Recall:

$A \in \mathbb{Z}^{m\times n}$ ($m \le n$) is unimodular if:

① every sub m columns of A has determinant $\{+1, 0, -1\}$.

to difficult to check

$\boxed{\text{we should find a simple sufficient condition (to check)}}$

relaxation

② $\min_x \; c^T x$ s.t. $Ax = b$, $x \ge 0$ $\Longleftrightarrow$ $\min \; c^T x$ s.t. $Ax = b$, $x \ge 0$, $x \in \mathbb{Z}^n$

③ extreme points of $P = \{x : Ax = b, x \ge 0\}$ are all integer for all int b

---

Total unimodularity. $\Rightarrow$ for a more general LP form

Defn: $A \in \mathbb{Z}^{m\times n}$ TU if:

the determinant of every SQUARED SUBMATRIX is in $\{+1, 0, -1\}$

Example: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rightsquigarrow$ squared-submatrix $\begin{cases} 1\times1\text{-submatrix} \to \text{entry.} \\ 2\times2\text{-sub matrix} \end{cases}$

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

determinant: -1 $\quad 1 \quad 1$

$\Downarrow$

TU

Example: $\begin{pmatrix} n+1 & n \\ n & n-1 \end{pmatrix}$ $\to$ unimodular

$\to$ not TU $(n \ne 0)$ $\Rightarrow$ the value of entry is not $\{-1, 0, +1\}$

$\to$ $n = 0 \Rightarrow$ TU

Note: if $A \in TU$, every entry of A must be $\{-1, 0, +1\}$

1×1 squared matrix of A.

Q: How to check a matrix is TU?

$\Downarrow$

it's easy to check if a matrix is not TU

$\begin{pmatrix} \cdots & 1 & \cdots & -1 & \cdots \\ \cdots & -1 & \cdots & -1 & \cdots \end{pmatrix} \longrightarrow$ not TU

[Thm]. Let $A \in \mathbb{Z}^{m\times n}$, then A is TU

$\Leftrightarrow$ $\{x : Ax \le b,\ x \ge 0\}$ is integral for all $b \in \mathbb{Z}^m$

$\hookrightarrow$ polyhedron $P(b)$ ($P(b)$ is nonempty)

Compare:

Last week, we have:

$A \to$ Unimodular $\Leftrightarrow$ $P(b) = \{x : Ax = b,\ x \ge 0\}$ is integral

Consider the ILP $-(1)$

$$\left.\begin{aligned} &\min\ c^T x \\ &\text{s.t}\ \ Ax \le b \\ &\qquad x \ge 0 \\ &\qquad x \in \mathbb{Z}^n \end{aligned}\right\} \to (1)$$

Suppose that $A \in TU$, $b \in \mathbb{Z}^m$, then the following (linear) relaxation of (1) is right!

$\hookrightarrow$ a sufficient condition

$\hookrightarrow$

$$\left.\begin{aligned} &\min\ c^T x \\ &\ \ x \end{aligned}\right\} \to (2)$$

(Since all the extreme points are all

s.t $Ax \le b$

$x \ge 0$

integer points)

① In particular, there exists an optimal solu., that is integer (provided the solution is finite)

② If A is not TU, then there exists a $b \in \mathbb{Z}^m$ and a $c \in \mathbb{R}^n$ such that $(2) < (1)$

$\Downarrow$

Actually exists a Polyhedron $P(b)$ not integral

$\downarrow$

define a $c^T x$ objective on $P(b)$ (construct)

s.t opt. attains in those non-int extreme point

$\Downarrow$

optimal attains at extreme point.

$\boxed{A}$

①

[Prop]. $A \in TU \iff [\overset{n}{A} \mid I_{m\times m}] \in U$

Hint: [Lemma] let $X = \begin{array}{cc} \left(\begin{array}{c:c} A & 0 \\ \hdashline B & I \end{array}\right) & \begin{array}{c} m \\ n-m \end{array} \\ \begin{array}{cc} m & n-m \end{array} & \end{array} \Rightarrow |X| = |A|$

Exercise!

Pf: $\begin{pmatrix} A & 0 \\ B & I \end{pmatrix} \longrightarrow \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$

②

[Prop]: Let b be a vector is integer entries.

$\{x: Ax \le b, x \ge 0\}$ is integral

$\Updownarrow$

$\{(x,y): Ax + Iy = b, x, y \ge 0\}$ is integral

slack variable

Exercise

① + ② $\Rightarrow$ [THM]

$\Downarrow$

Pf of [Thm]:

prop1: $A \in TU \Leftrightarrow [A \mid I_{m \times m}] \in U$

prop2: $\{x : Ax \leq b, x \geq 0\}$ is integral

$\Updownarrow$

$\{(x,y) : Ax + Iy = b, x, y \geq 0\}$ is integral

slack variable

Thm: $A \in TU \Leftrightarrow \{x : Ax \leq b, x \geq 0\}$ integral for all int b.

Pf: $A \in TU \Leftrightarrow [A \mid I_{m \times m}] \in U$

$\Leftrightarrow \{(x,y) : Ax + Iy = b, x, y \geq 0\}$ integral for all int b

$\Leftrightarrow \{x : Ax \leq b, x \geq 0\}$ integral for all int b.

Next, we will see an alternative characterization of TU

$\downarrow$

well suited for network application

Defn: (Equitable bicolouring)

$A^{m \times n}$

we say that A admits an equitable column bicolouring

if it is possible to partition the matrix A such that:

the difference between the sum of each column in each

partition is a vector $\in \{-1, 0, +1\}$ a vector

i.e. $\sum_{i \in J} A_i - \sum_{i \in J^c} A_i \in \{-1, 0, +1\}$

i.e $\left| \sum_{i \in J} A_i - \sum_{i \in J^c} A_i \right| \leq 1^m$

[Thm]: $A \in \mathbb{Z}^{m \times n}$ TU $\Leftrightarrow$ every sub-matrix attained by taking a subset of columns of A admits an equitable column bicolouring.

Example: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

- 1 column $\to \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \boxed{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$ → convention
- 2 columns $\to \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{J} - \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{J^c}$
- 3 columns $\to \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{J} - \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{J^c}$

guarantee that:

the entry of A just can be $\{-1, 0, +1\}$

Example: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightsquigarrow$ not TU since itself is not equitable column bicolouring

---

How to express a network/graph in an ILP?

↓

Given an undirected graph $G = (V, E)$, the node-vetex incidence matrix A is the matrix where:

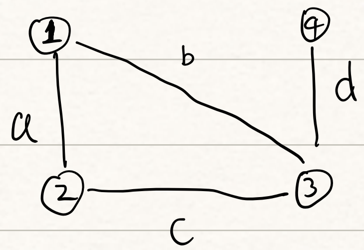
$$\begin{cases} \text{rows} = \text{vertices } V \\ \text{columns} = \text{edges } E \end{cases}$$

$$A_{\text{vertex } i, \text{ edge } j} = \begin{cases} 1, & \text{if vertex } i \text{ is in edge } j \\ 0, & \text{else} \end{cases}$$

edge

vertex

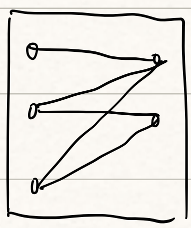
Example:



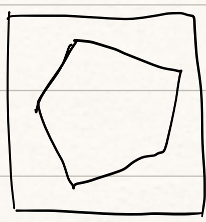
$\Leftrightarrow$ $A = \begin{array}{c|c} & \text{edge} \\ & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \text{vertex} & \end{array}$

Reminder: A Graph $G=(V,E)$ is bi-partite if:

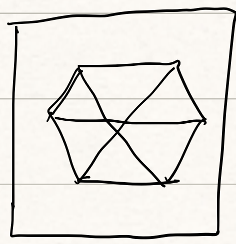
it is possible to partition $V = V_1 \cup V_2$ s.t there are no edges with V_1 or V_2



bi-partite



not bi-partite



bi-partite

[props]: The node-edge incidence matrix A of an undirected bipartite graph is TU

Pf: show that $A \in TU$

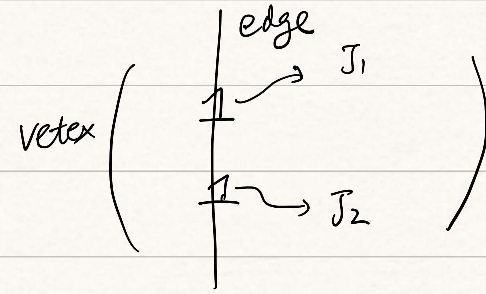
~~Let J be any subset of rows of A~~

> Observation: $A \in TU \Leftrightarrow A^T \in TU$
>
> Corollary: $A \in TU \Leftrightarrow$ Every submatrix attained by taking subsets of row of A admits an equitable row bicoloring

Suppose that the set of vertices can be patitioned to V_1 & V_2 and there are no edges in V_1 or V_2.

Define a subset J of rows of A

$$\boxed{\begin{aligned} J_1 &:= J \cap V_1 \\ J_2 &:= J \cap V_2 \end{aligned}}$$



guarantees the summation cannot reach 2

otherwise, this means there is an edge in V_1 !

basic idea

① use the row bi-colouring criterion

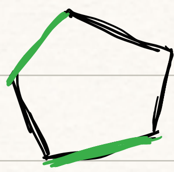
② $J_1 = J \cap V_1$ $J_2 = J \cap V_2$

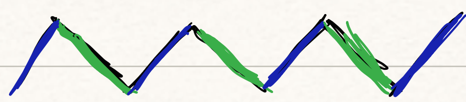
③ Check

Application (matching in graphs)

defn: matching is a subset of disjoint edges of a graph.

Example :





- larger matching

Problems of seeking the largest matching of a graph can be phrased as an ILP as follows

$$\begin{cases} E \\ V \\ \delta(v) \rightsquigarrow \text{the edges connected to } v \end{cases}$$

$$\max \sum_{e \in E} x_e$$

$$\text{s.t.} \quad \underbrace{\sum_{e \in \delta(v)} x_e \le 1}\quad \text{for all } v \in V$$

for one v, pick at most 1 edge

$\Downarrow$

match

$$\underbrace{x_e \in \{0,1\}}_{\text{binary}} \quad \text{for all } e \in E$$

variable according to edge

Actually, this prob. can be rewritten in terms of its incidence matrix A.

$\Downarrow$

$$\max \sum_{e \in E} x_e$$

$$\text{s.t.} \quad Ax \le \mathbb{1}$$

$$x_e \in \{0,1\}$$

edge

$A = \begin{pmatrix} \cancel{1 \quad 1 \quad 1} \end{pmatrix}$ vertex

$\boxed{x = (x_e)}$

count the # of vertex appears in our choice of edges

One can show that if the $\boxed{\text{graph is bi-partite}}$ then the following relaxation is tight

$\boxed{TU \Leftrightarrow A}$

$\Downarrow$

$\boxed{\begin{bmatrix} A \\ I \end{bmatrix} \rightsquigarrow TU}$

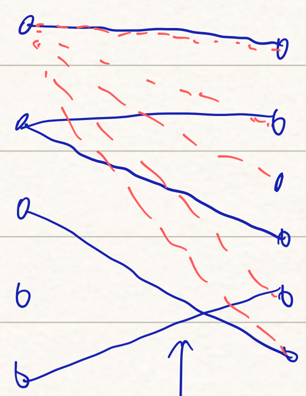
$$\max \sum x_e$$

$$\text{s.t.} \quad Ax \le \mathbb{1}$$

$$\boxed{0 \le x_e \le 1} \rightarrow \text{relaxation}$$

Note: In practice, we solve this by an augmenting path algorithm, rather then solving the LP directly.

[Rmk]: what kind of pratical case can be phrased as finding a matching? (Application)



the matching of a bi-partite graph.
drivers/taxi
people who need rides
edge means drivers
can offer ride to some passengers