Qualifying Examination Spring 2006, Section on Partial Differential Equations

April 5, 2006

(Solve three of the problems.)

1. Solve the initial value problem

$$u_t + x u_x = u$$

with $u(x,0) = x^2$. Describe and draw the characteristics.

2. Using separation of variables, find the eigenfunctions of the Laplace operator with Dirichlet boundary conditions on the rectangle $[0,\pi] \times [0,2\pi]$.

3. Assume u is twice continuously differentiable on $[0,1]^n \subset R^n$, that u is zero on the boundary of that domain and $|\Delta u| \leq 1$. Use the maximum principle to give an estimate of the size of the function u.

4. What is the proper weak solution (i.e. the solution fulfilling the Lax entropy condition) of the equation

$$u_t + u^3 \cdot u_x = 0$$

for the initial values

$$f_1(x) = \begin{cases} -2 \text{ for } x > 0 \\ 0 \text{ for } x \leq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 \text{ for } x > 0 \\ -2 \text{ for } x \leq 0 \end{cases} \quad ?$$

5. Compute the Fourier series

$$\sum_{k=0} a_k \cos(kx)$$

for the function

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in [0,\pi/2] \\ -1 & \text{for} \quad x \in (\pi/2,\pi] \end{cases}$$

on the interval $[0, \pi]$. Also solve the heat equation $u_t(x,t) = u_{xx}(x,t)$ on the square $[0,\pi] \times [0,\infty)$ with the initial value $u(x,0) = f(x)$ and the boundary condition $u_x(0,t) = u_x(\pi,t) = 0$. What does it converge to as $t \to \infty$?

Qualifying Exam: PDE, Fall, 2017

Choose any three out of the five problems. Please indicate your choice.
Show all your work.

1. Find the solution to the initial value problem

$$u_t + (\frac{u^2}{2})_x = 0, \quad x \in R, \ t > 0,$$

$$u(x,0) = \begin{cases} 2 & x \le 0 \\ 2 - x & 0 < x \le 2 \\ 1 & 2 < x \end{cases}.$$

2. Show that if the C^1 initial data $f(x)$ has $f'(x_0) < 0$ for some x_0 and that $F''(u) \geq 1$ for all u, then the C^1 solution of

$$u_t + F(u)_x = 0, \quad u(x, 0) = f(x)$$

must break down at some time $t > 0$.

3. Let $u(x,t)$ and $v(x,t)$ be solutions of the equation

$$u_t - ku_{xx} = q(x,t), \quad x \in R, \quad 0 < t \leq T$$

satisfy

$$u(x,0) = f(x),\ v(x,0) = g(x),\ x \in R$$

respectively, where $k > 0, T > 0$, and $f(x), g(x), q(x,t)$ are continuous and bounded functions. Suppose that $u(x,t)$ and $v(x,t)$ are continuous and bounded on $x \in R$, $0 \leq t \leq T$, and that

$$f(x) \leq g(x),\ x \in R.$$

Show that $u(x,t) \leq v(x,t)$ for $x \in R$, $0 \leq t \leq T$.

4. Solve the initial-boundary-value problem

$u_t = u_{xx}, \;\; 0 < x < 1, \; t > 0,$

$u(0,t) = 0, \;\; u(1,t) = 1, \;\; t > 0,$

$u(x,0) = x^2, \;\; 0 < x < 1.$

Also find a steady-state solution $U(x)$ of the above problem.

5. Consider the damped wave equation problem

$$u_{tt} + du_t - c^2 u_{xx} = 0, \quad x \in R, \ t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in R$$

where $c > 0$, $d > 0$ and f, g are smooth functions with compact support.
Define energy as $e(t) = \dfrac{1}{2}\displaystyle\int_R (u_t^2(x,t) + c^2 u_x^2(x,t))dx$.
Show that the energy is nonincreasing as t increases.

Qualifying Exam: PDE, Spring, 2018

Choose any three out of the five problems. Please indicate your choice. Show all your work.

1. (i) Solve the initial value problem

$$u_t - \frac{1}{2x} u_x = -u, \quad x > 0, \; t > 0,$$

$$u(x, 0) = \frac{1}{2 + x^2}, \quad x \geq 0.$$

Over what region in the first quarter of the x-t plane does the solution exist? Draw the characteristics on the x-t plane where the solution exists.

(ii) Write an upwind scheme for the above problem. What is the CFL condition for the scheme?

2. Find a weak solution for the nonlinear conservation law with the following Riemann initial data such that the discontinuous solutions satisfy the entropy condition

$$u_t + (u(1-u))_x = 0, \quad x \in \mathbb{R}, \ t > 0,$$

(i) with initial data

$$u(x,0) = \begin{cases} 3 & x < 0 \\ 2 & x \geq 0. \end{cases}$$

(ii) with initial data

$$u(x,0) = \begin{cases} 2 & x < 0 \\ 3 & x \geq 0. \end{cases}$$

3. Let $u(x,t)$ and $v(x,t)$ be solutions of the equation

$$u_t - u_{xx} = 2, \quad x \in \mathbb{R}, \quad 0 < t \le T$$

satisfying

$$u(x,0) = f(x),\ v(x,0) = g(x),\ x \in \mathbb{R}$$

respectively, where $T > 0$, and $f(x), g(x)$ are continuous and bounded functions. Suppose that $u(x,t)$ and $v(x,t)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \le t \le T$ and that

$$f(x) \le g(x),\ x \in \mathbb{R}.$$

Show that $u(x,t) \le v(x,t)$ for $x \in \mathbb{R}$, $0 \le t \le T$.

4. Solve the initial-boundary-value problem

$u_t = u_{xx}, \;\; 0 < x < 1, \; t > 0,$

$u(0,t) = 1, \;\; u(1,t) = 3, \;\; t > 0,$

$u(x,0) = x, \;\; 0 < x < 1.$

and also find the steady-state solution $U(x)$ of the above problem.

5. Solve the initial value problem of the wave equation

$$u_{tt} - u_{xx} = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = -e^{-x^2}, \quad u_t(x,0) = 6xe^{-x^2}, \quad x \in \mathbb{R}.$$

Qualifying Exam: PDE, Fall, 2019

Choose any **Four** out of the five problems. Please indicate your choice.
Show all your work.

1. Find a weak solution for the nonlinear conservation law with the following Riemann initial data such that the discontinuous solutions satisfy the entropy condition

$$u_t + (u(2-u))_x = 0, \quad x \in \mathbb{R}, \ t > 0,$$

(i) with initial data

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0; \end{cases}$$

and

(ii) with initial data

$$u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

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2. (i) Solve the initial value problem

$$u_t - 3x^2 u_x = -u, \quad x \in \mathbb{R}, \ t > 0,$$
$$u(x, 0) = e^{-2x^2}, \quad x \in \mathbb{R}.$$

(ii) Draw the characteristics and find the region in the x-t plane where the solution exists.

(iii) Write an upwind scheme for the above problem.

3. Let both $u(x,t)$ and $v(x,t)$ be solutions of the equation

$$u_t - ku_{xx} = q(x,t), \quad x \in \mathbb{R}, \quad 0 < t \le T$$

satisfying

$$u(x,0) = f(x) \text{ and } v(x,0) = g(x),\ x \in \mathbb{R}$$

respectively, where $k > 0$, $T > 0$, $f(x), g(x)$ and $q(x,t)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \le t \le T$.

Suppose that $u(x,t)$ and $v(x,t)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \le t \le T$, and that

$$f(x) \le g(x),\ x \in \mathbb{R}.$$

Show that $u(x,t) \le v(x,t)$ for $x \in \mathbb{R}$, $0 \le t \le T$.

4. (i) Solve the initial-boundary-value problem

$u_t = u_{xx}, \quad 0 < x < 1, \; t > 0,$

$u(0,t) = 1, \quad u(1,t) = 3, \quad t > 0,$

$u(x,0) = x^2 + x + 1, \quad 0 \le x \le 1.$

(ii) What is the limit of the solution as $t \to +\infty$?

5. Solve the following initial-boundary-value problem

$u_{tt} - u_{xx} = 0, \;\; x > 0, \; t > 0,$

$u(x,0) = f(x), \;\; u_t(x,0) = g(x), \;\; x \geq 0,$

$u_x(0,t) = 1, \;\; t > 0$

where f and g are smooth functions satisfying $f'(0) = 1$ and $g'(0) = 0$.

Qualifying Exam: PDE, Spring, 2019

Choose any **Four** out of the five problems. Please indicate your choice.
Show all your work.

1. Show that if the C^1 initial data $f(x)$ has $f'(x_0) < 0$ for some x_0, then the C^1 solution of

$$u_t + (u^2)_x = 0, \quad x \in \mathbb{R}, t > 0, \quad u(x,0) = f(x)$$

must break down at some time $t > 0$.

2. (i) Solve the initial value problem

$$u_t + x^2 u_x = -u, \quad x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = x^2, \quad x \in \mathbb{R}.$$

(ii) Over which region in the x-t plane does the solution exist?

(iii) Write an upwind scheme for the above problem.

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3. Solve the following initial boundary value problem

$u_t - u_{xx} = 0, \quad x > 0, \ t > 0,$

$u(x, 0) = f(x), \quad x \geq 0,$

$u(0, t) = 1, \quad t \geq 0$

where $f \in C^2[0, +\infty)$ is bounded and $f(0) = 1$.

4. (i) Solve the initial-boundary-value problem

$u_t = u_{xx}, \quad 0 < x < 1, \ t > 0,$

$u(0,t) = 0, \quad u(1,t) = 3, \quad t > 0,$

$u(x,0) = x^2 + 2x, \quad 0 \leq x \leq 1.$

(ii) What is the limit of the solution as $t \to +\infty$?

5. Consider the damped wave equation problem

$$u_{tt} + du_t - c^2 u_{xx} = 0, \quad x \in R, \ t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in R$$

where $c > 0$, $d > 0$ and f, g are smooth functions with compact support. Define energy as $e(t) = \dfrac{1}{2}\displaystyle\int_R (u_t^2(x,t) + c^2 u_x^2(x,t))dx$.

Show that the energy is nonincreasing as t increases.

Qualifying Examination on Differential Equations, Fall 2005

August 23, 2005

1 Section on ODE

(Solve three of the problems)

1. Find and classify all equilibria of the system of equations

$$\begin{aligned} x' &= x + x^2 + y + y^2, \\ y' &= -x - x^2 + y + y^2. \end{aligned}$$

2. Prove that the system of equations

$$\begin{aligned} x' &= y + x - x^3, \\ y' &= -x + y - y^3 \end{aligned}$$

has at least one non-constant periodic solution. You may assume that $(0, 0)$ is the only equilibrium point.

3. Consider the ode $y' = f(x)$ with a function $f : \mathbb{R} \to \mathbb{R}$ which is infinitely differentiable. Estimate the truncation error for one step of the second-order Taylor method for this equation.

4. Let y_1 and y_2 be two solutions of the equation $x' = -x^2 + t^2$, and let $y_1(0) = 1, y_2(0) = 2$. Prove that we have $0 < y_1(t) < y_2(t) < y_1(t) + 1$ for all $t > 0$.

2 Section on PDE

(Solve three of the problems)

1. Compute the Fourier series

$$\sum_{k=1}^{\infty} a_k \sin(kx)$$

for the function

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in [0, \pi/2] \\ 0 & \text{for} \quad x \in (\pi/2, \pi] \end{cases}$$

on the interval $[0, \pi]$. Also solve the heat equation $u_t(x,t) = u_{xx}(x,t)$ on the square $[0,\pi] \times [0,\infty)$ with the initial value $u(x,0) = f(x)$ and the boundary condition $u(0,t) = u(\pi,t) = 0$.

2. Using separation of variables, find the eigenfunctions of the Laplace operator with Neumann boundary conditions on the rectangle $[0,1] \times [0,1]$.

3. Let $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$. Show that if $u \in C^2(B) \cap C^0(\overline{B})$, $u(x) = 0$ for $|x| = 1$ and $|\Delta u| \leq K$, then also

$$-\frac{K}{2n} \leq u \leq \frac{K}{2n}.$$

Hint: Use maximum principle for a function $v = u - w$ where $w(x) = 0$ for $|x| = 1, \Delta w = \pm K$. Note that w is a simple polynomial.

4. What is the proper weak solution of the equation

$$u_t + u \cdot u_x = 0$$

for the initial values

$$f_1(x) = \begin{cases} 2 \text{ for } x > 0 \\ 0 \text{ for } x \leq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 \text{ for } x > 0 \\ 3 \text{ for } x \leq 0 \end{cases} ?$$

5. Solve the initial value problem

$$u_t + e^t u_x = u$$

with $u(x,0) = x$. Describe and draw the characteristics .

Qualifying Examination Fall 2006, Section on Partial Differential Equations

August 23, 2006

(Solve three of the problems.)

1. Solve the initial value problem

$$u_t + 3t^2 u_x = u$$

with $u(x,0) = x^2$. Describe and draw the characteristics.

2. Compute the Fourier series

$$\sum_{k=0}^{\infty} a_k \cos(kx)$$

for the function

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in [0, \pi/2] \\ 0 & \text{for} \quad x \in (\pi/2, \pi] \end{cases}$$

on the interval $[0,\pi]$. Also solve the heat equation $u_t(x,t) = u_{xx}(x,t)$ on the square $[0,\pi] \times [0,\infty)$ with the initial value $u(x,0) = f(x)$ and the boundary condition $u_x(0,t) = u_x(\pi,t) = 0$. What does it converge to as $t \to \infty$?

3. What is the proper weak solution (i.e. the solution fulfilling the Lax entropy condition) of the equation

$$u_t + u^9 \cdot u_x = 0$$

for the initial values

$$f_1(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x \le 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 1 \text{ for } x > 0 \\ -1 \text{ for } x \le 0 \end{cases} ?$$

4. Assume $u \in C^2\left\{x \in R^3 \mid |x| \le 1\right\}$, $\Delta u \le 6$ and $u(x) \ge 0$ for $|x| = 1$. How small can $u(0)$ become ?

5. Using separation of variables, find the eigenfunctions of the Laplace operator with Neumann boundary conditions on the rectangle $[0, 3\pi] \times [0, \pi]$.

Qualifying Exam: PDE, Fall, 2008

Choose any three out of the six problems.

1. (i) Solve the initial value problem for the linear equation

$u_t + (x^2 + 1)u_x = 0, \quad x \in R, \ t > 0, \ u(x, 0) = x^2, \quad x \in R.$

(ii) Over what region in the x-t plane does the solution exist? Draw the characteristics on the x-t plane where the solution exists.

2. (i) Find the bounded solution u to the following initial-boundary-value problem

$u_t - u_{xx} = 0, \quad x > 0, \ t > 0,$

$u(x, 0) = f(x), \quad x \geq 0, \ u(0, t) = 2, \quad t \geq 0$

where f is continuous on $[0, +\infty)$ satisfying $f(0) = 2$ and $\sup_{x \geq 0} |f(x)| = M < +\infty$.

(ii) Find the supremum of $|u(x, t)|$ for $x \geq 0$ and $t \geq 0$ in terms of the given data.

3. Compute the Fourier series

$\Sigma_{k=0}^{+\infty} a_k \cos(kx)$

for function

$$f(x) = \begin{cases} 1 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

on the interval $[0, \pi]$. Also solve the heat equation $u_t = u_{xx}$ on $[0, \pi] \times [0, +\infty)$ with the initial value $u(x, 0) = f(x)$ and the boundary conditions $u_x(0, t) = u_x(\pi, t) = 0$. What does the solution converge to as $t \to +\infty$?

4. Solve the following initial-boundary-value problem

$u_{tt} - u_{xx} = 0, \quad x > 0, \ t > 0,$

$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \geq 0,$

$u(0, t) = 0, \quad t > 0$

where f and g are smooth functions satisfying $f(0) = g(0) = 0$.

5. (i) Find a weak solution satisfying the entropy conditions for

$u_t + (\frac{u^2}{2})_x = 0, \quad x \in R, \ t > 0,$

with initial data $u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0 \end{cases}$

and with initial data $u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$.

(ii) Write an upwind scheme for the above problems. What is the CFL condition for the scheme?

6. Consider the wave equation problem

$u_{tt} - c^2 u_{xx} = q(x,t), \quad x \in R, \ t > 0,$

$u(x,0) = 0, \quad u_t(x,0) = 0, \quad x \in R$

where $c > 0$ and

$q(x,t) = \begin{cases} (1 - x^2)\sin t & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$.

Show that $u(x,t) = 0$ for $|x| > ct + 1$.

Qualifying Exam: ODE, Fall, 2008

Please choose 4 out of the 6 problems.

1. In each case, find the value of r at which bifurcations occur, classify types of bifurcations and sketch the bifurcation diagram of fixed points x^* vs r.

(a) $\dfrac{dx}{dt} = r + 2x - x^2$;

(b) $\dfrac{dx}{dt} = rx - x^3$.

2. Consider the flow on a circle given by

$$\frac{d\theta}{dt} = 1 + 2r\cos\theta.$$

(a) Draw a phase portrait on the circle for different cases of the control parameter r.

(b) Find all bifurcation values of r and draw a bifurcation diagram on the $r\theta$-plane.

(c) Compute the oscillation period when the system is an oscillator.

3. Consider the nonlinear system

$$\frac{dx}{dt} = r - x^2, \quad \frac{dy}{dt} = x - y.$$

Assume that $r > 0$.

(a) Find all fixed points and the linearized system at each fixed point.

(b) Find eigenvalues and corresponding eigenvectors for each linearized system.

(c) Classify each fixed point for the linearized system and for the given nonlinear system. Determine their stability.

(d) Sketch a phase portrait of the given nonlinear system.

4. Consider the following model of competition between two species, where $x, y \geq 0$. Find the fixed points, investigate their stability, draw the nullclines and sketch phase portraits. Indicate the basins of attraction of any stable fixed points.

$$\frac{dx}{dt} = x(3 - 2x - y)$$
$$\frac{dy}{dt} = y(2 - x - y).$$

5. Consider the system

$$\frac{d^2x}{dt^2} = x - 4x^3.$$

Find all the equilibrium points and classify them. Find a conserved quantity. Sketch the phase portrait.

6. Show that the system

$$\frac{dx}{dt} = x - y - x(x^2 + y^2)$$
$$\frac{dy}{dt} = x + y - y(2x^2 + y^2)$$

has a periodic solution.

Hint: Rewrite the system in polar coordinates and then construct a trapping region.

Qualifying Exam: PDE, Fall, 2020

Choose any **Four** out of the five problems. Please indicate your choice. Show all your work.

1. Find a weak solution for the nonlinear conservation law with the following Riemann initial data such that the discontinuous solutions satisfy the entropy condition

$$u_t + (u(1+u))_x = 0, \quad x \in \mathbb{R}, \ t > 0,$$

(i) with initial data

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0; \end{cases}$$

and

(ii) with initial data

$$u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

2. (i) Solve the initial value problem

$$u_t + x^2 u_x = -u, \quad x \in \mathbb{R}, \ t > 0,$$

$$u(x,0) = e^{-x^2}, \quad x \in \mathbb{R}.$$

(ii) Draw the characteristics and find the region in the x-t plane where the solution exists.

(iii) Write an upwind scheme for the above problem.

3. Let $u(x,t)$ and $v(x,t)$ be solutions of the heat equation

$$u_t - k u_{xx} = 1, \quad x \in \mathbb{R}, \quad 0 < t \leq T$$

satisfying

$$u(x,0) = f(x) \text{ and } v(x,0) = g(x), \ x \in \mathbb{R}$$

respectively, where $k > 0$, $T > 0$, $f(x)$ and $g(x)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \leq t \leq T$.

Suppose that $u(x,t)$ and $v(x,t)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \leq t \leq T$, and that

$$f(x) \leq g(x), \ x \in \mathbb{R}.$$

Show that $u(x,t) \leq v(x,t)$ for $x \in \mathbb{R}$, $0 \leq t \leq T$.

4. (i) Solve the initial boundary value problem

$u_t = u_{xx}, \quad 0 < x < 1, \; t > 0,$

$u(0,t) = 2, \quad u(1,t) = 3, \quad t > 0,$

$u(x,0) = x^2 + 2, \quad 0 \leq x \leq 1.$

(ii) What is the limit of the solution as $t \to +\infty$?

5. (i) Solve the initial boundary value problem

$u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \quad t \in \mathbb{R},$

$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \geq 0,$

$u_x(0,t) = 0, \quad t \in \mathbb{R}$

where $c > 0$, $f \in C^2$, $g \in C^1$, $f'(0) = 0$ and $g'(0) = 0$.

(ii) Assuming further that f, g are of compact support, i.e., $f(x) = g(x) = 0$ for $|x| > a$, for some $a > 0$, show that the energy

$$e(t) = \frac{1}{2} \int_0^{+\infty} (u_t^2(x,t) + c^2 u_x^2(x,t)) dx$$

is a conserved quantity as t varies.

Qualifying Exam: PDE, Fall, 2007

Choose any three out of the six problems.

1. Solve the initial value problem

$u_t + xu_x = 0, \quad x \in R, \ t > 0, \ u(x,0) = f(x), \quad x \in R$

where $f(x) \in C^1(R)$. Over what region in the x-t plane does the solution exist? Draw the characteristics on the x-t plane where the solution exists.

2. Find a weak solution satisfying the entropy conditions for

$u_t + (u^3)_x = 0, \quad x \in R, \ t > 0,$

with initial data $u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0 \end{cases}$

and with initial data $u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$.

(ii) Write an upwind scheme for the above problem. What is the CFL condition for the scheme?

3. Solve the following problem

$u_t - u_{xx} = 0, \quad x > 0, \ t > 0,$

$u(x,0) = f(x), \quad x \geq 0, \ u(0,t) = 2, \quad t \geq 0$

where $f(0) = 2$. Given that $f(x)$ is continuous and $|f(x)| \leq M$ for $x \geq 0$, find the maximum of $|u(x,t)|$ for $x \geq 0$ and $t \geq 0$.

4. Compute the Fourier series

$\Sigma_{k=0}^{+\infty} a_k \cos(kx)$

for function

$f(x) = \begin{cases} 2 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi] \end{cases}$

on the interval $[0,\pi]$. Also solve the heat equation $u_t = u_{xx}$ on $[0,\pi] \times [0,+\infty)$ with the initial value $u(x,0) = f(x)$ and the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$. What does the solution converge to as $t \to +\infty$?

5. Solve the initial-boundary-value problem for the wave equation on the half-line

$$u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \; t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \geq 0,$$
$$u_x(0,t) = 0, \quad t > 0.$$

6. Consider the wave equation with damping

$$u_{tt} + du_t - c^2 u_{xx} = 0, \quad x \in R, \; t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in R.$$

where $d > 0$, f and g are smooth functions with compact support. Show that the energy $e(t)$ decays as t increases.

Qualifying Exam: PDE, Spring, 2008

Choose any three out of the six problems.

1. (i) Find a weak solution satisfying the entropy conditions for

$$u_t + (2u^2)_x = 0, \quad x \in R, \; t > 0,$$

with initial data $u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0 \end{cases}$

and with initial data $u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$.

(ii) Write an upwind scheme for the above problem. What is the CFL condition for the scheme?

2. (i) Find the bounded solution u to the following initial-boundary-value problem

$$u_t - u_{xx} = 0, \quad x > 0, \; t > 0,$$

$$u(x,0) = f(x), \quad x \geq 0, \; u(0,t) = 1, \quad t \geq 0$$

where f is continuous on $[0, +\infty)$ satisfying $f(0) = 1$ and $|f(x)| \leq M$ for $x \geq 0$, where $M > 0$.

(ii) Find the supremum of $|u(x,t)|$ for $x \geq 0$ and $t \geq 0$ in terms of the given data.

3. Compute the Fourier series

$$\Sigma_{k=1}^{+\infty} b_k \sin(kx)$$

for function

$$f(x) = \begin{cases} 1 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

on the interval $[0, \pi]$. Also solve the heat equation $u_t = u_{xx}$ on $[0, \pi] \times [0, +\infty)$ with the initial value $u(x,0) = f(x)$ and the boundary conditions $u(0,t) = u(\pi, t) = 0$. What does the solution converge to as $t \to +\infty$?

4. Solve the following initial-boundary-value problem

$$u_{tt} - u_{xx} = 0, \quad x > 0, \; t > 0,$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \geq 0,$$

$$u_x(0,t) = 0, \quad t > 0$$

where f and g are smooth functions satisfying $f'(0) = 0$ and $g'(0) = 0$.

5. Solve the initial value problem

$u_t + 2xu_x = 0, \quad x \in R, \ t > 0, \ u(x, 0) = f(x), \quad x \in R$

where $f(x) \in C^1(R)$. Over what region in the x-t plane does the solution exist? Draw the characteristics on the x-t plane where the solution exists.

6. Consider the wave equation with damping

$u_{tt} + du_t - c^2 u_{xx} = 0, \quad x \in R, \ t > 0,$

$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in R.$

where $c > 0$, $d > 0$, and f, g are smooth functions with compact support. Define the energy $e(t)$ and show that the energy decays as t increases.

Qualifying Exam: PDE, Fall, 2020

Choose any **Four** out of the five problems. Please indicate your choice. Show all your work.

1. Find a weak solution for the nonlinear conservation law with the following Riemann initial data such that the discontinuous solutions satisfy the entropy condition

$$u_t + (u(1+u))_x = 0, \quad x \in \mathbb{R}, \ t > 0,$$

(i) with initial data

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0; \end{cases}$$

and

(ii) with initial data

$$u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

2. (i) Solve the initial value problem

$$u_t + x^2 u_x = -u, \quad x \in \mathbb{R}, \ t > 0,$$
$$u(x,0) = e^{-x^2}, \quad x \in \mathbb{R}.$$

(ii) Draw the characteristics and find the region in the x-t plane where the solution exists.

(iii) Write an upwind scheme for the above problem.

3. Let $u(x,t)$ and $v(x,t)$ be solutions of the heat equation

$$u_t - ku_{xx} = 1, \quad x \in \mathbb{R}, \quad 0 < t \leq T$$

satisfying

$$u(x,0) = f(x) \text{ and } v(x,0) = g(x), \ x \in \mathbb{R}$$

respectively, where $k > 0$, $T > 0$, $f(x)$ and $g(x)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \leq t \leq T$.

Suppose that $u(x,t)$ and $v(x,t)$ are continuous and bounded on $x \in \mathbb{R}$, $0 \leq t \leq T$, and that

$$f(x) \leq g(x), \ x \in \mathbb{R}.$$

Show that $u(x,t) \leq v(x,t)$ for $x \in \mathbb{R}$, $0 \leq t \leq T$.

4. (i) Solve the initial boundary value problem

$u_t = u_{xx}, \quad 0 < x < 1, \ t > 0,$

$u(0,t) = 2, \quad u(1,t) = 3, \quad t > 0,$

$u(x,0) = x^2 + 2, \quad 0 \le x \le 1.$

(ii) What is the limit of the solution as $t \to +\infty$?

5. (i) Solve the initial boundary value problem

$u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \quad t \in \mathbb{R},$

$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \ge 0,$

$u_x(0,t) = 0, \quad t \in \mathbb{R}$

where $c > 0$, $f \in C^2$, $g \in C^1$, $f'(0) = 0$ and $g'(0) = 0$.

(ii) Assuming further that f, g are of compact support, i.e., $f(x) = g(x) = 0$ for $|x| > a$, for some $a > 0$, show that the energy

$$e(t) = \frac{1}{2}\int_0^{+\infty} (u_t^2(x,t) + c^2 u_x^2(x,t))dx$$

is a conserved quantity as t varies.

Qualifying Exam: PDE, Fall, 2018

Choose any **Four** out of the five problems. Please indicate your choice.
Show all your work.

1. Show that if the C^1 initial data $f(x)$ has $f'(x_0) > 0$ for some x_0, then the C^1 solution of

$$u_t + (u(2-u))_x = 0, \quad x \in \mathbb{R}, t > 0, \quad u(x,0) = f(x)$$

must break down at some time $t > 0$.

2. (i) Solve the initial value problem

$$u_t - x^2 u_x = -u, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = f(x), \quad x \in \mathbb{R}$$

where $f \in C^1(\mathbb{R})$.

(ii) Over which region in the x-t plane does the solution exist?

(iii) Write an upwind scheme for the above problem.

.

3. Solve the following initial boundary value problem

$u_t - ku_{xx} = 0, \;\; x > 0, \; t > 0,$

$u(x, 0) = f(x), \;\; x \geq 0,$

$u_x(0, t) = 1, \;\; t \geq 0.$

where $k > 0$, $f \in C^2[0, +\infty)$, is bounded and $f'(0) = 1$.

4. (i) Solve the initial-boundary-value problem

$u_t = u_{xx}, \quad 0 < x < 1, \ t > 0,$

$u(0,t) = 1, \quad u(1,t) = 2, \quad t > 0,$

$u(x,0) = x^2 + 1, \quad 0 \le x \le 1.$

(ii) What is the limit of the solution as $t \to +\infty$?

5. Solve the initial value problem of the wave equation

$u_{tt} - 4u_{xx} = 0, \quad x \in \mathbb{R},\ t > 0,$

$u(x, 0) = -e^{-x^2}, \quad u_t(x, 0) = 12xe^{-x^2}, \quad x \in \mathbb{R}.$

Qualifying Exam for Math 5600

August 19, 2020

Dr. Zahra Aminzare

INSTRUCTION:

- The questions for this exam (Math 5600) are divided into two parts.

Answer both questions in Part I.

Answer only one question in Part II.

- If you work on more than one question in Part II, please state **clearly** which one should be graded.
- No additional credit will be given for more than one of the questions in Part II.
- If no choice between the questions is indicated, then the first optional question attempted will be the only one graded.
- All the questions have equal points.
- Please start a new page for every new question and put your name on each sheet.
- Please turn in the exam questions with your solutions.
- Please turn in the scratch papers. All scratch papers will be discarded.

Good Luck!

Part I. Please answer <u>BOTH</u> questions 1 and 2.

Question 1. Consider the motion of an undamped harmonic oscillator, given by the equation

$$\ddot{x} = -4x,$$

where $x(t)$ represents the location of the oscillator, and $\ddot{x}$ denotes the second derivative of x with respect to t.

(a) Introduce a new variable y for the velocity of the oscillator and formulate the motion of the harmonic oscillator as a two dimensional linear system of the form $\dot{X} = AX$, where $X = (x, y)^\top$ and A is a 2×2 matrix.

(b) Find the eigenvalues and eigenvectors of A.

(c) Show there exists an invertible matrix T so that $T^{-1}AT = B$ where B is in Jordan canonical form.

(d) Use the Jordan canonical form to find the general solution of the system $\dot{X} = AX$.

Question 2. Consider the system

$$\begin{aligned} \dot{x} &= y - ax \\ \dot{y} &= -ay + \frac{x}{1+x}, \end{aligned}$$

where a is a positive parameter. Answer the following questions.

(a) For each qualitatively different value of $a > 0$, find all equilibrium points. When the Hartman-Grobman theorem applies, classify each equilibrium point.

(b) Describe the bifurcation that occurs as a varies and find the critical value of a (call it a^*) at which the bifurcation occurs. (You do <u>not</u> need to compute the center manifold at $a = a^*$.)

(c) Sketch the phase plane (phase portrait) for $a > a^*$ which <u>qualitatively</u> describes the full dynamics of the system. *Hint: You should indicate the equilibrium points, a heteroclitic orbit, the stable and unstable curves, and six trajectories.*

Part II. Please answer <u>ONLY ONE</u> of the following questions.

Question 3. Consider

$$\begin{aligned}\dot{x} &= y - x\\ \dot{y} &= x - y - xz\\ \dot{z} &= xy - z.\end{aligned}$$

(a) State the definitions of a (Lyapunov) stable equilibrium point and an asymptotically stable equilibrium point.

(b) Is the origin $(0,0,0)^\top$ a (Lyapunov) stable equilibrium point or asymptotically stable equilibrium point?

(c) What is the basin of attraction?

Hint: Find an appropriate Lyapunov function and use LaSalle's invariance principle.

Question 4. Consider the system

$$\begin{aligned}\dot{x} &= x(1-r) - y\\ \dot{y} &= y(1-r) + x,\end{aligned}$$

where $r^2 = x^2 + y^2$. This system has a periodic solution $\gamma(t) = (\cos(t), \sin(t))^\top$. Determine the stability of $\gamma(t)$.

Hint: You may use ONE (and only one) of the following methods:

1. *Compute the characteristic multipliers for $\gamma(t)$. (Liouville's Formula may help.)*
2. *Compute the Poincaré map of $\gamma(t)$. (Polar coordinate transformation may help.)*