MAT 4375

Multivariate Statistical Methods

Study Guide

Fall 2024

Vectors and Matrices

- A matrix is a rectangular array of scalar elements

$$A_{p\times q} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{pmatrix}$$

$p\times q$ dimension of matrix

p: number of rows

q: number of columns

- Row vector: $a_{1j} = (a_{11}\ a_{12}\ a_{13} \cdots a_{1q})$
- Column vector:

$$a_{i1} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{p1} \end{pmatrix}$$

Elementary Matrix Operations

- Sum and difference: $A+B$ or $A-B$, done by doing element-wise addition and subtraction
- Multiplication (written AB)

$$AB = \left(\sum_k a_{ik} b_{kj}\right)$$

  - For multiplication to work, we need the matrices to be conformable
- Inner product: product of row vector and column vector, leads to scalar
- Outer product: product of column vector and row vector, leads to matrix
- Transpose: interchanging rows and columns of A, denoted A'
- A square matrix has same rows and columns. A square matrix A is symmetric if $A = A'$
- Properties of transpose:
  - $(A')' = A$
  - $(A+B)' = A' + B'$
  - $(AB)' = B'A'$
  - $(cA)' = cA'$
- A diagonal matrix is a square matrix with zero off-diagonal elements
- Upper triangular matrices have zero elements below the diagonal and lower triangular matrices have zero elements above the diagonal
- Kronecker Product:
  - $A = (a_{ij}) : p\times q$
  - $B = (b_{ij}) : m\times n$

$$A\otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ \vdots & \ddots & & \vdots \\ a_{p1}B & \cdots & \cdots & a_{pq}B \end{pmatrix},\ pm\times qn$$

- Properties of Kronecker Product
  - $(A+B)\otimes C = (A\otimes C) + (B\otimes C)$

    $A\otimes(B+C) = (A\otimes B) + (A\otimes C)$
  - $A\otimes B \neq B\otimes A$

    $(A\otimes B)(C\otimes D) = AC\otimes BD$
  - $(A\otimes B)' = A'\otimes B'$
- Vectorization, denote by $vec(A)$, is done by stacking the columns of A and we form a column vector

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$vec(A) = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}$$

Trace of Matrix

- Sum of diagonals of matrix

$$trace(A) = \sum_i a_{ii}$$

- Properties of trace
  - $trace(A') = trace(A)$
  - $trace(A+B) = trace(A) + trace(B)$
  - $trace(AB) = trace(BA)$
  - $trace(AA') = trace(A'A)$
  - $trace(ABC) = trace(CAB) = trace(BCA)$
  - $trace(A\otimes B) = trace(A)\,trace(B)$

Determinant of Matrix

- For 1×1 matrix, or scalar, the determinant is the value itself
- For 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the determinant, denoted by $|A|$ is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- For 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The determinant is calculated as

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- General formula

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \text{ for all } j$$

or

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \text{ for all } i$$