The Greedy Algorithm — Submodularity

$\begin{cases} \text{Let } N \text{ be a discrete set} \\ \text{Let } f: 2^N \to \mathbb{R} \text{ be some } f^{n} \end{cases}$

Consider this Maximum Prob. :

$$(1) \quad \left[\begin{array}{lll} \max & \sum c_j x_j & \boxed{j \in N} \\ \text{s.t} & \sum_{j \in T} x_j \le f(T) & \text{for all } T \subseteq N. \\ & x_j \ge 0 \quad \forall j & \end{array}\right.$$

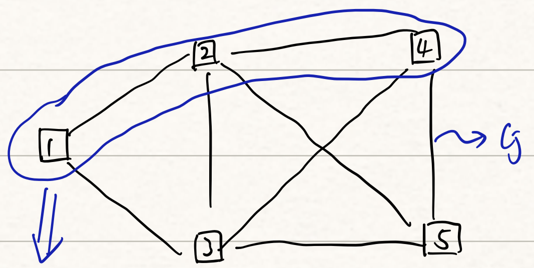
y_T

Maximum given by $f(T)$

Example :

Minimum Spanning True :

N = set of edges $\quad x_j \in \{0,1\}$



$\max \quad \sum c_j x_j$

$f(\{(1,2), (2,4)\}) = \dots$

tell us the constraint of pick rules (edge)

Greedy Alg.

Assume that c_j are sorted so that

$$c_1 \ge c_2 \ge \dots \ge c_k > 0 \ge c_{k+1} \ge \dots \ge c_n$$

$\boxed{\text{Greedy strategy}}$ is to start with choosing x_1 As LARGE As possible without violating constraints

$\Downarrow$

then Pick x_2 As LARGE AS Possible

x_3
x_4
$\vdots$
x_n

Suppose that f is (1) sub-modular (2) non-decreasing

(3) $f(\phi)=0$

Then Greedy Alg. Is Optimal ! optimal soln.

Define $S^j = \{1, 2, 3, \dots, j\}$ $(S^0 = \phi)$

The optimal soln is:

$$x_j = \begin{cases} f(S^j) - f(S^{j-1}) & \text{if: } 1 \le j \le K \\ 0 & j > K \end{cases}$$

$\Rightarrow$ $\boxed{\text{if } c_j < 0}$

$\Downarrow$

(P)

Note: our Maximum Prob. (1) is LP ! $\Rightarrow$ Dual Form (Problem)

$\downarrow$

Outline: we show a dual feasible soln that attains the SAME Objective Value !

The dual of (1) is: $\min_{T \subseteq N} \sum f(T) y_T$ $\Rightarrow$ corresponding to every subset $T \subseteq N$.

(1)-Dual

s.t $\sum_{T: j \in T} y_T \ge c_j$ $j \in N$

$y_T \ge 0$

The dual soln is: $y_T = \begin{cases} c_j - c_{j+1} & \text{if } T = S^j,\ 1 \le j < K \\ c_K & T = S^K \end{cases}$

||

$\Downarrow$

$\overset{\text{max}}{\Uparrow}$ (1) $\leq$ (1)-Dual $\overset{\text{min}}{\nearrow}$

| 0   Otherwise

(D)

$\Downarrow$

we want to find a feasible soln to make this inequality to be equality

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Proposition: Suppose f is : (1) submodular

(2) non-decreasing

(3) $f(\phi) = 0$

Consider (1) & (1)-D, then the solutions (P) and (D) are the respective opt. soln.

Pf:

① Primal Feasibility $\to$ check $\boxed{x_j \text{ is feasible}}$

$\Downarrow$

check that $\boxed{\sum_{j\in T} x_j \leq f(T)}$

①

$$\sum_{j\in T} x_j = \sum_{j\in T, j\leq k} [f(S^j) - f(S^{j-1})]$$

$\boxed{\text{defn of } x_j}$

$\boxed{\text{sub-modularity}}$

$$\leq \sum_{j\in T, j\leq k} [f(S^j \cap T) - f(S^{j-1} \cap T)]$$

$$= f(S^k \cap T) - f(S^0 \cap T)$$

$$= f(S^k \cap T)$$

$$\leq f(T)$$

$\boxed{\text{Non-decreasing}}$

② $x_j \geq 0$ Since f is Non-decreasing.

① + ② $\Rightarrow$ x_j is Primal Feasible

② Dual Feasibility : ① $\sum_{T: j \in T} y_T \leq c_j$

$\boxed{\text{defn of } y_T}$

For $j \leq K$ $\quad \sum_{T: j \in T} y_T = y_{S^j} + \cdots + y_{S^K}$

$= (c_j - c_{j+1}) + \cdots + (c_{K-1} - c_K) + c_K$

$= c_j \geq 0$ $\quad$ ($j \leq K$, c_j non-negative)

$\boxed{\text{defn of } y_T}$

For $j > K$ $\quad \sum_{T: j \in T} y_T = 0 + \cdots + 0$

$\geq c_j$ $\quad$ ($j > K$, c_j negative)

② $y_T \geq 0$ For all $T \subset N$.

easily!

① + ② $\Rightarrow$ y_T is dual Feasible.

③ Check Dual Obj. = Primal Obj.

$$\text{Dual Obj.} = \sum_{T \subseteq N} f(T)\, y_T$$

$$= \sum_{j=1}^{K-1} (c_j - c_{j+1}) f(S^j) + c_K f(S^K)$$

$$= \sum_{j=1}^{K} c_j f(S^j) - \sum_{j=0}^{K-1} c_{j+1} f(S^j)$$

$$= \sum_{j=1}^{K} c_j \left[f(S^j) - f(S^{j-1}) \right]$$

$$= \sum_{j=1}^{K} c_j x_j = \text{Primal Obj.}$$

#

Suppose $f: 2^N \to \mathbb{Z}$ is ~~(1) non-negative~~

(2) Submodular

(3) non-decreasing

(4) $f(\phi) = 0$

(5) Integral

Observation:

$$\Rightarrow x_j = \begin{cases} f(S^j) - f(S^{j-1}), & j \le K \\ 0, & j > K \end{cases}$$

always be INTEGER

Note:

[This is the soln obtained by Greedy Alg.

This means the Greedy Soln is always also the soln of:

$[(1) + \mathbb{Z}] \rightarrow$

$$\begin{aligned} \max \quad & \sum c_j x_j \\ \text{s.t} \quad & \sum_{j \in T} x_j \le f(T) \quad \text{for all } T \subseteq N \\ & x_j \ge 0 \\ & x_j \in \mathbb{Z} \end{aligned}$$

结论：如果 ① $f: 2^N \to \mathbb{Z}$, ② sub-moduler ③ non-decreasing ④ $f(\phi) = 0$

$\Rightarrow$ greedy soln is optimal for $\boxed{(1) + \mathbb{Z}}$

Matroid → Generalize linearly independence

$\Downarrow$

Graph

Defn: Independent System

$\begin{cases} N \to \text{finite set} \\ I \to \text{a collection of } N \end{cases}$

We say that (N, I) is an independence system

$\Leftrightarrow$ ① $\phi \in I$

② $A \in I \Rightarrow B \in I \quad \forall B \subset A$

遗传的

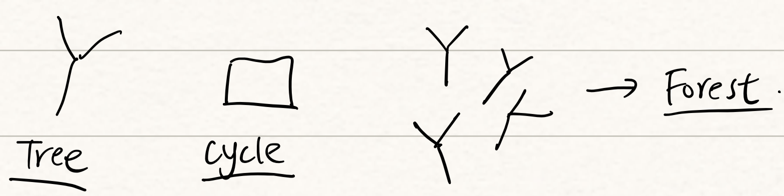
Note: ② is known as a hereditary property

① Example: Consider a Matrix.

$\begin{cases} N \to \text{the columns} \\ I \to \text{subsets of columns that are linearly independent} \end{cases}$

② Example: Consider an undirected graph

$\Rightarrow$ Forest → a subset of edges that do not have any circles



$\begin{cases} N \to \text{the set of edges of the Graph} \\ I \to \text{subset of edges that form a Forest.} \end{cases}$

Notes:

Let (N, I) be an independence System.

1. Sets in I are called independent sets
2. we call $A \subseteq N$ But $A \notin I$ $\to$ dependent set
3. We say $S \in I$ maximally independent with respect to $T \subseteq N$ if $S \cup \{j\}$ is dependent for $\forall j \in T \setminus S$

$\Downarrow$ Something like 极大线性无关组.

4. we call maximally independent subsets of T

$\Downarrow$

base of T

5. Define rank function $r(T)$ to be the size of the largest maximally independent subset of T.

$r(T) = n \Rightarrow \begin{cases} \alpha \subset T \\ \alpha \in I \\ |\alpha| = n \end{cases}$

Defn. Matroid.

We say an independence system (N, I) is a Matroid if Every maximal independence set in F has the same size as $\text{rank}(F)$ for all subsets $F \subseteq N$

Rmk: (Two basic observation about Rank function)

$\Downarrow$

for independence system

1) Rank function are non-decreasing

Pf: let $S \subseteq T$ Let $A \subseteq S$ which is a Basis.

$A \in I$. But $A \subseteq S \subseteq T \Rightarrow A \subseteq T$
①   ②

$\Downarrow$ ①+②

$\text{rank}(T) \geq |A| = \text{rank}(S)$

2) Given any subset $S \subseteq N$ and any $j \in N \setminus S$.

we have $\text{rank}(S \cup \{j\}) \leq \text{rank}\, S + 1$

Pf: pick a Basis Q of $S \cup \{j\}$

(a) if $j \in Q$ since $Q \in I$. then $Q \setminus \{j\} \in I$.

Now $Q \setminus \{j\} \subseteq S \Rightarrow \text{rank}(S) \geq |Q \setminus \{j\}|$

$= |Q| - 1$

$= \text{rank}(S \setminus \{j\}) - 1$

(b) if $j \notin Q$. Then $Q \subseteq S$ $Q \in I$.

$\text{rank}(S) \geq |Q| = \text{rank}(S \cup \{j\})$

$\Rightarrow \text{rank}\, S = \text{rank}(S \cup \{j\})$

☆☆☆ Important

> Thm: (N, I) Independence System.
>
> (N, I) is a matroid $\Leftrightarrow$ $r(\cdot)$ is submodular

Takeaway: If you have a matroid $\Rightarrow$ rank $f^{\underline{n}}$ is submodular

$\Rightarrow$ you can apply Greedy Alg.

The Greedy Alg. for max. index sets

→ Let (N, I) be an independence system

we have a set of weights $c_j \geq 0$

Rank function is $r(\cdot)$

Goal is to find the independent sets of Largest Weights

Can be formulated as follows:

Important Problem.

$$\left[\begin{array}{ll} \max & \sum_j c_j x_j \\ \text{s.t.} & \boxed{\begin{array}{l} \sum_{j \in T} x_j \leq r(T) \\ x_j \in \{0, 1\} \end{array}} \quad \forall T \subseteq N \end{array}\right.$$

⇓ Model: Problem of searching for independence sets

☆ This Problem is hard in General,

☆ But if (N, I) is a matroid, we can apply the Greedy algorithm:

⇓ tells us that the choice of optemal solution

$x_j = r(S^j) - r(S^{j-1}) \in \{0, 1\}$

⇓

the property of rank function

⇓

Greedy soln is feasible!

Punchline: if I have a matroid, then the problems of finding the maximal weight independence set can be SOLVED by the Greedy Algorithm.

Algorithm: (Greedy) specialized to Matroids

1. Sort the weights in decreasing Order

$$c_{j_1} \geq \cdots \geq c_{j_n}$$

2. Initialize $J \leftarrow \phi$

3. For $k = 1, \ldots, n$

   if $J \cup \{j_k\}$ is independent

   then $J \leftarrow J \cup \{j_k\}$

   else:

   $J \leftarrow J$.

Spanning Tree (Minimizing)

$\downarrow$

pick the Largest weight one by one

(if form a tree)

<u>Pf</u> of Matroid $\Leftrightarrow$ $r(\cdot)$ sub-modular.

"$\Rightarrow$" Assume (N, I) is a matroid.

$$\begin{cases} ① \ r(\phi) = 0 \\ ② \ r(\cdot) \text{ is non-decreasing} \end{cases}$$

we want to show that:

$$r(S \cup \{j\}) - r(S) \geq r(S \cup \{j, k\}) - r(S \cup \{k\})$$

holds for $\forall S \subseteq N$ and $j \neq k$, $j,k \notin S$

① if $r(S \cup \{j\}) - r(S) = 1$ ✓

② if $r(S \cup \{j\}) - r(S) = 0$, we need to show that $r(S \cup \{j,k\}) - r(S \cup \{k\}) = 0$

case (i): $r(S \cup \{k\}) = r(S)$

But $r(S) = r(S \cup \{j\}) = r(S \cup \{k\})$

$Q \rightarrow r(S)$

$Q \cup \{j\}$ & $Q \cup \{k\}$ → dependent

~~⇒ adding j or k to S is dependent~~

Matroid

Q is basis of $S \cup \{j,k\}$ ⇒ $r(S \cup \{j,k\}) = r(S) = r(S \cup \{k\})$

case (ii): $r(S \cup \{k\}) = r(S) + 1 = r(S \cup \{j\}) + 1$

$Q \rightarrow S \cup \{k\}$ $\quad Q \setminus \{k\} \in I$ and $Q \setminus \{k\} \subset S$

$\Rightarrow Q \setminus \{k\} \rightarrow S$ since $r(S) = r(S \cup \{k\}) - 1$

$\Rightarrow Q \setminus \{k\} \rightarrow S \cup \{j\} \Rightarrow Q \rightarrow S \cup \{j,k\}$

$\Rightarrow r(S \cup \{j,k\}) = r(S \cup \{k\})$

if $Q \cup \{j\}$ Independent

$\Rightarrow Q \cup \{j\} \setminus \{k\}$ Independent

Contradiction

⇐ Assume $r(\cdot)$ is sub-modular, we want to show (N, I) is matroid.

Suppose not, there is some set T for which there are maximal independent sets S_1 and S_2 $\quad S_1 \subseteq T$, $S_2 \subseteq T$ s.t. $|S_1| < |S_2|$

Because $r(\cdot)$ is submodular & non-decreasing, we have: $r(S_2) \leq r(S_1) + \sum_{j \in S_2 \setminus S_1} [r(S_1 \cup \{j\}) - r(S_1)]$

$$r(S_1) = |S_1| < |S_2| = r(S_2)$$

$\Rightarrow$ for some $\boxed{j}$ $r(S_1 \cup \{j\}) - r(S_1) > 0$

$j \in S_2 \subseteq T$

$\Downarrow$

$r(S_1 \cup \{j\}) - r(S_1) = 1$

$\Rightarrow$ $S_1 \cup \{j\}$ is independent

$\Rightarrow$ S_1 is not maximally independent

$\Downarrow$

Contradiction