Qualifying Exam —Analysis
Summer 2020

Professors Ionut Chifan and Lihe Wang

Rules of the exam

- You have 180 minutes to complete this exam.
- The exam contains a section of 5 problems in real analysis and a section of 5 problems in complex analysis. For maximum points you must submit solutions for 7 problems, at least 3 from each section.
- Please mark the problems to be graded on the first column of the grading table on page 2.
- Show your work! – any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- Do not forget to write your name, see page 2.
- You are not allowed to use a calculator, cell phone, ipad or any other internet browser device during the exam.

Good luck!

NAME (*PRINT*): ________________________________

Mark in the first column below which problems should be graded!

Your Choice	**Problem**	**Points**	**Your Score**
	R - I	25	
	R - II	25	
	R - III	25	
	R - IV	25	
	R - V	25	
	C - I	25	
	C - II	25	
	C - III	25	
	C - IV	25	
	C - V	25	
	Total	175	

Real Analysis

R - I: Solve at your choice ONE of the following problems:

a) Suppose for all any $x \in (0,1)$ and $\varepsilon > 0$ there exists $0 < r < \varepsilon$, such that $\int_{x-r}^{x+r} f(x)dx \geq 2r$. Show that $f \geq 1$ a.e for $x \in [0,1]$.

b) Is there a closed, uncountable subset of $\mathbb{R}$ containing no rational numbers? Justify your answer!

c) (True-False) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and denote by $A = \{x \in \mathbb{R} : m(f^{-1}(\{x\})) > 0\}$. Then $m(A) = 0$. If you believe is true provide a proof otherwise supply a counterexample.

R - II: (True-False) If f is integrable on $\mathbb{R}$ then $\lim_{x\to\infty} f(x) = 0$. If you believe it is true provide a proof, otherwise supply a counterexample.

R - III: Suppose E is a measurable set such that $m(E \cap (a,b)) \geq \frac{b-a}{2}$ for all $a < b$. Show that E is the whole axis except a measure zero set.

R - IV: Let A be a measurable subset of $[0,2]$ and define $f : \mathbb{R} \to \mathbb{R}$ by letting $f(x) = m((-\infty, x] \cap A)$, for every $x \in \mathbb{R}$; here m is the Lebesgue measure on $\mathbb{R}$.

1.) Show that f is absolutely continuous on $\mathbb{R}$, calculate f' and $\int_0^3 f'(x)dm(x)$, explaining your reasoning.

2.) Show that for every $0 < b < m(A)$ there exists $x_0 \in \mathbb{R}$ such that $b = m((-\infty, x_0] \cap A)$.

Make sure you state correctly all the results you use in the proof.

R - V: Let $1 \leq p < \infty$ and suppose that $f, f_k \in L^p(\mathbb{R})$ are functions satisfying $\lim_{k\to\infty} f_k(x) = f(x)$, for almost every $x \in \mathbb{R}$. Then prove that $\lim_{k\to\infty} \|f_k - f\|_{L^p} = 0$ if and only if $\lim_{k\to\infty} \|f_k\|_{L^p} = \|f\|_{L^p}$.

Complex analysis

C - I: Solve at your choice ONE of the following problems:

a) If $0 < a < 1$ then show that
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(a\pi)}.$$

b) (True-False) Let f be analytic on the open punctured unit disk $D(0,1) \setminus \{0\}$. Can f' have a polar singularity of order one at 0? If you believe it is true provide a proof, if not supply a counterexample. Also make sure you include all the details in your arguments.

c) Assume that $(a_n) = (1, 1, 2, 3, 5, 8, ...)$ is Fibonacci sequence. Consider the power series $f(z) = \sum_n a_n z^n$. Find the radius of convergence for $f(z)$ and determine a singularity point of the circle of convergence in case it is finite.

C - II: Find all entire functions f of finite order such that f has 2020 roots and f' has 2022 roots, counted with their multiplicities. State clearly all the theorems you are using.

C - III: Assume that f is an entire function such that $|f(z)| = 1$ when $|z| = 1$. Prove that $f(z) = az^n$ for some integer $n \geq 0$ and some $a \in \mathbb{C}$ with $|a| = 1$.

C - IV: Let $\mathcal{F}$ be the class of all $f \in H(D(0,1))$ such that $Ref > 0$ and $f(0) = 1$. Show $\mathcal{F}$ is a normal family.

C - V: Suppose that $f : D(0,1) \to P$ is a conformal mapping onto a regular pentagonal region P, with center at 0 such that $f(0) = 0$. Compute $f^{(2020)}(0)$.
(Here we denoted by $f^{(n)}$ the n-th derivative of f.)

R - I:Solve at your choice ONE of the following problems:

a) Suppose for all any $x \in (0,1)$ and $\varepsilon > 0$ there exists $0 < r < \varepsilon$, such that $\int_{x-r}^{x+r} f(x)dx \geq 2r$. Show that $f \geq 1$ a.e for $x \in [0,1]$.

b) Is there a closed, uncountable subset of $\mathbb{R}$ containing no rational numbers? Justify your answer!

c) (True-False) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and denote by $A = \{x \in \mathbb{R} : m(f^{-1}(\{x\})) > 0\}$. Then $m(A) = 0$. If you believe is true provide a proof, otherwise supply a counterexample.

Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: R - II:(True-False) If f is integrable on $\mathbb{R}$ then $\lim_{x\to\infty} f(x) = 0$. If you believe it is true provide a proof, otherwise supply a counterexample.
Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: R - III: Suppose E is a measurable set such that $m(E \cap (a, b)) \geq \frac{b-a}{2}$ for all $a < b$. Show that E is the whole axis except a measure zero set.
Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: R - IV: Let A be a measurable subset of $[0, 2]$ and define $f : \mathbb{R} \to \mathbb{R}$ by letting $f(x) = m((-\infty, x] \cap A)$, for every $x \in \mathbb{R}$; here m is the Lebesgue measure on $\mathbb{R}$.

1.) Show that f is absolutely continuous on $\mathbb{R}$, calculate f' and $\int_0^3 f'(x)dm(x)$, explaining your reasoning.

2.) Show that for every $0 < b < m(A)$ there exists $x_0 \in \mathbb{R}$ such that $b = m((-\infty, x_0] \cap A)$.

Make sure you state correctly all the theorems you use in the proof.

Solution:

PROBLEM: R - V: Let $1 \leq p < \infty$ and suppose that $f, f_k \in L^p(\mathbb{R})$ are functions satisfying $\lim_{k\to\infty} f_k(x) = f(x)$, for almost every $x \in \mathbb{R}$. Then prove that $\lim_{k\to\infty} \|f_k - f\|_{L^p} = 0$ if and only if $\lim_{k\to\infty} \|f_k\|_{L^p} = \|f\|_{L^p}$. Make sure you state correctly all the theorems you use in the proof.

Solution:

PROBLEM: C - I: Solve at your choice ONE of the following problems:

a) If $0 < a < 1$ then show that
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(a\pi)}.$$

b) (True-False) Let f be analytic on the open punctured unit disk $D(0,1) \setminus \{0\}$. Can f' have a polar singularity of order one at 0? If you believe it is true provide a proof, if not supply a counterexample. Also make sure you include all the details in your arguments.

c) Assume that $(a_n) = (1, 1, 2, 3, 5, 8, ...)$ is Fibonacci sequence. Consider the power series $f(z) = \sum_n a_n z^n$. Find the radius of convergence for $f(z)$ and determine a singularity point of the circle of convergence in case it is finite.

Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: C - II. Find all entire functions f of finite order such that f has 2020 roots and f' has 2022 roots, counted with their multiplicities. State clearly all the theorems you are using.
Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: C - III. Assume that f is an entire function such that $|f(z)| = 1$ when $|z| = 1$. Prove that $f(z) = az^n$ for some integer $n \geq 0$ and some $a \in \mathbb{C}$ with $|a| = 1$.
Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: C - IV. Let $\mathcal{F}$ be the class of all $f \in H(D(0,1))$ such that $Ref > 0$ and $f(0) = 1$. Show $\mathcal{F}$ is a normal family.
Make sure you state correctly all the results you use in the proof.

Solution:

PROBLEM: C - V. Suppose that $f : D(0,1) \to P$ is a conformal mapping onto a regular pentagonal region P, with center at 0 such that $f(0) = 0$. Compute $f^{(2020)}(0)$.
(Here we denoted by $f^{(n)}$ the n-th derivative of f.)
Make sure you state correctly all the results you use in the proof.

Solution:

Ph.D. Qualifying Examination in Analysis

Professors Ionut Chifan and Paul Muhly

August 15, 2018

Instructions. Be sure to put your name on each booklet you use.

This examination has a number of "true-false" questions in it. When a problem is a true-false problem, the operative statement will be preceded by **True-False?**. You are to decide whether it is true or false. If you think it is true, you must provide a proof. If you think it is false, you must provide a counter example or a proof of why it is false. No points will be given for a correct guess that the problem is true or false without any justification. Also, there will be no "Bankruptcy" points given.

The exam is divided into two parts. The first covers real analysis and the second covers complex analysis. Each part has 5 problems. You need only work 4 problems in each part. You must indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus, but still mark the four problems (in each part) that you think are your best work.

Part I

1. Let f be a real-valued function defined on the interval $[0,1]$. **True-False?** If f is *not* of bounded variation on $[0,1]$, then there is a point x_0 in $[0,1]$ such that on any open interval I about x_0, f fails to be of bounded variation on I.

2. A (parametrized) curve C in the plane is given by a pair of real-valued functions f and g defined on an interval $[a,b]$. (So as a point set, $C = \{(f(t), g(t)) \mid t \in [a,b]\}$.) The *length of* C is defined to be

$$\sup\{\sum_{i=1}^{n}[(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2]^{\frac{1}{2}}\},$$

   where the sup is taken over all partitions $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$. **True-False?** The length of C is finite if and only if f and g are of bounded variation.

3. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open sets in $[0,1]$. **True-False?** If the interior of $K := \cap_{n=1}^{\infty} U_n$ is empty, then the Lebesgue measure of K is zero.

4. Let f be a non-negative real-valued function defined on the interval $[0,1]$. **True-False?** f is measurable if and only if there is a (finite or infinite) sequence $\{E_n\}$ of measurable subsets of $[0,1]$ and a sequence of non-negative constants $\{c_n\}$ such that $f(x) = \sum c_n 1_{E_n}(x)$ for every $x \in [0,1]$.

5. Let $\{f_n\}_{n\geq 1}$ be a sequence of non-negative Lebesgue measurable functions defined on $\mathbb{R}$ that converges almost everywhere (with respect to Lebesgue measure m) to the function f. **True-False?** If $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm = 0$, then $f = 0$ a.e. with respect to m.

Part II

6. Let $P_n(z) := \sum_{k=0}^{n-1}(k+1)z^k$, $n = 1, 2, \cdots$ and let $0 < r < 1$. **True-False?** There is an n_0 such that for all $n > n_0$, P_n has no zero in the disc $\{|z| < r\}$.

7. Suppose a is an isolated singularity of f and suppose the real part of f, $\Re(f(z))$, satisfies the inequality $\Re(f(z)) \leq -m \ln|z-a|$ for some positive integer m and for z in some disc centered at a. What kind of singularity is a? (Is it removable, a pole, or essential?)

8. Let C be the circle $x^2 + y^2 = 2x$ oriented in the counter clockwise direction. Calculate $\int_C \frac{dz}{z^4+1}$.

9. Let Ω be a region in the plain, $\mathbb{C}$, and let $\mathcal{F}_\Omega = \cup_{n\geq 0}\{f \mid f|\Omega = z^n\}$. Identify all the regions Ω such that $\mathcal{F}_\Omega$ is a normal family.

10. Suppose the series $\sum_{k=0}^{\infty} b_n z^n$ converges in the open unit disc and that $b_n \geq 0$ for all n. Let

$$\mathcal{F} = \left\{ \sum_{k=0}^{\infty} a_n z^n \mid |a_n| \leq b_n \right\}.$$

True-False? $\mathcal{F}$ is a normal family.

Qualifying Exam —Analysis
Winter 2018

Professors Ionut Chifan and Paul Muhly

Rules of the exam

- You have 180 minutes to complete this exam.
- The exam contains a section of 4 problems in real analysis and a section of 4 problems in complex analysis. For maximum points you must submit solutions for 6 problems. Also you must attempt at least 2 problems in each section.
- Please mark the problems to be graded on the first column of the grading table on page 2.
- Show your work! – any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- Do not forget to write your name, see page 2.
- You are not allowed to use a cell phone or a calculator during the exam.

Real Analysis

R - I: Solve at your choice ONE of the following problems:

a) Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set such that $3\mu(E \cap (a,b)) \leq b - a$ for all $a < b$. Find $\mu(E)$. Make sure you include all the details in your arguments.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Show that f has at most countably many discontinuity points. Conversely, if $A \subset \mathbb{R}$ is a countable subset then there exists a increasing function f whose discontinuity points coincide with A. Make sure you include all the details in your arguments.

R - II: Let $p \geq 1$. Assume that f is an absolute continuous function on any compact interval and moreover $f' \in L^p(\mathbb{R}, \mu)$. Show that

$$\sum_{n \in \mathbb{Z}} |f(n+1) - f(n)|^p < \infty.$$

R - III: Let (X, d) be a compact metric space and let $f : X \to X$ be an isometry (i.e. $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$). Show that f is a homeomorphism (i.e. it is continuous, invertible, and the inverse is continuous as well).

R - IV: Let $f_n \in L^3((0,1))$ nonnegative functions such that $\|f_n\|_3 = 1$ for all n and $f_n \to 0$ almost everywhere as $n \to \infty$. Show that $\int_0^1 f_n d\mu \to 0$ as $n \to \infty$.

Complex analysis

C - I: Solve at your choice ONE of the following problems:

a) Compute the following integral
$$\int_0^\infty \frac{dx}{1+x^7}.$$

b) Construct a conformal map from the unit disk onto the infinite horizontal strip $|Im(z)| < 1$. Make sure you include all the details in your arguments.

c) TRUE-FALSE: Let f be analytic on the open punctured unit disk $\mathbb{D} \setminus \{0\}$. Can f' have a polar singularity of order one at 0? Make sure you include all the details in your arguments.

C - II: Suppose $f : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ is a nonconstant, real-valued harmonic function on the punctured plane. Prove that the image of f is all of $\mathbb{R}$.

C - III: Suppose f is a holomorphic function on $\{z \in \mathbb{C} \,|\, |z| < 1\}$, the open unit disk, with the property that $\mathrm{Re} f(z) > 0$ for every point z in the disk. Prove that $|f'(0)| \leq 2\mathrm{Re} f(0)$.

C - IV: Let $f : \mathbb{C} \to \mathbb{C}$ be an injective holomorphic function. Show there exists $a, b \in \mathbb{C}$ such that $f(z) = az + b$.

Qualifying Exam —Analysis
Winter 2017

Professors Ionut Chifan and Paul Muhly

Rules of the exam

- You have 180 minutes to complete this exam.
- The exam contains a section of 5 problems in real analysis and a section of 5 problems in complex analysis. For maximum points you must submit solutions for 7 problems. Also you must attempt at least 2 problems in each section.
- Please mark the problems to be graded on the first column of the grading table on page 2.
- Show your work! – any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- Do not forget to write your name, see page 2.
- You are not allowed to use a cell phone or a calculator during the exam.

Real Analysis

R - I: Solve at your choice ONE of the following problems:

a) Let E be a measurable set such that $0 < \mu(E) < \infty$. Show that the set $E - E = \{x - y \,:\, x, y \in E\}$ contains an nonempty open interval.

b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is measurable then the set $\{x \in \mathbb{R} \,:\, \mu(f^{-1}(x)) > 0\}$ has measure zero.

R - II: Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesque integrable on the real line. Show that

$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| d\mu(x) = 0.$$

R - III: Let (X, d) be a compact metric space and let $f : X \to X$ be a function such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Show that f has a unique fixed point.

R - IV: Let $f : [0, 1] \to \mathbb{R}_+$ be an integrable function and let $F_n \subseteq [0, 1]$ be a sequence of measurable sets such that $\int_{F_n} f d\mu \to 0$, as $n \to \infty$. Show that $\mu(F_n) \to 0$, as $n \to \infty$.

R - V: Let $F_k \subset [0, 1]$, $k \in \mathbb{N}$ be measurable sets, and there exists $\delta > 0$ such that $m(F_k) \geq \delta$ for all k. Assume the sequence $a_k \geq 0$ satisfies

$$\sum_{k=1}^{\infty} a_k \chi_{F_k}(x) < \infty \text{ for a.e. } x \in [0, 1].$$

Show that

$$\sum_{k=1}^{\infty} a_k < \infty.$$

Make sure you include all the details in your arguments.

Complex analysis

C - I: Define $D = \{z \in \mathbb{C} : |z| < 1\}$, $D_+ = \{z \in \mathbb{C} : |z - i|^2 < 2\}$, $D_- = \{z \in \mathbb{C} : |z + i|^2 < 2\}$, and let $\Omega = D_+ \cap D_-$. Construct a bi-holomophic map from Ω to D.

C - II: Let $f \in \mathbb{C}[z]$ be a polynomial of degree n. Let $\alpha_1, \ldots, \alpha_n$ be roots of $f(z)$ and let $\beta_1, \ldots, \beta_{n-1}$ be roots of $f'(z)$.

1. If for all $i \in \{1, \ldots, n\}$ we have $Re(\alpha_i) > 0$ then prove that $Re(\beta_j) > 0$ for all $j \in \{1, \ldots, n-1\}$;
2. If for all $i \in \{1, \ldots, n\}$ and $|\alpha_i| < 1$ then prove that $|\beta_j| < 1$ for all $j \in \{1, \ldots, n-1\}$.

C - III: Let $f(z)$ be a non-constant analytic function on $D_2 = \{z \in \mathbb{C} : |z| < 2\}$. If $|f(z)| \equiv 1$ for all z such that $|z| = 1$, then prove that f has at least one zero in $D = \{z \in \mathbb{C} : |z| < 1\}$.

C - IV: Compute the following integral

$$\int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^3 dx.$$

C - V: Show that all the roots of the equation $e^z = 3z^2$ in $D = \{z \in \mathbb{C} : |z| < 1\}$ are real.

Ph.D. Qualifying Examination in Analysis
Professors Bor-Luh Lin and Paul S. Muhly

August 24, 2007

Instructions. Be sure to put your name on each booklet you use.

Much of this examination is "true-false". When a problem begins with "True-false", you are to decide if the operative assertion is true or false. If you decide that it is "true", you are to give a proof, while if you decide that it is "false", you are to present a counter example or explain why it is false.

The exam is divided into two parts. The first tests your knowledge of real analysis and the second tests your knowledge of complex analysis. Each part has 5 problems. You need only work 4 problems in each part. Please indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus to be used to help us make our recommendations for the best performance on the Analysis Qual, but we would like you to indicate which four problems you are submitting as representing your best effort.

Part I

1. Let A and B be two subsets of $[0,1]$ whose union is all of $[0,1]$. Show that $m^*(A) \geq 1 - m^*(B)$. (Here, m^* denotes Lebesgue outer measure.)
2. Define the following function on $C[0,1] \times C[0,1]$: $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$. Show that d is a metric on $C[0,1]$ *and* determine whether $C[0,1]$ is complete in this metric.
3. Let A be a subset of $\mathbb{R}$ with the property that for each $\epsilon > 0$ there are (Lebesgue) measurable sets B and C such that
$$B \subset A \subset C$$
and $m(C \cap B^c) < \epsilon$. Show that A is measurable. (Here, m denotes Lebesgue measure.)
4. True-false: Let f be a non-negative continuous function on $\mathbb{R}$ and suppose $\int_{\mathbb{R}} f(x)\, dx < \infty$, then $\lim_{|x| \to \infty} f(x) = 0$.
5. True-False: If $\{f_n\}_{n \geq 1}$ is a sequence of (Lebesgue) measurable functions such that
$$0 \leq f_1 \leq f_2 \leq f_3 \leq \cdots,$$
if $\sup \int_{\mathbb{R}} f_n(x)\, dx < \infty$ and if $f(x) = \lim f_n(x)$ for all x, then $\{x \mid f(x) = \infty\}$ has measure zero.

Part II

1. Calculate the radius of convergence of the power series
$$\sum_{n=0}^{\infty} z^{n^2}.$$
2. Suppose f is analytic in the region $0 < |z| < 1$ and suppose there is a constant K such that
$$|f(z)| \leq K|z|^{-\frac{1}{2}}$$
there. What kind of isolated singularity does f have at zero? (Please prove your answer.)
3. For what values of z does the series
$$\sum_{n=0}^{\infty} \frac{z^n}{1 - z^n}$$
converge? Is the sum an analytic function of z?
4. True-false: There is a non-constant entire function f such $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all z in $\mathbb{C}$.
5. True-false: Suppose $\{f_n\}_{n=0}^{\infty}$ is a sequence of functions defined and analytic in the open unit disc, $|z| < 1$. Suppose also that the values of each f_n are contained in the upper half-plane, i.e., suppose that $\mathrm{Im}(\mathrm{f_n(z)}) \geq 0$ for all n and for all z, $|z| < 1$. Then $\{f_n\}_{n=0}^{\infty}$ is a normal family.

Ph.D. Qualifying Examination in Analysis

Professors Bor-Luh Lin and Paul S. Muhly

August 19, 2005

Instructions. Be sure to put your name on each booklet you use.

This examination is a true-false test. Each problem contains a statement that is either true or false. If you believe a statement is true, you must indicate so and give a proof. If you think it is false, you must indicate so and present a counter example.

The exam is divided into two parts. The first covers real analysis and the second covers complex analysis. Each part has 5 problems. You need only work 4 problems in each part. You must indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus.

Part I

1. Let f be a bounded measurable function on the interval $[0,1]$ and let $\epsilon > 0$ be given. Then there is a step function σ on $[0,1]$ such that $|f(x) - \sigma(x)| < \epsilon$ for all $x \in [0,1]$.

2. If $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence of nonnegative, Lebesgue integrable functions on $\mathbb{R}$ such that $\{f_n\}_{n\in\mathbb{N}}$ converges to zero pointwise on $\mathbb{R}$, then $\int_{\mathbb{R}} f_n(x)\, dx \to 0$.

3. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence of Lebesgue measurable functions defined on $[0,1]$ and for each $n \in \mathbb{N}$ and $x \in [0,1]$, let $F_n(x) = \int_0^x f_n(t)\, dt$. Then the sequence $\{F_n\}_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$.

4. The composition of two absolutely continuous functions on $\mathbb{R}$ is absolutely continuous; i.e., if f and g are two absolutely continuous functions defined on $\mathbb{R}$, then $f \circ g$ is absolutely continuous.

5. If f is a continuous, non-decreasing function defined on $[0,1]$ and if $E \subseteq [0,1]$ is a set of Lebesgue measure zero, then $f(E)$ is a set of Lebesgue measure zero.

Part II

6. The equation $\sin(z) = 2$ has no solutions in the complex plane.

7. Suppose f is analytic in a region G (in the complex plane) and that for some positive integer n, the n^{th} derivative of f achieves its maximum modulus at a point z_0 in G. Then f is a polynomial of degree at most n.

8. Let G be a region in the complex plane and let z_0 be a point in G. Suppose that f is a function defined and analytic on $G \setminus \{z_0\}$ and that f maps $G \setminus \{z_0\}$ into the upper half-plane. Then z_0 is a removable singularity of f.

9. The function $f(z) = \csc(z)$ has a simple pole at $z = 0$ and its residue there is 1.

10. Let G be the open upper half-plane and for $z \in G$ define

$$\psi_n(z) := \exp\{\frac{i-(z-n)}{i+(z-n)}\},$$

for $z \in G$ and $n \in \mathbb{N}$. Then $\{\psi_n\}_{n\in\mathbb{N}}$ is a normal family in $H(G)$ with no non-constant limit points.

Ph.D. Qualifying Examination in Analysis

Professors Ionut Chifan and Paul Muhly

January 9, 2019

Instructions. Be sure to put your name on each booklet you use.

This examination has a number of “true-false" questions in it. When a problem is a true-false problem, the operative statement will be preceded by **True-False?**. You are to decide whether it is true or false. If you think it is true, you must provide a proof. If you think it is false, you must provide a counter example or a proof of why it is false. No points will be given for a correct guess that the problem is true or false without any justification. Also, there will be no “Bankruptcy” points given.

The exam is divided into two parts. The first covers real analysis and the second covers complex analysis. Each part has 5 problems. You need only work 4 problems in each part. You must indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus, but still mark the four problems (in each part) that you think are your best work.

Part I

1. Let f be a real-valued continuous function mapping $[0, 1]$ to $[0, 1]$. **True-False?** f is absolutely continuous if and only if f maps Lebesgue null sets to Lebesgue null sets.
2. Suppose f is a non-negative Lebesgue integrable function defined on $[0, 1]$. **True-False?** There is a Lebesgue measurable set $E \subseteq [0, 1]$ such that $f = 1_E$ if and only if $\int_0^1 f^n \, d\mu = \int_0^1 f \, d\mu$ for all positive integers n.
3. For what values of $\alpha > 0$ is the function $x \to x^\alpha$ absolutely continuous on every bounded subinterval of $[0, \infty)$?
4. Let σ be the function defined by the formula
$$\sigma(x) := \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$
and let σ^* be the outer measure determined by σ. **True-False** Every subset of $\mathbb{R}$ is measurable with respect to σ^*.
5. **True-False** The function
$$f(x) := \sum_{k=1}^{\infty} \sin(kx)/k^m$$
is of bounded variation over every finite interval whenever $m > 2$.

Part II

6. Suppose f is holomorphic in the open unit disc $\mathbb{D}$. Suppose also that for each $z \in \mathbb{D}$ there is an integer $n(z)$ such that the derivative $f^{(n(z))}$ vanishes at z. **True-False?** f must be a polynomial.
7. **True-False?** There is a holomorphic function f on the closed unit disc such that $f(\frac{1}{n}) = \frac{1}{n+2}$, $n \geq 1$.
8. A function f defined and analytic on a region G is said to have a *fixed point* z in G if $f(z) = z$. If f is analytic in a region that contains the closed unit disc and if $|f(z)| < 1$ for all z, $|z| = 1$, how many fixed points does f have in the open unit disc?
9. **True-False?** The function of r
$$\varphi(r) := \int_{|z|=r} \frac{\sin z}{z^2 + 1} \, dz, \quad r \neq 1$$
can be extended to a continuous function defined on all of $[0, \infty)$.

10. Recall that a function f defined on the extended complex plane $\widehat{\mathbf{C}} := \mathbb{C} \cup \{\infty\}$ is said to be meromorphic on $\widehat{\mathbf{C}}$ in case f is meromorphic on $\mathbb{C}$ and $f(\frac{1}{z})$ has a non-essential singularity at $z = 0$. Show that a nonconstant function that is meromorphic on $\widehat{\mathbb{C}}$ has the same number of zeros and poles in $\widehat{\mathbf{C}}$.

Ph.D. Qualifying Exam in Analysis, by Paul Muhly and Lihe Wang

January 2020

The exam has two parts: real analysis and complex analysis. Each part has five problems. Please solve any four problems in each part.

If you want to try the fifth problem in either part, the exam will be scored on the best four scores from each part.

Real Analysis. Solve any four problems from this group of five problems.

1. Suppose $f_n(x), f(x)$ are functions in $L^1([0,1])$. Suppose $f_n(x) \to f(x)$ for every x, is it true that $\int_{(0,1)} f_n(x)dx \to \int_{(0,1)} f(x)dx$? Give a proof or a counterexample.

2. Suppose $f \in L^1(R^1)$ is it true that $\lim\limits_{x\to\infty} f(x) = 0$? Give a proof or a counterexample.

3. Suppose that a measurable set $E \subset (0,1)$ is such that $m(E \cap (r,s)) \geq \frac{s-r}{4}$ for all rational $0 < r < s < 1$. Compute $m(E)$.

4. Suppose f_k, f are functions in $L^2([0,1])$ such that $f_k(x) \to f(x)$, a.e., and that $\|f_k\|_{L^2} \to \|f\|_{L^2}$. Is it true that $f_k \to f$ in L^2? Give a proof or a counterexample.

5. Suppose f is absolutely continuous and that $f' \in L^1(R^1)$. Prove that $\lim\limits_{x\to+\infty} f(x)$ exists. Does the limit have to be zero?

Complex Analysis. Solve any four problems in this group of five problems.

1. Find all entire functions f for which there is a positive number C such that $|f(z)| \leq C(1 + |z|)$ for all z.

2. Prove the uniform limit of a sequence of holomorphic functions is holomorphic.

3. Suppose f is holomorphic on the unit disc and satisfies the inequality $|f(z)| \leq (1-|z|)^{-1}$ for all z in the disc. Prove that $|f'(z)| \leq C(1-|z|)^{-2}$ for some constant C.

4. Suppose f is entire and that its imaginary part satisfies the inequality $Im(f) \geq 0$. Show f is a constant.

5. Suppose f is analytic in the annular region $1 \leq |z| \leq 2$. Suppose also that $|f| \leq 1$ on the circle $|z| = 1$ and that $|f| \leq 4$ on the circle $|z| = 2$. Show that $|f(z)| \leq |z|^2$ for all $z, 1 \leq |z| \leq 2$.

Qualifying Exam —Analysis
Summer 2017

Rules of the exam

- You have 180 minutes to complete this exam.
- The exam contains a section of 4 problems in real analysis and a section of 4 problems in complex analysis. For maximum points you must submit solutions for 6 problems. Also you must attempt at least 2 problems in each section.
- Please mark the problems to be graded on the first column of the grading table on page 2.
- Show your work! – any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- Do not forget to write your name, see page 2.
- You are not allowed to use a cell phone or a calculator during the exam.

Good luck!

Real Analysis

R - I: Solve at your choice ONE of the following problems:

a) Let E be the subset of all elements in $[0,1]$ which do not contain the digits 3 and 9 in their decimal expansion. Is E Lebesgue measurable? If yes find its measure.

b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is measurable then the set $\{x \in \mathbb{R} : \mu(f^{-1}(x)) > 0\}$ has measure zero.

R - II: Let $f_n : [-1,1] \to \mathbb{R}$ be a sequence of Lebesque measurable functions that converges to f almost everywhere. If $\int_{[-1,1]} |f_n|^4 d\mu \leq 1$ for every n then show that $\int_{[-1,1]} |f_n - f| d\mu$ converges to 0.

R - III: Let (X,d) be a compact metric space and let $f : X \to X$ be a continuous function. Show that there exists $A \subseteq X$ a compact subset such that $f(A) = A$.

R - IV: Let $F_k \subset [0,1]$, $k \in \mathbb{N}$ be measurable sets, and there exists $\delta > 0$ such that $m(F_k) \geq \delta$ for all k. Assume the sequence $a_k \geq 0$ satisfies

$$\sum_{k=1}^{\infty} a_k \chi_{F_k}(x) < \infty \text{ for a.e. } x \in [0,1].$$

Show that

$$\sum_{k=1}^{\infty} a_k < \infty.$$

Make sure you include all the details in your arguments.

Complex analysis

C - I: Solve at your choice ONE of the following problems:

a) Compute the following integral

$$\int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^3 \, dx.$$

b) Let $\mathcal{P}$ be the open region determined by the pentagon with vertices at ω^k where $k = \overline{0,4}$ and $\omega = \cos(2\pi/5) + i\sin(2\pi/5)$. Let $f : \overline{\mathcal{P}} \to \mathbb{C}$ be a continuous function that is analytic on $\mathcal{P}$. Assume that for every $t \in (0,1)$ we have that $\lim_{z \to \frac{2-t+t\omega}{2}} f(z) = \lim_{z \to \frac{2-t\omega^2+t\omega^3}{2}} f(z) = 0$. Find f.

C - II: If we denote by $\mathcal{H} = \{z \in \mathbb{C} : |z - i| > 1\}$ then describe all analytic, bijective maps $f : \mathcal{H} \to \mathcal{H}$.

C - III: Let f be a non-constant, analytic function on the unit disk $\mathbb{D}$. If there exists a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=2}^{\infty} n|a_n| \leq |a_1|$ then show that f is injective.

C - IV: Let f be an analytic function on the open unit disk $\mathbb{D}$. Assume that for every $z \in (-1, 0]$ the power series expansion around z has a vanishing coefficient. Show that f is a a polynomial function.

Ph.D. Qualifying Exam in Analysis, by Paul Muhly and Lihe Wang

August 2019

The exam has two parts: real analysis and complex analysis. Each part has five problems and solve any four problems from five problems in each part.

If you want to try the fifth problems, the exam will be graded as the best four scores from the five.

Real Variables. Solve any four problems in these five problems from real variables.

1. Suppose $f_n(x), f(x)$ are measurable functions on $(0,1)$. Suppose $f_n(x) \to f(x)$, it is true that $\int_{(0,1)} f_n(x)dx \to \int_{(0,1)} f(x)dx$? Give a proof or a counterexample.

2. Suppose f is a function defined $[0,1]$ and suppose that $|f(x) - f(y)| \leq M|x-y|$ for all $x, y \in [0,1]$ where M is a fixed constant. Prove that f is differentiable a.e. and $|f'(x)| \leq M$.

3. Show that there is no measurable set such that that $m(E \cap (a,b)) = \frac{b-a}{2}$ for all $a < b$, here $m(A)$ is the Lebesgue measure of A.

4. Suppose f_k, f are functions in $L^1([0,1])$ such that $f_k(x) \to f(x)$, a.e., and that $\|f_k\|_{L^1} \to \|f\|_{L^1}$. Then $f_k \to f$ in L^1.

5. If $f \in L^1(R^1)$, show that $\sum_{n=-\infty}^{\infty} f(x+n)$ is convergent e.a to a function which has period 1.

Complex Variables. Solve any four problems in these five problems from complex variables.

1. Find all entire functions with the condition that $|f(z)| \leq A(1 + |z|^2)$ for some constant A.

2. Compute $\int_{\partial D(0,1)} (1 + \bar{z})^5 dz$.

3. Suppose f is holomorphic in the punctured disk $D(0, 1)\backslash\{0\}$. Suppose also that $|f| \leq \frac{1}{|z|^{0.5}}$. Prove f is differentiable at 0.

4. Suppose f is entire so that $Re(f) \geq 0$. Show f is a constant.

5. Suppose f is holomorphic in the unit disk. Prove that there exists z_n in the disk with $|z_n| \to 1$ that $f(z_n)$ is bounded.

Ph.D. Qualifying Examination in Analysis

Professors Bor-Luh Lin and Paul S. Muhly

August 18, 2006

Instructions. Be sure to put your name on each booklet you use.

Much of this examination is "true-false". When a problem begins with "True-false", you are to decide if the operative assertion is true or false. If you decide that it is "true", you are to give a proof, while if you decide that it is "false", you are to present a counter example.

The exam is divided into two parts. The first covers real analysis and the second covers complex analysis. Each part has 5 problems. You need only work 4 problems in each part. You must indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus.

Part I

1. True-false. Lebesgue measure is *continuous* in the sense that the Lebesgue measure of the closure of a set coincides with the Lebesgue measure of the set.

2. True-false. Let I_1 and I_2 be two disjoint open intervals, and for $i = 1, 2$, let A_i be an arbitrary subset of I_i. Then $m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2)$, where m^* denotes Lebesgue outer measure.

3. True-false. Let $\{f_n\}_{n\geq 0}$ be a sequence of non-negative integrable functions defined on $\mathbb{R}$ such that
$$0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \dots$$
and such that the sequence of numbers $\{\int_{\mathbb{R}} f_n(x)\,dx\}$ is bounded. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Then $f(x) < \infty$ for almost all x.

4. True-false. Let σ be the function on $\mathbb{R}$ that is zero to the left of 1, 1/2 for $1 \leq x < 2$, 3/4 for $2 \leq x < 3$, 7/8 for $3 \leq x < 4$, etc. (Thus, σ jumps by 2^{-n} at n for $n = 1, 2, \dots$, is constant between any two consecutive integers and is continuous from the right.) If $f(x) = x$ on $\mathbb{R}$, then f is integrable with respect to the Lebesgue-Stieltjes measure determined by σ.

5. Let $\{f_n\}$ be a uniformly bounded sequence of measurable functions defined on the interval $[0, 1]$ and let
$$F_n(x) = \int_0^x f_n(t)\,dt \qquad 0 \leq x \leq 1.$$
Show that there is a subsequence $\{F_{n_k}\}$ that converges uniformly on $[0, 1]$.

Part II

1. True-false. Suppose f is analytic in the region $0 < |z| < 1$ and suppose that for each r, $0 < r < 1$, the integral $\int_{C_r} f(z)\,dz = 0$, where C_r is the circle $|z| = r$. Then f is analytic on the open unit disc.

2. Suppose f is analytic in the annular region $1 - \epsilon < |z| < 2 + \epsilon$ for some positive ϵ. Suppose also that $|f| \leq 1$ on the circle $|z| = 1$ and that $|f| \leq 4$ on the circle $|z| = 2$. Show that $|f(z)| \leq |z|^2$ for all z, $1 < |z| < 2$

3. True-false. Let $\mathfrak{G}$ be a domain and let z_0 be a point in $\mathfrak{G}$. Suppose f is analytic in $\mathfrak{G}/\{z_0\}$ and that f takes values in the upper half-plane. Then z_0 is a removable singularity of f.

4. Find the Laurent series representation of the function
$$f(z) = \frac{1}{z^2(1-z)}$$
that is valid in the region $1 < |z| < \infty$.

5. Let f be analytic in the open unit disc $\mathbb{D}$ and let the Taylor series expansion for f be

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Suppose

(a) $f(\mathbb{D}) \subseteq \mathbb{D}$

(b) $a_0 = 0$

(c) $|a_1| = 1$

Calculate $\sup\{|a_n| \mid n \geq 2\}$.