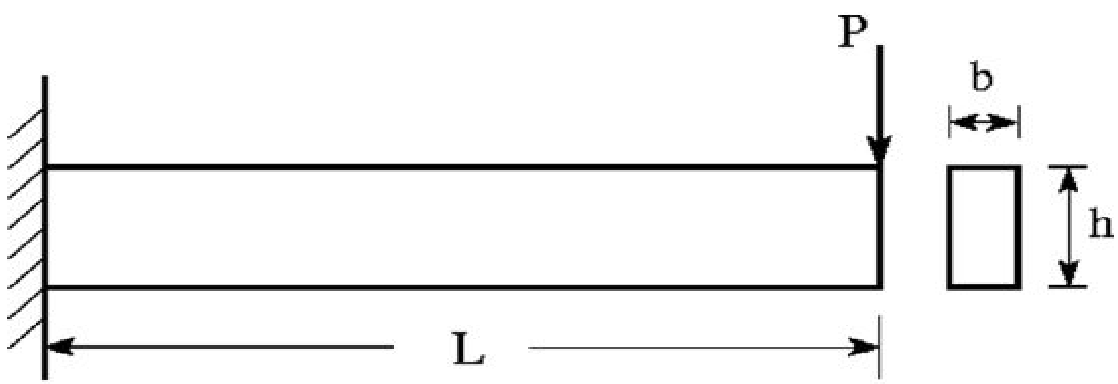
Portland State University

Analysis of a Cantilever Beam

Viscoelastic vs. Elastic



Joseph Shields

T. Gordon Chen

Joshua Schmidt

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ME510

Viscoelasticity

Abstract

This paper goes into detail to derive solutions to both the Euler-Bernoulli and Timoshenko beam bending theories, specifically, when applied to a free-vibrating cantilever beam. This is done to compare and discuss what differences there are in the theories and how they affect the end solution. When moving from an elastic to viscoelastic application, it is important to understand which beam bending theory most accurately represents the beam characteristics.

Main Objective

Explore the solutions to both Euler-Bernoulli and Timoshenko theories in the case of an elastic cantilever beam, then discuss the implications of how these theories differ and to which beam characteristics a specific theory should be applied.

Keywords: Cantilever Beam, Elastic, Viscoelastic, Timoshenko, Euler-Bernoulli, Free-Vibration

Introduction

Viscoelastic materials have been utilized in various engineering applications ranging from automotive to construction, and are generally used to isolate vibrations, dampen noises and absorbs shocks. For calculation simplicity in structural analysis, materials are often considered to be elastic. When solving for viscoelastic materials, it is important to understand which beam bending theory most accurately takes into account the effects of certain beam characteristics. The purpose of this paper is to compare the Euler-Bernoulli and Timoshenko beam bending theories in the case of a free-vibrating cantilever beam, and determine which theory best represents those beam characteristics.

Beam Bending Theory

Derivations and solutions to an elastic cantilever beam in free-vibration will be found for both the Euler-Bernoulli and Timoshenko beam bending theories. Through the process of deriving solutions to each theory, it will be discussed how one theory differs from another.

Euler-Bernoulli

The complete solutions to static end-loading and steady-state dynamic vibration of an elastic Euler-Bernoulli beam are found in appendix A.

This theory deals with the bending of long, slender beams ($length \gg height$ when vibrating in the direction of the height). The only relevant force is the elastic normal stress.

The static solution is solved simply by starting with the traction applied to the end of the beam and integrating until a solution for the displacement is reached, applying the boundary conditions along the way.

The dynamic steady state solution uses separation of variables to break the differential equation of motion of a control volume into differential equations for the shape and vibration of the beam separately. Each of these linear partial differential equations is then solved by creating a characteristic equation and applying the boundary conditions to solve for their coefficients.

Ultimately, there are infinitely many solutions to the dynamic steady state case, each one corresponding to a mode of vibration of the beam.

Timoshenko

The Timoshenko derivation and solution for a beam is seen in Appendix B. This model incorporates the shear deformation and rotational inertia effects when a beam experiences a stress application. This method is typically used in models that contain varying densities along the beam and composite materials such as viscoelastic materials. When comparing the two theories of Euler-Bernoulli and Timoshenko, we can see that Timoshenko incorporates both Euler-Bernoulli and Rayleigh theories on the deformation of beams.

In order to solve the Timoshenko fourth order equation, separation of variables and methods in differential equations were used to solve for the transverse displacement and slope of the centroidal axis.

Beam Characteristics

Due to the differences in the beam bending theories previously discussed, it is apparent that certain beam properties could influence which beam theory is most applicable under various situations. It will be discussed that factors such as beam dimensions and material properties should be taken into considerations before choosing an appropriate beam bending theory.

Beam Dimensions

While unable to investigate in great detail in this research paper, it can be shown from various literature [4,8] that the ratio of height versus length (h/l), when the height is in the same direction as the vibration, can have a significant impact to which beam theory is most accurate. When looking at a beam with h/l < ~10%, it has been shown that the Euler-Bernoulli theory is sufficient, however for h/l > ~10% the Timoshenko theory most accurately represents this case. [8] Conclusions can be drawn from the detailed derivations of this paper to support this claim. It is apparent that in Timoshenko theory, the inclusion of shear force and rotary inertia terms is especially important in regards to high h/l ratio beams, since a larger cross-sectional area in the shear direction will have greater influence on these additional terms. When the h/l ratio is low, it makes sense that these additional terms have a smaller effect, and therefore can be neglected by using the Euler-Bernoulli theory.

Viscoelastic Materials

As shown in the beam theories section of this paper, it was made apparent that the Timoshenko beam bending theory takes into account the effects of shearing force and rotary motion. In the case of a free-vibrating cantilever beam made of viscoelastic material, it can be theorized that these considerations are especially important. Viscoelastic material adds a time dependence in the displacement of the material when analyzing the dynamic response of a structure or beam utilizing viscoelastic material. This dependence is initially represented in the complex modulus.

In order to complete an accurate analysis of a beam with viscoelastic material properties, an elastic-viscoelastic analogy must be performed. This analogy is also known as the Elastic-Viscoelastic Correspondence Principle. Starting with the homogeneous Timoshenko equation (from Appendix B), one must perform a Laplace transformation to convert this equation into the Laplace domain. Once in the Laplace domain, both the Elastic and Shear Modulus (E and G, respectively) must be substituted by the equivalent Complex Moduli (E* and G*), also in terms of Laplace domain. Solve for the desired quantities, similar to what was done in the elastic solution. It is then required to perform an inverse Laplace transform, which will result in the viscoelastic solution of the problem as a function of time. [1]

Course Takeaways

The difference between viscoelastic and elastic materials is the relationship between stress and strain. Elastic behavior is a function of stress and deformations, while viscous behavior relates stress to strain rate. The ubiquitous method for analyzing viscoelastic materials is to use spring and dashpot models, in various combinations, to describe the relationship between elements of the strain tensor and the corresponding elements in the stress tensor. An elastic model contains Hooke’s elastic spring, while the Kelvin model and Maxwell models consist of a Newtonian dashpot and Hookean spring connected in parallel and series, respectively. Appendix C depicts the Rheological models for these three simple cases, along with the corresponding stress-strain relationships.

Elastic Model

An Elastic model is represented as a linear spring model. Hooke’s law is a first order linear approximation in analyzing the response of a spring to an applied force without a permanent deformation, defining a linear relationship between stress and strain. General application of this model involves the Euler beam bending method. This model is widely used across engineering applications to describe mechanical structures such as beams, plates and endless other items. This model is often used to represent the response of many metals, plastics and other materials considered as linear-elastic. This method is unable to accurately describe the response of materials that exhibit time and temperature dependence, such as viscoelastic materials. In many industry applications, this limitation is ignored for the benefit of simple analysis.

Kelvin Model

A Kelvin model is a solid undergoing reversible viscoelastic strain. It is represented as a parallel system containing both spring and dashpot elements. This model is considered a viscous solid, and therefore can be useful in describing the response of rubber, wood (in low loading conditions) and organic polymers. This model is unable to accurately describe stress relaxation because at constant strain a dashpot cannot relax.

Maxwell Model

When a Maxwell is loaded with a constant strain, the stress relaxes over time. It is represented as a system in series containing both spring and dashpot elements. This model is considered an elastic liquid, and therefore can be useful in describing the response of soft solids such as wet concrete, or metals and thermoplastic polymers near their melting temperature. This model is unable to accurately describe creep because at constant stress it shows a constant strain-rate, rather than a decreasing strain-rate.

Summary of Applications

The stress-strain relationships for Rheological models are analyzed in the form of Maxwell, Kelvin, Generalized Kelvin(GKM), Generalized Maxwell(GMM) and standard model. Simplifying a complex system can be performed by analyzing each sub-system in terms of Kelvin and Maxwell models. The Fourier and Laplace transformations can be applied to these models to find the dynamic response, complex modulus, storage and dissipation energy. An elastic and viscoelastic analogy for stress and strain can be derived by utilizing deviatoric operators and inverse Laplace transformation.

Conclusions

The derivation of solutions to the Euler-Bernoulli and Timoshenko theories, in context of a free-vibrating elastic cantilever beam, has given insight to the contributing terms in each theory. It has become apparent, through this process, that Timoshenko is a specialized case of Euler-Bernoulli, in which it incorporates additional terms such as shear forcing and rotary inertia. Through comparison, research, and general observation of these theories, it was concluded that Timoshenko is best applied to cantilever beams which are of Viscoelastic material, or in cases where the height to length (h/l) ratio is greater than 10%. [8]

Appendix A : Euler-Bernoulli Beams

Nomenclature

Symbol	Meaning	Symbol	Meaning
E	Elastic modulus	t	Time
I	Area moment of inertia	x	Distance from the root of the beam
$w(x,t)$	Vertical displacement	L	Length of the beam
w_0	End displacement (similar for other variables)	ω	Frequency
θ	w', angle of the beam	k	Wave number
M	EIw'', bending moment	A	Cross-sectional area of the beam
V	EIw''', total shear over the cross-section	X''	$\frac{dX}{dx}$, spatial derivative
$X(x)$	Vibratory mode shape function	$\ddot{X}$	$\frac{dX}{dt}$, time derivative

Static Case

Assumptions

- Only small deflections / long aspect ratio (small bending radius)
- Elastic material
- End loading
- Static (no time dependence)

Boundary Conditions

1. w(0)=0 (clamping position)
2. w'(0)=0 (clamping angle)
3. $w''(L) = M_0$ (end bending)
4. $w'''(L) = V_0$ (end shear)

Solution

In this case, some shear and bending moment are applied to the end of the beam. The solution is simply a matter of integrating the shear to get the displacement.

$$V = V_0$$

We integrate to get

$$M = V_0 x + C_1 .$$

We apply the value of the applied bending moment to get the value of C_1.

$$M_0 = V_0 L + C_1$$
$$C_1 = M_0 - V_0 L$$

So,

$$M = V_0 x + M_0 - V_0 L .$$

Integrating again, we get

$$EI\theta = V_0 x^2 + (M_0 - V_0 L)x + C_2 .$$

To find C_2, we apply the angle at the root of the beam.

$$EI\theta_{root} = V_0 0^2 + (M_0 - V_0 L)0 + C_2$$
$$C_2 = EI\theta_{root}$$

So,

$$EI\theta = V_0 x^2 + (M_0 - V_0 L)x + EI\theta_{root} .$$

We integrate one more time to get the equation for displacement.

$$EIw = V_0 x^3 + (M_0 - V_0 L)x^2 + EI\theta_{root} x + C_3$$

Finally, we apply the position of the root of the beam to get the value of C_3.

$$EI0 = V_0 0^3 + (M_0 - V_0 L)0^2 + EI\theta_{root} 0 + C_3$$
$$C_3 = 0$$

So, the complete solution for the statically loaded case is

$$EIw = V_0 x^3 + (M_0 - V_0 L)x^2 + EI\theta_{root} x .$$

Dynamic Case

Assumptions

- Only small vibrations / long aspect ratio (small bending radius)
- Elastic material
- No waves (initial displacement matches a modeshape)
- No loading

Boundary Conditions

5. X(0)=0 (clamping position)
6. X’(0)=0 (clamping angle)

7. X''(L)=0 (end bending)
8. X'''(L)=0 (end shear)
9. $\dot{w}(t=0)=0$ (initially static)
10. $w(t=0,\ x=L)=\ w_0$ (initial displacement)

Governing Equations

$F=m\cdot a$

$\sigma=\varepsilon\cdot E$

$V'=q$

Solution

Equation of Motion

For a control volume some distance x along the length of the beam, the shear acting on each side of the control volume will be the only force acting on it. Thus,

$$F=m\cdot a$$
$$-dx\ V'=\rho A dx\cdot\ddot{w}$$
$$-EIw^{iv}=\rho A\ddot{w}\,,$$

with w^{iv} being the fourth derivative of w with respect to x.

Separation of Variables

We begin by assuming the solution is separable in x and t, i.e.

$$w(x,t)=X(x)T(t)\,.$$

Manipulating the differential equation for w, we separate the space and time components.

$$-EIX^{iv}T=\rho AX\ddot{T}$$
$$\tfrac{-EI}{\rho A}\cdot\tfrac{X^{iv}}{X}=\tfrac{\ddot{T}}{T}=-\omega^2$$

Note that here we have arbitrarily chosen the constant value of $-\omega^2$. Right now, this is an arbitrary constant, like C_1, for which we will later find a value. We now have one differential equation in x, and another in t.

Mode Shape Function

Manipulating the equation in x, we get

$$X^{iv}=Ce^{sx}\,,$$

which yields the characteristic equation

$$Cs^4e^{sx}=\tfrac{\rho A\omega^2}{EI}Ce^{sx}\,.$$

This has solutions of

$$s_1=k\,,\ s_2=-k\,,\ s_3=ik\,,\text{ and }\ s_4=-ik\,,$$

where

$$i = \sqrt{-1} \text{ and } k = \sqrt[4]{\frac{\rho A \omega^2}{EI}}.$$

So, the solution for X takes the form

$$X = B_1 e^{kx} + B_2 e^{-kx} + B_3 e^{ikx} + B_4 e^{-ikx}.$$

Note that these are not the same C values as in the static case. Conveniently, this kind of linear combination of exponentials can be rephrased as

$$X = C_1 \cos kx + C_2 \sin kx + C_3 \cosh kx + C_4 \sinh kx,$$

so long as X is known to be only real, which it is.

To apply the boundary conditions, we will need to use the derivatives of X, which are

$$X' = -C_1 k \sin kx + C_2 k \cos kx + C_3 k \sinh kx + C_4 k \cosh kx,$$

$$X'' = -C_1 k^2 \cos kx - C_2 k^2 \sin kx + C_3 k^2 \cosh kx + C_4 k^2 \sinh kx, \text{ and}$$

$$X''' = C_1 k^3 \sin kx - C_2 k^3 \cos kx + C_3 k^3 \sinh kx + C_4 k^3 \cosh kx.$$

We begin with our first boundary condition, the clamping position of the beam.

$$0 = C_1 \cos 0 + C_2 \sin 0 + C_3 \cosh 0 + C_4 \sinh 0$$

$$C_3 = -C_1$$

Similarly, we can apply the second boundary condition, the clamping angle, to get a relationship between the other constants.

$$0 = -C_1 k \sin 0 + C_2 k \cos 0 + C_3 k \sinh 0 + C_4 k \cosh 0$$

$$C_4 = -C_2$$

This changes our X equation to

$$X = C_1 \cos kx + C_2 \sin kx - C_1 \cosh kx - C_2 \sinh kx.$$

Applying the third boundary condition, the lack of end bending, we get

$$0 = -C_1 k^2 \cos kL - C_2 k^2 \sin kL - C_1 k^2 \cosh kL - C_2 k^2 \sinh kL.$$

The fourth boundary condition, the lack of end shear, gives us

$$0 = C_1 k^3 \sin kL - C_2 k^3 \cos kL - C_1 k^3 \sinh kL - C_2 k^3 \cosh kL.$$

Together with the previous equation, we get the system of equations

$$C_1(-\cos kL - \cosh kL) = C_2(\sin kL + \sinh kL)$$

$$C_1(\sin kL) - \sinh kL = C_2(\cos kL + \cosh kL).$$

Multiplying them together, we get

$$-C_1 C_2(\cos kL + \cosh kL) = C_1 C_2(\sin kL - \sinh kL)(\sin kL + \sinh kL)$$

$$-\cos^2 kL - \cosh^2 kL - 2\cos kL \cosh kL = \sin^2 kL - \sinh^2 kL$$

$$-2\cos kL \cosh kL = (\sin^2 kL + \cos^2 kL) + (\cosh^2 kL - \sinh^2 kL)$$

$$\cos kL \cosh kL = -1.$$

This allows us to numerically find values for k, by numerically finding the roots of the above equation.

$kL = 1.875104,\ 4.694091,\ 7.854757,\ 10.995540,\ 14.137168,\ 17.278759,\ 20.420352,\ 23.561944, 26.703537\ ...$

There are also the negative versions of these roots, but they would result in non-physical solutions for $T(t)$. Now that we know the possible values of k, we also know the possible values of ω.

$$k_n = \sqrt[4]{\frac{\rho A \omega_n^2}{EI}}$$
$$k_n L^4 = L^4 \frac{\rho A \omega_n^2}{EI}$$
$$\omega_n^2 = \frac{EI}{\rho A} \cdot \frac{(k_n L)^4}{L^4}$$
$$\omega_n = \frac{(k_n L)^2}{L^2} \sqrt{\frac{EI}{\rho A}}$$

Time-Dependent Component

Now, we can address the equation in time.

$$\ddot{T} = -\omega_n^2 T$$

Assuming T takes the form

$$T = e^{\lambda t + \phi}$$

yields the characteristic equation

$$\lambda^2 T = -\omega_n^2 T\ ,$$

which implies

$$\lambda = \pm i\omega_0\ .$$

Again, our solution is simplified by converting the exponentials into a trigonometric function.

$$T = e^{i\omega_n t + \phi} + e^{-i\omega_n t + \phi}$$

becomes

$$T = \cos(\omega_n t + \phi)\ .$$

At this point, we can see why negative values for ω are not physically possible, since they would result in real exponential solutions, which would “blow up.” Applying our fifth boundary condition, no motion at $t = 0$,

$$0 = \frac{d}{dt}(X(L)T(0))$$
$$0 = \dot{T}(0)$$
$$0 = -\sin(\omega_n t + \phi)$$
$$\phi = 0\ .$$

Complete Solution

The values of C_1 and C_2 are still unknown. To find them, we begin by substituting the two equations generated by the third and fourth boundary conditions together to form

$$C_2 = C_1 \frac{-\cos k_n L - \cosh k_n L}{\sin k_n L + \sinh k_n L}\ .$$

Applying our final boundary condition, we can now get the value for C_1.

$$w_0 = C_1 \cos k_n L - C_1 \frac{\cos k_n L + \cosh k_n L}{\sin k_n L + \sinh k_n L} \sin k_n L - C_1 \cosh k_n L + C_1 \frac{\cos k_n L + \cosh k_n L}{\sin k_n L + \sinh k_n L} \sinh k_n L$$
$$C_1 = \frac{w_0}{\cos k_n L - \cosh k_n L + \frac{\cos k_n L + \cosh k_n L}{\sin k_n L + \sinh k_n L}(-\sin k_n L + \sinh k_n L)}$$

For brevity, we will continue to simply write C_1. We now have used all of our boundary conditions and found all of our constants. The solution is complete.

$$X(x) = \frac{w_0(\cos k_n x - \cosh k_n x + \frac{\cos k_n L + \cosh k_n L}{\sin k_n L + \sinh k_n L}(-\sin k_n x + \sinh k_n x))}{\cos k_n L - \cosh k_n L + \frac{\cos k_n L + \cosh k_n L}{\sin k_n L + \sinh k_n L}(-\sin k_n L + \sinh k_n L)}$$
$$T(t) = \cos(\omega_n t)$$

$$w(x,t) = \frac{w_0(\cos k_n x - \cosh k_n x + \frac{\cos k_n L + \cosh k_n L}{\sin k_n L + \sinh k_n L}(-\sin k_n x + \sinh k_n x))}{\cos k_n L - \cosh k_n L + \frac{\cos k_n L + \cosh k_n L}{\sin k_n L + \sinh k_n L}(-\sin k_n L + \sinh k_n L)} \cos(\omega_n t)$$

Plots

Below are plots of the different mode shapes taken at evenly spaced time intervals. The example used is an 1 m x 10 cm x 5 mm aluminum beam vibrating along its intermediate axis, with an initial displacement of 1 cm at the tip.

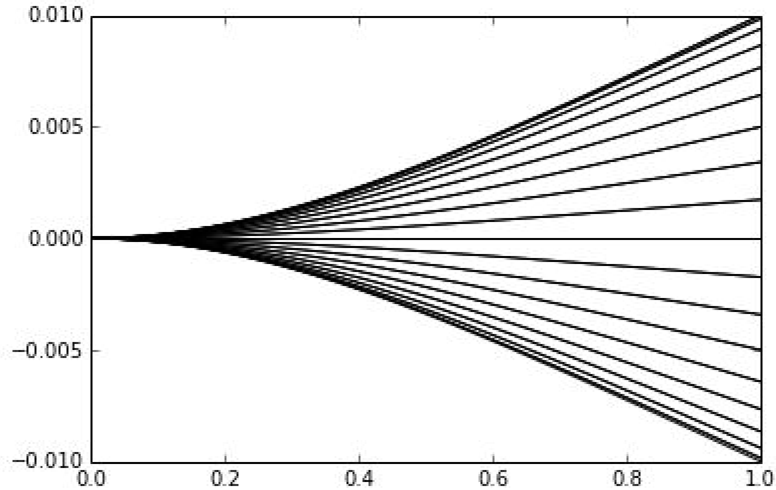


Figure E1. The fundamental mode of vibration for an Euler-Bernoulli beam (ω_0).

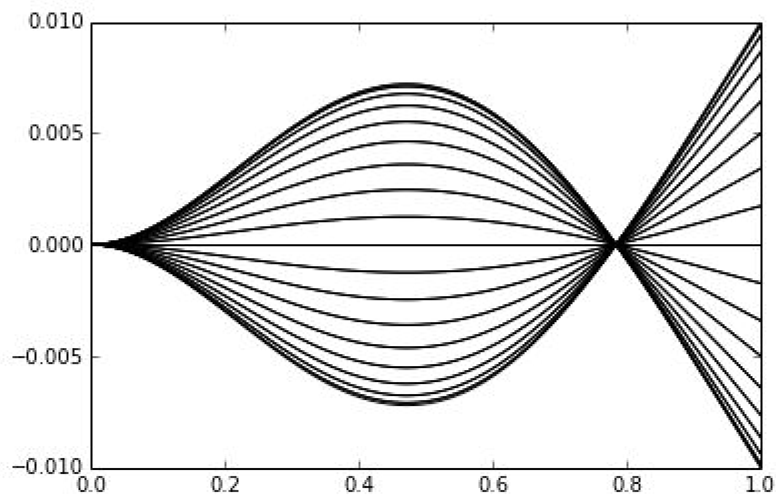


Figure E2. The first harmonic of an Euler-Bernoulli beam (ω_1).

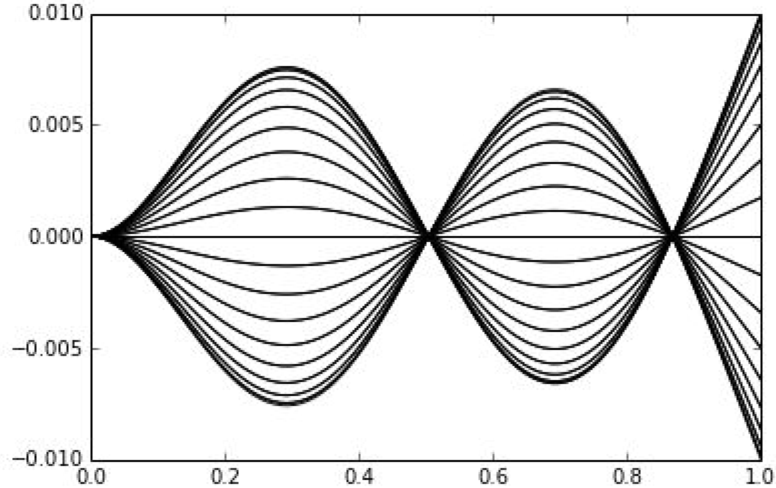


Figure E3. The second harmonic of an Euler-Bernoulli beam (ω_2).

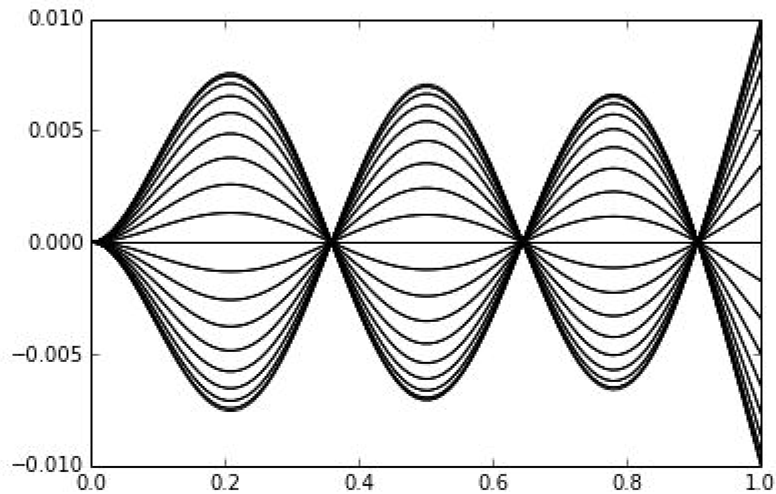


Figure E4. The third harmonic of an Euler-Bernoulli beam (ω_3).

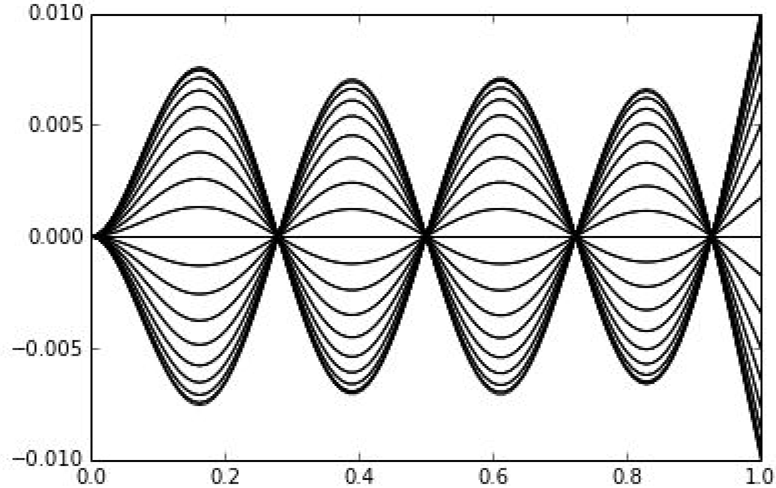


Figure E5. The fourth harmonic of an Euler-Bernoulli beam (ω_4).

Appendix B : Timoshenko

Assumptions

- Only small vibrations / long aspect ratio (small bending radius)
- Elastic material
- No waves

Boundary Conditions for Cantilever Beam

- $w(0,t) = 0$
- $\psi(0,t) = 0$
- $\partial\psi/\partial x(l,t) = 0$
- $\frac{1}{l}\frac{\partial w}{\partial x}(l,t) - \psi(l,t) = 0$

Governing Equations

The governing equations are derived by analyzing the equation of motion with the assumption of deformation in two planes. The elastic relationship is shown below:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \frac{\rho \partial u^2}{\partial t^2}$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} = \frac{\rho \partial u^2}{\partial t^2}$$

The shear relationship with the beam is derived by integrating across the cross sectional area.

$$\frac{\partial M}{\partial x} - Q - \int_{c_1}^{c_2} \tau_{xz} z \frac{\partial b}{\partial z} dz = \rho \int_{c_1}^{c_2} bz \frac{\partial^2 u}{\partial t^2} dz$$

$$q + \frac{\partial Q}{\partial x} - \int_{c_1}^{c_2} \sigma_z \frac{\partial b}{\partial z} dz = \rho \int_{c_1}^{c_2} b \frac{\partial^2 w}{\partial t^2} dz$$

u is porportional to $-z * \psi(x,t)$

$$\sigma_x = E\epsilon_x = E\frac{\partial u}{\partial x} = -Ez\frac{\partial \psi}{\partial x}$$

$$\sigma_z = 0 \; there\ is\ no\ external\ loads\ on\ the\ Z\ surface$$

$$M = \int_{c_1}^{c_2} \sigma_x \, bz \, dz = -E\frac{\partial \psi}{\partial x}\int_{c_1}^{c_2} z^2 b \, dz = -EI\frac{\partial \psi}{\partial x}$$

$$\tau_{xz} = G\gamma_{xz} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = G\left(\frac{\partial w}{\partial x} - \psi\right) = G\beta$$

k' ensures that gamma is constant across each cross section.

$$Q = \int_{c_1}^{c_2} \tau_{xz} \, bz \, dz = G\beta\int_{c_1}^{c_2} b \, dz = G\beta A = k'G\beta A$$

By transposing the two equations below; we are able to eliminate one of the dependent variables to find a solution and substituting it back in to solve for the other equation.

$$EI\frac{\partial^2 \psi}{\partial x^2} + k'GA\left(\frac{\partial w}{\partial x} - \psi\right) = \rho I\frac{\partial^2 \psi}{\partial t^2}$$

$$q + k'GA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) = \rho A\frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial \psi}{\partial x} = \frac{q}{k'GA} + \frac{\partial^2 w}{\partial x^2} - \frac{\rho}{k'G}\frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial^3 \psi}{\partial x^3} = \frac{1}{k'GA}\frac{\partial^2 q}{\partial x^2} + \frac{\partial^4 w}{\partial x^4} - \frac{\rho}{k'G}\frac{\partial^4 w}{\partial x^2 \partial t^2}$$

$$\frac{\partial^3 \psi}{\partial x \partial t^2} = \frac{1}{k'GA}\frac{\partial^2 q}{\partial t^2} + \frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{\rho}{k'G}\frac{\partial^4 w}{\partial t^4}$$

$$EI\frac{\partial^3 \psi}{\partial x^3} + k'GA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) = \rho I\frac{\partial^3 \psi}{\partial x \partial t}$$

Timoshenko Equations: To include shear corrections and rotary inertia terms

$$EI\frac{\partial^4 w}{\partial x^4} + A\rho\frac{\partial^2 w}{\partial t^2} - I\rho\left(\frac{E}{k'G} + 1\right)\left(\frac{\partial^4 w}{\partial x^2 \partial t^2}\right) + \frac{I\rho^2}{k'G}\frac{\partial^4 w}{\partial t^4} = q + \frac{EI}{k'GA}\left(\frac{\rho}{E}\frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2}\right)$$

$$EI\frac{\partial^4 \psi}{\partial x^4} + A\rho\left(\frac{\partial^2 \psi}{\partial t^2}\right) - I\rho\left(\frac{E}{k'G} + 1\right)\left(\frac{\partial^4 \psi}{\partial x^2 \partial t^2}\right) + \frac{\rho^2 I}{k'G}\frac{\partial^4 \psi}{\partial t^4} = \frac{\partial q}{\partial x}$$

Homogeneous Solutions

$$EI\frac{\partial^4 w}{\partial x^4}+A\rho\frac{\partial^2 w}{\partial t^2}-I\rho\left(\frac{E}{k'G}+1\right)\left(\frac{\partial^4 w}{\partial x^2\partial t^2}\right)+\frac{I\rho^2}{k'G}\frac{\partial^4 w}{\partial t^4}=0$$

$$EI\frac{\partial^4 \psi}{\partial x^4}+A\rho\left(\frac{\partial^2 \psi}{\partial t^2}\right)-I\rho\left(\frac{E}{k'G}+1\right)\left(\frac{\partial^4 \psi}{\partial x^2\partial t^2}\right)+\frac{\rho^2 I}{k'G}\frac{\partial^4 \psi}{\partial t^4}=0$$

Separation of Variables

$$Let \qquad w(x)=Y(x)T(t)$$

$$\frac{EI\frac{\partial^4 Y}{\partial x^4}}{Y}+\frac{\rho A\frac{\partial^2 T}{\partial t^2}}{T}-\frac{\rho I\left(1+\frac{E}{k'G}\right)\frac{\partial^2 Y\partial^2 T}{\partial x^2\partial t^2}}{YT}+\frac{\frac{\rho^2 I}{k'G}\frac{\partial^4 T}{\partial t^4}}{T}=0$$

$$\frac{d}{dx}EI\frac{\frac{\partial^4 Y}{\partial x^4}}{Y}-\frac{d}{dx}\left(\rho I\left(1+\frac{E}{k'G}\right)\frac{\frac{\partial^2 Y}{\partial x^2}}{Y}\right)\frac{\frac{\partial^2 T}{\partial t^2}}{T}=0$$

$$\frac{d}{dx}EI\frac{\frac{\partial^4 Y}{\partial x^4}}{Y}=\frac{d}{dx}\left(\rho I\left(1+\frac{E}{k'G}\right)\frac{\frac{\partial^2 Y}{\partial x^2}}{Y}\right)\frac{\frac{\partial^2 T}{\partial t^2}}{T}$$

$$\frac{\frac{\partial^2 T}{\partial t^2}}{T}=\frac{\frac{d}{dx}EI\frac{\frac{\partial^4 Y}{\partial x^4}}{Y}}{\frac{d}{dx}\left(\rho I\left(1+\frac{E}{k'G}\right)\frac{\frac{\partial^2 Y}{\partial x^2}}{Y}\right)}=-\lambda$$

$$\left(\frac{\partial^2 T}{\partial t^2}\right)=\lambda T(t)\rightarrow\left(\frac{\partial^4 T}{\partial t^4}\right)=\lambda^2 T(t)$$

$$EI\frac{\frac{\partial^4 Y}{\partial x^4}}{Y}-\rho A\lambda+\rho I\lambda\left(1+\frac{E}{k'G}\right)\frac{\frac{\partial^2 Y}{\partial x^2}}{Y}+\frac{\rho^2 I}{k'G}\lambda^2=0$$

$$\left(\frac{\partial^4 Y}{\partial x^4}\right)-\frac{\rho A}{EI}\lambda Y+\frac{\rho}{E}\lambda\left(1+\frac{E}{k'G}\right)\left(\frac{\partial^2 Y}{\partial x^2}\right)+\frac{\rho^2}{k'GE}\lambda^2 Y=0$$

$$\left(\frac{\partial^4 Y}{\partial x^4}\right)+\frac{\rho}{E}\lambda\left(1+\frac{E}{k'G}\right)\left(\frac{\partial^2 Y}{\partial x^2}\right)+\frac{\rho\lambda}{EI}\left(\frac{\rho I}{k'G}-A\right)Y=0$$

Solving for Roots of ODE

$$Y'''' = r^2$$

$$r^2 + \frac{\rho}{E}\lambda\left(1 + \frac{E}{k'G}\right)r + \frac{\rho\lambda}{EI}\left(\frac{\rho I}{k'G} - A\right) = 0$$

$$r = \frac{-\frac{\rho}{E}\lambda\left(1 + \frac{E}{k'G}\right) \pm \sqrt{\left(\frac{\rho}{E}\lambda\left(1 + \frac{E}{k'G}\right)\right)^2 - 4\left(\frac{\rho\lambda}{EI}\left(\frac{\rho I}{k'G} - A\right)\right)}}{2}$$

$$r_1 = \frac{-\frac{\rho}{E}\lambda\left(1 + \frac{E}{k'G}\right) + \sqrt{\left(\frac{\rho}{E}\lambda\left(1 + \frac{E}{k'G}\right)\right)^2 - 4\left(\frac{\rho\lambda}{EI}\left(\frac{\rho I}{k'G} - A\right)\right)}}{2}$$

$$r_2 = \frac{-\frac{\rho}{E}\lambda\left(1 + \frac{E}{k'G}\right) - \sqrt{\left(\frac{\rho}{E}\lambda\left(1 + \frac{E}{k'G}\right)\right)^2 - 4\left(\frac{\rho\lambda}{EI}\left(\frac{\rho I}{k'G} - A\right)\right)}}{2}$$

$For \ \frac{Ak'G}{\rho I} > 1$

$$Y = C_1 e^{rx} + C_2 e^{-rx} + C_3 \cos(r_2 x) + C_4 \sin(r_2 x)$$

$For \ \frac{Ak'G}{\rho I} < 1$

$$Y = C_1 \cos\,(r_1 x) + C_2 \sin\,(r_1 x) + C_3 \cos(r_2 x) + C_4 \sin(r_2 x)$$

$$\left(\frac{d^2T}{dt^2}\right) = -\lambda T(t) \rightarrow \left(\frac{d^4T}{dt^4}\right) = \lambda^2 T(t)$$

$$\ddot{T} = -\lambda T \quad \Rightarrow \quad T = C_5 e^{i\sqrt{\lambda}t} + C_6 e^{-i\sqrt{\lambda}t}$$

$$w(x,t) = Y(x)T(t)$$

For Elastic or Viscoelastic materials that share a ratio of cross sectional area, shear modulus, density, and 2nd moment of inertia, the dynamic response for displacement and slope of the centroidal axis will have the relationship shown below for any kind of beam.

.

$For\ \frac{Ak'G}{\rho I} > 1$

$$w(x,t) = Y(x)T(t)$$

$$w(x,t) = (C_1 e^{r_1 x} + C_2 e^{-r_1 x} + C_3 \cos(r_2 x) + C_4 \sin(r_2 x))(C_5 e^{i\sqrt{\lambda}t} + C_6 e^{-i\sqrt{\lambda}t})$$

$$\psi(x,t) = (C_1 e^{r_1 x} + C_2 e^{-r_1 x} + C_3 \cos(r_2 x) + C_4 \sin(r_2 x))(C_5 e^{i\sqrt{\lambda}t} + C_6 e^{-i\sqrt{\lambda}t})$$

$For\ \frac{Ak'G}{\rho I} < 1$

When the ratio is less than 1, all the roots become complex.

$$w(x,t) = Y(x)T(t)$$

$$w(x,t) = (C_1 \cos(r_1 x) + C_2 \sin(r_1 x) + C_3 \cos(r_2 x) + C_4 \sin(r_2 x))(C_5 e^{i\sqrt{\lambda}t} + C_6 e^{-i\sqrt{\lambda}t})$$

$$\psi(x,t) = (C_1 \cos(r_1 x) + C_2 \sin(r_1 x) + C_3 \cos(r_2 x) + C_4 \sin(r_2 x))(C_5 e^{i\sqrt{\lambda}t} + C_6 e^{-i\sqrt{\lambda}t})$$

Appendix C : Rheological Models

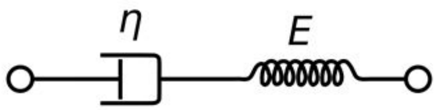
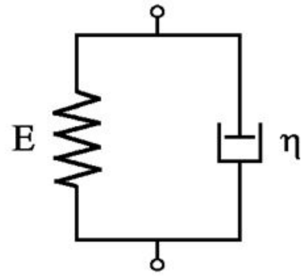
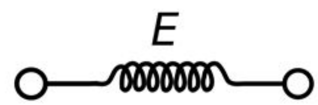
Maxwell	Kelvin	Elastic
		
$\sigma = E\epsilon = \eta\dot{\epsilon}$ $\epsilon = \epsilon^E + \epsilon^\eta$ $\dot{\epsilon} = \dot{\epsilon}^E + \dot{\epsilon}^\eta$	$\sigma = E\epsilon + \eta\dot{\epsilon}$ $\epsilon = \epsilon^E = \epsilon^\eta$ $\dot{\epsilon} = \dot{\epsilon}^E = \dot{\epsilon}^\eta$	$\sigma = E\epsilon$ $\epsilon = \epsilon^E$ $\dot{\epsilon} = \dot{\epsilon}^E$

Table 1: Rheological Models and the Corresponding Stress-Strain Relationships

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