Step 1: Reason and Decide

What do we know initially?

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(H_2) The car is behind door No. 2.
(H_3) The car is behind door No. 3.

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The three hypotheses are mutually exclusive and exhaustive:

$$P(H_1 \vee H_2 \vee H_3) = P(H_1) + P(H_2) + P(H_3) = 1$$

Step 2: Learn

What is it, precisely, that we learn?

Step 3: Reason and Decide

Shall we switch doors?

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Note: We use the notation $P(\mathrm{H},\mathrm{E}) = P(\mathrm{H} \wedge \mathrm{E})$ for the probability of the conjunction of two propositions.

Bayes Rule:

$$P(\mathrm{H}|\mathrm{E}) = \frac{P(\mathrm{E}|\mathrm{H})\, P(\mathrm{H})}{P(\mathrm{E})}$$

$P(\mathrm{H})$: prior probability of H

$P(\mathrm{E}|\mathrm{H})$: likelihood of E

$P(\mathrm{E})$: expectedness of E

$P(\mathrm{H}|\mathrm{E})$: posterior probability of H

From the Rule of Total Probability and the definition of conditional probability, we obtain for the expectedness of E:

$$
\begin{aligned}
P(\mathrm{E}) &= P(\mathrm{E}, \mathrm{H}) + P(\mathrm{E}, \neg\mathrm{H}) \\
&= P(\mathrm{E}|\mathrm{H})\, P(\mathrm{H}) + P(\mathrm{E}|\neg\mathrm{H})\, P(\neg\mathrm{H})
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If $\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_\mathrm{n}$ are mutually exclusive and exhaustive, then

$$\begin{aligned} P(\mathrm{E}) &= P(\mathrm{E}|\mathrm{H}_1)\,P(\mathrm{H}_1) + \cdots + P(\mathrm{E}|\mathrm{H}_\mathrm{n})\,P(\mathrm{H}_\mathrm{n}) \\ &= \sum_{i=0}^{n} P(\mathrm{E}|\mathrm{H}_\mathrm{i})\,P(\mathrm{H}_\mathrm{i}). \end{aligned}$$

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Remember: $P(\mathrm{H}_1) = P(\mathrm{H}_2) = P(\mathrm{H}_3) = 1/3$

$$P(\mathrm{E}) = 1 \cdot (1/3) + 1 \cdot (1/3) = 2/3$$

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$$
\begin{aligned}
&P(\mathrm{E}|\mathrm{H}_1) = 1\\
&P(\mathrm{E}|\mathrm{H}_2) = 1\\
&P(\mathrm{E}|\mathrm{H}_3) = 0\\
&P(\mathrm{E}) = 2/3\\
&P(\mathrm{H}_1) = P(\mathrm{H}_2) = P(\mathrm{H}_3) = 1/3
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
P_E(\mathrm{H}_1) = P_E(\mathrm{H}_2) &= \frac{1 \cdot 1/3}{2/3}\\
&= 1/2
\end{aligned}
$$

Reasoning 2:

F: Monty opens door 3.

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Reasoning 2:

F: Monty opens door 3.

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$$\begin{aligned} P(\mathrm{F}) &= P(\mathrm{F}|\mathrm{H}_1)P(\mathrm{H}_1) + P(\mathrm{F}|\mathrm{H}_2)P(\mathrm{H}_2) + P(\mathrm{F}|\mathrm{H}_3)P(\mathrm{H}_3) \\ &= (1/2)\cdot(1/3) + 1\cdot(1/3) \\ &= 1/2 \end{aligned}$$

Confirmation – The Bayesian Way

E confirms H iff $P_E(\mathrm{H}) > P(\mathrm{H})$.

E disconfirms H, iff $P_E(\mathrm{H}) < P(\mathrm{H})$.

E is irrelevant for H, iff $P_E(\mathrm{H}) = P(\mathrm{H})$.

Irrelevance:

$$P(\mathrm{H}|\mathrm{E}) = P(\mathrm{H}) \Leftrightarrow P(\mathrm{H}, \mathrm{E}) = P(\mathrm{H})P(\mathrm{E})$$

We assume: E is a deductive consequence of H.

Hence, $P(\mathrm{E}|\mathrm{H}) = 1$.

Using Bayes Theorem and Bayes Rule:

$$P_E(\mathrm{H}) = \frac{P(\mathrm{H})}{P(\mathrm{E})}$$

We conclude: The smaller $P(\mathrm{E})$, the the greater $P_E(\mathrm{H})$ for a fixed $P(\mathrm{H})$.

Let $P(\mathrm{H}_1) = h_1, P(\mathrm{H}_2) = h_2$, and $P(\mathrm{H}_3) = h_3$.

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Let $h_1 > h_2, h_3$.

$$P_F(\mathrm{H}_1) = \frac{1/2 \cdot h_1}{P(\mathrm{F})}$$

$$P_F(\mathrm{H}_2) = \frac{h_2}{P(\mathrm{F})}$$

$$P(\mathrm{F}) = 1/2 \cdot h_1 + h_2$$

$$P_F(\mathrm{H}_2) > P_F(\mathrm{H}_1) \text{ iff } h_2 > 1/2 \cdot h_1$$

$$P(\mathrm{F}|\mathrm{H}_1) =: a, \text{ with } 1/2 < a < 1$$

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$$\begin{aligned} P_F(\mathrm{H}_1) &= \frac{a \cdot 1/3}{P(\mathrm{F})} \\ P_F(\mathrm{H}_2) &= \frac{1/3}{P(\mathrm{F})} \\ P(\mathrm{F}) &= (1+a) \cdot 1/3 \end{aligned}$$

Hence, $P_F(\mathrm{H}_2) > P_F(\mathrm{H}_1)$. Note that this results holds for all $a \in (0,1)$.