CUTTING PLANES ⇒ Last lecture for exam

The cutting plane algorithm ⇒ solve General MILPs

(Greedy alg only works when matroid)

↳ not general method.

MILP ⇒ hard to solve , Thus, we don't expect the cutting plane to be tractable in the worst case.

Basic Strategy: Consider the MILP

$$\boxed{\begin{aligned} \min_{x,y}\ & c^T x + d^T y \\ \text{s.t}\ & Ax + By = b \\ & x \geq 0,\ y \geq 0 \\ & x \in \mathbb{Z} \end{aligned}} \quad \text{— (1)}$$

LR →

$$\boxed{\begin{aligned} \min_{x,y}\ & c^T x + d^T y \\ \text{s.t}\ & Ax + By = b \\ & x \geq 0,\ y \geq 0 \end{aligned}} \quad \text{— (2)}$$

Suppose we went ahead and solved (2).

① if $x_{opt} \in \mathbb{Z}$, then we have done!

② if $x_{opt} \in \mathbb{Z}$! we seek an inequality of the form

$$\boxed{\alpha^T x + \gamma^T y \leq \beta} \quad \text{— (3)}$$

↓

that excludes (x_{opt}, y_{opt}) but 我们不要!

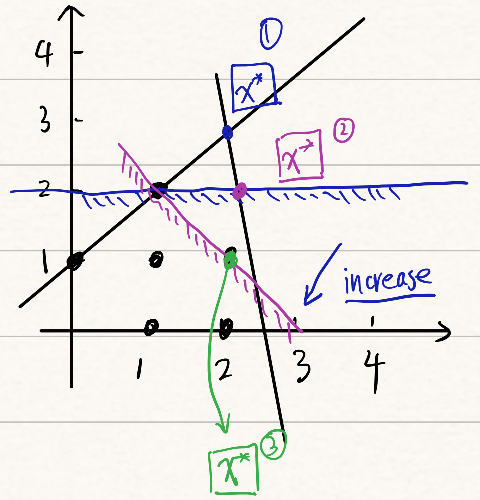
⇒ since $x_{opt} \notin \mathbb{Z}$

⇓

includes all feasible solutions of (1)

Once we obtain (3), we play it into (2) and solve again.
This will lead to an Improved soln.

[Example]



Question:
How to find the cut?

"Optimal"

one feasible soluton of MILP

Main Question:

→ How do we produce such cuts?

→ How do we ensure that our cuts are EFFECTIVE?

⇕

"Convergence"

Gomory cuts for IPs

Consider the following IP:

$$\boxed{\begin{aligned} \min \quad & c^T x \\ \text{s.t} \quad & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned}} \quad \text{— (4)}$$

Assume that A & b have rational entries

Let $\hat{x}$ be the optimal soln to the LR:

$$\boxed{\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}}$$

① if $\hat{x} \in \mathbb{Z}^n$, then we have solved (4)!

② if $\hat{x} \notin \mathbb{Z}^n$, let $\hat{x}$ be a BFS,

without loss of generality, we may assume the basis associated to $\hat{x}$ are the first m columns of A

$$\Downarrow$$

$$\hat{x} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \qquad A = [B \quad A_N]_{m \times n}$$

Now, by multipling the linear equation

$$Ax = b$$

with B^{-1} on the left, then

$$B^{-1}Ax = B^{-1}b \iff [I \ \bar{A}]\, x = \bar{b}$$

$$\begin{cases} \bar{A} = B^{-1}A_N \\ \bar{b} = B^{-1}b \end{cases}$$

$\bar{b} = B^{-1}b$ is not $\mathbb{Z}^n$

$\Downarrow$

use this to determine cutting plane

if x is feasible, we have

Now, each row of the above equation is of the form:

$$(7) \quad \boxed{x_h + \sum_{j \in N} \bar{a}_{hj} x_j = \bar{b}_h}$$

$h \in (B)$ index $(m \uparrow)$

Here, N denotes the subset of indices corresponding to $\bar{A}$ (A_N)

if we plug our optimal soln $\hat{x}$ into (7)

$\Downarrow$

$$\boxed{\hat{x}_h = \bar{b}_h}$$

But one of these must be non-integral

$\Rightarrow$ Pick one such h for which b_h is not integer!

$b_h \notin \mathbb{Z}$ to construct the cutting plane!

[Proposition]

Suppose $\bar{b}_h$ is not an integer.

then the following specifies a cutting plane:

$$\boxed{x_h + \sum_{j \in N} \lfloor \bar{a}_{hj} \rfloor x_j \leq \lfloor \bar{b}_h \rfloor} \quad \text{— (8)}$$

Exclude $\hat{x}$

Two Parts!

① $\hat{x}$ is CUT! $\Rightarrow$ $\hat{x}$ not satisfy (8)

② all integer vectors satisfy!

$\Leftrightarrow$ ① 把 $\hat{x}$ 切掉
② 保留所有整数解

Pf: Part 1

$x \geq 0$

$Ax = b$

IP

We show that feasible solution of problem satisfy (8)

Notice that since $x_j \geq 0$

$$\begin{aligned} & x_h + \sum_{j \in N} \lfloor \bar{a}_{hj} \rfloor x_j \\ & \leq x_h + \sum_{j \in N} \bar{a}_{hj} x_j \\ & = \bar{b}_h \end{aligned}$$

$\rightarrow$ Conclusion

$$\boxed{x_h + \sum_{j \in N} \lfloor \bar{a}_{hj} \rfloor x_j \leq b_h}$$

Now, if x is an $\left\{ \begin{array}{l} \text{feasible} \\ \text{integer} \end{array} \right.$ vector, then we have the following:

$$x_h + \sum_{j \in N} \lfloor \bar{a}_{hj} \rfloor x_j$$

$$= \left\lfloor x_h + \sum_{j \in N} \lfloor \bar{a}_{hj} \rfloor x_j \right\rfloor \quad \Rightarrow \text{since } x_h \in \mathbb{Z},\ x_j \in \mathbb{Z}$$

$\leq \lfloor \bar{b}_n \rfloor$ $\leftrightarrow$ since "floor" preserve inequality

Therefore, (8) holds for INTEGER VECTORS $\Rightarrow$ these cutting will not kick off any integer vector of IP

$\boxed{\text{Part 2}}$ we need to show $\hat{x}$ violates (8)

$$\hat{x}_n + \sum_{j \in N} \lfloor a_{nj} \rfloor \hat{x}_j = \hat{x}_n \quad (\text{since } \hat{x}_j = 0 \text{ for } j \in N)$$

$$= \bar{b}_n$$

$$> \lfloor \bar{b}_n \rfloor \rightarrow \text{since } b_n \notin \mathbb{Z}, \text{ strictly holds!}$$

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Solving IPs with Gomory Cuts in Dual Simplex Alg.

Suppose we solve the LR to optimality using the simplex algorithm.

The simplex tableau at optimality looks like:

Basic	x^T	Solutions
$-c_B^T x_B$	$c^T - c_B^T B^{-1} A$	$-c_B^T B^{-1} b$
x_B	$B^{-1}A$	$B^{-1}b$

Hence, B is the basis corresponding to $\hat{x}$

① If $\bar{b} = B^{-1} b \in \mathbb{Z}^n \Rightarrow$ solve!

② Search for a row of $\underbrace{\bar{b} = B^{-1}b}_{\bar{b}_n \notin \mathbb{Z}}$ for which the value is not an integer

⇒ Perform a cut!

Include a new basis variable, say s, and then we append the following at the bottom:

$$\sum_{j \in N} (\lfloor \bar{a}_{nj} \rfloor - \bar{a}_{nj}) x_j + s = \lfloor \bar{b}_n \rfloor - \bar{b}_n$$

where $\bar{b} = B^{-1}b$ $\quad \bar{A} = \underbrace{(\text{RHS}) \text{ of } B^{-1}A}_{B^{-1} *}$ $\quad A = (B \mid *)$

Notice that the only basis variable that appears is s

One we add a new row, we obtain a new tableau.

The solution is no longer primal feasible. But it is dual feasible.

And we can solve the problem with dual optimality via the dual simplex algorithm.

之前 → IP → cutting plane

↓

Now → MILP → Gomony cuts!

Consider the following MILP:

$$\boxed{\begin{aligned} \min \quad & c^T x + d^T y \\ \text{s.t} \quad & Ax + By = b \\ & x \geq 0, \quad y \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned}} \rightarrow (10)$$

Assume all variables are rational.

The LR is:

$$\boxed{\begin{array}{ll} \min & c^T x + d^T y \\ \text{s.t} & Ax + By = b \\ & x \geq 0,\ y \geq 0 \end{array}} \rightarrow (11)$$

we only want x to be $\mathbb{Z}$

Suppose we solve (11) via the simplex to optimality.

Let the optimal solution be $(\hat{x}, \hat{y})$

① $\hat{x} \in \mathbb{Z}^n \rightarrow$ fine!

② $\hat{x} \notin \mathbb{Z}^n$:

Consider the optimal tableau. The equality condition is of the form:

$$x_h + \sum_{j \in N} \bar{a}_{hj} x_j = \bar{b}_h$$

Pick h for which x_h is required to be an integer but $\bar{b}_h$ is not an integer.

(Notation: $N \rightarrow$ non-basic variables. and they combine the indices from x and y; x_j represents $\underbrace{\text{both } x \text{ and } y}_{\text{non-basic part}}$)

Denote: $J^+ = \{ j \in N : \bar{a}_{hj} \geq 0 \}$

$J^- = \{ j \in N : \bar{a}_{hj} < 0 \}$

$\beta = \bar{b}_h - \lfloor \bar{b}_h \rfloor > 0$

[Proposition] The following inequality specifies a cutting plane:

$$\boxed{-\sum_{j\in J^+} \bar{a}_{hj} x_j - \frac{\beta}{\beta-1} \sum_{j\in J^-} \bar{a}_{hj} x_j \leq -\beta} \quad -(12)$$

{ ① cuts the optimal soln of LR
② any feasible solution of MILP satisfy !

Pf:

$\boxed{\text{Part 1:}}$ Check that a feasible solution of MILP satisfy (12)

Note that:

$$x_h - \lfloor \bar{b}_h \rfloor = \bar{b}_h - \lfloor \bar{b}_h \rfloor - \sum_{j\in N} \bar{a}_{hj} x_j$$

$$= \beta - \sum_{j\in N} \bar{a}_{hj} x_j$$

Suppose that x_h is an integer $\Rightarrow$ $x_h - \lfloor \bar{b}_h \rfloor$ is integer

① Case: $x_h - \lfloor \bar{b}_h \rfloor \leq 0$ remaining part is all negative !

Then $\sum_{j\in J^+} \bar{a}_{hj} x_j \geq \boxed{\sum_{j\in N} \bar{a}_{hj} x_j} \geq \beta > 0$

Now, by noting that $\frac{\beta}{\beta-1} < 0$, $\bar{a}_{hj} < 0$ for $j \in J^-$, $x_j \geq 0$

we have $\frac{\beta}{\beta-1} \sum_{j\in J^-} \bar{a}_{hj} x_j \geq 0$

$$\Rightarrow -\sum_{j\in J^+} \bar{a}_{nj} x_j - \frac{\beta}{\beta-1} \sum_{j\in J^-} \bar{a}_{nj} x_j \leq -\beta$$

② Case $x_n - \lfloor \bar{b}_n \rfloor \geq 1$:

we then have $\boxed{\sum_{j\in N} \bar{a}_{nj} x_j \leq \beta - 1 < 0}$

Now, note that $0 < \beta < 1$, then

$$\frac{\beta}{\beta-1} \sum_{j\in J^-} \bar{a}_{nj} x_j \geq \frac{\beta}{\beta-1} \sum_{j\in N} \bar{a}_{nj} x_j$$

$$\geq \beta > 0$$

Next, by noting that $\bar{a}_{nj} \geq 0$ for $j \in J^+$ and $x_j \geq 0$

we have $\sum_{j\in J^+} \bar{a}_{nj} x_j \geq 0$

Thus, $-\sum_{j\in J^+} \bar{a}_{nj} x_j - \frac{\beta}{\beta-1} \sum_{j\in J^-} \bar{a}_{nj} x_j \leq -\beta$

PART 2 check that $(\hat{x}, \hat{y})$ gets cut away!

LHS of (12) $= 0$ because $j \in N$

RHS of (12) $= -\beta < 0$

$\Rightarrow$ inequality violates!

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When implementing the Gomory Cut, we would add

$$\boxed{-\sum_{j\in J^+} \bar{a}_{nj} x_j - \frac{\beta}{\beta-1} \sum_{j\in J^-} \bar{a}_{nj} x_j + s = -\beta}$$

$$s \geqslant 0$$

* Note *: In this case, we do not further constrain $s \in \mathbb{Z}$

$$\begin{cases} \text{MILP} \Rightarrow \text{Not Integer Coefficient} \\ \text{LP} \Rightarrow \text{Integer} \end{cases}$$