Notes From class:

a,b ∈ $Z$

The Division Algorithm (Actually a theorem):

Suppose a∈$Z$ and b ∈ $Z^+$. Then there exists <u>unique</u> integers q and r such that $a = bq + r$ and $0 <= r < b$

(Follows from the 'Well Ordering principle')

<u>unique:</u> x is unique if there is only one such value. x and y are <u>distinct</u> if x/=y.

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Definition: Suppose a∈$Z$. We say that a is even when there exists n∈$Z$ such that a=2n. We say that a is odd when there exists k∈$Z$ such that a=2k+1

Claim: Suppose a∈$Z$. Then a is even or a is odd, and a can't be both even and odd.

Could a be both even and odd? Suppose so. This means that there exists integers k and n such that a=2n = 2k+1. Then n=k+(1/2), and so n-k=(1/2). But since n and k are integers and $Z$ is closed under subtraction, n-k∈$Z$. This would make (1/2) an integer, which it isn't.

Recall:

Definition: Suppose a and b are integers. Then b divideds a, denoted b|a, when there exists an integer k such that a = bk.

Observation: b|a <=> the (unique) remainder when we divide a by b is 0.

Suppose b|a. this means a=bk for some k∈$Z$. In other words, a=bk+0 where b,0∈$Z$ and 0<=0<b. Thus the remainder when we divide a by b, the remainder is 0.

<== : Suppose that when we divide b by a using the division algorithm, the remainder is 0. In other words, there exists q∈$Z$ such that a=bq+0=bq. Thus, by definition of divisor, b|a. Therefore, b|a if and only if the remainder on dividing a by b is 0. ■

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Proposition: If n∈$Z$, then the remainder we get when we divide $n^2$ by 4 is either 0 or 1.

Case 1: Suppose a is an even integer. Then a=2n, and $a^2$ = $4n^2$. If we then divide $4n^2$ by 4 we get $n^2$. Let q = $n^2$, since $z$ is closed under multiplication and n∈$Z$, q∈$Z$. Thus giving us 4q+0, which leaves us with a remainder of 0. ■

Case 2: Suppose b is an odd integer. Then $b = 2k + 1$, and $b^2 = 4k^2 + 4k + 1$. We then can foil $4k^2 + 4k$ into $4(k^2 + k) + 1$. Let q = $k^2 + k$, since $Z$ is closed under addition and multiplication, and k∈$Z$, q∈$Z$. Thus we get 4q+1, leaving a remainder of 1. ■

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Other Proposition: If n∈$Z$, then the remainder when you divide $n^2$ by 3 is 0 or 1.

Case 1: Suppose c is an integer that is a multiple of 3, then $c = 3d$. With this we can square c, giving $c^2 = 9d^2$. We can then reformat to $3(3d^2)$. Let $m = 3d^2$, since $Z$ is closed under multiplication, $3d^2 \in Z$, $m \in Z$. This gives us 3m+0, thus leaving a remainder of 0. ■

Case 2: Suppose g is an even integer that isn't a multiple of 3, then $g = 2f$. We then square g, getting $4n^2$. Let $p = f^2$, since $Z$ is closed under multiplication and $g \in Z$, $f \in Z$. Thus when 3 goes into $4f^2$ we get a remainder of 1. ■

Case 3: Suppose ■ is an odd integer that isn't a multiple of 3, then $■ = 2s + 1$. We then square it getting $■^2 = 4s^2 + 4s + 1$.
Never ended up finishing this proof. Think I got the first part right but for case 2 it seemed to work out but wouldn't work for the odds definition. Below I give the correct proofs that the professor gives.

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Professors proof for 1:
Suppose n∈$Z$. Then we know n is even or n is odd.

Case 1: n is even, then $n = 2k$ for some k∈$Z$. Then $n^2 = 4k^2 = 4k^2 + 0$. Let q = $k^2$ and r=0, then q∈$Z$, since $Z$ is closed under multiplication and r∈$Z$, and $n^2 = 4q + r$ and $0 <= r < 4$. So the remainder when you divide $n^2$ by 4 is 0. ■

Case 2: n is odd, then $n = 2k + 1$ for some k∈$Z$. Then $n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. Let q = $k^2 + k$, then q∈$Z$ and $n^2 = 4q + 1$, where 0<=1<4. Thus the remainder when you divide $n^2$ by 4 is 1. ■

Professors proof for 2:
Suppose n is an integer.
By the division algorithm, there exists Integer k such that $n = 3k + r$ where r= 0, 1 or 2.

Case 1(r=0): n is divisible by 3. $n = 3k$ where k∈$Z$.
$n^2 = (3k)^2 = 9k^2 = 3(3k^2) + 0$. So if we let $q = 3k^2$ and r=0, then q∈$Z$($Z$ is closed under...) and r∈$Z$ and $n^2 = 3q + r$ and $0 <= r < 3$. So the remainder when you divide $n^2$ by 3 is 0. ■

Case 2(r=1): Then $n = 3k + 1$. So=,
$n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Let q = $3k^2 + 2k$, since $Z$ is closed under multiplication and n,k are integers, q is an integer, and r∈$Z$ and $n^2 = 3q + r$ and $0 <= r < 3.Sotheremainderfor\$n^2$ is 1.■

Case 3:(r=2) then $n = 3k + 2$.
$n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k) + 4 = 3(3k^2 + 4k + 1) + 1$. Let $q = 3k^2 + 4k + 1$,($Z$ is closed under...) and r∈$Z$ and $n^2 = 3q + r$ and $0 <= r < 3$. So the remainder for $n^2$ is 1. ■