Assignment 1 DSA5103 A0236307J

Q1. Convexity

(1) $f(x) = x^2$

Pf: $f(x)$ is convex $\Leftrightarrow$ $\forall x, y$, $\lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$$

Denote
$$\begin{aligned}
F(\lambda) &= f(\lambda x + (1-\lambda) y) - \lambda f(x) - (1-\lambda) f(y) \\
&= (\lambda x + (1-\lambda) y)^2 - \lambda x^2 - (1-\lambda) y^2 \\
&= \lambda^2 x^2 + 2\lambda(1-\lambda) xy + (1-\lambda)^2 y^2 - \lambda x^2 - (1-\lambda) y^2 \\
&= \lambda(\lambda-1) x^2 + 2\lambda(1-\lambda) xy + \lambda(\lambda-1) y^2 \\
&= \lambda(\lambda-1)\left[x^2 + 2xy + y^2\right] \\
&= \lambda(\lambda-1)(x-y)^2 \le 0 \quad \text{for } \forall x, y \text{ and } \lambda \in [0,1]
\end{aligned}$$

That is, we show that:

For arbitrary $x, y \in \mathbb{R}$, $\lambda \in [0,1]$, we all have

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$$

$\Leftrightarrow$ $f(x) = x^2$ is convex function.

(2) $f(x) = x_1^2 + x_2^2 + 2x_1 + 4$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Pf: denote $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\lambda x + (1-\lambda) y = \begin{bmatrix} \lambda x_1 + (1-\lambda) y_1 \\ \lambda x_2 + (1-\lambda) y_2 \end{bmatrix}$

we want to show:

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$$

This can be proved through conclusion of (1):

For simplicity, we denote $g(x) = x^2$ and $f(x) = g(x_1) + g(x_2) + 2x_1 + 4$

$$f(\lambda x + (1-\lambda) y) = (\lambda x_1 + (1-\lambda) y_1)^2 + (\lambda x_2 + (1-\lambda) y_2)^2 + 2(\lambda x_1 + (1-\lambda) y_1) + 4$$

$$= g(\lambda x_1 + (1-\lambda) y_1) + g(\lambda x_2 + (1-\lambda) y_2)$$
$$+ \lambda \cdot (2x_1) + (1-\lambda)\cdot(2y_1) + \lambda \cdot 4 + (1-\lambda)\cdot 4$$

From convexity of $g(x) = x^2$, which we have proved in (1)

$$\leq \lambda g(x_1) + (1-\lambda) g(y_1) + \lambda g(x_2) + (1-\lambda) g(y_2)$$
$$+ \lambda \cdot (2x_1) + (1-\lambda)\cdot(2y_1) + \lambda \cdot 4 + (1-\lambda)\cdot 4$$
$$= \lambda (g(x_1) + g(x_2) + 2x_1 + 4)$$
$$+ (1-\lambda) (g(y_1) + g(y_2) + 2y_1 + 4)$$

From $f(x) = g(x_1) + g(x_2) + 2x_1 + 4$ where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= \lambda \cdot f(x) + (1-\lambda) f(y)$$

That is, we show that:

For arbitrary $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$, $\lambda \in [0, 1]$, we all have:

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

$\Leftrightarrow$ $\underline{f(x) = x_1^2 + x_2^2 + 2x_1 + 4}$ is convex function