Additional Problem Set for Lecture 6: Decision

(1) (a) There are two acts to choose from. If you choose act 1, you will win 4\$ with probability p and 81\$ with probability $1-p$. If you choose act 2, you will win 16\$ with probability p and 49\$ with probability $1-p$. For which values of p does an expected utility maximizer choose a different act than an expected monetary value maximizer if the utility function is $u(x) = \sqrt{x}$ where x is the amount in \$. (b) Read about the **St. Petersburg Paradox** in the Stanford Encyclopedia of Philosophy (http://plato.stanford.edu/entries/paradox-stpetersburg/). Does it provide an argument against maximizing the expected monetary value?

(2) Risk aversion is an ubiquitous phenomenon. For example, someone is risk averse if she prefers 1000\$ for sure to 5000\$ if a fair coin lands heads. Clearly, in this case, the expected monetary value of the first option is lower (viz. 1000\$) than the expected monetary value of the second option (viz. 2500\$). And yet, it may be totally rational to have this preference if one uses a *concave* utility function. A utility function $u(x)$ is concave if the second derivative of u with respect to x is negative (i.e. if $u''(x) < 0$). (To learn more about concave functions, you might want to read http://en.wikipedia.org/wiki/Concave_function) (a) Show how risk aversion can be modeled in terms of concave utility functions. (b) Show that the standard utility functions $u_1(x) = \sqrt{x}$ and $u_2(x) = \log(1 + x)$ are concave.

(3) Your preferences are as follows: (i) $A \sim BpC$, (ii) $A \sim BqD$, and (iii) $B \succ C$ with p, q in $(0, 1)$. What do you prefer most, C or D, provided that all your preferences satisfy the von Neumann and Morgenstern axioms?

(4) A famous argument for transitivity is the **money pump argument**. Read about it in the Stanford Encyclopedia of Philosophy:
http://plato.stanford.edu/entries/dynamic-choice/
http://plato.stanford.edu/entries/preferences/
At the same time, it is understandable that people sometimes have intransitive preferences as the puzzle of the self-torturer and the example involving cups of coffee discussed in the above-mentioned SEP entries show. This raises the question whether transitive preferences should be a requirement of rationality. Discuss.

(5) A famous example for (and indeed one of the earliest examples of) decision making under uncertainty is **Pascal's Wager**, which is an argument to convince you to act so as to come to believe in God. Read http://plato.stanford.edu/entries/pascal-wager/ and critically discuss the argument.

Solutions

(1a) We first calculate they expected monetary values for both acts:

$$\begin{aligned} EV1 &= 4\,p + 81\,(1-p) = 81 - 77\,p \\ EV2 &= 16\,p + 49\,(1-p) = 49 - 33\,p \end{aligned}$$

Hence, $EV1 > EV2$ if and only if $81 - 77\,p > 49 - 33\,p$, i.e. if $p < p_v := 8/11 \approx 0.73$.
Next, we calculate the expected utility for both acts with $u(x) = \sqrt{x}$:

$$\begin{aligned} EU1 &= 2\,p + 9\,(1-p) = 9 - 7\,p \\ EU2 &= 4\,p + 7\,(1-p) = 7 - 3\,p \end{aligned}$$

Hence, $EU1 > EU2$ if and only if $9 - 7p > 7 - 3p$, i.e. if $p < p_u := 1/2$.
Note that $p_u < p_v$ in this example. We conclude that an expected utility maximizer chooses a different act than an expected monetary value maximizer if and only if $p_u < p < p_v$.
(1b) Read the text.

(2a) Read http://en.wikipedia.org/wiki/Risk_aversion.
(2b) We calculate: $u_1'(x) = 1/(2\sqrt{x})$ and $u_1''(x) = -1/(4\sqrt{x}^3) < 0$. Hence, $u_1(x)$ is concave. Similarly, we calculate $u_2'(x) = 1/(1+x)$ and $u_2''(x) = -1/(1+x)^2 < 0$. Hence, $u_2(x)$ is concave.
N.B.: To refresh your mathematics, I recommend Kevin Houston's *How to Think Like a Mathematician: A Companion to Undergraduate Mathematics*, Cambridge: Cambridge University Press 2009. – An excellent book!

(3) Let the corresponding utilities be $u(A), u(B), u(C)$, and $u(D)$. We apply the von Neumann and Morgenstern axioms and obtain from (i) and (ii)

$$p\,u(B) + (1-p)\,u(C) = q\,u(B) + (1-q)\,u(D),$$

which is equivalent to

$$(p-q)\,(u(B) - u(C)) = (1-q)\,(u(D) - u(C)).$$

(Show this!) From (iii) we infer that $u(B) > u(C)$. Hence, $u(D) > u(C)$ if $p > q$, $u(D) < u(C)$ if $p > q$, and $u(D) = u(C)$ if $p = q$. That is, you prefer D to C if $p > q$, C to D if $p < q$, and you are indifferent between C and D if $p = q$.

(4) Read the texts and make up your mind! Feel free to discuss your views in the forum.

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