An Explanation of the EPR Paradox

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1 Introduction

In 1935, Albert Einstein, Boris Podolsky, and Nathan Rosen (EPR) published a controversial paper [1] about the completeness of quantum mechanics (QM). They challenged the notion that QM described reality completely be developing a paradox between QM an relativity. It was later shown by John Bell that an experiment could test what the weak point in the paradox was [2]. He developed a set of inequalities which would narrow down the bad assumption to either local reality (the kind of reality assumed by EPR) or relativity, if they were experimentally violated. This report won't cover Bell's work though. It will be limited only to the EPR paper.

EPR did not use Dirac's bra-ket notation, nor did they distinguish between algebraic operators and quantum state operators the way Feynman does. This can make the paper difficult to read for someone accustomed to these comparatively easy-to-read forms of notation. They rely on determining what the eigenfunctions of operators are in order to determine what wave functions would describe a system in which some variable (the one corresponding to the chosen operator) is known exactly. For their purposes, this method of working with the wave function of a system turns out to be extremely convenient, since they don't need to worry about how they're representing the wave function in series of base states. At least, they don't have to explicitly worry about it in the notation. They simply treat the wave function as an amplitude probability density as a function of some variables. For the sake of this special convenience, this report will keep the notation used by EPR.

The EPR paper takes some previous results for granted and assumes the reader understands what is happening in between the equations. Even when the math is made clear, the core logic is a lot to keep in one mind at a point in time. This report will try to translate EPR into a more digestible form.

2 Their Definitions

EPR mainly focus on *completeness* and *reality*, and whether or not QM has these properties. Completeness refers to how the predictions of a theory match up with the phenomena observed. As they put it, "*every element of the physical* *reality must have a counterpart in the physical theory.*" Or in other words, if we observe some phenomenon in our experiments, a *complete* theory will predict that phenomenon.

Reality refers to the predictability of a physical system. As they put it, "*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*" In other words, an observable is only *real* if we can predict its future value exactly. If we can only state it probabilistically, then it has no physical reality. EPR require that reality be deterministic. At first, this might seem like EPR are demanding that physical reality conform to their own aesthetic desires. That's why they go to the trouble of describing their paradox. They are trying to demonstrate that QM conflicts with relativity.

3 Eigenfunctions

The EPR paper makes a big deal over the eigenfunctions of operators. All that means is

$$\hat{A}\{\psi(x)\} = a \cdot \psi(x). \tag{1}$$

When you apply an operator to one of its eigenfunctions, you wind up just multiplying that function by a constant. An example of something that isn't an eigenfunction would be

$$\hat{q}\{\psi(x)\} = x \cdot \psi(x), \tag{2}$$

since the result is not a constant multiple of the original function. Why is this math relevant? Generally, it comes up in QM when we try to find the expectation value for a particular observable. Consider the expectation value for an observable.

$$\begin{aligned} \langle a \rangle &= \int_{all} \psi^* \hat{A} \psi \, dx && (3) \\ &= a \int_{all} \psi^* \psi \, dx && (4) \\ &= a, && (5) \end{aligned}$$

for a normalized ψ. The part that's relevant to EPR is that this will give the same expectation value over *any* region, not just all space.

$$\begin{aligned} \langle a \rangle_{(x_1,x_2)} &= \frac{\int_{x_1}^{x_2} \psi^* \hat{A} \psi \, dx}{\int_{x_1}^{x_2} \psi^* \psi \, dx} && (6) \\ &= a \frac{\int_{x_1}^{x_2} \psi^* \psi \, dx}{\int_{x_1}^{x_2} \psi^* \psi \, dx} && (7) \\ &= a && (8) \end{aligned}$$

So, imagining x_1 and x_2 very close together, we can say that if the wave function ψ is an eigenfunction of $\hat{A}$, then a has the same value everywhere, or in other words a is known completely. (It's important to note that we don't necessarily need to converse in the space of position. If we wanted, we could run the integral over momentum, phase, or any other variable.)

Generally, we need to have *noncommuting* operators to demonstrate the paradox. However, we can avoid using that more abstract (and complicated) proof by finding two noncommuting operators and demonstrating the paradox with those. We only need $\hat{P}$ and $\hat{Q}$ for momentum and position, respectively. The goal is simply to demonstrate that there is something inconsistent going on. If we can do that for $\hat{P}$ and $\hat{Q}$, we've met our goal. There will be similar paradoxes for other pairs of noncommuting operators left unaddressed, but we will still have a paradox. For reference, noncommuting operators are just those which satisfy the inequality

$$\hat{A}\hat{B}\psi \neq \hat{B}\hat{A}\psi. \tag{9}$$

Let us take for granted the following forms of the operators and their eigenfunctions.

$$\hat{P} = \frac{\hbar}{i}\frac{\partial}{\partial x} \tag{10}$$

$$\psi_p = e^{(i/\hbar)p(x+x_0)} \tag{11}$$

$$\hat{Q} = x \tag{12}$$

$$\psi_q = \delta(x - x_0) = \delta(x_0 - x) \tag{13}$$

To clarify, the operator $\hat{Q}$ just means 'multiply by x'. $\delta(y)$ is the Dirac delta function. It's zero everywhere except for at $y = 0$, at which it equals ∞. If it's integrated over an interval containing $y = 0$, the integral takes the value of 1. So, for example,

$$\int_0^{\pi/2} \sin(\theta)\delta(\theta - \pi/6)\, d\theta = 1/2, \tag{14}$$

because $\sin(\pi/6) = 1/2$. (11) is easily confirmed by plugging by applying $\hat{P}$, but it may seem odd at first to claim (13). The justification is practically a mathematical pun. Recall that $\delta(x - x_0)$ is zero everywhere, except for when $x - x_0 = 0$. Therefore, $x\delta(x - x_0) = x_0\delta(x - x_0)$. x_0 is a constant. Therefore, $x\delta(x - x_0)$ is a scalar multiple of $\delta(x - x_0)$, making it an eigenfunction of $\hat{Q}$.

4 The Thought Experiment

Consider a system of two particles which interact at $t = 0$ and $x = 0$. The particles move out in the $+x$ and $-x$ directions. We then measure the properties of the particles at points in spacetime which are space-like separated. That is to say, we send the particles to observers who are so far apart that when the particles reach them, they aren't causally connected. No matter what relativistic reference frame we're in, there won't be time for a light ray to get from one measuring device to the other. Whatever we do at the left detector can't possibly effect events at the right detector (and vice versa). For simplicity, we will talk about the detections being at the same time – we will use a reference frame in which the detections appear simultaneous.

First, we suppose that the total wave function takes the form

$$\Psi(x_1, x_2) = \int_{all} e^{(i/\hbar)p(x_1 - x_2)}\, dp. \tag{15}$$

4.1 Measuring the Momentum

The operator for the momentum of the first particle, $\hat{P}_1 = (\hbar/i)\partial/\partial x_1$, will yield an eigenfunction of

$$u_p(x_1) = e^{(i/\hbar)px_1}, \tag{16}$$

with the eigenvalue p. We can write Ψ in a way that uses this eigenfunction.

$$\Psi(x_1, x_2) = \int_{all} u_p(x_1)\psi_p(x_2)\, dp, \tag{17}$$

where

$$\psi_p(x_2) = e^{(i/\hbar)p(-x_2)}. \tag{18}$$

Notice that by substituting (16) and (18) into (17) we get back to (15). Also notice that $\psi_p(x_2)$ happens to be an eigenfunction of $\hat{P}_2 = (\hbar/i)\partial/\partial x_2$, with the eigenvalue of $-p$.

In a one-particle system, if we measure some property of the particle, we disturb it (for example, by hitting it with a photon or impacting it on a screen) and therefore can't say anything about its future wave function. In our two particle system, if we measure the first particle we definitely disturb it. However, we don't disturb the second particle. Thus, the wave function of the second particle will be one in which the momentum of the first particle is known exactly, but the overall wave function remains unchanged.

That's exactly the form of Ψ we just wrote. Thus, once we measure the momentum of the first particle, we know the momentum of the second one to be *exactly* equal and opposite to the momentum of the first particle (since the eigenvalue of $\hat{P}_2$ is $-p$). In other words, the momenta of the two particles have simultaneous reality. The momentum of the first is real because we measured it, while the momentum of the second is real because we can predict it exactly.

4.2 Measuring the Position

The consideration of measuring x_1 takes a very similar form. We start with the fact that the eigenfunction of $\hat{Q}_1 = x_1$ (by which I mean the operator is just 'multiply by x_1') is

$$v_x(x_1) = \delta(x_1 - x), \tag{19}$$

as we concluded before. Notice that in this case, x is the constant and x_1 is the variable. So, the eigenvalue is x. We now say that

$$\Psi(x_1, x_2) = \int_{all} v_x(x_1)\phi_x(x_2)\,dx \tag{20}$$

and that

$$\phi_x(x_2) = \int_{all} e^{(i/\hbar)p(x-x_2)}\,dp. \tag{21}$$

Observe that when we plug (19) and (21) into (20), the $\delta(x_1 - x)$ within the integral over dx sets all instances of x (the only other one being contained in $\psi_x(x_2)$) to be x_1, and so we end up with (20) being equal to (15).

Now, we need to do some further manipulation on $\psi_x(x_2)$. We notice that when we write it as

$$\phi_x(x_2) = \int_{all} e^{2\pi ixp/h} e^{-2\pi ix_2p/h}\,dp, \tag{22}$$

that it's just a Fourier transform. Evaluating, we get

$$\phi_x(x_2) = h\delta(x - x_2). \tag{23}$$

This new form also happens to be an eigenfunction of $\hat{Q}_2 = x_2$ with the eigenvalue x. Thus Ψ takes the form

$$\Psi(x_1, x_2) = \int_{all} \delta(x_1 - x)h\delta(x - x_2)\,dx. \tag{24}$$

Notice that $\hat{Q}_1\Psi = x_2\Psi$ and $\hat{Q}_2\Psi = x_1\Psi$ – seemingly the wrong result! However, $x_1 \neq x_2 \Rightarrow \Psi = 0$, so $x_2 = x_1$.

Just as with the momentum, measuring the position of one particle completely determines the position of the other particle. The particle we measure is disturbed and the other is not.

5 Noncommuting Operators

This is where the business of noncommuting operators comes in. The *commutator* [3] of two operators, written $[\hat{A}, \hat{B}]$, is defined in the following way.

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{25}$$

We can actually prove that the uncertainty principle relates to the commutator [4] [5]. This will give a more general form than the typical version involving only p and x. We start with these two operators:

$$\Delta\hat{A} = \hat{A} - \langle a\rangle \tag{26}$$

$$\Delta\hat{B} = \hat{B} - \langle b\rangle. \tag{27}$$

Now, consider the state $\phi = (\Delta\hat{A} - i\lambda\Delta\hat{B})\psi$. Let's try dotting ϕ with itself. What does that mean? Imagine ϕ is a wave function of a particle. ϕ dotted with itself if the probability of finding it anywhere in space. That's not the physical meaning of ϕ though, so it won't necessarily be 1. Here will take for granted any state dotted with itself will be greater than zero, regardless of whether it's normalized (not too difficult to prove).

$$0 \leq \int_{all} \phi^* \phi \, dx \tag{28}$$

$$0 \leq \int_{all} \psi^* (\Delta\hat{A} + i\lambda\Delta\hat{B})(\Delta\hat{A} - i\lambda\Delta\hat{B})\psi \, dx \tag{29}$$

$$0 \leq \int_{all} \psi^* ((\Delta\hat{A})^2 + \lambda^2(\Delta\hat{B})^2 - i\lambda[\Delta\hat{A}, \Delta\hat{B}])\psi \, dx \tag{30}$$

$$0 \leq \langle(\Delta\hat{A})^2\rangle + \lambda^2\langle(\Delta\hat{B})^2\rangle - i\lambda\langle[\Delta\hat{A}, \Delta\hat{B}]\rangle \tag{31}$$

Notice that because $\langle a\rangle$ and $\langle a\rangle$ are constants, $[\Delta\hat{A}, \Delta\hat{B}] = [\hat{A}, \hat{B}]$, which simplifies things a little. Also, $\langle(\Delta\hat{A})^2\rangle$ is the mean square error of $\Delta\hat{A}$ which is just the variance of $\Delta\hat{A}$. So, we write $\langle(\Delta\hat{A})^2\rangle = \sigma_a^2$. With these renamings, the equation is written as

$$0 \leq \sigma_a^2 + \lambda^2\sigma_b^2 - i\lambda\langle[\hat{A}, \hat{B}]\rangle. \tag{32}$$

λ was part of our arbitrary choice for ϕ. So, we choose it to be whatever value(s) correspond to the extremes of this inequality, so that we will wind up with the strongest possible inequality. This works out to be

$$\lambda = \frac{\pm i\langle[\hat{A}, \hat{B}]\rangle}{2(\Delta\hat{B})^2}. \tag{33}$$

plugging into (32), we get

$$\sigma_a^2\sigma_b^2 \geq -\frac{1}{4}\langle[\hat{A}, \hat{B}]\rangle^2 \qquad \text{or} \qquad \sigma_a^2\sigma_b^2 \geq \frac{3}{4}\langle[\hat{A}, \hat{B}]\rangle^2 \tag{34}$$

$$\sigma_a\sigma_b \geq \frac{i}{2}|\langle[\hat{A}, \hat{B}]\rangle| \qquad \text{or} \qquad \sigma_a\sigma_b \geq \frac{\sqrt{3}}{2}|\langle[\hat{A}, \hat{B}]\rangle|. \tag{35}$$

plugging in $\hat{P}$ and $\hat{Q}$, the first option of (35) just becomes the uncertainty principle, $\sigma_p\sigma_x \geq \hbar/2$. Notice that if we make σ_a arbitrarily small, then σ_b becomes arbitrarily large. Taking the limit as $\sigma_a \to 0$ (meaning its value is known with certainty), we see that $\sigma_b \to \infty$ (meaning its value is uniformly distributed). These standard deviations of the observables are the standard deviations across the whole ensemble in question. If one of the standard deviations is non-zero, this implies that the value it corresponds to is not known with certainty. Using EPR's definition of reality, this means it is not real. So, if σ_a is real (known with certainty), then σ_b is not real (not known with certainty) and vice-versa. Thus, we have shown that if two operators don't commute (meaning their commutator is not zero), then only one of them can be real at a time!

6 The Paradox

Now, we return to the thought experiment and consider two possibilities: we measure either the momentum of the first particle or the position of the first particle. In both cases, we haven't disturbed the second particle. However, depending on our choice, the momentum of the second particle will be either real or not real. If we measure p_1, we make x_2 unreal. If we measure x_1, we make p_2 unreal. Remember though that the two are space-like separated. There is no way we could possibly alter what's going on with the second particle. Both possibilities for the second particle must correspond to its overall reality. Therefore, p_2 and x_2 must both be real. This conflicts with our previous (purely quantum-mechanical) result that p and x can't be simultaneously real. We have found something that must be an element of reality but is not reflected in quantum mechanics. Therefore, quantum mechanics must be incomplete.

6.1 Personal Perspective

As someone born 55 years after this paper was published, it was hard at first to see what the paradox was. That's partly because it's subtle, but also partly because all of my physics education has been taught by a generation of people who wanted a better incorporation of QM into physics classes. So, I've heard lots of different perspectives of how to reconcile the weirdness of QM in my terrestrial human mind.

Consider detectors for each particle. Each detector has a finite range. So, taken over all space, a particle whose observable is indeterminate will basically never be detected. Say we have detectors set up for each particle, and we can choose either one to be measuring one of two noncommuting operators (the classic example here being x and y polarizations). If we choose the same setting for each detector, we get the same result on each, either both detect or both don't detect. If we choose opposite settings, a detection on one implies a non-detection on the other and vice-versa.

We run the experiment and get a detection for the first detector. Now, what was the setting for the second detector? Was it the same as the setting for the first or was it different? We don't know. Either one is possible. The same is true if we don't get a detection for the first particle. We can't experimentally determine what happened with the second detector merely by what happens at the first. No information is transferred from the second particle to the first. They aren't causally connected, and the events at the second detector are not part of the reality of the first detector. They only become real once enough time has passed for them to become causally reconnected. By then, there is no more paradox, since the choice of detector positions has already been made.

This is how I preferred to think about the problem. From this perspective it didn't seem like much of a paradox.

References

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