A conditional is an 'If… then…' sentence: a sentence that is of the form

- If A, then B

or which has the same meaning as a sentence of that form, such as, e.g.,

- If A, B
- B if A
- In case that A it holds that B

  $\vdots$

A conditional is an 'If... then...' sentence: a sentence that is of the form

- If A, then B

or which has the same meaning as a sentence of that form, such as, e.g.,

- If A, B
- B if A
- In case that A it holds that B

  $\vdots$

In short:

- $A \to B$

For instance:

- If Conny comes to the party, then I will talk to her there.
- If Oswald did not kill Kennedy, someone else did.
- If Shackleton had known how to ski, then he would have reached the South Pole in 1909.
- If Oswald had not killed Kennedy, someone else would have.

Reconsider the *indicative*

- If Oswald did not kill Kennedy, someone else did.

  $\hookrightarrow$ Acceptable

in comparison with the *subjunctive*

- If Oswald had not killed Kennedy, someone else would have.

  $\hookrightarrow$ Not acceptable

$A \rightarrow B$:

- not (A and not B)
- (not A) or B
- (not A) or (A and B)

$A \to B$:

- not (A and not B)
- (not A) or B
- (not A) or (A and B)

- $\neg(A \wedge \neg B)$
- $\neg A \vee B$
- $\neg A \vee (A \wedge B)$

A expresses the set X of worlds.
B expresses the set Y of worlds.

The proposition expressed by

- $\neg(A \wedge \neg B)$

is nothing else than

- $W \setminus (X \cap (W \setminus Y))$

A expresses the set X of worlds.
B expresses the set Y of worlds.

The proposition expressed by

- $\neg A \vee B$

is nothing else than

- $(W \setminus X) \cup Y$

A expresses the set X of worlds.
B expresses the set Y of worlds.

The proposition expressed by

- $\neg A \vee (A \wedge B)$

is nothing else than

- $(W \setminus X) \cup (X \cap Y)$

$A \to B$:

- $\neg(A \wedge \neg B)$
- $\neg A \vee B$
- $\neg A \vee (A \wedge B)$

So ‘if A then B’ amounts to:

- Either case 1: The conditional does not apply ($\neg A$).
- Or case 2: The conditional does apply (A) and B is true (B).

Or more briefly:

$\neg A \vee (A \wedge B)$

$A \to B$:

- $\neg(A \land \neg B)$
- $\neg A \lor B$

Assume that a mathematician has proven $A \to B$, that is,

- $\neg A \lor B$

Case 1:

- She also proves $\neg A$.

  Nothing bad follows.

Assume that a mathematician has proven $A \to B$, that is,

- $\neg A \vee B$

Case 2:

- She also proves A.

$$\left.\begin{array}{l} A \\ A \to B \\ \hline B \end{array}\right\} \text{logically valid}$$

Assume that a mathematician has proven $A \to B$, that is,

- $\neg A \vee B$

Case 2:

- She also proves A.

$$\left.\begin{array}{l} A \\ \neg A \vee B \\ \hline B \end{array}\right\} \text{logically valid}$$

Assume that a mathematician has proven $A \to B$, that is,

- $\neg A \vee B$

Case 3:

- She neither proves A nor $\neg A$.

  Nothing bad follows.

$A \to B$:

- $\neg(A \wedge \neg B)$

- (case: $x =$ 'if' $+ y +$ 'then' $+ z$)

  if there is a sentence y of L and a sentence z of L, such that x is the result of putting together 'if', with y, with 'then', and with z, then
  x is true if and only if y is not true or z is true;

  (Equivalently:
  if x is the result of putting together 'if', with a sentence y of L, with 'then', and with a sentence z of L, then
  x is true if and only if y is not true or z is true.)

  (Equivalently, and most precisely:
  for all sentences y of L, for all sentences z of L, if x is the result of putting together 'if', with y, with 'then', and with z, then
  x is true if and only if y is not true or z is true.)

- If the moon is made of green cheese, then 2+2=4.
- If the moon is made of green cheese, then it is not the case that 2+2=4.
- If the moon is not made of green cheese, then 2+2=4.

- If $\underbrace{\text{the moon is made of green cheese}}_{A}$, then $\underbrace{\text{2+2=4}}_{B}$.
- If $\underbrace{\text{the moon is made of green cheese}}_{A}$, then $\underbrace{\text{not 2+2=4}}_{B}$.

correspond to, respectively:

- $\underbrace{\text{The moon is not made of green cheese}}_{\neg A}$ or $\underbrace{\text{2+2=4}}_{B}$.
- $\underbrace{\text{The moon is not made of green cheese}}_{\neg A}$ or $\underbrace{\text{not 2+2=4}}_{B}$.

- If $\underbrace{\text{the moon is not made of green cheese}}_{A}$, then $\underbrace{2{+}2{=}4}_{B}$.

corresponds to:

- $\underbrace{\text{The moon is not not made of green cheese}}_{\neg A}$ or $\underbrace{2{+}2{=}4}_{B}$.

$W = \{w_1, \ldots, w_8\}$.

$B(w_1) = 1/15$
$B(w_2) = 1/3$
$B(w_3) = 1/15$
$B(w_4) = 1/15$
$B(w_5) = 1/3$
$B(w_6) = 1/15$
$B(w_7) = 1/15$
$B(w_8) = 0$

For all propositions X (over W):

$$P(X) = \sum_{w \text{ in } X} B(w).$$

For all propositions X (over W):

$$P(X) = \sum_{w \text{ in } X} B(w).$$

- $P(W) = 1$;
- for all propositions X (over W): $P(X)$ is a real number, such that $0 \leq P(X) \leq 1$;
- for all propositions X, Y (over W): if $X \cap Y = \{\}$, then $P(X \cup Y) = P(X) + P(Y)$.

E.g., $P(\underbrace{\{w_1, w_2, w_4, w_5\}}_{X}) = B(w_1) + B(w_2) + B(w_4) + B(w_5) =$

$$= 2/3 \,+\, 2/15 = 0.8$$

$X = \{w_1, w_2, w_4, w_5\}$

$$
\begin{array}{lll}
w_1\colon 1/15 & \rightarrow & 1/15 \\
w_2\colon 1/3 & \rightarrow & 1/3 \\
w_3\colon 1/15 & \rightarrow & {\color{red}0} \\
w_4\colon 1/15 & \rightarrow & 1/15 \\
w_5\colon 1/3 & \rightarrow & 1/3 \\
w_6\colon 1/15 & \rightarrow & {\color{red}0} \\
w_7\colon 1/15 & \rightarrow & {\color{red}0} \\
w_8\colon 0 & \rightarrow & {\color{red}0}
\end{array}
$$

$X = \{w_1, w_2, w_4, w_5\}$

$$
\begin{array}{lllllll}
w_1\!: 1/15 & \rightarrow & 1/15 & \rightarrow & (1/15)/0.8 & = 0.08333\ldots \\
w_2\!: 1/3 & \rightarrow & 1/3 & \rightarrow & (1/3)/0.8 & = 0.41666\ldots \\
w_3\!: 1/15 & \rightarrow & 0 & \rightarrow & 0/0.8 & = 0 \\
w_4\!: 1/15 & \rightarrow & 1/15 & \rightarrow & (1/15)/0.8 & = 0.08333\ldots \\
w_5\!: 1/3 & \rightarrow & 1/3 & \rightarrow & (1/3)/0.8 & = 0.41666\ldots \\
w_6\!: 1/15 & \rightarrow & 0 & \rightarrow & 0/0.8 & = 0 \\
w_7\!: 1/15 & \rightarrow & 0 & \rightarrow & 0/0.8 & = 0 \\
w_8\!: 0 & \rightarrow & 0 & \rightarrow & 0/0.8 & = 0
\end{array}
$$

$X = \{w_1, w_2, w_4, w_5\}$

$$
\begin{array}{llllll}
w_1\colon 1/15 & \rightarrow & 1/15 & \rightarrow & (1/15)/0.8 & = 0.08333\ldots \\
w_2\colon 1/3 & \rightarrow & 1/3 & \rightarrow & (1/3)/0.8 & = 0.41666\ldots \\
w_3\colon 1/15 & \rightarrow & 0 & \rightarrow & 0/0.8 & = 0 \\
w_4\colon 1/15 & \rightarrow & 1/15 & \rightarrow & (1/15)/0.8 & = 0.08333\ldots \\
w_5\colon 1/3 & \rightarrow & 1/3 & \rightarrow & (1/3)/0.8 & = 0.41666\ldots \\
w_6\colon 1/15 & \rightarrow & 0 & \rightarrow & 0/0.8 & = 0 \\
w_7\colon 1/15 & \rightarrow & 0 & \rightarrow & 0/0.8 & = 0 \\
w_8\colon 0 & \rightarrow & 0 & \rightarrow & 0/0.8 & = 0
\end{array}
$$

Initial degree of belief function: P

Final degree of belief function: P_X

Let W be a non-empty and finite set of possible worlds.

Let P be a probability measure (over W), and assume P to be determined uniquely by B as explained in the previous lecture.

Then we can define: for all propositions X with $P(X) > 0$, for all worlds w in W,

- $B_X(w) = 0$ if w is not a member of X;
- $B_X(w) = \frac{B(w)}{P(X)}$ if w is a member of X.

For all propositions Y:

$$P_X(Y) = \sum_{w \text{ in } Y} B_X(w).$$

$$P_X(Y) = \sum_{w \text{ in } X \cap Y} \frac{B(w)}{P(X)} + \sum_{w \text{ not in } X} 0$$

$$P_X(Y) = \sum_{w \text{ in } X \cap Y} \frac{B(w)}{P(X)} + \sum_{w \text{ not in } X} 0$$

$$P_X(Y) = \frac{1}{P(X)} \cdot \sum_{w \text{ in } X \cap Y} B(w)$$

$$P_X(Y) = \sum_{w \text{ in } X \cap Y} \frac{B(w)}{P(X)} + \sum_{w \text{ not in } X} 0$$

$$P_X(Y) = \frac{1}{P(X)} \cdot \sum_{w \text{ in } X \cap Y} B(w)$$

$$P_X(Y) = \frac{1}{P(X)} \cdot P(X \cap Y)$$

$$P_X(Y) = \sum_{w \text{ in } X \cap Y} \frac{B(w)}{P(X)} + \sum_{w \text{ not in } X} 0$$

$$P_X(Y) = \frac{1}{P(X)} \cdot \sum_{w \text{ in } X \cap Y} B(w)$$

$$P_X(Y) = \frac{1}{P(X)} \cdot P(X \cap Y)$$

$$P_X(Y) = \frac{P(X \cap Y)}{P(X)}$$

Definition:

For all probability measures P (over W), for all propositions X with $P(X) > 0$, for all propositions Y:

$$P(Y|X) = \frac{P(X \cap Y)}{P(X)}$$

In words: The conditional probability of Y given X as determined by P is $P(X \cap Y)/P(X)$.

(If $P(X) > 0$:)

$$P_X(Y) = P(Y|X) = \frac{P(X \cap Y)}{P(X)}$$

(If $P(X) > 0$:)

$$P_X(X) = P(X|X) = \frac{P(X \cap X)}{P(X)}$$

(If $P(X) > 0$:)

$$P_X(X) = P(X|X) = \frac{P(X \cap X)}{P(X)}$$

$$= \frac{P(X)}{P(X)} = 1$$

E.g., by

$$P(Y|X) = \frac{P(X \cap Y)}{P(X)}$$

it holds:

$$P(X|X \cap Y) = \frac{P(X \cap Y \cap X)}{P(X \cap Y)} = \frac{P(X \cap Y)}{P(X \cap Y)} = 1$$

E.g., by

$$P(Y|X) = \frac{P(X \cap Y)}{P(X)}$$

we also have:

$$P(X|\neg X \cap Y) = \frac{P(\neg X \cap Y \cap X)}{P(\neg X \cap Y)} = \frac{P(\{\})}{P(\neg X \cap Y)} = 0$$

Initial degree of belief in $Y \cap Z = \{w_5, w_6\}$:

$P(Y \cap Z) = P(\{w_5, w_6\}) = 1/3 + 1/15 = 0.4$

Degree of belief in $Y \cap Z = \{w_5, w_6\}$
on the supposition of X:

$$P_X(Y \cap Z) = P(Y \cap Z | X) = P(X \cap Y \cap Z)/P(X)$$

$$= P(w_5)/0.8 = \frac{1/3}{0.8} = 0.416666\ldots$$

Degree of belief in $Y \cap Z = \{w_5, w_6\}$ on the supposition of

$$Z = \{w_4, w_5, w_6, w_7\}:$$

$$P_Z(Y \cap Z) = P(Y \cap Z|Z) = P(Z \cap Y \cap Z)/P(Z)$$

$$= P(Y \cap Z)/0.53333\ldots = \frac{0.4}{0.53333...} = 0.75$$

Conditionalizing P on X leads to: P_X

Thesis 1:

(i) There is a conditional operation $\to$ that can take any two propositions as input, which maps them to a proposition as output, and which has the following property:

For every indicative conditional $A \to B$, where A expresses the proposition X, and B expresses the proposition Y, it holds that:

$$A \to B \text{ expresses the proposition } X \to Y.$$

Thesis 1: [CONTINUED]

(ii) For every indicative conditional $A \to B$, where A expresses the proposition X, and B expresses the proposition Y, and for every probability measure P on propositions, it holds that:

the degree of acceptability for the indicative conditional $A \to B$ (rel. to P) is identical to $P(X \to Y)$

where $X \to Y$ is the proposition expressed by $A \to B$ as explained in (i).

Frank P. Ramsey:

> *If two people are arguing 'If* p *will* q*?' and are both in doubt as to* p*, they are adding* p *hypothetically to their stock of knowledge and arguing on that basis about* q*.*

Thesis 2:

For every indicative conditional $A \rightarrow B$, where A expresses the proposition X, and B expresses the proposition Y, and for every probability measure P on propositions, it holds that:

the degree of acceptability for the indicative conditional $A \rightarrow B$ *(rel. to* P*) is identical to* $P(Y|X)$ *(or* $P_X(Y)$*).*

How acceptable is $A \to B$ to me?

- Thesis 1: $P(X \to Y)$.
- Thesis 2: $P(Y|X)$.

How acceptable is $A \to B$ to me?

- Thesis 1: $P(X \to Y)$.
- Thesis 2: $P(Y|X)$.
- Theses 1 and 2 taken together entail:

$$P(X \to Y) = P(Y|X).$$

(David Kellogg Lewis)

Theorem:

Let W be a given non-empty set of possible worlds. By propositions we mean subsets of W again, as usual.

(Ass. 1) There is a conditional operation $\rightarrow$ that can take any two propositions over W as input and which maps them to a uniquely determined proposition over W as output.

(Ass. 2) Every rational degree of belief function P (on W) is a probability measure.

(Ass. 3) Every rational degree of belief function P (on W) satisfies:

For all propositions X, Y: $P(X \rightarrow Y) = P(Y|X)$,

where $X \rightarrow Y$ is as described by Ass. 1.

(Ass. 4) The set of all rational degree of belief functions P (on W) is closed under conditionalization: that is,

for every rational degree of belief function P (on W), for every proposition X for which $P(X) > 0$, the conditionalization P_X of P on X is a rational degree of belief function (on W) as well.

And we presuppose that P_X was defined like this:

(If $P(X) > 0$:)

For all propositions Y: $P_X(Y) = P(Y|X) = P(X \cap Y)/P(X)$.

Conclusion 1:

For all propositions X, Y, Z, for all rational degree of belief functions P (on W) for which it holds that $P(X) > 0$, $P(X \cap Y) > 0$:

$$P(Y \to Z|X) = P(Z|X \cap Y).$$

Conclusion 2:

For all propositions X, Y, Z, for all rational degree of belief functions P (on W) for which it holds that $P(Z) > 0$, $P(\neg Z) > 0$, $P(Z \cap Y) > 0$, $P(\neg Z \cap Y) > 0$:

$$P(Y \to Z) = P(Z).$$

Proof: Concerning Conclusion 1.

$$
\begin{aligned}
P(Y \to Z|X) &= P_X(Y \to Z) && \text{(def. of } P_X\text{)} \\
&= P_X(Z|Y) && \text{(Ass. 4 and 3)} \\
&= \frac{P_X(Y \wedge Z)}{P_X(Y)} && \text{(def. of cond. prob.)} \\
&= \frac{P(Y \wedge Z|X)}{P(Y|X)} && \text{(def. of } P_X\text{)} \\
&= \frac{\frac{P(X \wedge Y \wedge Z)}{P(X)}}{\frac{P(X \wedge Y)}{P(X)}} && \text{(def. of.cond. prob.)} \\
&= \frac{P(X \wedge Y \wedge Z)}{P(X \wedge Y)} && \text{(calculate)} \\
&= P(Z|X \wedge Y) && \text{(def. of cond. prob.)}
\end{aligned}
$$

Concerning Conclusion 2.

$$
\begin{aligned}
P(Y \to Z) &= P(Z \wedge (Y \to Z)) + P(\neg Z \wedge (Y \to Z)) && \text{(Add. Th., by Ass. 2)} \\
&= P((Y \to Z)|Z)P(Z) + \\
&\quad P((Y \to Z)|\neg Z)P(\neg Z) && \text{(def. of cond. prob., calc.)} \\
&= P(Z|Z \wedge Y)P(Z) + \\
&\quad P(Z|\neg Z \cap Y)P(\neg Z) && \text{(Conclusion 1)} \\
&= 1 \cdot P(Z) + 0 \cdot P(\neg Z) && \text{(def. of cond. prob., Ass. 2)} \\
&= P(Z) && \text{(calculate)}
\end{aligned}
$$

(Ass. 3) Every rational degree of belief function P (on W) satisfies:

For all propositions X, Y: $P(X \to Y) = P(Y|X)$,

where $X \to Y$ is as described by Ass. 1.

Thesis 1:

(i) There is a conditional operation $\rightarrow$ that can take any two propositions as input, which maps them to a proposition as output, and which has the following property:

For every indicative conditional $A \rightarrow B$, where A expresses the proposition X, and B expresses the proposition Y, it holds that:

$$A \rightarrow B \text{ expresses the proposition } X \rightarrow Y.$$

Thesis 1: [CONTINUED]

(ii) For every indicative conditional $A \rightarrow B$, where A expresses the proposition X, and B expresses the proposition Y, and for every probability measure P on propositions, it holds that:

the degree of acceptability for the indicative conditional $A \rightarrow B$ *(rel. to* P*) is identical to* $P(X \rightarrow Y)$

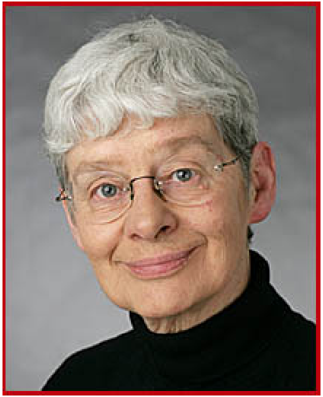
where $X \rightarrow Y$ is the proposition expressed by $A \rightarrow B$ as explained in (i).

Thesis 2:

For every indicative conditional $A \to B$, where A expresses the proposition X, and B expresses the proposition Y, and for every probability measure P on propositions, it holds that:

the degree of acceptability for the indicative conditional $A \to B$ *(rel. to* P*) is identical to* $P(Y|X)$ *(or* $P_X(Y)$*).*

(Ernest W. Adams
we thank UC Berkley for kindly allowing us to use their picture of Prof. Ernst Adams)



(Dorothy Edgington
we thank Prof. Dorothy Edgington for kindly allowing us to use her picture)