$\boxed{\text{Incomplete}}$ Information Static Game

$\downarrow$

$\boxed{\text{payoff } f^{\frown}} \to$ random

$\boxed{\text{Example}}$

Firm 1 — Firm 2

$\begin{cases} C_H \\ C_L \end{cases}$ $\qquad C_2$

$\Downarrow \qquad\qquad \downarrow$

$\left.\begin{cases} \pi_1(q_1, q_2; C_H) \\ \pi_1(q_1, q_2; C_L) \end{cases} \quad \pi_2(q_1, q_2; C_2) \right\} \to \boxed{\text{payoff}}$

From the perspective of $\boxed{\text{each player's type}}$ (choice)

i.e., $\begin{cases} C_H \\ C_L \end{cases}$ & C_2

$\Rightarrow$ Type Space of Firm 1: $T_1 = \{C_H, C_L\}$

Type Space of Firm 2: $T_2 = \{C_2\}$

$\boxed{\text{What is Belief}}$

$\to$ given that he knows his own type, his guess on other players' type space

$\boxed{\text{从玩家角度}}$

$\Uparrow$

$\Rightarrow \begin{cases} A_1 = [0, +\infty] \quad A_2 = [0, +\infty) \\ T_1 = \{C_H, C_L\} \quad T_2 = \{C_2\} \\ \boxed{\text{Belief}}: P_1(\cdot), \quad P_2(\cdot) \\ \qquad \begin{cases} \cancel{P_1(C_H) = p} \\ \cancel{P_1(C_L) = 1-p} \end{cases} \quad \cancel{P_2(C_2) = 1} \\ \qquad \boxed{P_1(C_2) = 1} \quad \boxed{\begin{matrix} P_2(C_H) = p \\ P_2(C_L) = 1-p \end{matrix}} \\ \text{payoff} \begin{cases} \pi_1(\cdot, \cdot; C_H) \quad \pi_1(\cdot, \cdot; C_L) \\ \pi_2(\cdot, \cdot; C_2) \end{cases} \end{cases} \longrightarrow \boxed{G = \{A; T; P; U\}}$

$\boxed{\text{one type specify 1 payoff}}$

$\Longrightarrow \boxed{\text{Bayesian NE}}$

1.

$\to$ First, define strategy for each player: (i-th player)

$\boxed{S_i: T_i \to A_i} \to$ a complete plan for $\boxed{\text{Type Space}}$

$T_i = \{t_{i1}, \ldots, t_{ik_i}\}$

$T_i = \{t_{i1}, \ldots t_{iK_i}\}$

→ 2. Bayesian NE

$\boxed{\text{n-players game}} \longrightarrow (s_1^*, \ldots, s_n^*)$ is BNE

$\iff$

$$s_i^*(t_{ij}) = \underset{a_i \in A_i}{\text{argmax}}\ \mathbb{E}_{t_{-i}}\ u_i(a_i, s_{-i}^*(t_{-i}); t_{ij})$$

$\sum_{t_{-i}} p_i(t_{-i})\, \underbrace{u_i(a_i, s_{-i}^*(t_{-i}); t_{ij}, t_{-i})}$

for $j = 1, 2, \ldots, K_i$

and $i = 1, 2, \ldots, n$

Sometimes $u_i(\cdot;\cdot)$ only related to its own type t_{ij}

→ $u_i(a_i, s_{-i}^*(t_{-i}); t_{ij}, t_{-i})$

$\equiv u_i(a_i, s_{-i}^*(t_{-i}); t_{ij})$

$\boxed{t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)}$

↓

$\boxed{\text{random variable}}$

↓

$\mathbb{P}(t_i = t_{ij})$ is given by $P(t_{ij})$

$\Downarrow$

$$\mathbb{E}_{t_{-i}}[u_i(a_i, s_{-i}^*(t_{-i}); t_{ij})]$$

$$= \sum \mathbb{P}(t_{-i} = t_{-ik} \mid t_i = t_{ij})\ u_i(a_i, s_{-i}^*(t_{-ik}); t_{ij})$$

t_{-ik} is all the value can be achieved by t_{-i}

why conditional probability → $\boxed{\text{natural choose}}$

$$= \sum_{t_{-ik}} \frac{\mathbb{P}(t_{-i} = t_{-ik}, t_i = t_{ij})}{\mathbb{P}(t_i = t_{ij})}\ u_i(a_i, s_{-i}^*(t_{-ik}); t_{ij})$$

↳ if independent assump.

→ then $= \underline{\mathbb{P}(t_{-i} = t_{-ik})}$

$\Rightarrow \boxed{\text{Model}}$:

	C	D
C	$1-c_1, 1-c_2$	$1-c_1, 1$
D	$1, 1-c_2$	$0, 0$

$\begin{cases} c_1 = \{0.5, 0.8\} \\ c_2 = \cdots \end{cases}$

→ n players → $\begin{cases} P_i(\cdot) = \\ A_i \\ T_i = \{t_{ij} : j = 1, 2, \ldots, K_i\} \\ U = u_i(a; t_{ij}) \end{cases}$ → $\boxed{\text{use this way}}$

② Nature Selects Game

$\{ G, P, (T_1, \dots, T_n) \}$

1. $G = \{ \text{game} \} \rightarrow \boxed{g_1, \dots, g_I}$
2. $P \rightarrow$ prob. distribution on G
3. $T_i \rightarrow$ partition of G $\Rightarrow$ type space

(4) $b_i : G \longrightarrow T_i$ $\Rightarrow$ $\boxed{\text{type map}}$

$$\Rightarrow \tilde{u}_i (a_i, s_{-i}^* ; t_{ij})$$

$$= \sum_g P(g \mid t_{ij})\, u_i(a_i, s_{-i}^*(b_{-i}(g)); g)$$

$$= P(t_{ij})^{-1} \sum_{g \in t_{ij}} P(g)\, u_i(a_i, s_{-i}^*(b_{-i}(g)); g)$$

$(s_1^*, \dots, s_n^*) \in$ BNE

$$\Leftrightarrow \quad s_i^*(t_{ij}) = \underset{a_i \in A_i}{\operatorname{argmax}}\ \tilde{u}_i (a_i, s_{-i}^* ; t_{ij})$$

!! if

	C	P
C	$1-c_1-c_2; 1-c_2$	$1-c_1-c_2; 1$
D	$1, 1-c_2$	$0, 0$

, then we can generate 4 games according to $\begin{cases} c_1 = \{ *, * \} \\ c_2 = \{ *, * \} \end{cases}$

$\Downarrow$

using this formulation

try the previous formulation

for example, $\boxed{c_{1e} = 0.5}$

conside $\mathbb{E}_{t_2} [u_1(a_1, s_2^*(t_2); c_{1e})]$

$= P_{2e}\, u_1(a_1, s_2^*(c_{2L}); c_{1e}, c_{2e})$

$+ P_{2h}\, u_1(a_1, s_2^*(c_{2h}); c_{1e}, c_{2h})$

if the Game is like this $\{G, P, \{T_1, \dots, T_n\}\}$

use this way

More complicated way

Generally, Static Incomplete Information Game

$\Downarrow$

① First-Price Auction → linear BNE

② Second-Price Auction → weakly dominant

③ Double Auction $\begin{cases} \text{linear Eq.} \\ \text{one-price Eq.} \end{cases}$

④ Relation with Mix-strategy

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① <u>First-Price Auction</u>: → $\boxed{\text{linear BNE}}$

Problem setting:
① 2 players
② Action space: $[0, +\infty) \to b_i$
③ Type Space: $V_i \in [0,1] \Rightarrow$ uniformly distributed
④ payoff function:

determine Strategy Space

$$u_i(b_i, b_j; V_i) = \begin{cases} V_i - b_i, & b_i > b_j \quad \cancel{V_i > V_j} \\ \frac{V_i - b_i}{2}, & \cancel{V_i = V_j} \quad b_i = b_j \\ 0, & \cancel{V_i < V_j} \quad b_i < b_j \end{cases}$$

$\boxed{\text{Solving}:}$ Linear BNE $\Leftrightarrow$ $b_i(V_i) = a_i + c_i V_i$ ① $\boxed{\begin{cases} c_i > 0 \\ a_i > 0 \end{cases}}$

② $\boxed{b_i(V_i) \in [a_j, a_j + c_j]}$ → ☆ $\boxed{\text{Necessary Condition}}$

$$\tilde{u}_i(b_i, b_j(V_j); V_i)$$

$$= \underbrace{(V_i - b_i)}_{\text{fix}} \cdot \mathbb{P}(b_i > b_j V_j = a_j + c_j V_j)$$

$$= (V_i - b_i)\ \mathbb{P}(a_j + c_j V_j < b_i)$$

$$= (v_i - b_i)\,\mathbb{P}\left(v_j < \frac{b_i - a_j}{c_j}\right)$$

$$= \frac{b_i - a_j}{c_j}(v_i - b_i) \longrightarrow \underline{\text{unconstraint minimizer} = \frac{v_i + a_j}{2}}$$

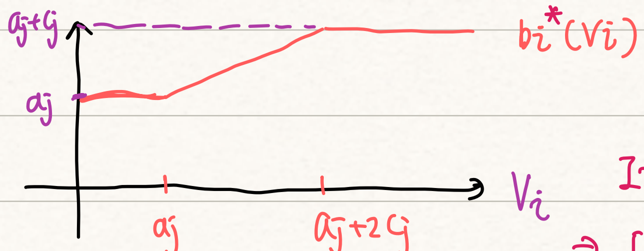
$$\underline{b_i^* = \underset{\boxed{a_j \le b_i \le a_j + c_j}}{\operatorname{argmax}}\ \hat{u}_i(b_i, b_j(v_j); v_i)}$$

$\rightarrow$ necessary condition for b_i

$\Downarrow$

constrained minimizer

$$b_i^* = \begin{cases} \dfrac{v_i + a_j}{2} & a_j \le \dfrac{v_i + a_j}{2} \le a_j + c_j \iff a_j \le v_i \le a_j + 2c_j \\ a_j & \dfrac{v_i + a_j}{2} < a_j \iff a_j > v_i \\ a_j + c_j & \dfrac{v_i + a_j}{2} > a_j + c_j \iff a_j + 2c_j < v_i \end{cases}$$



$a_j \Downarrow a_j \ge 0$ $\qquad$ $a_j + 2c_j \Downarrow a_j + 2c_j > 0$

If we want linear BNE

$\Rightarrow [0,1] \subset [a_j, a_j + 2c_j]$

$\Rightarrow b_i^* = \frac{1}{2}v_i + \frac{1}{2}a_j$

$= c_i v_i + a_i$

$\Rightarrow \begin{cases} c_i = \frac{1}{2} \\ a_i = \frac{1}{2}a_j \end{cases}$ $\boxed{\text{for } i = 1, 2}$

$\Rightarrow \begin{cases} c_i = c_j = \frac{1}{2} \\ a_i = a_j = 0 \end{cases}$ #

The process is somehow like 'finding candidates then varifying'.

$\begin{cases} b_1 = \frac{1}{2}v_1 \\ b_2 = \frac{1}{2}v_2 \end{cases}$ is actually a candidate

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② Second-Price Auction

Problem Setting:

① n players

② Action Space $[0, +\infty)$

③ Type Space $[0, 1]$

Strategy Space

④ payoff function :

$$u_i(b_i, r_i; v_i) = \begin{cases} v_i - r_i, & b_i > r_i \\ \frac{v_i - r_i}{k}, & b_i = r_i \\ 0, & o/w \end{cases}$$

More steady | More Reasonable

Analyze : $\boxed{b_i^*(v_i) = v_i\ !}$ → weakly dominant solution

1. when player knows that r_i & v_i

↳ if $v_i > r_i$, then $u_i \le v_i - r_i$

if $v_i \le r_i$ then $u_i = 0$

$$\Rightarrow u_i \le \max\{v_i - r_i, 0\}$$

2. $u_i(b_i^*(v_i), r_i; v_i)$

$= u_i(v_i, r_i; v_i)$

$= \max\{v_i - r_i, 0\}$

Rmk: Actually, Best Response should be :

if $\begin{cases} v_i > r_i, & \text{then } b_i(v_i) \text{ should be as big as possible} \\ v_i \le r_i, & \text{then } b_i(v_i) \text{ should be as small as possible} \end{cases}$

⇑ v_i is enough

⇓ v_i is enough

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③ Double Auction

1. One-price BNE
2. Linear BNE

Problem Setting

① 2 players $\begin{cases} \text{Seller} \\ \text{buyer} \end{cases}$

② Action Space ~~[0,+∞)~~ [0,1]

③ Type Space [0,1]

④ payoff function

$\boxed{\text{成交价}}$

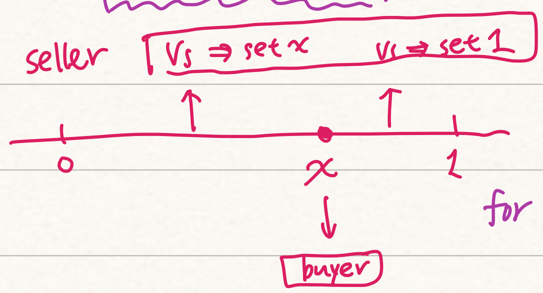
$$\boxed{\bar{p} = \frac{p_s+p_b}{2}}$$

$\boxed{\text{Trade happen when } p_b \ge p_s}$

1. $\pi_s(p_s, p_b; v_s) = \begin{cases} \bar{p} - v_s & \text{if } p_b \ge p_s \\ 0 & \text{o/w} \end{cases}$

2. $\pi_b(p_b, p_s; v_b) = \begin{cases} v_b - \bar{p} & \text{if } p_b \ge p_s \\ 0 & \text{o/w} \end{cases}$

1. $\boxed{\text{One-Price BNE}}$ $\Leftrightarrow$ for seller, her strategy is:



$$p_s(v_s) = \begin{cases} x, & \text{when } v_s \le x \\ 1, & \text{when } v_s > x \end{cases}$$

for buyer, his strategy is:

$$p_b(v_b) = \begin{cases} y, & \text{when } v_b \ge y \\ 0, & \text{when } v_b < y \end{cases}$$

2. $\boxed{\text{Linear BNE}}$ → Calculation!

$$\begin{cases} p_s(v_s) = c_s v_s + a_s \\ p_b(v_b) = c_b v_b + a_b \end{cases}$$

$$\Rightarrow \tilde{\pi}_s(p_s, p_b; v_s)$$

$$= \mathbb{E}_{v_b}\left[\pi_s[p_s, \overbrace{\boxed{p_b(v_b)}}^{c_b v_b + a_b}; v_s]\right]$$

$$= \int_{\{1 \ge v_b \ge \frac{p_s - a_b}{c_b}\}} \left(\frac{p_s + c_b v_b + a_b}{2} - v_s\right) \cdot dv_b$$

$$u = c_b v_b + a_b = \int_{p_s}^{a_b+c_b} \left(\frac{p_s+u}{2} - v_s\right) du \cdot \frac{1}{c_b}$$

$$= \left[\frac{1}{4}u^2 + \frac{p_s}{2}u - v_s u\right]\Big|_{p_s}^{a_b+c_b} \cdot \frac{1}{c_b}$$

$$\tilde{\pi}_s(p_s, p_b; v_s)$$

$$= \mathbb{E}_{v_b}[\pi_s(p_s, p_b(v_b); v_s, v_b)]$$

$$= \mathbb{E}_{v_b}[\pi_s(p_s, p_b(v_b); v_s, v_b) \cdot \mathbb{1}\{p_s \le p_b(v_b)\}]$$

$$= \mathbb{E}_{v_b}[\pi_s(p_s, p_b(v_b); v_s, v_b) \mid p_s \le p_b(v_b)] \cdot \mathbb{P}(\cdots)$$

$$= \mathbb{E}_{v_b}\left[\frac{p_b(v_b)+p_s}{2} - v_s \mid p_s \le p_b(v_b)\right] \cdot \mathbb{P}(p_s \le p_b(v_b))$$

$$p_s \le c_b v_b + a_b \le c_b + a_b$$

$$\Leftrightarrow v_b \ge \frac{p_s - a_b}{c_b}$$

$$= \left[\mathbb{E}_{v_b}\left[\frac{p_b(v_b)}{2} \mid p_b(v_b) \ge p_s\right] + \frac{p_s}{2} - v_s\right] \cdot \mathbb{P}(\cdot)$$

$$= \left[\frac{\frac{p_s + a_b + c_b}{2} + p_s}{2} - v_s\right] \cdot \frac{a_b + c_b - p_s}{c_b}$$

$$= \frac{1}{c_s}\left[\frac{1}{4}(a_b+c_b)^2 + \frac{p_s}{2}(a_b+c_b) - v_s(a_b+c_b) \right] - \frac{1}{c_s}\left[\frac{1}{4}p_s^2 + \frac{1}{2}p_s^2 - v_s p_s \right]$$

$$= c_s^{-1}\left[\frac{1}{4}(a_b+c_b+p_s)(a_b+c_b-p_s) + \frac{1}{2}p_s(a_b+c_b-p_s) - v_s(a_b+c_b-p_s) \right]$$

$$= c_s^{-1}\left[(a_b+c_b-p_s)\left(\frac{1}{4}(a_b+c_b+p_s) + \frac{1}{2}p_s - v_s \right) \right]$$

$$= \frac{a_b+c_b-p_s}{c_s}\left[\frac{1}{4}(a_b+c_b) + \frac{3}{4}p_s - v_s \right] \qquad \frac{-p_s}{c_s}\left[-\frac{2}{4}(a_b+c_b) + \frac{3}{4}p_s - v_s - \cancel{\frac{2}{4}(a_b+c_b)} \right]$$

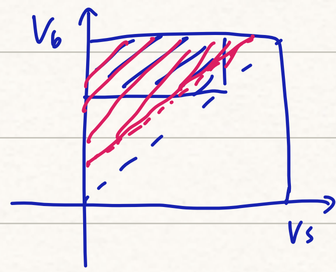
$$\Rightarrow \text{Best Response} \quad p_s^*(v_s) = \frac{\frac{2}{4}(a_b+c_b) + v_s}{\frac{3}{2}} = \frac{(a_b+c_b) + 2v_s}{3}$$

$$\Rightarrow \begin{cases} a_s = (a_b+c_b)/3 \\ c_s = \frac{2}{3} \end{cases}$$

fixed x

$\boxed{\text{For One-Price Eq}}$ $\begin{cases} v_s \le x \\ v_b \ge x \end{cases}$

Trade Occurs $\Leftrightarrow \{ v_s \le x \le v_b \}$



Note: $v_b - v_s$ can be viewed as total profit

$\boxed{\text{For Linear Eq}}$

Trade Occur $\Leftrightarrow$ $\boxed{p_b \ge p_s}$

$$\Leftrightarrow \frac{2}{3}v_b + \frac{1}{12} \ge \frac{2}{3}v_s + \frac{1}{4}$$

$$\Leftrightarrow v_b - v_s \ge \frac{1}{4}$$

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④ Mix-strategy → Opera & Football Game

Use private information to interpret mix-strategy

⇓

add uncertainty to payoff table

PRIMAL

M \ P	O	F
O	2,1	0,0
F	0,0	1,2

payoff table

	O	F
O	$2+t_m$,1	0,0
F	0,0	1,$2+t_p$

$$\begin{cases} t_m : U[0,x] \\ t_p : U[0,x] \end{cases}$$

$$\begin{cases} S_m : [0, x] \longrightarrow A = \{0, F\} \\ S_p : [0, x] \longrightarrow A = \{0, F\} \end{cases}$$

$\boxed{\text{we want to find BNE}}$

① given Peter's strategy is S_p, find Mary's Best Response

$$\tilde{u}_m(S, S_p; t_m) = \mathbb{E}[u_m(a, S_p; t_m)]$$

$$= \mathbb{P}(F) \cdot u_m(a, F; t_m) + \mathbb{P}(0)\, u_m(a, 0; t_m)$$

$$= \begin{cases} \mathbb{P}(0)\,(2 + t_m), & a = 0 \\ \mathbb{P}(F) \cdot 1, & a = F \end{cases}$$

$\boxed{S_p}$

$$\boxed{\begin{array}{l} \mathbb{P}(0) := 1 - \delta_F \\ \mathbb{P}(F) := \delta_F \end{array}} \Rightarrow \begin{cases} (1-\delta_F)(2+t_m) > \delta_F \Rightarrow \text{choose } 0 \\ (1-\delta_F)(2+t_m) < \delta_F \Rightarrow \text{choose } F \end{cases}$$

$$\Rightarrow \begin{cases} t_m > \frac{\delta_F}{1-\delta_F} - 2 \Rightarrow \boxed{\text{choose } 0} \\ t_m < \frac{\delta_F}{1-\delta_F} - 2 \Rightarrow \boxed{\text{choose } F} \end{cases}$$

→ Mary's Best response to $\boxed{S_p} \Rightarrow \delta_F$

$$\boxed{r_m^*(S_p) = \begin{cases} 0, & t_m > \frac{\delta_F}{1-\delta_F} - 2 \\ F, & t_m < \frac{\delta_F}{1-\delta_F} - 2 \end{cases}}$$

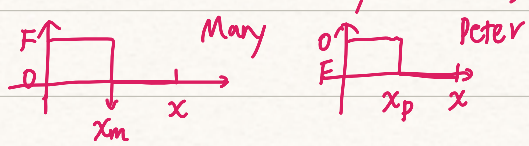
Peter's Best Response to $S_m \Rightarrow \delta_0$

$$\boxed{r_p^*(S_m) = \begin{cases} F, & t_p > \frac{\delta_0}{1-\delta_0} - 2 \\ 0, & t_p < \frac{\delta_m}{1-\delta_0} - 2 \end{cases}}$$

Therefore, if $(S_p^*, S_m^*) \in$ BNE

then it must be 分段 (necessary condition)

suppose that



$$\boxed{\text{interest is } \begin{cases} x_m \\ x_p \end{cases}} \Rightarrow ① \begin{cases} \delta_0 = \frac{x - x_m}{x} \\ \delta_F = \frac{x - x_p}{x} \end{cases} \Rightarrow \begin{cases} x_m = \frac{x - x_p}{x_p} - 2 \\ x_p = \frac{x - x_m}{x_m} - 2 \end{cases}$$

$$② \begin{cases} x_m = \frac{\delta_F}{1-\delta_F} - 2 \\ x_p = \frac{\delta_0}{1-\delta_0} - 2 \end{cases}$$

$\frac{1}{2} x^{-\frac{1}{2}}$

$x^{\frac{1}{2}}$

$$② \begin{cases} \delta_0 = \frac{x - \frac{\delta_F}{1-\delta_F} + 2}{x} \\ \delta_F = \frac{x - \frac{\delta_0}{1-\delta_0} + 2}{x} \end{cases}$$

$$\Rightarrow \begin{cases} x(\delta_O - 1) = -\frac{\delta_F}{1-\delta_F} + 2 \\ x(\delta_F - 1) = -\frac{\delta_O}{1-\delta_O} + 2 \end{cases}$$

$$\Rightarrow \begin{cases} \delta_O = \delta_F \\ 3\delta_F = 2 \end{cases} \Rightarrow \begin{cases} \delta_O = \frac{2}{3} \\ \delta_F = \frac{2}{3} \end{cases}$$

$\Rightarrow x_m, x_p$ is the solution of equ.

$$\boxed{u^2 + 3u - x = 0}$$

$$\Rightarrow u = \frac{-3 + \sqrt{9+4x}}{2}$$

$$\frac{\sqrt{9+4x} - 3}{2} = \frac{4x}{2(\sqrt{9+4x}+3)}$$

$$\Rightarrow x_m = x_p = \frac{\sqrt{9+4x} - 3}{2}$$

$$\Rightarrow \boxed{\delta_O = \delta_F = \frac{x - \frac{\sqrt{9+4x}-3}{2}}{x} \to \boxed{\frac{2}{3}}}$$

$\delta_O \downarrow$ Mary $\Downarrow$ O

$\delta_F \downarrow$ Peter $\Downarrow$ F

$\boxed{\text{Mix-strategy solution agreement}}$