Approximation of Lévy Processes

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Overview

Introduction

A stochastic process $X = \{X(t)\}_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a **Lévy process** if it satisfies the following properties:

1. $\mathbb{P}(X(0) = 0) = 1$
2. For $0 \leq s \leq t$, $X(t) - X(s) \sim X(t - s)$
3. For $0 \leq s \leq t$, $X(t) - X(s)$ is independent of $\{X(u) : u < s\}$
4. X is stochastically continuous: $\forall \epsilon > 0$, $\forall s > 0$

$$\lim_{t \to s} \mathbb{P}\Big(|\, X(t) - X(s) \,| > \epsilon \Big) = 0.$$

Figure: Path of standard Brownian motion.

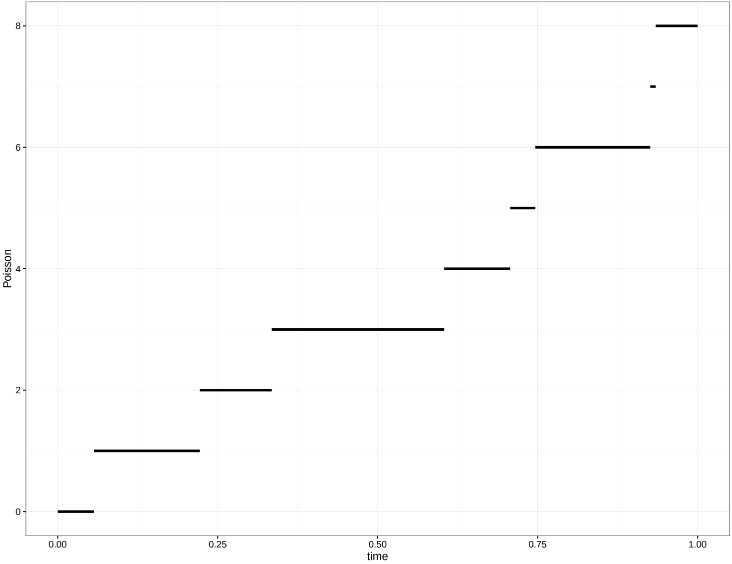

Figure: Path of a Poisson process.

Definition

A real-valued r.v. Y *is said to have* **infinitely divisible distribution** *if* $\forall n \in \mathbb{N} \; \exists \, Y_{1,n}, \ldots, Y_{n,n}$ *i.i.d. such that*

$$Y \sim Y_{1,n} + \cdots + Y_{n,n}.$$

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$$Y \sim Y_{1,n} + \cdots + Y_{n,n}.$$

$$X(t) = X(t/n) + \Big(X(2t/n) - X(t/n)\Big) + \cdots + \Big(X(t) - X((n-1)t/n)\Big)$$

Lévy-Khintchine formula

$$\Psi_Y(\theta) = -\log(\mathbb{E}(e^{i\theta Y}))$$

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Theorem

The probability distribution μ of a real-valued r.v. is infinitely divisible if and only if $\exists\,(a, b, Q)$ where $a \in \mathbb{R}$, $b \geq 0$, Q is a measure concentrated on $\mathbb{R} \setminus \{0\}$ (called Lévy measure) satisfying $\int_{\mathbb{R}} (1 \wedge x^2)\, Q(dx) < \infty$, such that for every $\theta \in \mathbb{R}$

$$\Psi(\theta) = ia\theta + \frac{1}{2} b^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x \mathbf{1}_{(|x|<1)}\right) Q(dx).$$

Linear Brownian motion

$$X(t) = \sigma W(t) + ct \sim \mathcal{N}(c, \sigma^2\, t)$$

Lévy-Khintchine formula

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Compound Poisson Process

$$\{\beta_i\}_{i\geq 1} \sim_{iid} F$$
$$N(t) \sim \text{Poisson}(\lambda t)$$
$$X(t) = \sum_{i=1}^{N(t)} \beta_i$$

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$$\Psi(\theta) = -\log \mathbb{E}\left(e^{i\theta \sum_{i=1}^{N(1)} \beta_i}\right) = \lambda \int_{\mathbb{R}} (1 - e^{i\theta x}) F(dx).$$

with Lévy measure $Q(dx) = \lambda F(dx)$.

$$
\begin{aligned}
\Psi(\theta) = & \left(ia\theta + \frac{1}{2}b^2\theta^2 \right) \\
& + \left(Q(\mathbb{R} \setminus (-1,1)) \int_{|x|>1} (1 - e^{i\theta x}) \frac{Q(dx)}{Q(\mathbb{R} \setminus (-1,1))} \right) \\
& + \left(\int_{0<|x|<1} (1 - e^{i\theta x} + i\theta x) Q(dx) \right).
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$$(\dots) = \sum_{n \geq 0} \left(\lambda_n \int_{A_n} (1 - e^{i\theta x}) F_n(dx) + i\theta\lambda_n \int_{A_n} x F_n(x) dx \right)$$

where $A_n = [2^{-(n+1)}, 2^{-n})$ and $F_n(dx) = \lambda_n^{-1} Q(dx)|_{A_n}$.

$$\int_{\mathbb{R}} (1 \wedge x^2)\, Q(dx) < \infty$$

for each Borel set B such that $0 \in B \setminus \partial B$, e.g. $B = (-\epsilon, \epsilon)$

- $Q(\mathbb{R} \setminus B) < \infty$
- $Q(B)$ can be finite or infinite
  if $Q(B) = \infty$ then $\forall \epsilon > 0 \quad Q((-\epsilon, \epsilon)) = \infty$
  $\Rightarrow$ there are countably many small jumps of arbitrarily small size!

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- Pure jump part reduces to a compound Poisson process when Q is finite.
- Q can only fail to be finite in a neighbourhood of 0.
- So as an approximation we can leave out jumps in some neighbourhood of 0.

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$N^\epsilon(t)$ is a Poisson process with (finite) rate $\|\tilde{Q}\|$ and jump distribution $\tilde{Q}/\|\tilde{Q}\|$ independent of the standard Brownian motion W.

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Theorem

[1] Assume Q has no atoms in a neighbourhood of 0. Restricting our processes to $t \in [0,1]$, $\sigma^{-1}(\epsilon) X_\epsilon \xrightarrow{D} W$ as $\epsilon \to 0$ if and only if $\sigma(\epsilon)/\epsilon \to \infty$ as $\epsilon \to 0$.

Second Approximation

- If the errors are approximately normal, then we can consider the following improved approximation:

$$X_2^{\epsilon}(t) = \mu_{\epsilon} t + (b^2 + \sigma^2(\epsilon))^{1/2} W(t) + N^{\epsilon}(t)$$

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Theorem

[1]

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\big(X_2^\epsilon(1) \leq x\big) - \mathbb{P}\big(X(1) \leq x\big) \right| \leq 0.8\sigma^{-3}(\epsilon) \int_{|x|<\epsilon} |x|^3 \, Q(\mathrm{d}x).$$

Stable process

- **Stable process** - Lévy process whose Lévy measure is given by:

$$Q(dx) := c_1 x^{-1-\alpha} \mathbf{1}_{x>0} dx + c_2 |x|^{-1-\alpha} \mathbf{1}_{x<0} dx,$$
$$\alpha \in (0,1), \quad c_1, c_2 \geq 0, \quad c_1 + c_2 > 0$$

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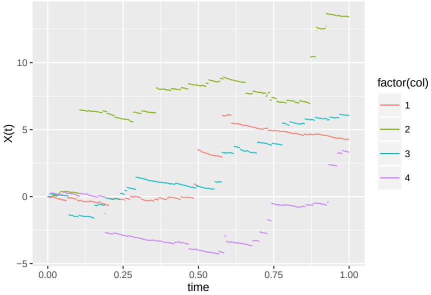

Figure: Sample paths of a stable process, parameters: $\alpha = 0.9$, $c_1 = 2$, $c_2 = 0.4$.

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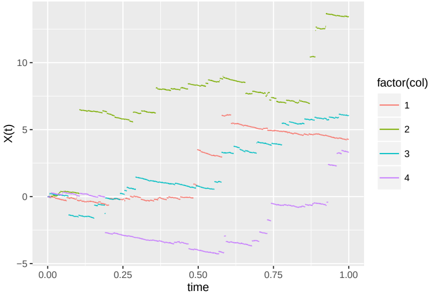

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- Conditions of theorem satisfied - approximation $X_2^\epsilon(t)$ is valid.

Normality of the error terms (stable process)

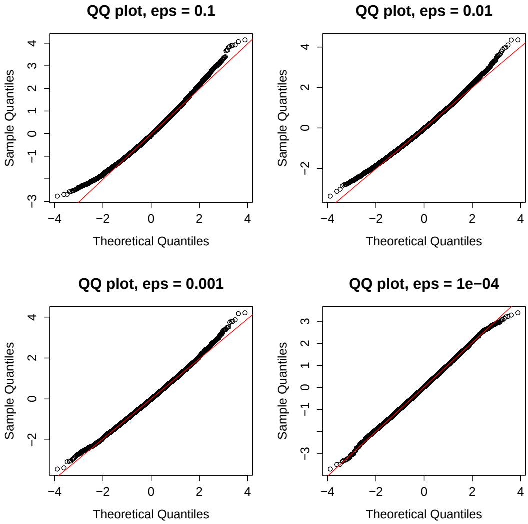

Figure: QQ-plots of $\sigma^{-1}(\epsilon)X_\epsilon(1)$ against standard normal.

Convergence (stable process)

Compare densities, and $E_\epsilon(x) := |\hat{P}(X_2^\epsilon(1) \leq x) - \hat{P}(X(1) \leq x)|$.

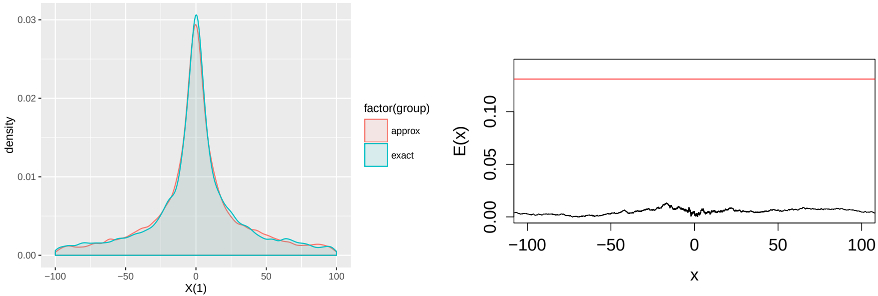

Figure: Marginal densities of $X_2^\epsilon(1)$ and $X(1)$ (left) and $E(x)$ and $E_\epsilon(x)$ (right).

Speed of Convergence (stable process)

Compare Berry–Esseen bound with empirical Kolmogorov–Smirnov distance $\sup_x E_\epsilon(x)$ as a function of ϵ

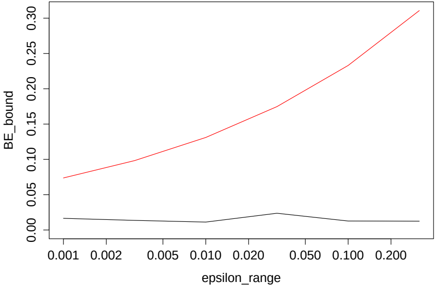

Figure: BE bound (red) and Kolmogorov–Smirnov distance (black).

Gamma and Generalised Gamma processes

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$$Q(dx) = cx^{-1}e^{-\tau x}\mathbf{1}_{\{x>0\}}dx, \quad c, \tau > 0$$

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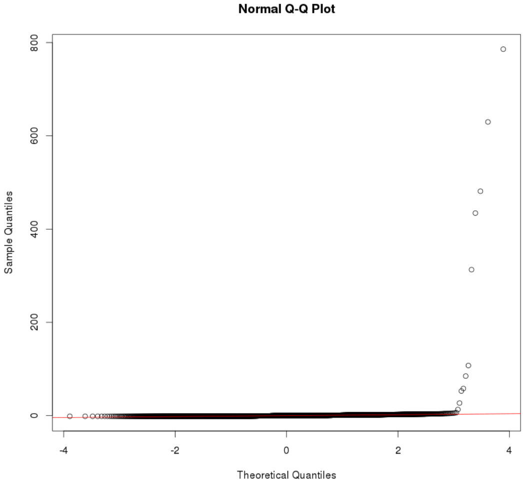
- Condition for the asymptotic normality of the error terms $X_\epsilon(t)$ fails in the case of Gamma process and is satisfied in the case of Generalised Gamma process.

Normality of the error terms (Gamma process)



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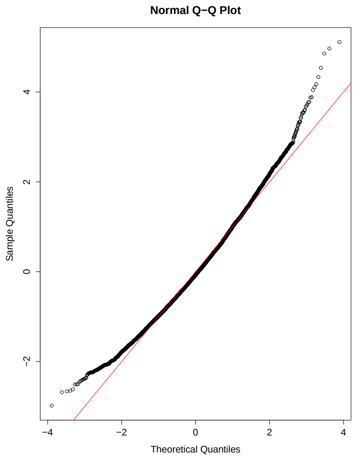
Normality of the error terms (Generalised Gamma process)



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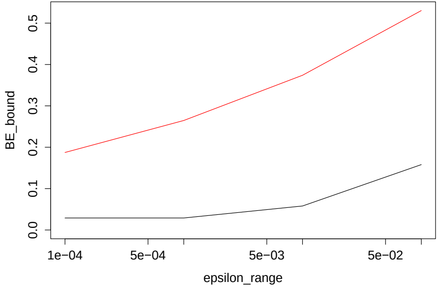

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Final Thoughts

When things go well

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When things go bad...

Theorem 4.1 in [1]: $\exists$ Lévy process X with Lévy measure Q such that for every other Lévy process Y with $\mathcal{L}(Y(1)) \in \mathcal{I}_0$ there exists a sequence $\epsilon_n \downarrow 0$ such that $\sigma(\epsilon_n)^{-1}X_{\epsilon_n}(1) \xrightarrow{D} Y(1)$.

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