$$\vec{y} = X\vec{\beta} + \vec{\varepsilon} \qquad \vec{\varepsilon} \sim N_n(\vec{0}, \sigma_n^2 I_n)$$

$$f(\vec{y} \mid X, \beta, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}(\vec{y} - X\vec{\beta})^T(\vec{y} - X\vec{\beta})\right)$$

Assume priors

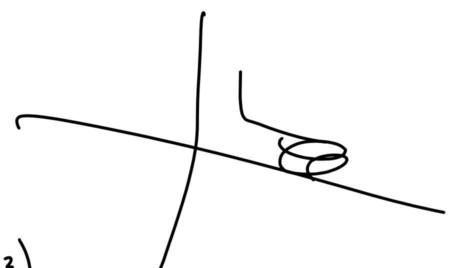
$$\vec{\beta} \mid \sigma^2 \sim N_k(\underbrace{\vec{m}}_{k\times 1}, \sigma^2 \underbrace{V}_{k\times k})$$

$$\sigma^2 \sim IG(a, b)$$

$$g(\beta \mid \sigma^2) = \frac{1}{(2\pi\sigma^2)^{k/2}} |V|^{-\frac{k}{2}} \exp\left(-\frac{1}{2\sigma^2}(\vec{\beta} - \vec{m})^T V^{-1}(\vec{\beta} - \vec{m})\right)$$

$$g(\sigma^2) = \frac{b^a}{\Gamma(a)}(\sigma^2)^{-(a+1)} \exp\left(-\frac{b}{\sigma^2}\right)$$

$$g(\beta, \sigma^2) = g(\beta \mid \sigma^2)\, g(\sigma^2)$$



The posterior is

$$h(\vec{\beta}, \sigma^2 \mid \vec{y}, X) \propto f(\vec{y} \mid X, \vec{\beta}, \sigma^2)\, g(\vec{\beta}, \sigma^2)$$

$$\propto f(\vec{y} \mid X, \vec{\beta}, \sigma^2)\, g(\vec{\beta} \mid \sigma^2)\, g(\sigma^2)$$

$$\propto (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}(\vec{y} - X\vec{\beta})^T(\vec{y} - X\vec{\beta})\right)(2\pi\sigma^2)^{-\frac{k}{2}} |V|^{-\frac{k}{2}}$$

$$\exp\left(-\frac{1}{2\sigma^2}(\vec{\beta} - \vec{m})^T V^{-1}(\vec{\beta} - \vec{m})\right) \cdot \frac{b^a}{\Gamma(a)}(\sigma^2)^{-(a+1)} \exp\left(-\frac{b}{\sigma^2}\right)$$

$$\propto (\sigma^2)^{-\frac{n}{2} - \frac{k}{2} - (a+1)} \exp\left(-\frac{1}{2\sigma^2}\left[(\vec{y} - X\vec{\beta})^T(\vec{y} - X\vec{\beta}) + (\vec{\beta} - \vec{m})^T V^{-1}(\vec{\beta} - \vec{m}) + 2b\right]\right)$$

Let $A = (\vec{y} - X\vec{\beta})^T(\vec{y} - X\vec{\beta}) + (\vec{\beta} - \vec{m})^T V^{-1}(\vec{\beta} - \vec{m}) + 2b$

$$= \vec{y}^T\vec{y} - \vec{y}^T X\vec{\beta} - \vec{\beta}^T X^T \vec{y} + \vec{\beta}^T X^T X \vec{\beta} + \vec{\beta}^T V^{-1} \vec{\beta} - \vec{\beta}^T V^{-1} \vec{m} - \vec{m}^T V^{-1} \vec{\beta} + \vec{m}^T V^{-1} \vec{m} + 2b$$

$$= \vec{\beta}^T(X^TX+V^{-1})\vec{\beta} - \vec{\beta}^T(X^T\vec{y}+V^{-1}\vec{m}) + (\vec{m}^TV^{-1}\vec{m} + 2b + \vec{y}^T\vec{y}) - (\vec{y}^TX + \vec{m}^TV^{-1})\vec{\beta}$$

$$\Lambda = (X^TX+V^{-1})^{-1} \qquad k\times k$$

$$\vec{\mu} = (X^TX+V^{-1})^{-1}(X^T\vec{y}+V^{-1}\vec{m}) \qquad k\times 1$$

$$A = \vec{\beta}^T\Lambda^{-1}\vec{\beta} - \vec{\beta}^T\Lambda^{-1}\vec{\mu} - \mu^T\Lambda^{-1}\vec{\beta} + \vec{m}^TV^{-1}\vec{m} + 2b + \vec{y}^T\vec{y}$$

$$= (\vec{\beta}-\vec{\mu})^T\Lambda^{-1}(\vec{\beta}-\vec{\mu}) - \vec{\mu}^T\Lambda^{-1}\vec{\mu} + m^TV^{-1}\vec{m} + 2b + \vec{y}^T\vec{y}$$

Then the joint posterior is

$$h(\vec{\beta},\sigma^2) \propto (\sigma^2)^{-\frac{k}{2}} \cdot \exp\left(\frac{(\vec{\beta}-\vec{\mu})\Lambda^{-1}(\vec{\beta}-\vec{\mu})}{2\sigma^2}\right) \cdot (\sigma^2)^{-(\frac{n}{2}+a+1)} \exp\left(\frac{\vec{m}^TV^{-1}\vec{m} - \vec{\mu}^T\Lambda^{-1}\vec{\mu} + 2b + \vec{y}^T\vec{y}}{2\sigma^2}\right)$$

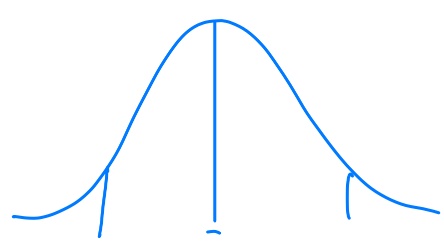
$$\vec{X} \sim N_n(\vec{\mu}, \Sigma)$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_n\left(\begin{pmatrix} \mu_1 \\ \cancel{\mu_2} \\ \cancel{\mu_3} \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cancel{\Sigma_{13}} \\ \cancel{\Sigma_{21}} & \cancel{\Sigma_{22}} & \cancel{\Sigma_{23}} \\ \cancel{\Sigma_{31}} & \cancel{\Sigma_{32}} & \cancel{\Sigma_{33}} \end{pmatrix}\right)$$

$$X_1 \sim N_n(\mu_1, \Sigma_{11})$$

$IG(\frac{n}{2}+a, \beta)$

$\sigma^2 \Lambda \qquad \mu$



$$\beta_i \sim N(\mu_i, \sigma^2\lambda_{ii})$$

$$P(\hat{y}_{n+1} \mid \vec{y}, \vec{\beta}, X)$$

$\beta | \sigma^2, \vec{y}, X \sim N_n(\vec{\mu}, \sigma^2 \Lambda)$

$\sigma^2 | \vec{y}, X \sim IG$

$\sigma^2 \sim IG$ $\quad\quad$ $\sigma^2 = (\cdot \; .. \; - - - \; . \; .)$

$\beta_1 \sim$ rnorm $\quad\quad$ $\beta_1 = (, - - \; . \; - \; . \; -)$

$\beta_2 \sim$ rnorm