Panel Econometrics, Problem Set 3

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1. I did the computations for this one in Julia. The code file is attached.

   (a)

   $$\hat{\alpha} = 0.57$$
   $$\hat{\beta} = 1.01$$

   (b)

   $$\hat{\alpha} \ \text{mean bias} \ = 0.0064$$
   $$\hat{\beta} \ \text{mean bias} \ = -0.0069$$
   $$\hat{\alpha} \ \text{MSE} \ = 0.015$$
   $$\hat{\beta} \ \text{MSE} \ = 0.017$$

   (c)

   $$\hat{\alpha}_{100} \ \text{MSE} \ = 0.015$$
   $$\hat{\beta}_{100} \ \text{MSE} \ = 0.017$$
   $$\hat{\alpha}_{200} \ \text{MSE} \ = 0.0067$$
   $$\hat{\beta}_{200} \ \text{MSE} \ = 0.0078$$
   $$\hat{\alpha}_{400} \ \text{MSE} \ = 0.0036$$
   $$\hat{\beta}_{400} \ \text{MSE} \ = 0.0039$$

   MSE decreases as the number of observations increases! That's not surprising, given that MLE is a consistent estimator.

2. m is the median of the random variable X if $P(X \leq m) \geq 1/2 \ \wedge \ P(X \geq m) \geq 1/2$. If X is continuously distributed, then

   $$\int_{-\infty}^{m} f(x) \, dx = \int_{m}^{+\infty} f(x) \, dx = \frac{1}{2}.$$

   Show that

(a) $\min_a \mathbb{E}[|X - a|] = \mathbb{E}[|X - m|]$.

First, note that
$$\mathbb{E}[|X - a|] = \begin{cases} \mathbb{E}[X] - a, & \text{if } X \geq a \\ a - \mathbb{E}[X], & \text{if } X < a \end{cases}$$

Suppose $a = \arg\min_a \mathbb{E}[|X - a|]$, but $a \neq m$. Then $P(X \leq a) < 1/2 \ \vee \ P(X \geq a) < 1/2$.

If $P(X \leq a) < 1/2$, then $\mathbb{E}[X] > a$, so there exists an $\tilde{a} > a$ such that $\mathbb{E}[|X - \tilde{a}|] < \mathbb{E}[|X - a|]$, so $a \neq \arg\min_a \mathbb{E}[|X - a|]$.

If $P(x \geq a) < 1/2$, then $\mathbb{E}[X] < a$, so there exists an $\tilde{a} < a$ such that $\mathbb{E}[|X - \tilde{a}|] < \mathbb{E}[|X - a|]$, so $a \neq \arg\min_a \mathbb{E}[|X - a|]$.

Thus, we must have $P(X \leq a) \geq 1/2 \ \wedge \ P(X \geq a) \geq 1/2$, so $a = m$. ∎

(b) Let $Y = g(X)$, where $g(\cdot)$ is weakly monotonic. Show that $g(m)$ is the median for Y.

To start, we know that g is weakly monotone, but not whether it is nondecreasing or nonincreasing. If g is nondecreasing, then
$$g(X) \leq g(m) \iff X \leq m,$$
$$g(X) \geq g(m) \iff X \geq m.$$

Likewise, if g is nonincreasing, then
$$g(X) \leq g(m) \iff X \geq m,$$
$$g(X) \geq g(m) \iff X \leq m.$$

Then, if g is nondecreasing,
$$P(Y \leq g(m)) = P(g(X) \leq g(m)) = P(X \leq m) \geq 1/2$$
$$\wedge$$
$$P(Y \geq g(m)) = P(g(X) \geq g(m)) = P(X \geq m) \geq 1/2$$

And if g is nonincreasing, we have
$$P(Y \leq g(m)) = P(g(X) \leq g(m)) = P(X \geq m) \geq 1/2$$
$$\wedge$$
$$P(Y \geq g(m)) = P(g(X) \geq g(m)) = P(X \leq m) \geq 1/2$$

Thus the median of Y is $g(m)$. ∎

(c) Suppose that we are interested in working with the αth quantile of X, for some $\alpha \in (0, 1)$, denoted by m_α and assumed to satisfy

$$P(X \leq m_\alpha) \geq \alpha \ \wedge \ P(X \geq m_\alpha) \geq 1 - \alpha$$

i. Repeat part (a), replacing the absolute value function with the following "check function"
$$\rho_\alpha(x) = |x| + (2\alpha - 1)x.$$

We want to show that $\min_a \mathbb{E}[\rho_\alpha(X - a)] = \mathbb{E}[\rho_\alpha(X - m_\alpha)]$. We'll follow the same strategy as in part (a), and begin by noting that

$$\mathbb{E}[\rho_\alpha(X - a)] = \begin{cases} 2\alpha(\mathbb{E}[X] - a), & \text{if} \quad X \geq a \\ (2\alpha - 2)(\mathbb{E}[X] - a), & \text{if} \quad X < a \end{cases}$$

Suppose that $a = \arg\min_a \mathbb{E}[\rho_\alpha(X - a)]$, but that $a \neq m_\alpha$. Then $P(X \leq a) < \alpha \ \vee \ P(X \geq a) < 1 - \alpha$.

If $P(X \leq a) < \alpha$, then there is a $\tilde{a} > a$ such that $\mathbb{E}[\rho_\alpha(X - \tilde{a})] < \mathbb{E}[\rho_\alpha(X - a)]$, so $a \neq \arg\min_a \mathbb{E}[\rho_\alpha(X - a)]$.

If $P(X \geq a) < 1 - \alpha$, then there exists a $\tilde{a} < a$ such that $\mathbb{E}[\rho_\alpha(X - \tilde{a})] < \mathbb{E}[\rho_\alpha(X - a)]$, so $a \neq \arg\min_a \mathbb{E}[\rho_\alpha(X - a)]$.

Thus, is must be that $P(X \leq m_\alpha) \geq \alpha \wedge P(X \geq m_\alpha) \geq 1 - \alpha$, so $a = m_\alpha$. $\blacksquare$

ii. Does the "equivariance property" you proved in part (b) hold?

Suppose $Y = g(X)$, where $g(\cdot)$ is weakly monotonic. The equivariance property holds if $g(m_\alpha)$ is the αth quantile of Y, which is the case if and only if
$$P(Y \leq g(m_\alpha)) \geq \alpha \ \wedge \ P(Y \geq g(m_\alpha)) \geq 1 - \alpha$$

Monotonicity of g has the same inequality implications as in part (b), so if g is nondecreasing, we have

$$P(Y \leq g(m_\alpha)) = P(g(X) \leq g(m_\alpha)) = P(X \leq m_\alpha) \geq \alpha$$
$$\wedge$$
$$P(Y \geq g(m_\alpha)) = P(g(X) \geq g(m_\alpha)) = P(X \geq m_\alpha) \geq 1 - \alpha$$

Which satisfies the equivariance condition. But if g is nonincreasing, then

$$P(Y \leq g(m_\alpha)) = P(g(X) \leq g(m_\alpha)) = P(X \geq m_\alpha) \geq 1 - \alpha$$
$$\wedge$$
$$P(Y \geq g(m_\alpha)) = P(g(X) \geq g(m_\alpha)) = P(X \leq m_\alpha) \geq \alpha$$

Which does not satisfy the equivariance condition. So the equivariance property only holds for $\alpha = 1/2$, or with an arbitrary α when g is non-decreasing. ∎

3. Assume that ϵ_1 and ϵ_2 are i.i.d. continuouslu distributed random variables, with common density function f. Let $\epsilon_3 = \epsilon_2 - \epsilon_1$.

   (a) Show that ϵ_3 is symmetrically distributed around zero.

   ϵ_3 is symmetrically distributed around 0 iff $P(\epsilon_3 \leq x) = P(\epsilon_3 \geq -x)$ for all x.

   Let F_{12} denote the CDF of $\epsilon_1 - \epsilon_2$ and let F_{21} denote the CDF of $\epsilon_2 - \epsilon_1$. Since ϵ_1 and ϵ_2 follow the same distribution, these CDFs must be identical: $F_{12}(x) = F_{21}(x)$ for all x. Take some $x \in \mathbb{R}$. Then

   $$
   \begin{aligned}
   P(\epsilon_3 \leq x) &= P(\epsilon_2 - \epsilon_1 \leq x) \\
   &= F_{21}(x) \\
   &= F_{12}(x) \\
   &= P(\epsilon_1 - \epsilon_2 \leq x) \\
   &= P(\epsilon_3 \geq -x) \quad ∎
   \end{aligned}
   $$

   (b) Suppose now that ϵ_1 and ϵ_2 are independently and symmetrically distributed around 0, but they follow different distributions. Show that ϵ_3 is still symmetrically distributed around 0.

   Notationally, I'm going to let ϵ denote an unrealized random variable, and ε denote a point. $\epsilon_3 = \epsilon_2 - \epsilon_1$ is the convolution of two random variables. If f_1 is the PDF of ϵ_1 and f_2 is the PDF of ϵ_2, then the PDF of ϵ_3 is

   $$
   f_3(\varepsilon_3) = \int_{-\infty}^{\infty} f_2(\varepsilon_3 + \varepsilon_1) f_1(-\varepsilon_1) \, d\varepsilon_1
   $$

   ε_3 is symmetrically distributed around 0 iff $f_3(x) = f_3(-x)$ for all x. Take some $x \in \mathbb{R}$, and let $\eta = -\varepsilon_1$. Then

   $$
   \begin{aligned}
   f_3(x) &= \int_{-\infty}^{\infty} f_2(x + \varepsilon_1) f_1(-\varepsilon_1) \, d\varepsilon_1 \\
   &= \int_{-\infty}^{\infty} f_2(-x - \varepsilon_1) f_1(\varepsilon_1) \, d\varepsilon_1 \\
   &= \int_{-\infty}^{\infty} f_2(-x + \eta) f_1(-\eta) \, d\eta \\
   &= f_3(-x) \quad ∎
   \end{aligned}
   $$

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4. Show equivalence between the following versions of the maximum score estimator:

(a) $\hat{\beta}_a = \arg\max_{||\beta||=1} \frac{1}{n} \sum_{i=1}^{n} y_i \mathbb{1}(x_i'\beta > 0) + (1 - y_i)\mathbb{1}(x_i'\beta < 0)$

(b) $\hat{\beta}_b = \arg\max_{||\beta||=1} \frac{1}{n} \sum_{i=1}^{n} (2y_i - 1)\mathbb{1}(x_i'\beta > 0)$

(c) $\hat{\beta}_c = \arg\max_{||\beta||=1} \frac{1}{n} \sum_{i=1}^{n} |y_i - \mathbb{1}(x_i'\beta > 0)|$

(d) $\hat{\beta}_d = \arg\max_{||\beta||=1} \frac{1}{n} \sum_{i=1}^{n} (y_i - \mathbb{1}(x_i'\beta > 0))^2$

$\hat{\beta}_a = \hat{\beta}_b$:

$$\hat{\beta}_a = \arg\max_{||\beta||=1} \frac{1}{n} \sum_{i=1}^{n} y_i \mathbb{1}(x_i'\beta > 0) + (1 - y_i)\mathbb{1}(x_i'\beta < 0)$$

$$= \arg\max_{||\beta||=1} \frac{1}{n} \sum_{i=1}^{n} y_i \mathbb{1}(x_i'\beta > 0) + (1 - y_i)(1 - \mathbb{1}(x_i'\beta > 0))$$

$$= \arg\min_{||\beta||=1} \frac{1}{n} \sum_{i=1}^{n} (2y_i - 1)\mathbb{1}(x_i'\beta > 0) - y_i + 1$$

$$= \arg\min_{||\beta||=1} \frac{1}{n} \sum_{i=1}^{n} (2y_i - 1)\mathbb{1}(x_i'\beta > 0)$$

$$= \hat{\beta}_b$$

$\hat{\beta}_a = \hat{\beta}_c$:
Let $A = y_i \mathbb{1}(x_i'\beta > 0) + (1 - y_i)\mathbb{1}(x_i'\beta < 0)$ and let $C = |y_i - \mathbb{1}(x_i'\beta > 0)|$. y_i and $\mathbb{1}(x_i'\beta > 0)$ only take values in $\{0, 1\}$, so consider the following four cases:

y_i	$\mathbb{1}(x_i'\beta > 0)$	A	C
0	0	1	0
0	1	0	1
1	0	0	1
1	1	1	0

Since $\hat{\beta}_a$ is an argmax and $\hat{\beta}_c$ is an argmin, A and C taking opposite values means that $\hat{\beta}_a$ and $\hat{\beta}_c$ will take the same value.

$\hat{\beta}_c = \hat{\beta}_d$:
Since $(y_i - \mathbb{1}(x_i'\beta > 0)) \in \{-1, 0, 1\}$, the absolute value is equivalent to the square. ∎