$\boxed{\text{Firstly.}}$ confusion $\Rightarrow$

① 对于 1 player, play several times, how to formulate (generally) its strategy space

② As for strategy NE, the N & S condition of Best Response $\boxed{\tilde{R}_1(S_2^*)}$

especially $\tilde{R}_2(T_1)$ & $\tilde{R}_2(NC_1)$

Summary of $\boxed{\text{Dynamic Game}}$

- Complete Information
- Perfect Information

$\boxed{\text{Something Important:}}$

Extensive Form Repres.

① action space $\leftrightarrow$ related to each STEP

we need m actions to determine payoff

② Information set

$\boxed{\text{n-player, m-step game}}$

③ payoff (for action) $u_i(a_1, \dots, a_m)$ $i = 1, 2, \dots, n$

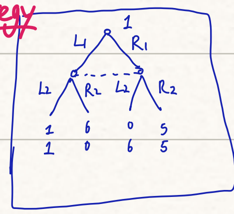
Motivation: These defns might not give a "sol" to some complicated problem. Then we need 'Strategy'!

$\boxed{\text{Note:}}$ 对于一个 Static Game, we still can formulate it as Dynamic Game with imperfect information, i.e., Prisoners' Delimma

	L_2	R_2
L_1	1,1	6,0
R_1	0,6	5,5

strategy

strategy



Normal Repres.

④ strategy space $\leftrightarrow$ related to each player's own plan (complete plan)

⑤ $\boxed{\text{payoff (for strategy)}}$ $\to$ $\boxed{\text{induced}}$ by payoff (for action)

$$\hat{u}_i(\underbrace{s_1, \dots, s_n}_{s}) = \underbrace{u_i(a_1(s), \dots, a_m(s))}_{\text{since for } s = (s_1, \dots, s_n)}$$

we should identify strategy space
↓
may not easy

Static Game ⇒ NE $(s_1^*, \dots, s_n^*)$
↓
strategy

we can determine $\boxed{\text{one}}$ action route $(a_1, \dots, a_m)$
↓
payoff (for action)

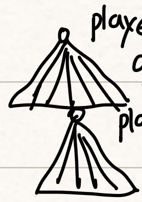
Example:

① BIO → $\boxed{\text{outcome}}$ → $\boxed{\text{an action route}}$

example: $\boxed{\text{Stackelberg Model of Duopoly}}$ ⟷ Cournot Duopoly

$\pi_1(q_1, q_2) = q_1 (P(Q) - c)$

$P(Q) = \max(a - q_1 - q_2, 0)$



$\left(\frac{a-c}{2}, \frac{a-c}{4}\right)$ $\left(\frac{a-c}{3}, \frac{a-c}{3}\right)$

$$R_2(q_1) = \begin{cases} \frac{a-c-q_1}{2}, & 0 < q_1 < a-c \\ 0, & q_1 \geq a-c \end{cases}$$

$\boxed{\text{all} < 0}$

相当于redundant的范围.

$$q_1^* = \arg\max_{q_1} u_1(q_1, R_2(q_1))$$

在argmax可忽略

$$= \arg\max_{q_1} \begin{cases} \boxed{u_1(q_1, 0) \quad q_1 \geq a-c} \\ u_1\left(q_1, \frac{a-c-q_1}{2}\right) \quad 0 < q_1 < a-c \end{cases}$$

扔掉

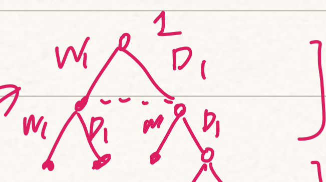
$$R_1\left(q_1, \frac{a-c-q_1}{2}\right) = q_1\left(a - c - \frac{a-c+q_1}{2}\right)$$

$$= q_1\left(\frac{a-c}{2} - \frac{q_1}{2}\right)$$

$$\Rightarrow \boxed{q_1^* = \frac{a-c}{2}, \quad q_2^* = \frac{a-c}{4}} \to \text{BIO}$$

---

② Subgame Perfect Outcome

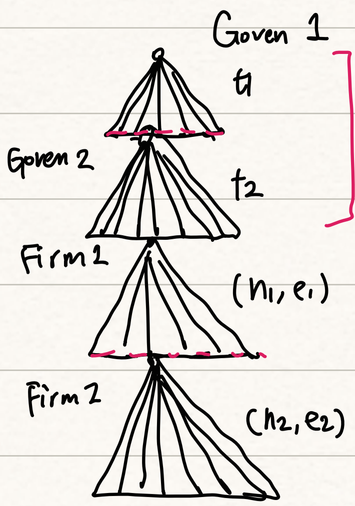


$t=2$

Example 1. $\boxed{\text{Bank Run}}$ → $\boxed{\text{easy task}}$

Example 2. $\boxed{\text{Tariff Game}}$ → consider $\boxed{\text{backwards}}$

$\Rightarrow t=1$



$\boxed{\text{Firm } i}$

$t=1$ $\quad \pi_i(t_i, t_j; h_i, e_i, h_j, e_j)$

$$= (a-h_i-e_j)h_i + (a-h_j-e_i)e_i - c(h_i+e_i) - t_j e_i$$

$t=2$ $$= (a-h_i-e_j-c)h_i + (a-h_j-e_i-c-t_j)e_i$$

$\boxed{\text{Goven } i}$

$$W_i(t_i, t_j; e_i, h_i, e_j, h_j)$$

$$= \frac{1}{2}Q_i^2 + \pi_i + e_i t_j$$

$$= \frac{1}{2}(h_i+e_j)^2 + \pi_i + e_i t_j$$

$\boxed{\text{Analysis:}}$ subgame $(h_1, e_1; h_2, e_2)$

$$(h_1^*, e_1^*) = \underset{(h_1, e_1)}{\arg\max}\ \pi_1(h_1, e_1, h_2, e_2; t_1, t_2)$$

① $\boxed{\text{Analyse the NE for subgame}}$

$$= \underset{h_1}{\arg\max}\ (a-h_1-e_2-c)h_1 \ \& \ \underset{e_1}{\arg\max}\ (a-h_2-e_1-c-t_2)e_1$$

$$\Rightarrow h_1^* = \begin{cases} \frac{a-c-e_2}{2}, & 0 < e_2 \le a-c \\ 0, & e_2 > a-c \end{cases} \qquad e_1^* = \begin{cases} \frac{a-c-t_2-h_2}{2}, & 0 < t_2+h_2 \le a-c \\ 0, & t_2+h_2 \ge a-c \end{cases}$$

Best Response Analysis → $R_1(h_2, e_2; t_1, t_2)$ & $R_2(h_1, e_1; t_1, t_2)$

similarly: $$h_2^* = \begin{cases} \frac{a-c-e_1}{2} & 0 < e_1 \le a-c \\ 0, & e_1 > a-c \end{cases} \qquad e_2^* = \begin{cases} \frac{a-c-t_2-h_1}{2} & 0 < t_1+h_1 < a-c \\ 0, & t_1+h_1 \ge a-c \end{cases}$$

Here, $\boxed{t_1, t_2 \text{ are fixed}}$ $\boxed{\text{what the real process then?}}$

Here we can actually achieve the NE for each fix pair (t_1^*, t_2^*)

i.e., $$\begin{cases} (h_1^*, e_1^*)(t_1^*, t_2^*) \\ (h_2^*, e_2^*)(t_1^*, t_2^*) \end{cases}$$

Then, consider $\boxed{W_1^*(t_1, t_2)}$ firm 1 response (best)

$= W_1(t_1, t_2; (h_1^*, e_1^*)(t_1, t_2), (h_2^*, e_2^*)(t_1, t_2))$

firm 2 Best Response

$\boxed{W_2^*(t_1, t_2)}$

$\begin{cases} W_1^* \\ W_2^* \end{cases} \to$ step 1-game $\Rightarrow$ keep finding Nash Eq.

② Remember: $h_1^* = \begin{cases} \frac{a-c-e_2}{2}, & 0 < e_2 \leq a-c \\ 0, & e_2 > a-c \end{cases}$ $\quad e_1^* = \begin{cases} \frac{a-c-h_2-t_2}{2}, & \text{else} \\ 0, & h_2+t_2 > a-c \end{cases}$

$h_2^* = \begin{cases} \frac{a-c-e_1}{2}, & 0 < e_1 \leq a-c \\ 0, & e_1 > a-c \end{cases}$ $\quad e_2^* = \begin{cases} \frac{a-c-h_1-t_1}{2}, & \text{else} \\ 0, & h_1+t_1 > a-c \end{cases}$

want to find $\boxed{\text{NE}}$ for $\boxed{\text{Subgame}}$ $(h_1^*, e_1^*; h_2^*, e_2^*)$ for (t_1, t_2)

$\boxed{\text{Actually}}$ → for different (t_1, t_2), the NE can be very different.

→ for example, if $t_1, t_2 > a-c$, then

(just example) $\begin{cases} e_2^* = 0, \ e_1^* = 0 \\ h_1^* = \frac{a-c}{2}, \ h_2^* = \frac{a-c}{2} \end{cases} \Rightarrow$ NE for $\begin{cases} t_1 \\ t_2 \end{cases}$ large

→ Here, we suppose the (t_1, t_2) s.t NE is:

$$\begin{cases} h_1^* = \frac{a-c-e_2^*}{2} \\ h_2^* = \frac{a-c-e_1^*}{2} \\ e_1^* = \frac{a-c-h_2^*-t_2}{2} \\ e_2^* = \frac{a-c-h_1^*-t_1}{2} \end{cases} \Rightarrow \underbrace{\begin{cases} e_2^* = \frac{a-c-2t_1}{3} \\ e_1^* = \frac{a-c-2t_2}{3} \\ h_2^* = \frac{a-c+t_2}{3} \\ h_1^* = \frac{a-c+t_1}{3} \end{cases}}_{\text{our interested NE}}$$

③ 1-stage game NE

Now we have for some range of t_1 & t_2,

the subgame NE is
$$\begin{cases} e_1^* = \dfrac{a-c-2t_2}{3} \\ e_2^* = \dfrac{a-c-2t_1}{3} \\ h_1^* = \dfrac{a-c+t_1}{3} \\ h_2^* = \dfrac{a-c+t_2}{3} \end{cases}$$

then $W_1^*(t_1,t_2) = W_1(t_1, t_2 ; (e_1^*, e_2^* ; h_1^*, h_2^*))$

$$= \frac{1}{2}Q_1^2 + \pi_1 + t_1 e_2^*$$

$$= \frac{1}{2}(e_2^* + h_1^*)^2 + (a - h_1^* - e_2^* - c)h_1^* + (a - h_2^* - e_1^* - c - t_2)e_1^* + t_1 e_2^*$$

$$= \frac{1}{2}\left(\frac{2a-2c-t_1}{3}\right)^2 + \left(\frac{a-c+t_1}{3}\right)\frac{a-c+t_1}{3} + \left(\frac{a-c-2t_2}{3}\right)\frac{a-c-2t_2}{3} + t_1\frac{a-c-2t_1}{3}$$

$R_1^*(t_2)$
$\|$
$t_1^* = \arg\max_{t_1} W_1^*(t_1, t_2)$

$$\Rightarrow \frac{dW_1^*}{dt_1} = \left(\frac{2a-2c-t_1}{3}\right)\left(-\frac{1}{3}\right) + 2\cdot\left(\frac{a-c+t_1}{3}\right)\left(\frac{1}{3}\right) + \cancel{2\left(\frac{a-c-2t_2}{3}\right)\cdot\left(-\frac{2}{3}\right)} + \frac{a-c-4t_1}{3}$$

$$= \frac{t_1-2a+2c}{9} + \frac{2a-2c+2t_1}{9} + \frac{a-c-4t_1}{3}$$

$$= \frac{a-c-3t_1}{3} = 0 \Rightarrow t_1 = \frac{a-c}{3}$$

similarly: $t_2 = \frac{a-c}{3}$

$\Rightarrow (t_1^*, t_2^*) = \left(\frac{a-c}{3}, \frac{a-c}{3}\right)$

Comment

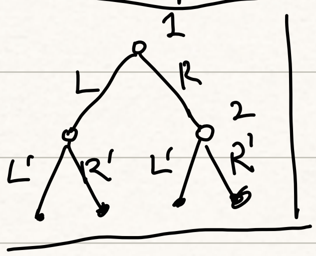
$\Rightarrow$ At least when we fix the range of (t_1, t_2) to some specific range (intervals),

then $\left(\frac{a-c}{3}, \frac{a-c}{3} ; \frac{4(a-c)}{9}, \frac{a-c}{9} ; \frac{4(a-c)}{9}, \frac{a-c}{9}\right)$ is S-P Outcome!

$\quad t_1 \quad t_2 \quad h_1 \quad e_1 \quad h_2 \quad e_2$

接下来是关于 Strategy → 有点微妙

Example 1



S_1: Starting $\to A_1$ $\Rightarrow$ $S_1 = \{L, R\}$

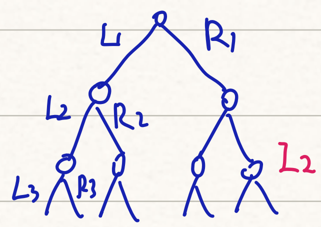
S_2: $A_1 \xrightarrow{I_1} A_2$ $\Rightarrow$ $S_2 = \{(*,*),\ * \in \{L, R\}\}$

$L \mapsto *$

$R \mapsto *$

→ translate to static game

Example 2



①: player 1 → 2 → 3.

$\hat{S}_1 = \{L_1, R_1\}$ 确定一个 elem.

$\hat{S}_2$: $A_1 \xrightarrow{I_1} A_2$

$L_1 \mapsto *$

$R_1 \mapsto *$

确定 2个 elem

→ 对于全连接 & no block, 这种 strategy construction 比较合适

$\hat{S}_3$: $A_1 \times A_2 \xrightarrow{I_2} A_3$

$(*_1, *_2) \mapsto *_3$

确定 4个 elem

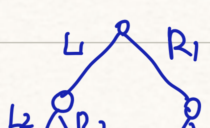
② player 1 → 2 → 1.

$S_1 = \{\hat{S}_1, \hat{S}_3\}$

$S_2 = \{\hat{S}_2\}$

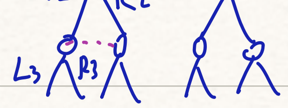
→ ☆ the construction for one player plays many times

Example 3 → 我理解: information set is just a local block



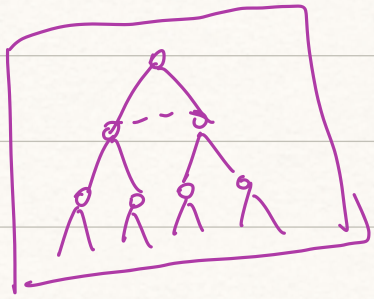
$S_1 = \hat{S}_1$?

$S_2 = \hat{S}_2$ I_2



S_3 : ~~[illegible]~~ $\longrightarrow A_3$

$\begin{cases} (L_1, *_2) \longmapsto *_3 & \text{(Both)} \\ (R_1, *_2) \longmapsto x_3 & \text{(respectively)} \end{cases}$ → 3个 need to be determined



或许：本质上来说，strategy should be defined on information set, i.e., $\boxed{s_k : I_k \longrightarrow A}$

但是，对于 perfect information 博弈.

I_k just is $\boxed{A_1 \times \dots \times A_k}$

Stackelberg's model of Duopoly → 1 BIO

→ infinitely many strategy NE

for $\forall\, \bar{q}_1 \in (0, a-c)$, we can construct

$s_2^*: A_1 \longrightarrow A_2$

$\bar{q}_1 \longmapsto R_2(\bar{q}_1)$

$q \longmapsto a$ other q

if $(s_1, s_2) \in NE$

$\Leftrightarrow \begin{cases} s_1 \in \tilde{r}_1(s_2) \\ s_2 \in \tilde{r}_2(s_1) \end{cases}$

$\tilde{r}_2(s_1) = \underset{s_2 \in S_2}{\operatorname{argmax}}\ \tilde{u}_2(s_1, s_2)$

$= \operatorname{argmax}\ u_2(s_1, s_2(s_1))$

$\Leftrightarrow s_2 : s_1 \longmapsto R_2(s_1)$

prove: ① $\bar{q}_1 \in \bar{R}_1(s_2^*)$

② $s_2^* \in \bar{R}_2(\bar{q}_1)$

①
$\bar{R}_1(s_2^*) = \underset{q}{\operatorname{argmax}}\ \bar{u}_1(q, s_2^*)$

$= \operatorname{argmax}\ u_1(q, s_2^*(q))$

since $u_1(\bar{q}_1, s_2^*(\bar{q}_1))$

$= u_1(\bar{q}_1, R_2(\bar{q}_1))$

≥ 0

other $u_1(q, s_2^*(q))$

$= u_1(q, a) < 0$

②
$s_2^* \in \underset{s_2}{\operatorname{argmax}}\ \bar{u}_2(\bar{q}_1, s_2)$

$\Leftrightarrow s_2^* \in \underset{s_2}{\operatorname{argmax}}\ u_2(\bar{q}_1, s_2(\bar{q}_1))$

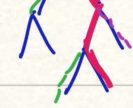
$\Leftrightarrow s_2(\bar{q}_1) = R_2(\bar{q}_1)$

$\Leftrightarrow$ map $\bar{q}_1$ to $R_2(\bar{q}_1)$

Connection (Relation) between Outcome & NE (Strategy)

(2- stage game)

s_1 不可能



$(s_1^* \times s_3^*, s_2^*) \to$ BIO strategy

↓ deviate

$(\tilde{s}_1 \times \tilde{s}_3, \tilde{s}_2) \to$ larger payoff for P1 or P2

① only change $s_1^* \Rightarrow$ impossible

② exists change on s_2^* & s_3^*

① For BIO outcome $(a_1^*, R_2(a_1^*))$

↓

$(a_1^*, R_2(\cdot))$ is a NE

$\boxed{\text{Pf}}$

$(a_1^*, R_2(\cdot))$ is NE

$$\Leftrightarrow \begin{cases} a_1^* \in \overline{R}_1(R_2(\cdot)) \\ R_2(\cdot) \in \overline{R}_2(a_1^*) \end{cases}$$

the choice of a_1^* when we use BIO

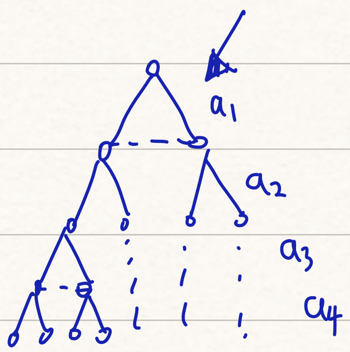
$$\Leftrightarrow \begin{cases} a_1^* \in \arg\max_{a_1} \bar{u}_1(a_1, R_2(\cdot)) = \arg\max_{a_1} u_1(a_1, R_2(a_1)) \\ R_2(\cdot) \in \arg\max_{s_2} \bar{u}_2(a_1^*, s_2) = \arg\max_{s_2} u_2(a_1^*, s_2(a_1^*)) \end{cases}$$

$$= \arg\max_{a_2} u_2(a_1^*, a_2)$$

$$= \{s_2 : a_1^* \mapsto R_2(a_1^*)\}$$

只要保证这映射的值

② for 2-stage game with $\begin{cases}\text{complete information} \\ \text{imperfect}\end{cases}$



(seems not proof)

$\boxed{\text{Result:}}$ if $(a_3^*(a_1,a_2), a_4^*(a_1,a_2))$ is the NE

$\updownarrow$

$(a_3, a_4): A_1 \times A_2 \to A_3 \times A_4$

and (a_1^*, a_2^*) is the NE of

$$\to \begin{cases} u_1(a_1, a_2, a_3^*(a_1,a_2), a_4^*(a_1,a_2)) \\ u_2(a_1, a_2; a_3^*(a_1,a_2), a_4^*(a_1,a_2)) \end{cases}$$

Then $(a_1^*, a_2^*, a_3^*(a_1^*, a_2^*), a_4^*(a_1^*, a_2^*))$

is the Subgame-Perfect Outcome

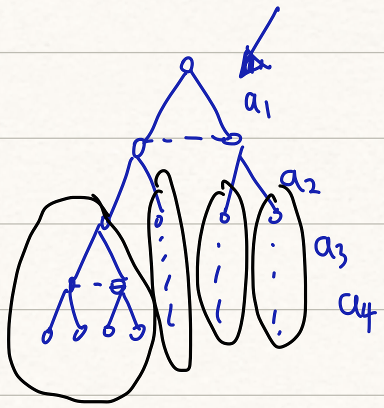
AND! $\boxed{(a_1^*, a_2^*; a_3^*(\cdot,\cdot), a_4^*(\cdot,\cdot))}$

is the strategy NE

Actually it is one strategy (a complete plan)

since there are too many NE!

$\boxed{\text{Subgame - Perfect NE}}$



$\boxed{\text{4 subgames!}}$

For ONE strategy for WHOLE GAME,

↓

can decompose to one sub-strategy for one sub-game

⇓

check whether this sub-strategy is NE for the sub-game

Some Results about infinite Game

① Sequential Bargain

↓

Outcome

② (infinite) 2-person Prisoners' Dilemma

↓

Strategy

⇓

NE → { (NC_i, NC_i), (T_i, T_i)

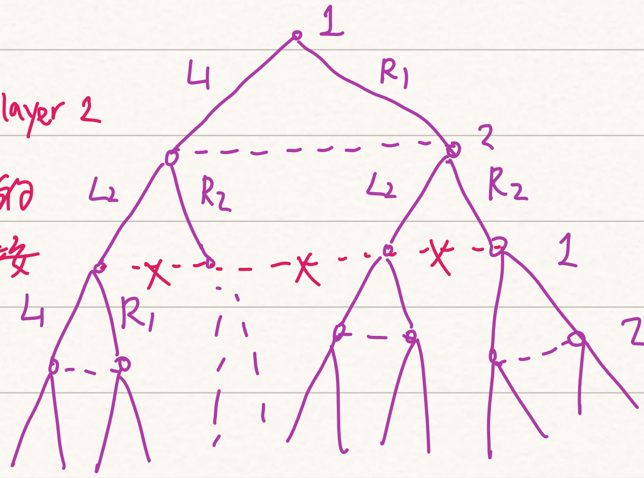
have condition

↓

δ should be relatively Big

The simplest subgame perfect game

如果要表示 player 2 看不到第一轮的 action, then 按红色连起来



$\hat{S}_1 = \{L_1, R_1\}$

$\hat{S}_2 = \{L_2, R_2\}$

$\hat{S}_3 = \{****\}$, 16个

$\hat{S}_4 = \{****\}$, 16个.

player 1's strategy: $S_1 = \hat{S}_1 \times \hat{S}_3$

player 2's strategy: $S_2 = \hat{S}_2 \times \hat{S}_4$

→ 每一轮的 strategy → information set 的 correspondance

① 在完全树情况, 可记为

stage t+1, player 1: $\{\underbrace{A_1 \times \dots \times A_t}_{\text{history}} \longrightarrow A_{t+1,1}\} \rightarrow \boxed{S_{t+1,2}}$

stage t+1, player 2: $\{\underbrace{A_1 \times \dots \times A_t \times A_{t+1,1}}_{\text{history}} \longrightarrow A_{t+1,2}\} \rightarrow \boxed{S_{t+1,2}}$

$\boxed{A_t = A_{t,1} \times A_{t,2}}$ ↗

strategy for player 1 $S_1 = S_{1,2} \times \dots \times S_{t,1} \times \dots$

$S_2 = S_{1,2} \times \dots \times S_{t,2} \times \dots$

② infinitely repeat game

stage t player 1: $\{\boxed{A_1 \times \dots \times A_{t-1}} \longrightarrow A_{t,1}\} := S_{t,2}$ → information set

stage t: player 2: $\{\boxed{A_1 \times \dots \times A_{t-1}} \longrightarrow A_{t,2}\} := S_{t,2}$

↳ Since stage t's game is Static Game

Strategy for player 1 → $S_1 = S_{1,1} \times \dots \times S_{t,2} \times \dots$

Strategy for player 2 → $S_2 = S_{1,2} \times \dots \times S_{t,2} \times \dots$

① when information set (number) is small then we may define strategy (stage)

as $S: I \longrightarrow$ Action Space

② when information set tend to be more

(as infinitely many repeated game)

it may be more convenient to use

$\boxed{A_1 \times \cdots \times A_{t-1}} \longrightarrow$ Action Space

$\Rightarrow$ is actually another expression of information set

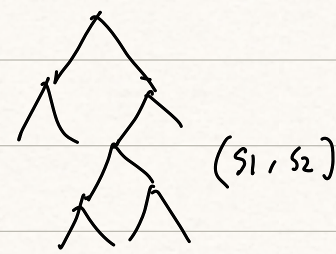
---

想法: $(s_1^*, s_2^*) \in NE$ (S-P Strategy) $\rightarrow$ $\boxed{\text{Perfect Game}}$

$\rightarrow$ Since Subgame-Perfect Game,

then In the Smallest Game (sub),

we should play the Best Response

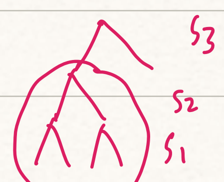


$\Rightarrow$ Strategy in last stage must be $\overset{\hat{s}_1}{\boxed{\text{Best Response}}}$

$\rightarrow$ last second stage sub-game $(\boxed{\hat{s}_1}, \hat{s}_2) \in NE$ (subgame) fix

Best response $\hat{s}_1 \Rightarrow \hat{s}_1 \in$ Best Response $(\hat{s}_2)$

we still need $\hat{s}_2 \in$ Best Response $(\hat{s}_1)$



$(s_1 \times s_3, s_2)$

⇒ Large payoff than

NE (s_1, s_2)

NE ⇒ no deviation

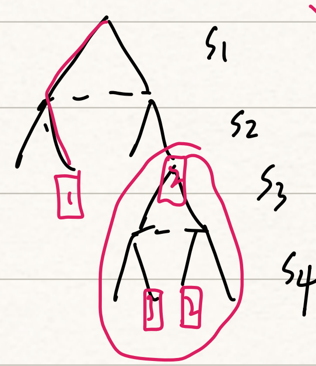
NE → no deviation

since NE has large payoff

$(s_1, s_2) \in NE$

↓

not deviate



$(s_1 \times s_3,\ s_2 \times s_4) \in NE$

$s_3 \quad s_4$

$(s_1 \times s_3,\ s_2 \times s_4) \in NE$

Analysis

→ ① NE ⇒ no deviation to ②

↓

NE of subgame

if ② → ③ ⇒ contradiction to ② NE