ME: 442
Take-home #2: Estimating the Average Surface Temperature of the Earth

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1 Problem

Use a heat transfer calculation to estimate the *average* surface temperature of the Earth.

2 Background

First, we know that the net heat flux of the Earth is zero, since the average temperature is (mostly) constant. Also, we know the physical parameters for this problem are:

```
Tsun <- 5778 #K (effective photosphere temp)
Rsun <- 696e6 #m, radius of Sun
sigma <- 5.67e-8 #W/m^2/K^4
#m, average orbital radius of Earth:
AU <- 1.496e11
Rearth <- 6.371e6 #m, radius of Earth
# IR emissivity for the atmosphere:
eps.atm.ir <- 0.78
tau.atm.ir <- 1-eps.atm.ir
eps.earth.ir <- 0.94
rho.bar <- 0.3 #average terrestrial albedo
#K, measured average surface temp of Earth:
Tearth.meas <- 14.0+273.15
#W, total radiogenic heat flux:
q.geo <- 20e12
```

sources for weird values:

- average albedo of the Earth: [1]
- average surface temperature of the Earth: [2]

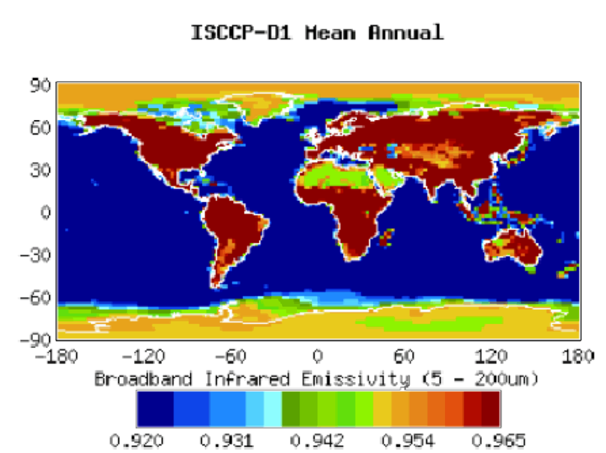

Figure 1: Emissivity atlas of the Earth. (NASA, ISCCP)

- radiogenic heating: [3]
- emissivity of Earth's surface: figure 1

Finding these values was really the hard part of this problem...

I was tempted to include radiogenic heating from the Earth's mantle and core. However, it turns out to be 10^{-3} times smaller than the other heat fluxes involved, so it's negligible. I'll justify this after the calculation.

Since the average temperature of space is only 2.7 K, and the temperatures we're working with are

Figure 2: Image of the Earth's atmosphere from on-edge. (NASA)

ranging from $\sim 300\,\mathrm{K}$ to $5800\,\mathrm{K}$, and thermal radiation goes with T^4, we can very reasonably neglect heating from outer space. In other words, heating from space will be on the order of

```
formatC(300^4/2.7^4)

## [1] "1.524e+08"
```

times smaller than the heat flux leaving the Earth, so we neglect it.

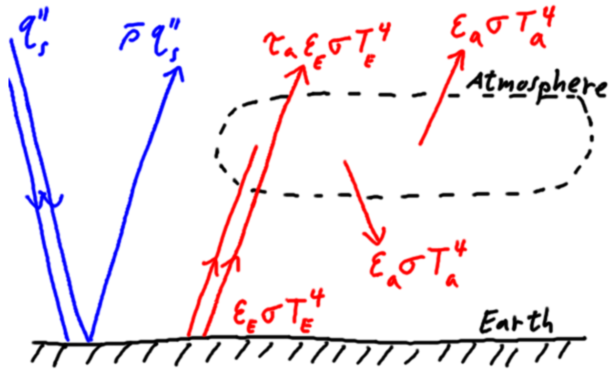

Figure 3: Diagram of the solar (blue), and infrared (red) radiation passing through the atmosphere. Note that ϵ_a is the *hemispherical* emissivity, so I'm not missing a 1/2 there.

3 Model

So, the general model is that the Earth is an opaque sphere with the atmosphere being a single spherical layer around it, of essentially the same radius. Additionally, the atmosphere is thin (fig 2), so half of the radiation emitted by the atmosphere goes inward towards Earth and the other half goes towards space.

Of course, we could start worrying about reflections between the Earth and the atmosphere. However, the Earth has a relatively high IR emissivity, which means it also has a high IR absorbance. So, most of the IR emitted from the atmosphere will just be absorbed by the Earth.

The most important part of the model is that the solar radiation either passes through the atmosphere without interacting or is reflected. Aka, its absorbance is 0. The really nice thing about having a value for the average albedo of the Earth[1] is that it completely wraps up that part of the calculation. That one number tells us how much of the irradiation from the Sun gets bounced back towards space. Meanwhile the IR radiation from the Earth does interact with the atmosphere[4].

So, our model ends up looking like figure 3. The solar irradiance is partially absorbed by the Earth system; the Earth emits IR radiation which is partially absorbed by the atmosphere; and the atmosphere emits some IR radiation to Earth and to space.

3.1 Assumptions

1. heat flux from space is negligible
2. radiogenic heat flux from the Earth is negligible
3. reflection of IR from the Earth is neglected
4. solar irradiance isn't absorbed by the atmosphere

4 Calculation

First, we can get a relationship between T_E and T_a by drawing our system boundary around the atmosphere and summing up the total heat flux.

$$0 = +\epsilon_a \epsilon_E \sigma T_E^4 - 2\epsilon_a \sigma T_a^4 \tag{1}$$

$$T_a = \left(\frac{\epsilon_a \epsilon_E \sigma}{2\epsilon_a \sigma}\right)^{1/4} T_E \tag{2}$$

$$T_a = (\epsilon_E/2)^{1/4} T_E \tag{3}$$

The other useful thing to draw a system boundary around is the whole planet, including the atmosphere (all the terms at the top of figure 3).

$$0 = +q_s'' - \bar{\rho} q_s'' - \tau_a \epsilon_E \sigma T_E^4 - \epsilon_a \sigma T_a^4 \quad (4)$$

$$0 = +q_s'' - \bar{\rho} q_s'' - \tau_a \epsilon_E \sigma T_E^4 - \epsilon_a \sigma \left((\epsilon_E/2)^{1/4} T_E \right)^4 \quad (5)$$

$$(\tau_a \epsilon_E + \epsilon_a \epsilon_E/2) \sigma T_E^4 = (1 - \bar{\rho}) q_s'' \quad (6)$$

$$T_E = \left(\frac{(1-\bar{\rho}) q_s''}{\sigma(\tau_a \epsilon_E + \epsilon_a \epsilon_E/2)} \right)^{1/4} \quad (7)$$

So, we start by finding what q_s'' is.

$$q_s'' = \text{solar constant} \cdot \frac{\text{frontal area of Earth}}{\text{surface area of Earth}} \quad (8)$$

$$q_s'' = \text{solar constant} \cdot \frac{\pi R_E^2}{4\pi R_E^2} \quad (9)$$

$$q_s'' = \text{solar constant}/4 \quad (10)$$

```
#watts from the whole Sun:
q.sun <- 4*pi*Rsun^2*sigma*Tsun^4
#watt/m^2 from Sun, out at 1 AU:
qA.AU <- q.sun/(4*pi*AU^2)
#avg solar flux over Earth's surface (q"_S):
qA.avg <- qA.AU/4
cat(qA.avg)# W/m^2

## 341.9704
```

We could have just looked up the solar constant, but we might as well calculate it.

Now, we plug everything into equation 7 to find the temperature of the Earth.

```
Tearth <- (
        ((1-rho.bar)*qA.avg)
        /(sigma*(
                tau.atm.ir*eps.earth.ir
                +eps.atm.ir*eps.earth.ir/2
                ))
        )^(1/4)
```

5 Results

```
cat(Tearth) # K, calculated temperature

## 292.9284

cat(Tearth.meas) # K, measured temperature

## 287.15

perc.err <- 100*
        (Tearth-Tearth.meas)/Tearth.meas
cat(perc.err) # percent error

## 2.012321
```

So, we wind up with an estimate of 292 K (15 °C), which represents only a 2% error. For a first order approximation, this seems unusually good. I think this can be attributed to the fact that pretty much every number we used was a "fudge factor", designed to make the first-approximation equations give the right answer. In reality, we would be doing integrations on the average absorbtion and emission spectra of everything.

So, let's check that some of our approximations were good...

```
qA.geo <- q.geo/(4*pi*Rearth^2)
cat(qA.geo)

## 0.03921073

cat(qA.avg)

## 341.9704

cat(qA.avg/qA.geo)

## 8721.347
```

Yup, the radiogenic heat flux is about eight thousand times smaller than the heat flux from the Sun. What about the n^{th} order reflections?

```r
#IR emission from Earth:
cat(eps.earth.ir*sigma*Tearth^4)

## 392.425

#IR emission from atmosphere:
cat(
        (1-eps.atm.ir)
        *eps.earth.ir*sigma*Tearth^4
        )

## 86.33351

#IR reflection from Earth:
cat(
        (1-eps.earth.ir)
        *(1-eps.atm.ir)
        *eps.earth.ir*sigma*Tearth^4
        )

## 5.180011
```

Yeah, the second order reflection of $5\,\mathrm{W/m^2}$ is pretty wimpy compared to the initial $392\,\mathrm{W/m^2}$ emission, or even the $86\,\mathrm{W/m^2}$ re-emission from the atmosphere.

References

[1] P. Goode, J. Qiu, V. Yurchyshyn, J. Hickey, M.-C. Chu, E. Kolbe, C. Brown, and S. Koonin, "Earthshine observations of the earth's reflectance," *Geophysical Research Letters*, vol. 28, no. 9, pp. 1671–1674, 2001.

[2] P. D. Jones, M. New, D. E. Parker, S. Martin, and I. G. Rigor, "Surface air temperature and its changes over the past 150 years," *Reviews of Geophysics*, vol. 37, no. 2, pp. 173–199, 1999.

[3] K. Collaboration *et al.*, "Partial radiogenic heat model for earth revealed by geoneutrino measurements," *Nature Geoscience*, vol. 4, no. 9, pp. 647–651, 2011.

[4] T. Bergman, A. Lavine, and F. Incropera, *Fundamentals of Heat and Mass Transfer, 7th Edition.* John Wiley & Sons, Incorporated, 2011.