

Comparing the Bayesian Linear Regression Model to Ordinary Least Squares

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Abstract

Linear regression is often used to model relationships between a response variable and one or more explanatory variables. In traditional settings, the ordinary least squares method is used to estimate the model. In this paper, our goal is to develop a linear model by estimating the model parameters under a Bayesian framework and compare the model against the traditional ordinary least squares model using a dataset.

Introduction

To model the relationship between a response variable and one or more explanatory variables, the simplest way to estimate a model is to assume that there is a linear relationship between the response variable and the predictor variables. The formulation of the model is

$$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where X is an $n \times k$ matrix with the observations of the explanatory variables, \mathbf{y} is an $n \times 1$ vector of observations of the response variables, $\boldsymbol{\beta}$ is a $k \times 1$ vector of model parameters, and $\boldsymbol{\epsilon}$ is an $n \times 1$ vector of random errors with each observation, n is the number of observations, and k is the number of parameters. The goal is to estimate $\boldsymbol{\beta}$ and then construct and evaluate the model from the estimated values. In other words, we have

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{k-1} \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

The most well-known way is to use the ordinary least squares method to estimate the parameters. The goal of this paper is to compare the ordinary least squares method with the Bayesian method of estimating parameters.

Ordinary Least Squares Regression Revisited

Estimation of Model Parameters

Consider a multiple linear regression model

$$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

assume that the random errors have mean 0, variance σ^2 , are uncorrelated, and they follow a normal distribution. In other words, we assume that $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 I)$, where $\mathbf{0}$ is the zero vector, and I is the identity matrix.

To estimate the coefficients, the following quantity should be minimized:

$$S(\boldsymbol{\beta}) = \sum_{i=1}^n \epsilon_i = \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = (\mathbf{y} - X\boldsymbol{\beta})^T (\mathbf{y} - X\boldsymbol{\beta})$$

We need to take derivatives of $S(\boldsymbol{\beta})$ with respect to $\boldsymbol{\beta}$. We get that

$$\frac{\partial S(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = -2X^T \mathbf{y} + 2X^T X \boldsymbol{\beta}$$

If we set this equation to 0 and we solve for $\boldsymbol{\beta}$, we get

$$X^T X \boldsymbol{\beta} = X^T \mathbf{y}$$

If we multiply both sides by the inverse of $X^T X$, assuming that it is invertible, then the least squares estimator of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = (X^T X)^{-1} X^T \mathbf{y}$$

The least squares estimator of $\boldsymbol{\beta}$ has the following properties:

- $\mathbb{E}[\hat{\boldsymbol{\beta}}] = \boldsymbol{\beta}$
- $Var[\hat{\boldsymbol{\beta}}] = \sigma^2 (X^T X)^{-1}$

Estimation of Variance

In the frequentist approach, we estimate σ^2 using the residual sum of squares. We have

$$SS_{res} = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n e_i^2 = \mathbf{e}^T \mathbf{e}$$

where \hat{y}_i are the fitted values of y_i using the least squares model and $e_i = y_i - \hat{y}_i$ are the residuals. \mathbf{e} is the vector of the residuals. If we simplify further, we get

$$SS_{res} = \mathbf{y}^T \mathbf{y} - \hat{\boldsymbol{\beta}}^T X^T \mathbf{y}$$

The residual sum of squares has $n - p$ degrees of freedom. The residual mean square is

$$MS_{res} = \frac{SS_{res}}{n - p}$$

We can show that

$$\mathbb{E}[MS_{res}] = \sigma^2$$

Therefore, an unbiased estimator for σ^2 is given by

$$\hat{\sigma}^2 = MS_{res} = \frac{\mathbf{y}^T \mathbf{y} - \hat{\boldsymbol{\beta}}^T X^T \mathbf{y}}{n - p}$$

Inference on model parameters

After obtaining the estimates of the model parameters, we need to make inference on them. The main goal of inference is to determine if all the explanatory variables are significant to the model. Sometimes when there are too many variables, the accuracy of predictions using the model can decrease significantly. Making inferences can help decide if some variables should not be included in the model.

We have assumed that $\epsilon \sim N(\mathbf{0}, \sigma^2 I)$. This consequently leads to that $\mathbf{y} \sim N(X\beta, \sigma^2 I)$. Under this assumption, we can also get that

$$\hat{\beta} \sim N(\beta, \sigma^2 (X^T X)^{-1})$$

Prediction

Bayesian Linear Regression

Application

Conclusion

References

https://en.wikipedia.org/wiki/Bayesian_linear_regression