Introduction to Mathematical Philosophy

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Contents

Overview v

6 Decision 1

6.1 Introduction (10:31) 1

6.2 Expected Utility (9:20) 5

6.3 Von Neumann Morgenstern (12:40) 9

6.4 Alleged Counterexamples (12:23) 13

6.5 Final Remarks (7:56) 18

A Quiz Solutions Week 6: Decision 21

Week 6: Overview

Overview of Lecture 6: Decision

6.1 Introduction: We set up the decision problem of decision making under risk. There are a number of acts to choose from, and there are a number of outcomes that occur with a certain probability once a certain act is chosen. Each outcome has an associated utility value which specifies how much we desire the corresponding outcome. We then present and discuss the Principle of Maximizing Expected Utility.

6.2 Expected Utility: We illustrate the Principle of Maximizing Expected Utility by considering the question as to whether we should insure our car. This example also serves as an illustration of the difference between the expected monetary value and the expected utility of an act.

6.3 Von Neumann Morgenstern: We introduce the axioms of von Neumann and Morgenstern for preference orderings and show that a preference relation that satisfies these axioms can be represented by an ordinal utility function which is unique up to a positive linear transformation.

6.4 Alleged Counterexamples: We discuss the Allais Pradox and the Ellsberg Paradox and show that the decisions many people make violate the Independence Axiom of von Neumann and Morgenstern.

6.5 Final Remarks: We outline further topics in decision theory. We ask, for example, how we should rationally decide if no or only some of the relevant probabilities are known. Moreover, it is interesting to explore what the utilities actually mean. Are they merely representational devices (i.e. to represent our preferences), or do they exist in some sense?

Chapter 6

Week 6: Decision

6.1 Introduction (10:31)

Welcome to Lecture 6 of our Introduction to Mathematical Philosophy! In Lecture 3, Hannes introduced the concepts of full belief and partial belief and showed how partial belief can be explicated in terms of probabilities.

In the previous lecture, I spoke about the concept of confirmation, which plays an important role in reasoning. We discussed how new evidence changes our degrees of belief in various propositions that we entertain, and we explicated the concept of confirmation in terms of probability raising.

To do so, the Monty Hall Problem served as an illustration. We imagined that we find ourselves in a game show hosted by Monty Hall, and want to win a car. The car is behind one of three doors. Remember that initially, all doors were equally likely to have the car behind it. And so we assign a prior probability of 1/3 to each door and randomly choose door 1. This random selection is justified as there is no reason to prefer one door over the other. Next, we learn that Monty opens door 3 and instead of a car, we find a goat behind it. We are then asked if we want to switch from door 1 to door 2. Our analysis showed that the new probability, i.e. the probability we assign after learning that Monty opened door 3 to the proposition that the car is behind door 1 is the same as before, viz. 1/3. However, the new probability that the car is behind door 2 is 2/3 (and, of course, the new probability that the car is behind door 3 is 0). From this we concluded that we should switch doors and move from door 1 to door 2. The act of switching doors is preferable to the act of not switching doors and so we should decide to switch. But why is this so?

Answering this question requires not only theoretical reasoning, i.e. reasoning about the probabilities of various outcomes, but also practical reasoning. We have to ask ourselves how valuable the various outcomes are for us. If we want to make a rational decision these value judgments have to be considered. We have to consider the various outcomes that may result from an act and judge how valuable or desirable they are for us. Technically speaking, we have to specify the utilities of the various outcomes. To make a rational decision, we have to consider the probabilities and the utilities of various possible outcomes, given a certain act.

More formally, our decision problem can be modeled as follows:

First, there are two possible acts:

(Slide 1)

There are two possible *acts*:

- Act 1: Switch

- Act 2: Not-Switch

Second:

(Slide 2)

There are two possible *outcomes*:

- Outcome 1: You get a goat.
- Outcome 2: You get the car.

The relevant probabilities can be summarized in the following table (see Figure 6.1):

	Get the car	Get the goat
Switch	$2/3$	$1/3$
Not switch	$1/3$	$2/3$

Figure 6.1: Table 1: Probabilities

We can draw a similar table with the corresponding utilities (see Figure 6.2):

	Get the car	Get the goat
Switch	U	u
Not switch	U	u

Figure 6.2: Table 2: Utilities

Here U is the utility that we associate with the car, and u is the utility of owning a goat. Given that we consider the car to be more desirable than the goat, we have $U > u$.

To make a decision, we calculate the expected utility for each of the two acts. That is, we calculate, for each act, the sum of the products of the utility of a given outcome and its probability. Hence,

(Slide 3)

We calculate the expected utilities of both acts:

$$\begin{aligned} EU(\text{Switch}) &= (2/3) \cdot U + (1/3) \cdot u \\ EU(\text{Not}-\text{Switch}) &= (1/3) \cdot U + (2/3) \cdot u \end{aligned}$$

(Slide 4)

As $U > u$, we see that

$$EU(\text{Switch}) - EU(\text{Not}-\text{Switch}) = (1/3) \cdot (U - u) > 0.$$

and hence (because $U > u$) $EU(\text{Switch}) > EU(\text{Not}-\text{Switch})$. We maximize the expected utility if we choose act 1, i.e. if we switch. Generally speaking, the Principle of Maximizing Expected Utility tells us to choose the act that maximizes the expected utility.

Note that the example we considered is a decision problem where the probabilities of the various outcomes are known. We are dealing with a problem of decision making under risk. Generally speaking, such problems can be formulated as follows.

(Slide 5)

The decision problem

1. There are a number of acts $A_1, A_2, \ldots, A_n$ of which we have to choose one.
2. There are a number of mutually exclusive and exhaustive outcomes $O_1, O_2, \ldots, O_m$, i.e. one of these outcomes occurs.
3. If we choose an act, outcome i occurs with probability $P(O_i)$ and has a utility $U(O_i)$.

(Slide 6)

The expected utility of an act A is then given by

$$EU(A) = \sum_{i=1}^{m} P(O_i)U(O_i).$$

where the sum runs over all possible outcomes.

The Principle of Maximizing Expected Utility in particular and decision theory in general then recommend that we should choose the act that maximizes the expected utility. This is the normative reading of decision theory, which is defended by many philosophers. Philosophers are interested in identifying rationality principles such as the Principle of Maximizing Expected Utility. Knowing these principles will hopefully help us makeing better decisions. Psychologists, on the other hand, are mostly studying the empirical question how real people make decisions. Do they follow principles like the Principle of Maximizing Expected Utility? Or do they violate it? One can learn a lot from these studies and it is perhaps not too surprising that one finds that real people often don't decide as a rational agent would. The famous studies by the psychologists Amos Tversky and Daniel Kahneman are a case in point. They initiated a debate about the status of the principles of decision theory to which we will return later. At this point I would like to stress that decision making is an interdisciplinary research field to which philosophers contribute side by side with psychologists, economists, statisticians and even computer scientists. Mathematical philosophy has much to gain from the empirical sciences.

Quiz 44:
The utility of an outcome specifies how much we desire it. It is a subjective concept and the hypothetical unit of measurement is the util. For more about this, see
http://www.amosweb.com/cgi-bin/awb_nav.pl?s=wpd&c=dsp&k=util
There are two acts, A and B. If you choose act A, then you get 14 utils with probability p and 1 util with probability $1-p$. If you choose act B, then you get 10 utils with probability p and 2 utils with probability $1-p$. Which act maximizes the expected utility?
Solution

6.2 Expected Utility (9:20)

Let us assume that we switched doors and that we were fortunate enough to win the desired car. We plan to use the car for a road trip to St. Petersburg in Russia. Before we depart, however, we have to decide whether to insure the car or not. Let us now analyze this decision problem. To begin with, we identify two acts, i.e.

(Slide 7)

There are two possible acts.

- Act 1: We insure the car.
- Act 2: We do not insure the car.

Next, we assume, for simplicity, that there are only two states of the world:

(Slide 8)

There are two possible *states of the world*.

- State 1: The car breaks in an accident.
- State 2: The car does not break in an accident.

This is clearly a simplification as other things could happen to the car, but we do not want to make things overly complicated at this point.

Depending on the chosen act and the state of the world, there are four possible outcomes which can be depicted in the decision tree (see Figure 6.3):

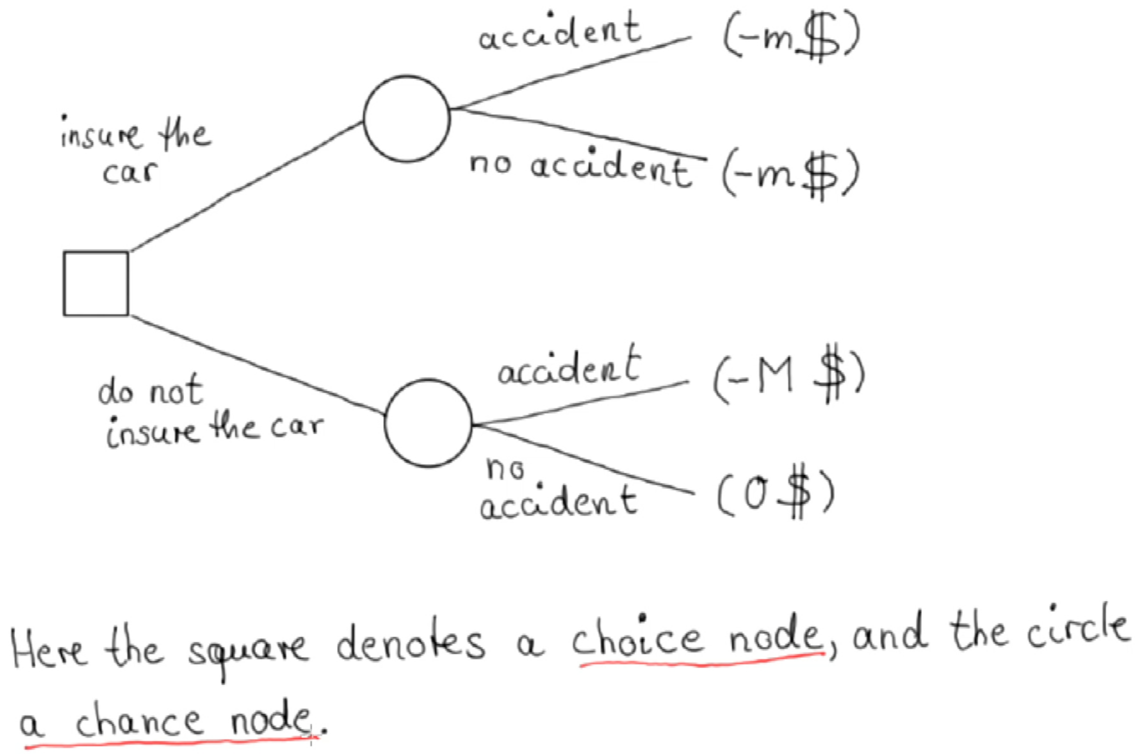

Figure 6.3: Decision tree 1

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If we insure the car and the car breaks, we are only left with the fee of the insurance ($-m$\$). The car is replaced by the insurance.

If we insure the car and the car does not break, we are left with the fee of the insurance ($-m$\$).

If we do not insure the car and the car breaks, we are left with no car ($-M$\$).

If we do not insure the car and the car does not break, nothing changes.(0$)

Let us assume that the probability of an accident is p.

We then obtain the expected monetary value (EV) of act 1 by summing over both states of the world, and similarly for the monetary value of act 2:

(Slide 9)

The expected monetary values:

$$\begin{aligned} EV(A_1) &= p\,(-m\$) + (1-p)\,(-m\$) = -m\$ \\ EV(A_2) &= p\,(-M\$) + (1-p)\,(0\$) = -p\,M\$ \end{aligned}$$

There is hence an expected monetary gain to buying insurance whenever $m \leq p\,M$.

Note that an insurance company makes a strictly positive profit only when $m \geq p\,M$, that is, whenever the expected income in fees is greater than the expected payment. However, the expected monetary value of buying the insurance is only positive for you when $m \leq p\,M$. So how can a deal be struck when the insurance company wants to make a profit? To answer this question, we have to remember that what we want to maximize is not the expected monetary value of an act, but rather its expected utility.

As before, we obtain the expected utility of act 1 by summing over both states of the world, and similarly for the expected utility of act 2.

(Slide 10)

The expected utilities:

$$\begin{aligned} EU(A_1) &= p\,U(-m\$) + (1-p)\,U(-m\$) = U(-m\$) \\ EU(A_2) &= p\,U(-M\$) + (1-p)\,U(0\$) = p\,U(-M\$) \end{aligned}$$

Here we have set $U(0\$) = 0$.

Clearly, whether $EU(A_1) > EU(A_2)$ depends not only on the values of m, M and p, but also on your utility function.

In general, people will be happy to pay more than the 'fair' price $m \leq p\,M$ of insurance whenever their utility function expresses risk-aversion – that is, whenever people want to avoid the risk of loosing something they value very much. Ideally, then, both parties can win in this business.

What does the utility function have to look like for you to be risk averse and buy insurance at a price that is higher than the fair prize? Think about this.

Here we are in a situation of decision making under risk, that is, we know the probabilities of the states of the world. These probabilities matter, because clearly, if there is no accident, it is better not to buy insurance – and if you will certainly be involved in an accident, then you'd better buy insurance. So what is best depends on how likely it is that there will be an accident.

Note that in this case no act dominates the other. Here dominance means the following:

(Slide 11)

Dominance: An act A dominates an act B if and only if the outcome of A will be as good as the outcome of B no matter which state of the world happens to be the true one, and strictly better under at least one state of the world.

It is easy to see that if act A dominates act B, then the expected utility of A is greater than the expected utility of B. Show this!

We assume that there are n states of the world $S_1, ..., S_n$. An act and a state of the world determine an outcome. We then calculate the expected utilities of acts A and B.
$EU(A) = P(S_1)U(A|S_1) + ... + P(S_n)U(A|S_n)$
$EU(B) = P(S_1)U(B|S_1) + ... + P(S_n)U(B|S_n)$
where we denoted the utility of the outcome that results from act A if the state of the world is S_i by $U(A|S_i)$. Next, we calculate
$EU(A) - EU(B) = P(S_1)[U(A|S_1) - U(B|S_1)] + ... + P(S_n)[U(A|S_n) - U(B|S_n)]$.
As A dominates B, $U(A|S_i)$ is never smaller than $U(B|S_i)$ and greater than $U(B|S_i)$ for at least one i. Hence, $EU(A) > EU(B)$.
Note that we have assumed here that the probabilities of the states of the world are not affected by the chosen act. This may sound like an innocent assumption. However, it turns out to be false as Newcomb's Paradox and related real-life problems show. If you want to learn more about this, have a look at
http://plato.stanford.edu/entries/decision-causal/

It is also easy to see that there are various equivalent ways to represent a utility function. Let u be a utility function and let

$$u'_i = a \cdot u_i + b \text{ with } a > 0.$$

(Slide 12)

Utility functions are only unique up to a positive linear transformation:

$$u_i \to u'_i = a \cdot u_i + b \text{ with } a > 0.$$

We then show that if act A has a higher expected utility than B under utility function u, then it also has a higher expected utility under the utility function u'.

(Slide 13)

EU is the utility function under u, and EU' is the utility function under u'. We then obtain:

$$\begin{aligned} EU'(A) &= \sum p_i\, u'_i = \sum p_i\, (a \cdot u_i + b) \\ &= a \cdot (\sum p_i\, u_i) + b \\ &= a \cdot EU(A) + b \end{aligned}$$

Hence, $EU'(A) > EU'(B)$ if and only if $EU(A) > EU(B)$.

where we have used that all p_i sum up to 1.

We conclude, then, that utility functions are only unique up to a positive linear transformation.

Quiz 45:
Imagine that you are very sick. Without further treatment, you will die in about four months. The only alternative is a risky operation. If you survive the operation, then you will live for about one year. The probability that you will not survive the operation is 20%.
(1): Draw a decision tree for this decision problem using choice nodes and chance nodes.
(2): As utilities are only unique up to a positive linear transformation, we can set the utilities u(live for 12 months) $= 1$ and u(live for 0 months) $= 0$. (Show this!) How low can your utility for living four months be and still have the operation preferred?
Solution

6.3 Von Neumann Morgenstern (12:40)

We have been speaking rather casually about utilities so far, and have been assigning it intuitive numerical values. You may have been wondering where these numbers come from, since we don't ordinarily express how much we like goats and cars in terms of numbers. The good news is that given certain axioms, utilities can be derived from preferences between lotteries – which you may find to be a more basic and accessible notion.

Let us consider:

(Slide 14)

A set of basic prizes $X = \{A, B, C, \dots\}$.

Each basic prize is a lottery which you may win.

Choose a lottery in which you win A with probability p and B with a probability $1-p$. That is, if A and B are lotteries, then

$$p\,A + (1-p)\,B$$

is also a lottery.

We denote the latter lottery by ApB.

Next, we define a preference relation over lotteries.

(Slide 15)

- $A \succ B$: you strictly prefer lottery A to lottery B.
- $A \prec B$: you strictly prefer lottery B to lottery A.
- $A \sim B$: you are indifferent between lotteries A and B.

We now formulate four axioms that hold for preferences over lotteries. Let the set of lotteries that we consider be $\mathcal{L}$.

(Slide 16)

The von Neumann Morgenstern Axioms

Consider a set $\mathcal{L}$ of lotteries.

1. Completeness: For all lotteries A, B in $\mathcal{L}$, $A \succ B$ or $A \prec B$ or $A \sim B$.

This axiom demands that all lotteries that we consider can be ordered. In our example, this means that you either prefer to win the car to winning the goat, or you prefer to win the goat rather than the car, or that you are indifferent between the two options.

(Slide 17)

2. Transitivity: For all lotteries A, B, C in $\mathcal{L}$, if $A \succ B$ and $B \succ C$, then $A \succ C$.

This means that if you prefer to win an expensive car to winning a cheap car, and if you prefer to win a cheap car to winning a goat, then you prefer to win the expensive car to winning a goat.

(Slide 18)

3. Continuity: For all lotteries A, B, C in $\mathcal{L}$, if $A \succ B \succ C$, then there are probabilities p and q such that $ApC \succ B \succ AqC$.

This axiom is required for mathematical reasons. It is very weak and can, in fact, not be falsified as p can (for example) be as close to 1 as you like and q can be as close as 0 as you like, which makes it difficult to argue that the axiom should not be accepted.

That is, if you prefer to win an expensive car to winning a cheap car to winning a goat, then there are probabilities p and q such that you prefer to win an expensive car (with a presumably high probability p) or a goat (with probability $1-p$) to winning a cheap car, and that you prefer to win a cheap car to winning an expensive car (with presumably low probability q) or a goat (with probability $1-q$).

(Slide 19)

4. Independence: For all lotteries A, B, C in $\mathcal{L}$, $A \succ B$ if and only if $ApC \succ BpC$.

This sounds reasonable. If you prefer to win an expensive car to a cheap car, then you prefer to win the expensive car (with probability p) or the goat (with probability $1-p$) to winning the cheap car (with probability p) or the goat (with probability $1-p$).

Now von Neumann and Morgenstern were able to prove the following theorem.

(Slide 20)

The Neumann and Morgenstern representation theorem.

A preference relation $\succ$ satisfies the axioms 1 to 4, if and only if there exists a utility function u such that

(i) if $A \succ B$, then $u(A) > u(B)$,

(ii) $u(ApB) = p\,u(A) + (1-p)\,u(B)$,

(iii) for every other function u' that satisfies (i) and (ii), there are numbers $a > 0$ and b such that $u' = a\,u + b$.

Note that the utility function u is defined over lotteries.

We say that the utilities represent our preferences. The theorem is called a representation theorem because it provides necessary and sufficient conditions for the existence of a representation of preferences in terms of numbers.

Here are three more remarks.

First, the utility function one obtains is not unique. It is only unique up to a positive linear transformation. We have seen already that this is in accordance with the principle of maximizing expected utility. Utilities are defined on an interval scale. This means that the absolute values of the utility do not matter, but the differences matter. To check this, we note that

(Slide 21)

$$\begin{aligned} u(A) - u(B) &= u(C) - u(D) \\ &\Leftrightarrow \\ u'(A) - u'(B) &= u'(C) - u'(D) \end{aligned}$$

with u' defined in our theorem. It is a nice exercise to prove this! So take a moment and prove this claim!

We use
$u'_i = au_i + b$
to rewrite the equation
$u'(A) - u'(B) = u'(C) - u'(D)$.
Hence,
$u'(A) - u'(B) = u'(C) - u'(D)$
$(a \cdot u(A) + b) - (a \cdot u(B) + b) = (a \cdot u(C) + b) - (a \cdot u(D) + b)$
This is equivalent to
$a(u(A) - u(B)) = a(u(C) - u(D))$.
As $a > 0$, this is equivalent to
$u(A) - u(B) = u(C) - u(D)$, which is what we wanted to show.

Second, the axioms and the representation theorem show us how to construct utilities. To do so, let us first identify the optimal element O in the set, i.e. the lottery which is not preferred by any other lottery. (If it is not unique, then pick one of them.) Next we identify the worst lottery W, i.e. the lottery which is preferred by all others, and if we are indifferent between some of them that are preferred by all others, we pick one of them. We now assign

(Slide 22)

O: optimal element in the lottery

W: worst element in the lottery

We set:

$$\begin{aligned} u(O) &= 1 \\ u(W) &= 0 \end{aligned}$$

This is possible as utilities are only unique up to a linear transformation. To fix the utility we assign to a remaining lottery A, we determine the value p such that we are indifferent between A and a lottery between O (with probability p) and W (with $1 - p$).

Then we set:

(Slide 23)

If $OpW \sim A$, then $u(A) = p$.

This can be done for all remaining lotteries.

Third, we note that the von Neumann and Morgenstern representation theorem justifies the principle of maximizing expected utility as the preferred lottery is always the one with the highest expected utility. That is, maximizing expected utility is not a consequence of our egoism, it is rather demanded by rationality. If our preferences are rational in the sense of the axioms laid out by von Neumann and Morgenstern, then we will automatically choose according to the principle of maximizing expected utility.

Quiz 46:
Let u be an ordinal utility function and A, B, C, and D four acts to choose from. Your utility assignments are as follows: $u(A) = 0.5, u(B) = 0.9, u(C) = 0.2$, and $u(D) = .65$. Which preference ordering does this utility function represent?
Solution

6.4 Alleged Counterexamples (12:23)

While the von Neumann and Morgenstern axioms may be requirements of rationality, it is an altogether different question whether real people follow them. And indeed, economists and psychologists have challenged the axioms in many empirical studies. Let us look at the two most famous cases.

1. The Allais paradox

Consider the following choice between two lotteries:

(Slide 24)

The Allais Paradox

Choice 1:

A: a cheap car for sure

B: nothing (1%) or an expensive car (10%) or a cheap car (89%)

Here most people choose lottery A over lottery B. They might reason as follows: In lottery A I get the cheap car for sure, whereas in lottery B there is a (if only slight) chance to end up with nothing. So I better choose option A.

Next, consider the choice between lotteries E and F.

(Slide 25)

The Allais Paradox

Choice 2:

E: a cheap car (11%) or nothing (89%)

F: an expensive car (10%) or nothing (90%)

Here most people choose lottery F. They might reason as follows: Most likely I will end up with nothing. However, in lottery F there is a chance to win an expensive car whereas in lottery E I can only win a cheap car with a probability which is only slightly higher than the probability of winning an expensive car in lottery F. So I choose lottery F.

It is easy to see that there is no utility function that is consistent with choosing A over B and F over E. To see this, we set

(Slide 26)

$u(\text{nothing}) = 0$

$u(\text{expensive car}) = 1$

$u(\text{cheap car}) = x$

We then calculate the difference between the utilities for the lotteries A and B and the difference between the utilities of the lotteries E and F. Here is what we obtain:

(Slide 27)

$$\begin{aligned} u(A) - u(B) &= x - (.1 + .89\,x) = .11\,x - .1 \\ u(E) - u(F) &= .11\,x - .1 \\ &= u(A) - u(B) \end{aligned}$$

Hence, if there is a utility function u, then $A \succ B \Leftrightarrow E \succ F$.

Hence, $u(A) > u(B)$ iff $u(E) > u(F)$. That is, if we choose A over B we have to choose E over F if we want our decision to respect the principle of maximizing expected utility. People who choose A over B and F over E must therefore violate at least one of the von

Neumann and Morgenstern axioms. To see which, we represent the lotteries in the following way (see Figure 6.4):

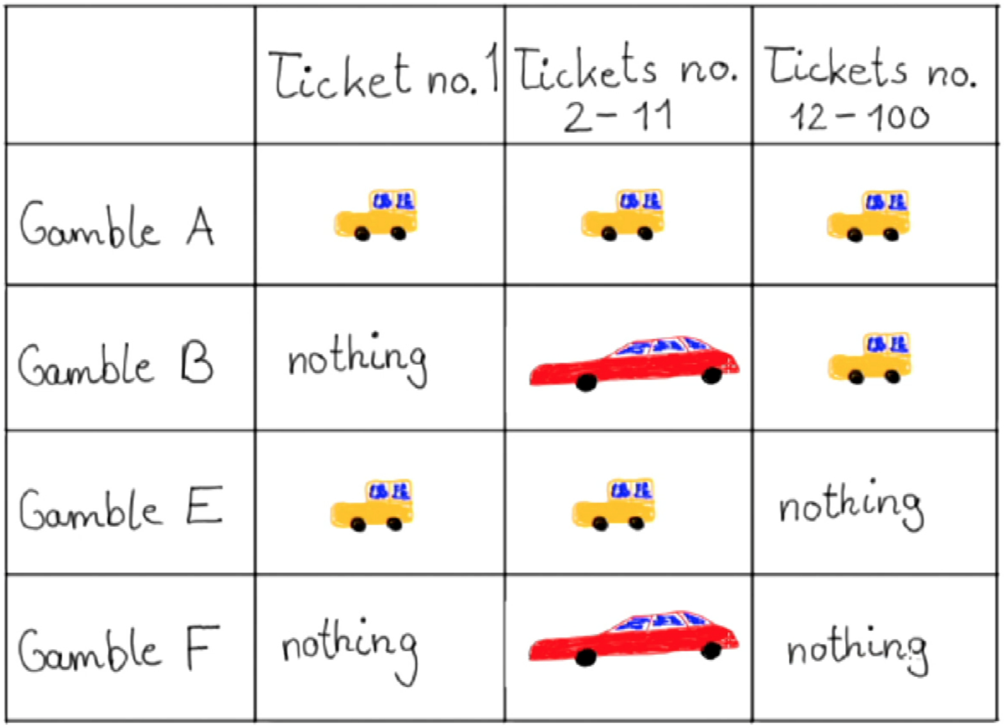

Figure 6.4: Table 3: Lotteries A, B, E, and F

The table shows that the Independence axiom is violated as a look at the third column indicates. Note that for tickets 12 to 100 the utility of the outcome is the same for lotteries A and B (in which case we win the cheap car) and for lotteries E and F (in which case we end up winning nothing). Hence, according to the Independence axiom, the choice between A and B (as well as the choice between E and F) should not depend on the third column. If we disregard the third column, then the choice between lotteries A and B is exactly the same as the choice between lotteries E and F. And so if you prefer A over B, you should prefer E over F. And if you prefer B over A, you should prefer F over E. Some people, who think about the decision situation in this way, admit that their previous choice was irrational and change their mind. Others remain unconvinced. Are they irrational?

Let us now consider the second example:

2. The Ellsberg Paradox

We are again confronted with the choice between two lotteries. This time the lotteries involve an urn which contains 90 balls. 30 of these balls are red. The remaining 60 balls are either blue or yellow. You are then ask to choose one of the following two lotteries:

(Slide 28)

The Ellsberg Paradox

An urn contains contains 90 balls. 30 of these balls are red. The remaining 60 balls are either blue or yellow.

Choice 1:

G: three nights in a luxury hotel in St. Petersburg if a red ball is drawn

H: three nights in a luxury hotel in St. Petersburg if a blue ball is drawn

Faced with this choice, most people choose lottery G. They might reason as follows: In lottery G I know the probability of winning (viz. 1/3), whereas I do not know the probability of winning in lottery H. This probability might be anywhere between 0 and 2/3. So it could be really small. And so I choose lottery G where I am certain about the probability

Note, however, that if we use the principle of indifference, then the probability of winning if we choose lottery H is also 1/3. And so we should be indifferent between lotteries G and H (as the prize is the same in both cases). However, there seems to be a difference between the two cases which cannot be represented in a way that it can be fed in the decision theory developed so far. Those who value certainty about probability assignments will choose option G.

Consider now another choice between two lotteries:

(Slide 29)

Choice 2:

K: three nights in a luxury hotel in St. Petersburg if a red or yellow ball is drawn

L: three nights in a luxury hotel in St. Petersburg if a blue or yellow ball is drawn

Here most people choose option L although the probability of winning is the same (viz. 2/3) in both cases if we apply the principle of indifference to lottery K. Again, people prefer the act that leads to an outcome with a known probability.

It turns out that choosing G over H and L over K is not consistent with the von Neumann Morgenstern axioms. There is no utility function that accounts for these choices. As in the Allais Paradox, the Independence axiom is violated as the following table illustrates (see Figure 6.5):

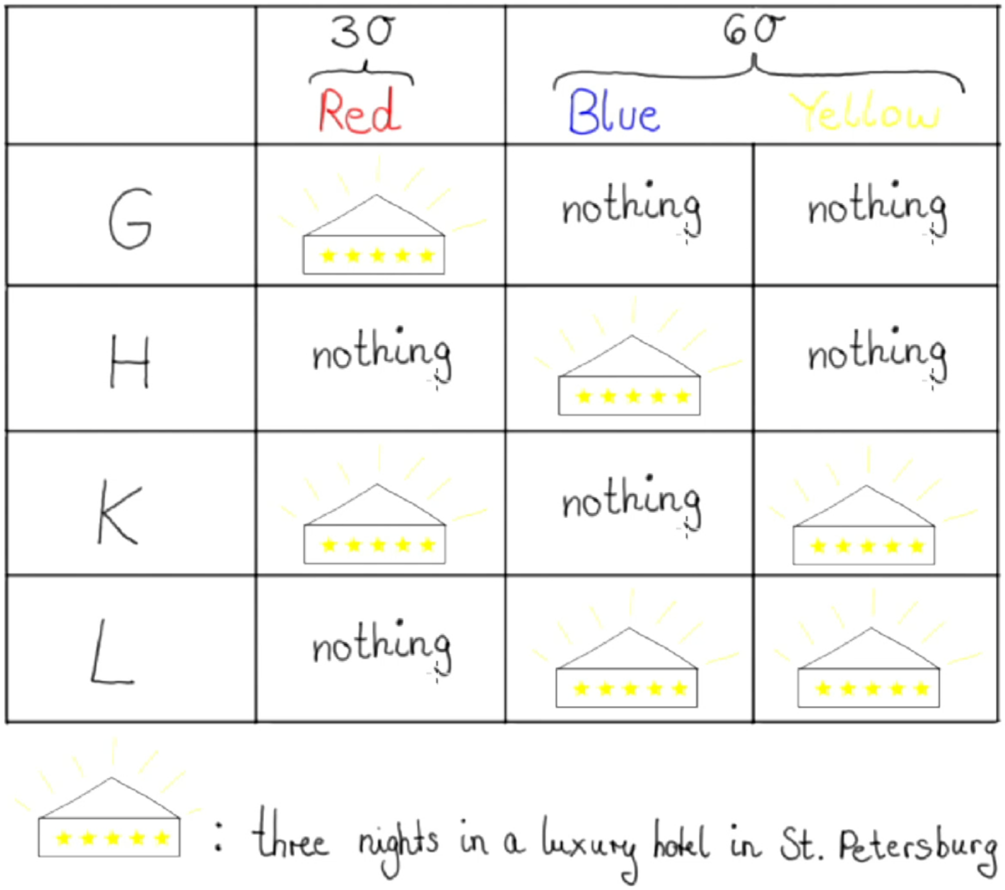

Figure 6.5: Table 4: Lotteries G, H, K, and L

Note, again, that the entries in the third column are the same for lotteries G and H (you loose in both cases) and they are the same for lotteries K and L (you win in both cases).

When confronted with this reasoning, many people revise their decisions and arrive at consistent choices, while other people remain unimpressed and stick to their original choice. They must have their reasons, and it is – at least in some cases – interesting to study their reasoning in more detail. This is one of the things psychologists do. But be this as it may, I think that it is fair to conclude that people who are asked to reconsider their choices in the light of the axioms will come to a better understanding of their choices. Reflecting on the principles of decision making will eventually lead to making better decisions.

Both examples we have discussed show violations of the Independence Axiom. It is worth noting that the Completeness Axiom and the Transitivity Axiom, too, have been challenged. Take a moment and try to come up with counterexamples to them!

Quiz 47:
Here are four lotteries.
In L_1, you win prize A with a probability of 1/4, prize C with a probability of 1/4, and prize D with a probability of 1/2.
In L_2, there is a fifty-fifty chance of wining prize B or prize D.
In L_3, you win prize B for sure, and
in L_4 there is a fifty-fifty chance of wining prize A or prize C.
You prefer L_1 to L_2 and L_3 to L_4. Is there a utility function that represents your preferences? If not, which of the von Neumann and Morgenstern axioms is violated?
Solution

6.5 Final Remarks (7:56)

Let us now step back and conclude this lecture with a number of final remarks.

In the concluding remarks of the previous lecture, I stressed the importance of formulating a model whenever we are engaged in theoretical reasoning. The same holds for practical reasoning. When we are in a decision situation, we start with representing the decision situation in a mathematical model. Recall the example where we deliberated whether to buy an insurance for our recently won sports car or not. Here we first specified the two possible acts, i.e. to buy an insurance or not to buy an insurance. Next, we considered possible states of the world. Simplifying quite dramatically, we only assumed two possible states of the world, i.e. one in which the car will break in an accident and one in which the car will not break in an accident. However, it is needless to say that there are many other possibilities we can think of. We might, for example, lose the car in a bet on our way to St. Petersburg. Or the car will be damaged while parking in front of the luxury hotel. Finally, we have to state the possible outcomes and fix the corresponding probabilities. Once we have specified the model in this way, we can apply the decision theoretical machinery and determine the act that maximizes the expected utility. It is important to note, though, that which act we choose might well depend on the assumptions we made when we formulated the model. Excluding, for example, certain possible states of the world might have a big impact on our decision.

Here is another worry. We may not be able to fix the relevant probabilities of the outcomes in a given decision problem. This may be because we do not have access to huge sets of statistical data (as insurance companies do) or we may not trust our subjective judgments on the matter. In many cases, these probabilities are not known. What, for example, is the probability that I will also enjoy my sports car in five years? Perhaps I will want to lead a quieter life then and enjoy myself with a goat, pigs and cows on a farm in the mountains instead of rushing through Eastern Europe with my sports car. In some cases, I do not have any ideas what the relevant probabilities are. In this case I am in a situation of decision making under uncertainty as opposed to decision making under risk, where the probabilities are known, as in the examples discussed so far. Note that many real life situations are like this. We often do not know the relevant probabilities, but we have to make a decision anyway. How shall we proceed in these cases?

If the probabilities of the outcomes are not known at all, then principles such as the above mentioned Dominance Principle can be applied. If the Dominance Principle does not apply (and it often does not apply), then other decision rules have been used (such as the maximin rule and the minimax regret rule). There is, however, a lot of controversy about which of these decisions rules is the right one in a given decision situation.

It is harder to develop a decision theory for situations in which we have some probabilistic information, but not enough to feed the machinery we have developed so far. We may, for example, know that a certain probability is in a certain interval, say between 0 and 2/3, as in the Ellsberg example. Other probabilities may be totally unknown. We then ask: which rationality principles can be formulated for the decision-making in such situations? While there is a considerable interest in these questions, it seems that there is still a lot to do. Mathematical philosophers, along with their colleagues in economics, psychology, statistics and computer science, will keep being busy for a while.

Besides these questions of profound practical and theoretical importance, there are also more philosophical problems and questions that decision theory raises. One question concerns the status of the utilities. What do they mean? Are they simply representational devices that help us making sure that our preferences and choices are consistent? That is, are they only mathematical functions that represent the preferences of people? According to the theory of revealed preferences, people reveal their preferences in their choices. This is what can be observed, and the utilities are simply book keeping devices, but nothing more. Alternatively, one might want to argue that utilities exist in some sense. But in which sense? Perhaps at some point it will be possible to identify certain configurations in our brains where the utilities sit. Perhaps modern technology such as fMRI scans will be helpful here. While this would surely be very interesting, from the point of view of mathematical philosophy it is important to note that what matters in the first place are the underlying rationality principles. These principles provide normative constraints on our actions. At the same time, they can be used to explain our actions. Identifying them and proving representation theorems for them is an important task to which the mathematical philosopher can contribute.

I hope that this introduction made you curious about decision theory and that you plan to get deeper into this exciting and theoretically and practically important field of study!

The best introduction to decision theory for philosophers is
Peterson, M., *An Introduction to Decision Theory*, Cambridge: Cambridge University Press, 2009.
Here is another well-known introductory text:
Resnik, M., Choices: *An Introduction to Decision Theory*, Minneapolis: University of Minnesota Press, 1987.
Hacking's book on Probability and Inductive Logic, which I mentioned in the last lecture, also contains a number of highly readable chapters on decision theory.
There is also a very readable introduction to decision theory, written by an economist, which contains many examples and problems that will sharpen your understanding of various decision-situations:
Gilboa, I., *Making Better Decisions: Decision Theory in Practice*, Wiley-Blackwell 2010.
Here are some more advanced texts that you will enjoy reading.
Bermudez, J. L., *Decision Theory and Rationality*, Oxford: Oxford University Press, 2011.
Binmore, K., *Rational Decisions*, Princeton: Princeton University Press, 2011.
Gilboa, I. *Theory of Decision under Uncertainty*, Cambridge: Cambridge University Press, 2009.
Jeffrey, R., *The Logic of Decision*, Chicago: The University of Chicago Press, 1990.
Joyce, J., *The Foundations of Causal Decision Theory*, Cambridge: Cambridge University Press, 2008.
Luce, R.D. and Raiffa, H., *Games and Decisions: Introduction and Critical Survey*, Dover Publications, 1989. (This is a classic textbook. It first appeared in 1957 and is still worth reading.)
Weirich, P., *Realistic Decision Theory: Rules for Nonideal Agents in Nonideal Circumstances*, Oxford: Oxford University Press, 2004.
Not published yet, but Lara Buchak's forthcoming book *Risk and Rationality* (Oxford: Oxford University Press, 2014) promises to be an excellent read.
Here are some relevant entries from the Stanford Encyclopedia:
http://plato.stanford.edu/entries/decision-causal/
http://plato.stanford.edu/entries/practical-reason/
http://plato.stanford.edu/entries/consequentialism/
http://plato.stanford.edu/entries/consequentialism-rule/
The books by Gigerenzer and Kahneman, recommended in the last lecture, are also worth reading in the context of decision theory.
Finally, I recommend this paper:
Briggs, R., "Decision-Theoretic Paradoxes as Voting Paradoxes", *Philosophical Review* 119 (2010), 1-30.

Appendix A

Quiz Solutions Week 6: Decision

Quiz 44:

The utility of an outcome specifies how much we desire it. It is a subjective concept and the hypothetical unit of measurement is the util. For more about this, see

http://www.amosweb.com/cgi-bin/awb_nav.pl?s=wpd&c=dsp&k=util

There are two acts, A and B. If you choose act A, then you get 14 utils with probability p and 1 util with probability $1-p$. If you choose act B, then you get 10 utils with probability p and 2 utils with probability $1-p$. Which act maximizes the expected utility?

SOLUTION Quiz 44:

The expected utility (in utils) of act A is $EU(A) = 14p + 1 \cdot (1-p) = 13p + 1$. Similarly, we calculate $EU(B) = 10p + 2 \cdot (1-p) = 8p + 2$. Hence, $EU(A) > EU(B)$ if and only if $13p + 1 > 8p + 2$, i.e. if $p > 1/5$. We conclude that act A maximizes the expected utility if $p > 1/5$ and act B maximizes the expected utility if $p < 1/5$. If $p = 1/5$, then both acts have the same expected utility.

Back to quiz

Quiz 45:

Imagine that you are very sick. Without further treatment, you will die in about four months. The only alternative is a risky operation. If you survive the operation, then you will live for about one year. The probability that you will not survive the operation is 20%.

(1): Draw a decision tree for this decision problem using choice nodes and chance nodes.

(2): As utilities are only unique up to a positive linear transformation, we can set the utilities u(live for 12 months) $= 1$ and u(live for 0 months) $= 0$. (Show this!) How low can your utility for living four months be and still have the operation preferred?

SOLUTION Quiz 45:

Solution (1): see Figure A.1).

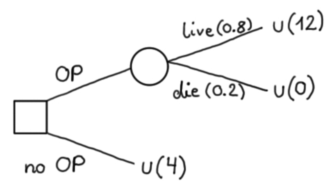

Figure A.1: Decision tree 2

Solution (2): Let u(live for 4 months) $=: U$. Hence, EU(no-OP) $= U$. Furthermore, $EU(\text{OP}) = .8 \cdot 1 + .2 \cdot 0 = .8$. We conclude that $EU(\text{OP}) > EU(\text{no-OP})$ if and only if $U < .8$.

Back to quiz

Quiz 46:

Let u be an ordinal utility function and A, B, C, and D four acts to choose from. Your utility assignments are as follows: $u(A) = 0.5, u(B) = 0.9, u(C) = 0.2$, and $u(D) = .65$. Which preference ordering does this utility function represent?

SOLUTION Quiz 46:

$B \succ D \succ A \succ C$.

Back to quiz

Quiz 47:

Here are four lotteries.

In L_1, you win prize A with a probability of 1/4, prize C with a probability of 1/4, and prize D with a probability of 1/2.

In L_2, there is a fifty-fifty chance of wining prize B or prize D.

In L_3, you win prize B for sure, and

in L_4 there is a fifty-fifty chance of wining prize A or prize C.

You prefer L_1 to L_2 and L_3 to L_4. Is there a utility function that represents your preferences? If not, which of the von Neumann and Morgenstern axioms is violated?

SOLUTION Quiz 47:

We calculate the expected utilities of the four lotteries.

$EU(L_1) = .25u(A) + .25u(C) + .5u(D)$ $EU(L_2) = .5u(B) + .5u(D)$ $EU(L_3) = u(B)$ $EU(L_4) = .5u(A) + .5u(C)$

We now calculate

$EU(L_1) - EU(L_2) = .25u(A) - .5u(B) + .25u(C)$

and

$EU(L_3) - EU(L_4) = -.5u(A) + u(B) - .5u(C) = -2(EU(L_1) - EU(L_2))$.

Hence, an expected utility maximizer should prefer L_1 to L_2 if and only if she prefers L_4 to L_3, and so there is no utility function that represents your preferences. Drawing a similar table as we did for the Allais paradox, it is easy to see that the Independence axiom is (again!) violated.

Back to quiz

List of Figures

6.1 Table 1: Probabilities 2
6.2 Table 2: Utilities 3
6.3 Decision tree 1 6
6.4 Table 3: Lotteries A, B, E, and F 15
6.5 Table 4: Lotteries G, H, K, and L 17

A.1 Decision tree 2 22