- I believe *that Bayern Munich wins the Champions League next year.*
- I hope *that Bayern Munich wins the Champions League next year.*
- I desire *that Bayern Munich wins the Champions League next year.*
- Others fear *that Bayern Munich wins the Champions League next year.*

  $\vdots$

First role of propositions:

- They are the meanings of descriptive sentences:

  ‘Bayern Munich wins the Champions League next year’ expresses the same proposition as ‘Bayern München gewinnt die Champions League nächstes Jahr’.

  Both express *that Bayern Munich wins the Champions League next year.*

Second role of propositions:

- They are bearers of truth values:

  That Bayern Munich wins the Champions League next year is true or false.

Third role of propositions:

- They are the contents of belief:

  I believe *that Bayern Munich wins the Champions League next year.*

The proposition *that Socrates is a philosopher* is true at the actual world, but it is not true at every possible world.

- ($\mathrm{Id}_{\mathsf{Worlds}}$) For all possible worlds w, for all possible worlds w': $w = w'$ if and only if for all propositions X: X is true at w if and only if X is true at w'.
- ($\mathrm{Id}_{\mathsf{Prop}}$) For all propositions X, for all propositions Y: $X = Y$ if and only if for all possible worlds w: X is true at w if and only if Y is true at w.

Definition:

Let W be a given non-empty set of possible worlds.

(i) X is a proposition (over W) if and only if X is a subset of W.

(ii) If X is a proposition (over W) and w is a world in W, then X is true at w if and only if w is a member of X.

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- ($\mathrm{Id}'_{\mathsf{Worlds}}$) For all members w of W, for all members w' of W: $w = w'$ if and only if for all subsets X of W: w is a member of X if and only if w' is a member of X.
- ($\mathrm{Id}'_{\mathsf{Prop}}$) For all subsets X of W, for all subsets Y of W: $X = Y$ if and only if for all members w of W: w is a member of X if and only if w is a member of Y.

If X is a proposition, that is, a subset of W, let us denote its *negation* by:

$$\neg X$$

$\neg X$ is the set of possible worlds in W that are not members of X.

Set-theoretically: $\neg X$ is the complement of X (with respect to W)

Equivalently: $\neg X = \{w \text{ in } W\text{: } w \text{ is not a member of } X\}$

Equivalently: $\neg X = W \setminus X$ (read: 'W without X')

If X and Y are propositions, that is, subsets of W, let us denote their *conjunction* by:

$$X \wedge Y$$

$X \wedge Y$ is the set of possible worlds in W that are members of both X and Y.

Set-theoretically: $X \wedge Y$ is the intersection of X and Y

Equivalently: $X \wedge Y = \{w$ in W: w is a member of X and w is a member of $Y\}$

Equivalently: $X \wedge Y = X \cap Y$ (read: 'X intersected with Y')

If X and Y are propositions, that is, subsets of W, let us denote their *disjunction* by:

$$X \vee Y$$

$X \vee Y$ is the set of possible worlds in W that are members of X or Y

Set-theoretically: $X \vee Y$ is the union of X and Y.

Equivalently: $X \vee Y = \{w \text{ in } W\text{: } w \text{ is a member of } X \text{ or } w \text{ is a member of } Y\}$

Equivalently: $X \vee Y = X \cup Y$ (read: 'X united with Y')

- *negation* (of descriptive sentences) corresponds to *complement* (of propositions/sets)
- *conjunction* (of descriptive sentences) corresponds to *intersection* (of propositions/sets)
- *disjunction* (of descriptive sentences) corresponds to *union* (of propositions/sets)

$\mathit{Prop}(A)$: the proposition expressed by sentence A

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- $Prop(\neg A) = \neg Prop(A) = W \setminus Prop(A)$

$\mathit{Prop}(A)$: the proposition expressed by sentence A

- $\mathit{Prop}(\neg A) = \neg \mathit{Prop}(A) = W \setminus \mathit{Prop}(A)$
- $\mathit{Prop}(A \wedge B) = \mathit{Prop}(A) \wedge \mathit{Prop}(B) = \mathit{Prop}(A) \cap \mathit{Prop}(B)$
- $\mathit{Prop}(A \vee B) = \mathit{Prop}(A) \vee \mathit{Prop}(B) = \mathit{Prop}(A) \cup \mathit{Prop}(B)$

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$$

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$$W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$$

$$\begin{aligned} X &= \{w_1, w_2, w_4, w_5\} \\ Y &= \{w_2, w_3, w_5, w_6\} \\ Z &= \{w_4, w_5, w_6, w_7\} \\ \neg X &= \{w_3, w_6, w_7, w_8\} \end{aligned}$$

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X &= \{w_1, w_2, w_4, w_5\} \\
Y &= \{w_2, w_3, w_5, w_6\} \\
Z &= \{w_4, w_5, w_6, w_7\} \\
\neg X &= \{w_3, w_6, w_7, w_8\} \\
X \wedge Y &= \{w_2, w_5\} \\
X \vee Y &= \{w_1, w_2, w_3, w_4, w_5, w_6\} \\
\neg X \wedge Z &= \{w_6, w_7\} \\
(X \wedge Y) \wedge Z &= \{w_5\} \\
\neg X \wedge \neg Y \wedge \neg Z &= \{w_8\}
\end{aligned}$$

$$X \vee \neg X = X \cup \neg X = W$$

$$X \vee \neg X = X \cup \neg X = W$$

$$X \wedge \neg X = X \cap \neg X = \{\}$$

(P1) $\ldots$

(P2) $\ldots$

$\vdots$

(Pn) $\ldots$

(C) $\ldots$

The argument

$$P1, P2, ..., Pn. \text{ Therefore: } C$$

is logically valid.

Equivalently:

$$P1, P2, ..., Pn \text{ logically imply } C.$$

The argument

$$P. \text{ Therefore: } C$$

is logically valid.

Equivalently:

$$P \text{ logically implies } C.$$

A logically implies $A \vee B$

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X is a subset of $X \cup Y$

- *negation* (of descriptive sentences) corresponds to *complement* (of propositions/sets)
- *conjunction* (of descriptive sentences) corresponds to *intersection* (of propositions/sets)
- *disjunction* (of descriptive sentences) corresponds to *union* (of propositions/sets)
- *logical implication* (of descriptive sentences) corresponds to the *subset relation* (between propositions/sets)

A logically implies B if and only if $\mathsf{Prop}(\mathsf{A})$ is a subset of $\mathsf{Prop}(\mathsf{B})$.

A logically implies B if and only if $\mathit{Prop}(A)$ is a subset of $\mathit{Prop}(B)$.

For all propositions X, Y:

X logically implies Y if and only if X is a subset of Y.

$A \wedge B$ logically implies A

$X \cap Y$ is a subset of X

Belief aims at the truth.

X: that the weather will be nice tomorrow

- Believed: X

X: that the weather will be nice tomorrow
Y: that the weather will be nice on the day after tomorrow

- Believed: X
- Believed: $\neg X \vee (X \wedge \neg Y)$

X: that the weather will be nice tomorrow
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- Believed: X
- Believed: $\neg X \vee (X \wedge \neg Y)$
- Believed: $X \wedge \neg Y$

X: that the weather will be nice tomorrow
Y: that the weather will be nice on the day after tomorrow

- Believed: X
- Believed: $\neg X \vee (X \wedge \neg Y)$
- Believed: $X \wedge \neg Y$
- Believed: $\neg Y$

- Rational Belief 1: If a person is inferentially perfectly rational (with W as her set of entertainable possible worlds), then she believes W.

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- Rational Belief 1: If a person is inferentially perfectly rational (with W as her set of entertainable possible worlds), then she believes W.
- Rational Belief 2: If a person is inferentially perfectly rational, then she does not believe $\{\}$.
- Rational Belief 3: If a person is inferentially perfectly rational, if she believes X, and if X is a subset of Y, then she also believes Y.
- Rational Belief 4: If a person is inferentially perfectly rational, if she believes X, and if she believes Y, then she also believes $X \cap Y$.

Theorem:

Let W be a finite, non-empty set of possible worlds.

If the Rational Belief postulates 1–4 are the case (for the given set W of possible worlds), then for every inferentially perfectly rational person p (for whom W is the set of entertainable possible worlds), there is a non-empty proposition B_W, such that the following holds:

For all propositions X (over W),

person p believes X if and only if B_W is a subset of X.

Proof:

Let B_W be the conjunction of all propositions believed by p.

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Let B_W be the conjunction of all propositions believed by p.

(By postulate Rational Belief 1, person p believes at least one proposition.)

Law of commutativity:

- $X \cap Y = Y \cap X$

Law of associativity:

- $X \cap (Y \cap Z) = (X \cap Y) \cap Z$

Law of commutativity:

- $X \cap Y = Y \cap X$

Law of associativity:

- $X \cap (Y \cap Z) = (X \cap Y) \cap Z$

Compare:

$2 \cdot 3 = 3 \cdot 2$
$2 \cdot (3 \cdot 5) = (2 \cdot 3) \cdot 5$

$$X_1 \cap X_2 \cap \dots \cap X_n$$

$$X_1 \cap X_2 \cap \ldots \cap X_n$$

$$X_2 \cap X_1 \cap \ldots \cap X_n$$

$$X_n \cap X_{n-1} \cap \ldots \cap X_1$$

We need to show:

For all propositions X (over W),

$$\text{person } p \text{ believes } X \text{ if and only if } B_W \text{ is a subset of } X.$$

Assume the left-hand side: person p believes X.

We need to show: B_W is a subset of that X.

If $X_1, \ldots, X_n$ are all the propositions believed by p, then

$$B_W = X_1 \cap X_2 \cap ... \cap X_n$$

where one of $X_1, \ldots, X_n$ is X.

If $X_1, \ldots, X_n$ are all the propositions believed by p, then

$$B_W = X_1 \cap X_2 \cap ... \cap X_n$$

where one of $X_1, \ldots, X_n$ is X.

E.g.:

$$B_W = X \cap X_2 \cap ... \cap X_n$$

In any case: B_W is a subset of X. ✓

Now for the other direction of:

For all propositions X (over W),

$$\text{person } p \text{ believes } X \text{ if and only if } B_W \text{ is a subset of } X.$$

Assume the right-hand side: B_W is a subset of X.

We need to show: person p believes X.

We have:

$$B_W = X_1 \cap X_2 \cap X_3 \cap ... \cap X_n$$

where $X_1, \ldots, X_n$ are all the propositions believed by p.

We can reformulate this:

$$B_W = (\ldots((X_1 \cap X_2) \cap X_3) \cap \ldots \cap X_n)$$

$$B_W = (\ldots((X_1 \cap X_2) \cap X_3) \cap \ldots \cap X_n)$$

By postulate Rational Belief 4:

- Since X_1 is believed by p, X_2 is believed by p, also $(X_1 \cap X_2)$ is believed by p.
- Since $X_1 \cap X_2$ is believed by p, and X_3 is believed by p, also $((X_1 \cap X_2) \cap X_3)$ is believed by p.
- Since $((X_1 \cap X_2) \cap X_3)$ is believed by p, and X_4 is believed by p, also $(((X_1 \cap X_2) \cap X_3) \cap X_4)$ is believed by p.

  $\vdots$
- $\underbrace{(...((X_1 \cap X_2) \cap X_3) \cap ... \cap X_n)}_{B_W}$ is believed by p.

By postulate Rational Belief 3:

Since B_W is believed by p, and B_W is a subset of X, also X is believed by p. ✓

By postulate Rational Belief 3:

Since B_W is believed by p, and B_W is a subset of X, also X is believed by p. ✓

Additionally, by postulate Rational Belief 2: $\{\}$ is not believed by p.

Therefore, B_W is not the empty set.

$X = \{w_1, w_2, w_4, w_5\}$
$Y = \{w_2, w_3, w_5, w_6\}$
$Z = \{w_4, w_5, w_6, w_7\}$

E.g., assume belief to be generated by $B_W = X \cap Y = \{w_2, w_5\}$.

Believed: X, Y, $X \cup Y, \ldots$

$X = \{w_1, w_2, w_4, w_5\}$
$Y = \{w_2, w_3, w_5, w_6\}$
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E.g., assume belief to be generated by $B_W = X \cap Y = \{w_2, w_5\}$.

Believed: X, Y, $X \cup Y, \ldots$
Not believed: Z, $\{w_5\}$, $\{w_8\}, \ldots$

Assume belief to be generated by B_W.

If X is a proposition (a subset of W), then there are three possible cases:

- B_W is a subset of X: X is believed.
- $B_W \cap X = \{\}$: $\neg X$ is believed (X is disbelieved).
- B_W is not a subset of X, $B_W \cap X \neq \{\}$:

  Neither X is believed, nor $\neg X$ is believed.

$X = \{w_1, w_2, w_4, w_5\}$
$Y = \{w_2, w_3, w_5, w_6\}$
$Z = \{w_4, w_5, w_6, w_7\}$

E.g., assume belief to be generated by $B_W = X \cap Y = \{w_2, w_5\}$.

Believed: e.g., X
Disbelieved: e.g., $\{w_4, w_7\}$
Neither believed nor disbelieved: e.g., Z, $\neg Z$

Boolean algebra of sets

Boolean algebra of sets

Filter

Boolean algebra of sets

Filter

Ultrafilter

- Area 1: $A(W) = 1$.

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- Area 2: For all propositions (regions) X, $0 \leq A(X) \leq 1$.

- Area 1: $A(W) = 1$.
- Area 2: For all propositions (regions) X, $0 \leq A(X) \leq 1$.
- Area 3: For all propositions (regions) X, for all propositions (regions) Y, if $X \cap Y = \{\}$, then $A(X \cup Y) = A(X) + A(Y)$.

- Area 1: $A(W) = 1$.
- Area 2: For all propositions (regions) X, $0 \leq A(X) \leq 1$.

- Area 3: For all propositions (regions) X, for all propositions (regions) Y, if $X \cap Y = \{\}$, then $A(X \cup Y) = A(X) + A(Y)$.

$X \cap \neg X = \{\}$.

$X \cup \neg X = W$.
$A(X \cup \neg X) = A(W) = 1$.

By Area 3: $A(X \cup \neg X) = A(X) + A(\neg X)$.

Therefore: $1 = A(X) + A(\neg X)$.
Equivalently: $A(\neg X) = 1 - A(X)$.

- Rational Degree of Belief 1: If a person is inferentially perfectly rational (with W as her set of entertainable possible worlds), then her degree of belief function P is such that: $P(W) = 1$.
- Rational Degree of Belief 2: If a person is inferentially perfectly rational (with W as her set of entertainable possible worlds), then her degree of belief function P is such that:
  For all propositions X (for all subsets X of W): $P(X)$ is a real number, such that $0 \leq P(X) \leq 1$.

- Rational Degree of Belief 3: If a person is inferentially perfectly rational (with W as her set of entertainable possible worlds), then her degree of belief function P is such that:
  For all propositions X (for all subsets X of W), for all propositions Y (for all subsets Y of W):
  if $X \cap Y = \{\}$, then $P(X \cup Y) = P(X) + P(Y)$.

Different kinds of probability:

- Statistical probability.
- Objective single-case probability.
- Geometrical probability.

  ⋮

Different kinds of probability:

- Statistical probability.
- Objective single-case probability.
- Geometrical probability.

  $\vdots$
- Subjective probability.

(i) $P(\neg X) = 1 - P(X)$.

(ii) $P(\{\}) = 0$.

(iii) $P(Y) = P(X \cap Y) + P(\neg X \cap Y)$.

(iv) If X is a subset of Y, then $P(X) \leq P(Y)$.

(i) Show: $P(\neg X) = 1 - P(X)$.

By Rational Degree of Belief 3: $P(W) = P(X) + P(\neg X)$.
By Rational Degree of Belief 1: $P(W) = 1$.
Hence: $1 = P(X) + P(\neg X)$.
That is: $P(\neg X) = 1 - P(X)$.

(ii) Show: $P(\{\}) = 0$.

Plug in 'W' for 'X' in (i): $P(\neg X) = 1 - P(X)$.
$P(\neg W) = 1 - P(W)$.
Since $\neg W = \{\}$, and $P(W) = 1$ by Rational Degree of Belief 1:
$P(\{\}) = 1 - 1 = 0$.

(iii) Show: $P(Y) = P(X \cap Y) + P(\neg X \cap Y)$.

Since $(X \cap Y) \cap (\neg X \cap Y) = \{\}$, it follows from Rational Degree of Belief 3:

$$P((X \cap Y) \cup (\neg X \cap Y)) = P(X \cap Y) + P(\neg X \cap Y).$$

$$P(Y) = P(X \cap Y) + P(\neg X \cap Y).$$

(iv) Show: If X is a subset of Y, then $P(X) \leq P(Y)$.

By (iii): $P(Y) = P(X \cap Y) + P(\neg X \cap Y)$.

If X is a subset of Y, then $X \cap Y = X$.
So: $P(Y) = P(X) + P(\neg X \cap Y)$.

By Rational Degree of Belief 2: $0 \leq P(\neg X \cap Y) \leq 1$.
Hence: $P(Y) = P(X) + P(\neg X \cap Y) \geq P(X)$.
That is: $P(X) \leq P(Y)$.

Theorem:

Let W be a finite, non-empty set of possible worlds.

If the Rational Degree of Belief postulates 1–3 are the case (for the given set W of possible worlds), then for every inferentially perfectly rational person's degree of belief function P (on W as her set of entertainable possible worlds), there is a function B, such that the following holds:

– B assigns to each world w in W a non-negative real number $B(w)$.

– The sum of all the values of B on worlds in W is 1: that is, if $W = \{w_1, \ldots, w_n\}$, then $B(w_1) + \ldots + B(w_n) = 1$.

– For all propositions X (over W), $P(X)$ is the sum of the values of B on worlds in X; that is:

$$P(X) = \sum_{w \text{ in } X} B(w).$$

E.g., assume P to be generated by:

$B(w_1) = 1/15$
$B(w_2) = 1/3$
$B(w_3) = 1/15$
$B(w_4) = 1/15$
$B(w_5) = 1/3$
$B(w_6) = 1/15$
$B(w_7) = 1/15$
$B(w_8) = 0$

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$B(w_5) = 1/3$
$B(w_6) = 1/15$
$B(w_7) = 1/15$
$B(w_8) = 0$

$P(\{w_1\}) = B(w_1) = 1/15$

E.g., assume P to be generated by:

$B(w_1) = 1/15$
$B(w_2) = 1/3$
$B(w_3) = 1/15$
$B(w_4) = 1/15$
$B(w_5) = 1/3$
$B(w_6) = 1/15$
$B(w_7) = 1/15$
$B(w_8) = 0$

$$P(\underbrace{\{w_1, w_2, w_4, w_5\}}_{X}) = B(w_1) + B(w_2) + B(w_4) + B(w_5) =$$

$$= 2/3 + 2/15 = 0.8$$

I buy a bet on the proposition X:

- First of all, I pay $q \cdot S$ Euro.
- If X is false, my net loss is $q \cdot S$ Euro.
- If X is true, my net profit is $S - q \cdot S$ Euro.

I buy a bet on the proposition X:

- First of all, I pay $q \cdot S$ Euro.
- If X is false, my net loss is $q \cdot S$ Euro.
- If X is true, my net profit is $S - q \cdot S$ Euro.

S: *stake* of the bet; $S = (S - q \cdot S) + q \cdot S$.

I buy a bet on the proposition X:

- First of all, I pay $q \cdot S$ Euro.
- If X is false, my net loss is $q \cdot S$ Euro.
- If X is true, my net profit is $S - q \cdot S$ Euro.

S: *stake* of the bet; $S = (S - q \cdot S) + q \cdot S$

q: *betting quotient* of the bet; $q = \frac{q \cdot S}{S}$

I regard a bet on X as fair if and only if my degree of belief $P(X)$ in X equals the betting quotient q of the bet:

$$q = P(X)$$

I regard a bet on X as fair if and only if my degree of belief $P(X)$ in X equals the betting quotient q of the bet:

$$q = P(X)$$

E.g., $P(X) = 1/2$, $q = 1/2$:

- First of all, I pay $q \cdot S = S/2$ Euro.
- If X is true, my net profit is $S - q \cdot S = S/2$ Euro.
- If X is false, my net loss is $q \cdot S = S/2$ Euro.

I regard a bet on X as fair if and only if my degree of belief $P(X)$ in X equals the betting quotient q of the bet:

$$q = P(X)$$

E.g., $P(X) = 1$, $q = 1$:

- First of all, I pay $q \cdot S = S$ Euro.
- If X is true, my net profit is $S - q \cdot S = 0$ Euro.
- If X is false, my net loss is $q \cdot S = S$ Euro.

If I sell a bet on the proposition X, things are reversed:

- First of all, my opponent pays $q \cdot S$ Euro.
- If X is true, my net loss is $S - q \cdot S$ Euro.
- If X is false, my net profit is $q \cdot S$ Euro.

Assume two propositions X and Y to have empty intersection, and:

$P(X) = 0.4$, $P(Y) = 0.4$, but $P(X \cup Y) = 0.7$ (!)

Assume two propositions X and Y to have empty intersection.

$P(X) = 0.4$, $P(Y) = 0.4$, but $P(X \cup Y) = 0.7$ (!)

- Bet on X: $P(X) = 0.4$, $S_X = 1$ Euro, $q_X = 0.4$.
  X is true: my net profit is $S_X - q_X \cdot S_X = 1 - q_X = 0.6$ Euro.
  $\neg X$ is true: my net loss is $q_X \cdot S_X = q_X = 0.4$ Euro.

Assume two propositions X and Y to have empty intersection.

$P(X) = 0.4$, $P(Y) = 0.4$, but $P(X \cup Y) = 0.7$ (!)

- Bet on Y: $P(Y) = 0.4$, $S_Y = 1$ Euro, $q_Y = 0.4$.
  Y is true: my net profit is $S_Y - q_Y \cdot S_Y = 1 - q_Y = 0.6$ Euro.
  $\neg Y$ is true: my net loss is $q_Y \cdot S_Y = q_Y = 0.4$ Euro.

Assume two propositions X and Y to have empty intersection.

$P(X) = 0.4$, $P(Y) = 0.4$, but $P(X \cup Y) = 0.7$ (!)

- Bet on $X \cup Y$: $P(X \cup Y) = 0.7$, $S_{X \cup Y} = 1$ Euro, $q_{X \cup Y} = 0.7$.
  $X \cup Y$ is true: my net loss is $S_{X \cup Y} - q_{X \cup Y} \cdot S_{X \cup Y} = 1 - q_{X \cup Y} = 0.3$ Euro.
  $\neg(X \cup Y)$ is true: my net profit is $q_{X \cup Y} \cdot S_{X \cup Y} = q_{X \cup Y} = 0.7$ Euro.

Say, I bet on X, I bet on Y, and someone else bets on $X \cup Y$ against me; my degrees of belief tell me that all of these bets are fair.

- Case 1: X is true and Y is true, hence $X \cup Y$ is true:
  logically impossible, since X and Y have empty intersection!
- Case 2: X is true and Y is false, and hence $X \cup Y$ is true:
  $0.6 - 0.4 - 0.3 = -0.1$. Overall I lose 0.1 Euro.
- Case 3: X is false and Y is true, and hence $X \cup Y$ is true:
  $-0.4 + 0.6 - 0.3 = -0.1$. Overall I lose 0.1 Euro.
- Case 4: X is false and Y is false, and hence $X \cup Y$ is false:
  $-0.4 - 0.4 + 0.7 = -0.1$. Overall I lose 0.1 Euro.

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Consider a lottery with 1000000 tickets:

(P1)

$P(\textsf{not ticket 1 wins} \ \wedge\ \textsf{not ticket 2 wins} \ \wedge \ldots \wedge \textsf{not ticket 1000000 wins}) = 0.$

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(P2)

$$P(\textsf{ticket 1 wins}) = 1/1000000$$
$$P(\textsf{ticket 2 wins}) = 1/1000000$$
$$\vdots$$
$$P(\textsf{ticket 1000000 wins}) = 1/1000000$$

(P3) For every proposition X:

I believe X if and only if $P(X) \geq 0.9$.

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(P5) For every proposition X:

$P(\neg X) = 1 - P(X)$.

(And: $P(X) = 1 - P(\neg X)$.)

(C)

$P(\textsf{not ticket 1 wins } \wedge \textsf{ not ticket 2 wins } \wedge \ldots \wedge \textsf{ not ticket 1000000 wins}) \geq 0.9$,

and

$P(\textsf{not ticket 1 wins } \wedge \textsf{ not ticket 2 wins } \wedge \ldots \wedge \textsf{ not ticket 1000000 wins}) = 0$.

The two premises

(P2)

$P(\text{ticket 1 wins}) = 1/1000000$

$\vdots$

$P(\text{ticket 1000000 wins}) = 1/1000000$

(P5) For every proposition X:

$P(\neg X) = 1 - P(X)$.

taken together entail:

for each i, $P(\text{not ticket } i \text{ wins}) = 999999/1000000$.

Therefore, by premise

> (P3) For every proposition X:
> I believe X if and only if $P(X) \geq 0.9$.

it follows:

I believe that not ticket 1 wins.

$\vdots$

I believe that not ticket 1000000 wins.

So, by applying premise

> (P4) For every proposition X, for every proposition Y:
>
> if I believe X and I believe Y, then (if I am perfectly rational) I also believe $X \wedge Y$.

multiple times, it follows:

- I believe that (not ticket 1 wins $\wedge$ not ticket 2 wins $\wedge \ldots \wedge$ not ticket 1000000 wins).

- I believe that (not ticket 1 wins $\wedge$ not ticket 2 wins $\wedge \ldots \wedge$ not ticket 1000000 wins).

with the two premises

(P1)

P(not ticket 1 wins $\wedge$ not ticket 2 wins $\wedge \ldots \wedge$ not ticket 1000000 wins) $= 0$.

(P3) For every proposition X:

I believe X if and only if $P(X) \geq 0.9$.

leads to the contradiction C.

(P1) $P(\text{not ticket 1 wins } \wedge \text{ not ticket 2 wins } \wedge \ldots$
$\wedge \text{ not ticket 1000000 wins}) = 0.$

(P2)

$P(\text{ticket 1 wins}) = 1/1000000$

$\vdots$

$P(\text{ticket 1000000 wins}) = 1/1000000$

(P3) For every proposition X:

I believe X if and only if $P(X) \geq 0.9$.

(P4) For every proposition X, for every proposition Y:

if I believe X and I believe Y, then I believe $X \wedge Y$.

(P5) For every proposition X:

$P(\neg X) = 1 - P(X)$.