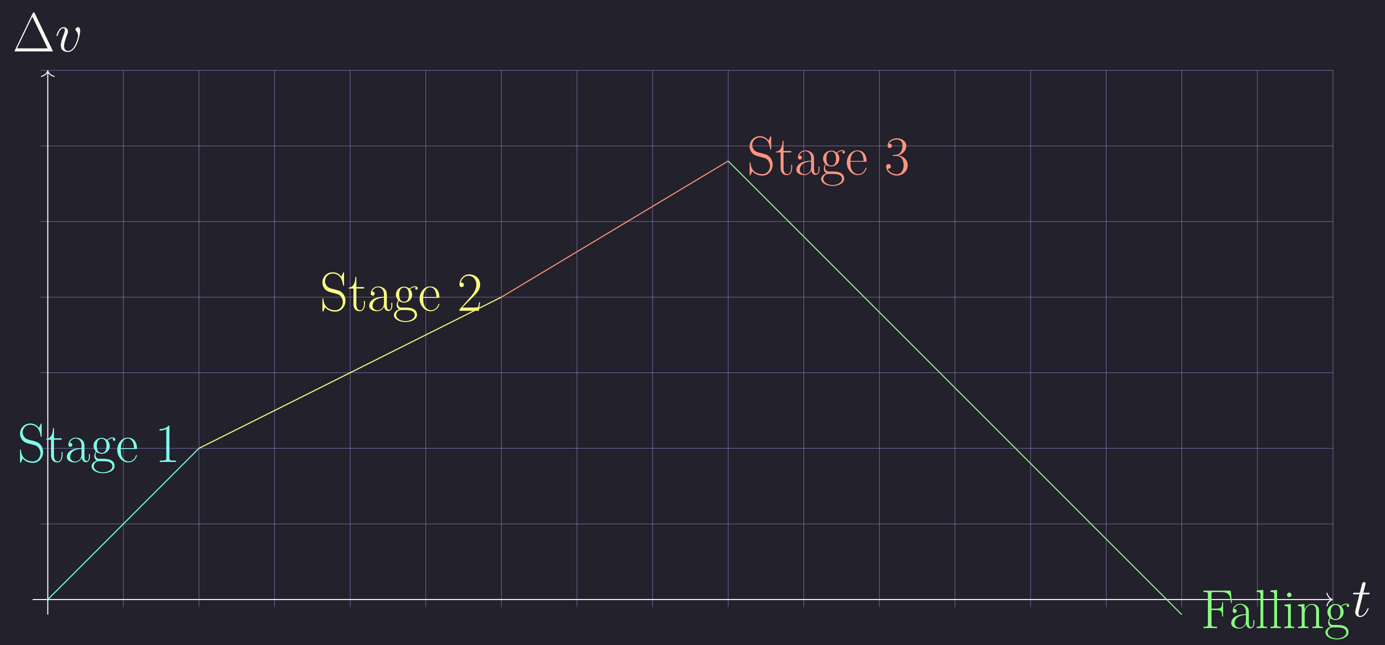
Types of Functions

Piecewise Functions

Notation

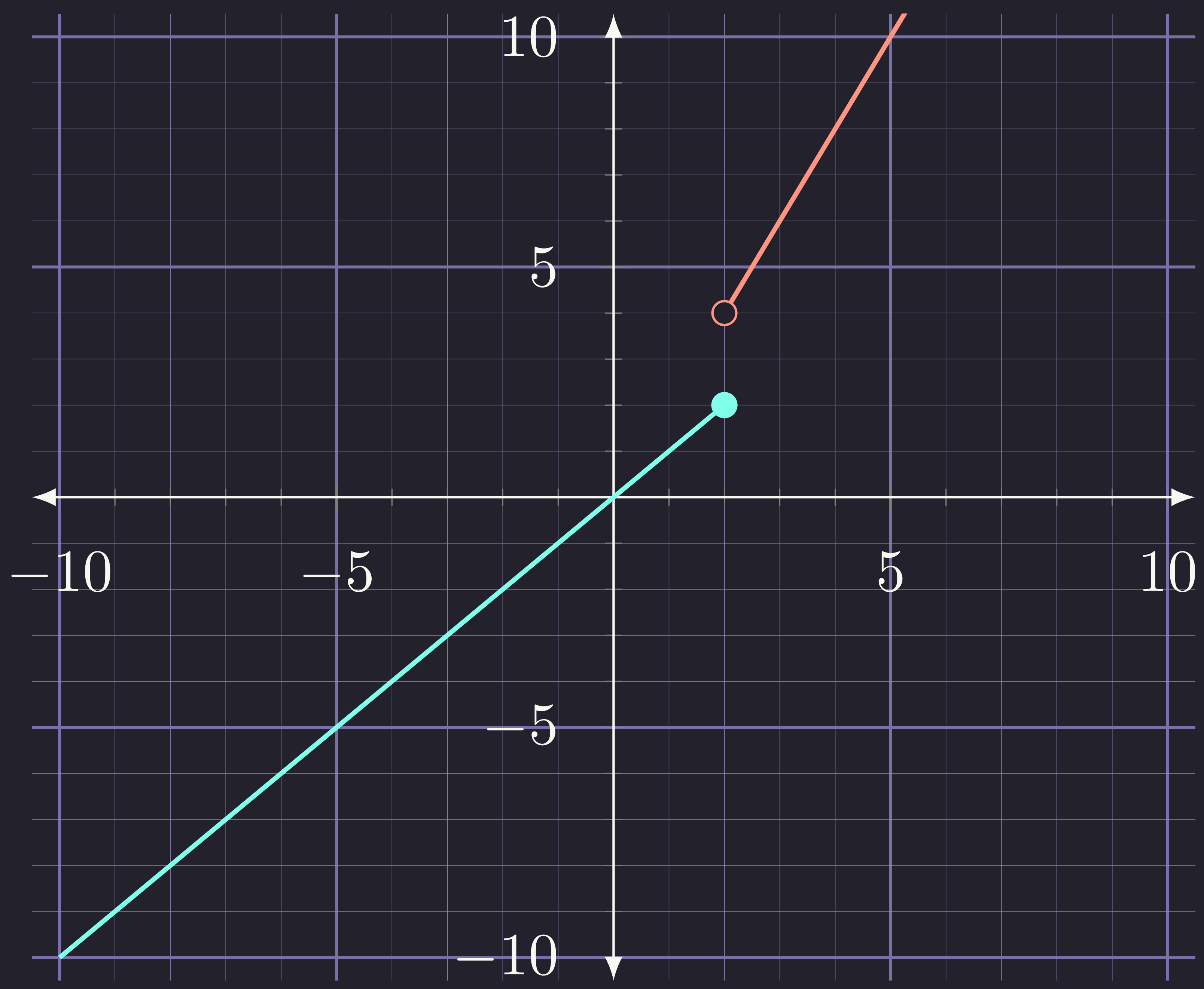
$$f(x) = \begin{cases} \text{function}, & \text{condition} \\ \vdots & \vdots \\ \text{function}, & \text{condition} \end{cases}$$

Example:



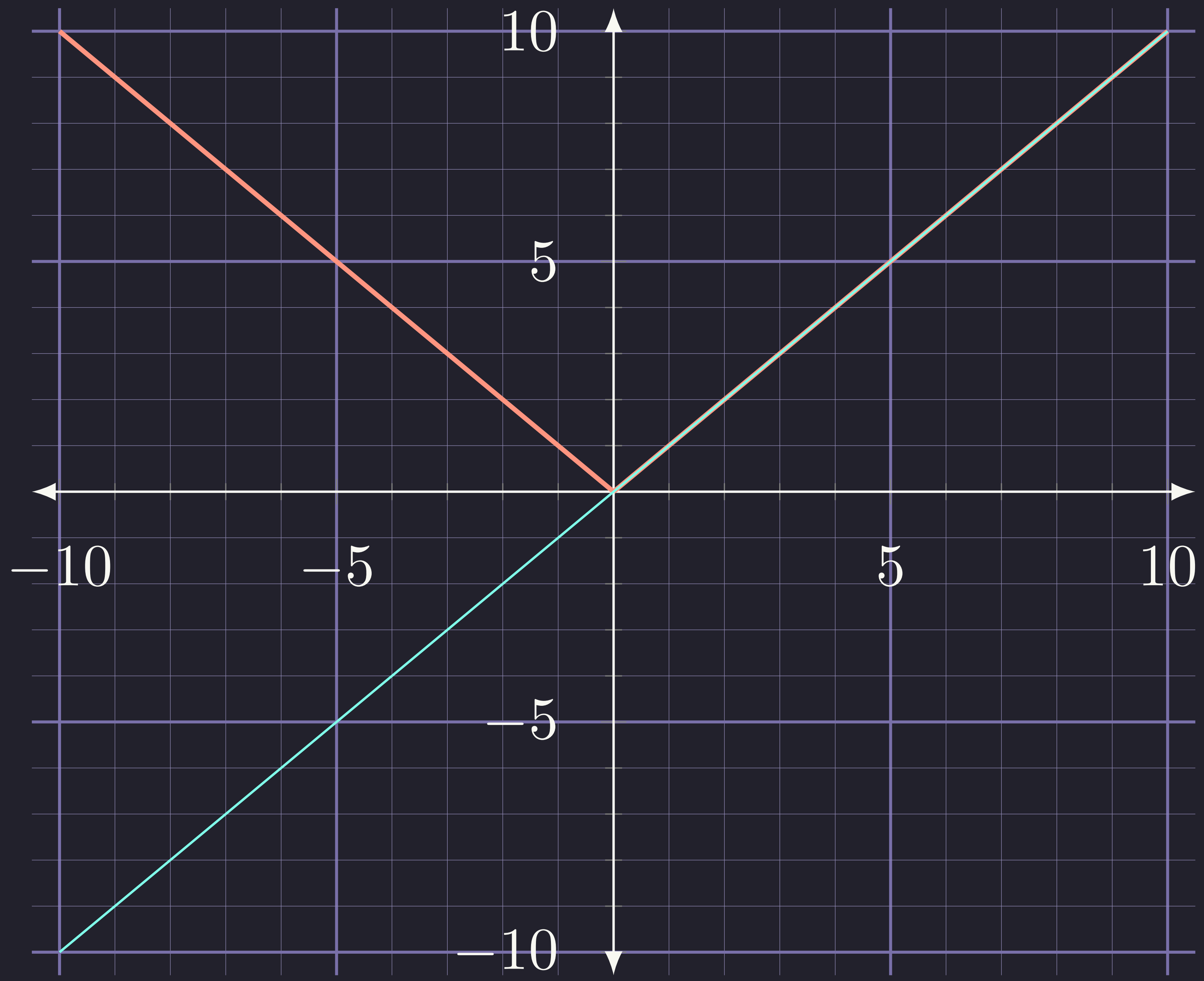
$$f(x) = \begin{cases} x, & [0,2) \\ \frac{1}{2}x, & (2,6] \\ \frac{3}{5}x, & (6,9] \\ -x, & (9,15] \end{cases}$$

Example:

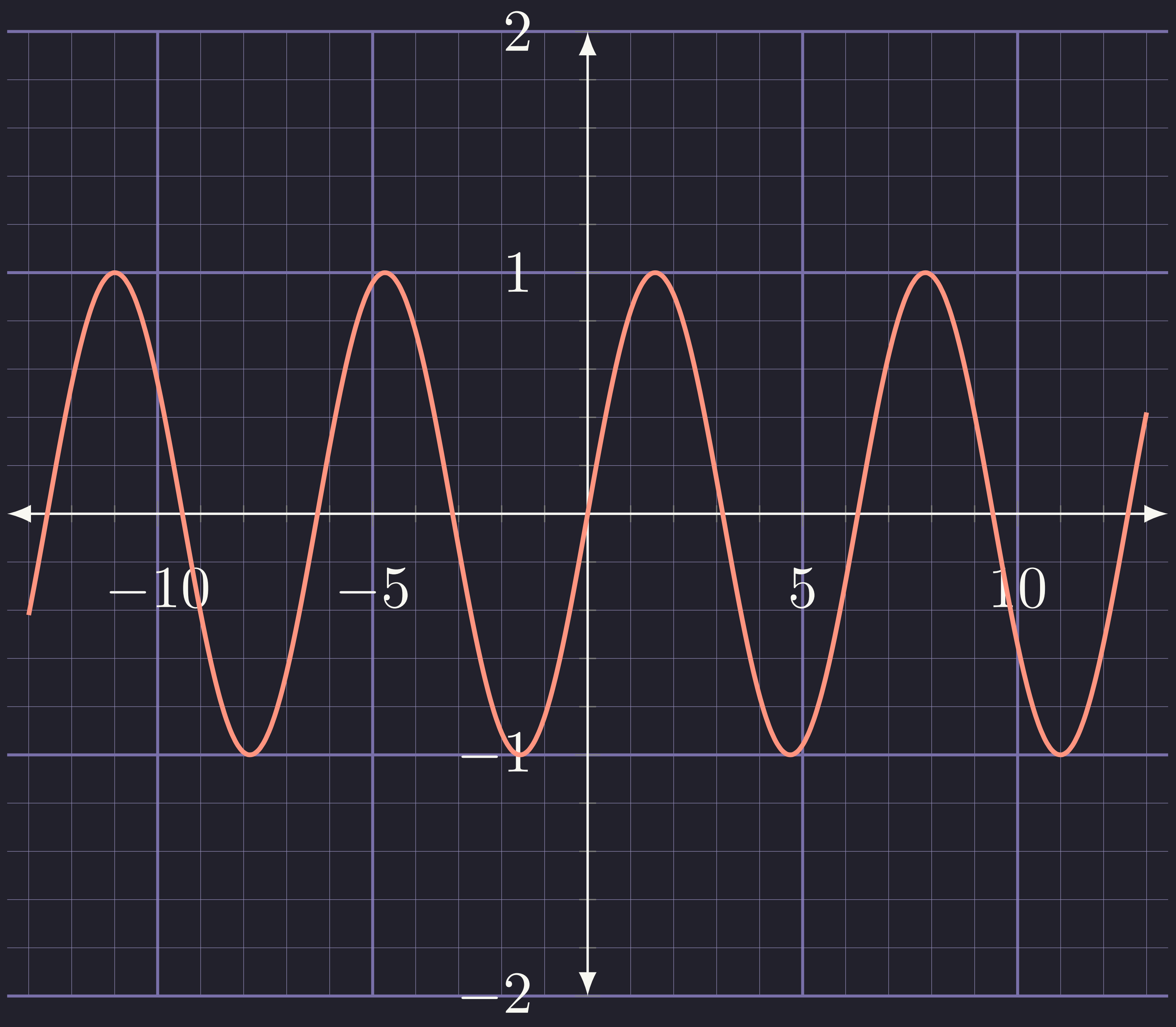


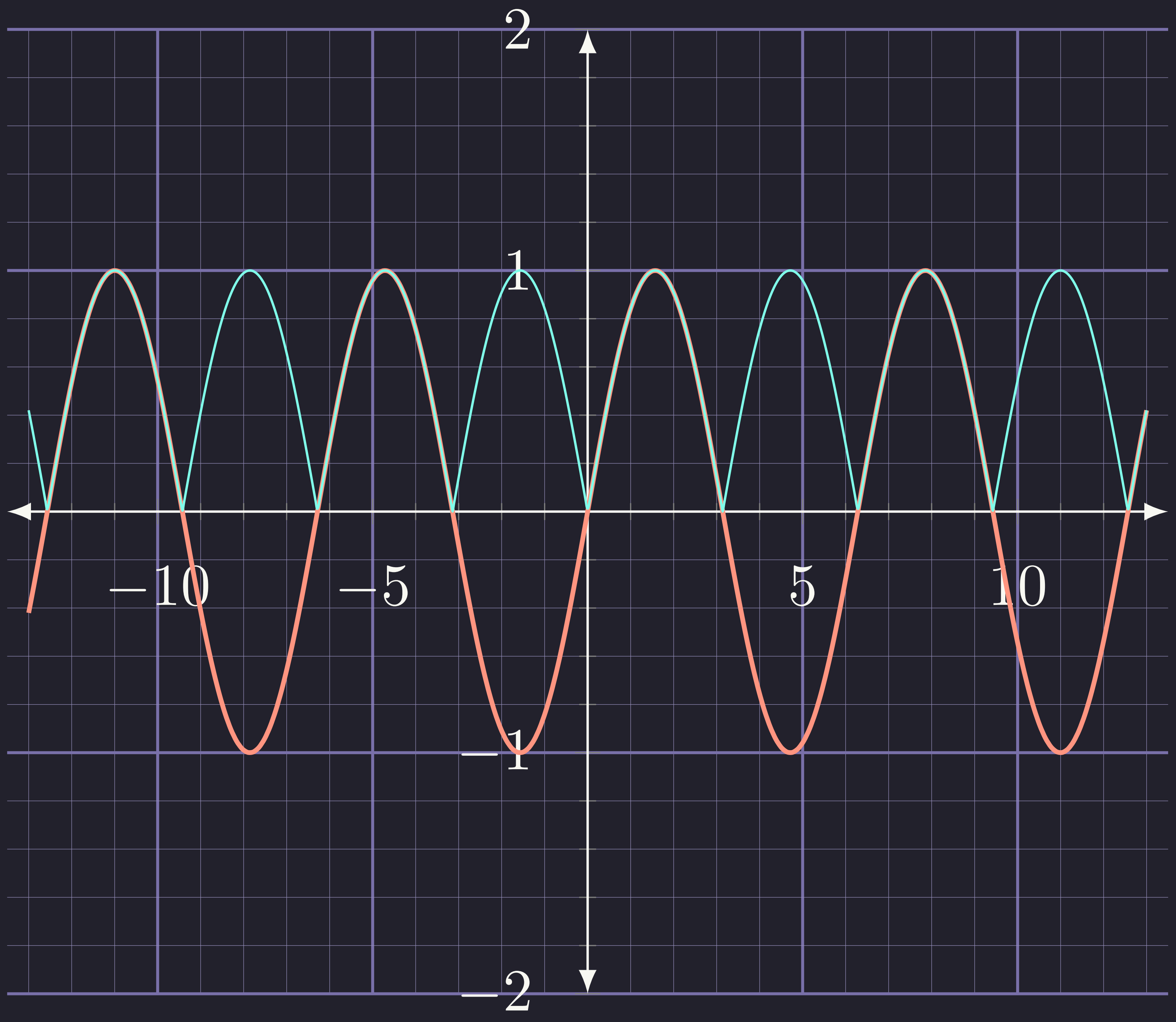
Absolute Value

Example:



Example:





2
1
−1
−2
−10
−5
5
10

Even and Odd

Definition:

A function $f(x)$ is called:

Even: If $\forall x \in D$ we have $f(\text{-}x) = f(x)$

Odd: If $\forall x \in D$ we have $f(\text{-}x) = \text{-}f(x)$

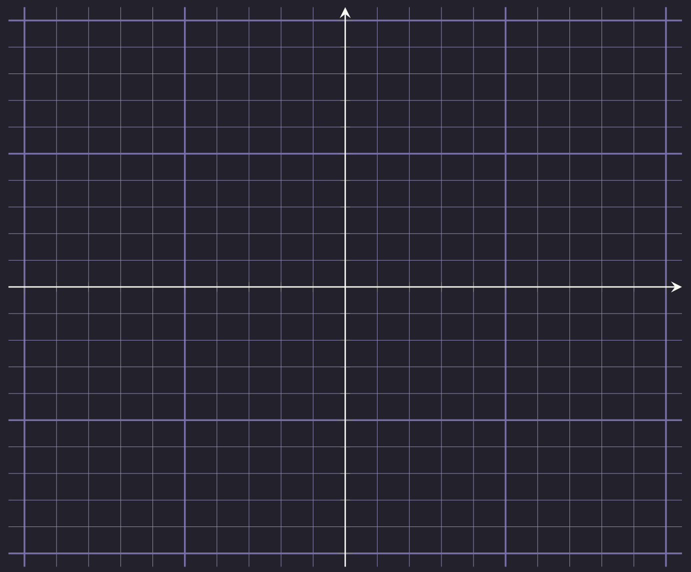
Neither:

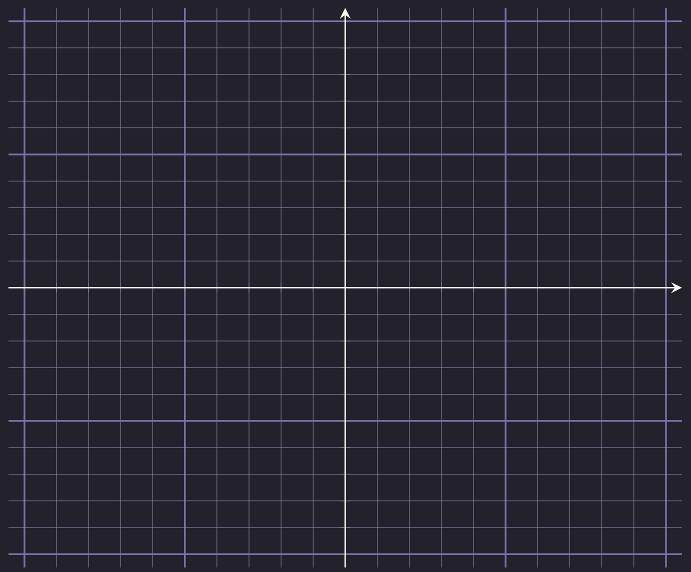
Symmetries:

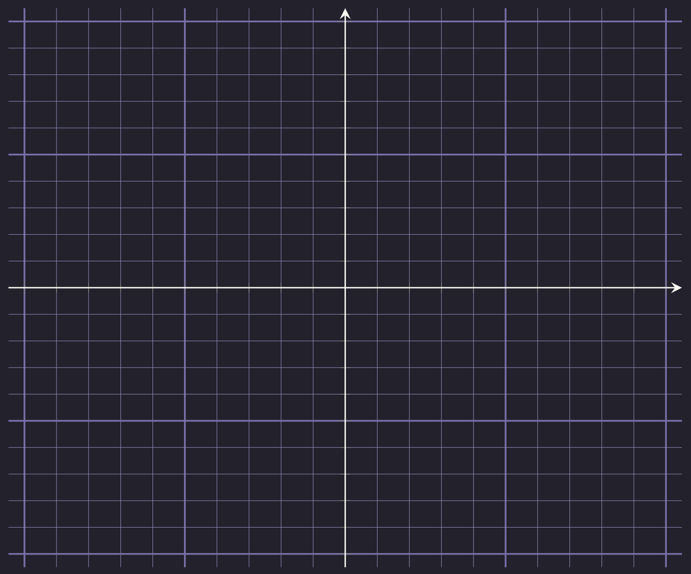
Even:

Odd:

Neither:







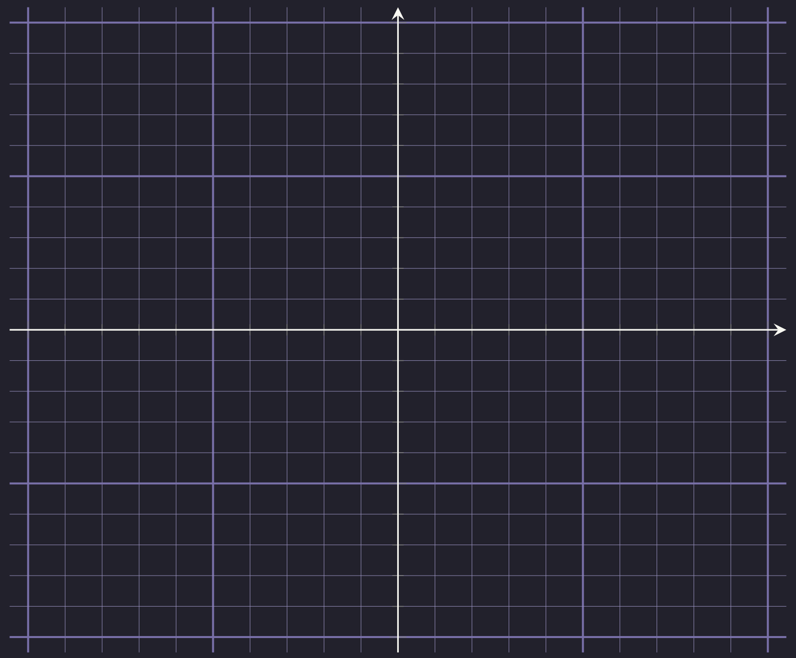
Increasing Decreasing

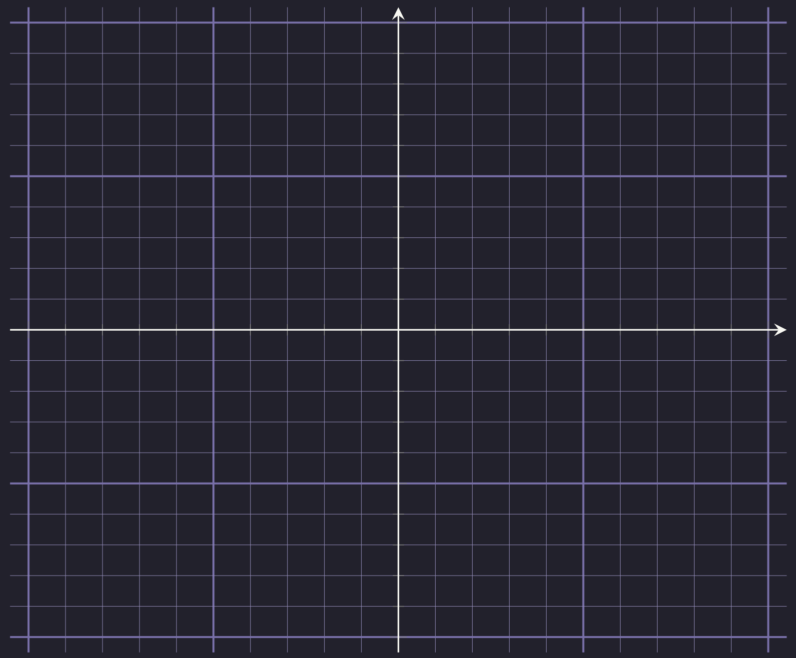
Definition:

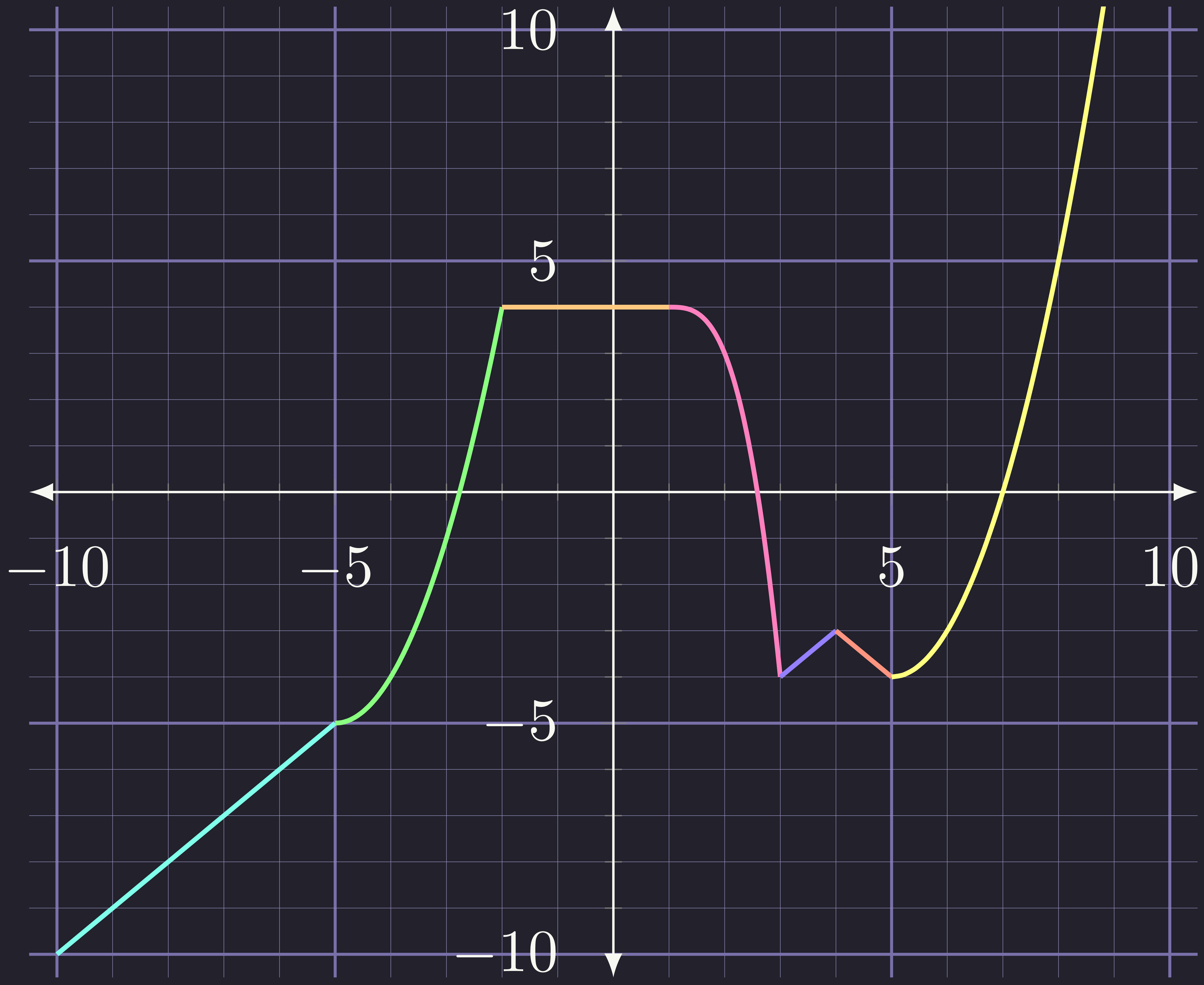
A function $f(x)$ on an inteval I is called:

Increasing: $\forall a, b \in I$ if $a < b$ then $f(a) < f(b)$

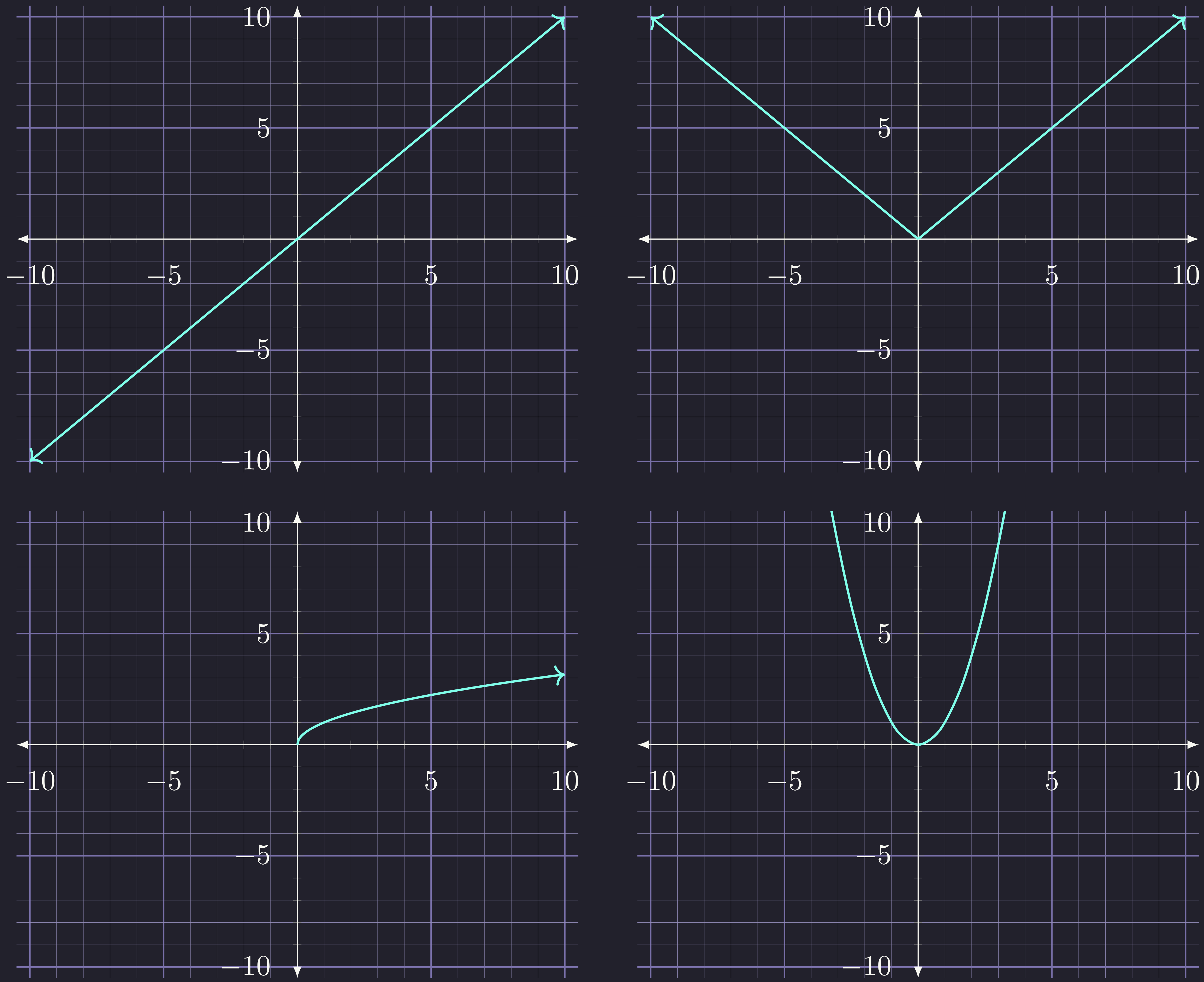
Decreasing: $\forall a, b \in I$ if $a < b$ then $f(a) > f(b)$

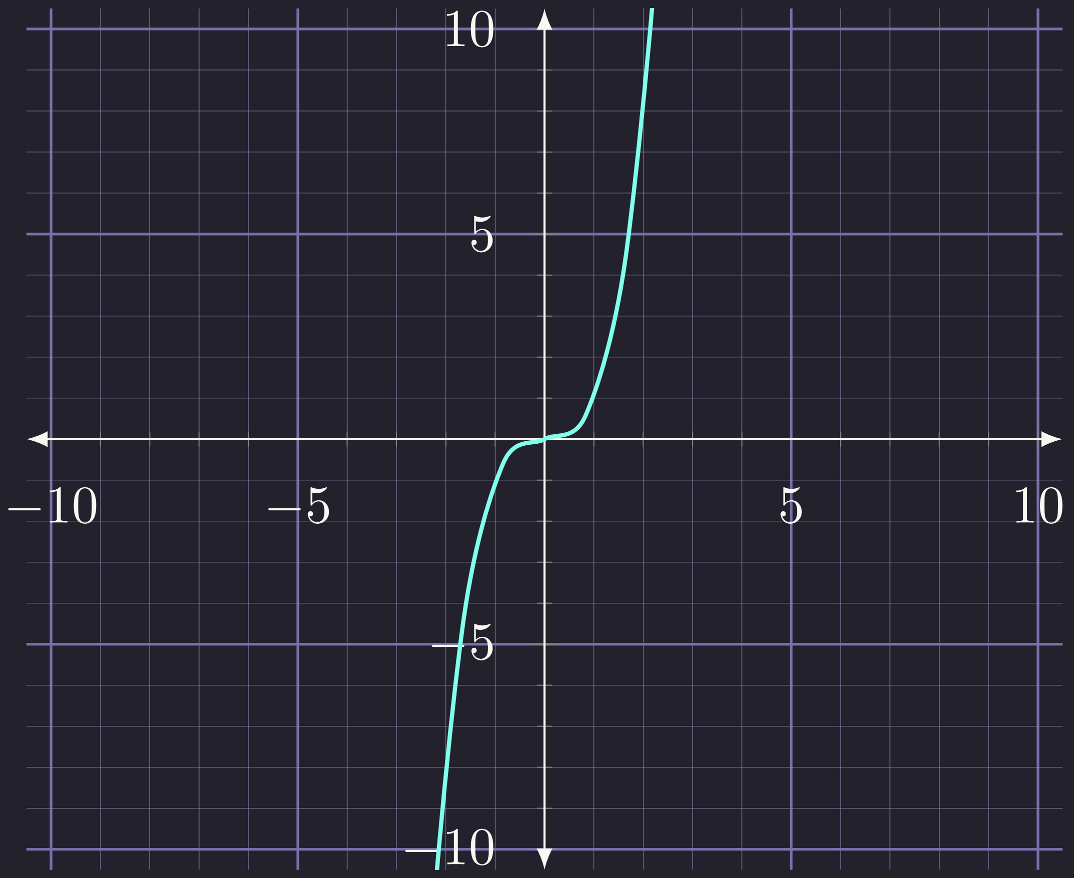





10
5
−5
−10
−10
−5
5
10

"Parent" Functions




10
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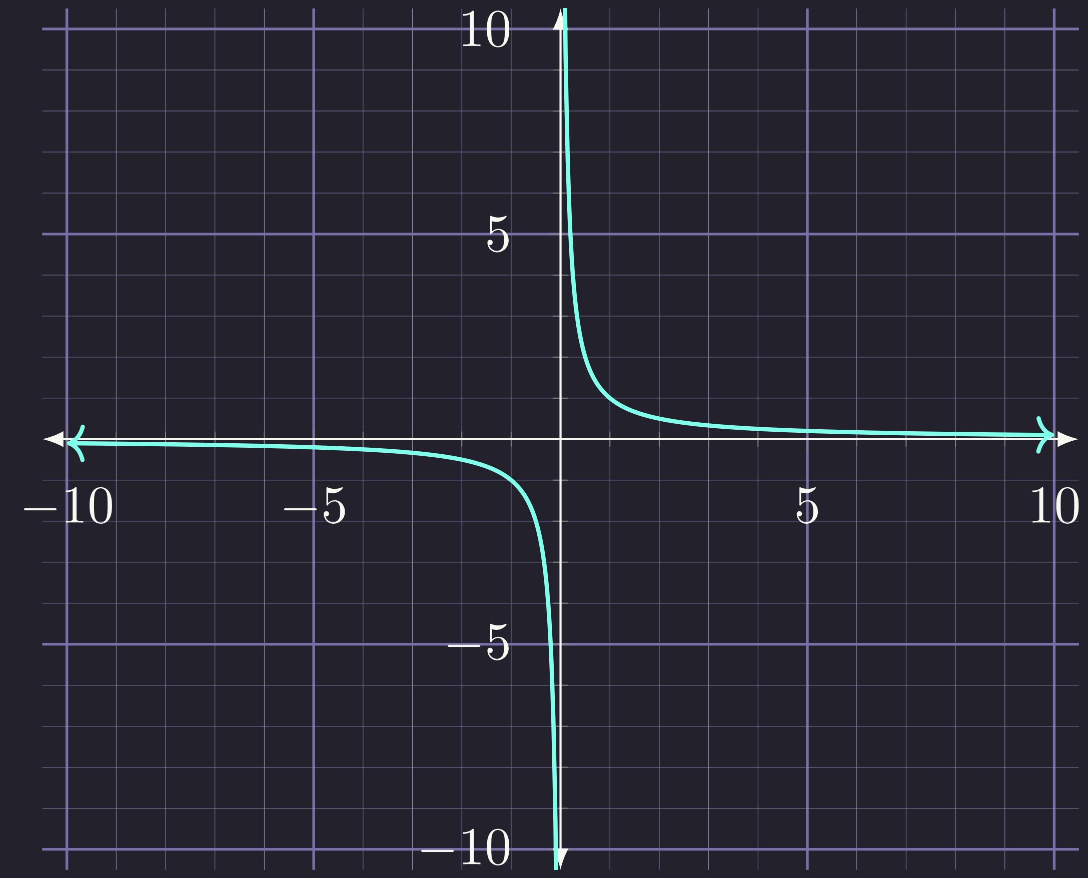
10
−5
−10


10
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10
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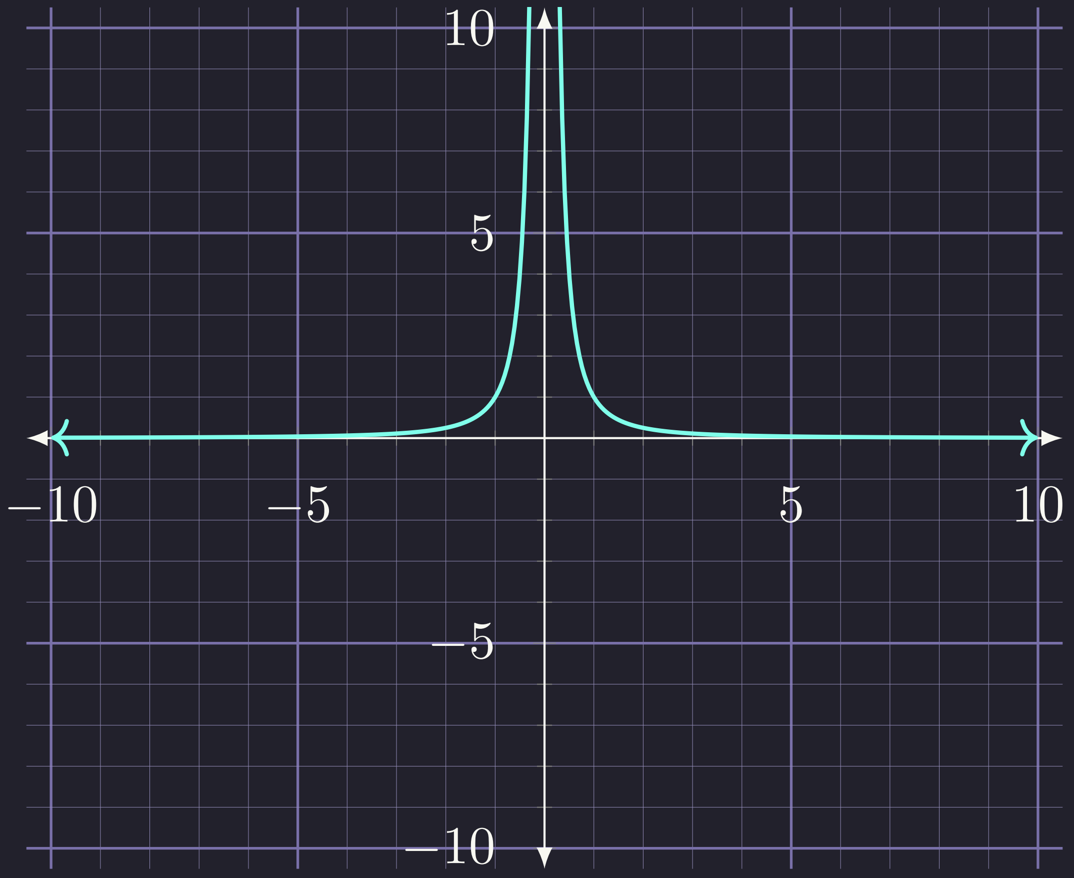
−10


10
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−10
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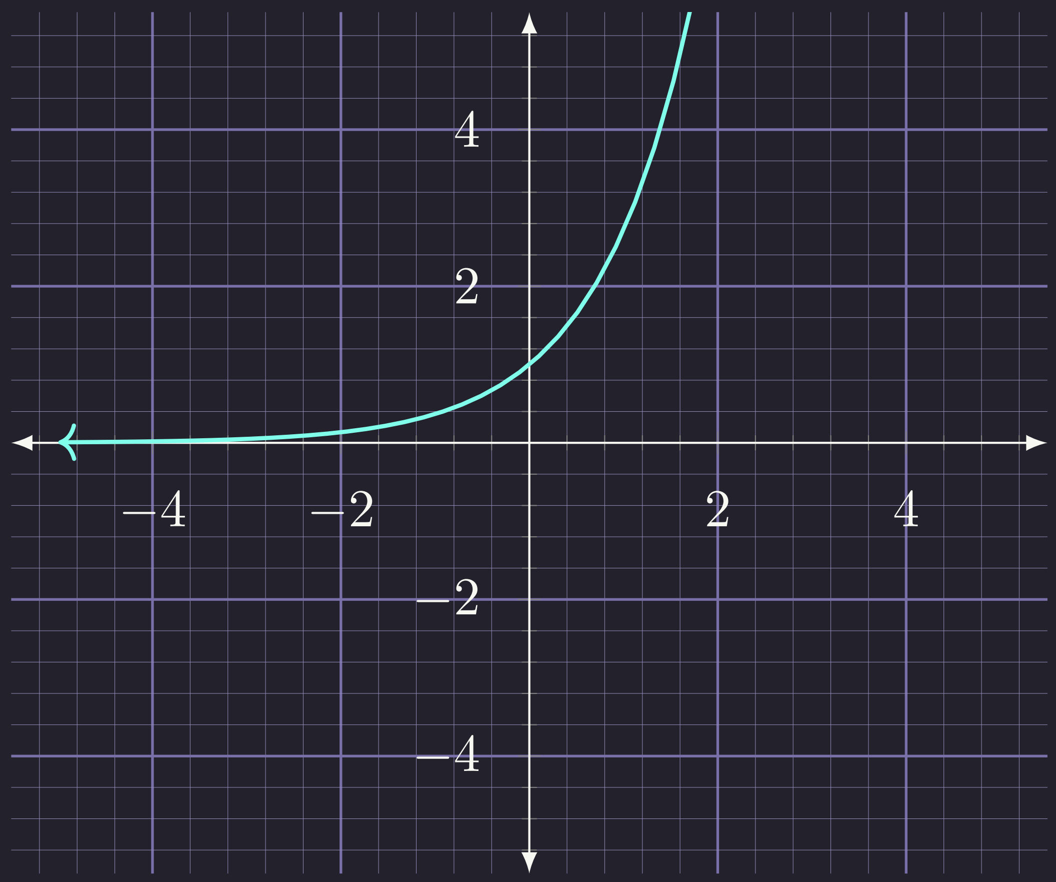
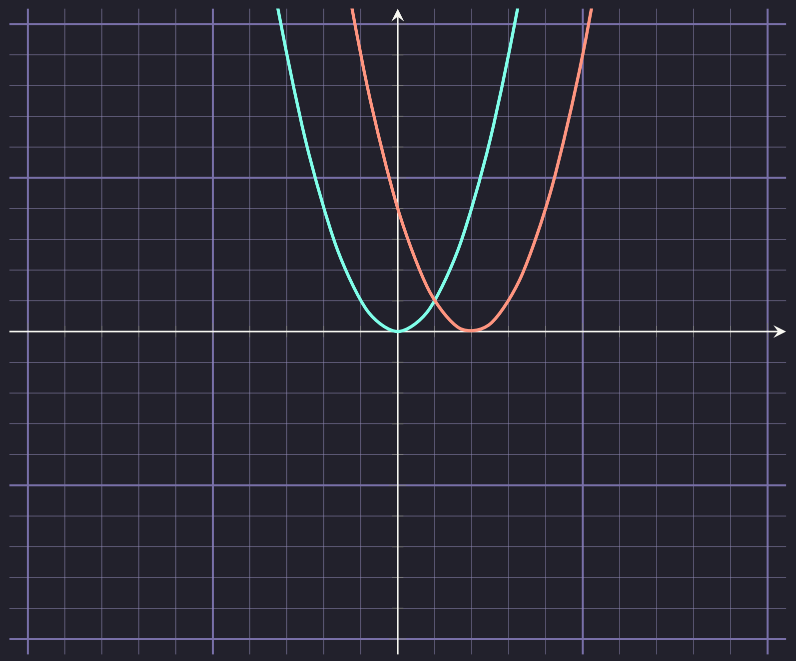
−10


4
2
−4
−2
2
4
−2
−4

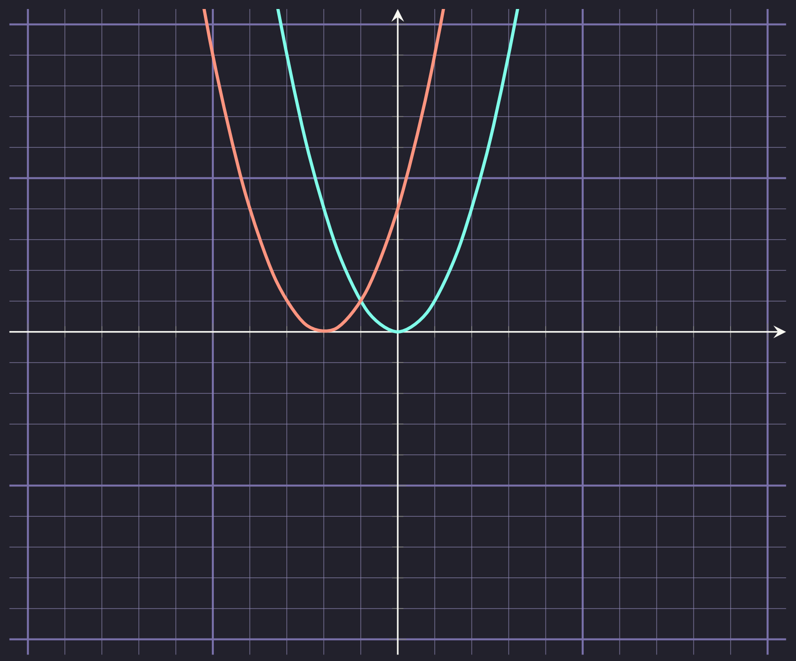
Translation of functions

Shifts:

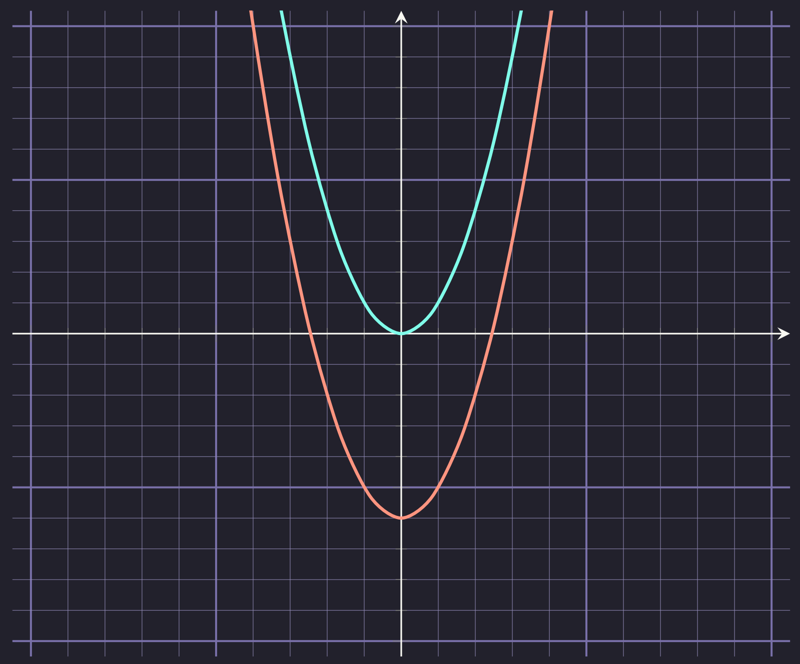
__________: $f(x) \to f(x-c)$



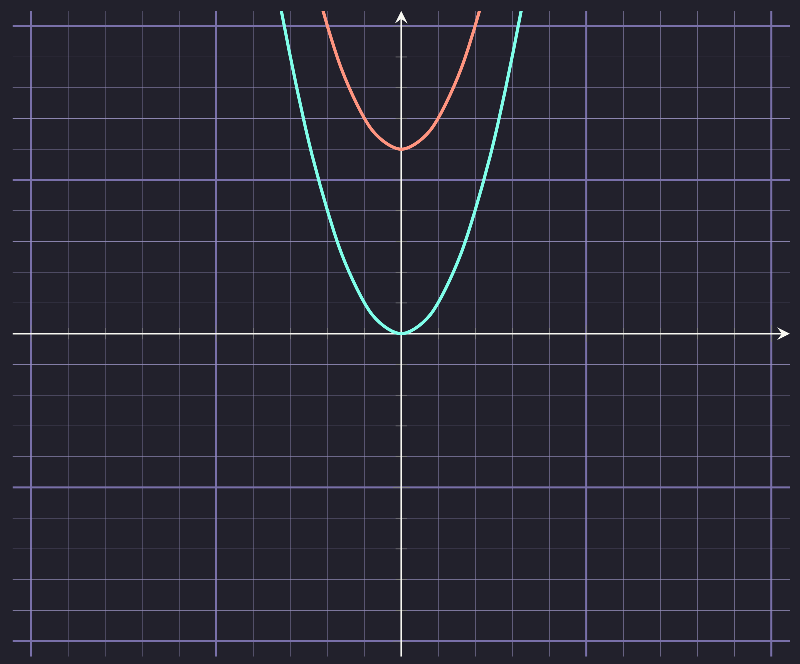
__________: $f(x) \to f(x+c)$



__________: $f(x) \to f(x) - c$



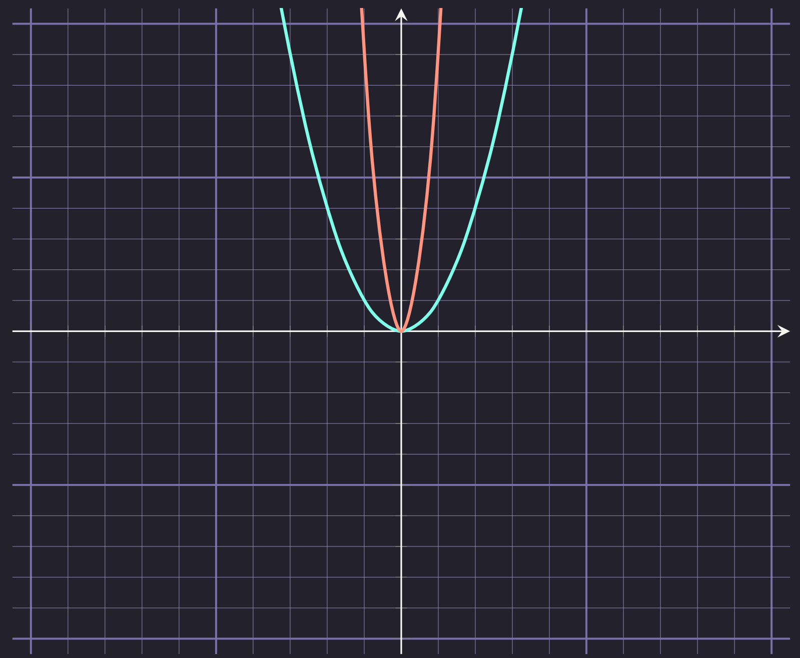
__________: $f(x) \to f(x) + c$



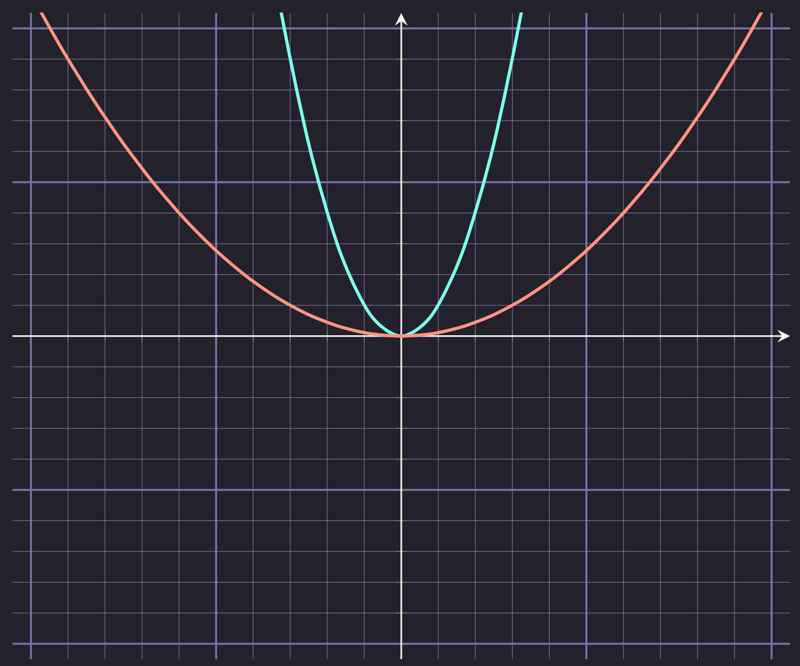
Stretching and reflection:

For $1 < c$

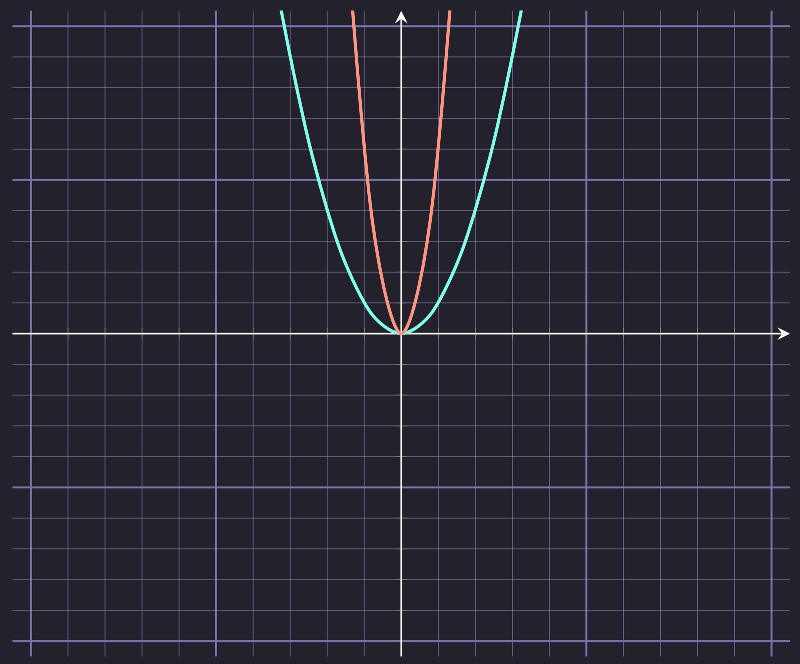
__________: $f(x) \to f(cx)$



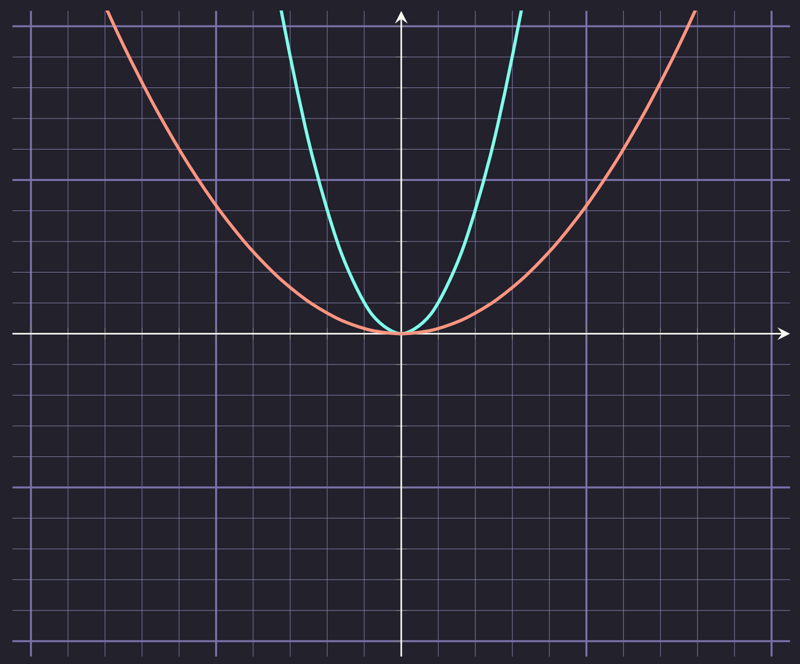
__________: $f(x) \to f(\frac{1}{c}x)$



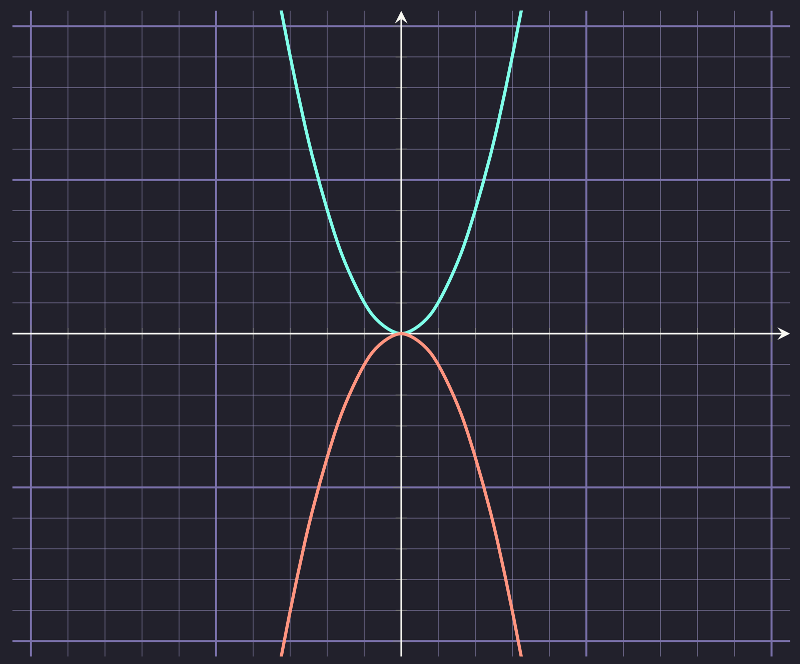
__________: $f(x) \to cf(x)$



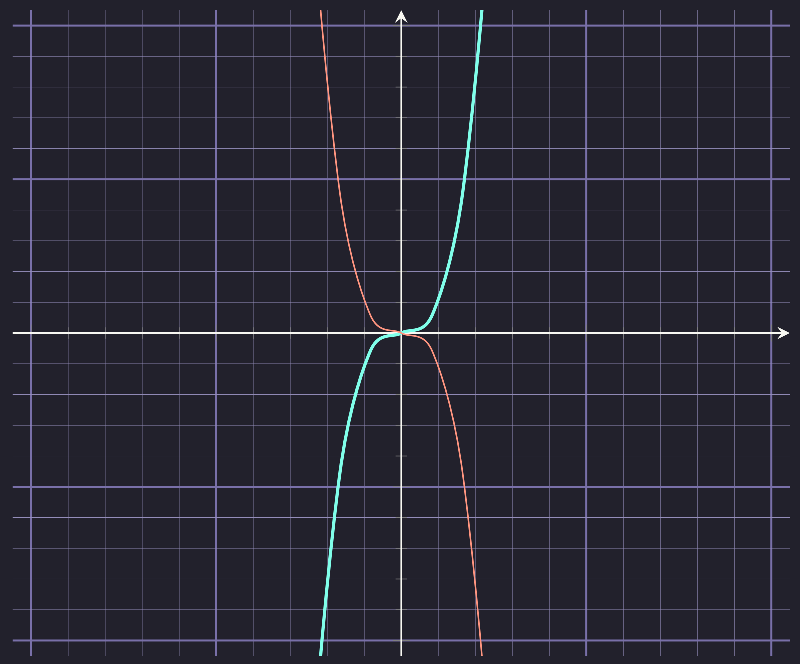
__________: $f(x) \to \frac{1}{c}f(x)$



__________: $f(x) \to -f(x)$



__________: $f(x) \to f(-x)$



Function Arithmetic

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example:

If we let $f(x) = x^2$ and $g(x) = 4x^3$

$(f+g)(x) =$ ________________

$(f-g)(x) =$ ________________

$(f \cdot g)(x) =$ ________________

$\left(\frac{f}{g}\right)(x) =$ ________________

Domain

Addition, multiplication $+ \;\; - \;\; \cdot$

The domain of $f+g$, $f-g$, and $f \cdot g$ is the intersection of the domains of f and g. In other words, $f+g$, $f-g$, and $f \cdot g$ are defined wherever both f and g are defined.

Division

For $\frac{f}{g}$, the domain is the intersection of the domains of f and g excluding x where $g(x)$ is 0.

Range

Function Composition

We let $f(x)$ and $g(x)$ be two functions with "compatable" domain and codomain

the compostion of $f(x)$ and $g(x)$ written

$$(f \circ g)(x)$$

is defined to be

$$f(g(x))$$

Example:

Let $f(x) = x^2$ and $g(x) = \sqrt{x} =$ ______

$(f \circ g)(x) =$

$(g \circ f)(x) =$

Example:

Let $f(x) = x^2$ and $g(x) = x + 1$

$(f \circ g)(x) =$

$(g \circ f)(x) =$

Domain

The domain of $f \circ g$ is all x in the domain of g so that $g(x)$ is in the domain of f.

Note:

The easiest way to find the domain is usually to write an expression for $(f \circ g)(x)$ and find its domain without simplifying.

Example:

Let $f : \mathbb{R} \to \mathbb{R}$ where $f : x \mapsto \frac{1}{x}$, and $g : (0, \infty) \to \mathbb{R}$ where $g : x \mapsto x^2$.

Find the domain of $(f \circ g)(x)$:

Note:

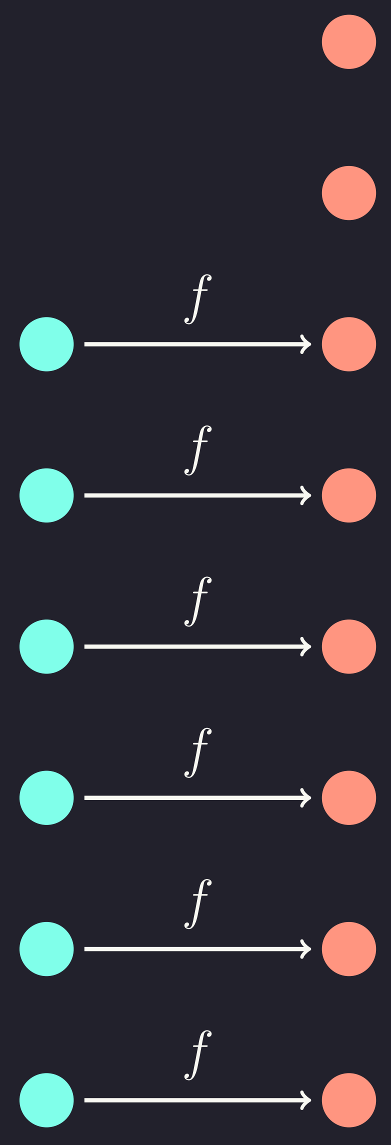
This is one of the most important skills you NEED to have for calculus 1.

Easy problems $>$ hard problems

further

2 Easy problems $>$ a hard problem

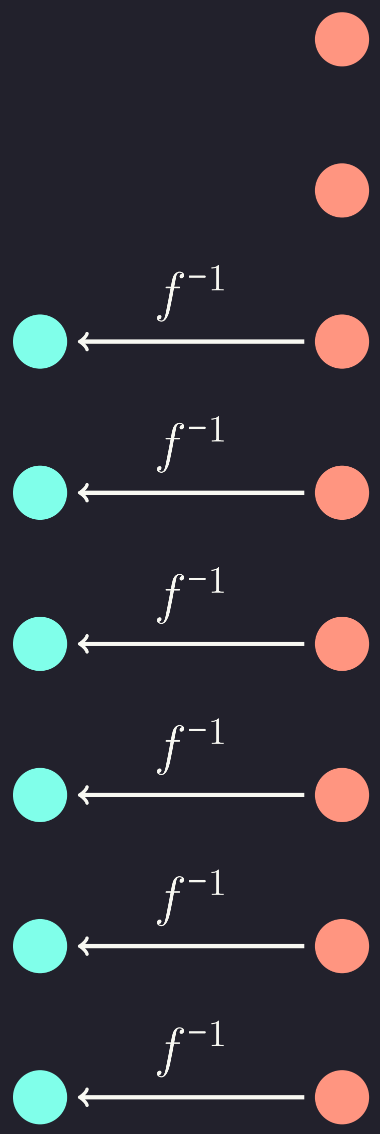
1:1 (Injective)



Formally:

A function f is said to be injective if for $a, b \in D$, with $f(a) = f(b)$ then $a = b$.

Inverse Functions



Identity Function

We the function $f : D \to D$ with $x \mapsto x$ the "Identity" function

$$f(x) = x$$

$$id_D(x) = x$$

Inverting a function

Let $f : D \to R$ be an injective function. Then $g : R \to D$ is an inverse of f if

$$(f \circ g)(r) = r$$

and

$$(g \circ f)(d) = d$$

we write g as f^{-1}.

Tests for injectivity

Let $f : A \to B$

I Algebraic

1. Proving something is NOT injective, find a counterexample:

   find $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$ BUT $x_1 \neq x_2$

2. Proving something is injective: Find a contradiction:

   (a) Assume $3x_1, x_2 \in A, x_1 \neq x_2$ but $f(x_1) = f(x_2)$

   (b) Write $f(x_1) = f(x_0)$

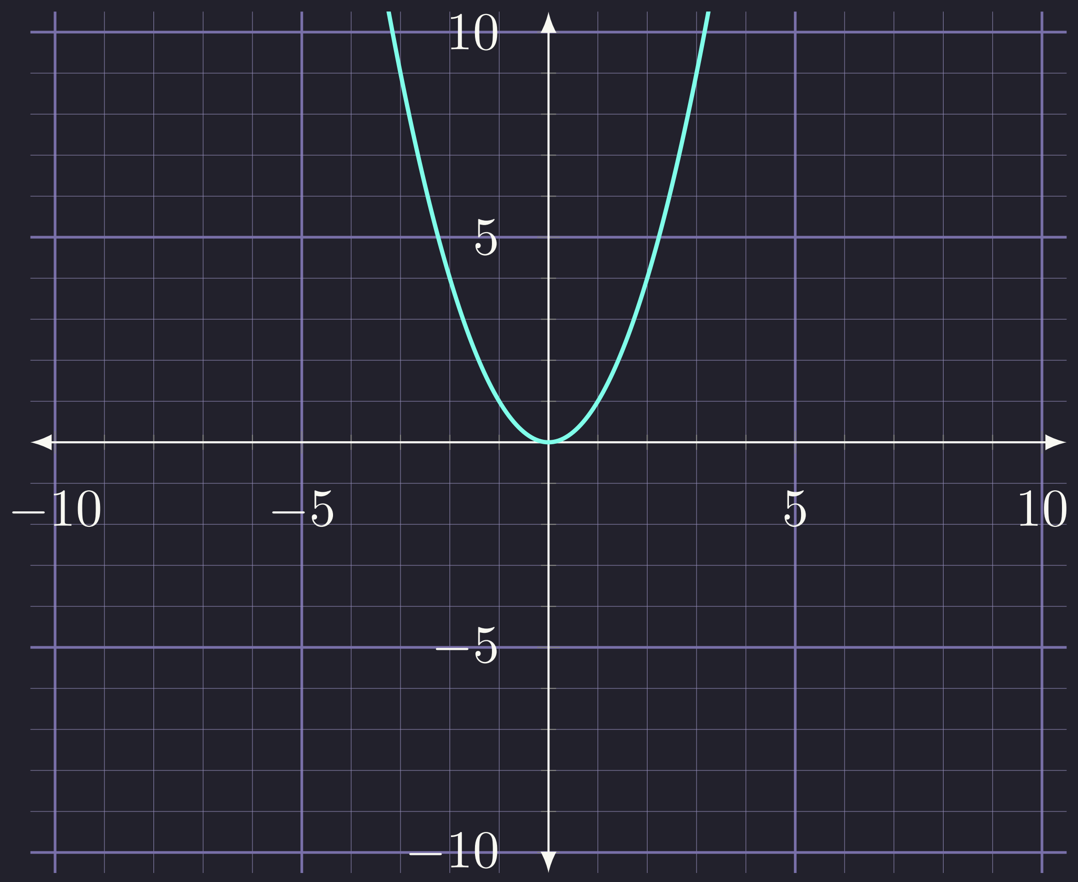
   (c) Simplify until you find a contradiction.

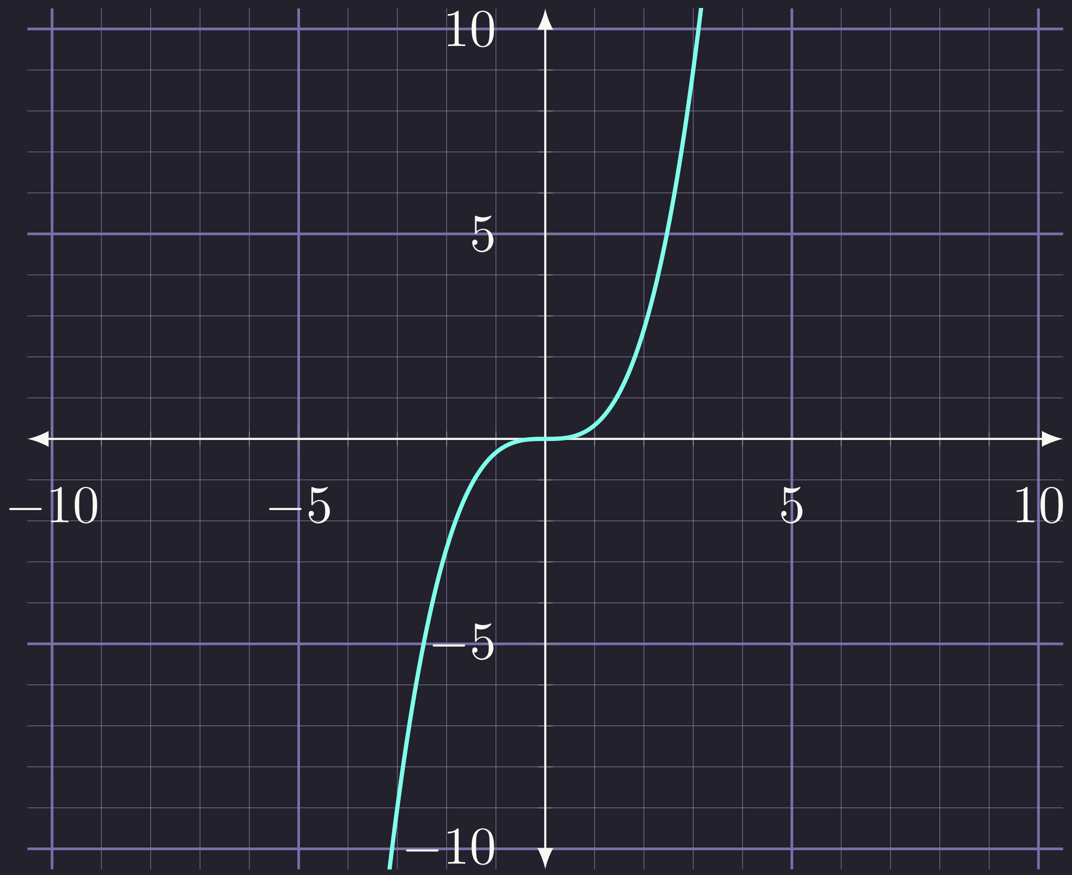
II Graphical

1. Horizontal line test

   (a) Graph the function

   (b) Run a Horizontal line across the graph.




10
5
−10
−5
5
10
−5
−10

Finding an Inverse

1. Algebraic:
   (a) Verify f is injective.
   (b) write $f(x)$ as y
   (c) exchange x and y
   (d) solve for x
   (e) exchange x and y
   (f) write y as f^{-1}

Example:

$f(x) = e^x$

Example:

$f(x) = \frac{x+1}{x-1}$

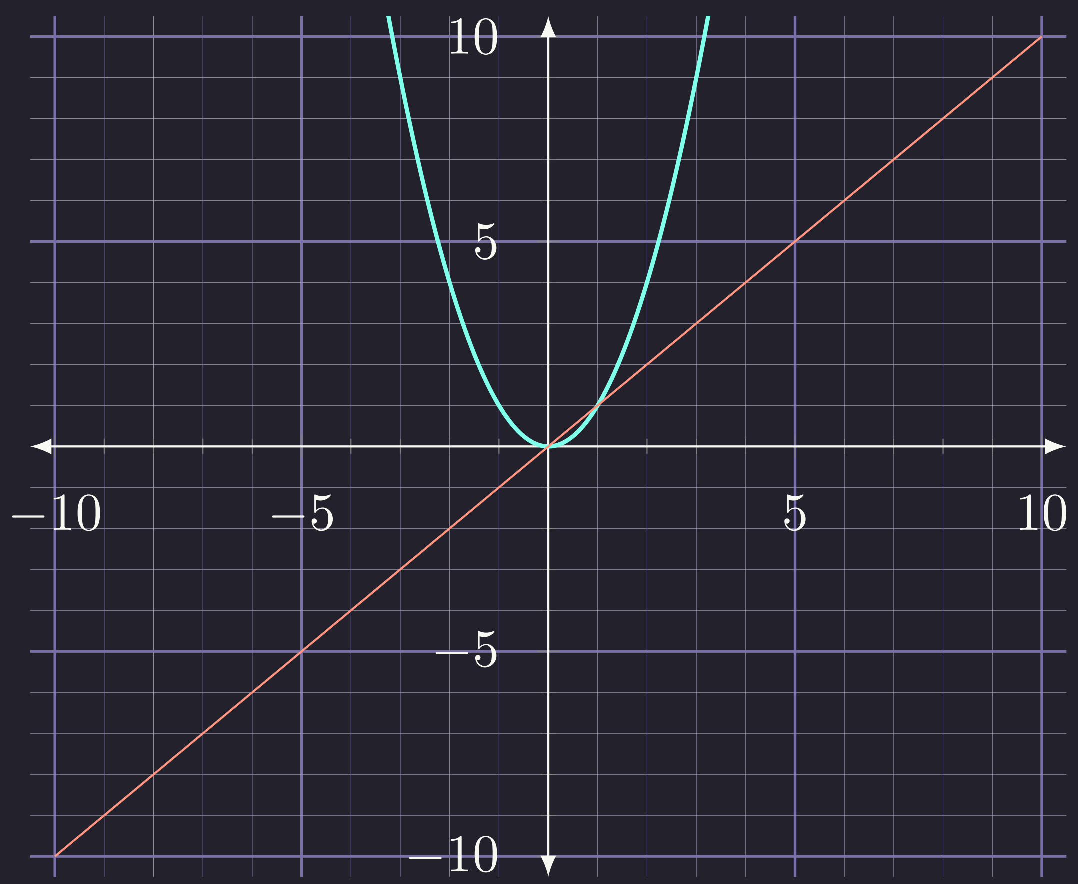
2. Geometric:

   (a) Graph f

   (b) Verify f is injective.

   (c) Reflect f across id_D

Example:



Example: