Summary: Incomplete Info Dynamic

Recall: Incomplete Info Static ← BNE

we consider the expected payoff with respect to prior belief → Fixed

$\Leftrightarrow \tilde{u}_i(a_i, s_{-i}; t_i) := \underset{t_{-i}}{E}[a_i, s_{-i}(t_{-i}); t_i, t_{-i}]$

$\Leftrightarrow$ best response $s_i^* \in r_i^*(s_{-i}^*) = \arg\max_{s_i} \tilde{u}_i(a_i \cdots)$

$\Leftrightarrow (s_1^*, \dots, s_n^*) \in BNE$

graph

更侧重 Dynamic 的角度

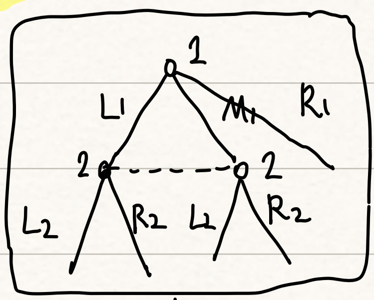
Here, Incomplete Info Dynamic ← PBE (Perfect Bayesian Equilibrium)

也可以为 Dynamic Game 提供一个更高质量的解 (Compared with Sub-game Perfect)

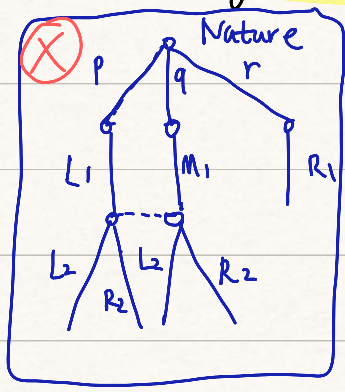
PBE

底层逻辑: when considering NE & S-P NE, we always assume that we know all other players' strategies and find the Best Response w.r.p others' strategies. → We will (more or less) neglect the DYNAMIC information.

(*)



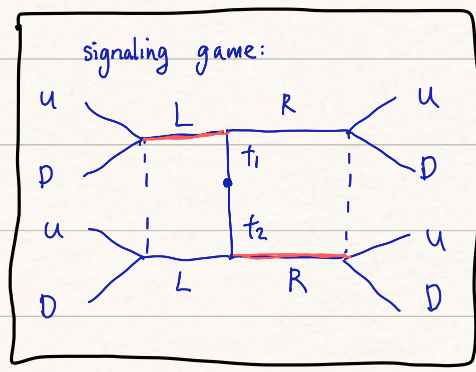
不好



应理解成 player 1 只有一个 type

Punchline: NE → 只能把握静态信息, 保证都是 Best response; 但是, 对于像 (*) 这种问题, (没有 subgame), (R_1, R_2) 中,

non-credible threat: { R_2 尝试 threat 使得 player 1 不敢 play L_1 or M_1



但是, 一旦 player 1 play L_1 or M_1, player 2 将 play L_2 (dominate)

⇒ 我们应该 exclude 掉 (R_1, R_2)

$\boxed{e_s}$ or $\boxed{e_c}$ intuitively

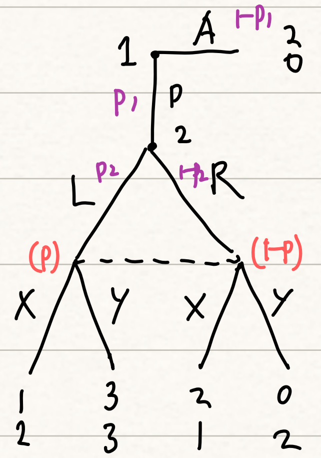
$\boxed{\text{倾向于给低工资}}$ → force (e_c, e_c) or (e_s, e_s)

Framework 1: $\boxed{\text{NE}} \to \boxed{\text{Belief}} \to \boxed{\text{Sequential Rationality}} \to \boxed{\text{Verification}}$ according to belief

Framework 2: $\boxed{\text{TRY}} \to \boxed{\cancel{\text{Action}}\ \text{near Belief}} \to \boxed{\text{Verification of Best Response}}$

$\boxed{\text{strategy}}$

$\boxed{\text{Example}}$:



① $\boxed{L} \Rightarrow p=1 \Rightarrow \boxed{Y}$

sequential rationality: $\boxed{Y} \Rightarrow \boxed{L} \Rightarrow \boxed{D}$

$\Rightarrow (DY, L)$

② $\boxed{R} \Rightarrow p=0 \Rightarrow \boxed{X}$

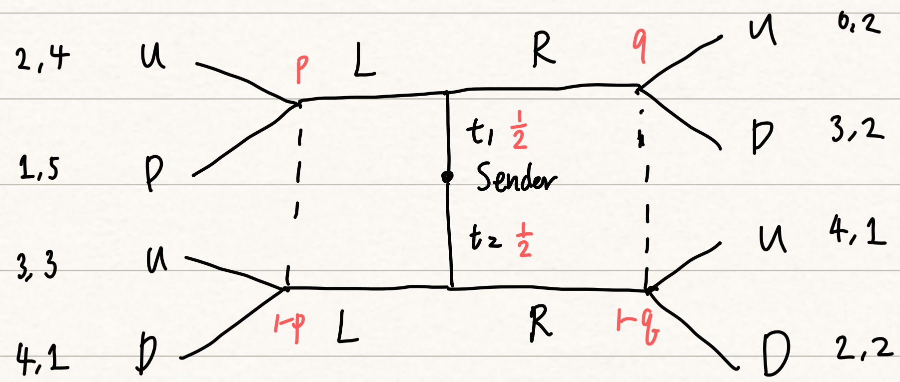
sequential rationality: $\boxed{X} \Rightarrow \boxed{L}$ ✗

$\Rightarrow$ the only PBE: $\boxed{(DY, L),\ p=0}$

$\boxed{\text{Belief}}$

$$\begin{cases} (p) = \dfrac{p_1 p_2}{p_1} = p_2 \\ (1-p) = \dfrac{p_1(1-p_2)}{p_1} = 1-p_2 \end{cases}$$

$\boxed{\text{Example}}$ Signaling game



① $LL \Rightarrow \boxed{p=\frac{1}{2}} \Rightarrow$ uu or uD

→ (LL, uD) ⇒ L L not best response Best response is RL ✗

→ (LL, uu) ⇒ L L → not best response Best response is LR ✗

② LR $\Rightarrow$ $\boxed{p=1 \quad q=0}$ $\Rightarrow$ DD $\Rightarrow$ Best response is RL ✗

③ RL $\Rightarrow$ $\boxed{p=0 \quad q=1}$ $\Rightarrow$ UU or UD

UU $\rightsquigarrow$ Best Response LR ✗

UD $\rightsquigarrow$ Best Response RL ☑

④ RR $\Rightarrow$ $q=\frac{1}{2}$ $\Rightarrow$ UD or DD

DD $\rightarrow$ Best Response is RL ✗

UD $\rightsquigarrow$ Best Response is RL ✗

$\Rightarrow$ PBE $= \{ (RL, UD), \ p=0, \ q=1 \}$

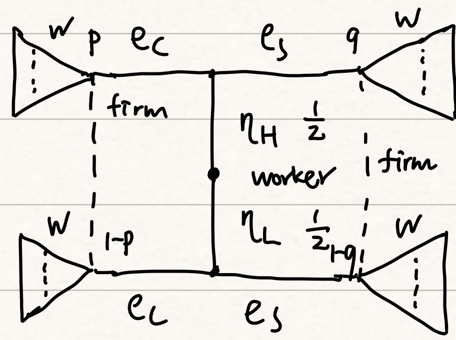
$$\boxed{\frac{\partial x^T(t) P(t) x(t)}{\partial t}}$$

$$= \boxed{\frac{\partial x^T(t) P(t) x(t)}{\partial x}} \boxed{\frac{\partial x}{\partial t}}$$

$$= 2x^T P \cdot \frac{\partial x}{\partial t}$$

$$= 2x^T P \frac{\partial x}{\partial t}$$

$\boxed{\text{Application}}$: $\boxed{\text{Job-Market}}$ $\longleftarrow$ $\boxed{\text{Signaling Model}}$



$$w - \overbrace{\boxed{c(\eta, e)}}^{\text{education cost}}, \quad -(\underbrace{\boxed{y(\eta, e)}}_{\text{output of worker}} - w)^2$$

Consider wage w is continuous case: (assume $c(\eta, e) = c_1(\eta) + c_2(e)$)

$\Rightarrow$ $\boxed{\text{claim:}}$ best response for worker must be pooling strategy.

$\rightarrow$ consider $\underline{w - c(\eta, e_S)}$ & $\underline{w - c(\eta, e_C)}$

$$\Delta(\eta) = w - c(\eta, e_S) - w + c(\eta, e_C)$$

$$= w^*(e_S) - c_2(e_S) - w^*(e_C) - c_2(e_C)$$

$\Rightarrow$ that is, given firm's strategy $w^*(e_C)$ & $w^*(e_S)$

$$\Delta(\eta_H) = \Delta(\eta_L)$$

$\Rightarrow$ for worker, he will make pooling strategy w.r.p to each type

① (e_c, e_c)

consider the best response for player 2 (firm) w.r.p (e_i, r)

$$w^*(e_i, r) = \arg\max \left\{-r[y(\eta_H, e_i) - w]^2 - (1-r)[y(\eta_L, e_i) - w]^2\right\}$$

$$= r\, y(\eta_H, e_i) + (1-r)\, y(\eta_L, e_i)$$

Up to now, we use $\boxed{\text{framework 2}}$

→ $\boxed{\text{TRY Strategy near belief}}$ → $\boxed{\text{Determine w.r.p to belief}}$

→ we still need to verify 'Best response' of (e_c, e_c)

s.t e_c is best response

select 要符的

check player 1's payoff There are infinitely many feasible $\boxed{w^*(e_s, q)}$

$$\to u_1(e_c, w^*(e_c, \tfrac{1}{2})) = \tfrac{1}{2} y(\eta_H, e_c) + \tfrac{1}{2} y(\eta_L, e_c) - c(\eta_H, e_c)$$

$$\to u_1(e_s, w^*(e_s, q)) = q\, y(\eta_H, e_s) + (1-q)\, y(\eta_L, e_s) - c(\eta_H, e_s)$$

Define $B(\eta, e) = y(\eta, e) - c_2(e)$

$$\Rightarrow \begin{cases} u_1(e_c, w^*(e_c, \tfrac{1}{2})) = \tfrac{1}{2} B(\eta_H, e_c) + \tfrac{1}{2} B(\eta_L, e_c) \\ u_1(e_s, w^*(e_s, q)) = q\, B(\eta_H, e_s) + (1-q)\, B(\eta_L, e_s) \end{cases}$$

$\boxed{\text{Aim:}}$ e_c is best response for η_H (also η_L)

$$\Leftrightarrow u_1(e_c, w^*(e_c, \tfrac{1}{2})) \geq u_1(e_s, w^*(e_s, q)) \quad (*)$$

$$\Leftrightarrow \tfrac{1}{2} B(\eta_H, e_c) + \tfrac{1}{2} B(\eta_L, e_c) \geq B(\eta_L, e_s)$$

↳ let $\bar{q}$ to be the biggest q s.t $(*)$ holds

$$\Rightarrow \{(e_c, e_c), (w^*(e_c, p), w^*(e_s, q)) : p = \tfrac{1}{2}, q \in [0, \bar{q}]\} \in PBE$$

② (e_s, e_s)

similarly, $\{(e_s, e_s), (w^*(e_c, p), w^*(e_s, q)) : p \in [0, \bar{p}], q = \tfrac{1}{2}\} \in PBE$

$\bar{p}$ is the biggest p satisfies :

$$u_1(e_c, w^*(e_c, p)) \leq u_1(e_s, w^*(e_s, \tfrac{1}{2}))$$

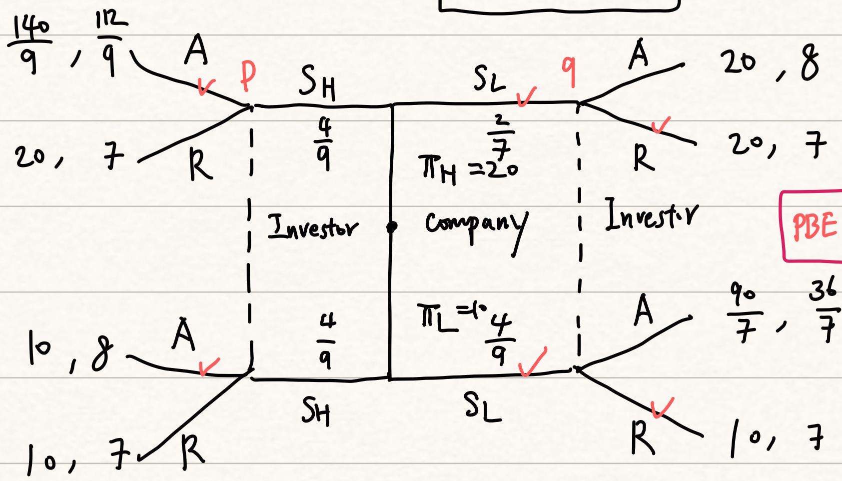
the existence of $\bar{p}$ $\Longleftrightarrow$ $B(\eta_L, e_c) \leq \tfrac{1}{2} B(\eta_H, e_s) + \tfrac{1}{2} B(\eta_L, e_s)$

Application 2 Corporate Investment & Captical Structure

Signaling game

$\boxed{A > R}$

$\boxed{I=5}$ $\boxed{R=8}$ $\boxed{r=0.4}$



PBE $= \{ (S_L, S_L), (A, R), p \in [0,1], q = \frac{1}{2} \}$

payoff
- Accept $\boxed{(1-s)(\pi+R) \ \& \ s(\pi+R)}$
- Reject $\pi, \ (1+r)I$

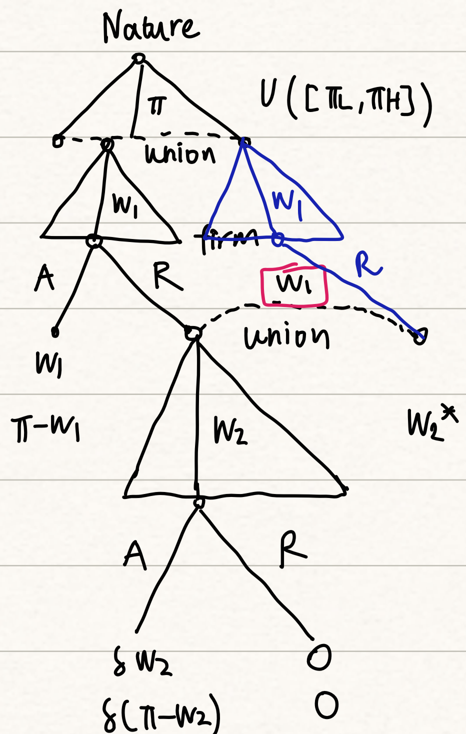
Model 2 (wage bargaining) sequential with asymmetric information

$\boxed{\text{union}} \longleftrightarrow \boxed{\text{firm}}$

Problem Setting

① firm $\xrightarrow{\text{profit } \pi}$ $\boxed{U([\pi_L, \pi_H])}$ (private information)

②



} Stage 0

} Stage 1

$\boxed{\text{Analysis}}$: $\boxed{\text{Strategy space}}$: $\boxed{\text{union}}$ S_u : starting $\times W_1 \longrightarrow W_1 \times W_2$

$\boxed{\text{firm}}$ S_f : $[\pi_L, \pi_H] \times W_1 \times [\pi_L, \pi_H] \times W_2 \longrightarrow A \times A$

$\boxed{\text{Actually}}$ $\begin{cases} S_{u1} : \text{starting} \longrightarrow W_1 \\ S_{u2} : W_1 \longrightarrow W_2 \end{cases}$

$\begin{cases} S_{f1} : [\pi_L, \pi_H] \times W_1 \longrightarrow A \\ S_{f2} : [\pi_L, \pi_H] \times W_2 \longrightarrow A \end{cases}$

① $(S_u^*, S_f^*) \in PBE$, then $(S_u^*, S_f^*) \in NE$ (necessary condition)

$\Rightarrow$ $\boxed{S_f^* \in r^*(S_u^*)}$ (Best response)

suppose $\boxed{S_{u1}^* : \text{starting} \rightarrow W_1^*}$ $\boxed{S_{u2}^* : W_1^* \longrightarrow W_2^*}$

$\begin{cases} \text{if } S_{f1}^* \text{ Accept} \rightarrow \text{payoff} = \pi - W_1^* \\ \text{if } S_{f1}^* \text{ Reject}, \ S_{f2}^* \text{ Accept} \rightarrow \text{payoff} = \delta(\pi - W_2^*) \\ \text{if both Reject} \rightarrow \text{payoff} = 0 \end{cases}$

$\Rightarrow$ compare $\pi - W_1^* > \delta(\pi - W_2^*)$, firm prefer accept W_1^*

$\Leftrightarrow \pi > \dfrac{W_1^* - \delta W_2^*}{1-\delta} := \pi_1^*$

we just require S_{f1}^* at optimal W_1^* & W_2^* for necessity

$\Leftrightarrow$ $S_{f1}^*(W_1^*, W_2^*, \pi) = \begin{cases} A & \pi > \pi_1^* \\ R & \pi < \pi_1^* \end{cases}$ $\boxed{W_2^* = W_2^*(W_1^*)}$

Necessary condition for S_{f1}^*

$\boxed{\text{only restrict } S_{f1}^* \text{ 在 } W = W_1^* \text{ 时的 strategy}}$

② given that $S_{f1}^*(W_1^*, W_2^*, \pi) = \begin{cases} A & \pi > \pi_1^* \\ R & \pi < \pi_1^* \end{cases}$ & $\boxed{S_{f2}^*}$ $\begin{cases} A, & \pi > W_2^* \\ R, & \pi < W_2^* \end{cases}$

$\boxed{\text{at information set } (W_1^*, R)}$

then $\pi_{posterior} \sim U[0, \pi_1]$ (Belief at stage 2)

Therefore, consider for $\boxed{S_{u2}^*(W_1^*)}$ w.r.p to belief,

then we have $S_{u2}^*(W_1^*) = \arg\max \mathbb{E}_{\pi_{post}}[\text{payoff}_u(W_1^*)]$

since the only updated belief is in (W_1^*, R)

$= \underset{W_2}{\arg\max} \ W_2 \cdot \mathbb{P}(\text{firm accept } W_2)$

$= \arg\max_{W_2} \ W_2 \cdot \dfrac{\pi_1^* - W_2}{\pi_1^*}$

$$= \frac{\pi_1^*}{2}$$

$$\Rightarrow \boxed{W_2^* = \frac{\pi_1^*}{2}} \rightarrow \boxed{\text{necessary condition}}$$

③ given that $S_{f_1}^* (w_1^*, W_2^*(w_1^*), \pi) = \begin{cases} A & \pi > \pi_1^* \\ R & \pi < \pi_1^* \end{cases}$

$$S_{u_2}^* (w_1^*) = \frac{\pi_1^*}{2}$$

develop the last necessary condition (Back to stage 1)

$\rightarrow$ w.r.p to union, $S_{u_1}^* (\text{start}) = W_1^*$

$$\Leftrightarrow w_1^* = \underset{w_1}{\operatorname{argmax}} \; w_1 \mathbb{P}(\text{firm accept } w_1) + \delta w_2 \mathbb{P}(\text{firm reject } w_1 \text{ accept } w_2)$$

$$= \underset{W_1}{\operatorname{argmax}} \; w_1 \mathbb{P}(\pi > \pi_1) + \delta W_2 \mathbb{P}(\pi < \pi_1, \pi > W_2)$$

☆ Actually, previously our analysis is all necessary condition.

$S_{f_1}^* (w_1^*, W_2^*(w_1^*), \pi) = \begin{cases} A & \pi > \pi_1^* \\ R & \pi < \pi_1^* \end{cases}$

$S_{u_2}^* (w_1^*) = \frac{\pi_1^*}{2}$

From our analysis, we can see actually

we can choose $\begin{cases} S_{f_1}^* (w_1, w_2(w_1), \pi) = \begin{cases} A, & \pi > \pi_1 \\ R, & \pi < \pi_1 \end{cases} \\ S_{u_2}^* (W_1) = \frac{\pi_1}{2} \end{cases}$

$\boxed{\pi_1 = \frac{W_1 - \delta W_2}{1-\delta}}$

which is a complete strategy (sufficient condition)

$$= \underset{w_1}{\operatorname{argmax}} \; W_1 \frac{\pi_H - \pi_1}{\pi_H} + \delta W_2 \frac{\pi_1 - W_2}{\pi_H}$$

$$= \underset{w_1}{\operatorname{argmax}} \; \frac{1}{\pi_H} \left(\pi_H W_1 - \frac{2W_1^2}{2-\delta} + \frac{\delta W_1^2}{(2-\delta)^2} \right)$$

$$\pi_1 = \frac{W_1 - \frac{\delta}{2}\pi_1}{1-\delta} \Rightarrow \boxed{W_1^* = \frac{(2-\delta)^2}{2(4-3\delta)} \pi_H}$$

$$(1-\delta)\pi_1 = W_1 - \frac{\delta}{2}\pi_1$$

$\Rightarrow$ $\boxed{\text{Complete Strategy:}}$

$S_{u_1}^*$: $\boxed{W_1^*} = \frac{(2-\delta)^2}{2(4-3\delta)} \pi_H$

$S_{f_1}^*$: $W_1 \times \Pi \longrightarrow A$

$(W_1, \pi) \longmapsto \begin{cases} A, & \pi > \pi_1 \\ R, & \pi < \pi_1 \end{cases}$

where $\pi_1 = \frac{2}{2-\delta} W_1$

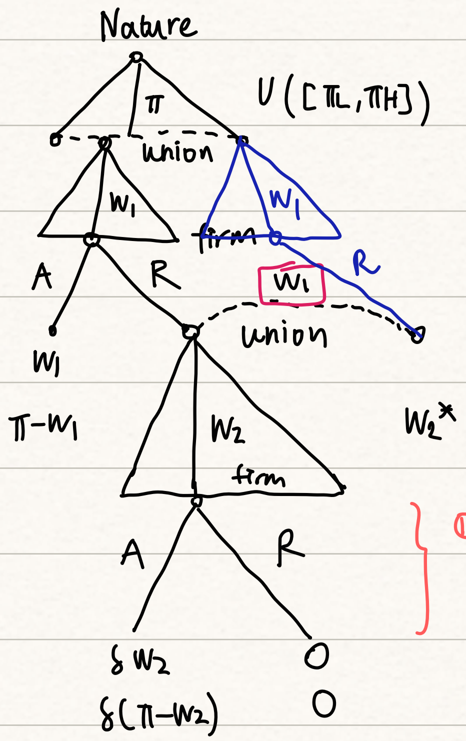
$S_{u_2}^*$: $W_1 \times \{R\} \longrightarrow W_2$

$$(w_1, R) \longmapsto \frac{1}{2-\delta} w_1$$

$$S^*_{f_2}: \quad W_2 \times \Pi \longrightarrow A$$

$$(w_2, \pi) \longmapsto \begin{cases} A, & \pi > w_2 \\ R, & \pi < w_2 \end{cases}$$

$\boxed{\text{Summary}}$



$\boxed{\text{Stage 0}}$

$\boxed{\text{Stage 1}}$

$\boxed{\text{use a sufficient condition}}$

④ solve for w_1^* by $\boxed{\text{sequential rationality}}$

② $S^*_{f_1}: \begin{cases} A, & \pi > \pi_1 \\ R, & \pi < \pi_1 \end{cases}$ given w_1, w_2

③ $\boxed{\text{update belief}}$, $S^*_{u_2}(w_1, R) = \frac{\pi_1}{2}$

↳ cosistence

① $S^*_{f_2}: \begin{cases} A & \pi > w_2 \\ R & \pi < w_2 \end{cases}$