Limits

Algebra

Limit Laws

1. $$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

2. $$\lim_{x \to a} (f - g)(x) = \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

3. $$\lim_{x \to a} (fg)(x) = \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

4.

$$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ if } \lim_{x\to a} g(x) \neq 0$$

5.

$$\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$$

6.

$$\lim_{x\to a} f(y) = f(y)$$

7.

$$\lim_{x\to c} x = a$$

Example:

let $f(x) = \sin(x)$, $g(x) = x$, and $h(x) = \cos(\pi/4 - x)$

1. $$\lim_{x \to 0} \left(\frac{f}{g}\right)(x) = ?$$

2. $$\lim_{x \to 0} h(x) = ?$$

3. $$\lim_{x \to 0} \left[\frac{f(x)h(x)}{g(x)}\right] = ?$$

4. $$\lim_{x \to 0} \left[\frac{f(x)}{g(x)} + h(x)\right] = ?$$

5. $$\lim_{x \to 0} \left[\frac{f(x)}{g(x)h(x)}\right] = ?$$

Example:

True/False:

$$\left(\lim_{x\to a} f(x)\right)^n = \lim_{x\to a} (f(x))^n$$

Convince me.

Example:

True/False:

$$\left(\lim_{x \to a} f(x)\right)^{\frac{1}{n}} = \lim_{x \to a} \left(f(x)\right)^{\frac{1}{n}}$$

Convince me.

Direct Substitution

Sometimes we can evaluate a limit by direct substitution, ie: by simply playing in a for x

Example:

$f(x) = x^2 + x + 4$

$$\lim_{x \to 1} f(x) =$$

Example:

Convince me that a polynomial $p(x)$ we can always solve

$$\lim_{x \to a} p(x)$$

by substitution.

Cancelation

What about $f(x) = x - 1$

$$\lim_{x \to 1} \frac{f(x)}{f(x)} = ?$$

When we have a rational expression, we can cancel like terms in the numerator and denominator.

Example:

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 + x - 6} = \lim_{x\to 2} \frac{(x-2)(x+1)}{(x-2)(x+3)}$$

$$= \lim_{x\to 2} \frac{x+1}{x+3}$$

$$= \frac{3}{5}$$

Example:

$$\lim_{x\to -2} \frac{x^2 + x - 2}{x^2 + 2x} =?$$

"Simplification"

Change the way the problem looks

Conjugate

Difference of squares

$$(\underline{\hspace{3cm}})(\underline{\hspace{3cm}}) = \underline{\hspace{3cm}}$$

gives us that

$$\frac{1}{a+\sqrt{b}} = \frac{a-\sqrt{b}}{a^2 - \left(\sqrt{b}\right)^2}$$

NOTE: This is super common in limit problems

Example:

$$\lim_{h+0} \frac{\sqrt{h^2+4}-3}{h^2}$$

DCT

Let $f(x) \leq g(x) \quad \forall x \in \text{Domain}(f)$. Let $a \in \text{Domain}(f) \cap \text{Domain}(g)$. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

Note: We allow $\infty \leq \infty$ and $-\infty \leq \infty$ but $\infty \nleq -\infty$

Squeeze

- Let $f(x), g(x), h(x)$ all have a in their domain.

- Suppose the limit $x \to a$ exists for $f(x), g(x), h(x)$ with

$$\lim_{x \to a} f(x) = L_1$$

$$\lim_{x \to a} h(x) = L_2$$

- If $f(x) \leq g(x) \leq h(x)$ then

$$L_1 \leq \lim_{x \to a} g(x) \leq L_2$$

- In particular, if $L_1 = L_2 = L$ then $\lim_{x \to a} g(x) = L$

NOTE: This is often used for trig functions.