ZOMBIES AND KNOTS

CONSTRUCTING THE JONES POLYNOMIAL TO SAVE THE WORLD

DAY 0: SMALLTOWNS-VILLE, IA:

A man picks up his daily slice of breakfast pizza from his local gas station. What he doesn’t know is that it’s his last. By mid-day he feels terrible, by the time he’s ready to go home for dinner he’s already feasting on brains.

DAY 4: CDC HEADQUARTERS

The Z-virus has spread midwest-wide. You’re working at the CDC as an expert in microscopy. You’re working frantically to get any information on the Z-virus you can. In furtherance of that goal, you decide to image the DNA of the Z-Virus.

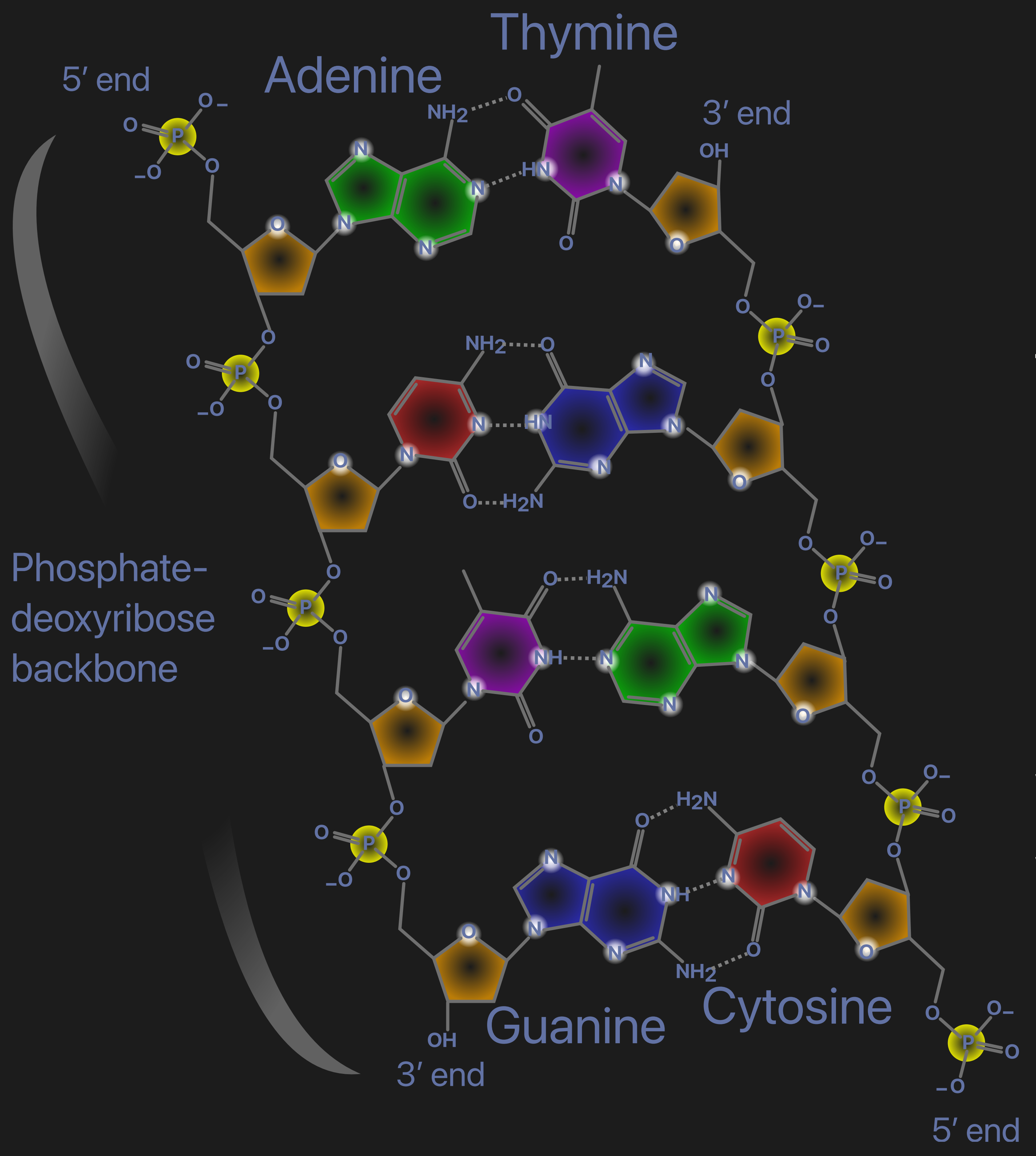
DNA knot as seen under the electron microscope. - Image Credit: Javier Arsuaga, CC BY-ND

DNA

Deoxyribonucleic acid (abbreviated DNA) is the molecule that carries genetic information for the development and functioning of an organism.

DNA is made of two linked strands that wind around each other to resemble a twisted ladder — a shape known as a double helix.

Deoxyribonucleic acid (DNA). (n.d.). Genome.gov. https://www.genome.gov/genetics-glossary/Deoxyribonucleic-Acid. Accessed 3 October 2023



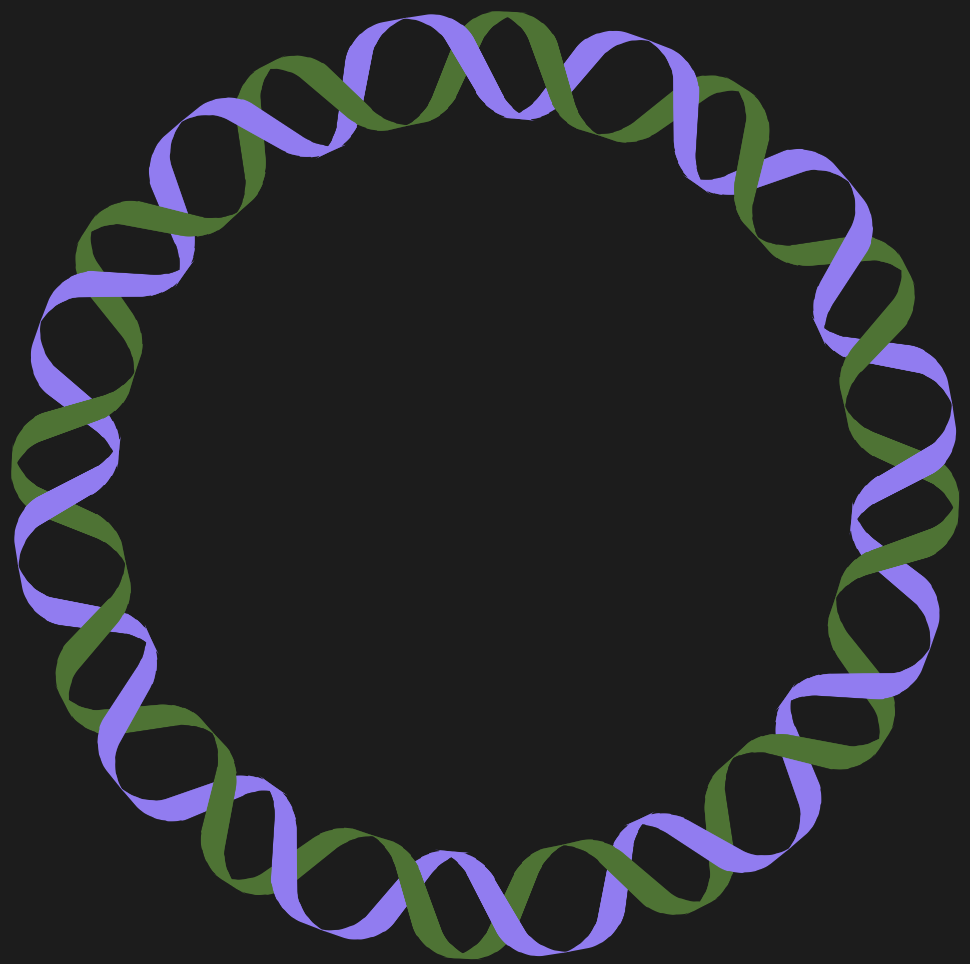
Each strand has a backbone. Attached to each sugar is one of four bases: adenine (A), cytosine (C), guanine (G) or thymine (T). The two strands are connected by chemical bonds between the bases: adenine bonds with thymine, and cytosine bonds with guanine.

Deoxyribonucleic acid (DNA). (n.d.). Genome.gov. https://www.genome.gov/genetics-glossary/Deoxyribonucleic-Acid. Accessed 3 October 2023

MACRO STRUCTURE

CIRCULAR DNA

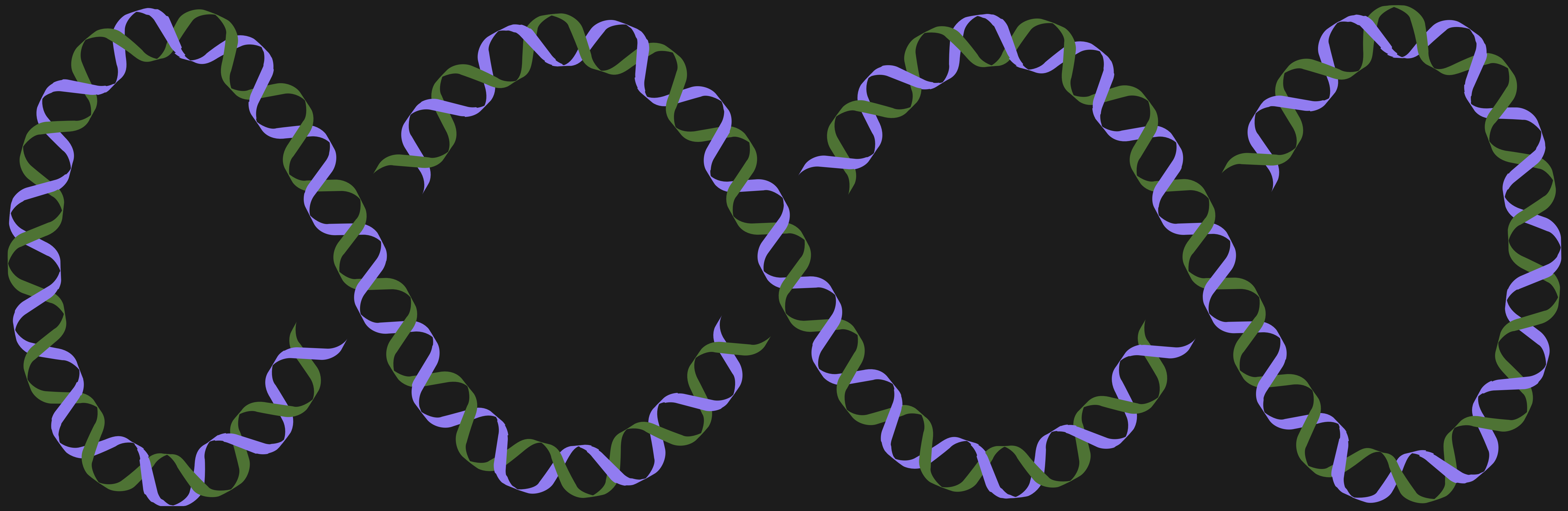
Dulbecco and Vogt (1963) and Weil and Vinograd (1963) discovered that double-stranded DNA of the polyoma virus is circular.



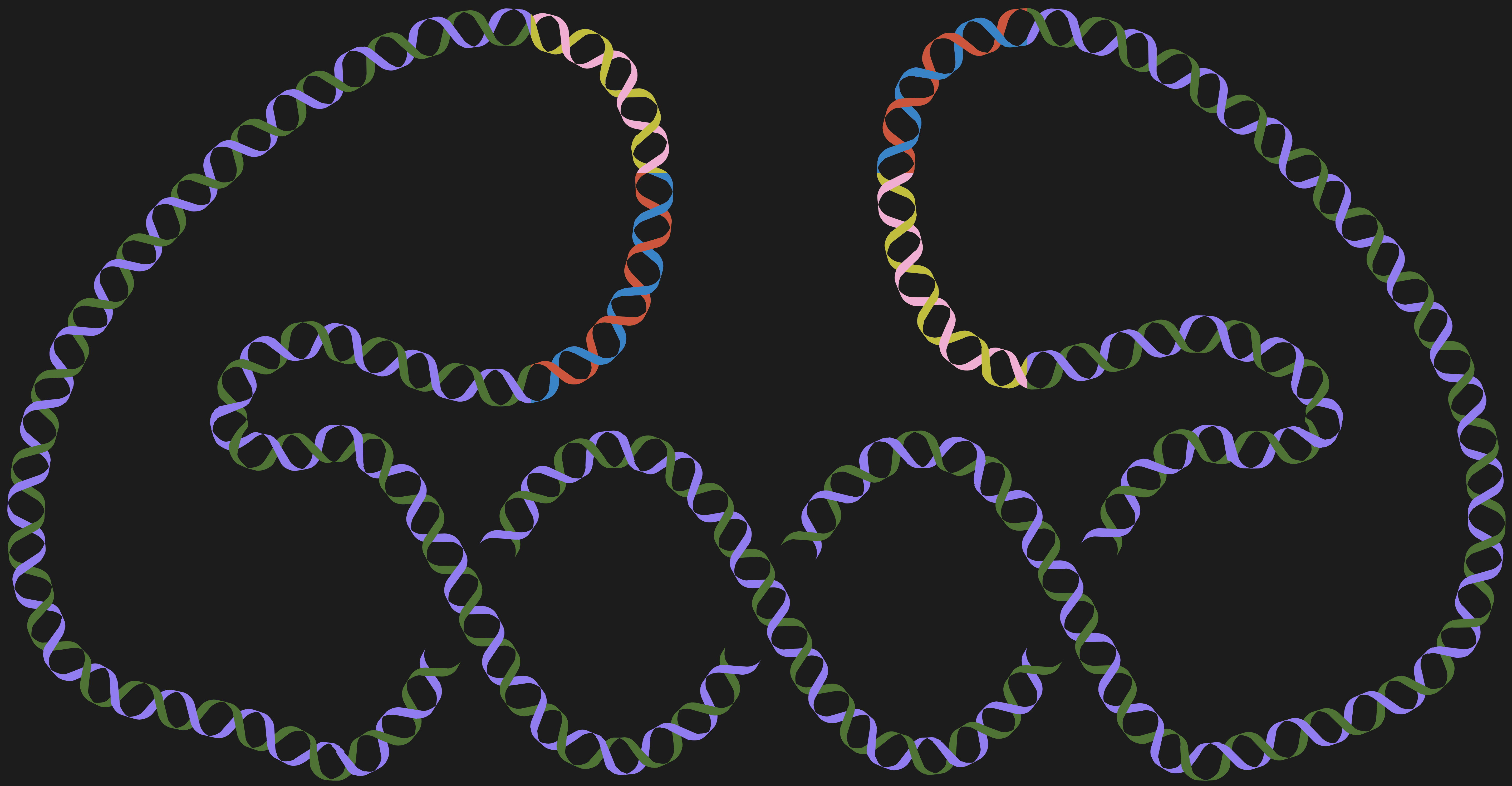
Vologodskii, A. V. (n.d.). Circular DNA. In Cyclic Polymers (pp. 47-83). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47117-5_2

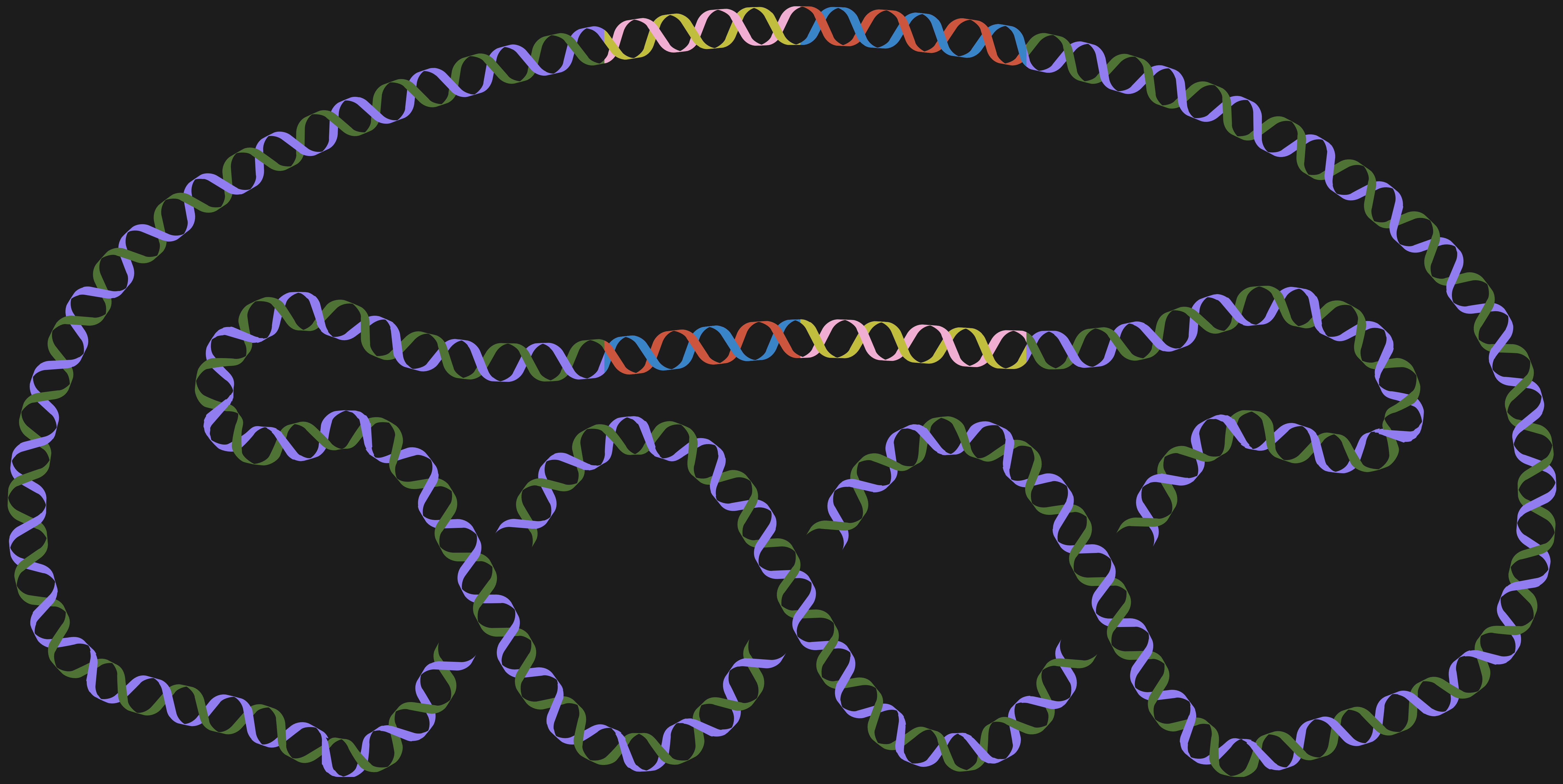
“SUPERCOILED” DNA

Vinograd, J., Lebowitz, J., Radloff, R., Watson, R., & Laipis, P. (1965) discover that double-stranded DNA can “supercoil”.



Vinograd, J., Lebowitz, J., Radloff, R., Watson, R., & Laipis, P. (1965). The twisted circular form of polyoma viral DNA. In Proceedings of the National Academy of Sciences (Vol. 53, Issue 5, pp. 1104-1111). Proceedings of the National Academy of Sciences. https://doi.org/10.1073/pnas.53.5.1104





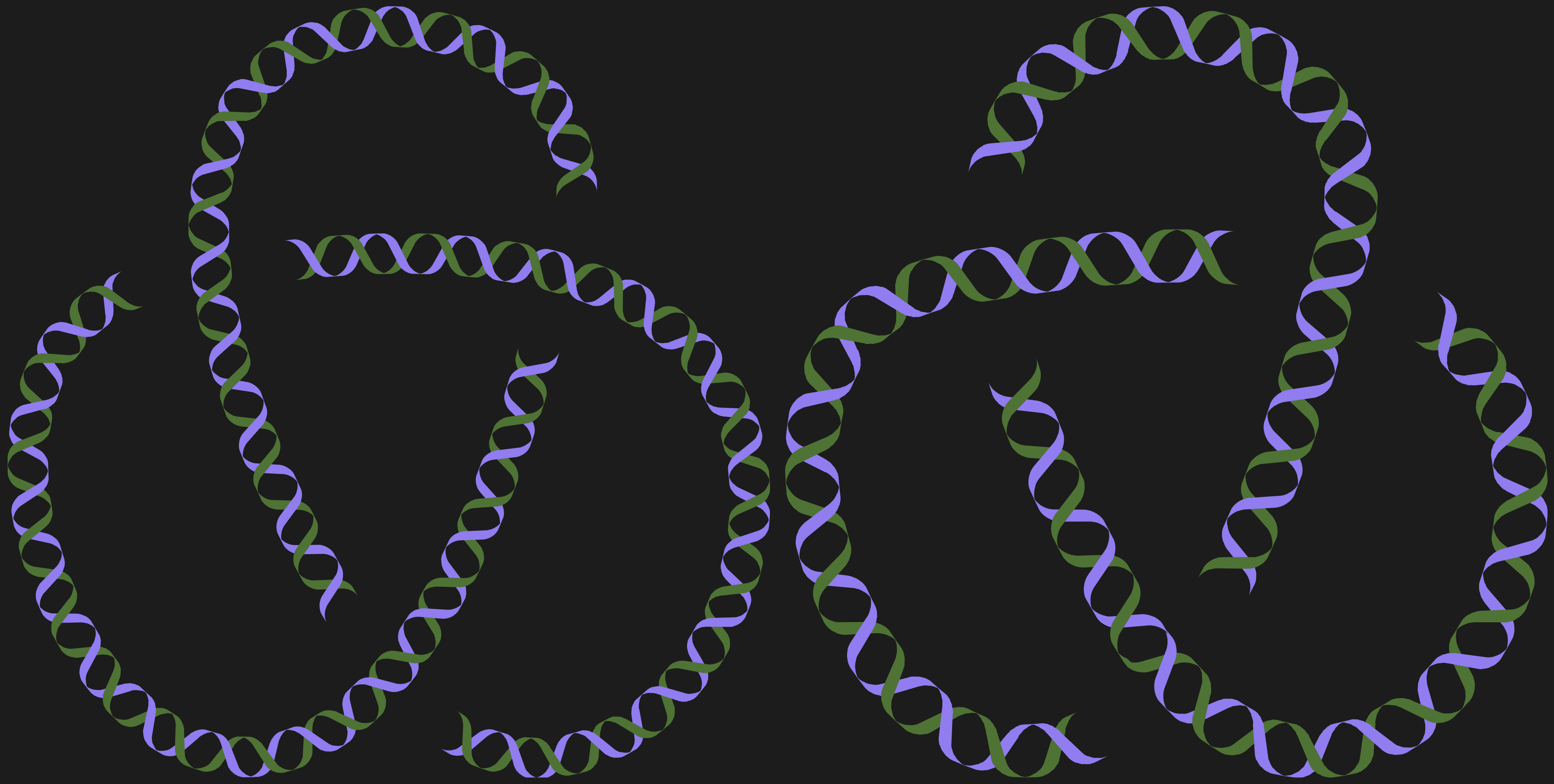
DAY 7: CDC HEADQUARTERS

The spread is now nation wide but still under some control.

You've successfully imaged the DNA of the Z-virus and found DNA with a knot. Your CDC coworkers are using your findings to construct an anti-Z-virus. The anti-virus is the mirror of the DNA knot you've found. This will allow the human body to build anti-bodies for the Z-virus.

The CDC now needs you to verify that the DNA knot they've produced truly is the mirror of the Z-virus.

ANTI-KNOT



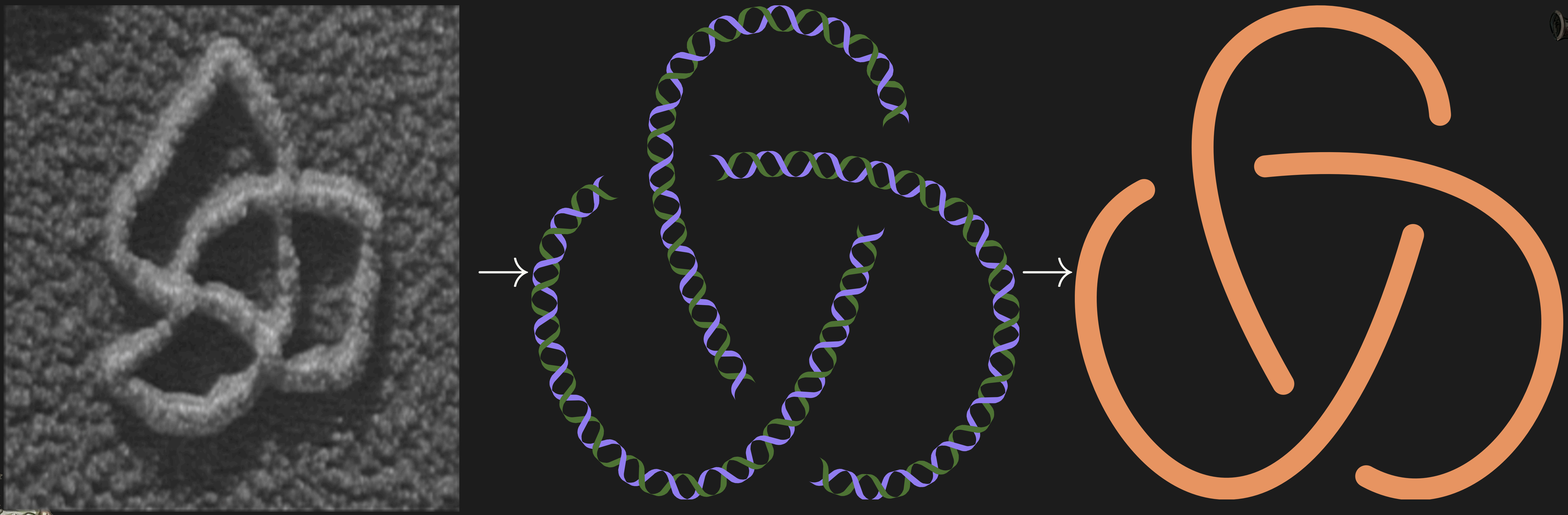

MATHEMATICAL KNOTS


*"A **knot** is a smooth embedding of a circle S^1 into Euclidean 3-dimensional space $\mathbb{R}^3$ (or the 3-dimensional sphere S^3)."*

Jablan, S., & Sazdanović, R. (2007). Linknot. In Series on Knots and Everything. WORLD SCIENTIFIC. https://doi.org/10.1142/6623

https://www.knotplot.com/

DIAGRAMS FOR KNOTTED DNA

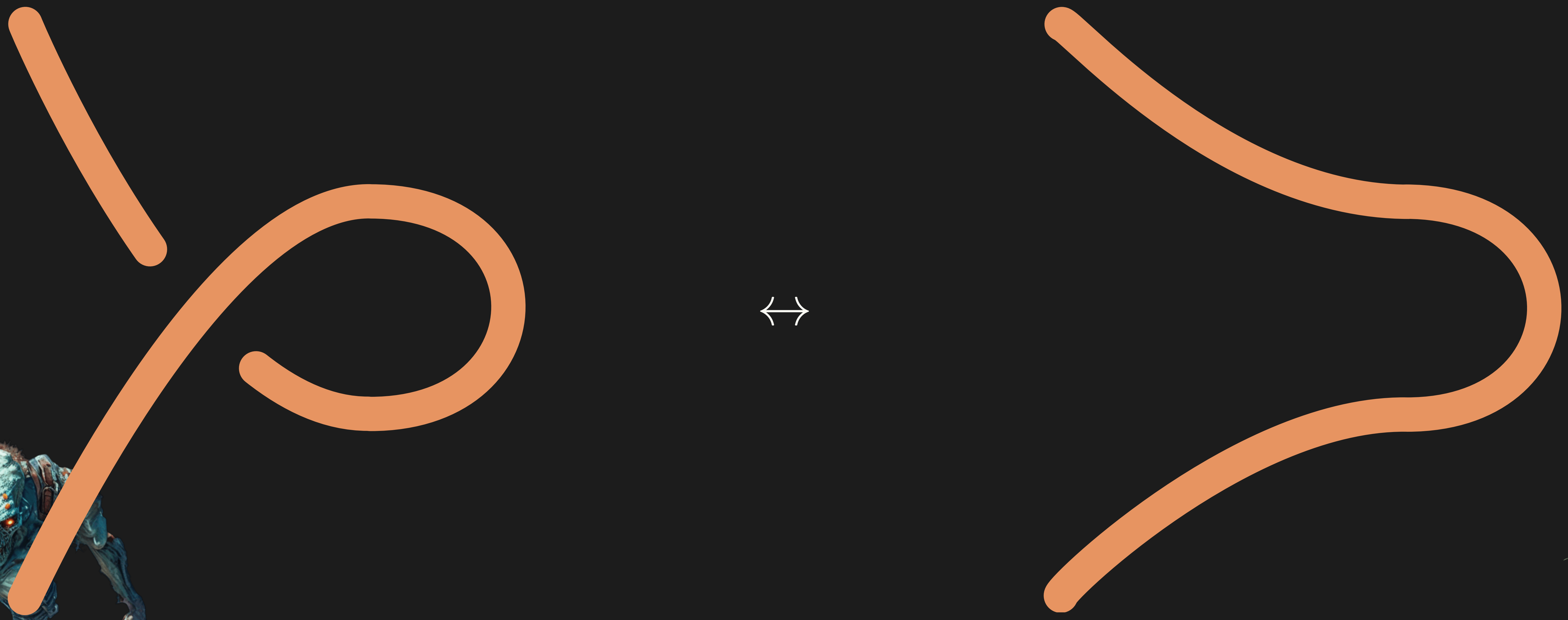


DNA knot as seen under the electron microscope. - Image Credit: Javier Arsuaga, CC BY-ND

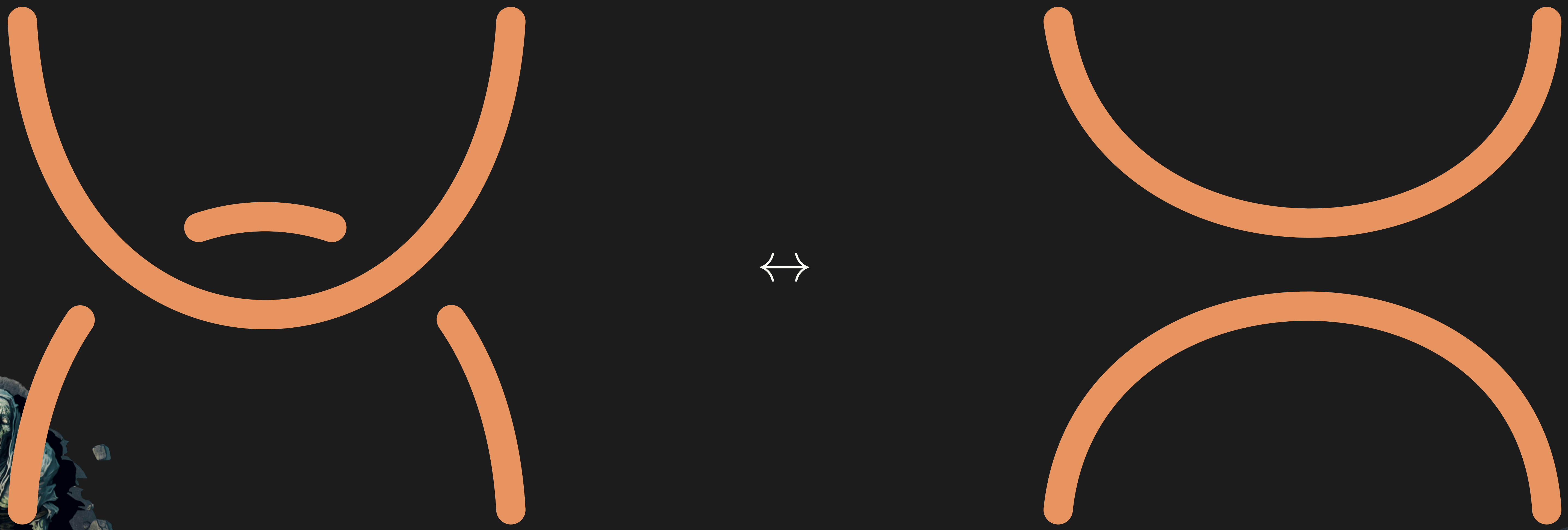
KNOT EQUIVALENCE

REIDEMEISTER MOVES

TYPE I



TYPE II



TYPE III

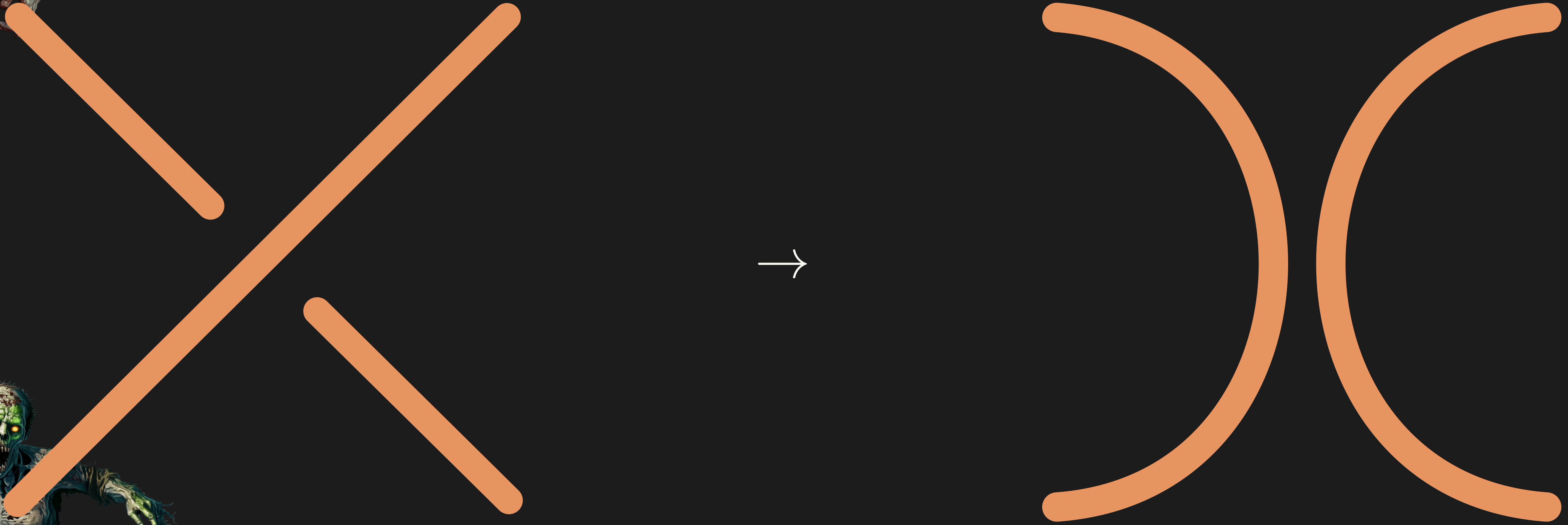
$\leftrightarrow$

EQUAL?

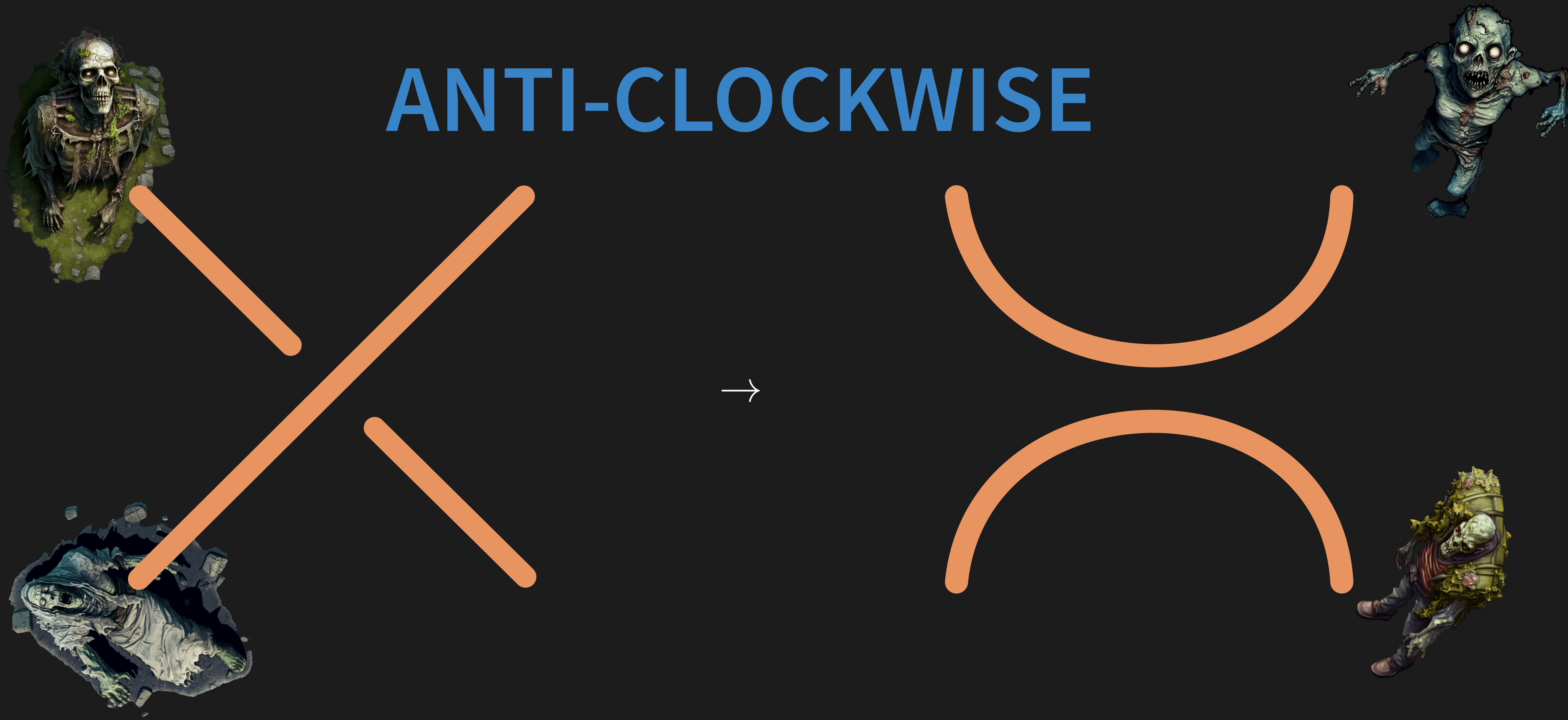
PLAYING WITH DIAGRAMS

What’s the important information inside a knot diagram?

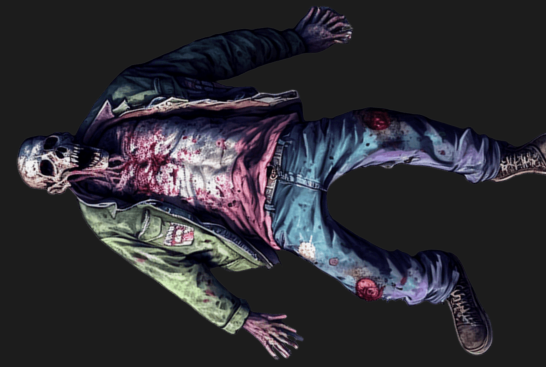
CLOCKWISE



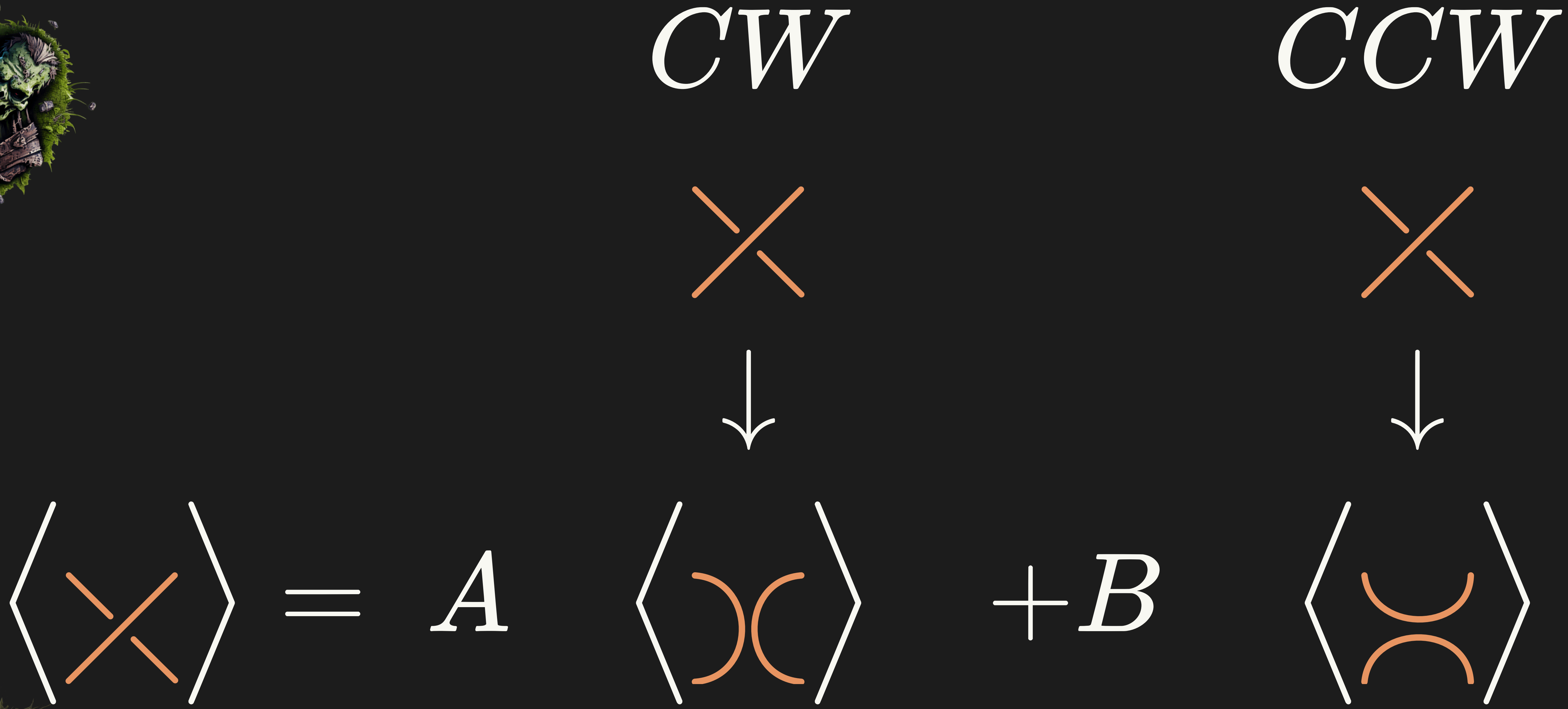
ANTI-CLOCKWISE



PICTURES ARE HARD LETS LEVERAGE ALGEBRA



SKEIN RELATION



WHAT ARE WE LOOKING FOR?

We want to use our bracket to build a polynomial that can tell two knots apart. In particular, we want to differentiate a knot and its “anti-knot”(mirror).

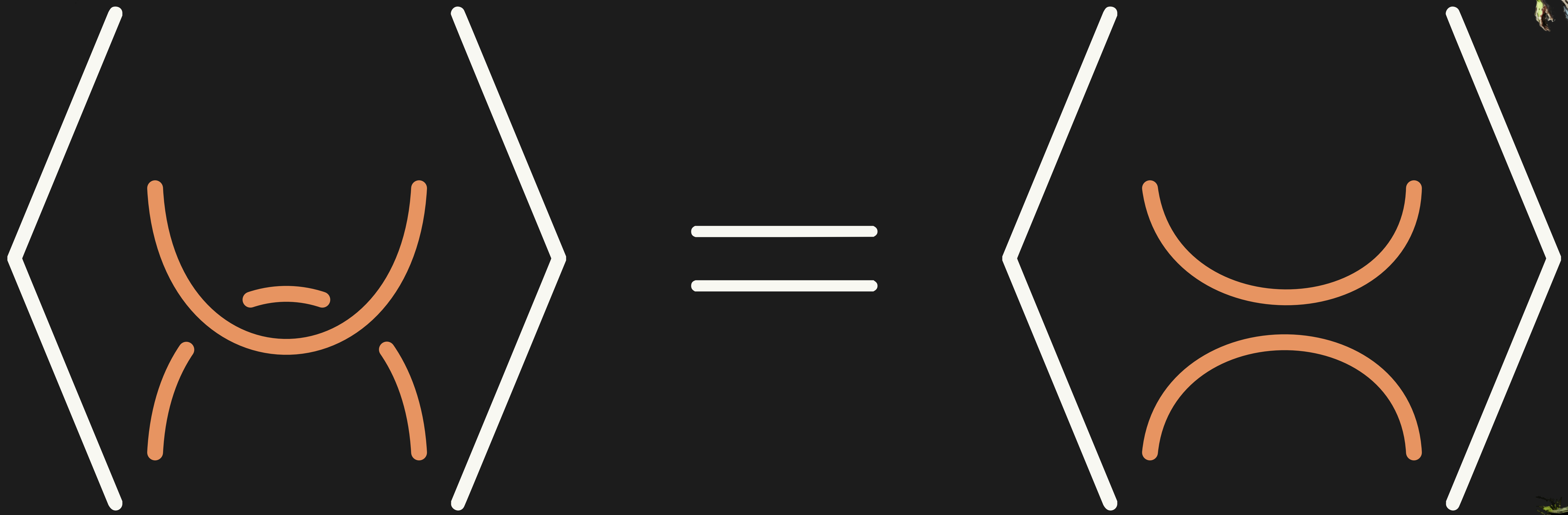

PUTTING PIECES TOGETHER

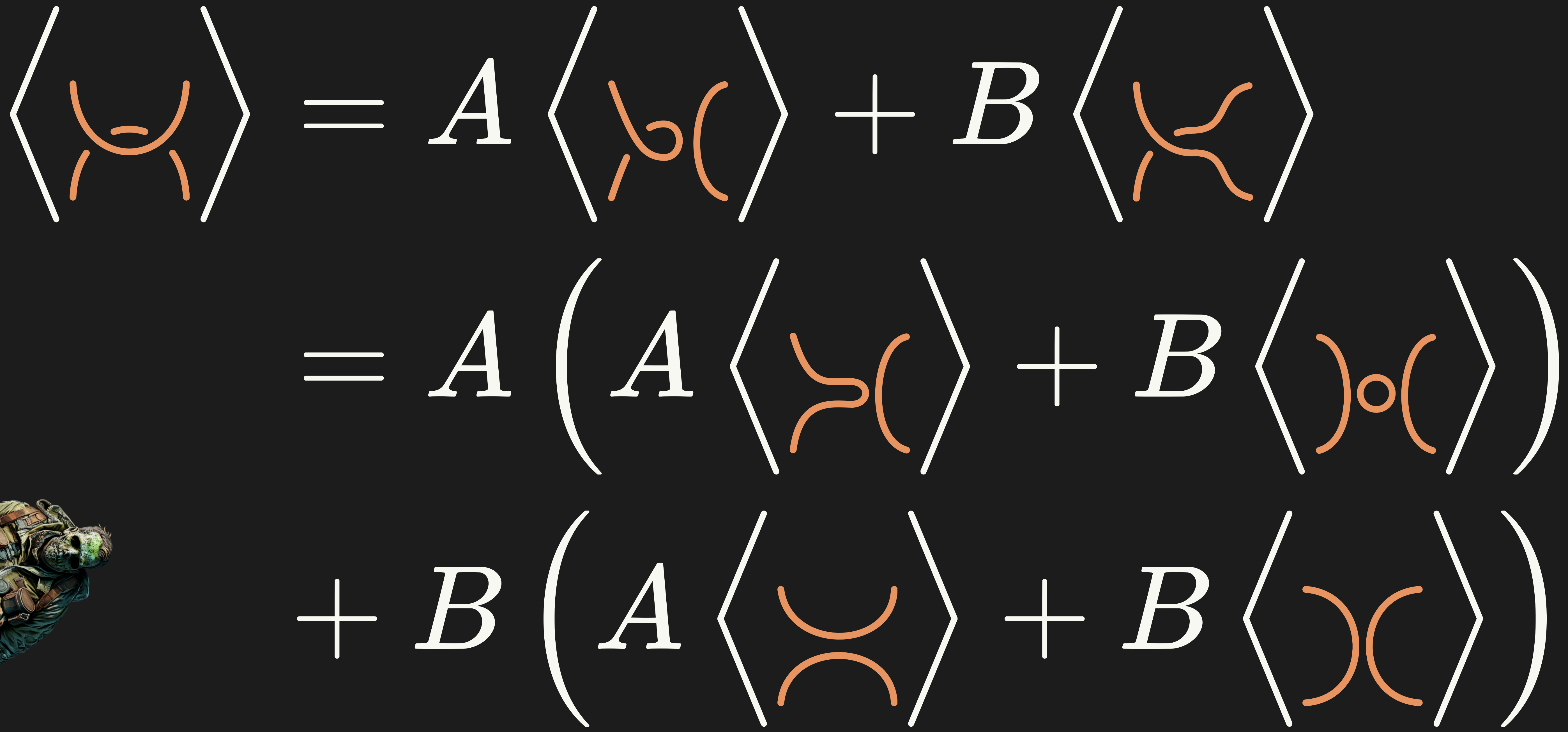
- How can we tell two knots apart?
- How can we use that and our bracket to build our polynomial?


CHECK WHAT HAPPENS UNDER REIDEMEISTER MOVES

If our bracket “respects” Reidemeister moves it respects knot “equivalence”.

TYPE II





A PROBLEM

$$B\left\langle \,)\circ(\, \right\rangle$$

1. $\left\langle \bigcirc \right\rangle = 1$
2. $\left\langle P \sqcup \bigcirc \right\rangle = C \left\langle P \right\rangle$

BACK TO COMPUTING

$$
\begin{aligned}
A\left(A\left\langle \succ (\right\rangle + B\left\langle) \circ (\right\rangle\right) + B\left(A\left\langle \asymp \right\rangle + B\left\langle)(\right\rangle\right) &= A\left(A\left\langle)(\right\rangle + BC\left\langle)(\right\rangle\right) \\
&+ B\left(A\left\langle \asymp \right\rangle + B\left\langle)(\right\rangle\right)
\end{aligned}
$$

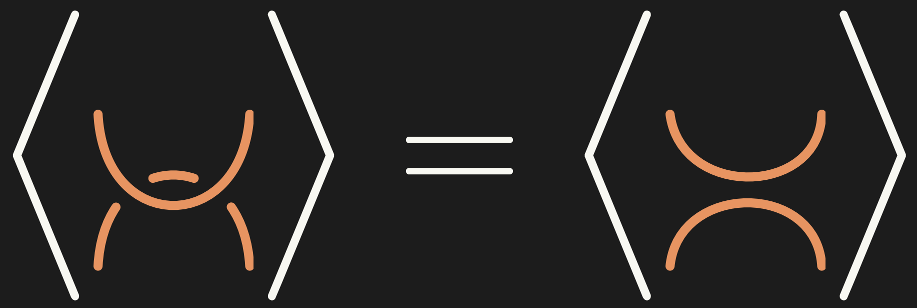
$$= A^2 \left\langle \,)(\, \right\rangle + ABC \left\langle \,)(\, \right\rangle$$

$$+ BA \left\langle \asymp \right\rangle + B^2 \left\langle \,)(\, \right\rangle$$

$$= (A^2 + ABC + B^2) \left\langle \,)(\, \right\rangle$$

$$+ BA \left\langle \asymp \right\rangle$$

WHAT WE WANTED



WHAT WE HAVE

SO WE NEED

PUTTING PIECES TOGETHER

How can we select A, B, and C to get equality?

$$\left(A^2 + ABC + B^2\right) \left\langle)(\right\rangle + BA \left\langle \asymp \right\rangle = \left\langle \asymp \right\rangle$$

$$B = A^{-1}$$

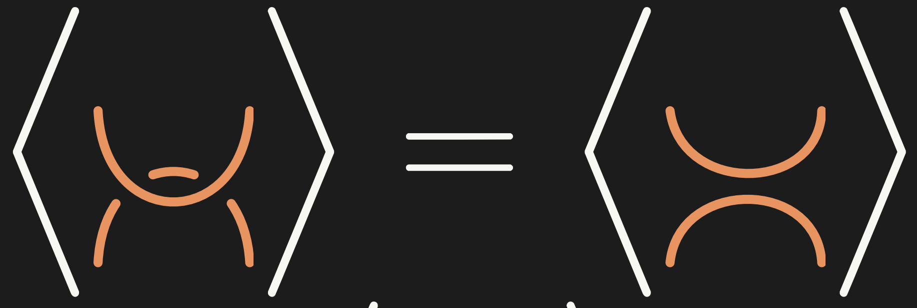
$$(A^2 + ABC + B^2)\left\langle)(\right\rangle + BA\left\langle \asymp \right\rangle = \left\langle \asymp \right\rangle$$

$$(A^2 + C + A^{-2})\left\langle)(\right\rangle + \left\langle \asymp \right\rangle = \left\langle \asymp \right\rangle$$

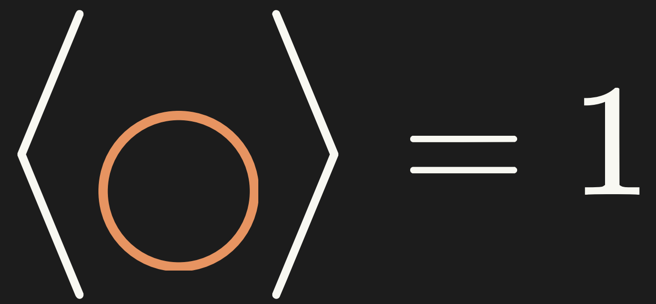
$$C = -A^{-2} - A^{2}$$

$$\left(A^{2} + C + A^{-2}\right)$$

TYPE II

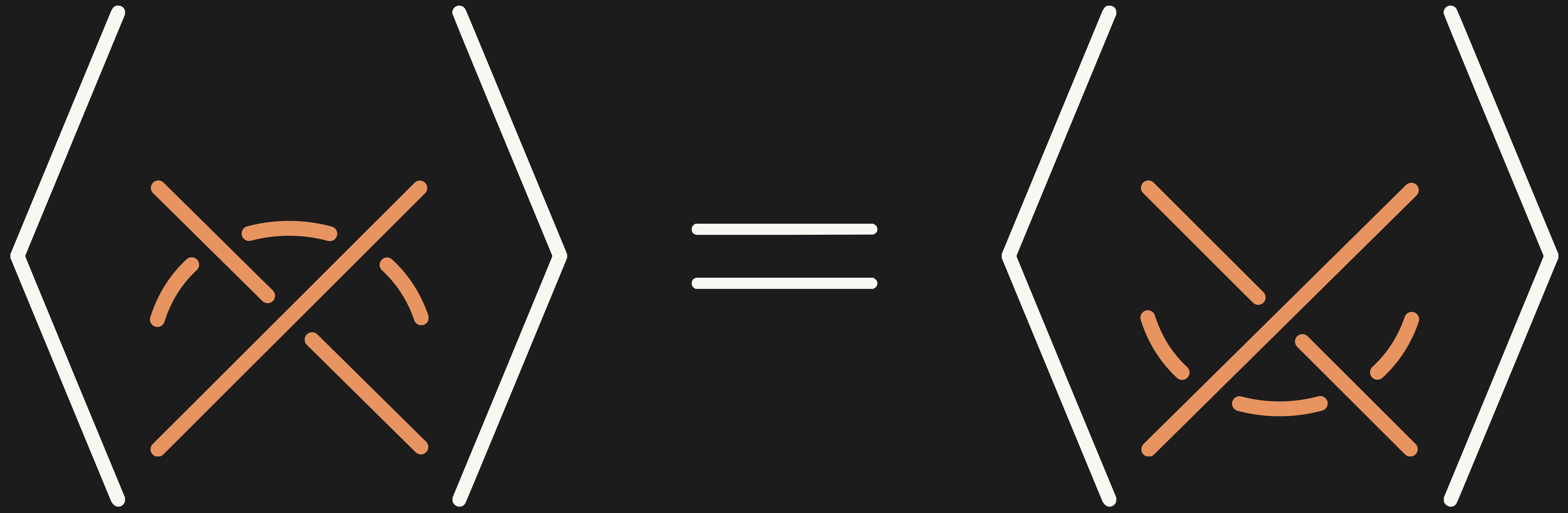


1. 

2. 

3. $$\left\langle P \sqcup \bigcirc \right\rangle = \left(-A^{-2} - A^{2}\right) \langle P \rangle$$

Exercise: Type III



TYPE I

$$
\begin{aligned}
\left\langle \quad \right\rangle &= A \left\langle \quad \right\rangle + A^{-1} \left\langle \quad \right\rangle \\
&= A \left\langle \quad \right\rangle \\
&\quad + A^{-1}\left(-A^{-2} - A^{2}\right) \left\langle \quad \right\rangle
\end{aligned}
$$

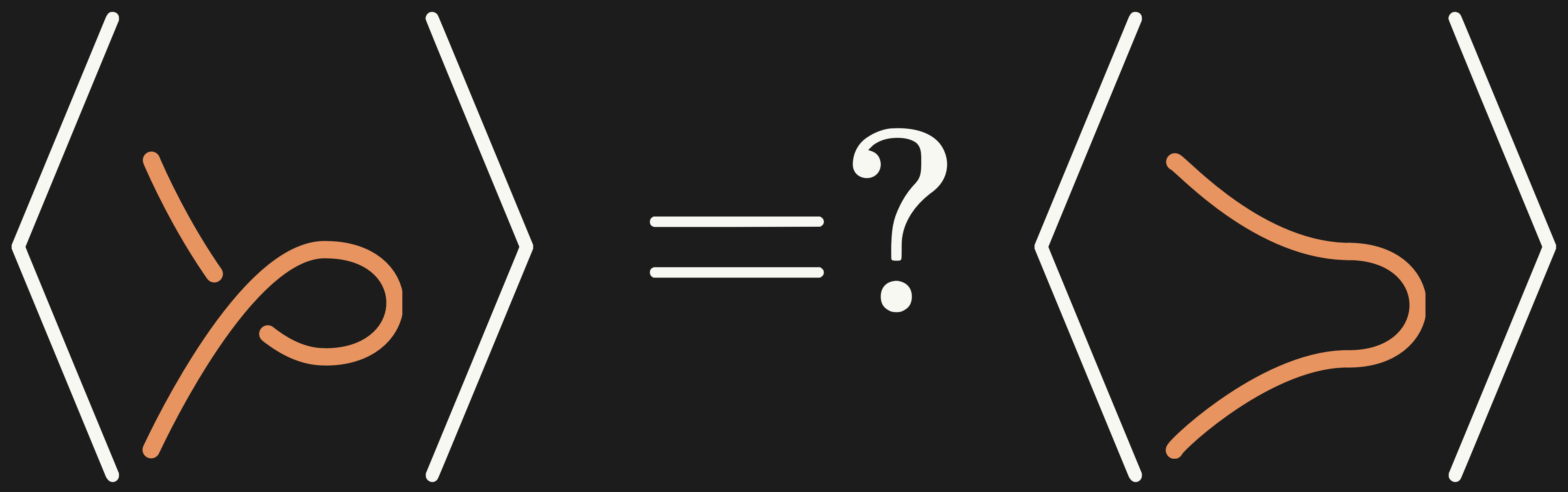
$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle$

$+ A^{-1}\left(-A^{-2} - A^{2}\right) \Big\langle$  $\Big\rangle$

$= \left(A - A^{-3} - A\right) \Big\langle$  $\Big\rangle$

$\langle$  $\rangle = -A^{-3} \langle$  $\rangle$

Exercise: Compute bracket for the other Type I



$\Big\langle$  $\Big\rangle = -A^3 \Big\langle$  $\Big\rangle$

WHY IS THIS A PROBLEM?

$\Big\langle$  $\Big\rangle = -A^{-3} \Big\langle$  $\Big\rangle$

$\Big\langle$  $\Big\rangle = -A^{3} \Big\langle$  $\Big\rangle$

WHAT DO WE HAVE SO FAR?

For Type II and III everything “works” with the rules:

1. $\langle \bigcirc \rangle = 1$

2. 

3. $\langle P \sqcup \bigcirc \rangle = \left(-A^{-2} - A^{2}\right) \langle P \rangle$

but Type I is “broken”:

DAY 53

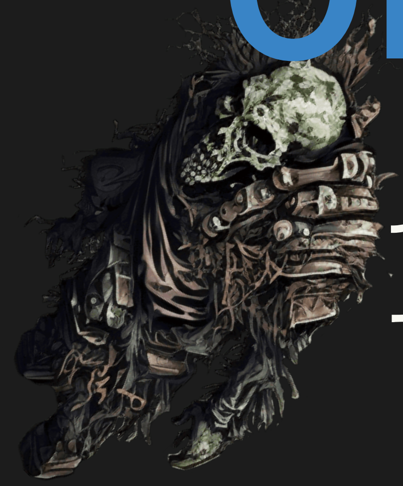
Time is running out. With your preliminary results in hand the vaccine is being produced. The future of the world is now on your shoulders waiting for your results.

HOW CAN WE FIX TYPE I

$\Big\langle$  $\Big\rangle = -A^{-3}\Big\langle$  $\Big\rangle$

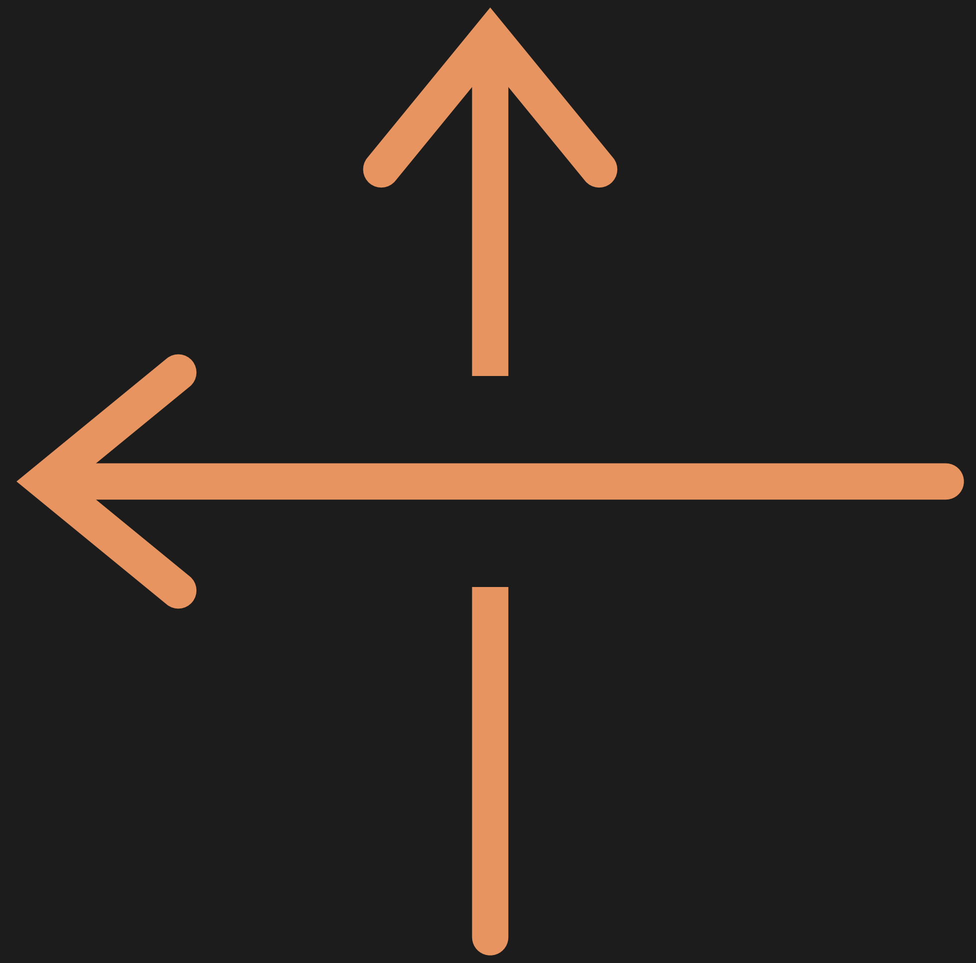
$\Big\langle$  $\Big\rangle = -A^{3}\Big\langle$  $\Big\rangle$

ORIENTATION OF A CROSSING



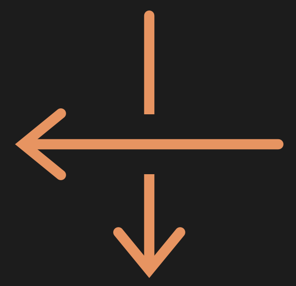
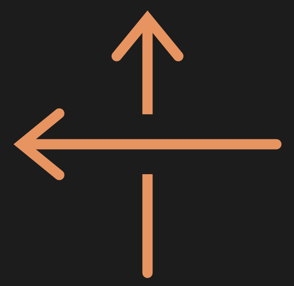
1. Positive

2. Negative



WRITHE OF A KNOT

The writhe $w(P)$ of a diagram P of an oriented link is the sum of the signs of the crossings of P.

$w(P) = \#\Big($  $\Big) - \#\Big($  $\Big)$

Exercise: Compute the writhe

$w\Bigg($  $\Bigg)$

FIXING TYPE I

$-A^{-3w}$() $\langle$  $\rangle$

$$-A^{-3w(\;)}\left\langle\;\right\rangle = -A^{-3(-1)}\left(-A^{-3}\right)\left\langle\;\right\rangle$$

$$= -A^{3}\left(-A^{-3}\right)\left\langle\;\right\rangle$$

$$= \left\langle\;\right\rangle$$

Exercise: Verify the other type I move

$-A^{-3w}\Big($  $\Big) = \Big\langle$  $\Big\rangle$

WHAT DO WE HAVE?

For Type I, II, and III everything "works" for the polynomial

$$V(P) = -A^{-3w(P)} \langle P \rangle$$

with the rules:

1. $\left\langle \bigcirc \right\rangle = 1$
2. $\left\langle \times \right\rangle = A \left\langle)(\right\rangle + A^{-1} \left\langle \asymp \right\rangle$
3. $\left\langle P \sqcup \bigcirc \right\rangle = \left(-A^{-2} - A^{2}\right) \langle P \rangle$

WE CAN NOW COMPUTE

$V\Big($  $\Big) = -A^{-3w\left(\right.}$  $^{\left.\right)}\Big\langle$  $\Big\rangle$

$-A^{-3w\left(\right.}$  $^{\left.\right)}\left\langle\right.$  $\left.\right\rangle$

$= -A^{-3\cdot-3}\Big(A\Big\langle$  $\Big\rangle + A^{-1}\Big\langle$  $\Big\rangle\Big)$

$= -A^{9}\Big(A\Big\langle$  $\Big\rangle + A^{-1}\Big\langle$  $\Big\rangle\Big)$

$\Big\langle$  $\Big\rangle = -A^3 \Big\langle$  $\Big\rangle$

$$= -A^3 \left(-A^3\right) \Big\langle \bigcirc \Big\rangle$$

$$= A^6$$

$$-A^{-3w\left(\right.}$$  $$\left.\right)} \left\langle \right.$$  $$\left.\right\rangle$$

$$-A^{9}\left(A\left\langle\right.\right.$$  $$\left.\right\rangle + A^{-1}\left\langle\right.$$  $$\left.\left.\right\rangle\right)$$

$$= -A^{9}\left(A\left(A^{6}\right) + A^{-1}\left\langle\right.\right.$$  $$\left.\left.\right\rangle\right)$$

$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$$= A\left(-A^{3}\left\langle \bigcirc \right\rangle\right)$$

$$+ A^{-1}\left(-A^{-3}\left\langle \bigcirc \right\rangle\right)$$

$$= -A^{4} - A^{-4}$$

$-A^{-3w(}$  $^{)}\left\langle\right.$  $\left.\right\rangle$

$= -A^{9}\Big(A\left\langle\right.$  $\left.\right\rangle + A^{-1}\left\langle\right.$  $\left.\right\rangle\Big)$

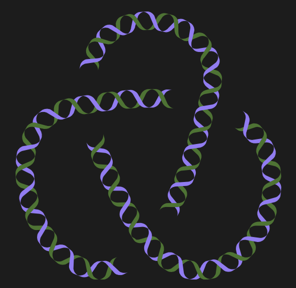
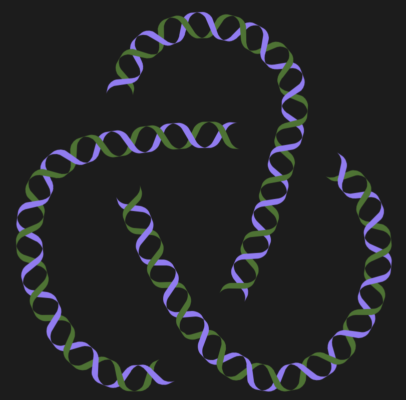
$= -A^{9}\Big(A\left(A^{6}\right) + A^{-1}\left\langle\right.$  $\left.\right\rangle\Big)$

$$= -A^{9}\left(A\left(A^{6}\right) + A^{-1}\left(-A^{4} - A^{-4}\right)\right)$$

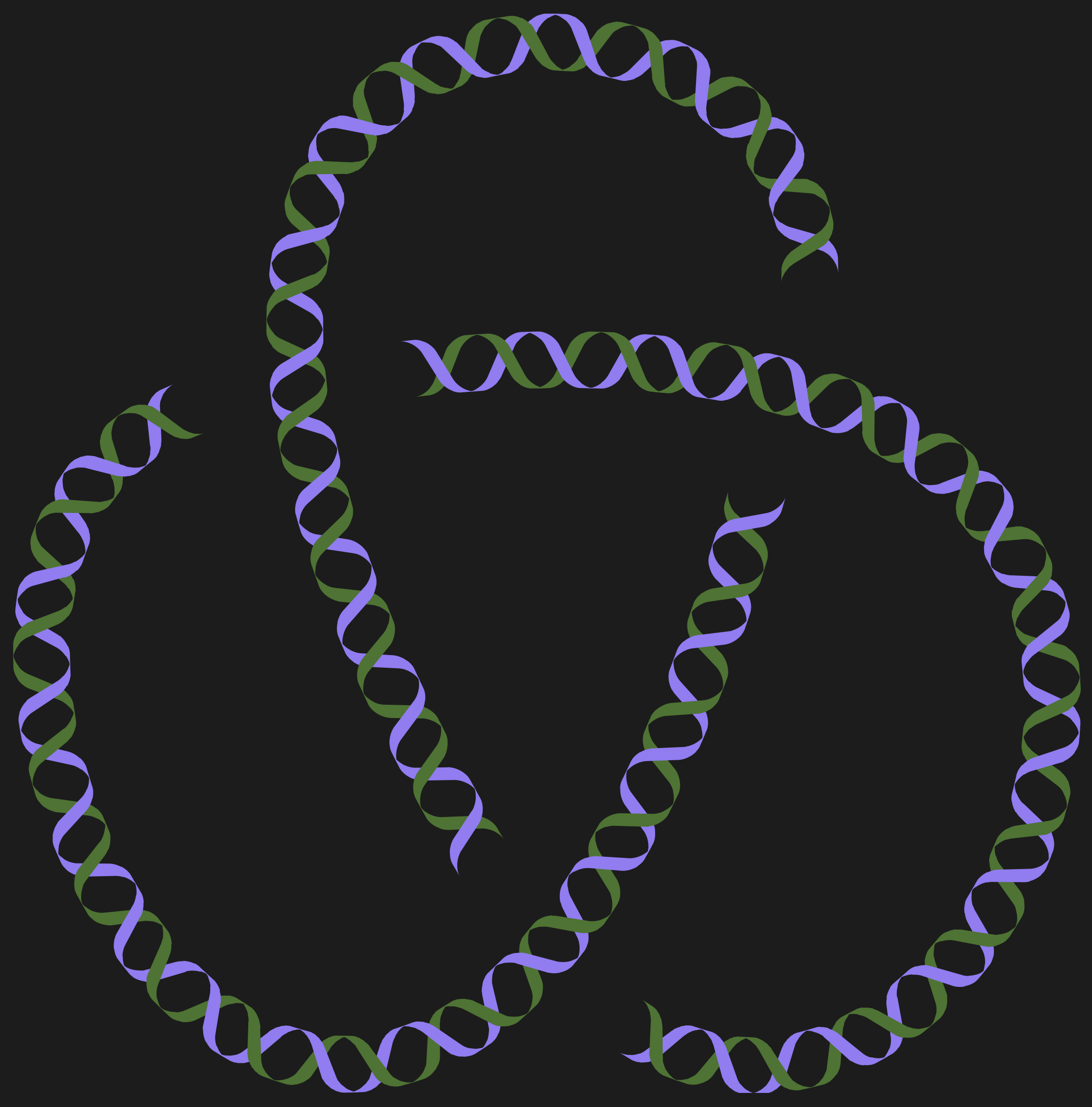
$$= -A^{9}\left(A^{7} - A^{3} - A^{-5}\right)$$

$$= -A^{16} + A^{12} + A^{4}$$

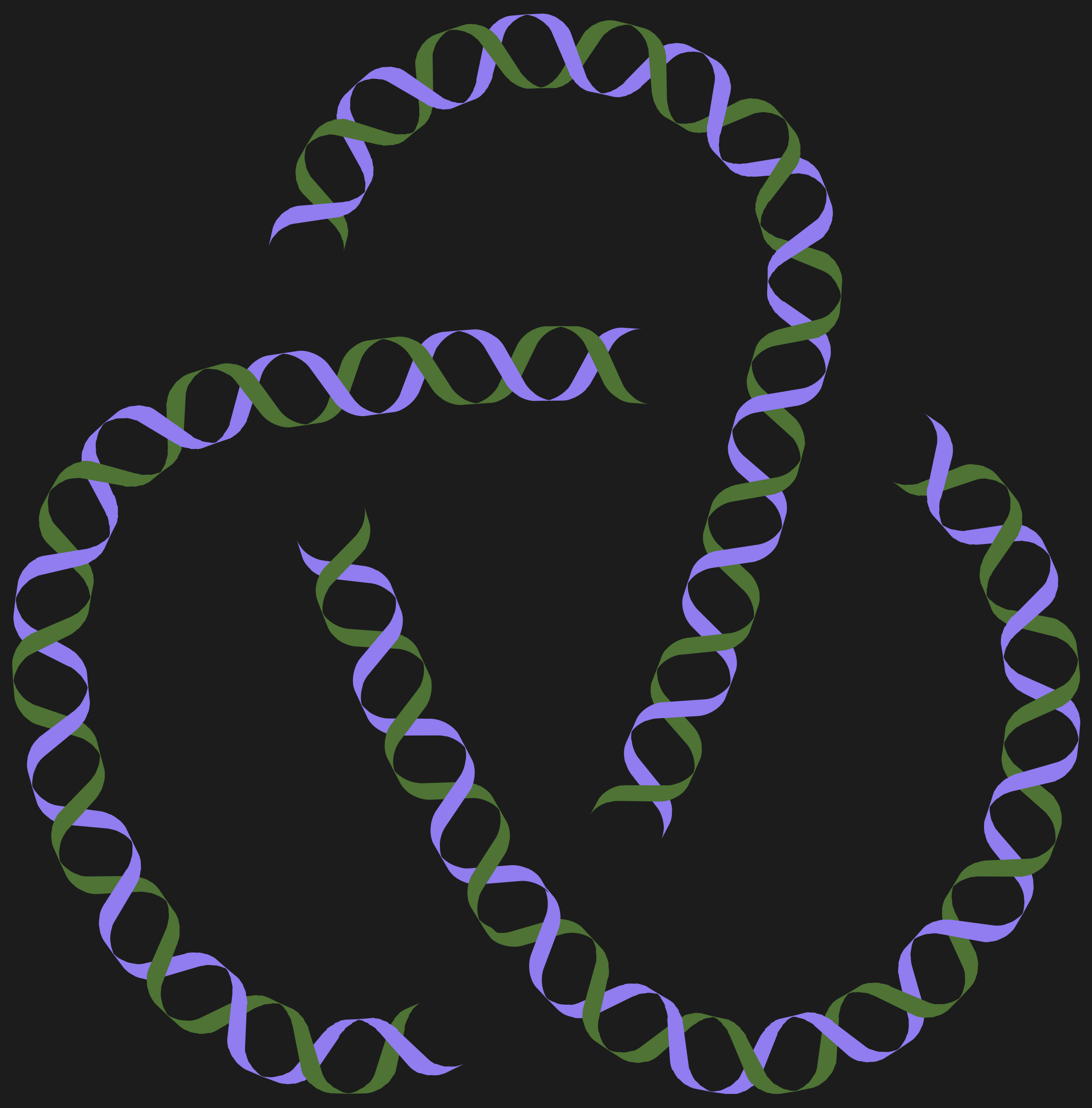
Exercise: Compute the bracket on the anti-knot

$-A^{-3w\left(\right.}$  $^{\left.\right)}$ $\Big\langle$  $\Big\rangle$

ANTI-KNOT



$$-A^{16} + A^{12} + A^{4}$$



$$-A^{-16} + A^{-12} + A^{-4}$$


DAY 121

With the successful completion of your work the vaccine is being administer world wide. The President congratulates you for your work and the world is optimistic.


DAY 300

The virus is completely controlled and you win every prize in every field imaginable!

THE JONES POLYNOMIAL

The Jones Polynomial $V(\mathscr{K})$ of an oriented knot $\mathscr{K}$ is the **Laurent polynomial** with integer coefficients in $t^{1/2}$.

Defined by $V(\mathscr{K}) = \left((-A)^{-3w(P)} \langle P \rangle\right)_{t^{1/2}=A^{-2}}$ where P is any oriented diagram for $\mathscr{K}$.

$V(\mathscr{K}) = \Big(-A^{-3w\big(}$  $^{\big)}\Big\langle$  $\Big\rangle\Big)_{t^{1/2}=A^{-2}}$

$= \Big(-A^{-3\cdot -3}\Big\langle$  $\Big\rangle\Big)_{t^{1/2}=A^{-2}}$

$$= \left(-A^{9}\left(A^{7} - A^{3} - A^{-5}\right)\right)_{t^{1/2}=A^{-2}}$$

$$= \left(-A^{16} + A^{12} + A^{-4}\right)_{t^{1/2}=A^{-2}}$$

$$= -t^{-4} + t^{-3} + t^{-1}$$

Worksheet

1. Livingston, C. (1993). Knot Theory. Mathematical Association of America. https://doi.org/10.5948/UPO9781614440239
2. Dale Rolfsen, Knots and links, Mathematics Lecture Series, vol. 7, Publish or Perish, Inc., Houston, TX, 1990, Corrected reprint of the 1976 original.
3. Robert Glenn Scharein. Interactive topological drawing. ProQuest LLC, Ann Arbor, MI, 1998. Thesis Ph.D. The University of British Columbia (Canada). URL: https://www.knotplot.com/.
4. Jablan, S., & Sazdanović, R. (2007). Linknot. In Series on Knots and Everything. WORLD SCIENTIFIC. https://doi.org/10.1142/6623
5. Vaughan Jones. The Jones polynomial for dummies. https://math.berkeley.edu/~vfr/jonesakl.pdf WebArchive
6. Deoxyribonucleic acid (DNA). (n.d.). Genome.gov. https://www.genome.gov/genetics-glossary/Deoxyribonucleic-Acid. Accessed 3 October 2023
7. DNA knot as seen under the electron microscope. - Image Credit: Javier Arsuaga, CC BY-ND
8. Vinograd, J., Lebowitz, J., Radloff, R., Watson, R., & Laipis, P. (1965). The twisted circular form of polyoma viral DNA. In Proceedings of the National Academy of Sciences (Vol. 53, Issue 5, pp. 1104-1111). Proceedings of the National Academy of Sciences. https://doi.org/10.1073/pnas.53.5.1104
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