By Heart

The Polar representation of complex numbers

The Polar representation of complex numbers

Given a point $z = x + yi$ in the complex plane. The point has a polar representation $(r,\ \theta) :\ \ x = r\cos\theta,\ y = r\sin\theta$, where $r = |z|$ and θ is the angle between the positive real axis and the line segment from 0 to z.

Roots of complex numbers

Roots of complex numbers

Given a complex number $a = |a| \operatorname{cis}(\alpha) \neq 0$ and an integer $n \geq 2$, a n^{th} root of a is a number

$$|a|^{\frac{1}{n}} \operatorname{cis}\left(\frac{1}{n}(\alpha + 2\pi k)\right)$$

where $0 \leq k \leq n-1$

Lines in $\mathbb{C}$

Lines in $\mathbb{C}$

A line in $\mathbb{C}$ is of the form

$$L = \{z = a + tb \mid -\infty < t < \infty\}$$

for $a, b \in \mathbb{C}$ or

$$L = \left\{z : \mathrm{Im}\left(\frac{z-a}{b}\right) = 0\right\}$$

Half planes in $\mathbb{C}$

Half planes in $\mathbb{C}$

For $a, b \in \mathbb{C}$W e are "walking along L in the direction of b." If we put

$$H_a = \left\{ z : \operatorname{Im}\left(\frac{z-a}{b}\right) > 0 \right\}$$

then it is easy to see that $H_a = a + H_0 \equiv \{a + w : w \in H_0\}$; that is, H_a is the translation of H_0 by a. Hence, H_a is the half plane lying to the left of L. Similarly,

$$K_a = \left\{ z : \operatorname{Im}\left(\frac{z-a}{b}\right) < 0 \right\}$$

is the half plane on the right of L.

The triangle inequality in $\mathbb{C}$

The triangle inequality in $\mathbb{C}$

$$|z+w| \leq |z| + |w|, \ (z, w \in \mathbb{C})$$

where

$$|z| = \left(x^2 + y^2\right)^{\frac{1}{2}}$$

The Weierstrass M-test for series of functions

The Weierstrass M-test for series of functions

Let $u_n : X \to \mathbb{C}$ be a function such that $|u_n(x)| \leq M_n$ for every x in X and suppose the constants satisfy $\sum_{n=1}^{\infty} M_n < \infty$. Then $\sum_1^{\infty} u_n$ is uniformly convergent.

The Heine-Borel Theorem

The Heine-Borel Theorem

A subset K of $\mathbb{R}^n$, $n \geq 1$ is compact iff K is closed and bounded.

The Cantor Intersection Theorem

The Cantor Intersection Theorem

Let X be a metric space. Then X is complete if and only if whenever $\{F_n\}_{n=1}^{\infty}$ is a contracting sequence of nonempty closed subsets of X, there is a point $x \in X$ for which $\bigcap_{n=1}^{\infty} F_n = \{x\}$

The Cauchy Convergence Criterion

The Cauchy Convergence Criterion

If (X, d) has the property that each Cauchy sequence has a limit in X then (X, d) is complete.

The Intermediate Value Theorem

The Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \leq \xi \leq f(b)$ then there is a point $x, a \leq x \leq b$, with $f(x) = \xi$.

Morera's Theorem

Morera's Theorem

Let G be a region and let $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path T in G; then f is analytic in G.

Cauchy’s Theorem (Second Version)

Cauchy's Theorem (Second Version)

If $f : G \to \mathbb{C}$ is an analytic function and γ is a closed rectifiable curve in G such that $\gamma \sim 0$, then

$$\int_\gamma f = 0$$

Cauchy's Theorem (Fourth Version)

Cauchy's Theorem (Fourth Version)

If G is simply connected then $\int_\gamma f = 0$ for every closed rectifiable curve and every analytic function f.

Open Mapping Theorem

Open Mapping Theorem

Let G be a region and suppose that f is a non constant analytic function on G. Then for any open set U in G, $f(U)$ is open.

Goursat's Theorem

Goursat's Theorem

Let G be an open set and let $f : G \to \mathbb{C}$ be a differentiable function; then f is analytic on G.

Laurent series development of an analytic function in an annulus

Laurent series development of an analytic function in an annulus

Let f be analytic in the annulus ann $(a; R_1, R_2)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where the convergence is absolute and uniform over ann $(a; r_1, r_2)^{-}$ if $R_1 < r_1 < r_2 < R_2$. Also the coefficients a_n are given by the formula

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz$$

where γ is the circle $|z-a|= r$ for any $r, R_1 < r < R_2$. Moreover, this series is unique.

Residue Theorem

Residue Theorem

Let f be analytic in the region G except for the isolated singularities

$a_1, a_2, \ldots, a_m$. If γ is a closed rectifiable curve in G which does not pass through any of the points a_k and if $\gamma \approx 0$ in G then

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^{m} n\left(\gamma; a_k\right) \operatorname{Res}\left(f; a_k\right)$$

The Argument Principle

The Argument Principle

Let f be meromorphic in G with poles $p_1, p_2, \ldots, p_m$ and zeros $z_1, z_2, \ldots, z_n$ counted according to multiplicity. If γ is a closed rectifiable curve in G with $\gamma \approx 0$ and not passing through $p_1, \ldots, p_m; z_1, \ldots, z_n$; then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{n} n\left(\gamma; z_k\right) - \sum_{j=1}^{m} n\left(\gamma; p_j\right).$$

Rouché’s Theorem

Rouché's Theorem

Suppose f and g are meromorphic in a neighborhood of $\overline{B}(a;R)$ with no zeros or poles on the circle $\gamma = \{z : |z - a| = R\}$. If Z_f, Z_g (P_f, P_g) are the number of zeros (poles) of f and g inside γ counted according to their multiplicities and if

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

on γ, then

$$Z_f - P_f = Z_g - P_g.$$

Schwarz's Lemma

Schwarz's Lemma

Let $D = \{z : |z| < 1\}$ and Suppose f is analytic on D with

(a) $|f(z)| \leq 1$ for z in D,

(b) $f(0) = 0$.

Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all z in the disk D. Moreover if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some $z \neq 0$ then there is a constant $c, |c| = 1$, such that $f(w) = cw$ for all w in D.

The Arzelà-Ascoli Theorem

The Arzelà-Ascoli Theorem

A set $\mathscr{F} \subset C(G, \Omega)$ is normal iff the following two conditions are satisfied:

(a) for each z in G, $\{f(z) : f \in \mathscr{F}\}$ has compact closure in Ω;

(b) $\mathscr{F}$ is equicontinuous at each point of G.

Hurwitz's Theorem

Hurwitz's Theorem

Let G be a region and suppose the sequence $\{f_n\}$ in $H(G)$ converges to f. If $f \not\equiv 0$, $\overline{B}(a;R) \subset G$, and $f(z) \neq 0$ for $|z-a|= R$ then there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a;R)$.

Montel's Theorem

Montel's Theorem

A family $\mathscr{F}$ in $H(G)$ is normal iff $\mathscr{F}$ is locally bounded.

The Riemann Mapping Theorem

The Riemann Mapping Theorem

Let G be a simply connected region which is not the whole plane and let $a \in G$. Then there is a unique analytic function $f : G \to \mathbb{C}$ having the properties:

(a) $f(a) = 0$ and $f'(a) > 0$;

(b) f is one-one;

(c) $f(G) = \{z : |z| < 1\}$.

Gauss's Formula

Gauss's Formula

For $z \neq 0, -1, \ldots$

$$\Gamma(z) = \lim_{n \to \infty} \frac{n!\, n^z}{z(z+1)\ldots(z+n)}$$

Functional Equation for Γ

Functional Equation for Γ

For $z \neq 0, -1, \ldots$

$$\Gamma(z+1) = z\Gamma(z)$$

Mean Value Theorem for harmonic functions

Mean Value Theorem for harmonic functions

If $u : G \to \mathbb{R}$ is a harmonic function and $\overline{B}(a; r)$ is a closed disk contained in G, then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u\left(a + re^{i\theta}\right) d\theta$$

Maximum Principle (First Version) for harmonic functions

Maximum Principle (First Version) for harmonic functions

Let G be a region and suppose that u is a continuous real valued function on G with the MVP. If there is a point a in G such that $u(a) \geq u(z)$ for all z in G then u is a constant function.

Minimum Principle for harmonic functions

Minimum Principle for harmonic functions

Let G be a region and suppose that u is a continuous real valued function on G with the MVP. If there is a point a in G such that $u(a) \leq u(z)$ for all z in G then u is a constant function.

Harnack's Theorem

Harnack's Theorem

Let G be a region. (a) The metric space Har(G) is complete. (b) If $\{u_n\}$ is a sequence in Har(G) such that $u_1 \leq u_2 \leq \ldots$ then either $u_n(z) \to \infty$ uniformly on compact subsets of G or $\{u_n\}$ converges in Har (G) to a harmonic function.

The field axioms

The field axioms

Commutativity of Addition: For all real numbers a and b,

$$a + b = b + a$$

Associativity of Addition: For all real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

The Additive Identity: There is a real number, denoted by 0 , such that

$$0 + a = a + 0 = a \quad \text{for all real numbers } a$$

The Additive Inverse: For each real number a, there is a real number b such that

$$a + b = 0$$

Commutativity of Multiplication: For all real numbers a and b,

$$ab = ba$$

Associativity of Multiplication: For all real numbers a, b, and c,

$$(ab)c = a(bc)$$

The Multiplicative Identity: There is a real number, denoted by 1 , such that

$$1a = a1 = a \quad \text{for all real numbers } a.$$

The Multiplicative Inverse: For each real number $a \neq 0$, there is a real number b such that

$$ab = 1.$$

The Distributive Property: For all real numbers a, b, and c,

$$a(b + c) = ab + ac$$

The Nontriviality Assumption: $1 \neq 0$.

The positivity axioms

The positivity axioms

P1 If a and b are positive, then ab and $a + b$ are also positive.

P2 For a real number a, exactly one of the following three alternatives is true:

$$a \text{ is positive}, \qquad -a \text{ is positive}, \qquad a = 0.$$

The completeness axiom

The completeness axiom

Let E be a nonempty set of real numbers that is bounded above. Then among the set of upper bounds for E there is a smallest, or least, upper bound.

Principle of mathematical induction

Principle of mathematical induction

For each natural number n, let $S(n)$ be some mathematical assertion. Suppose $S(1)$ is true. Also suppose that whenever k is a natural number for which $S(k)$ is true, then $S(k+1)$ is also true. Then $S(n)$ is true for every natural number n.

Archimedean Property

Archimedean Property

For each pair of positive real numbers a and b, there is a natural number n for which $na > b$.

The pigeonhole principle

The pigeonhole principle

The first observation regarding equipotence (In the preliminaries we called two sets A and B equipotent provided there is a one-to-one mapping f of A onto B.) is that for any natural numbers n and m, the set $\{1, \ldots, n+m\}$ is not equipotent to the set $\{1, \ldots, n\}$.

The Nested Set Theorem

The Nested Set Theorem

Let $\{F_n\}_{n=1}^{\infty}$ be a descending countable collection of nonempty closed sets of real numbers for which F_1 bounded. Then

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset$$

The Bolzano-Weierstrass Theorem

The Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

The Extreme Value Theorem

The Extreme Value Theorem

A continuous real-valued function on a nonempty closed, bounded set of real numbers takes a minimum and maximum value.

Every interval is a measurable set.

Every interval is a measurable set.

true

The translate of a measurable set is measurable.

The translate of a measurable set is measurable.

true

Continuity of measure

Continuity of measure

Lebesgue measure possesses the following continuity properties:

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m\left(A_k\right)$$

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets and $m\left(B_1\right) <$

∞, then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m\left(B_k\right)$$

The Borel-Cantelli Lemma

The Borel-Cantelli Lemma

Lemma Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbf{R}$ belong to at most finitely many of the E_k 's.

Vitali's Theorem

Vitali's Theorem

Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

A measurable set that is not Borel

A measurable set that is not Borel

There is a measurable set, a subset of the Cantor set, that is not a Borel set.

The function ψ maps a non-Borel measurable set to a nonmeasurable set.

The function ψ maps a non-Borel measurable set to a nonmeasurable set.

Let φ be the Cantor-Lebesgue function and define the function ψ on $[0, 1]$ by

$$\psi(x) = \varphi(x) + x \text{ for all } x \in [0, 1]$$

Then ψ is a strictly increasing continuous function that maps $[0, 1]$ onto $[0, 2]$,

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

A continuous function defined on a measurable set is measurable.

A continuous function defined on a measurable set is measurable.

true

A monotone function defined on an interval is measurable.

A monotone function defined on an interval is measurable.

true

The composition of a continuous function and a measurable function is measurable.

The composition of a continuous function and a measurable function is measurable.

true

The pointwise limit of a sequence of measurable function is measurable.

The pointwise limit of a sequence of measurable function is measurable.

true

The Vitali Covering Lemma

The Vitali Covering Lemma

Let E be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that covers E in the sense of Vitali. Then for each $\epsilon > 0$, there is a finite disjoint subcollection $\{I_k\}_{k=1}^n$ of $\mathcal{F}$ for which

$$m^*\left[E \sim \bigcup_{k=1}^{n} I_k\right] < \epsilon$$

Jordan decomposition of a function of bounded variation

Jordan decomposition of a function of bounded variation

We call the expression of a function of bounded variation f as the difference of increasing functions a Jordan decomposition of f.

Indefinite integral of a Lebesgue integrable function over a closed, bounded interval.

Indefinite integral of a Lebesgue integrable function over a closed, bounded interval.

We here call a function f on a closed, bounded interval $[a, b]$ the indefinite integral of g over $[a, b]$ provided g is Lebesgue integrable over $[a, b]$ and

$$f(x) = f(a) + \int_a^x g \text{ for all } x \in [a, b]$$

Additivity over domains of integration

Additivity over domains of integration

Let f be integrable over E. Assume A and B are disjoint measurable subsets of E. Then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Lebesgue's Theorem on Riemann integrability

Lebesgue's Theorem on Riemann integrability

Let f be a bounded function on the closed, bounded interval $[a, b]$. Then f is Riemann integrable over $[a, b]$ if and only if the set of points in $[a, b]$ at which f fails to be continuous has measure zero.

Jordan's Theorem

Jordan's Theorem

A function f is of bounded variation on the closed, bounded interval $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$.

Lebesgue Decomposition for a function of bounded variation

Lebesgue Decomposition for a function of bounded variation

The above decomposition of a function of bounded variation f as the sum $g+h$ of two functions of bounded variation, where g is absolutely continuous and h is singular, is called a Lebesgue decomposition of f.

Young's Inequality

Young's Inequality

For $1 < p < \infty, q$ the conjugate of p, and any two positive numbers a and b,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Holder’s Inequality

Holder's Inequality

Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. If f belongs to $L^p(E)$ and g belongs to $L^q(E)$, then their product $f \cdot g$ is integrable over E and Hölder's Inequality Moreover, if $f \neq 0$, the function $^2 f^* = \|f\|_p^{1-p} \cdot \operatorname{sgn}(f) \cdot |f|^{p-1}$ belongs to $L^q(X, \mu)$,

$$\int_E f \cdot f^* = \|f\|_p \text{ and } \|f^*\|_q = 1$$

It is convenient, for $f \in L^p(E)$, $f \neq 0$, to call the function f^* defined above the conjugate function of f.

Minkowski's Inequality

Minkowski's Inequality

Let E be a measurable set and $1 \leq p \leq \infty$. If the functions f and g belong to $L^p(E)$, then so does their sum $f+g$ and, moreover,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

The Cauchy-Schwarz Inequality

The Cauchy-Schwarz Inequality

Let E be a measurable set and f and g measurable functions on E for which f^2 and g^2 are integrable over E. Then their product $f \cdot g$ also is integrable over E and

$$\int_E |fg| \leq \sqrt{\int_E f^2} \cdot \sqrt{\int_E g^2}$$

The Riesz-Fischer Theorem

The Riesz-Fischer Theorem

Let E be a measurable set and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in $L^p(E)$, a subsequence of $\{f_n\}$ converges pointwise a.e. on E to f.

The Riesz Representation Theorem for the Dual of $L^p(E)$

The Riesz Representation Theorem for the Dual of $L^p(E)$

Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. For each $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by

$$\mathcal{R}_g(f) = \int_E g \cdot f \text{ for all } f \text{ in } L^p(E)$$

Then for each bounded linear functional T on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which

$$\mathcal{R}_g = T, \text{ and } \|T\|_* = \|g\|_q.$$

$\epsilon - \delta$ Criterion for Continuity

$\epsilon - \delta$ Criterion for Continuity

A mapping f from a metric space (X, ρ) to a metric space (Y, σ) is continuous at the point $x \in X$ if and only if for every $\epsilon > 0$, there is $a\delta > 0$ for which if $\rho\left(x, x'\right) < \delta$, then $\sigma\left(f(x), f\left(x'\right)\right) < \epsilon$, that is,

$$f(B(x, \delta)) \subseteq B(f(x), \epsilon)$$

The Lebesgue Covering Lemma

The Lebesgue Covering Lemma

Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of a compact metric space X. Then there is a number $\epsilon > 0$, such that for each $x \in X$, the open ball $B(x, \epsilon)$ is contained in some member of the cover.

Define

Connectedness

Connectedness

A metric space (X, d) is connected if the only subsets of X which are both open and closed are $\square$ and X. If $A \subset X$ then A is a connected subset of X if the metric space (A, d) is connected.

Cauchy sequence

Cauchy sequence

A sequence $\{x_n\}$ is called a Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Uniform convergence

Uniform convergence

Let X be a set and (Ω, ρ) a metric space and suppose $f, f_1, f_2, \ldots$ are functions from X into Ω. The sequence $\{f_n\}$ converges uniformly to f-written $f = u - \lim f_n$-if for every $\epsilon > 0$ there is an integer N (depending on ϵ alone) such that $\rho(f(x), f_n(x)) < \epsilon$ for all x in X, whenever $n \geq N$.

Analytic function

Analytic function

A function $f : G \to \mathbb{C}$ is analytic if f is continuously differentiable on G.

Principal branch of the logarithm

Principal branch of the logarithm

If G is an open connected set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ is a continuous function such that $z = \exp f(z)$ for all z in G then f is a branch of the logarithm. We designate the particular branch of the logarithm defined above on $\mathbb{C} - \{z : z \leq 0\}$ to be the principal branch of the logarithm.

Definition of Mobius map

Definition of Mobius map

A mapping of the form $S(z) = \frac{az+b}{cz+d}$ is called a linear fractional transformation. If a, b, c, and d also satisfy $ad - bc \neq 0$ then $S(z)$ is called a Mobius transformation.

Symmetry Principle

Symmetry Principle

If a Mobius transformation T takes a circle Γ_1 onto the circle Γ_2 then any pair of points symmetric with respect to Γ_1 are mapped by T onto a pair of points symmetric with respect to Γ_2.

Orientation Principle

Orientation Principle

Let Γ_1 and Γ_2 be two circles in $\mathbb{C}_\infty$ and let T be a Mobius transformation such that $T(\Gamma_1) = \Gamma_2$. Let (z_1, z_2, z_3) be an orientation for Γ_1. Then T takes the right side and the left side of Γ_1 onto the right side and left side of Γ_2 with respect to the orientation (Tz_1, Tz_2, Tz_3).

Riemann-Stieltjes integral

Riemann-Stieltjes integral

Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation and suppose that $f : [a, b] \to \mathbb{C}$ is continuous. Then there is a complex number I such that for every $\epsilon > 0$ there is a $\delta > 0$ such that when $P = \{t_0 < t_1 < \ldots < t_m\}$ is a partition of $[a, b]$ with $\|P\| = \max\{(t_k - t_{k-1}) : 1 \leq k \leq m\} < \delta$ then

$$\left| I - \sum_{k=1}^{m} f(\tau_k) \left[\gamma(t_k) - (t_{k-1})\right] \right| < \varepsilon$$

for whatever choice of points $\tau_k, t_{k-1} \leq \tau_k \leq t_k$. This number I is called the integral of f with respect to γ over $[a, b]$ and is designated by

$$I = \int_a^b f d\gamma = \int_a^b f(t) d\gamma(t).$$

Cauchy's Estimate

Cauchy's Estimate

Let f be analytic in $B(a;R)$ and suppose $|f(z)| \leq M$ for all z in $B(a;R)$. Then

$$\left| f^{(n)}(a) \right| \leq \frac{n!\, M}{R^n}$$

Liouville's Theorem

Liouville's Theorem

If f is a bounded entire function then f is constant.

Fundamental Theorem of Algebra

Fundamental Theorem of Algebra

If $p(z)$ is a non constant polynomial then there is a complex number a with $p(a) = 0$.

Maximum Modulus Theorem

Maximum Modulus Theorem

If G is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point a in G with $|f(a)| \geq |f(z)|$ for all z in G, then f is constant.

Index of a closed rectifiable curve γ in $\mathbb{C}$ with respect to a point $a \notin \gamma$

Index of a closed rectifiable curve γ in $\mathbb{C}$ with respect to a point $a \notin \gamma$

If γ is a closed rectifiable curve in $\mathbb{C}$ then for $a \notin \{\gamma\}$

$$n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma (z-a)^{-1} dz$$

is called the index of γ with respect to the point a. It is also sometimes called the winding number of γ around a.

When is an open set simply connected?

When is an open set simply connected?

An open set G is simply connected if G is connected and every closed curve in G is homotopic to zero.

A rectifiable curve homologous to zero

A rectifiable curve homologous to zero

If G is an open set then γ is homologous to zero, in symbols $\gamma \approx 0$, if $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$.

Removable singularity of an analytic function at a point $z = a$

Removable singularity of an analytic function at a point $z = a$

A function f has an isolated singularity at $z = a$ if there is an $R > 0$ such that f is defined and analytic in $B(a; R) - \{a\}$ but not in $B(a; R)$. The point a is called a removable singularity if there is an analytic function $g : B(a; R) \to \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z - a| < R$.

Pole of a function

Pole of a function

If $z=a$ is an isolated singularity of f then a is a pole of f if $\lim_{z\to a}|f(z)|=\infty$. That is, for any $M>0$ there is a number $\epsilon>0$ such that $|f(z)|\geq M$ whenever $0<|z-a|<\epsilon$. If an isolated singularity is neither a pole nor a removable singularity it is called an essential singularity.

Cauchy's Integral Formula (First Version)

Cauchy's Integral Formula (First Version)

Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$, then for a in $G - \{\gamma\}$

$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} dz.$$

Cauchy's Integral Formula (Second Version)

Cauchy's Integral Formula (Second Version)

Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If $\gamma_1, \ldots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$ for all w in $\mathbb{C} - G$, then for a in $G - \cup_{k=1}^{m} \{\gamma_k\}$

$$f(a) \sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z - a} dz.$$

If f is analytic on an open connected set G and f is ____, then for each a in G with ____ there is an integer $n \geq 1$ and an ____ $g : G \to \mathbb{C}$ such that ____ and for all z in G. That is, ____ If f is analytic on an open connected set G and f is ____, then for each a in G with ____ there is an integer $n \geq 1$ and an ____ $g : G \to \mathbb{C}$ such that ____ and for all z in G. That is, ____

If f is analytic on an open connected set G and f is not identically zero, then for each a in G with $f(a) = 0$ there is an integer $n \geq 1$ and an analytic function $g : G \to \mathbb{C}$ such that $g(a) \neq 0$ and

$$f(z) = (z - a)^n g(z)$$

for all z in G. That is, each zero of f has finite multiplicity.

If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then

$$\_\_\_\_$$

is an ____.

If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then

$$\underline{\qquad}$$

is an _____.

If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

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is an integer.

Let γ be a closed rectifiable curve in $\mathbb{C}$. Then

(i) $n(\gamma; a)$ ____ of $G = \mathbb{C} - \{\gamma\}$; and

(ii) $n(\gamma; a) = 0$ for a belonging ____ G.

Let γ be a closed rectifiable curve in $\mathbb{C}$. Then

(i) $n(\gamma; a)$ ____ of $G = \mathbb{C} - \{\gamma\}$; and

(ii) $n(\gamma; a) = 0$ for a belonging ____ G.

Let γ be a closed rectifiable curve in $\mathbb{C}$. Then

(i) $n(\gamma; a)$ is constant for a belonging to a component of $G = \mathbb{C} - \{\gamma\}$; and

(ii) $n(\gamma; a) = 0$ for a belonging to the unbounded component of G. is an integer.

Cauchy's Theorem for functions analytic in a disk

Cauchy's Theorem for functions analytic in a disk

if G is an open disk then

$$\int_\gamma f = 0$$

for any analytic function f on G and any closed rectifiable curve γ in G.

Cauchy’s Integral Formula for Derivatives

Cauchy's Integral Formula for Derivatives

Let G be an open subset of the plane and $f : G \to \mathbb{C}$ be an analytic function. Let $\gamma_1, \ldots, \gamma_m$ be closed rectifiable curves in G such that

$$n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$$

for all w in $\mathbb{C} - G$. Then for a in $G - \{\gamma\}$ and $k \geq 1$, we have

$$f^{(k)}(a) \sum_{j=1}^{m} n(\gamma_j; a) = k! \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma,} \frac{f(z)}{(z-a)^{k+1}} dz$$

A region G is _____ if and only if $\mathbb{C}_\infty - G$, its complement in the extended plane, is connected in $\mathbb{C}_\infty$.

A region G is ____ if and only if $\mathbb{C}_\infty - G$, its complement in the extended plane, is connected in $\mathbb{C}_\infty$.

A region G is simply connected if and only if $\mathbb{C}_\infty - G$, its complement in the extended plane, is connected in $\mathbb{C}_\infty$.

Let G be ____ and let $f : G \to \mathbb{C}$ be an analytic function such the $f(z) \neq 0$ for any z in G. Then there exists an analytic function $g : G \to \mathbb{C}$ such that ____. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose g such that ____

Let G be ____ and let $f : G \to \mathbb{C}$ be an analytic function such the $f(z) \neq 0$ for any z in G. Then there exists an analytic function $g : G \to \mathbb{C}$ such that ____. If $z_0 \in G$ and $e^{w_0} = f\left(z_0\right)$, we may choose g such that ____

Let G be simply connected and let $f : G \to \mathbb{C}$ be an analytic function such the $f(z) \neq 0$ for any z in G. Then there exists an analytic function $g : G \to \mathbb{C}$ such that $f(z) = \exp g(z)$. If $z_0 \in G$ and $e^{w_0} = f\left(z_0\right)$, we may choose g such that $g\left(z_0\right) = w_0$

Let $z = a$ be an isolated singularity of f and let $f(z) = \sum_{-\infty}^{\infty} a_n(z-a)^n$ be its Laurent Expansion in ann $(a; 0, R)$. Then:

(a) $z = a$ is a ____ if and only if $a_n = 0$ for $n \leq -1$;

(b) $z = a$ is a ____ of order m if and only if $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$;

(c) $z = a$ is an____ if and only if $a_n \neq 0$ for infinitely many negative integers n.

Let $z = a$ be an isolated singularity of f and let $f(z) = \sum_{-\infty}^{\infty} a_n(z-a)^n$ be its Laurent Expansion in ann $(a; 0, R)$. Then:

(a) $z = a$ is a _____ if and only if $a_n = 0$ for $n \leq -1$;

(b) $z = a$ is a _____ of order m if and only if $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$;

(c) $z = a$ is an_____ if and only if $a_n \neq 0$ for infinitely many negative integers n.

Let $z = a$ be an isolated singularity of f and let $f(z) = \sum_{-\infty}^{\infty} a_n(z-a)^n$ be its Laurent Expansion in ann $(a; 0, R)$. Then:

(a) $z = a$ is a removable singularity if and only if $a_n = 0$ for $n \leq -1$;

(b) $z = a$ is a pole of order m if and only if $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$;

(c) $z = a$ is an essential singularity if and only if $a_n \neq 0$ for infinitely many negative integers n.

Suppose f has a pole of order m at $z = a$ and put $g(z) =$ ____; then

$$\underline{\hspace{3em}}$$

Suppose f has a pole of order m at $z = a$ and put $g(z) =$ ____; then

____

Suppose f has a pole of order m at $z = a$ and put $g(z) = (z-a)^m f(z)$; then

$$\operatorname{Res}(f;a) = \frac{1}{(m-1)!} g^{(m-1)}(a)$$

Polynomially convex hull of a compact set

Polynomially convex hull of a compact set

Let K be a compact subset of the plane; the polynomially convex hull of K, denoted by $\hat{K}$, is defined to be the set of all points w such that for every polynomial p

$$|p(w)| \leq \max\{|p(z)| : z \in K\}.$$

That is, if the right hand side of this inequality is denoted by $\|p\|_K$, then

$$\hat{K} = \{w : |p(w)| \leq \|p\|_K \text{ for all polynomials } p\}.$$

harmonic conjugate

harmonic conjugate

If $f : G \to \mathbb{C}$ is an analytic function then $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are called harmonic conjugates.

Harmonic function

Harmonic function

If G is an open subset of $\mathbb{C}$ then a function $u : G \to \mathbb{R}$ is harmonic if u has continuous second partial derivatives and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$C(G, \Omega)$

$C(G, \Omega)$

If G is an open set in $\mathbb{C}$ and (Ω, d) is a complete metric space then designate by $C(G, \Omega)$ the set of all continuous functions from G to Ω.

equicontinuous

equicontinuous

A set $\mathscr{F} \subset C(G, \Omega)$ is equicontinuous at a point z_0 in G iff for every $\epsilon > 0$ there is a $\delta > 0$ such that for $|z - z_0| < \delta$,

$$d\left(f(z), f\left(z_0\right)\right) < \epsilon$$

for every f in $\mathscr{F}$. $\mathscr{F}$ is equicontinuous over a set $E \subset G$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for z and z' in E and $|z - z'| < \delta$,

$$d\left(f(z), f\left(z'\right)\right) < \epsilon$$

normal

normal

A set $\mathscr{F} \subset C(G, \Omega)$ is normal if each sequence in $\mathscr{F}$ has a subsequence which converges to a function f in $C(G, \Omega)$.

Series representation for e^z

Series representation for e^z

$$\sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Series representation for $\log(z)$

Series representation for $\log(z)$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(z-1)^k}{k}$$

Series representation for $\sin(z)$

Series representation for $\sin(z)$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

Series representation for $\cos(z)$

Series representation for $\cos(z)$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

Automorphisms of the unit disk

Automorphisms of the unit disk

$$f(z) = \frac{z_1 - z}{1 - \overline{z}_1 z} e^{i\theta}$$

sends z_1 to zero

The Simple Approximation Lemma

The Simple Approximation Lemma

Let f be a measurable real-valued function on E. Assume f is bounded on E, that is, there is an $M > 0$ for which $|f| \leq M$ on E. Then for each $\epsilon > 0$, there are simple functions φ_ϵ and ψ_ϵ defined on E which have the following approximation properties:

$$\varphi_\epsilon \leq f \leq \psi_\epsilon \text{ and } 0 \leq \psi_\epsilon - \varphi_\epsilon < \epsilon \text{ on } E$$

The Simple Approximation Theorem.

The Simple Approximation Theorem.

An extended real-valued function f on a measurable set E is measurable if and only if there is a sequence $\{\varphi_n\}$ of simple functions on E which converges pointwise on E to f and has the property that

$$|\varphi_n| \leq |f|$$

on E for all n. If $f \geq 0$, we may choose $\{\varphi_n\}$ to be increasing.

Egoroff's Theorem

Egoroff's Theorem

Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\epsilon > 0$ there is a closed set F contained in E for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \sim F) < \epsilon.$$

Lusin's Theorem

Lusin's Theorem

Let f be a real-valued measurable function on E. Then for each $\epsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on F and $m(E \sim F) < \epsilon$.

Continuous, Borel, and measurable functions

Continuous, Borel, and measurable functions

f is continuous $\iff$ for every open set $\mathcal{O}$ we have $f^{-1}(\mathcal{O})$ is open.

f is Borel $\iff$ for every open set $\mathcal{O}$ we have $f^{-1}(\mathcal{O})$ is Borel.

f is measurable $\iff$ for every open set $\mathcal{O}$ we have $f^{-1}(\mathcal{O})$ is measurable.

The Bounded Convergence Theorem

The Bounded Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E, that is, there is a number $M \geq 0$ for which

$$|f_n| \leq M$$

on E for all n. If $\{f_n\} \to f$ pointwise on E, then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Chebychev's Inequality

Chebychev's Inequality

Let f be a nonnegative measurable function on E. Then for any $\lambda > 0$,

$$m(\{x \in E : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \cdot \int_E f$$

Fatou's Lemma

Fatou's Lemma

Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then

$$\int_E f \leq \liminf \int_E f_n$$

The Monotone Convergence Theorem

The Monotone Convergence Theorem

Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. Assume that $f_n \to f$ pointwise a.e. on E. Then

$$\lim_n \int_E f_n = \int_E f.$$

Beppo Levi's Lemma

Beppo Levi's Lemma

Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If the sequence of integrals $\left\{\int_E f_n\right\}$ is bounded, then $\{f_n\}$ converges pointwise on E to a measurable function f that is finite a.e. on E and

$$\lim_n \int_E f_n = \int_E f < \infty$$

The Vitali Convergence Theorem

The Vitali Convergence Theorem

Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E. If $f_n \to f$ pointwise a.e. on E, then f is integrable over E and

$$\lim_n \int_E f_n = \int_E f$$

The Lebesgue Dominated Convergence Theorem

The Lebesgue Dominated Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that

$$|f_n(x)| \leq g(x)$$

for all $x \in E$ and for all $n \in \mathbb{N}$. If $f_n \to f$ pointwise a.e. on E, then f is integrable over E and

$$\lim_n \int_E f_n = \int_E f.$$

(Riesz)

(Riesz)

If $\{f_n\} \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f.

Lebesgue's Theorem

Lebesgue's Theorem

If the function f is monotone on the open interval (a, b), then it is differentiable almost everywhere on (a, b).

Theorem 6.8 Let the function f be absolutely continuous on the closed, bounded interval $[a, b]$.

Theorem 6.8 Let the function f be absolutely continuous on the closed, bounded interval $[a, b]$.

Then f is the difference of increasing absolutely continuous functions and, in particular, is of bounded variation.

Theorem 6.11 A function f on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if

Theorem 6.11 A function f on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if

it is an indefinite integral over $[a, b]$.

Theorem 6.14 Let f be integrable over the closed, bounded interval $[a, b]$.

Theorem 6.14 Let f be integrable over the closed, bounded interval $[a, b]$.

Then $\frac{d}{dx}\left[\int_a^x f\right] = f(x)$ for almost all $x \in (a, b)$.

Theorem 7.6 Let E be a measurable set and $1 \leq p \leq \infty$.

Theorem 7.6 Let E be a measurable set and $1 \leq p \leq \infty$.

Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.

Theorem 7.7 Let E be a measurable set and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$.

Theorem 7.7 Let E be a measurable set and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$.

Then

$$\{f_n\} \to f \text{ in } L^p(E) \iff \lim_n \int_E |f_n|^p = \int_E |f|^p.$$

Theorem 7.12 Let E be a measurable set and $1 \leq p < \infty$.

Theorem 7.12 Let E be a measurable set and $1 \leq p < \infty$.

Then $C_c(E)$ is dense in $L^p(E)$.

Theorem 9.12 The following are complete metric spaces

Theorem 9.12 The following are complete metric spaces

(i) Each nonempty closed subset of Euclidean space $\mathbb{R}^n$.

(ii) For E a measurable set of real numbers and $1 \leq p \leq \infty$, each nonempty closed subset of $L^p(E)$.

(iii) Each nonempty closed subset of $C[a, b]$.

Theorem 9.16 (Characterization of Compactness for a Metric Space)

Theorem 9.16 (Characterization of Compactness for a Metric Space)

For a metric space X, the following three assertions are equivalent: (i) X is complete and totally bounded;

(ii) X is compact;

(iii) X is sequentially compact.

Theorem 9.27 The following are separable metric spaces:

Theorem 9.27 The following are separable metric spaces:

(i) Each nonempty subset of Euclidean space $\mathbb{R}^n$;

(ii) For E a Lebesgue measurable set of real numbers and $1 \leq p < \infty$, each nonempty subset of $L^p(E)$;

(iii) Each nonempty subset of $C[a, b]$.

The Baire Category Theorem Let X be a complete metric space.

The Baire Category Theorem Let X be a complete metric space.

(i) Let $\{\mathcal{O}_n\}$ be a countable collection of open dense sets of X. Then the intersection $\bigcap_n \mathcal{O}_n$ is also dense. (ii) Let $\{F_n\}$ be a countable collection of closed hollow subsets of X. then the union $\bigcup_n F_n$ is also hollow.

The Banach Contraction Principle

The Banach Contraction Principle

Let X be a complete metric space and the mapping $T : X \to X$ be a contraction. Then T has exactly one fixed point.

(The cross ratio of four points)

(The cross ratio of four points)

If $z_1 \in \mathbb{C}_\infty$ then (z_1, z_2, z_3, z_4) is the image of z_1 under the unique Möbius transformation which takes z_2 to 1, z_3 to 0 , and z_4 to ∞.

If z_2, z_3, z_4 are distinct points and T is any Möbius transformation then

If z_2, z_3, z_4 are distinct points and T is any Möbius transformation then

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$$

for any point z_1. (Möbius transformations preserve cross ratios.)

(Unique Interpolation)

(Unique Interpolation)

If z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and $\omega_2, \omega_3, \omega_4$ are also distinct points of $\mathbb{C}_\infty$, then there is one and only one Möbius transformation S such that $Sz_2 = \omega_2, Sz_3 = \omega_3, Sz_4 = \omega_4$.

Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$.

Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$.

Then (z_1, z_2, z_3, z_4) is a real number if and only if all four points lie on a circle.

For any given circles Γ and Γ' in $\mathbb{C}_\infty$ there is a Möbius transformation T such that

For any given circles Γ and Γ' in $\mathbb{C}_\infty$ there is a Möbius transformation T such that

$T(\Gamma) = \Gamma'$. Furthermore we can specify that T take any three points on Γ onto any three points of Γ'. If we do specify Tz_j for $j = 2, 3, 4$ (distinct z_j in Γ) then T is unique.

Casorati-Weierstrass Theorem.

Casorati-Weierstrass Theorem.

Suppose that f has an essential singularity at $z = a$, and let $\delta > 0$. Then

$$\{f[\operatorname{ann}(a; 0, \delta)]\}^- = \mathbb{C}.$$

Maximum Modulus Theorem-First Version.

Maximum Modulus Theorem-First Version.

If f is analytic in a region G and a is a point in G with $|f(a)| \geq |f(z)|$ for all z in G then f must be a constant function.

Maximum Modulus Theorem-Second Version.

Maximum Modulus Theorem-Second Version.

Let G be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on G^{-}which is analytic in G. Then

$$\max\left\{|f(z)|: z \in G^{-}\right\} = \max\{|f(z)|: z \in \partial G\}.$$

Maximum Modulus Theorem-Third Version.

Maximum Modulus Theorem-Third Version.

Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose there is a constant M such that $\limsup |f(z)| \leq M$ for all a in $\partial_\infty G$. Then $|f(z)| \leq M$ for all z in G.

If G is open in $\mathbb{C}$ then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$. Moreover, the sets K_n can be chosen to satisfy the following conditions:

If G is open in $\mathbb{C}$ then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$. Moreover, the sets K_n can be chosen to satisfy the following conditions:

(a) $K_n \subset \text{int } K_{n+1}$;

(b) $K \subset G$ and K compact implies $K \subset K_n$ for some n;

(c) Every component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$.

The Weierstrass Factorization Theorem.

The Weierstrass Factorization Theorem.

Let f be an entire function and let $\{a_n\}$ be the non-zero zeros of f repeated according to multiplicity; suppose f has a zero at $z = 0$ of order $m \geqslant 0$ (a zero of order $m = 0$ at $z = 0$ means $f(0) \neq 0$). Then there is an entire function g and a sequence of integers $\{p_n\}$ such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

If f is a meromorphic function on an open set G then

If f is a meromorphic function on an open set G then

there are analytic functions g and h on G such that $f = g/h$.

$$\boxed{\sin \pi z =}$$

$$\sin \pi z =$$

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

The gamma function, $\Gamma(z)$, is the meromorphic function on with simple poles at $z = 0, -1, \ldots$ defined by

The gamma function, $\Gamma(z)$, is the meromorphic function on with simple poles at $z = 0, -1, \ldots$ defined by

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

where γ is a constant chosen so that $\Gamma(1) = 1$.

Bohr-Mollerup Theorem

Bohr-Mollerup Theorem

Let f be a function defined on $(0, \infty)$ such that $f(x) > 0$ for all $x > 0$. Suppose that f has the following properties:

(a) $\log f(x)$ is a convex function;

(b) $f(x+1) = xf(x)$ for all x;

(c) $f(1) = 1$.

Then $f(x) = \Gamma(x)$ for all x.

If $\operatorname{Re} z > 0$ then

If $\operatorname{Re} z > 0$ then

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Runge's Theorem.

Runge's Theorem.

Let K be a compact subset of $\mathbb{C}$ and let E be a subset of $\mathbb{C}_\infty - K$ that meets each component of $\mathbb{C}_\infty - K$. If f is analytic in an open set containing K and $\epsilon > 0$ then there is a rational function $R(z)$ whose only poles lie in E and such that

$$|f(z) - R(z)| < \epsilon$$

Mittag-Leffler's Theorem.

Mittag-Leffler's Theorem.

Let G be an open set, $\{a_k\}$ a sequence of distinct points in G without a limit point in G, and let $\{S_k(z)\}$ be the sequence of rational functions given by equation (3.1). Then there is a meromorphic function f on G whose poles are exactly the points $\{a_k\}$ and such that the singular part of f at a_k is $S_k(z)$.

The function P_r

The function P_r

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta},$$

for $0 \leq r < 1$ and $-\infty < \theta < \infty$, is called the Poisson kernel.

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = \mathrm{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right)$$

The Poisson kernel satisfies the following:

The Poisson kernel satisfies the following:

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1$

(b) $P_r(\theta) > 0$ for all θ, $P_r(-\theta) = P_r(\theta)$, and P_r is periodic in θ with period 2π;

(c) $P_r(\theta) < P_r(\delta)$ if $0 < \delta < |\theta| \leq \pi$;

(d) for each $\delta > 0$, $\lim_{r\to 1^-} P_r(\theta) = 0$ uniformly in θ for $\pi \geq |\theta| \geq \delta$.

Let $D = \{z : |z|< 1\}$ and suppose that $f : \partial D \to \mathbb{R}$ is a continuous function. Then

Let $D = \{z : |z|< 1\}$ and suppose that $f : \partial D \to \mathbb{R}$ is a continuous function. Then

there is a continuous function $u : D^- \to \mathbb{R}$ such that

(a) $u(z) = f(z)$ for z in ∂D;

(b) u is harmonic in D.

Moreover u is unique and is defined by the formula 2.5

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f\left(e^{it}\right) dt$$

for $0 \leq r < 1, 0 \leq \theta \leq 2\pi$.

If $u : D^{-} \to \mathbb{R}$ is a continuous function that is harmonic in D

If $u : D^{-} \to \mathbb{R}$ is a continuous function that is harmonic in D

then

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u\left(e^{it}\right) dt$$

for $0 \leq r < 1$ and all θ. Moreover, u is the real part of the analytic function

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u\left(e^{it}\right) dt$$

If $u : G \to \mathbb{R}$ is a continuous function which has the MVP then

If $u : G \to \mathbb{R}$ is a continuous function which has the MVP then

u is harmonic.

Harnack's Inequality

Harnack's Inequality

If $u : \overline{B}(a; R) \to \mathbb{R}$ is continuous, harmonic in $B(a; R)$, and $u \geq 0$ then for $0 \leq r < R$ and all θ

$$\frac{R-r}{R+r}u(a) \leq u\left(a + re^{i\theta}\right) \leq \frac{R+r}{R-r}u(a)$$

σ-algebra

σ-algebra

A collection of subsets of $\mathbf{R}$ is called an σ-algebra provided it contains $\mathbf{R}$ and is closed with respect to the formation of complements and countable unions; by De Morgan's Identities, such a collection is also closed with respect to the formation of countable intersections. The preceding proposition tells us that the collection of measurable sets is a σ-algebra.

Borel set

Borel set

The intersection of all the σ-algebras of subsets of $\mathbf{R}$ that contain the open sets is a σ-algebra called the Borel σ-algebra; members of this collection are called Borel sets.

Measurable set

Measurable set

The collection $\mathcal{M}$ of measurable sets is a σ-algebra that contains the σ-algebra $\mathcal{B}$ of Borel sets. Each interval, each open set, each closed set, each G_δ set, and each F_σ set is measurable.

A property that holds almost everywhere

A property that holds almost everywhere

For a measurable set E, we say that a property holds almost everywhere on E, or it holds for almost all $x \in E$, provided there is a subset E_0 of E for which $m(E_0) = 0$ and the property holds for all $x \in E \sim E_0$.

The Cantor set

The Cantor set

We define the Cantor set $\mathbf{C}$ by

$$\mathbf{C} = \bigcap_{k=1}^{\infty} C_k.$$

The collection $\{C_k\}_{k=1}^{\infty}$ possesses the following two properties:

(i) $\{C_k\}_{k=1}^{\infty}$ is a descending sequence of closed sets;

(ii) For each k, C_k is the disjoint union of 2^k closed intervals, each of length $1/3^k$.

The Cantor-Lebesgue function φ

The Cantor-Lebesgue function φ

The Cantor-Lebesgue function φ is an increasing continuous function that maps $[0,1]$ onto $[0,1]$. Its derivative exists on the open set $\mathcal{O}$, the complement in $[0,1]$ of the Cantor set,

$$\varphi' = 0 \text{ on } \mathcal{O} \text{ while } m(\mathcal{O}) = 1$$

Measurable function

Measurable function

Let the function f have a measurable domain E. Then the following statements are equivalent:

(i) For each real number c, the set $\{x \in E \mid f(x) > c\}$ is measurable.

(ii) For each real number c, the set $\{x \in E \mid f(x) \geq c\}$ is measurable.

(iii) For each real number c, the set $\{x \in E \mid f(x) < c\}$ is measurable.

(iv) For each real number c, the set $\{x \in E \mid f(x) \leq c\}$ is measurable.

Each of these properties implies that for each extended real number c, the set $\{x \in E \mid f(x) = c\}$ is measurable.

Upper and lower Riemann integrals of a bounded function

Upper and lower Riemann integrals of a bounded function

Define the lower and upper Darboux sums for f with respect to P, respectively, by

$$L(f, P) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$$

and

$$U(f, P) = \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}) \, ,$$

where, [1] for $1 \leq i \leq n$,

$$m_i = \inf \{ f(x) \mid x_{i-1} < x < x_i \} \text{ and } M_i = \sup \{ f(x) \mid x_{i-1} < x < x_i \} \, .$$

We then define the lower and upper Riemann integrals of f over $[a, b]$, respectively, by $(R)\underline{\int_a^b} f = \sup\{L(f, P) \mid P \text{ a partition of } [a, b]\}$ and $(R)\overline{\int_a^b} f = \inf\{U(f, P) \mid P \text{ a partition of } [a, b]\}$

Upper and lower Lebesgue integrals of a measurable function

Upper and lower Lebesgue integrals of a measurable function

Let f be a bounded real-valued function defined on a set of finite measure E. By analogy with the Riemann integral, we define the lower and upper Lebesgue integral, respectively, of f over E to be

$$\sup\left\{\int_E \varphi \mid \varphi \text{ simple and } \varphi \leq f \text{ on } E,\right\}$$

and

$$\inf\left\{\int_E \psi \mid \psi \text{ simple and } f \leq \psi \text{ on } E\right\}$$

Lebesgue integrability of a bounded function on a measurable set of finite measure

Lebesgue integrability of a bounded function on a measurable set of finite measure

A bounded function f on a domain E of finite measure is said to be Lebesgue integrable over E provided its upper and lower Lebesgue integrals over E are equal. The common value of the upper and lower integrals is called the Lebesgue integral, or simply the integral, of f over E and is denoted by $\int_E f$.

Convergence in measure

Convergence in measure

Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite a.e. on E. The sequence $\{f_n\}$ is said to converge in measure on E to f provided for each $\eta > 0$,

$$\lim_{n\to\infty} m\{x \in E || f_n(x) - f(x) |> \eta\} = 0$$

Upper and lower derivatives of a function at a point in its domain

Upper and lower derivatives of a function at a point in its domain

For a real-valued function f and an interior point x of its domain, the upper derivative of f at x, $\overline{D}f(x)$ and the lower derivative of f at x, $\underline{D}f(x)$ are defined as follows:

$$\overline{D}f(x) = \lim_{h\to 0}\left[\sup_{0<|t|\leq h}\frac{f(x+t)-f(x)}{t}\right]$$

$$\underline{D}f(x) = \lim_{h\to 0}\left[\inf_{0<|t|\leq h}\frac{f(x+t)-f(x)}{t}\right].$$

Differentiability of a function at a point in its domain

Differentiability of a function at a point in its domain

We say that f is difierentiable at x and define $f'(x)$ to be the common value of the upper and lower derivatives.

Variation of function with respect to a partition

Variation of function with respect to a partition

Let f be a real-valued function defined on the closed, bounded interval $[a, b]$ and $P = \{x_0, \ldots, x_k\}$ be a partition of $[a, b]$. Define the variation of f with respect to P by

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

Total variation of a function

Total variation of a function

$$TV(f) = \sup\{V(f, P) \mid P \text{ a partition of } [a, b]\}$$

Function of bounded variation

Function of bounded variation

A real-valued function f on the closed, bounded interval $[a, b]$ is said to be of bounded variation on $[a, b]$ provided

$$TV(f) < \infty$$

Absolute continuity of a function on a closed, bounded interval

Absolute continuity of a function on a closed, bounded interval

A real-valued function f on a closed, bounded interval $[a, b]$ is said to be absolutely continuous on $[a, b]$ provided for each $\epsilon > 0$, there is $a\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{n}$ of open intervals in (a, b),

$$\text{if } \sum_{k=1}^{n} [b_k - a_k] < \delta, \text{ then } \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$$

Lipschitz function

Lipschitz function

The function f is said to be Lipschitz provided there is a $c \geq 0$ for which

$$|f(x') - f(x)| \leq c \cdot |x' - x| \text{ for all } x', x \in E.$$

Total variation function

Total variation function

Therefore the function $x \mapsto TV\left(f_{[a,x]}\right)$, which we call the total variation function for f, is a real-valued increasing function on $[a,b]$. Moreover, for $a \le u < v \le b$, if we take the crudest partition $P = \{u,v\}$ of $[u,v]$, we have

$$f(u) - f(v_0) \le |f(v) - f(u)| = V\left(f_{[u,v]}, P\right) \le TV\left(f_{[u,v]}\right) = TV\left(f_{[a,v]}\right) - TV\left(f\right.$$

Thus

$$f(v) + TV\left(f_{[a,v]}\right) \ge f(u) + TV\left(f_{[a,u]}\right) \text{ for all } a \le u < v \le b$$

Divided difference function and Average value function

Divided difference function and Average value function

Let f be integrable over the closed, bounded interval $[a, b]$. Extend f to take the value $f(b)$ on $(b, b+1]$. For $0 < h \leq 1$, define the divided difierence function Diff h and average value function $\mathrm{Av}_h f$ of $[a, b]$ by $\mathrm{Diff}_h\, f(x) = \frac{f(x+h)-f(x)}{h}$ and $\mathrm{Av}_h\, f(x) = \frac{1}{h} \cdot \int_x^{x+h} f$ for all $x \in [a, b]$ By a change of variables in the integral and cancellation, for all $a \leq u < v \leq b$,

$$\int_u^v \mathrm{Diff}_h\, f = \mathrm{Av}_h f(v) - \mathrm{Av}_h f(u)$$

A cover of a set in the sense of Vitali

A cover of a set in the sense of Vitali

A collection $\mathcal{F}$ of closed, bounded, nondegenerate intervals is said to cover a set E in the sense of Vitali provided for each point x in E and $\epsilon > 0$, there is an interval I in $\mathcal{F}$ that contains x and has $\ell(I) < \epsilon$.

Uniform integrability

Uniform integrability

A family $\mathcal{F}$ of measurable functions on E is said to be uniformly integrable over E provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \int_A |f| < \epsilon$$

Essentially bounded function

Essentially bounded function

We call a function $f \in \mathcal{F}$ essentially bounded provided there is some $M \geq 0$, called an essential upper bound for f, for which

$$|f(x)| \leq M \text{ for almost all } x \in E$$

Essential supremum of a function

Essential supremum of a function

For a function f in $L^\infty(E)$, define $\|f\|_\infty$ to be the infimum of the essential upper bounds for f. We call $\|f\|_\infty$ the essential supremum of f and claim that $\|\cdot\|_\infty$ is a norm on $L^\infty(E)$.

Linear functional on a linear space

Linear functional on a linear space

A linear functional on a linear space X is a real-valued function T on X such that for g and h in X and α and β real numbers,

$$T(\alpha \cdot g + \beta \cdot h) = \alpha \cdot T(g) + \beta \cdot T(h)$$

Bounded linear functional on a normed linear space

Bounded linear functional on a normed linear space

For a normed linear space X, a linear functional T on X is said to be bounded provided there is an $M \geq 0$ for which

$$|T(f)| \leq M \cdot \|f\| \text{ for all } f \in X$$

The infimum of all such M is called the norm of T and denoted by $\|T\|_*$.

Metric on a nonempty set

Metric on a nonempty set

Let X be a nonempty set. A function $\rho : X \times X \to \mathbf{R}$ is called a metric provided for all x, y, and z in X

(i) $\rho(x, y) \geq 0$;

(ii) $\rho(x, y) = 0$ if and only if $x = y$;

(iii) $\rho(x, y) = \rho(y, x)$;

(iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Point of closure of a subset E of a metric space X

Point of closure of a subset E of a metric space X

For a set E of real numbers, a real number x is called a point of closure of E provided every open interval that contains x also contains a point in E. The collection of points of closure of E is called the closure of E and denoted by $\overline{E}$.

Separable metric space

Separable metric space

A subset D of a metric space X is said to be dense in X provided every nonempty open subset of X contains a point of D. A metric space X is said to be separable provided there is a countable subset of X that is dense in X.

Contracting sequence of subsets of a metric space

Contracting sequence of subsets of a metric space

For a nonempty subset E of a metric space (X, ρ), we define the diameter of E, diam E, *by*

$$\operatorname{diam} E = \sup\{\rho(x, y) \mid x, y \in E\}$$

We say E is bounded provided it has finite diameter. A descending sequence $\{E_n\}_{n=1}^{\infty}$ of nonempty subsets of X is called a contracting sequence provided

$$\lim_{n \to \infty} \operatorname{diam}(E_n) = 0$$

Compact metric space

Compact metric space

A metric space X is called compact provided every open cover of X has a finite subcover. A subset K of X is called compact provided K, considered as a metric subspace of X, is compact.

Sequentially compact metric space

Sequentially compact metric space

A metric space X is said to be sequentially compact provided every sequence in X has a subsequence that converges to a point in X.

Totally bounded metric space

Totally bounded metric space

A metric space X is said to be totally bounded provided for each $\epsilon > 0$, the space X can be covered by a finite number of open balls of radius ϵ. A subset E of X is called totally bounded provided that E, considered as a subspace of the metric space X, is totally bounded.

Lebesgue number for an open cover

Lebesgue number for an open cover

If $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is an open cover of a metric space X, then each point $x \in X$ is contained in a member of the cover, $\mathcal{O}_\lambda$, and since $\mathcal{O}_\lambda$ is open, there is some $\epsilon > 0$, such that

$$B(x,\epsilon) \subseteq \mathcal{O}_\lambda$$

In general, the ϵ depends on the choice of x. The following proposition tells us that for a compact metric space this containment holds uniformly in the sense that we can find ϵ independently of $x \in X$ for which the inclusion (4) holds. A positive number ϵ with this property is called a Lebesgue number for the cover $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$.

Nowhere dense subset of a metric space

Nowhere dense subset of a metric space

A subset E of a metric space X is called nowhere dense provided its closure $\overline{E}$ is hollow.