Static Game → { IESDS, NE }

↓

$\boxed{\text{Strategy Space}}$

Prisoners' Dilemma

	H	C
H	-1, -1	-20, 0
C	0, -20	-5, -5

Example

Split the Bill

	C	E
C	2, 2	-3, 3
E	3, -3	-2, -2

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1. Normal-form of GAME

① n player

② Strategy space $S_1, \ldots, S_n$ $\Rightarrow$ $\boxed{G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}}$

③ payoff function $u_1, \ldots, u_n$

2. Strictly dominate

→ s_{11} is strictly dominated by s_{12} $\quad s_{11}, s_{12} \in S_1$

$\Leftrightarrow$ $u_1(s_{11}, s_{-1}) < u_1(s_{12}, s_{-1}) \quad \forall s_{-1} \in S_{-1}$

⇒ 3. IESDS

⇒ 4. Nash Equilibria $(s_1^*, \ldots, s_n^*)$

① Best Response $R_i(s_{-i})$

$$= \underset{s_i \in S_i}{\operatorname{argmax}}\ u_i(s_i, s_{-i})$$

$\boxed{\text{can be}}$
{ empty set, singleton, finite | infinite

② NE $(s_1^*, \dots, s_n^*)$ $\boxed{s_{-i}^* = (s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_n^*)}$

$\Leftrightarrow \quad s_i^* \in R_i(s_{-i}^*)$

$\Leftrightarrow \quad u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*)$

③ use GRAPH to find NE $G(R_i)$ ⇒ Bimatrix Method

$G(R_i) := \{ (s_i, s_{-i}) : s_i \in R_i(s_{-i}) \quad s_{-i} \in S_{-i} \}$

Then $(s_1^*, \dots, s_n^*) \in NE$

$\Leftrightarrow \quad s_i^* \in R_i(s_{-i}^*) \qquad i = 1, 2, \dots, n$

$\Leftrightarrow \quad (s_i^*, s_{-i}^*) \in G(R_i) \qquad i = 1, 2, \dots, n$

$\Leftrightarrow \quad (s_1^*, \dots, s_n^*) \in \bigcap_i G(R_i)$

Battle of Sexes

Some examples

	O	F
O	2,1	0,0
F	0,0	1,2

3 - players game → tri-matrix

		A3	B3
A1	A2	-1, 0, 2	3, 5, 1
	B2	5, -2, 3	-2, 1, 5
B1	A2	0, 2, 1	2, 2, 5
	B2	2, -3, -3	$\boxed{-1, 4, 0}$ → NE

Result

① → $\boxed{NE \subseteq IESDS}$

→ Application: we can do IESDS first (in order to find NE quickly)

Pf: (KEY POINT) suppose that $\hat{s}$ is the first dominated strategy in NE $s^* = (s_1^*, \ldots, s_n^*)$

② $|IESDS| = 1 \Rightarrow \boxed{NE = IESDS}$ (unique NE)

Example:

① SPLIT ONE DOLLAR

$S_1 = [0, 1]$

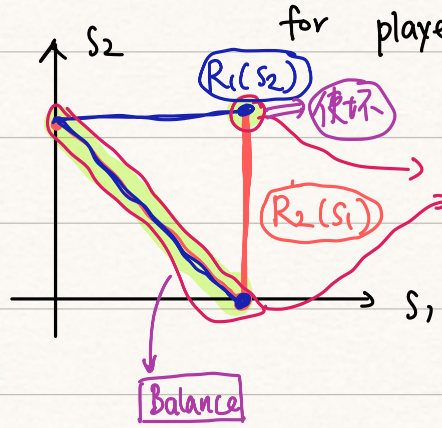
$S_2 = [0, 1]$

$s_1 + s_2 > 1 \rightarrow$ Both 0

$s_1 + s_2 < 1 \rightarrow \begin{cases} \pi_1 : s_1 \\ \pi_2 : s_2 \end{cases}$

Best Response for player 1: $R_1(s_2) = \begin{cases} 1 - s_2 & 0 \le s_2 < 1 \\ [0,1] & s_2 = 1. \end{cases}$

for player 2: $R_2(s_1) = \begin{cases} 1 - s_1 & 0 \le s_1 < 1 \\ [0,1] & s_1 = 1. \end{cases}$



② $\boxed{\text{Cournot Duopoly}}$

$\pi_1 = q_1 (P(Q) - c)$

$\pi_2 = q_2 (P(Q) - c)$

$$\boxed{P(Q) = \begin{cases} a - Q & Q < a \\ 0 & Q \ge a \end{cases}}$$

☆ if $(q_1^*, q_2^*) \in NE$

we must have $q_1^* \,\&\, (q_2^*)$

Best Response for Player 1:

$\in [0, a-q_2^*]$

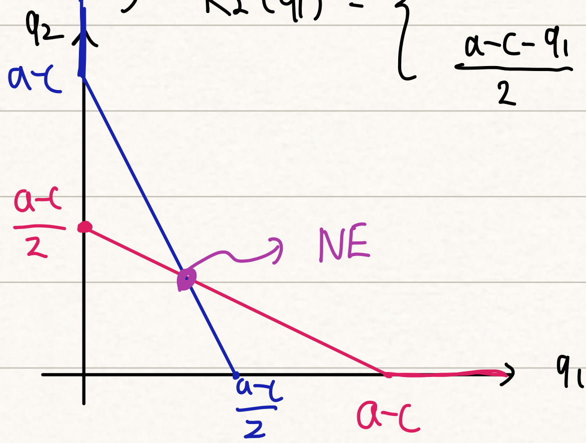
$([0, a-q_1^*])$

$$\rightarrow R_1(q_2) = \begin{cases} 0 , & q_2 \geq a-c \\ \frac{a-c-q_2}{2} , & 0 < q_2 < a-c \end{cases}$$

divide into

$$\begin{cases} q_2 > a \rightarrow \pi_1 = q_1 \cdot (-c) \\ a-c < q_2 < a \rightarrow \pi_1 = q_1(a-q_1-q_2-c) \\ 0 < q_2 < a-c \rightarrow \pi_1 = q_1(a-q_1-q_2-c) \end{cases}$$

$$\rightarrow R_2(q_1) = \begin{cases} 0 , & q_1 \geq a-c \\ \frac{a-c-q_1}{2} , & 0 < q_1 < a-c \end{cases}$$



$$\begin{cases} q_1 = \frac{a-c-q_2}{2} \\ q_2 = \frac{a-c-q_1}{2} \end{cases} \Rightarrow q_1^* = q_2^* = \frac{a-c}{3}$$

③ $\boxed{\text{Bertrand Model}}$ $\Rightarrow$ $\boxed{\text{not useful}}$

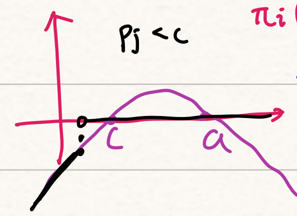
consider $\boxed{\text{homogeneous Products}}$ $\rightarrow q_i(p_i, p_j) = \begin{cases} a-p_i & p_i < p_j \\ 0 & p_i > p_j \\ \frac{a-p_i}{2} , & p_i = p_j \end{cases}$

$\boxed{\pi_i(p_i, p_j)}$

$$\pi_i = q_i(p_i - c)$$

$$= \begin{cases} (a-p_i)(p_i-c) & p_i < p_j \\ 0 & p_i > p_j \\ \frac{(a-p_i)(p_i-c)}{2} & p_i = p_j \end{cases}$$

$\pi_i(p_i)$



$\Rightarrow p_j < c$ $\quad \phi$

$p_j = c$ $\quad [c, +\infty)$

$p_j > c$ & $p_j \leq \frac{a+c}{2}$ $\quad \phi$

$p_j > \frac{a+c}{2}$ $\quad \frac{a+c}{2}$

从 Deviate 的角度分析会容易些

First, if $(p_1^*, p_2^*) \in NE$, then $c \leq p_i^* \leq a$

if not, W.L.O.G $\boxed{p_1^* < c}$ or $\boxed{p_1^* > a}$

if one player's payoff

Case ① $p_1^* < c$ if $p_1^* \leq p_2^*$

then $\pi_1(p_1^*, p_2^*) < 0 \Rightarrow$ of course not NE is negative, we can always change his own price to make it positive

if $p_1^* > p_2^*$, $\boxed{p_2^* < c}$

$\pi_1(p_1^*, p_2^*) = 0$

$\pi_2(p_1^*, p_2^*) < 0 \Rightarrow$ of course not NE

case ② $p_1^* > a$ if $p_1^* \le p_2^*$

then $\pi_1(p_1^*, p_2^*) < 0 \Rightarrow$ of course not NE

if $p_1^* > p_2^*$

then $\boxed{\pi_1(p_1^*, p_2^*) = 0}$

$\hookrightarrow$ we can decrease the value of p_1^* to make π_1 positive

$\Downarrow$

$\boxed{\text{not NE}}$

$\boxed{\text{in the sense of NE}}$

$\boxed{\text{Therefore, we only need to consider } S_i = [c, a] \text{ case}}$

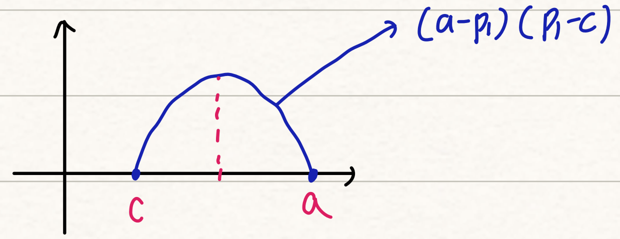
Then consider Best Response for $\boxed{\text{Player 1}}$

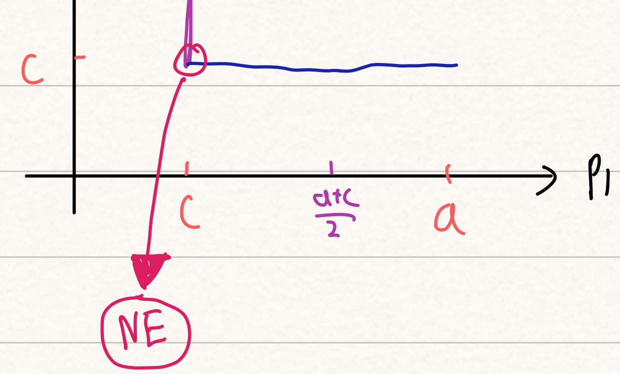
$$R_1(p_2) = \underset{p_1}{\text{argmax}}\ \pi_1(p_1, p_2) = \begin{cases} \frac{a+c}{2}, & a \geq p_2 > \frac{a+c}{2} \\ \phi, & c < p_2 \leq \frac{a+c}{2} \\ [c, a], & p_2 = c \end{cases}$$

$$= \underset{c \leq p_1 \leq a}{\text{argmax}} \begin{cases} (a-p_1)(p_1-c) & p_1 < p_2 \\ \frac{1}{2}(a-p_1)(p_1-c) & p_1 = p_2 \\ 0, & p_1 > p_2 \end{cases}$$

Actually we can analyze the Best Response for $p_2 \in [0, +\infty)$

(here we only attain for $p_2 \in [a, c]$





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④ Problem of commons

1. $n \to$ farmers $\Rightarrow$ # of players

2. $\boxed{u_i(g_i, g_{-i}) = g_i\left(v(g_i + g_{-i}) - c\right)}$ $\Rightarrow$ payoff function

Best Response for player i:

$$R_i(g_{-i}) = \arg\max_{g_i} u_i(g_i, g_{-i})$$

$$= \arg\max_{g_i} g_i V(g_i + g_{-i}) - g_i c$$

if $(g_1^*, \dots, g_n^*) \in NE$,

then $g_i^* \in \arg\max_{g_i} g_i V(g_i + g_{-i}^*) - g_i c$

$\Rightarrow$ (a necessary condition) $g_i^* v'(g_i^* + g_{-i}^*) + v(g_i^* + g_{-i}^*) - c = 0$

$\Rightarrow$ $\boxed{G^* v'(G^*) + nV(G^*) - nc = 0}$ $\Rightarrow$ Comparison Between Social Optima

for social optima, i.e., maximize $\sum_{i=1}^{n} g_i V(g_i + g_{-i}) - g_i c$

$\Leftrightarrow$ max $G V(G) - G \cdot c$

if $G^{**} = \arg\max G V(G) - G c$

then (a necessary condition) $\boxed{V(G^{**}) + G^{**} v'(G^{**}) - c = 0}$

strictly ↗

for NE, $V(G^*) + G^*/n \, v'(G^*) - c = 0$

strictly ↗

for Social Optima, $V(G^{**}) + G^{**} v'(G^{**}) - c = 0$

if $G^* \leq G^{**}$, then

$$\begin{cases} ① \ V(G^*) \geq V(G^{**}) \\ ② \ \boxed{V'(G^{**}) \leq V'(G^*)} \leq 0 \end{cases}$$

since $V'' \leq 0 \Rightarrow V'$ decreasing

$$G^{**} \geq G^* \rightarrow G^{**} > \frac{G^*}{n}$$

$$\Rightarrow \quad V'(G^{**})\, G^{**} < V'(G^*) \cdot \frac{G^*}{n}$$

$\Rightarrow$ Contradiction

$\Rightarrow \boxed{G^* > G^{**}} \Rightarrow$ social optima has fewer goats

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Final-Offer Arbitration

strategy space

$$\begin{cases} \text{Firm} \rightarrow W_f \\ \text{union} \rightarrow W_u \end{cases}$$

judgement $x \sim F(\cdot)$

$(F(x_m) = \frac{1}{2}) \rightarrow x_m$: ½分位数

tricky part → define payoff

Expectation of wage : $\phi(W_f, W_u) = W_f \cdot P(W_f \text{ is chosen}) + W_u P(W_u \text{ is chosen})$

$$= W_f \cdot \left(1 - F\left(\frac{W_f + W_u}{2}\right)\right) + W_u \cdot F\left(\frac{W_f + W_u}{2}\right)$$

$$\Rightarrow \boxed{\begin{cases} W_f^* \in \underset{W_f}{\operatorname{argmax}}\ \phi(W_f, W_u^*) \\ W_u^* \in \underset{W_u}{\operatorname{argmin}}\ \phi(W_f^*, W_u) \end{cases}} \Leftrightarrow (W_f^*, W_u^*) \in NE$$

$$\Rightarrow \begin{cases} W_f^* \cdot \frac{1}{2}\left(-f\left(\frac{W_f^* + W_u^*}{2}\right)\right) + \left(1 - F\left(\frac{W_f^* + W_u^*}{2}\right)\right) + W_u^* f\left(\frac{W_f^* + W_u^*}{2}\right) \cdot \frac{1}{2} = 0 \\ W_f^* \cdot \frac{1}{2}\left(-f\left(\frac{W_f^* + W_u^*}{2}\right)\right) + W_u^* \cdot f\left(\frac{W_f^* + W_u^*}{2}\right) \cdot \frac{1}{2} + F\left(\frac{W_f^* + W_u^*}{2}\right) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{2} f\left(\frac{W_f^* + W_u^*}{2}\right) \cdot \left[W_f^* - W_u^*\right] = 1 - F\left(\frac{W_f^* + W_u^*}{2}\right) \\ \frac{1}{2} f\left(\frac{W_f^* + W_u^*}{2}\right) \cdot \left[W_f^* - W_u^*\right] = F\left(\frac{W_f^* + W_u^*}{2}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{W_f^* + W_u^*}{2} = x_m \\ W_f^* - W_u^* = \frac{1}{f(x_m)} \end{cases} \Rightarrow \begin{cases} W_f^* = x_m + \frac{1}{2f(x_m)} \\ W_u^* = x_m - \frac{1}{2f(x_m)} \end{cases}$$

Matching Pennies

	H (p)	T ($1-p$)
(q) H	$\underline{1}, -1$	$-1, \underline{1}$
($1-q$) T	$-1, \underline{1}$	$\underline{1}, -1$

→ No $\boxed{\text{nash Equillibria}}$ ⇒ introduce $\boxed{\text{mix-strategy}}$

$\boxed{\text{Mix strategy}}$

1. n player
2. Strategy Space

$$\begin{cases} S_1 = \{S_{11}, \dots, S_{1k_1}\} \to k_1 \text{个 strategy} \\ \vdots \\ S_n = \{S_{n1}, \dots, S_{nk_n}\} \to k_n \text{个 strategy} \end{cases}$$

pure strategy

Mix-strategy

$$\begin{cases} P_1 = \{P_{11}, \dots, P_{1k_1}\} \\ \vdots \\ P_n = \{P_{n1}, \dots, P_{nk_n}\} \end{cases} \qquad \boxed{\sum_{k=1}^{k_i} P_{ik} = 1}$$

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2 players Mix-strategy Nash Equilibria

推广至 n-player

$$\begin{cases} S_1 = \{S_{11}, \dots, S_{1K}\} \\ S_2 = \{S_{21}, \dots, S_{2J}\} \end{cases} \Rightarrow \begin{cases} P_1 = \{P_{11}, \dots, P_{1K}\} \\ P_2 = \{P_{21}, \dots, P_{2J}\} \end{cases}$$

$$V_1(P_1, \dots, P_n) = \sum_{i_1 \dots i_n} P_{1i_1} \cdots P_{ni_n} u_1(S_{1i_1} \dots S_{ni_n})$$

payoff $\begin{cases} u_1(S_1, S_2) \\ u_2(S_1, S_2) \end{cases}$ ⇒ expected payoff

$$\begin{cases} V_1(P_1, P_2) = \sum_{k=1}^{K} P_{1k} V_1(S_{1k}, P_2) \\ \quad = \sum_{k=1}^{K} \sum_{j=1}^{J} P_{1k} P_{2j} u_1(S_{1k}, S_{2j}) \\ V_2(P_1, P_2) = \sum_{j=1}^{J} P_{2j} V_2(P_1, S_{2j}) \end{cases}$$

$$= \sum_{j=1}^{J} \sum_{k=1}^{K} p_{2j}\, p_{1k}\, u_2(s_{1k}, s_{2j})$$

$(p_1^*, p_2^*) \in NE$

$$\Leftrightarrow \begin{cases} p_1^* \in \underset{p_1}{\arg\max}\ V_1(p_1, p_2^*) \\ p_2^* \in \underset{p_2}{\arg\max}\ V_2(p_1^*, p_2) \end{cases} \qquad \begin{cases} p_1 = (p_{11}, \dots, p_{1K}) & p_{11} + \dots + p_{1K} = 1 \\ p_2 = (p_{21}, \dots, p_{2J}) & p_{21} + \dots + p_{2J} = 1 \end{cases}$$

Re-consider Pennies Matching

$$\begin{array}{cc|cc} & & q & 1-q \\ & & H & T \\ p & H & 1,-1 & -1,1 \\ 1-p & T & -1,1 & 1,-1 \end{array}$$

$p_1 = (p, 1-p)$

$p_2 = (q, 1-q)$

①

$$\Rightarrow V_1(p_1, p_2) = p\, V_1(H, p_2) + (1-p) V_1(T, p_2)$$
$$= p(q+q-1) + (1-p)(-q-q+1)$$

② $V_2(p_1, p_2) = q\, V_2(p_1, H) + (1-q) V_2(p_1, T)$

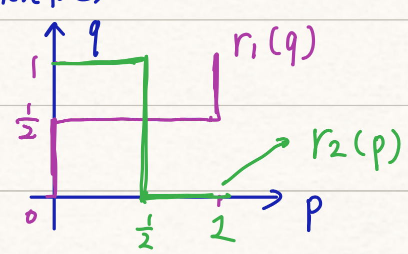
$$= q(-p+1-p) + (1-q)(p+p-1)$$
$$= q(1-2p) + (1-q)(2p-1)$$

Best Response for player 1:

$$r_1(p_2) = r_1(q) = \underset{p_1}{\arg\max}\ V_1(p_1, p_2)$$
$$= \underset{0 \le p \le 1}{\arg\max}\ p(2q-1) + (1-p)(1-2q)$$
$$= \begin{cases} [0,1] & q = \frac{1}{2} \\ 1 & q > \frac{1}{2} \\ 0 & q < \frac{1}{2} \end{cases}$$

$$r_2(p_1) = r_2(p) = \underset{p_2}{\arg\max}\ V_2(p_1, p_2)$$
$$= \underset{0 \le q \le 1}{\arg\max}\ V_2(p_1, p_2)$$
$$= \begin{cases} [0,1] & p = \frac{1}{2} \\ 1 & p < \frac{1}{2} \\ 0 & p > \frac{1}{2} \end{cases}$$

Therefore,



$$\Rightarrow NE = \{ \tfrac{1}{2}H + \tfrac{1}{2}T,\ \tfrac{1}{2}H + \tfrac{1}{2}T \}$$

or $NE = \{ p_1^* = (\frac{1}{2}, \frac{1}{2}),\ p_2^* = (\frac{1}{2}, \frac{1}{2}) \}$

Another Result:

if pure strategy $s_{kj} \in S_j$ is strictly dominated by some strategy, then the strategy will be play with probability 0 in any mix-strategy NE. (i.e., $p^*_{kj} = 0$)

Motivation: Analysis: $NE = \{p_1^*, \dots, p_n^*\}$

Given $p_{-1}^* = (p_2^*, \dots, p_n^*)$

$$V_1(p_1^*, p_{-1}^*) = p_{11}^* \underbrace{V_1(s_{11}, p_{-1}^*)}_{\text{const}} + \cdots + p_{1K}^* \underbrace{V_1(s_{1K}, p_{-1}^*)}_{\text{const}}$$

$$\downarrow$$

$$p_1^* = \arg\max_{p_1} V_1(p_1, p_{-1}^*)$$

if $p_{1i}^* > 0$, then $V_1(s_{1i}, p_{-1}^*)$ must be the biggest!

(may have many)

$\downarrow$

is strictly dominated means this term cannot be the biggest one!

Rigorous Pf (By contradiction)

Suppose that $\exists\, p_{1i}^* > 0$, but it is eliminated by IESDS.

W.L.O.G Then we pick the FIRST eliminated one in NE with non-zero probability, namely $\boxed{s_{1i}^*}$ and $\boxed{p_{1i}^* > 0}$

we suppose the strategy is player 1's

$s_{1i}^* \in S_1$

since it is elimated, then there exists $\hat{s}_1 \in S_1$, s.t

$u_1(\hat{s}_1, s_{-1}) > u_1(s_{1i}^*, s_{-1})$ $\forall s_{-1} \in S_{-1}$ for those not eliminated strategy

→ then consider all combination of strategies in mix-strategy whose probability is $\boxed{\text{non-zero}}$ ⇒ these combination hasn't been eliminated yet

$\Rightarrow \boxed{u_1(\hat{s}_1, s_{-1}^*) > u_1(s_{1i}^*, s_{-1}^*)}$ $\boxed{s_{-1}^* \text{ is } \boxed{\text{such combination}}}$

$J_2 = \{i;\ p_{2i}^* \neq 0\}$

$\vdots$

$J_n = \{i:\ p_{ni}^* \neq 0\}$

suppose in p_1^*, $\hat{s}_1 \to$ probability $= \tilde{p}_1$, $s_{1i} = p_{1i}^*$

Consider $\tilde{p}_1^* = \begin{cases} p_1^* & s_1 \neq \hat{s}_1 \ \&\ s_{1i} \\ 0 & s_1 = s_{1i} \\ \tilde{p}_1 + p_{1i}^* & s_1 = \hat{s}_1 \end{cases}$, then $\tilde{p}_1^*$ is still a probability distribution

$s_{-1}^* = \{(s_{2i_2}, \dots, s_{ni_n}) \in S_2 \times \cdots \times S_n :\ i_2 \in J_2, \dots, i_n \in J_n\}$

moreover, $u_1(\tilde{p}_1^*, p_{-1}^*) > u_1(p_1^*, p_{-1}^*)$ must hold

since $u_1(\tilde{p}_1^*, p_{-1}^*) = \sum_i \tilde{p}_{1i}^* v_1(s_1, p_{-1}^*)$

Pf of $V_1(\tilde{s}_1, p_{-1}^*) > V_1(s_{1i}^*, p_{-1}^*)$

$\rightarrow V_1(\tilde{s}_1, p_{-1}^*) = \sum_{j_2 \dots j_n \in J} p_{2j_2}^* \cdots p_{nj_n}^* u_1(\tilde{s}_1, s_{2j_2}, \dots s_{nj_n})$

$< \sum_{j_2 \dots j_n \in J} p_{2j_2}^* \dots p_{nj_n}^* u_1(s_{1i}^*, s_{-1})$

non-zero

$= \sum_{e_2 \dots e_n} p_{2e_2}^* \cdots p_{ne_n}^* u_1(s_{1i}^*, s_{-1})$

including 0

$= V_1(s_{1i}^*, p_{-1}^*)$

same part

$\boxed{\text{Const}} + (\hat{p}_1 + p_{1i}^*) \cdot u_1(\tilde{s}_1, p_{-1}^*)$

same part

$> \boxed{\text{const}} + \tilde{p}_1 u_1(s_i^*, p_{-i}^*) + p_{1i}^* u_1(s_{1i}, p_{-1}^*)$

$= u_1(p_1^*, p_{-1}^*)$

So, when we focus on NE (or mix-strategy NE)

first, we can do IESDS

$\begin{cases} \text{for pure-strategy NE} \rightarrow \text{nothing will be changed} \\ \text{for mix-strategy NE} \rightarrow \text{we should remember to} \end{cases}$

assign 0-probability to those

eliminated strategies

Betrand Model: (Non-homogeneous Products)

① Player # = 2

② $S_1 = [0, +\infty)$; $S_2 = [0, +\infty)$

③ $u_1 = q_1(p_1 - c)$

$= (a - p_1 + b p_2)(p_1 - c)$

$u_2 = q_2(p_2 - c)$

$= (a - p_2 + b p_1)(p_2 - c)$

$\longrightarrow$ figure out Best Response $R_1(s_2)$ & $R_2(s_1)$

$$R_1(s_2) = \underset{s_1 \in [0,+\infty)}{\arg\max}\ u_1(s_1, s_2)$$

$$= \underset{s_1}{\arg\max}\ (a - s_1 + b s_2)(s_1 - c)$$

$$a s_1 - s_1^2 + b s_2 s_1$$

$$\cancel{-ac} + c s_1 - \cancel{bcs_2}$$

$$(a + c + b s_2) s_1 - s_1^2$$

$$= \frac{a + c + b s_2}{2}$$

$$R_2(s_1) = \frac{a + c + b s_1}{2}$$

then, $(s_1^*, s_2^*) \in NE$

$$\Leftrightarrow \begin{cases} s_1^* \in R_1(s_2^*) \Leftrightarrow s_1^* = \dfrac{a + c + b s_2^*}{2} \\ s_2^* \in R_2(s_1^*) \Leftrightarrow s_2^* = \dfrac{a + c + b s_1^*}{2} \end{cases} \Rightarrow \begin{cases} s_1^* = \dfrac{a + c}{2 - b} \\ s_2^* = \dfrac{a + c}{2 - b} \end{cases}$$