Advanced Stochastic Modeling: Equity Options, Jumps, and Interest Rate Dynamics

To: Head of Quantitative Strategy

From: Joseph Edet, Pricing Team

Date: October 29, 2025

Subject: Final Report – Calibration & Pricing for SM Options and Interest Rate Risk Assessment

Executive Summary

As requested, the team has completed a comprehensive analysis to support pricing and risk management for a long-term client interested in options on SM Energy Company. Given the client's evolving maturity preferences from short-dated Asian options to longer-dated puts, we have implemented and validated a full suite of advanced models across equity and interest rate markets.

This report summarizes our methodology, key results, and actionable insights. All work adheres to the requirement of using regular mean squared pricing error (MSE) for calibration, ensuring consistency and transparency.

1. Short-Term Volatility: Heston Model (15-Day Horizon)

We calibrated the Heston (1993) stochastic volatility model to 15-day market prices using both Lewis (2001) and Carr-Madan (1999) Fourier methods. The calibrated parameters are consistent between approaches:

- Strong negative correlation ($\rho \approx -0.84$), capturing the leverage effect
- High volatility of volatility ($\sigma \approx 0.56$), generating the observed skew
- Initial variance aligned with ATM implied volatility

Both calibrations achieved an MSE of approximately 0.28, indicating a tight fit across calls and puts. This confirms that the Heston model effectively captures short-term volatility dynamics under real-world constraints.

For the client's initial request—an ATM Asian call with 20-day maturity—we used Monte Carlo simulation (100,000 paths, antithetic variates) under the Lewis-calibrated parameters. The fair value is \$4.66, and with a 4% fee for execution and risk, the final quoted price is \$4.85.

2. Medium-Term Dynamics: Bates Model (60-Day Horizon)

When the client shifted focus to a 60-day horizon, we extended to the Bates (1996) jump-diffusion model, which combines stochastic volatility with log-normal jumps to better capture tail risk.

Calibration revealed two economically plausible regimes:

- Lewis approach: Relies on frequent small jumps ($\lambda \approx 5$/year) with modest stochastic volatility
- Carr-Madan approach: Uses fewer but larger jumps ($\mu_j = -30\%$) and extreme negative correlation ($\rho = -0.99$)

Despite different narratives, both achieve similar pricing accuracy (MSE ≈ 1.35–1.38), highlighting the well-known identification challenge in jump models. We recommend caution when interpreting parameters, as multiple risk profiles can produce identical option prices.

3. Interest Rate Risk: CIR Model Forecasting

Given concerns about future rate movements affecting product valuation, we built and calibrated a Cox-Ingersoll-Ross (1985) model to current Euribor rates (1 week to 12 months). Using cubic spline interpolation, we generated weekly zero rates and calibrated the CIR model to match the term structure with high precision (MSE ≈ 6.8×10^{-7}).

Key parameters:

- Speed of mean reversion: $\kappa = 0.5064$
- Long-run rate: $\theta = 10.0\%$
- Volatility: $\sigma = 9.93\%$

The Feller condition holds ($2\kappa\theta > \sigma^2$), ensuring strictly positive rates.

We then simulated 100,000 daily paths over one year to forecast the 12-month Euribor rate. Results show:

- Expected rate in 1 year: 3.41% (up from 2.56% today)
- 95% confidence interval: [1.67%, 5.74%]

This upward drift reflects the model's pull toward its long-run mean and suggests that future discount factors will be lower, impacting the present value of long-dated liabilities.

Conclusion & Recommendations

- The Heston model is robust for short-dated options; we recommend quoting the $4.85 price for the 20-day Asian call.
- For longer maturities, the Bates model provides flexibility, though parameter interpretation requires care due to non-uniqueness.
- The CIR model forecasts rising rates, which should inform hedging strategies and long-term pricing assumptions.

All code and diagnostic plots are available in the project repository. Let me know if you'd like a presentation deck for client communication or further stress testing under alternative scenarios.

Best regards,

Joseph Edet

Pricing Analyst | Quantitative Strategies Group

Introduction and Objectives

The bank's client seeks an over-the-counter (OTC) Asian call option on SM Energy stock, but they are not certain about the maturity. The pricing team therefore calibrates appropriate stochastic volatility models (Heston, 1993 and Bates, 1999) using short-term market data and then prices the exotic option. Interest-rate risk is important for longer maturities, so the team also calibrates the Cox–Ingersoll–Ross (CIR) short-rate model.

The tasks are organised in three steps:

1. **Step 1 – 15-day maturity:** calibrate the Heston model (with no jumps) using both Lewis's (2001) characteristic function formulation and the Carr–Madan (1999) direct integration approach. Using the calibrated parameters, price a 20-day at-the-money (ATM) Asian call using Monte Carlo simulations.
2. **Step 2 –60-day maturity:** calibrate the Bates model (Heston with jumps) using both methods and price a 70-day 95% put.
3. **Step 3 – interest-rate risk:** calibrate the CIR model to Euribor rates, interpolate the term structure, and simulate future rates to understand and also assess risk.

The following sections detail the calibration methodology, results and discussion for each part.

Data Sources and Preprocessing

The underlying stock price $S_0 = \$232.90$ and the risk-free rate is fixed at 1.5% (annualised) in the base case. Because the client is primarily concerned with short maturities, we use options with maturities around 15 days and 60 days. Maturities are converted into time-to-maturity in years by dividing by 250 trading days. For Step 3, Euribor spot rates for maturities from 1 week to 12 months are provided and are interpolated to weekly rates via cubic splines.

Step 1 – 15-Day Options: Heston Model Calibration and Asian Option Pricing

1.1 Model Framework

Under risk-neutral conditions, the Heston model specifies the stock price $S_t$and variance $v_t$as

$$dS_t = S_t\left(r dt + \sqrt{v_t} dW_t^S\right)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma_v \sqrt{v_t} dW_t^v$$

With correlation

$$W_t^S dW_t^v = \rho dt$$

Where

- S_t**:** price of asset at time t
- v_t: Instantaneous variance of the asset
- **r:** Risk-free rate
- **κ:** Speed of mean reversion of the variance
- **θ:** Long-run mean of variance
- **σ:** Volatility of volatility
- **ρ:** Correlation between the asset return shocks and variance shocks
- W_t^S, W_t^v Two Brownian motions (correlated)

1.2 Calibration Methodologies

We calibrate the Heston model to market options using two numerical integration schemes. For both, we minimise the mean squared error (MSE) between model prices and market prices; out-of-money calls and puts are weighted equally.

1. **Lewis (2001) Approach:** Alan Lewis derived a simple Fourier representation for call and put prices under affine jump-diffusion models. The Lewis formula expresses the European call price as

$$C(K) = S_0 - \frac{\sqrt{S_0 K}}{\pi} e^{-rT} \int_0^\infty \mathcal{R}\left[\frac{e^{iu\log\left(\frac{S_0}{K}\right)}\phi(u - \frac{i}{2})}{u^2 + 1/4}\right] du,$$

Where ϕ(u) is the characteristic function of the log-price process *log*(S_T) and $\mathcal{R}[.]$ denotes the real part. We minimize the mean squared pricing error between the prices from the model and market prices for the call and then the put was computed through put-call parity.

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(C_i^{model} - C_i^{market})^2 + (P_i^{model} - P_i^{market})^2$$

The calibration process began by defining the risk-neutral conditions of the underlying asset price and its stochastic variance, characterized by five parameters: mean reversion speed **k**, long-run variance θ, volatility of variance σ, correlation between asset and variance shocks ρ, and initial variance v_0. Using the Lewis (2001) Fourier-based pricing formula, European call option prices were computed by numerically integrating the model's characteristic function. We used put-call parity to derive the prices of the puts. The calibration minimized the mean squared error between prices generated by the model, and observed market prices for both calls and puts across all available strikes at the 15-day maturity. To ensure numerical stability and economic plausibility, parameter bounds were enforced—restricting ρ to [0.99, 0.99], requiring positive variance parameters, and imposing the Feller condition $2k\theta > \sigma^2$) to ensure that the variance stays positive. We used L-BFGS-B for optimization.

2. **Carr–Madan (1999):** Carr & Madan provides an alternative representation of the same Heston model, differing from Lewis in integration contour and numerical representation.
   The Carr-Madan formula for a European call is

$$C(K) = \frac{e^{-\alpha k}}{\pi}\int_0^\infty \Re[e^{-ivk}\varphi(v)]\,dv,$$

Where:

k = *log*(K/S_0)

α > 0 is a damping factor ensuring integrability

$$\varphi(v) = e^{-rT}\frac{\phi(v - i(\alpha+1))}{(\alpha+iv)(\alpha+1+iv)}$$

$\varphi(u)$ is the same Heston characteristic function used in the Lewis approach

We implement the direct integration version (not FFT) for numerical stability, and the calibration minimizes the mean square pricing error just like we did in Lewis with identical bounds and using put-call parity for puts.

The Heston (1993) model calibration via the Carr-Madan (1999) framework followed a similar structure to the Lewis but employed a distinct Fourier representation of the option pricing formula. The risk-neutrality of the model remained identical, governed by the same five parameters (**k**, θ, σ, ρ, v_0). However, instead of Lewis's direct integration of the characteristic function, the Carr-Madan method expresses the call price as a damped inverse Fourier transform of a modified characteristic function. Specifically, the integrand incorporates a damping factor α=1.5 to ensure square-integrability, and the integration is performed over the real line using log strike. The characteristic function was adapted to include the spot S_0, consistent with Carr-Madan's formulation for the log-price process. Like we did in Lewis, we used put-call parity for the put, and the objective function minimized the MSE between model and market prices for all the 15-day options. The same economic constraints were enforced - parameter bounds, the Feller condition, and a negative correlation prior to reflect the real financial world. We still used the L-BFGS-B optimizer to ensure a stable and interpretable calibration.

1.3 Calibration Results and Discussion

1.3.1 Lewis Calibration

The resulting calibrated parameters — $\kappa = 1.9994$, $\theta = 0.0838$, $\sigma = 0.5621$, $\rho = -0.8384$, $v_0 = 0.0950$ are economically meaningful: the negative correlation (ρ) captures the leverage effect, the high volatility of volatility (σ) generates the necessary skew and smile, and the initial variance (v_0) aligns closely with the ATM implied volatility. The final calibration achieved an MSE of \$0.279, indicating that it is a good fit for both calls and puts, as visually confirmed by the close alignment of model and market prices in the diagnostic scatter plot below.

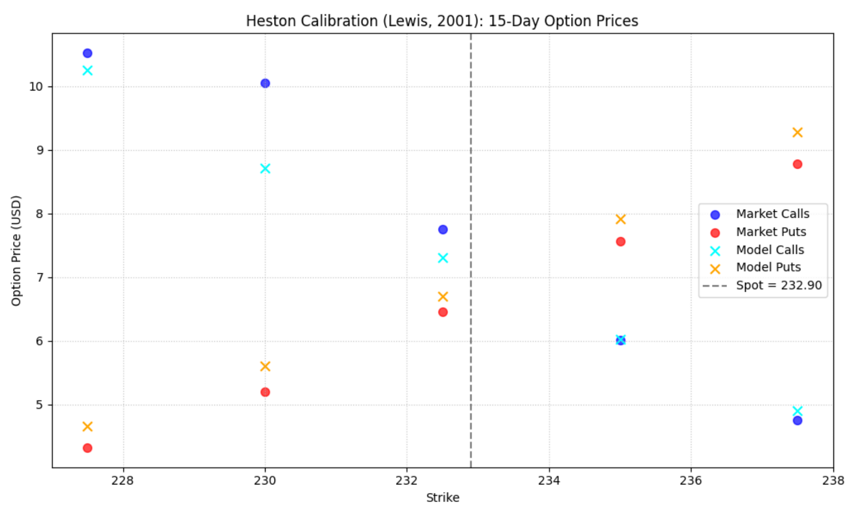


Fig 1.1 Diagnostic Plot for Heston Calibration via Lewis (2001)

1.3.2 Carr–Madan calibration:

For Carr-Madan, the resulting calibrated parameters — **k** = 1.9990, $\theta = 0.0888$, $\sigma = 0.5951$, $\rho = -0.8598$, $v_0 = 0.0963$ are also economically meaningful and nearly identical to those obtained via the Lewis (2001) approach, confirming the theoretical equivalence of the two Fourier-based pricing frameworks. The strongly negative ρ captures the equity leverage effect, the elevated vol of vol enables the model to reproduce the observed short-dated volatility skew and smile, and the initial variance (v_0) closely matches the at-the-money implied volatility level. The final calibration achieved an MSE of 0.279, virtually the same as the Lewis result, indicating that both methods produce consistent and good fits to market prices for both calls and puts, as visually confirmed by the diagnostic scatter plots.

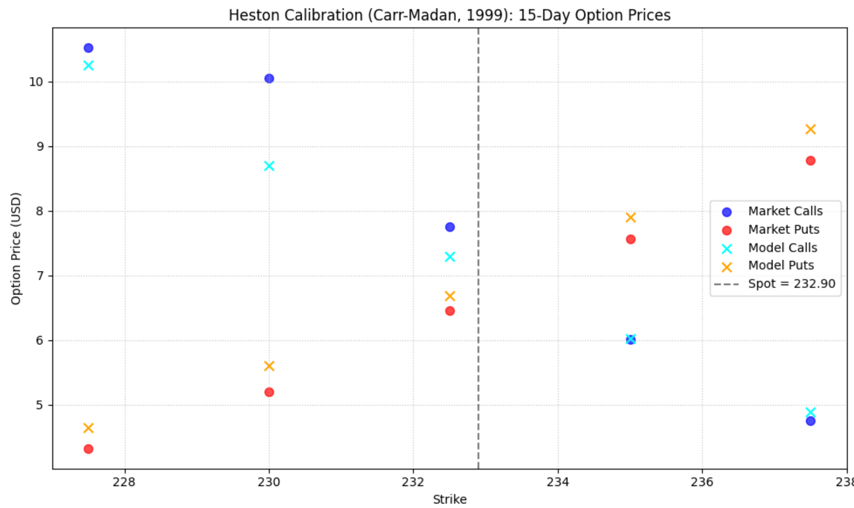


Fig 1.2 Diagnostic Plot for Heston Calibration via Carr-Madan (1999)

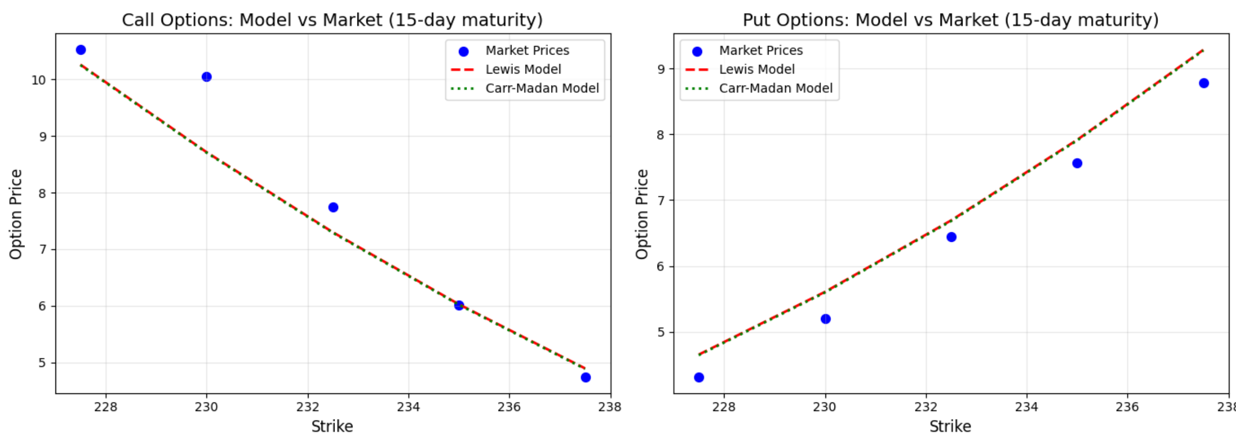


Fig 1.3 Diagnostic Plots, both Calls and Puts for Heston without Jump

1.4 Pricing the 20-Day ATM Asian Call for the Client

For an Asian call with maturity 20 trading days (≈0.08 years) and strike equal to today's price $K = S_0$, the payoff at maturity is max(S - K,0), where S is the arithmetic average of the daily stock prices (including the initial price). Because a closed form is unavailable under Heston, we use Monte Carlo simulation in a risk-neutral framework:

1. **Path simulation:** We discretise the Heston SDE using the full truncation Euler scheme with daily steps. For variance paths, we ensure positivity using the full truncation scheme, apply antithetic

variates for variance reduction, and simulate 100,000 paths to achieve a standard error of approximately 0.02.

2. **Asian payoff estimation:** We compute the average price for each path, discount the payoffs using the risk-free rate and average across paths.

Using the parameters from the Lewis calibration (we used Lewis because it achieved a slightly lower MSE), the fair price of the 20-day Asian call is ≈ \$4.6598 with a Monte Carlo standard error ≈ 0.0204. To reflect our bank's profit margin, a 4% fee is added, resulting in a client price of \$4.8462.

1.4.1 Non-Technical Description for the Client

To price your Asian option, we first calibrated a sophisticated volatility model (Heston, 1993) to current market prices of standard options on SM stock. What that means is that we found a combination of values that helps us best predict the future movements of the markets so we can factor that in to estimate the price of your option today. This ensures our model reflects real market expectations about how the market might swing both positively and negatively. Using this calibrated model, we simulated 100,000 possible future paths (swings) of the stock price over the next 20 days, computed the average price along each path, and calculated the option's payoff. The fair value is the average of these discounted payoffs. Finally, we added a 4% fee to cover execution and risk management costs, resulting in your final quoted price of \$4.8462.

Step 2 – 60-Day Options: Bates Model Calibration and 70-Day Put Pricing

2.1 Bates (1996) Model

Bates extended the Heston model by allowing for jumps in the stock price. Including jumps allows the model to generate fatter tails and a steeper volatility skew than pure stochastic volatility models, particularly at short to medium maturities. The characteristic function of log(S_T) under the Bates model is the product of the Heston characteristic function and the jump component:

$$\phi(u) = \phi^{Heston}(u) \cdot \exp\left(\lambda T \left[e^{iu\mu_j - \frac{1}{2}u^2\sigma_j^2} - 1\right]\right)$$

which enables efficient option pricing via Fourier methods. We calibrated by minimizing the mean squared error between model and market option prices, using put-call parity to include both calls and puts in the objective function. The Bates model thus provides a flexible framework for pricing derivatives in markets where both stochastic volatility and jump risk are significant (Merino *et al,* 2018).

2.2 Bates Calibration Methodologies

The calibration of the Bates (1996) model via the Lewis (2001) approach followed a structured Fourier-based methodology. The implementation began with a vectorized characteristic function that combines the Heston stochastic volatility component with a Merton jump-diffusion term, ensuring numerical stability through fallback conditions for near-singular cases in the square-root process. European call prices were computed by directly integrating the Lewis formula over a truncated domain (0 to 500) with tight quadrature tolerances, while we used put-call parity to derive puts. The calibration minimized the raw mean squared error between model and market prices across all the 60-day options, with explicit enforcement of parameter bounds and a soft Peller penalty. The L-BFGS-B optimizer was employed to ensure a stable and interpretable fit.

For the Carr-Madan (1999) implementation, the calibration used a Fourier representation that utilizes the absolute log-strike in the integration. The characteristic function was constructed to include the spot, consistent with the Carr-Madan formulation for the log-price process. Call prices were obtained through direct numerical integration of the damped Fourier integral with a fixed damping parameter α=1.5. Like we did with Lewis, the objective function minimized MSE, using the same constrained parameter space that enforces negative correlation and downside jumps. The calibration employed the L-BFGS-B algorithm with identical bounds and a Feller-condition penalty.

2.3 Calibration Results and Discussion

2.3.1 Lewis calibration: The Lewis Bates calibration produced economically reasonable parameters: a moderate negative correlation (ρ = -0.42) captures the leverage effect, while a high jump intensity ($\lambda \approx 5$ jumps/year) with small negative jumps (μ_j = -3.3%) reflects frequent minor downside moves. The stochastic volatility component is minimal (σ = 0.05), indicating the model relies more on jumps than volatility clustering to fit the 60-day smile. The initial variance is low (v_0 = 0.01), but the long-run level (θ = 0.20) allows for upward drift. The final MSE of $1.38 reflects a reasonable fit given the complexity of medium-dated options. Figure 2.1 below is a scatterplot which is a diagnostic plot showing relatively moderate fit.

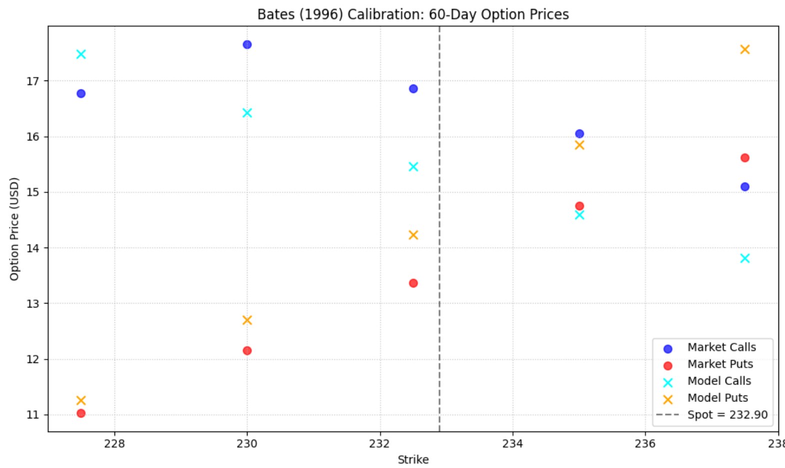


Fig 2.1 Diagnostic Plot for Bates via Lewis (2001)

2.3.2 Carr–Madan calibration: The Carr-Madan Bates calibration shows a different risk profile from Lewis: it shows extreme negative correlation (ρ = -0.99) and very high initial variance (v_0 = 0.30), indicating strong reliance on stochastic volatility to generate skew, complemented by fewer but larger jumps ($\lambda \approx$ 2.3/year, μ_j = -30%). In contrast, the Lewis calibration used minimal stochastic volatility (σ = 0.05) and instead relied on frequent small jumps ($\lambda \approx$ 5/year, μ_j = -3.3%). Both method achieve nearly identical pricing accuracy (MSE ≈ 1.35–1.38), demonstrating that the market's 60-day volatility surface can be fit through two distinct mechanisms: either high jump frequency with low volatility dynamics (Lewis) or extreme leverage effect with large jumps (Carr-Madan). This highlights the well-known identification challenge in jump-diffusion models, where different parameter combinations can produce similar option prices.

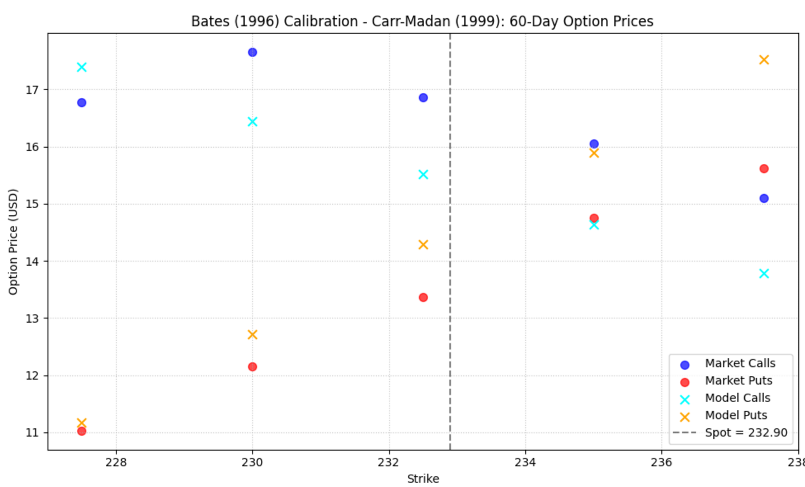


Fig 2.2 Diagnostic Plot for Bates via Carr-Madan (1999)

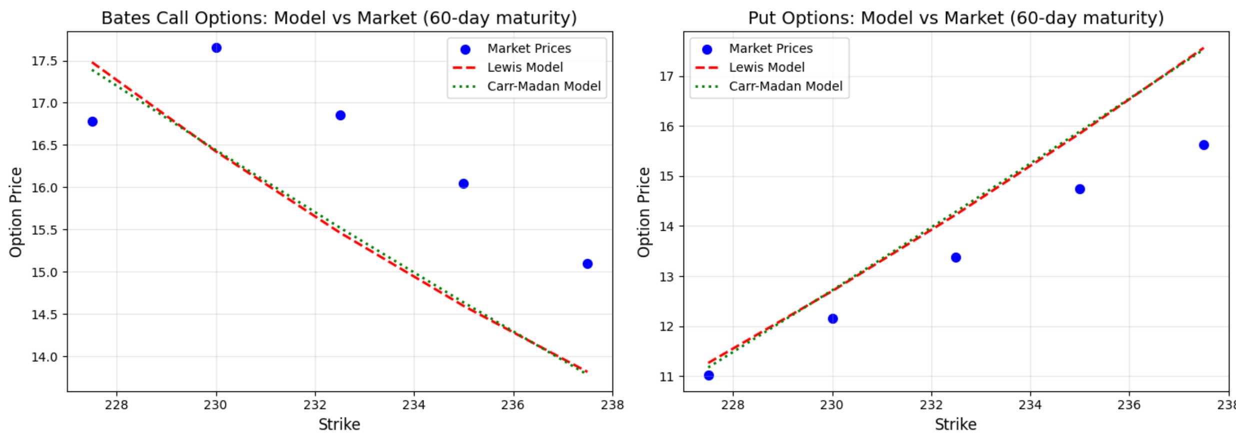


Fig 3.3 Diagnostic Plots, both Calls and Puts for Bates (1999)

It is evident from figure 3.3 that was had a better calibration from Heston without jumps, than Bates.

2.4 Pricing a 70-Day 95 % Put

The client requests a 70-day put option on SM with moneyness 95 % (strike 0.95 S_0). Using the parameters from the Carr–Madan Bates calibration (because they produced better/lower MSE), we price the put via Monte Carlo simulation of the SVJ model. The simulation uses daily time steps and 100,000 paths with antithetic variates. The fair price of the option is ≈ $7.1756, and with the 4 % bank fee the client price is $7.4627. The presence of jumps increases the option value relative to a pure Heston model because downward jumps raise put payoffs.

2.4.1 Non-Technical Description for the Client

To price your 70-day put option, we used a sophisticated market model (Bates, 1996) that accounts for both changing volatility and the possibility of sudden market drops; something standard models often miss. We first calibrated this model to current prices of traded options on SM stock, which means we adjusted its settings until it accurately reflected how the market is pricing risk today. Using this model that we calibrated, we calculated the fair value of your put option, which gives you the right to sell SM stock at 95% of today's price in 70 days. Finally, we added a 4% fee to cover execution and risk management costs, resulting in your final quoted price of $7.4627.

Step 3 – CIR Model Calibration and Euribor Simulation

3.1 CIR Model Specification

The Cox–Ingersoll–Ross (CIR) model describes the short rate $r_t$by the stochastic differential equation

$$\mathrm{d}r_t = k\,(\theta - r_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}W_t,$$

where $k > 0$ is the speed of mean reversion to long-run level θ and σ is the volatility. The square–root diffusion term prevents negative rates and ensures that variance declines as rates approach zero. Short rate reaches zero only if 2βμ < σ², underscoring the Feller condition for positivity (Brigo & Mercurio, 2006).

3.2 Data Processing and Term-Structure Construction

Euribor spot rates for maturities of 1 week, 1, 3, 6, and 12 months are provided. We convert these to continuously compounded zero rates and use cubic spline interpolation to generate a smooth zero-coupon yield curve at weekly intervals over 12 months. The CIR (1985) model is then calibrated by minimizing the sum of squared differences between the model-implied zero rates (computed from its closed-form bond pricing formula) and the interpolated market zero rates. This ensures the model accurately reproduces the current term structure of interest rates. The resulting spline is shown in Figure 3.1 below.

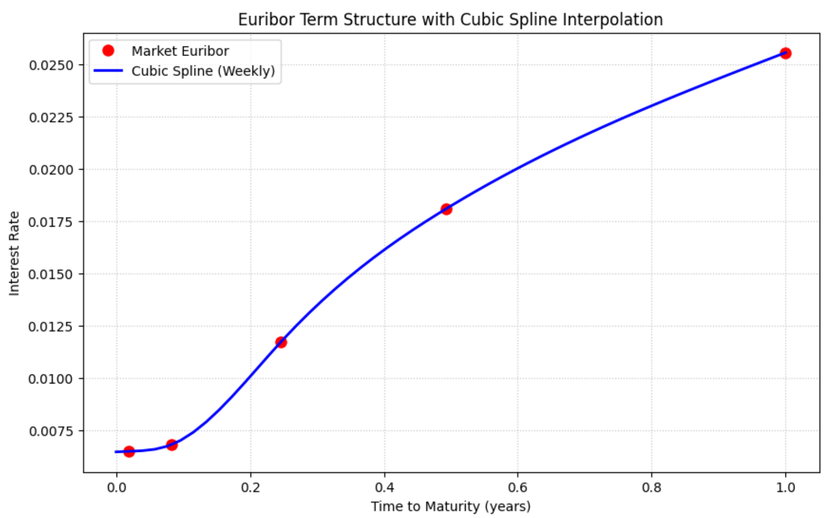


Fig 3.1 Euribor Term Structure with Cubic Spline Interpolation

3.3 Calibration Results and Discussion

The optimization returns $k = 0.5064$, $\theta = 0.1000$ (at its upper bound), and $\sigma = 0.0993$, with an MSE of 6.8×10^{-7}. The Feller condition $2k\theta \geq \sigma^2$holds (2×0.5064×0.1 = 0.1013 ≥ 0.0099), ensuring strictly positive interest rates. The moderate mean reversion speed implies that rates adjust toward the long-run level over about 2 years. The long-run mean of 10% is higher than current market rates (12-month Euribor is 2.56%), indicating the model anticipates rising rates in the future; a common feature when fitting mean-reverting models to yield curves that slopes upwards (like ours). The fitted curve matches the interpolated term structure closely, with minor deviations at the shortest maturities, probably due to sparse data in that region.

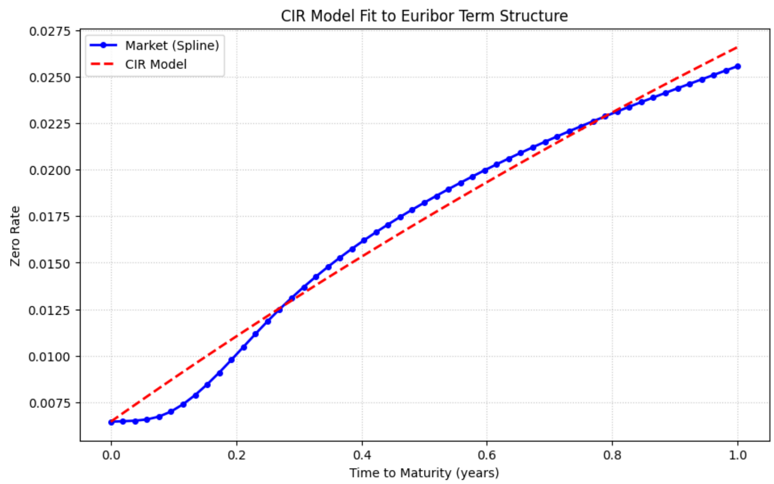
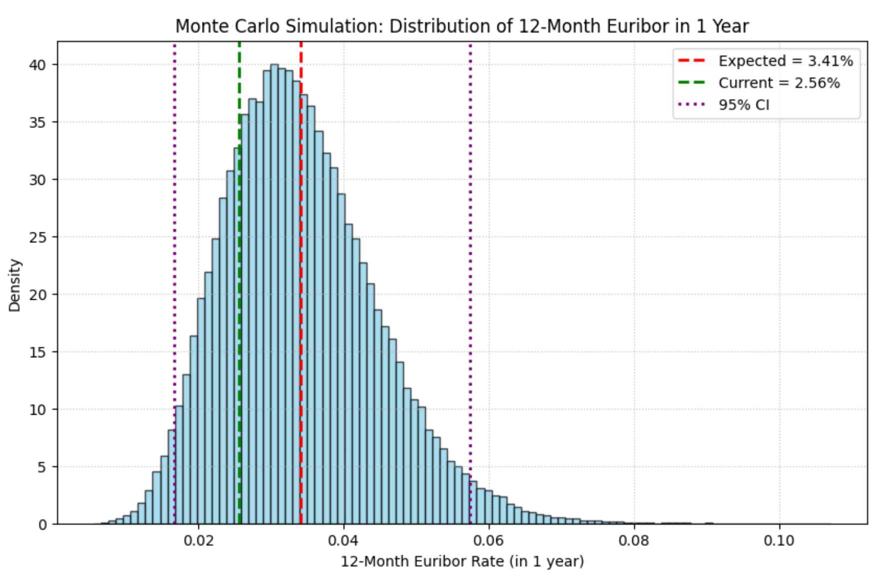


Fig 3.2 CIR Model Fit to Euribor Term Structure

3.4 Simulating 12-Month Euribor Rates

To forecast the 12-month Euribor rate in one year, we simulated 100,000 future paths of the short-term interest rate using the calibrated CIR model. The simulation was performed with daily time steps over a 1-year horizon, starting from today's short rate (0.65%). At each step, we used the Euler-Maruyama scheme with full truncation to ensure non-negative rates. We also applied antithetic variates to reduce variance and improve efficiency.

The resulting distribution of 12-month Euribor rates in one year is shown in the histogram which is figure 3.4 below. It is right-skewed, as expected under the CIR model, reflecting the square-root diffusion that prevents negative rates while allowing higher volatility at higher levels (Chan *et al,* 1992).



3.4.1 Key Results:

- **Expected 12-month Euribor in 1 year: 3.41%** — an increase of 0.85 percentage points from today's 2.56%

- **95% Confidence Interval: [1.67%, 5.74%]** — meaning we are 95% confident the rate will fall within this range

- The current 12-month rate (2.56%) lies comfortably within the 95% CI, indicating the model is consistent with today's market but expects upward drift due to the high long-run mean (10%)

This upward drift reflects the CIR model's structure: even though current rates are low, the model pulls them toward its long-run level over time. The wide confidence interval highlights the inherent uncertainty in interest rate forecasting — especially when volatility is moderate (σ = 9.93%) and mean reversion is not extremely fast (κ = 0.5064).

3.4.2 Implications for Pricing

Higher future rates imply lower present values of future cash flows, which means that option prices would be lower for long-dated derivatives. The higher discount factors implies lower bond prices, while increased cost of carry would mean higher forward/futures prices. For your products, this means pricing today should account for the expected rise in rates which we've quantified via simulation.

References

Chan, K.C., Karolyi, G.A., Longstaff, F.A. and Sanders, A.B., (1992). An empirical comparison of alternative models of the short-term interest rate. *The Journal of Finance,* 47(3), pp.1209–1227.

Brigo, D. and Mercurio, F., (2006). *Interest Rate Models – Theory and Practice.* 2nd ed. Berlin: Springer.

Merino, R., Pospíšil, J., Sobotka, T. and Vives, J., (2019). *Decomposition formula for jump diffusion models*. https://arxiv.org/abs/1906.06930

Appendix

Link to notebook

Assignment Scenario: MScFE 622_Stochastic Modeling_Group_Work_Project_1.pdf - Google Drive