Panel Econometrics, Problem Set 3

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1. I did the computations for this one in Julia. The code file is attached.

   (a)

$$\hat{\alpha} = 0.57$$
$$\hat{\beta} = 1.01$$

   (b)

$$\begin{aligned} \hat{\alpha} \text{ mean bias } &= 0.0064 \\ \hat{\beta} \text{ mean bias } &= -0.0069 \\ \hat{\alpha} \text{ MSE } &= 0.015 \\ \hat{\beta} \text{ MSE } &= 0.017 \end{aligned}$$

   (c)

$$\begin{aligned} \hat{\alpha}_{100} \text{ MSE } &= 0.015 \\ \hat{\beta}_{100} \text{ MSE } &= 0.017 \\ \hat{\alpha}_{200} \text{ MSE } &= 0.0067 \\ \hat{\beta}_{200} \text{ MSE } &= 0.0078 \\ \hat{\alpha}_{400} \text{ MSE } &= 0.0036 \\ \hat{\beta}_{400} \text{ MSE } &= 0.0039 \end{aligned}$$

   MSE decreases as the number of observations increases! That's not surprising, given that MLE is a consistent estimator.

2. m is the median of the random variable X if $P(X \leq m) \geq 1/2 \wedge P(X \geq m) \geq 1/2$. If X is continuously distributed, then

$$\int_{-\infty}^{m} f(x)\,\mathrm{d}x = \int_{m}^{+\infty} f(x)\,\mathrm{d}x = \frac{1}{2}.$$

   Show that

(a) $\min_a \mathbb{E}[|X-a|] = \mathbb{E}[|X-m|]$.

First, note that

$$\mathbb{E}[|X-a|] = \begin{cases} \mathbb{E}[X]-a, & \text{if } X \geq a \\ a-\mathbb{E}[X], & \text{if } X < a \end{cases}$$

Suppose $a = \arg\min_a \mathbb{E}[|X-a|]$, but $a \neq m$. Then $P(X \leq a) < 1/2 \ \vee \ P(X \geq a) < 1/2$.

If $P(X \leq a) < 1/2$, then $\mathbb{E}[X] > a$, so there exists an $\tilde{a} > a$ such that $\mathbb{E}[|X-\tilde{a}|] < \mathbb{E}[|X-a|]$, so $a \neq \arg\min_a \mathbb{E}[|X-a|]$.

If $P(x \geq a) < 1/2$, then $\mathbb{E}[X] < a$, so there exists an $\tilde{a} < a$ such that $\mathbb{E}[|X-\tilde{a}|] < \mathbb{E}[|X-a|]$, so $a \neq \arg\min_a \mathbb{E}[|X-a|]$.

Thus, we must have $P(X \leq a) \geq 1/2 \ \wedge \ P(X \geq a) \geq 1/2$, so $a = m$. ■

(b) Let $Y = g(X)$, where $g(\cdot)$ is weakly monotonic. Show that $g(m)$ is the median for Y.

To start, we know that g is weakly monotone, but not whether it is nondecreasing or nonincreasing. If g is nondecreasing, then

$$\begin{aligned} g(X) \leq g(m) &\iff X \leq m, \\ g(X) \geq g(m) &\iff X \geq m. \end{aligned}$$

Likewise, if g is nonincreasing, then

$$\begin{aligned} g(X) \leq g(m) &\iff X \geq m, \\ g(X) \geq g(m) &\iff X \leq m. \end{aligned}$$

Then, if g is nondecreasing,

$$\begin{aligned} &P(Y \leq g(m)) = P(g(X) \leq g(m)) = P(X \leq m) \geq 1/2 \\ &\wedge \\ &P(Y \geq g(m)) = P(g(X) \geq g(m)) = P(X \geq m) \geq 1/2 \end{aligned}$$

And if g is nonincreasing, we have

$$\begin{aligned} &P(Y \leq g(m)) = P(g(X) \leq g(m)) = P(X \geq m) \geq 1/2 \\ &\wedge \\ &P(Y \geq g(m)) = P(g(X) \geq g(m)) = P(X \leq m) \geq 1/2 \end{aligned}$$

Thus the median of Y is $g(m)$. ■

(c) Suppose that we are interested in working with the α^{th} quantile of X, for some $\alpha \in (0,1)$, denoted by m_α and assumed to satisfy

$$P(X \leq m_\alpha) \geq \alpha \;\; \wedge \;\; P(X \geq m_\alpha) \geq 1-\alpha$$

i. Repeat part (a), replacing the absolute value function with the following "check function"

$$\rho_\alpha(x) = |x| + (2\alpha - 1)x.$$

We want to show that $\min_a \mathbb{E}[\rho_\alpha(X-a)] = \mathbb{E}[\rho_\alpha(X-m_\alpha)]$. We'll follow the same strategy as in part (a), and begin by noting that

$$\mathbb{E}[\rho_\alpha(X-a)] = \begin{cases} 2\alpha(\mathbb{E}[X]-a), & \text{if} \quad X \geq a \\ (2\alpha-2)(\mathbb{E}[X]-a), & \text{if} \quad X < a \end{cases}$$

Suppose that $a = \arg\min_a \mathbb{E}[\rho_\alpha(X-a)]$, but that $a \neq m_\alpha$. Then $P(X \leq a) < \alpha \vee P(X \geq a) < 1-\alpha$.

If $P(X \leq a) < \alpha$, then there is a $\tilde{a} > a$ such that $\mathbb{E}[\rho_\alpha(X-\tilde{a})] < \mathbb{E}[\rho_\alpha(X-a)]$, so $a \neq \arg\min_a \mathbb{E}[\rho_\alpha(X-a)]$.

If $P(X \geq a) < 1-\alpha$, then there exists a $\tilde{a} < a$ such that $\mathbb{E}[\rho_\alpha(X-\tilde{a})] < \mathbb{E}[\rho_\alpha(X-a)]$, so $a \neq \arg\min_a \mathbb{E}[\rho_\alpha(X-a)]$.

Thus, is must be that $P(X \leq m_\alpha) \geq \alpha \wedge P(X \geq m_\alpha) \geq 1-\alpha$, so $a = m_\alpha$. ■

ii. Does the "equivariance property" you proved in part (b) hold?

Suppose $Y = g(X)$, where $g(\cdot)$ is weakly monotonic. The equivariance property holds if $g(m_\alpha)$ is the α^{th} quantile of Y, which is the case if and only if

$$P(Y \leq g(m_\alpha)) \geq \alpha \;\; \wedge \;\; P(Y \geq g(m_\alpha)) \geq 1-\alpha$$

Monotonicity of g has the same inequality implications as in part (b), so if g is nondecreasing, we have

$$\begin{aligned} &P(Y \leq g(m_\alpha)) = P(g(X) \leq g(m_\alpha)) = P(X \leq m_\alpha) \geq \alpha \\ &\wedge \\ &P(Y \geq g(m_\alpha)) = P(g(X) \geq g(m_\alpha)) = P(X \geq m_\alpha) \geq 1-\alpha \end{aligned}$$

Which satisfies the equivariance condition. But if g is nonincreasing, then

$$\begin{aligned} &P(Y \leq g(m_\alpha)) = P(g(X) \leq g(m_\alpha)) = P(X \geq m_\alpha) \geq 1-\alpha \\ &\wedge \\ &P(Y \geq g(m_\alpha)) = P(g(X) \geq g(m_\alpha)) = P(X \leq m_\alpha) \geq \alpha \end{aligned}$$

Which does not satisfy the equivariance condition. So the equivariance property only holds for $\alpha = 1/2$, or with an arbitrary α when g is non-decreasing. ■

3. Assume that ϵ_1 and ϵ_2 are i.i.d. continuouslu distributed random variables, with common density function f. Let $\epsilon_3 = \epsilon_2 - \epsilon_1$.

   (a) Show that ϵ_3 is symmetrically distributed around zero.

   ϵ_3 is symmetrically distributed around 0 iff $P(\epsilon_3 \leq x) = P(\epsilon_3 \geq -x)$ for all x.

   Let F_{12} denote the CDF of $\epsilon_1 - \epsilon_2$ and let F_{21} denote the CDF of $\epsilon_2 - \epsilon_1$. Since ϵ_1 and ϵ_2 follow the same distribution, these CDFs must be identical: $F_{12}(x) = F_{21}(x)$ for all x. Take some $x \in \mathbb{R}$. Then

$$\begin{aligned} P(\epsilon_3 \leq x) &= P(\epsilon_2 - \epsilon_1 \leq x) \\ &= F_{21}(x) \\ &= F_{12}(x) \\ &= P(\epsilon_1 - \epsilon_2 \leq x) \\ &= P(\epsilon_3 \geq -x) \quad \blacksquare \end{aligned}$$

   (b) Suppose now that ϵ_1 and ϵ_2 are independently and symmetrically distributed around 0, but they follow different distributions. Show that ϵ_3 is still symmetrically distributed around 0.

   Notationally, I'm going to let ϵ denote an unrealized random variable, and ε denote a point. $\epsilon_3 = \epsilon_2 - \epsilon_1$ is the convolution of two random variables. If f_1 is the PDF of ϵ_1 and f_2 is the PDF of ϵ_2, then the PDF of ϵ_3 is

$$f_3(\varepsilon_3) = \int_{-\infty}^{\infty} f_2(\varepsilon_3 + \varepsilon_1) f_1(-\varepsilon_1) \, \mathrm{d}\varepsilon_1$$

   ε_3 is symmetrically distributed around 0 iff $f_3(x) = f_3(-x)$ for all x. Take some $x \in \mathbb{R}$, and let $\eta = -\varepsilon_1$. Then

$$\begin{aligned} f_3(x) &= \int_{-\infty}^{\infty} f_2(x + \varepsilon_1) f_1(-\varepsilon_1) \, \mathrm{d}\varepsilon_1 \\ &= \int_{-\infty}^{\infty} f_2(-x - \varepsilon_1) f_1(\varepsilon_1) \, \mathrm{d}\varepsilon_1 \\ &= \int_{-\infty}^{\infty} f_2(-x + \eta) f_1(-\eta) \, \mathrm{d}\eta \\ &= f_3(-x) \quad \blacksquare \end{aligned}$$

4. Show equivalence between the following versions of the maximum score estimator:

   (a) $\hat{\beta}_a = \text{arg}\max_{||\beta||=1} \frac{1}{n}\sum_{i=1}^n \; y_i \mathbb{1}(x_i'\beta > 0) + (1-y_i)\mathbb{1}(x_i'\beta < 0)$

   (b) $\hat{\beta}_b = \text{arg}\max_{||\beta||=1} \frac{1}{n}\sum_{i=1}^n \; (2y_i - 1)\mathbb{1}(x_i'\beta > 0)$

   (c) $\hat{\beta}_c = \text{arg}\min_{||\beta||=1} \frac{1}{n}\sum_{i=1}^n \; |y_i - \mathbb{1}(x_i'\beta > 0)|$

   (d) $\hat{\beta}_d = \text{arg}\min_{||\beta||=1} \frac{1}{n}\sum_{i=1}^n \; (y_i - \mathbb{1}(x_i'\beta > 0))^2$

   $\hat{\beta}_a = \hat{\beta}_b$:

$$
\begin{aligned}
\hat{\beta}_a &= \text{arg}\max_{||\beta||=1} \frac{1}{n}\sum_{i=1}^n \; y_i \mathbb{1}(x_i'\beta > 0) + (1-y_i)\mathbb{1}(x_i'\beta < 0) \\
&= \text{arg}\max_{||\beta||=1} \frac{1}{n}\sum_{i=1}^n \; y_i \mathbb{1}(x_i'\beta > 0) + (1-y_i)(1 - \mathbb{1}(x_i'\beta > 0)) \\
&= \text{arg}\max_{||\beta||=1} \frac{1}{n}\sum_{i=1}^n (2y_i - 1)\mathbb{1}(x_i'\beta > 0) - y_i + 1 \\
&= \text{arg}\max_{||\beta||=1} \frac{1}{n}\sum_{i=1}^n (2y_i - 1)\mathbb{1}(x_i'\beta > 0) \\
&= \hat{\beta}_b
\end{aligned}
$$

   $\hat{\beta}_b = \hat{\beta}_c$:
   Let $B = (2y_i - 1)\mathbb{1}(x_i'\beta > 0)$ and let $C = |y_i - \mathbb{1}(x_i'\beta > 0)|$. y_i and $\mathbb{1}(x_i'\beta > 0)$ only take values in $\{0, 1\}$, so consider the following four cases:

y_i	$\mathbb{1}(x_i'\beta > 0)$	B	C
0	0	1	0
0	1	0	1
1	0	0	1
1	1	1	0

   Since $\hat{\beta}_b$ is an argmax and $\hat{\beta}_c$ is an argmin, B and C taking opposite values means that $\hat{\beta}_b$ and $\hat{\beta}_c$ will take the same value.

   $\hat{\beta}_c = \hat{\beta}_d$:
   Since $(y_i - \mathbb{1}(x_i'\beta > 0)) \in \{-1, 0, 1\}$, the absolute value is equivalent to the square. ■