- ▶ I believe that Bayern Munich wins the Champions League next year.
- ▶ I hope that Bayern Munich wins the Champions League next year.
- ▶ I desire that Bayern Munich wins the Champions League next year.

▶ Others fear that Bayern Munich wins the Champions League next year.

## First role of propositions:

▶ They are the meanings of descriptive sentences:

'Bayern Munich wins the Champions League next year' expresses the same proposition as 'Bayern München gewinnt die Champions League nächstes Jahr'.

Both express that Bayern Munich wins the Champions League next year.

Second role of propositions:

► They are bearers of truth values:

That Bayern Munich wins the Champions League next year is true or false.

Third role of propositions:

▶ They are the contents of belief:

I believe that Bayern Munich wins the Champions League next year.

The proposition $that\ Socrates\ is\ a\ philosopher$ is true at the actual world, but it is not true at every possible world.

► (Id<sub>Worlds</sub>) For all possible worlds w, for all possible worlds w':

true at w.

- w = w' if and only if for all propositions X: X is true at w if and only if X is
- true at w'.
  ▶ (Id<sub>Prop</sub>) For all propositions X, for all propositions Y:
  X = Y if and only if for all possible worlds w: X is true at w if and only if Y is

## Definition:

Let W be a given non-empty set of possible worlds.

X is true at w if and only if w is a member of X.

- (i) X is a proposition (over W) if and only if X is a subset of W. (ii) If X is a proposition (over W) and w is a world in W, then

## Definition:

 $X = \{w: w \text{ is a member of } X\}$ 

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$$X = \{w: w \text{ is a member of } X\}$$

 $X = \{w: X \text{ is true at } w\}$ 

- $(\mathrm{Id}'_{Worlds})$  For all members w of W, for all members w' of W:

w is a member of Y.

w = w' if and only if for all subsets X of W: w is a member of X if and only if w' is a member of X.  $ightharpoonup (\mathrm{Id}'_{\mathsf{Prop}})$  For all subsets X of W, for all subsets Y of W:

 $X = \dot{Y}$  if and only if for all members w of W: w is a member of X if and only if

If X is a proposition, that is, a subset of W, let us denote its negation by:

 $\neg X$  is the set of possible worlds in W that are not members of X.

Set-theoretically:  $\neg X$  is the complement of X (with respect to W)

Equivalently:  $\neg X = \{ w \text{ in } W \text{: } w \text{ is not a member of } X \}$ 

Equivalently:  $\neg X = W \setminus X$  (read: 'W without X')

If X and Y are propositions, that is, subsets of W, let us denote their conjunction by:

 $X \wedge Y$ 

 $X \wedge Y$  is the set of possible worlds in W that are members of both X and Y.

Equivalently:  $X \wedge Y = \{ w \text{ in } W \text{: } w \text{ is a member of } X \text{ and } w \text{ is a member of } Y \}$ 

Set-theoretically:  $X \wedge Y$  is the intersection of X and Y

Equivalently:  $X \wedge Y = X \cap Y$  (read: 'X intersected with Y)

If $X$ and $Y$ are propositions, that is, subsets of $W$ , let us denote their $disjunction$ by:
$X \lor Y$

Equivalently:  $X \vee Y = \{w \text{ in } W: w \text{ is a member of } X \text{ or } w \text{ is a member of } Y\}$ 

 $X \vee Y$  is the set of possible worlds in W that are members of X or Y

Set-theoretically:  $X \vee Y$  is the union of X and Y.

Equivalently:  $X \lor Y = X \cup Y$  (read: 'X united with Y')

- negation (of descriptive sentences) corresponds to complement (of propositions/sets)
- conjunction (of descriptive sentences) corresponds to intersection (of
- propositions/sets)

▶ disjunction (of descriptive sentences) corresponds to union (of

propositions/sets)

# Prop(A): the proposition expressed by sentence A

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$$Prop(\neg A) \equiv \neg Prop(A) \equiv W \setminus Prop(A)$$

$$Prop(\neg A) = \neg Prop(A) = W \setminus Prop(A)$$

 $ightharpoonup Prop(A \land B) = Prop(A) \land Prop(B) = Prop(A) \cap Prop(B)$ ▶  $Prop(A \lor B) = Prop(A) \lor Prop(B) = Prop(A) \cup Prop(B)$ 



$$W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$$

$$v_1 = v_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8$$

 $X = \{w_1, w_2, w_4, w_5\}$   $Y = \{w_2, w_3, w_5, w_6\}$   $Z = \{w_4, w_5, w_6, w_7\}$ 

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$$

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 $X = \{w_1, w_2, w_4, w_5\}$  $Y = \{w_2, w_3, w_5, w_6\}$  $Z = \{w_4, w_5, w_6, w_7\}$  $\neg X = \{w_3, w_6, w_7, w_8\}$ 

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$$

$$X = \{w_1, w_2, w_4, w_5\}$$

$$Y = \{w_2, w_3, w_5, w_6\}$$

$$Z = \{w_4, w_5, w_6, w_7\}$$

$$\neg X = \{w_3, w_6, w_7, w_8\}$$

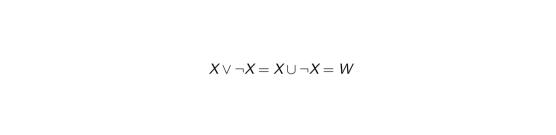
$$X \land Y = \{w_2, w_5\}$$

$$X \lor Y = \{w_1, w_2, w_3, w_4, w_5, w_6\}$$

$$\neg X \land Z = \{w_6, w_7\}$$

$$(X \land Y) \land Z = \{w_5\}$$

 $\neg X \land \neg Y \land \neg Z = \{w_8\}$ 



 $X \vee \neg X = X \cup \neg X = W$ 

 $X \land \neg X = X \cap \neg X = \{\}$ 

```
(P1) ...
(P2) ...
:
:
(Pn) ...
(C) ...
```

The argument

*P*1, *P*2, ..., *Pn*. Therefore: *C* 

P1, P2, ..., Pn logically imply C.

is logically valid.

Equivalently:

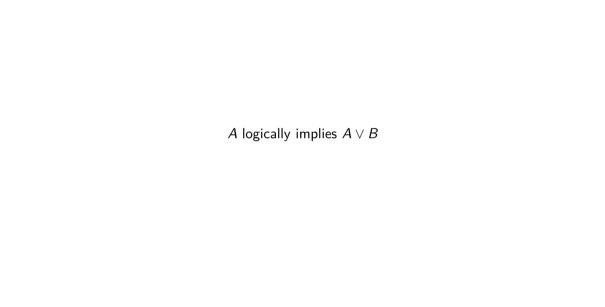
The argument

P. Therefore: C

is logically valid.

Equivalently:

P logically implies C.



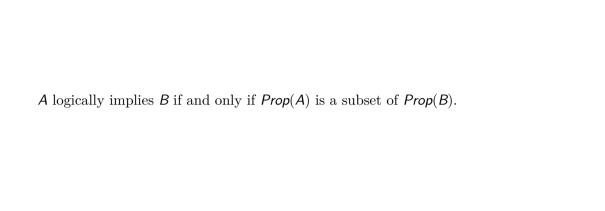
A logically implies $A \lor B$	
$X$ is a subset of $X \cup Y$	

- ▶ negation (of descriptive sentences) corresponds to complement (of propositions/sets)
- ▶ conjunction (of descriptive sentences) corresponds to intersection (of propositions/sets)
- ▶ disjunction (of descriptive sentences) corresponds to union (of

▶ logical implication (of descriptive sentences) corresponds to the subset relation

propositions/sets)

(between propositions/sets)

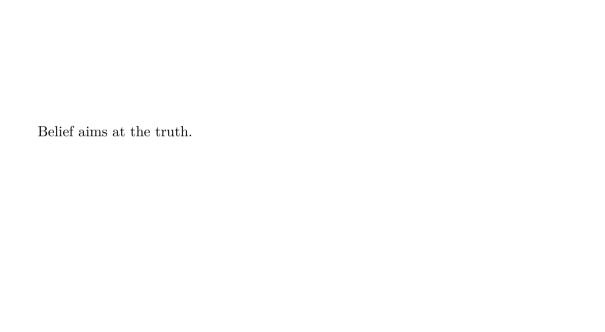


A logically implies B if and only if Prop(A) is a subset of Prop(B).

For all propositions X, Y:

X logically implies Y if and only if X is a subset of Y.

# $A \wedge B$ logically implies A $X \cap Y$ is a subset of X



X: that the weather will be nice tomorrow

► Believed: X

X: that the weather will be nice tomorrowY: that the weather will be nice on the day after tomorrow

- ► Believed: X
  - ▶ Believed:  $\neg X \lor (X \land \neg Y)$

X: that the weather will be nice tomorrow

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 $\triangleright$  Believed:  $X \land \neg Y$ 

## X: that the weather will be nice tomorrow

- Y: that the weather will be nice on the day after tomorrow
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▶ Believed:  $X \land \neg Y$ ▶ Believed:  $\neg Y$ 

▶ Rational Belief 1: If a person is inferentially perfectly rational (with W as her set of entertainable possible worlds), then she believes W.

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not believe {}.

▶ Rational Belief 2: If a person is inferentially perfectly rational, then she does

not believe {}.

- ▶ Rational Belief 2: If a person is inferentially perfectly rational, then she does

Rational Belief 1: If a person is inferentially perfectly rational (with W as her

▶ Rational Belief 3: If a person is inferentially perfectly rational, if she believes

set of entertainable possible worlds), then she believes W.

X, and if X is a subset of Y, then she also believes Y.

- ▶ Rational Belief 1: If a person is inferentially perfectly rational (with W as her set of entertainable possible worlds), then she believes W. ▶ Rational Belief 2: If a person is inferentially perfectly rational, then she does
- not believe {}.
- ▶ Rational Belief 3: If a person is inferentially perfectly rational, if she believes
- X, and if X is a subset of Y, then she also believes Y.

▶ Rational Belief 4: If a person is inferentially perfectly rational, if she believes

X, and if she believes Y, then she also believes  $X \cap Y$ .

Theorem:

Let W be a finite, non-empty set of possible worlds.

If the Rational Belief postulates 1–4 are the case (for the given set W of possible worlds), then for every inferentially perfectly rational person p (for whom W is the set of entertainable possible worlds), there is a non-empty proposition  $B_W$ , such that the following holds:

For all propositions X (over W),

person p believes X if and only if  $B_W$  is a subset of X.

Proof	:

Let  $B_W$  be the conjunction of all propositions believed by p.

Proof:

Let  $B_W$  be the conjunction of all propositions believed by  $\rho$ .

(By postulate Rational Belief 1, person p believes at least one proposition.)

Let DW be the conjunction of an propositions believed by p

Law of commutativity:

$$X \cap Y = Y \cap X$$

 $ightharpoonup X \cap (Y \cap Z) = (X \cap Y) \cap Z$ 

Law of commutativity:

$$X \cap Y = Y \cap X$$

Law of associativity:

 $2 \cdot (3 \cdot 5) = (2 \cdot 3) \cdot 5$ 

$$\blacktriangleright X \cap (Y \cap Z) = (X \cap Y) \cap Z$$

, , , , ,

 $2 \cdot 3 = 3 \cdot 2$ 



$$X_1 \cap X_2 \cap ... \cap X_n$$

 $X_n \cap X_{n-1} \cap ... \cap X_1$ 

 $X_2 \cap X_1 \cap ... \cap X_n$ 

We need to show:

For all propositions X (over W),

person p believes X if and only if  $B_W$  is a subset of X.

Assume the left-hand side: person p believes X.

We need to show:  $B_W$  is a subset of that X.

If  $X_1, \ldots, X_n$  are all the propositions believed by p, then

$$X_1, \ldots, X_n$$
 are all the propositions believed by  $p$ , the

where one of  $X_1, \ldots, X_n$  is X.

 $B_W = X_1 \cap X_2 \cap ... \cap X_n$ 

If  $X_1, \ldots, X_n$  are all the propositions believed by p, then

$$B_W = X_1 \cap X_2 \cap ... \cap X_n$$

where one of  $X_1, \ldots, X_n$  is X.

where one of 
$$\chi_1, \ldots, \chi_n$$
 is  $\chi$ .

E.g.: $B_W = X \cap X_2 \cap ... \cap X_n$ 

In any case: 
$$B_W$$
 is a subset of  $X$ .  $\checkmark$ 

Now for the other direction of:

For all propositions X (over W),

person p believes X if and only if  $B_W$  is a subset of X.

Assume the right-hand side:  $B_W$  is a subset of X.

We need to show: person p believes X.

We have:

$$B_W = X_1 \cap X_2 \cap X_3 \cap ... \cap X_n$$

where  $X_1, \ldots, X_n$  are all the propositions believed by p.

We can reformulate this:

$$B_W = (\dots((X_1 \cap X_2) \cap X_3) \cap \dots \cap X_n)$$

$$B_W = (\dots ((X_1 \cap X_2) \cap X_3) \cap \dots \cap X_n)$$

By postulate Rational Belief 4:

- ▶ Since  $X_1$  is believed by p,  $X_2$  is believed by p, also  $(X_1 \cap X_2)$  is believed by p.
- ▶ Since  $X_1 \cap X_2$  is believed by p, and  $X_3$  is believed by p, also  $((X_1 \cap X_2) \cap X_3)$  is believed by p.
- ▶ Since  $((X_1 \cap X_2) \cap X_3)$  is believed by p, and  $X_4$  is believed by p, also  $(((X_1 \cap X_2) \cap X_3) \cap X_4)$  is believed by p.
- :
- $\underbrace{\left(...((X_1\cap X_2)\cap X_3)\cap...\cap X_n\right)}_{\text{is believed by }p.$

By postulate Rational Belief 3:	

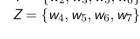
Since  $B_W$  is believed by p, and  $B_W$  is a subset of X, also X is believed by p.  $\checkmark$ 

Therefore,  $B_W$  is not the empty set.

By postulate Rational Belief 3:

Additionally, by postulate Rational Belief 2:  $\{\}$  is not believed by p.

Since  $B_W$  is believed by p, and  $B_W$  is a subset of X, also X is believed by p.  $\checkmark$ 





$$X = \{w_1, w_2, w_4, w_5\}$$
  
 $Y = \{w_2, w_3, w_5, w_6\}$ 

Believed:  $X, Y, X \cup Y, \dots$ 













$$X = \{w_1, w_2, w_4, w_5\}$$
  
 $Y = \{w_2, w_3, w_5, w_6\}$ 





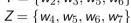


















Believed:  $X, Y, X \cup Y, \dots$ 

Not believed: Z,  $\{w_5\}$ ,  $\{w_8\}$ , ...

E.g., assume belief to be generated by  $B_W = X \cap Y = \{w_2, w_5\}$ .



Assume belief to be generated by  $B_W$ .

If X is a proposition (a subset of W), then there are three possible cases:

- $\triangleright$   $B_W$  is a subset of X: X is believed.
- ▶  $B_W \cap X = \{\}: \neg X \text{ is believed } (X \text{ is disbelieved}).$
- ▶  $B_W \cap X = \{\}$ : ¬X is believed (X is disbelieved). ▶  $B_W$  is not a subset of X,  $B_W \cap X \neq \{\}$ :

Neither X is believed, nor  $\neg X$  is believed.

 $Y = \{w_2, w_3, w_5, w_6\}$ 

 $Z = \{w_4, w_5, w_6, w_7\}$ 



 $X = \{w_1, w_2, w_4, w_5\}$ 





Believed: e.g., X

Disbelieved: e.g.,  $\{w_4, w_7\}$ 

Neither believed nor disbelieved: e.g., Z,  $\neg Z$ 

E.g., assume belief to be generated by  $B_W = X \cap Y = \{w_2, w_5\}$ .



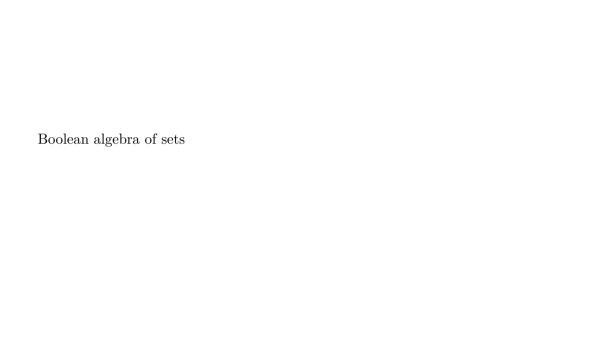












Boolean algebra of sets			
Filter			

Boolean algebra of sets
Filter
Ultrafilter

• Area 1: A(W) = 1.

▶ Area 2: For all propositions (regions) X,  $0 \le A(X) \le 1$ .

▶ Area 1: A(W) = 1.

- Area 1: A(W) = 1.
- ▶ Area 2: For all propositions (regions) X,  $0 \le A(X) \le 1$ .

 $X \cap Y = \{\}, \text{ then } A(X \cup Y) = A(X) + A(Y).$ 

▶ Area 3: For all propositions (regions) X, for all propositions (regions) Y, if

▶ Area 2: For all propositions (regions) X,  $0 \le A(X) \le 1$ .

▶ Area 1: A(W) = 1.

Area 3: For all propositions (regions) X, for all propositions (regions) Y, if  $X \cap Y = \{\}, \text{ then } A(X \cup Y) = A(X) + A(Y).$ 

 $X \cap \neg X = \{\}.$ 

 $X \cup \neg X = W$ 

 $A(X \cup \neg X) = A(W) = 1.$ 

Therefore:  $1 = A(X) + A(\neg X)$ . Equivalently:  $A(\neg X) = 1 - A(X)$ .

By Area 3:  $A(X \cup \neg X) = A(X) + A(\neg X)$ .

▶ Rational Degree of Belief 1: If a person is inferentially perfectly rational (with W as her set of entertainable possible worlds), then her degree of belief function P is such that: P(W) = 1.

that  $0 \leq P(X) \leq 1$ .

▶ Rational Degree of Belief 2: If a person is inferentially perfectly rational (with

W as her set of entertainable possible worlds), then her degree of belief

function P is such that:

For all propositions X (for all subsets X of W): P(X) is a real number, such

- ▶ Rational Degree of Belief 3: If a person is inferentially perfectly rational (with W as her set of entertainable possible worlds), then her degree of belief function P is such that:
  - function P is such that: For all propositions X (for all subsets X of W), for all propositions Y (for all

subsets Y of W):

if  $X \cap Y = \{\}$ , then  $P(X \cup Y) = P(X) + P(Y)$ .

## Different kinds of probability:

- ► Statistical probability.
- ► Objective single-case probability.
- ► Geometrical probability.

## Different kinds of probability:

- ► Statistical probability.
- ▶ Objective single-case probability.
- Geometrical probability.
  - :
- ► Subjective probability.

- (i)  $P(\neg X) = 1 P(X)$ .
- (ii)  $P(\{\}) = 0$ .
- (iii)  $P(Y) = P(X \cap Y) + P(\neg X \cap Y)$ .

(iv) If X is a subset of Y, then  $P(X) \leq P(Y)$ .

(i) Show: 
$$P(\neg X) = 1 - P(X)$$

(i) Show:  $P(\neg X) = 1 - P(X)$ .

By Rational Degree of Belief 3:  $P(W) = P(X) + P(\neg X)$ .

Hence:  $1 = P(X) + P(\neg X)$ . That is:  $P(\neg X) = 1 - P(X)$ .

By Rational Degree of Belief 1: P(W) = 1.

(ii) Show: 
$$P(\{\}) =$$

(ii) Show:  $P(\{\}) = 0$ .

Plug in 'W' for 'X' in (i):  $P(\neg X) = 1 - P(X)$ .

 $P(\neg W) = 1 - P(W).$ 

 $P(\{\}) = 1 - 1 = 0.$ 

Since  $\neg W = \{\}$ , and P(W) = 1 by Rational Degree of Belief 1:

(iii) Show: 
$$P(Y) = P(X \cap Y) + P(\neg X \cap Y)$$

(iii) Show:  $P(Y) = P(X \cap Y) + P(\neg X \cap Y)$ .

Since 
$$(X \cap Y) \cap (\neg X \cap Y) = \{\}$$
, it follows from Rational Degree of Belief 3:

 $P(Y) = P(X \cap Y) + P(\neg X \cap Y).$ 

 $P((X \cap Y) \cup (\neg X \cap Y)) = P(X \cap Y) + P(\neg X \cap Y).$ 

(iv) Show: If 
$$X$$
 is a subset of  $Y$ , then  $P(X) \leq P(Y)$ .

By (iii):  $P(Y) = P(X \cap Y) + P(\neg X \cap Y)$ .

By (iii): 
$$P(Y) = P(X \cap Y) + P(\neg X \cap Y)$$
.

So:  $P(Y) = P(X) + P(\neg X \cap Y)$ .

If X is a subset of Y, then  $X \cap Y = X$ .

Hence:  $P(Y) = P(X) + P(\neg X \cap Y) > P(X)$ .

That is: P(X) < P(Y).

By Rational Degree of Belief 2:  $0 \le P(\neg X \cap Y) \le 1$ .

Theorem:

Let W be a finite, non-empty set of possible worlds.

If the Rational Degree of Belief postulates 1–3 are the case (for the given set W of possible worlds), then for every inferentially perfectly rational person's degree of belief function P (on W as her set of entertainable possible worlds), there is a function B, such that the following holds:

- -B assigns to each world w in W a non-negative real number B(w).
- The sum of all the values of B on worlds in W is 1: that is, if  $W = \{w_1, \ldots, w_n\}$ , then  $B(w_1) + \ldots + B(w_n) = 1$ .
- For all propositions X (over W), P(X) is the sum of the values of B on worlds in X; that is:

$$P(X) = \sum_{w \text{ in } X} B(w).$$

```
E.g., assume P to be generated by:
```

$$P(\dots) = 1/15$$

$$B(w_1) = 1/15$$

 $B(w_3) = 1/15$  $B(w_4) = 1/15$  $B(w_5) = 1/3$  $B(w_6) = 1/15$  $B(w_7) = 1/15$  $B(w_8) = 0$ 

$$B(w_1) = 1/15$$

$$B(w_1) = 1/15$$
  
 $B(w_2) = 1/3$ 

$$B(w_1) = 1/15$$

$$S(w_1) = 1/15$$

$$f(\mathbf{w}_1) = 1/15$$

$$(w_1) = 1/15$$

$$(w_1) = 1/15$$

$$w_1$$
) = 1/15

$$(1) = 1/15$$

$$(\mathbf{w}_1) = 1/15$$

$$\mathbf{v}_1) = 1/15$$

$$(w_1) = 1/15$$

$$(4) - 1/15$$

assume 
$$P$$
 to be generated by:

$$g$$
., assume  $P$  to be generated by:

assume 
$$P$$
 to be generated by:

$$p$$
,, assume  $p$  to be generated by:

assume 
$$P$$
 to be generated by:

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$$P$$
 to be generated by:

, assume 
$$P$$
 to be generated by:

assume 
$$P$$
 to be generated by:

E.g., assume P to be generated by:

$$B(w_1) = 1/15$$
  
 $B(w_2) = 1/3$ 

$$B(w_3) = 1/15$$
  
 $B(w_4) = 1/15$ 

$$B(w_4) = 1/15$$
  
 $B(w_5) = 1/3$ 

$$B(w_4) = 1/15$$
  
 $B(w_5) = 1/3$ 

 $B(w_7) = 1/15$  $B(w_8) = 0$ 

 $P(\{w_1\}) = B(w_1) = 1/15$ 

$$B(w_5) = 1/3$$
  
 $B(w_6) = 1/15$ 

$$B(w_5) = 1/3$$

E.g., assume P to be generated by:

$$B(w_1) = 1/15$$
  
 $B(w_2) = 1/3$   
 $B(w_3) = 1/15$ 

$$B(w_3) = 1/15$$
  
 $B(w_4) = 1/15$ 

 $B(w_5) = 1/3$  $B(w_6) = 1/15$  $B(w_7) = 1/15$  $B(w_8) = 0$ 

$$B(w_2) = 1/3$$
  
 $B(w_3) = 1/15$   
 $B(w_4) = 1/15$ 

$$B(w_3) = 1/15$$
  
 $B(w_4) = 1/15$ 

$$(w_1) = 1/13$$
  
 $(w_2) = 1/3$   
 $(w_1) = 1/15$ 

 $P(\underbrace{\{w_1, w_2, w_4, w_5\}}_{X}) = B(w_1) + B(w_2) + B(w_4) + B(w_5) =$ 

= 2/3 + 2/15 = 0.8

I buy a bet on the proposition X:

- First of all, I pay  $q \cdot S$  Euro.
- ▶ If X is false, my net loss is  $q \cdot S$  Euro.
- If X is raise, my net loss is q ⋅ S Euro.
  If X is true, my net profit is S q ⋅ S Euro.

I buy a bet on the proposition X:

- First of all, I pay  $q \cdot S$  Euro.
- ▶ If X is false, my net loss is  $q \cdot S$  Euro.
  - If X is false, my flet loss is  $q \cdot S$  Euro.

▶ If X is true, my net profit is  $S - q \cdot S$  Euro.

S: stake of the bet;  $S = (S - q \cdot S) + q \cdot S$ .

I buy a bet on the proposition X:

- $\triangleright$  First of all, I pay  $q \cdot S$  Euro.

S: stake of the bet;  $S = (S - q \cdot S) + q \cdot S$ 

q: betting quotient of the bet;  $q = \frac{q \cdot S}{S}$ 

- ▶ If X is false, my net loss is  $a \cdot S$  Euro.
- ▶ If X is true, my net profit is  $S q \cdot S$  Euro.

I regard a bet on X as fair if and only if my degree of belief P(X) in X equals the

betting quotient q of the bet:

q = P(X)

$$(X)$$
 in  $(X)$  equals the

I regard a bet on X as fair if and only if my degree of belief P(X) in X equals the betting quotient q of the bet:

$$q = P(X)$$

E.g., 
$$P(X) = 1/2$$
,  $q = 1/2$ :

- First of all, I pay  $q \cdot S = S/2$  Euro.
  - ▶ If X is true, my net profit is  $S q \cdot S = S/2$  Euro.
  - If X is true, my net pront is S q ⋅ S = S/2 Euro.
    If X is false, my net loss is q ⋅ S = S/2 Euro.

I regard a bet on X as fair if and only if my degree of belief P(X) in X equals the betting quotient q of the bet:

$$q = P(X)$$

E.g., 
$$P(X) = 1$$
,  $q = 1$ :

- First of all, I pay  $q \cdot S = S$  Euro.
  - ▶ If X is true, my net profit is  $S q \cdot S = 0$  Euro.
  - If X is false, my net loss is  $g \cdot S = S$  Euro.

If I sell a bet on the proposition X, things are reversed:

- $\triangleright$  First of all, my opponent pays  $q \cdot S$  Euro.
- ▶ If X is true, my net loss is  $S q \cdot S$  Euro.
- ▶ If X is false, my net profit is  $q \cdot S$  Euro.

Assume two propositions X and Y to have empty intersection, and:

P(X) = 0.4, P(Y) = 0.4, but  $P(X \cup Y) = 0.7$  (!)

Assume two propositions X and Y to have empty intersection.

Assume two propositions 
$$\lambda$$
 and  $\gamma$  to have empty intersection.

P(X) = 0.4, P(Y) = 0.4, but  $P(X \cup Y) = 0.7$  (!)

X is true: my net profit is  $S_X - q_X \cdot S_X = 1 - q_X = 0.6$  Euro.

▶ Bet on X: P(X) = 0.4,  $S_X = 1$  Euro,  $g_X = 0.4$ .

 $\neg X$  is true: my net loss is  $q_X \cdot S_X = q_X = 0.4$  Euro.

Assume two propositions X and Y to have empty intersection.

Assume two propositions 
$$\lambda$$
 and  $\gamma$  to have empty intersection.

▶ Bet on Y: P(Y) = 0.4,  $S_Y = 1$  Euro,  $q_Y = 0.4$ .

 $\neg Y$  is true: my net loss is  $q_Y \cdot S_Y = q_Y = 0.4$  Euro.

P(X) = 0.4, P(Y) = 0.4, but  $P(X \cup Y) = 0.7$  (!)

Y is true: my net profit is  $S_Y - q_Y \cdot S_Y = 1 - q_Y = 0.6$  Euro.

Assume two propositions X and Y to have empty intersection.

Assume two propositions 
$$X$$
 and  $Y$  to have empty intersection.

P(X) = 0.4, P(Y) = 0.4, but  $P(X \cup Y) = 0.7$  (!)

▶ Bet on  $X \cup Y$ :  $P(X \cup Y) = 0.7$ ,  $S_{X \cup Y} = 1$  Euro,  $q_{X \cup Y} = 0.7$ .

 $X \cup Y$  is true: my net loss is  $S_{X \cup Y} - g_{X \cup Y} \cdot S_{X \cup Y} = 1 - g_{X \cup Y} = 0.3$  Euro.

 $\neg (X \cup Y)$  is true: my net profit is  $q_{X \cup Y} \cdot S_{X \cup Y} = q_{X \cup Y} = 0.7$  Euro.

Say, I bet on X, I bet on Y, and someone else bets on  $X \cup Y$  against me; my degrees of belief tell me that all of these bets are fair.

- $\blacktriangleright$  Case 1: X is true and Y is true, hence  $X \cup Y$  is true:
  - logically impossible, since X and Y have empty intersection!
  - ▶ Case 2: X is true and Y is false, and hence  $X \cup Y$  is true: 0.6 - 0.4 - 0.3 = -0.1. Overall I lose 0.1 Euro.
  - ▶ Case 3: X is false and Y is true, and hence  $X \cup Y$  is true: -0.4 + 0.6 - 0.3 = -0.1. Overall I lose 0.1 Euro.
- ▶ Case 4: X is false and Y is false, and hence  $X \cup Y$  is false: -0.4 - 0.4 + 0.7 = -0.1. Overall I lose 0.1 Euro.



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Consider a lottery with 1000000 tickets:

(D1)

(P1)  $P(\text{not ticket 1 wins } \land \text{ not ticket 2 wins } \land \dots \land \text{ not ticket 1000000 wins}) = 0.$ 

```
Consider a lottery with 1000000 tickets: 
 (P1) P(\text{not ticket 1 wins } \land \text{ not ticket 2 wins } \land \dots \land \text{ not ticket 1000000 wins}) = 0.
```

P(ticket 1000000 wins) = 1/1000000

```
(P2)

P(\text{ticket 1 wins}) = 1/1000000

P(\text{ticket 2 wins}) = 1/1000000
```

(P3) For every proposition X:

I believe X if and only if  $P(X) \ge 0.9$ .

(P3) For every proposition X:

 $X \wedge Y$ .

I believe X if and only if  $P(X) \ge 0.9$ .

if I believe X and I believe Y, then (if I am perfectly rational) I also believe

(P4) For every proposition X, for every proposition Y:

(P3) For every proposition X:

(P5) For every proposition X:

(And:  $P(X) = 1 - P(\neg X)$ .)

 $P(\neg X) = 1 - P(X)$ .

 $X \wedge Y$ 

I believe X if and only if P(X) > 0.9.

(P4) For every proposition X, for every proposition Y:

if I believe X and I believe Y, then (if I am perfectly rational) I also believe

(C)

 $P(\text{not ticket 1 wins } \land \text{ not ticket 2 wins } \land \dots \land \text{ not ticket 1000000 wins}) \ge 0.9,$  and

and  $P(\text{not ticket 1 wins } \land \text{ not ticket 2 wins } \land ... \land \text{ not ticket 1000000 wins}) = 0.$ 

```
The two premises (P2)
```

P(ticket 1 wins) = 1/1000000

P(ticket 1 wins) = 1/1000000

$$P(\text{ticket } 1000000 \text{ wins}) = 1/1000000$$

(P5) For every proposition X:

 $P(\neg X) = 1 - P(X).$ 

taken together entail:

for each i, P(not ticket i wins) = 999999/1000000.

Therefore, by premise

Promis

(P3) For every proposition X:

I believe X if and only if  $P(X) \ge 0.9$ . it follows:

I believe that not ticket 1 wins.

believe that not ticket I wins.

: I believe that not ticket 1000000 wins.

So, by applying premise

(P4) For every proposition X, for every proposition Y: if I believe X and I believe Y, then (if I am perfectly rational) I also believe

multiple times, it follows:

 $X \wedge Y$ .

▶ I believe that (not ticket 1 wins  $\land$  not ticket 2 wins  $\land \ldots \land$  not ticket 1000000 wins).

▶ I believe that (not ticket 1 wins  $\wedge$  not ticket 2 wins  $\wedge \ldots \wedge$  not ticket 1000000 wins).

with the two premises (P1)

 $P(\text{not ticket } 1 \text{ wins } \land \text{ not ticket } 2 \text{ wins } \land \dots \land \text{not ticket } 1000000 \text{ wins}) = 0.$ (P3) For every proposition X:

I believe X if and only if  $P(X) \ge 0.9$ .

leads to the contradiction C.

$$\begin{array}{ll} \text{(P1)} \ \textit{P}(\text{not ticket 1 wins} \ \land \ \text{not ticket 2 wins} \ \land \dots \\ \land \ \text{not ticket 1000000 wins)} = 0. \end{array}$$

(P2) 
$$P(\text{ticket 1 wins}) = 1/1000000$$

P(ticket 1 wins) = 1/1000000

(P5) For every proposition X:

 $P(\neg X) = 1 - P(X)$ .

$$(\text{ticket } 1000000 \text{ wins}) = 1/1000000$$

I believe X if and only if  $P(X) \ge 0.9$ .

(P3) For every proposition 
$$X$$
:  
I believe  $X$  if and only if  $P(X) \ge 0.9$ .  
(P4) For every proposition  $X$ , for every proposition  $Y$ :

if I believe X and I believe Y, then I believe  $X \wedge Y$ .

P(ticket 1000000 wins) = 1/1000000(P3) For every proposition X: