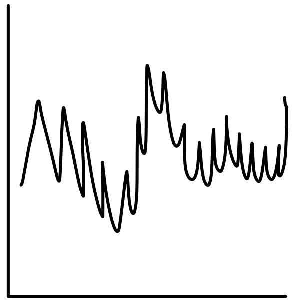
MAT 3379

Time Series Analysis

Study Guide

Winter 2024



$$X_t = Y_t + m_t + S_t$$

Examples

- White noise: $\{Z_t\}$ sequence of independent random variables with mean 0 and variance 1
- Random Walk: $\{Z_t\}$ sequence of iid random variables with mean 0 and variance σ_Z^2. $X_t = \sum_{i=1}^{t} Z_t$, $t=1,2,\ldots$
- Model with trend: Sometimes a trend is present in time series. $X_t = 1+2t+Z_t$, $t=1,2,\ldots$ Trend is $m_t = 1+2t$
- Economics Trend: $X_t = P_t e^{rt}$ P_t is real price, r is interest rate

Time Series Structure

$$X_t = m_t + Y_t + S_t$$

- m_t is a trend
- S_t is a seasonal part
- Y_t is a stationary part

Eliminate Trend

- Differencing: Compute $\nabla X_t = X_t - X_{t-1}$, $t=2,\ldots,n$
- Polynomial Fitting: Assume $m_t = a+bt$. Estimate a and b by minimizing $\sum_{t=1}^{n}(X_t-a-bt)^2$, $\hat{m}_t = \hat{a}+\hat{b}t$. Detrended time series: $\hat{Y} = X_t - \hat{m}_t$
- Exponential Smoothing: $\alpha \in (0,1)$ Trend: $\hat{m}_1 = X_1$, $\hat{m}_t = \alpha X_t + (1-\alpha)\hat{m}_{t-1}$, $t=2,\ldots,n$ De-trended time series: $\hat{Y}_t = X_t - \hat{m}_t$
- Moving Average Smoothing: $q \in \mathbb{Z}^+$ $\hat{m}_t = (2q+1)^{-1}\sum_{j=-q}^{q} X_{t+j}$ $q+1 \le t \le n-q$ Detrended time series: $\hat{Y}_t = X_t - \hat{m}_t$

Mean Function

- $\mu_X(t) = E[X_t]$

Covariance Function

- $\gamma_X(t,s) = Cov(X_t, X_s) = E[X_t X_s] - E[X_t]E[X_s]$

Note: $\gamma_X(t,t) = Var(X_t)$

Properties of Covariance

- For $a \in \mathbb{R}$, $Cov(X,a) = 0$
- For $a,b \in \mathbb{R}$, $Cov(X, aU+bV) = a\,Cov(X,U) + b\,Cov(X,V)$
- $Cov(X,Y)^2 \le Var(X)Var(Y)$

Stationary Time Series

- $\mu_X(t)$ does not depend on t
- $\gamma_X(t,s)$ depends only on $h = t-s$
- Covariance function is non-negative definite

Some useful Properties

- $Cov(A,B) = E[AB] - E[A]E[B]$
- If A,B are independent, then $Cov(A,B) = 0$
- $E[aA+bB] = aE[A] + bE[B]$
- $Cov(A,A) = Var(A)$
- $Var(A+bB) = Var(A) + b^2 Var(B)$ if A,B are independent

MA(1) Model

$\{Z_t\}$ white noise
$\theta \in \mathbb{R}$ $(\theta \neq 0)$
$X_t = Z_t + \theta Z_{t-1}$

- $\mu_X(t) = 0$
- $\gamma_X(t,t+h) = \begin{cases} (1+\theta^2)\sigma_Z^2; & h=0 \\ \theta\sigma_Z^2; & h=1 \\ 0; & h\ge 2 \end{cases}$

Autocorrelation

- $\rho_X(h) = \dfrac{\gamma_X(h)}{\gamma_X(0)} = \dfrac{Cov(X_0, X_h)}{Var(X_0)}$

Partial Autocorrelation

- Correlation between X_t and X_{t+h} after conditioning out the "in-between" variables $X_{t+1},\ldots,X_{t+h-1}$

PACF between X_1 and X_2 when conditioning out X_3:

$$\rho_{12.3} = \frac{Corr(X_1,X_2) - Corr(X_1,X_3)Corr(X_2,X_3)}{\sqrt{1-Corr^2(X_2,X_3)}\sqrt{1-Corr^2(X_1,X_3)}}$$

Sample Mean, Sample Autocovariance, Sample Autocorrelation

- Sample Mean: $\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
- Sample Variance: $\hat{\sigma}_X^2 = \hat{\gamma}_X(0) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$
- Sample Autocovariance: $\hat{\gamma}_X(h) = \frac{1}{n-1}\sum_{t=1}^{n-h}(X_t-\bar{X})(X_{t+h}-\bar{X})$
- Sample Autocorrelation: $\hat{\rho}_X(h) = \dfrac{\hat{\gamma}_X(h)}{\hat{\gamma}_X(0)}$
- Sample PACF at lag 2:

$$\hat{\alpha}(2) = \frac{\hat{\rho}_X(2) - \hat{\rho}_X(1)}{1-\hat{\rho}_X^2(1)}$$

Linear Processes

Let $\{Z_t\}$ be a white noise
Let $\{\psi_j\}$ be a sequence of constants

$$X_t = \sum_{j=0}^{\infty}\psi_j Z_{t-j}, \quad t=1,2,\ldots$$

The model is called:

- linear model
- (infinite order) moving average
- (causal) moving average
- (one-sided) moving average

The linear process is well-defined if $\sum_{j=0}^{\infty}|\psi_j| < \infty$

The linear process is also stationary with

- $E[X_t] = 0$
- $\gamma_X(h) = \sigma_Z^2 \sum_{j=0}^{\infty}\psi_j\psi_{j+h}$

Difference and Backward operators

- Difference operator ∇ defined as $\nabla X_t = X_t - X_{t-1}$
- Backward Operator B defined as $BX_t = X_{t-1}$

ARMA models

Let $\{Z_t\}$ be a white noise. Then ARMA(p,q) mode is a stationary solution to

$$\phi(B)X_t = \theta(B)Z_t \quad (*)$$

where

$$\phi(z) = 1-\phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p$$
$$\theta(z) = 1+\theta_1 z + \theta_2 z^2 + \ldots + \theta_q z^q$$

- $\phi(z)$ autoregressive polynomial
- $\theta(z)$ moving average polynomial
- $(*)$ does not guarantee a stationary solution or a causal solution

Stationarity and Causality

The existence of causal and stationary solution means that ARMA model can be written as

$$X_t = \sum_{j=0}^{\infty}\psi_j Z_{t-j}$$

How to check if there is a stationary and causal solution?

- Take AR polynomial $\phi(z)$
- Find roots of ϕ: $\phi(z) = 0$
- If all roots are such that $|z| \neq 1$, then the model is stationary
- If all roots are such that $|z| > 1$, then the model is causal

Linear Representations

Given an ARMA model. If it is stationary and causal, then we can write it as

$$X_t = \sum_{j=1}^{\infty}\psi_j Z_{t-j}$$

The linear representation of the AR(1) model is

$$X_t = \sum_{j=1}^{\infty}\phi^j Z_{t-j} \quad |\phi| < 1$$

The linear representation of the ARMA(1,1) model is

$$X_t = \sum_{j=0}^{\infty}\psi_j Z_{t-j}$$

where $\psi_0 = 1$ and $\psi_j = \phi^{j-1}(\theta+\phi)$

Recursive approach to ACF

To find a recursive definition for the covariance of an AR(p) model, first multiply every term by X_{t-h} in the model. Then apply expected value to all terms and use that to find the recursive definition and initial cases for the covariance.

For AR(1), the ACF is

$$\gamma_X(0) = \sigma_Z^2 \frac{1}{1-\phi^2}, \quad \gamma_X(h) = \phi\gamma_X(h-1), \ h \ge 1$$

ACF and PACF of ARMA(p,q) models

For AR(p) models, the autocorrelation function is never 0 for all lags h. The partial autocorrelation function is 0 for all $h > p$.

For MA(q) models, the autocorrelation function is 0 for all $h > q$. The partial autocorrelation function is never 0.

Yule-Walker Formula for Forecasting

Consider a stationary sequence with $\mu_X = E[X_t]$ and autocovariance $\gamma_X(h)$. We denote $P_n X_{n+k}$, the predicted value of X_{n+k} given that we have n observations $X_1,\ldots,X_n$.

We use linear predictors:

$$P_n X_{n+k} = a_0 + \sum_{i=1}^{n} a_i X_{n+1-i}$$

We minimize the mean square error

$$E[(X_{n+k} - P_n X_{n+k})^2]$$

Goal: estimate $a_0, a_1, \ldots, a_n$

We obtain $a_0 = \mu\left(1-\sum_{i=1}^{n} a_i\right)$

If we assume $\mu = 0$, then $a_0 = 0$.

We would then obtain

$$\underline{a_n} = \Gamma_n^{-1}\gamma(n;k)$$

where

$$\Gamma_n = [\gamma_X(i-j)]_{i,j=1}^{n}$$
$$\gamma(n;k) = (\gamma_X(k), \ldots, \gamma_X(k+n-1))^T$$
$$\underline{a_n} = (a_1, \ldots, a_n)$$

This equation is called the Yule-Walker formula for prediction.

Mean Squared Error:

$$MSPE_n(k) = \gamma_X(0) - \underline{a_n}^T\gamma(n;k)$$

Durbin-Levinson Algorithm

$$P_n X_{n+1} = \phi_{n1}X_n + \ldots + \phi_{nn}X_1$$
$$a_1 = \phi_{n1}, \ldots, a_n = \phi_{nn}$$

The coefficients $\phi_{n1},\ldots,\phi_{nn}$ are given by

$$\phi_{nn} = \left[\gamma(n) - \sum_{j=1}^{n-1}\phi_{n-1,j}\gamma(n-j)\right]V_{n-1}^{-1}$$

$$\begin{bmatrix}\phi_{n,1}\\ \vdots \\ \phi_{n,n-1}\end{bmatrix} = \begin{bmatrix}\phi_{n-1,1}\\ \vdots \\ \phi_{n-1,n-1}\end{bmatrix} - \phi_{nn}\begin{bmatrix}\phi_{n-1,n-1}\\ \vdots \\ \phi_{n-1,1}\end{bmatrix}$$

$$V_n = V_{n-1}[1-\phi_{nn}^2]; \quad V_0 = \gamma(0)$$
$$\phi_{11} = \rho(1)$$

Partial Autocorrelation

$$\alpha(0) = 1; \quad \alpha(k) = \phi_{kk}; \ k \ge 1$$

Prediction Intervals

A prediction interval for X_{n+k} is

$$P_n X_{n+k} \pm Z_{\frac{\alpha}{2}}\sqrt{MSPE_n(k)}$$

For linear process, we have $MSPE_n(k) = \sigma_Z^2\sum_{j=1}^{k-1}\psi_j^2$

for AR(1) with $k=1$, we have

CI: $\phi X_n \pm Z_{\frac{\alpha}{2}}\sigma_Z$

for AR(1) with $k=2$, we have

CI: $\phi^2 X_n \pm Z_{\frac{\alpha}{2}}\sigma_Z\sqrt{\frac{1+\phi^{2k}}{1-\phi^2}}$

Estimation of the Mean: iid case

We estimate the population mean μ using the sample mean

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \,, \quad E[\bar{X}] = \mu$$

Using independence, we get

$$\mathrm{Var}[\bar{X}] = \frac{\gamma_X(0)}{n}$$

Central Limit Theorem: Assume $X_1, \dots, X_n$ are i.i.d. with mean μ and variance $\gamma_X(0)$. Then

$$\frac{\bar{X}-\mu}{\sqrt{\frac{\gamma_X(0)}{n}}} \longrightarrow N(0,1)$$

The 95% confidence interval for the mean μ is

$$\left(\bar{X} - 1.96\sqrt{\frac{\gamma_X(0)}{n}},\ \bar{X} + 1.96\sqrt{\frac{\gamma_X(0)}{n}}\right)$$

Estimation of the mean: Time Series Case

We still have that $E[\bar{X}] = \mu$.

However, $\mathrm{Var}(\bar{X}) = \gamma_X(0) + 2\sum_{h=1}^{\infty}\gamma_X(h)$ whenever the time series is short-range dependent $\sum_{h=-\infty}^{\infty}|\gamma_X(h)| < \infty$

Central Limit Theorem: Assume that $X_1, \dots, X_n$ is a stationary time series with mean μ, variance $\gamma_X(0)$, and the covariance function $\gamma_X(h)$. Then

$$\frac{\bar{X}-\mu}{\frac{v}{\sqrt{n}}} \longrightarrow N(0,1)$$

where $v^2 = \gamma_X(0) + 2\sum_{h=1}^{\infty}\gamma_X(h)$

The 95% confidence interval for the mean μ is

$$\left(\bar{X} - 1.96\frac{v}{\sqrt{n}},\ \bar{X} + 1.96\frac{v}{\sqrt{n}}\right)$$

Yule-Walker Estimator

For AR(p) models, we have

$$\Gamma_p = \left[\gamma_X(i-j)\right]_{i,j=1}^{p}$$

$$\boldsymbol{\phi}_p = \begin{pmatrix}\phi_1\\ \vdots \\ \phi_p\end{pmatrix} \qquad \gamma_{X,p} = \begin{pmatrix}\gamma_X(1)\\ \vdots \\ \gamma_X(p)\end{pmatrix}$$

We then get

$$\boldsymbol{\phi}_p = \Gamma_p^{-1}\gamma_{X,p}$$

$$\sigma_Z^2 = \gamma_X(0) - \boldsymbol{\phi}^T\gamma_{X,p}$$

The Yule-Walker estimators are

$$\hat{\boldsymbol{\phi}}_p = \hat{\Gamma}_p^{-1}\hat{\gamma}_{X,p}$$

$$\hat{\sigma}_Z^2 = \hat{\gamma}_X(0) - \hat{\boldsymbol{\phi}}\hat{\gamma}_{X,p}$$

$\hat{\Gamma}_p$ and $\hat{\gamma}_{X,p}$ obtained by replacing γ_X with $\hat{\gamma}_X$

Confidence Intervals for AR(p)

For a big sample size, the Yule-Walker estimators are approximately normal:

$$\hat{\boldsymbol{\phi}}_p \sim N_p\left(\boldsymbol{\phi}_p,\ \frac{1}{n}\sigma_Z^2\,\Gamma_p^{-1}\right)$$

For $p=1$:

$$\hat{\phi} \sim N\left(\phi,\ \frac{1}{n}\sigma_Z^2\,\gamma_X^{-1}(0)\right)$$

Practical CI for ϕ of AR(1):

$$\hat{\phi} \pm Z_{\frac{\alpha}{2}}\frac{1}{\sqrt{n}}\hat{\sigma}_Z\sqrt{\hat{\gamma}^{-1}(0)}$$

$$\hat{\phi} = \frac{\hat{\gamma}_X(1)}{\hat{\gamma}_X(0)} \,, \quad \hat{\sigma}_Z^2 = \hat{\gamma}_X(0) - \hat{\phi}\hat{\gamma}_X(1)$$

Practical CI for ϕ_1, ϕ_2 of AR(2):

$$\hat{\phi}_1 \pm Z_{\frac{\alpha}{2}}\frac{1}{\sqrt{n}}\sqrt{1-\hat{\phi}_2^2}$$

$$\hat{\phi}_2 \pm Z_{\frac{\alpha}{2}}\frac{1}{\sqrt{n}}\sqrt{1-\hat{\phi}_2^2}$$