Plateau-Rayleigh Instability in a Lava Lamp

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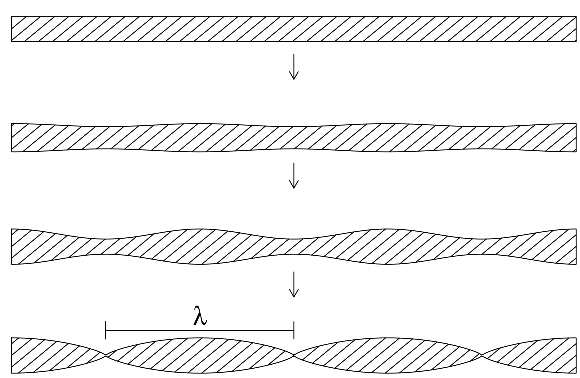
I chose to examine one of the phases exhibited in a lava lamp. It's something I'd found odd for a long time, but only recently realized I had the intellectual tools to analyze. After learning a little bit about the Plateau-Rayleigh instability on my own, I stored that information away. Later, I realized that I could combine what I'd learned about the Plateau-Rayleigh instability and what I'd learned about the Navier-Stokes equation to better understand what was going on during that particular phase of a lava lamp.

This requires that I import some math and physics from outside of the Advanced Fluids class. I discuss this outside material before addressing the flow in the lava lamp, because this is a better approximation to my own journey than to have the information on the Plateau-Rayleigh instability come later. This also has the nice effect of making the lava lamp seem like a mystery that needs explaining, rather than feeling like some math is pulled out of thin air to explain what we see.

Figure 1 progression of an unstable fluid column into discrete droplets with periodicity λ.



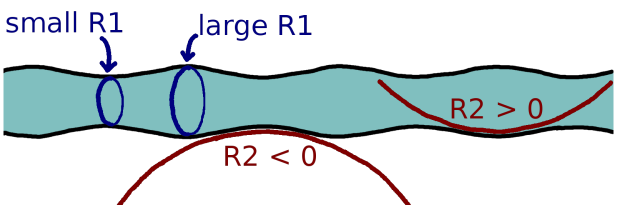
1 Plateau-Rayleigh Instability

The Plateau-Rayleigh instability occurs when a jet or column of fluid is in another medium with which it is immiscible. The obvious example would be a jet of water leaving a shower head. The initially cylindrical jet takes on distortions, narrowing in some regions and growing in others. These distortions grow rapidly until the jet breaks into many droplets (see figure 1). Due to the comparatively low viscosity and density of the air, the problem is considered as a stationary column of water with some surface tension. This assumption will end up being justified in section 2.1, and is used in the classical instability problem[1].

Early experiments established that a fluid cylinder held between two discs with an aspect ratio above about 3.1 would be unstable, breaking into two separate blobs[2]. This is closely analogous to the situation of a jet or column. As we will see, only certain distortion wavelengths are unstable – those above some critical value. So, if the cylinder is bound between two discs whose distance is shorter than the critical wavelength, the cylinder can't accommodate any unstable wavelength, and the instability doesn't form.

At the interfaces between immiscible fluids (air and water, or mercury and most other fluids, for example) there is a layer of molecules which are at a higher energy than the rest of the fluid. This is usually because at least one of the fluids forms strong attractive

Figure 2 a diagram explaining the meaning of R_1 and R_2 in the Young-Laplace equation (3). For the problem of an unstable jet, $1/R_2$ will initially be quite small and fluctuating around 0, while $1/R_1$ will be large and fluctuating.



bonds with itself. The molecules at the boundary do not have anything to bond with on one side, and therefore are at a higher energy. Thus, the total energy of the system increases with the surface area, according to

$$E_s = \gamma A, \tag{1}$$

where E_s is the surface energy, γ is the surface tension (in N/m or equivalently J/m^2), and A is the surface area. Using equation (1), one can derive[3] the Young-Laplace equation,

$$\begin{aligned} \Delta P &= -\gamma \nabla \cdot \vec{n} &(2)\\ &= -\gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right), &(3) \end{aligned}$$

where ΔP is the negative gauge pressure within the fluid, relative to its surroundings, $\vec{n}$ is the normal vector to the surface, and R_1 and R_2 are the principal radii of curvature (see figure 2). The Δ runs in the same direction as $\vec{n}$. So for example, spheres have *positive* internal gauge pressure.

1.1 Surface Perturbations

To analyze the instability, we start by considering a cylinder with small sinusoidal perturbations in its radius. So, the radius as a function of the distance x along the axis is

$$r = r_0 + \epsilon \cos(kx), \tag{4}$$

where r_0 is the mean radius, ϵ is some small length, and $k = 2\pi/\lambda$. Notice that if we just assumed r_0 to be constant, the volume of the cylinder would depend on ϵ. Since we will be making considerations with respect to ϵ, this is unacceptable. We need the volume of the infinite cylinder to remain constant. The volume-per-length averaged over one wavelength is

$$S = \frac{1}{\lambda} \int_0^\lambda \pi (r_0 + \epsilon \cos(kx))^2 \, dx \tag{5}$$

$$S = \pi r_0^2 + \frac{1}{2} \pi \epsilon^2. \tag{6}$$

From that, we find that r_0 is

$$r_0 = \sqrt{\frac{S}{\pi} - \frac{\epsilon^2}{2}}. \tag{7}$$

The next part is really laborious and not really worth including. We would start by finding the area of column as a function of ϵ:

$$\sigma = \frac{1}{\lambda} \int_0^\lambda 2\pi \sqrt{r^2 + \left(r \frac{dr}{dx} \right)^2} \, dx \tag{8}$$

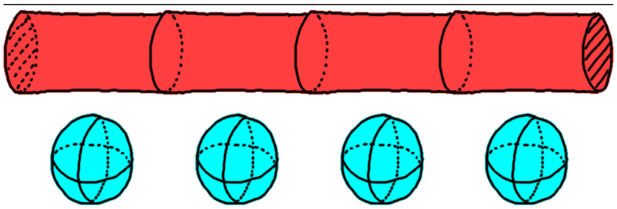
This is a difficult integral to solve. When its solution is approximated to the fourth order (cut off terms above ϵ^4) using Taylor expansion, we arrive at, with some rearranging,

$$\sigma = 2\sqrt{\pi S} + \frac{\pi \epsilon^2}{2 r_0} (k^2 r_0^2 - 1). \tag{9}$$

From here, notice that if $k^2 r_0^2 - 1 < 0$, then the surface area will decrease with ϵ. In other words, when $\lambda > 2\pi r_0$, the surface area decreases. Increasing the amplitude of such a perturbation decreases the overall energy of the system. So, we should expect that perturbations with λ greater than π times the diameter of the column will be unstable and grow.

It's important to note that we're neglecting the mechanical energy here, and assuming it doesn't significantly contribute to $dE/d\epsilon$. Additionally, it may seem odd to assume the initial perturbations are harmonic. For some arbitrary perturbations, we can consider their Fourier transform. It can be shown[1] that the harmonic components of the original perturbations

Figure 3 a diagram of how an infinite cylinder can be partitioned into equal spheres.



will grow exponentially. Importantly, they will grow exponentially with different characteristic times. So, there will be some fastest-growing component that eventually dominates the others, making the overall distortion nearly harmonic.

1.2 Quasi-Static Analysis

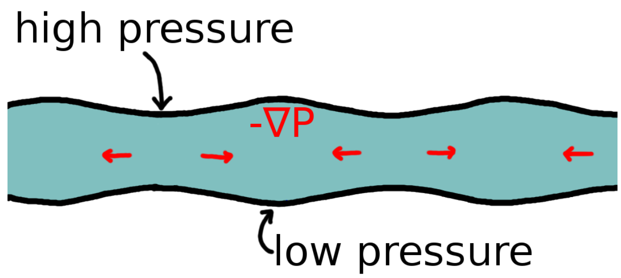
There is a much easier, although not quite correct, way of thinking about the Plateau-Rayleigh instability[4]. One can consider the overall energy of the starting cylinder and ending spheres (see figure 3). We only require that the total volume not change, and that the total energy decrease. The point at which the energies are equal will be our estimate of the minimum unstable wavelength. This method is based off of the notion that systems will move from higher energy states to lower energy states, dissipating the difference through some loss mechanism, in this case viscosity.

setting the volumes equal, we get

$$\pi r_c^2 \lambda = \frac{4}{3}\pi r_s^3, \tag{10}$$

where r_c is the cylinder radius, λ is the length of the cylinder segment, and r_s is the radius of the sphere. There one other energy we need to consider, other than E_s. We need to consider the mechanical energy, $E_m = P \cdot V$, where V is the volume of the body in question. To get the pressure of the cylinders and spheres, we'll use the Young-Laplace equation (3).

Figure 4 a diagram of how the surface curvature of a distorted column effects the pressure gradient within the column.



Setting the energies equal, we get

$$\begin{aligned} E_{\text{s,cyl}} \quad &+ E_{\text{m,cyl}} \quad &= E_{\text{s,sph}} \quad &+ E_{\text{m,sph}} \qquad (11)\\ \gamma(2\pi r_c \lambda) \quad &+ \frac{\gamma}{r_c}(\pi r_c^2 \lambda) \quad &= \gamma(4\pi r_s^2) \quad &+ \frac{2\gamma}{r_s}\left(\frac{4}{3}\pi r_c^3\right) \qquad (12)\end{aligned}$$

combining (10) and (12) we get, after some algebra,

$$\lambda = \frac{500}{81} r_c \approx 1.96\pi r_c. \tag{13}$$

So, we find that the unstable length is about 3.1 times the diameter of the cylinder. That's not quite the π it's supposed to be, but it's pretty close and conceptually much easier to think about.

1.3 Young-Laplace Explanation

The above explanations of the instability rely on the notion that a system will move from high energy states to low energy states, but it would be nice to have some sort of force-based explanation too. This comes pretty easily from the Young-Laplace equation (3). We've already seen that the $1/R_2$ term is negligible and that $1/R_1$ dominates for long wavelengths (see figure 2). So in the narrow sections, there will be a higher pressure than in the wider sections. We therefore have a $-\nabla P$ pointing away from the narrow regions and towards the wide regions (see figure 4), driving even more fluid away from the narrow regions and increasing the distortions.

2 Flow in a Lava Lamp

A lava lamp is a novel kind of lamp which showcases some interesting fluid dynamics. The basic design is a glass bottle filled with water, wax, a small piece of metal, and a small amount of air, with a heat lamp underneath the bottle. The bottle is typically conical, being wider at the base (see figure 5). The densities of the water and wax are tuned so that the wax is just slightly more dense than the water at room temperature. This could be done by adding alcohol to the water to lower its density, for example. As the bulb heats the wax at the bottom of the bottle, the wax expands decreasing its density. When the density of the wax is less than that of the water, the wax floats to the top of the bottle via the buoyant force and is cooled by the surrounding water. It then sinks back down to the bottom of the bottle, and the cycle repeats. It's just convection with two immiscible fluids.

Although the above description is always the overall behavior of the lamp, its actual manifestation can change dramatically. The lamp will go through different phases, depending on the temperatures of the water and wax.

Starting at room temperature, there's a single slug of wax sitting at the bottom of the lamp. As it's heated by the bulb, the bottom portion of that slug melts first, creating an unstable configuration. The molten wax erupts around the side of the still-solid disk. As it rises through the cold water, the outer layer of the hot wax cools and solidifies, creating a tube. As it reaches the top of the bottle, this tube snakes around filling the top portion of the bottle. What's left is a structure that looks somewhat like a brain. Eventually, the bottom portion of the tube melts and the whole structure falls.

A few minutes later, the bottom of the lamp is filled with molten wax which is not quite warm enough to rise. As it heats up, it forms a central bulge, which gradually turns into a column (see figure 5). Sometimes this column will attach to the top surface of the bottle, since the wax-air interface is much lower energy than the water-air interface. The column will usually have a diameter of 2 cm to 4 cm with brief necking events where the smallest diameter of the

Figure 5 the single-column phase of a lava lamp. This column is exhibiting the brief "necking" which occrs intermitantly. Note the narrow section in the middle. See the additional media for video.

column drops to a centimeter or less. Such an event can be seen in figure 5. This column phase consistently lasts about 5 minutes before the wax breaks into separate droplets. Although, it is somewhat ambiguous when it begins since the transition form a molten slug at the bottom of the bottle to a column is gradual. The column phase can be reliably reproduced by heating up the lamp past the column phase, shutting off the power, waiting for the wax to form a single blob at the bottom of the bottle, and then turning the power back on.

After the column phase, the wax breaks up into many blobs via the Plateau-Rayleigh instability. These blobs then convect between the bottom and top of the bottle, with some wax always attached to the top surface and to the small piece of metal at the bottom. This is presumably why the metal is added, so that the returning blobs will reform with the wax on the metal, hugging them close to the bulb. As the temperature continues to increase, the blobs gradually become smaller and more numerous.

The column phase is what's most interesting, considering what we've just discussed about the Plateau-Rayleigh instability. During this phase, the wax flows through the column in an unsteady way. Sometimes it's entirely flowing up or down through the column, and other times the flow is up on one side and down on the other. The thickness and shape of the column vary significantly. Although the overall aspect ratio of the lamp is about 3 (less than the unstable aspect ratio), sections of the column can often be found with a super-critical aspect ratio. So the question becomes, "how do we explain these super-critical regions?"

2.1 Canonical solutions

There's one canonical solution that's involved which we implicitly used to extend the above analysis to jets, not just columns. Consider two infinite plates with layers of fluid in between (figure 6). One of the plates moves with a constant velocity perpendicular to its normal vector, and the two fluids are identical except for their viscosities. We assume there is no variation except for in the normal direction, the flow is laminar and steady, there is no pressure gradient applied, and the fluids are Newtonian. We can solve for the velocity field in this scenario exactly. We find that the velocity profile is piecewise linear, and steeper in the low viscosity fluid. Although the shear is constant in this scenario, the low viscosity fluid seems to insulate the high viscosity fluid against the motion of the plates.

Figure 6 velocity profile of fluids with different viscosities between two shearing plates. The low viscosity fluid has an insulating effect on the high viscosity fluid.

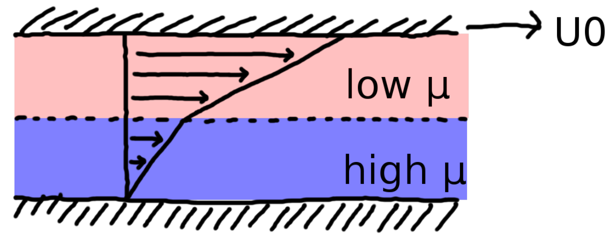
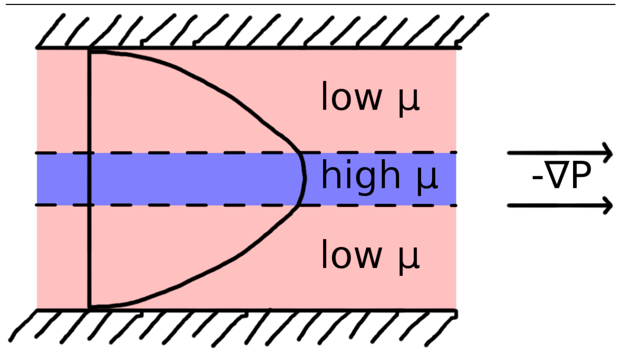


Figure 7 velocity profile of a high viscosity fluid layer between two low viscosity lubricating layers in a pressure driven flow. The low viscosity fluid has an insulating effect, much like what's shown in figure 6.



The problem remains solvable if we introduce multiple layers, introduce a pressure gradient, and remove the relative velocities of the plates (see figure 7), although the solution is not very concise. We choose the layers such that the high viscosity fluid sits between two layers of low viscosity fluid. In this situation, the velocity field is piecewise parabolic, with the steeper regions again being in the low viscosity fluid. This time however, most of the shear is also occurring in the low viscosity fluid. Notice that if we make the viscosity of the outer fluid *much lower* than the central fluid the velocity profile of the central fluid will be nearly constant. We can achieve the same effect if we make the thickness of the outer fluid much greater than the thickness of the inner fluid.

This effect of the low viscosity fluid insulating the inner fluid from the walls is how we justify including fluid jets in the same analysis as fluid columns. The shear from the air is just very low, and so we consider the velocity profile of the jet to be constant. However, these canonical solutions are for *sheets* of fluid. The problems we're trying to analyze are axisymmetric. So, shouldn't we try to find a canonical solution that uses coaxial layers of fluid? Well, yes. We could work out such solutions, but the solutions for fluid sheets are readily available and we wouldn't get much more insight out of the coaxial solutions. These solutions are just used to justify the notion that if we're talking about a fluid within a medium which has a much lower viscosity than its own, we can neglect the shear at the interface between the two.

The material properties of the fluids in a lava lamp are unknown, since they are a proprietary mixture. However, they should fall within certain ranges. If we consider the minimum and maximum reasonable values for each, we can obtain estimates for the extremes of what might be going on inside the lamp. For example, the viscosity of the wax should fall somewhere within the range of commercially available waxes when molten[5]. Using this min/max approach, we expect that the ratio of the viscosities should lie between 6 and 30. We measure the ratio of the diameters to lie between 1 and 10, with 1 representing the attached region at the top of the lamp and 10 representing a thin section of the column near the bottom. So, for the super-critical regions that we're interested in, this ratio will likely be around 7 or 8. Thus, we consider the assumption that there is no wall shear in the wax column to be reasonable.

2.2 Nondimensional Groups

Trying to nondimensionalize the Navier-Stokes and Young-Laplace equations together is difficult, since they only combine at the fluid interface. However, we can still use the Buckingham-Pi method to find insightful dimensionless groups. As above, we can use the min/max reasonable values to estimate the range of these dimensionless parameters.

Measuring the velocity of the wax column by tracking small water bubbles within it, we find that the velocity of the column is between 1 cm/s and 2 cm/s. Using this as our velocity scale, the diameter of the column as our length scale, and the various min/max reasonable parameters, we can estimate the Reynolds number,

$$\mathrm{Re} \equiv \frac{\rho v L}{\mu}, \tag{14}$$

to be on the order of 10^{-4} to 10^{-5}. This indicates that momentum is not playing a significant role in the flow of the wax, compared to its viscosity.

One of the available dimensionless groups is the Weber number, defined as

$$\mathrm{We} \equiv \frac{\rho v^2 L}{\gamma}. \tag{15}$$

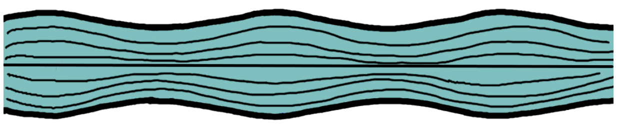
This compares surface energy to kinetic energy. With our min/max estimates, the Weber number should be on the order of 10^{-2} to 10^{-1}. So, the main contribution to the total energy of the wax column will be from the surface tension.

We also have the Ohnesorge number, which compares viscosity to inertia and surface tension. It's defined as

$$\mathrm{Oh} \equiv \frac{\mu}{\sqrt{\rho \gamma L}} = \frac{\sqrt{\mathrm{We}}}{\mathrm{Re}}. \tag{16}$$

Plugging in the min/max values, we find that Oh is on the order of 10^3 to 10^4, meaning that viscous effects are dominating inertia and surface tension. If we notice the low value of the Reynolds number, we can guess that it's the very low inertia which is bringing this number up so much.

Figure 8 a depiction of what the stream lines might look like in a column of fluid which flows relative to the surface perturbations.



2.3 Hard Approximation

The approximation I would have liked to use is shown in figure 8. It's a column of fluid with surface perturbations, much like the original Plateau-Rayleigh scenario. The difference is that instead of being static, relative to the surface perturbations, the fluid flows in the axial direction. This is meant to be analogous to the flow in the lava lamp, since the overall shape of the wax column changes rather slowly relative to the motion of the wax.

For this approximation, we would have the following conditions: the fluid interface is of the form

$$r = r_0 + \epsilon \sum_{n=1}^{\infty} a_n \cos(nkx); \tag{17}$$

no shear at the fluid interface; no divergence of the velocity field (ρ is constant); and the flow is axisymmetric (no variation in the circumferential direction). To solve it we would need to first find the velocity field, find the total energy as a function of ϵ, and then look to see if there is some value of ϵ that minimizes the total energy.

It's important to note here that ϵ might not be small. Additionally, $a_{n>1}$ may not be zero. In other words, we have no idea what the surface interface looks like. Trying to find the velocity field for such boundary conditions would be hard enough, but we need to find a combination of an interface shape and a velocity field for which Navier-Stokes and Young-Laplace give the same pressure field at the interface! Also, because of the assumptions we made about the lack of wall shear and not explicitly including the buoyant force (in reality, the viscosity is balancing the buoyant force), the flow rate is indeterminate. So, the solution we're looking for is a stable interface shape which is a function of flow rate. This problem feels like it should be solvable, but my attempts immediately led to very nasty integrals.

2.4 Easy Approximation

An even simpler approximation of the problem would be a cylinder (no surface perturbations) with a uniform velocity profile inside. If we set the flow rate to be constant, the energy of the flow and the energy of the surface become functions of the radius of the cylinder. Importantly, they have different forms. If we look at the pressure as a function of the radius, we see that the pressure of the fluid goes like $P \sim C - r^{-4}$ while the pressure drop across the interface goes like $P \sim r^{-1}$, so we should expect that there will be some positive radius at which they agree.

We can write the average energy per length as

$$\begin{aligned} e_{tot} &= e_{surf} + e_{kin} + e_{mech} && (18) \\ &= \frac{\gamma A_L}{L} + \frac{m_L v^2}{2L} + \frac{P V_L}{L} && (19) \\ &= 2\pi\gamma r + \frac{\dot{m}^2}{2\pi\rho r^2} + \pi\gamma r \quad . && (20) \end{aligned}$$

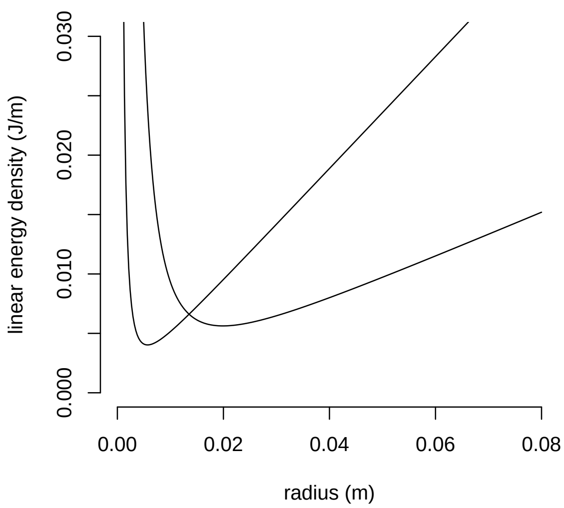
Setting ${}^{de_{tot}}/_{dr} = 0$ to find the radius of minimum energy, we get

$$r = \left(\frac{\dot{m}^2}{3\pi^2\gamma\rho} \right)^{1/3} . \tag{21}$$

If we plug in the min/max reasonable material properties, we find that the stable radius should be between 0.5 cm and 2 cm.

If we examine the energy as a function of radius (see figure 9), we find that the gradient is very steep for undersized radii (those smaller than the stable radius), and much shallower for oversized radii. So, if the system is disturbed such that the radius is undersized, it will return to equilibrium quickly, experiencing a large restoring "force." And, if it's disturbed such that the radius is oversized, it will return to equilibrium slowly, experiencing a small restoring "force." So, we should expect that the column will regularly be found to have radii greater than a few centimeters, with brief periods of extreme necking, which is what we observe.

Figure 9 plots of the energy per unit length, using parameters which minimize and maximize the stable radius. This is closely analogous to the effective-potential plot of a planet's orbit.



Additional Media

For videos of the single-column phase of a lava lamp, please visit these URLs:
`https://youtu.be/Ife6cmf4CRQ`
`https://youtu.be/XSbiSGerqyY`
`https://youtu.be/fGuJTH9QLPM`
`https://youtu.be/tuevg4sRzDc`

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