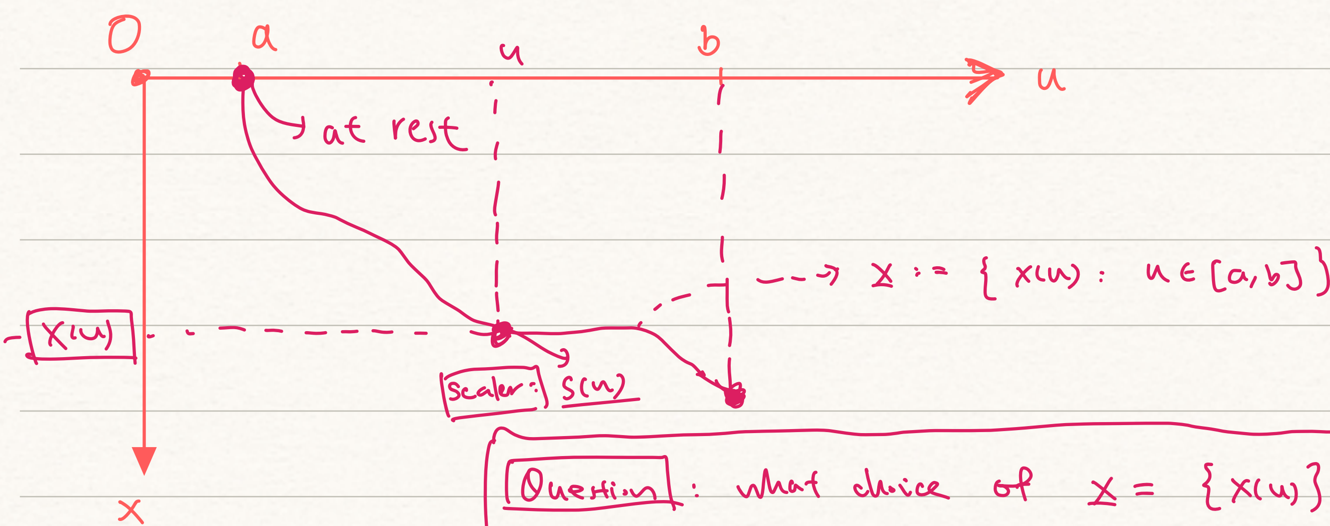
$$J[x] = \int_a^b L(u, x(u), x'(u))\, du$$

L → $\boxed{\text{Lagrangian}}$

$L : [a,b] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$

$\boxed{\text{Background}}$

$\boxed{\text{Example}}$: (Brachistochrone)



$\boxed{\text{Question}}$: what choice of $x = \{x(u)\}$ minimizes the time taken from a to b

↓

$\boxed{\text{formulate}}$

$\min J[x]$

x

Method: define
$$\begin{cases} \bullet \text{ speed at } u: \ s(u) \rightarrow \boxed{\text{scaler}} \\ \bullet \text{ height at } u: \ -x(u) \end{cases}$$

$$\Big\Downarrow \ \underline{\text{Conservation law}}$$

Energy conservation

$$\underbrace{\frac{1}{2} m \cdot s(u)^2 - 0}_{\substack{\text{gain kinetic} \\ \text{energy}}} = \underbrace{mg(0 - (-x(u)))}_{\substack{\text{loss in potential} \\ \text{energy}}}$$

$$\Rightarrow \frac{1}{2} s(u)^2 = g\, x(u)$$

$$\Rightarrow s(u) = \sqrt{2g \cdot x(u)} \qquad \boxed{\text{speed}}$$

$$\text{time} = \frac{\text{length}}{\text{speed}}$$

$$\Rightarrow T[x] = \int_a^b \frac{\text{arglength}(u)}{s(u)} \, du \qquad \boxed{\text{curve}} : (u, x(u))$$

$$= \int_a^b \frac{\sqrt{1 + x'(u)^2}}{\sqrt{2g\, x(u)}} \, du$$

$\Rightarrow$ B - Problem

$$\min_{x} T[x] = \int_a^b \underbrace{\frac{\sqrt{1 + x'(u)^2}}{\sqrt{2g\, x(u)}}}_{L(\cancel{u},\, x(u),\, x'(u))} \, du$$

actually only depends on $x(u)$ & $x'(u)$

$\boxed{\text{Infinite-dimensional optimization}}$

How to solve? $\rightarrow$ $\boxed{\text{E-L Equation}}$ $\leftarrow$ $\boxed{\text{Answer}}$

Euler - Lagrangian Method (Equation)

Recall: we want to solve:

$$\min_{x} J[x] = \int_a^b L(u, x(u), x'(u))\, du$$

$$\downarrow$$

$$x \in C^1([a,b], \mathbb{R}^d)$$

(Euler-Lagrangian)

~~we call~~: THM: If we have a extremer (x), then it satisfies

$\boxed{\text{Necessary Condition}}$

$$\boxed{\nabla_{x} L(u, \boxed{x(u)}, x'(u)) = \frac{d}{du}\left(\nabla_{x'} L(u, x(u), \boxed{x'(u)})\right)}$$

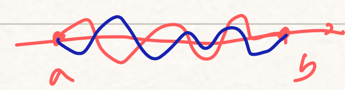
$\boxed{\text{candidate}}$

$$\Downarrow$$

Euler - Lagrangian Equation

$\boxed{\text{Pf}}$: Consider $\boxed{\text{Pertubation}}$ → arbitrary pertubation will be higher value

$$y_\varepsilon(u) = x(u) + \varepsilon \eta(u)$$



where $\begin{cases} \eta \in C^1 \\ \eta(a) = \eta(b) = 0 \end{cases}$

check, $\begin{cases} y_\varepsilon(a) = x(a) \\ y_\varepsilon(b) = x(b) \end{cases}$ since $\eta(a) = \eta(b) = 0$

check $\boxed{J[y_\varepsilon]} = \int_a^b L(u, \eta_\varepsilon(u), \eta_\varepsilon'(u))\, du$

fix η $\Downarrow$

$$= \int_a^b L(u, x(u) + \varepsilon\eta(u), x'(u) + \varepsilon\eta'(u))\, du$$

$\boxed{\text{fix } \eta}$

$F(\varepsilon)$

since $\underline{x}$ is a minima $\Rightarrow$ $\boxed{\frac{d}{d\varepsilon} F(\varepsilon)\Big|_{\varepsilon=0} = 0}$

Hence, $\frac{d}{d\varepsilon}\left(F(\varepsilon)\right)\Big|_{\varepsilon=0}$ $\rightarrow$ Integral & derivative change order $\Rightarrow$ $\boxed{\text{regularity}}$

$$= \int_a^b \nabla_x L(u, x(u)+\varepsilon\eta(u), x'(u)+\varepsilon\eta'(u)) \cdot \eta(u) + \nabla_{x'} L(u, x(u)+\varepsilon\eta(u), x'(u)+\varepsilon\eta'(u))\, \eta'(u)\, du \Big|_{\varepsilon=0}$$

$$= \int_a^b \nabla_x L(u, x(u), x'(u))\, \eta(u) + \nabla_{x'} L(u, x(u), x'(u))\, \eta'(u)\, du$$

Integral by part

$$= \int_a^b \nabla_x L(u, x(u), x'(u))\, \eta(u)\, du + \int_a^b \nabla_x L(u, x(u), x'(u))\, d\eta(u)$$

$$= * + \eta(u) \nabla_{x'} L(u, x(u), x'(u))\Big|_a^b - \int_a^b \eta(u) \cdot \frac{d}{du}(\nabla_x L)\, du$$

$\boxed{\eta(\cdot) \text{ is arbitrary}}$

$$= \int_a^b \boxed{\eta(u)} \cdot \left(\nabla_x L - \frac{d}{du}(\nabla_{x'} L) \right) du$$

$$\Downarrow$$

$$\equiv 0$$

Thus, $\nabla_x L(u, x(u), x'(u)) = \frac{d}{du}\left[\nabla_{x'} L(u, x(u), x'(u))\right]$ #

$\boxed{\text{Summary}}$ Key idea

we can only guarantee smooth pertubation

- consider pertubation $\begin{cases} \varepsilon \\ \eta(u) \rightarrow \boxed{\text{smooth}} \end{cases}$

- extremer property
- get Euler-Lagrangian Necessary condition

Normally, we just solve E-L Equation → we always get the solutions we want

Weak & strong Minima (Extremer)

- Local minima in finite-dimensional opt. problems

$$\min_{x \in \mathbb{R}^d} \Phi(x)$$

- x^* is a local minima of Φ if $\exists \delta > 0$, s.t

$\Phi(x^*) \le \Phi(x)$ for all $x \in \mathbb{R}^d$ s.t $\boxed{\|x - x^*\| \le \delta}$ → do not specify ⇒ all norms are equivalent → finite-dimensional space

推广 → infinite-dimensional space

- Local minima in infinite-dimensional opt. problems

$$\min_{\underline{x} \in X} J[\underline{x}]$$

- $\underline{x}^*$ is a local minima of J if $\exists \delta > 0$, s.t

$J[\underline{x}^*] \le J[\underline{x}]$ $\forall\ \underline{x} \in X$ s.t $\boxed{\|\underline{x} - \underline{x}^*\| \le \delta}$ → norms are not equivalent in infinite-dimensional space

⇓

minima depends on norm!

→ weak minima

→ strong minima

or something

Euclidean norm

- $\boxed{\text{weak minima}}$ (1-norm)

$$\boxed{\|\underline{x} - \underline{x}^*\|_1} := \sup_{u\in[a,b]} \|x(u) - x^*(u)\|$$

derivative

$$+ \sup_{u\in(a,b)} \|x'(u) - x^{*\prime}(u)\|$$

$\downarrow$

$\boxed{\text{BIGGER}}$

- $\boxed{\text{Strong minima}}$ (0-norm)

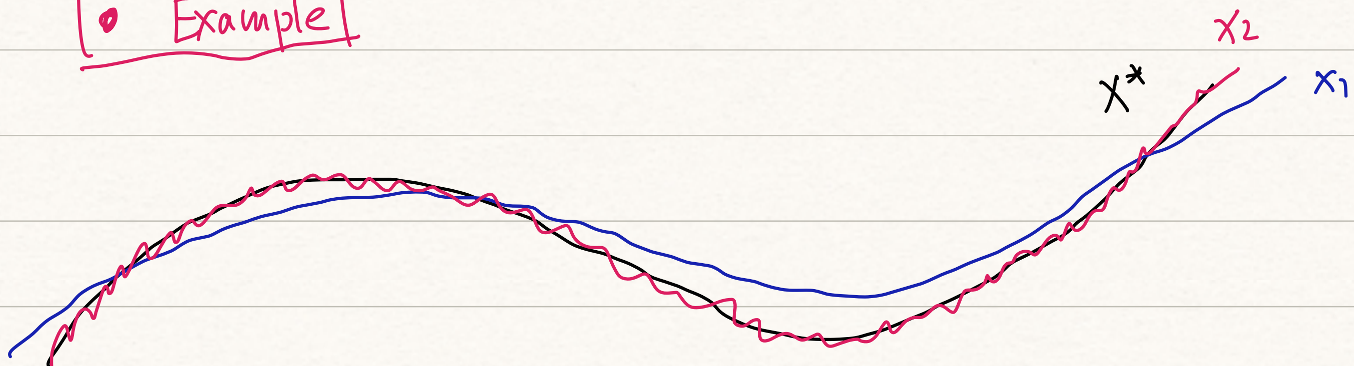
$$\boxed{\|\underline{x} - \underline{x}^*\|_0} = \sup_{u\in[a,b]} \|x(u) - x^*(u)\|$$

$\downarrow$

$\boxed{\text{smaller}}$

since we allow more pertubation (norm is small)

• Example



$\underline{x}_2 \to$ not close to x^*

in the norm-1 sense

$\to$ is close to x^*

in the norm-0 sense

$\Rightarrow$ more kinds of pertubation are allowed for norm-0 case

$\Downarrow$

strong minima

If x^* is a weak minima, then $J[x^*] \le J[x_1]$

If x^* in a strong minima, then $\begin{cases} J[x^*] \le J[x_1] \\ J[x^*] \le J[x_2] \end{cases}$

$\boxed{\text{E-L Equation}}$ $\Leftarrow$ only guarantee weak minima

(smooth perturbation)

$\boxed{\text{Example}}$: $J[x] = \int_{-1}^{1} (x(u))^2 (x'(u)-1)^2 \, du \ge 0$

$$\min_{\text{s.t. } \begin{cases} x \in X \\ x(-1)=0 \\ x(1)=1 \end{cases}} J[x]$$

constant ~~co~~ curve

① $x(u) \equiv 0$

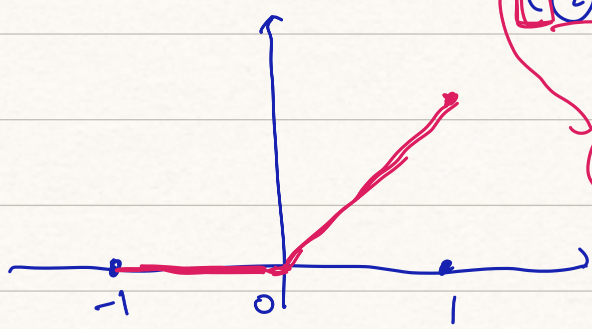
② $x'(u) \equiv 1$

finite 间断

global minima

$\Downarrow$

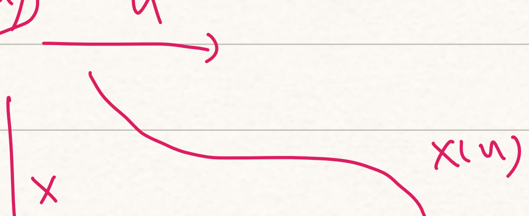
$\forall \delta$



$\Rightarrow$ extend E-L Equation $\Rightarrow$ optimal control theory

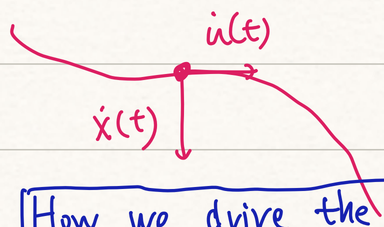
Revisit B-Problem (shortest time)

previously



$\rightarrow$ 坐标系 $(u, x(u))$

now



How we drive the ball

Move general

$(x(t), u(t))$

reformulate the problem

$\downarrow$

generalize E-L equation

→ Reformulation

speed $\boxed{S(u)} \longrightarrow S(t)$

$$S(t) = \sqrt{\dot{x}(t)^2 + \dot{u}(t)^2}$$

Energy Conservation $\quad \frac{1}{2} m \cdot S(t)^2 = mg\, x(t)$

$$\Downarrow$$

$$2g\, x(t) = \dot{x}(t)^2 + \dot{u}(t)^2$$

Define Control

$$\theta_1(t) := \frac{\dot{u}(t)}{\sqrt{2g\, x(t)}} \qquad \theta_2(t) := \frac{\dot{x}(t)}{\sqrt{2g\, x(t)}}$$

something like force !

and $\boxed{\theta_1^2(t) + \theta_2^2(t) = 1}$

apply forces with some constraints!

let gravity pull it to move

the force has condition

$$(*)\quad \Rightarrow \begin{cases} \dot{u}(t) = \theta_1(t)\sqrt{2g\,x(t)} \\ \dot{x}(t) = \theta_2(t)\sqrt{2g\,x(t)} \\ \theta_1^2(t) + \theta_2^2(t) \equiv 1 \end{cases}$$

$\boxed{\text{Dynamic system}}$

$\rightarrow$ $\boxed{\text{energy conservation}}$

$\Rightarrow$ initial state $\quad ((u(0), x(0)) = (a, 0)$

terminal condition $\quad u(T) = b$

$$J[(\underline{\theta}_1, \underline{\theta}_2)] = \int_0^T dt = T$$

s.t these holds.

$\Rightarrow$ Reformulation

$$\min_{\underline{\theta}} J[\underline{\theta}] \rightarrow (\text{variation on } \underline{\theta})$$

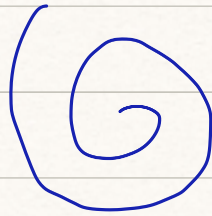
s.t $(*)$ holds

previously $\quad u, x(u)$ must be a function

$\theta_1, \theta_2 \Rightarrow$ not needs to be

like this

like this



a broader class

but $(x(t), u(t))$ can characterize this shape

Optimal Control Problem Formulation:

define Dynamics: $\dot{x}(t) = f(t, \underbrace{x(t)}_{\text{state}}, \underbrace{\theta(t)}_{\boxed{\text{control}}})$

$t \in [0, T]$ $\quad \underline{x}(0) = x_0$

for each t, $\theta(t) \in \Theta \subset \mathbb{R}^p$

Assume
- Θ control set is closed
- $f(t, x, \theta)$ is continuous in t, θ for all x
- $f(t, x, \theta)$ is C^1 in x for all t, θ

Not assuming that
- f is differentiable with respect θ → non-smooth control
- $t \mapsto \theta(t)$ is regular (or even

⇒ $\boxed{\theta \text{ can be very small}}$

cost functional

$$J[\theta] = \Phi(x(T)) + \int_0^T L(t, x(t), \theta(t))\, dt$$

- Φ → terminate cost $\quad \Phi : \mathbb{R}^d \to \mathbb{R}$
- L is the running cost

$L : [0,T] \times \mathbb{R}^d \times \Theta \to \mathbb{R}$

$\mathbb{R}^d \downarrow$ state, $\Theta \downarrow$ control

$\dot{x}(t) = f(t, x(t), \theta(t))$

$\downarrow$

$\boxed{\text{given } \theta(t), \text{ all is determined}}$

$\Downarrow$

$J[\theta]$ is determined only on $\underline{\theta}$

Bolza Problem of Optimal control

$$\min_{\underbrace{\theta \in L^\infty([0,T] \to \Theta)}_{\text{admissible set}}} J[\theta] = \Phi(x(T)) + \int_0^T L(t, x(t), \theta(t))\,dt$$

$\boxed{\text{essentially bounded}}$

$$\text{s.t. } \begin{cases} \dot{x}(t) = f(t, x(t), \theta(t)) \\ x(0) = x_0 \\ \theta(t) \in \Theta \end{cases}$$

- Mayer Problem : $L \equiv 0 \to$ only terminal cost
- Lagrange problem: $\Phi \equiv 0$

$\boxed{\text{very generalization}}$

In fact, $\boxed{\text{Bolza Problem}}$ generalizes these 2.

(calculus of variation)

$\Downarrow$ usual case

$$\min_{x} \int_0^T L(t, x(t), \dot{x}(t))\,dt \quad \begin{pmatrix} u \to t \\ a \to 0 \\ b \to T \end{pmatrix}$$

actually a special case

derive necessary condition if we define $\dot{x}(t) = \theta(t)$

for Optimal Control problem

$f(t, x, \theta) = \theta$

next time

Bolza Problem

determine $\theta(t)$

can uniquely determine $x(t)$