# Introduction to Mathematical Philosophy

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# Week 7: Overview

Overview of Lecture 7: Voting

7.1 Introduction: A group has to rank a number of options. To do so, each group member submits a linear preference ordering, which have to be combined to a group ordering. But how? It turns out that there are several prima facie plausible aggregation procedures which, however, do not always lead to the same group ordering.

7.2 Arrow: We discuss the axiomatic approach to voting theory. It turns out that for two options the (simple) majority rule is the only aggregation procedure that satisfies three reasonable requirements. For more than two options, the Noble Prize-winning economist Kenneth Arrow has shown that there is no aggregation procedure that satisfies the conditions Universal Domain, Pareto, Independence of Irrelevant Alternatives, and Non-Dictatorship.

7.3 Judgement Aggregation: We consider the aggregation of judgments on logically interconnected propositions. Here the discursive dilemma shows up, and we present two ways out of the dilemma: the Premise-Based Procedure and the Conclusion-Based Procedure.

7.4 Epistemic Considerations: Sometimes, decision making is an epistemic endeavor. The group wants to make the right decision, which raises the question which aggregation procedure is best to reach this goal. We present the Condorcet Jury Theorem which shows that majority voting is, given certain conditions, truth- conducive. Next, we apply these ideas to judgment aggregation and show that the Premise-Based Procedure does well epistemically (at least for the example we discuss).

7.5 Final Remarks: We discuss two open questions. First, voting procedures do not allow for an opinion-change of the group members in the light of what the other group members thing. Second, we have left out strategic considerations, which often play an important role.

# Chapter 7

# Week 7: Voting

## 7.1 Introduction (14:39)

Welcome to Lecture 7 of our Introduction to Mathematical Philosophy! In the previous lecture, I spoke about individual decision making. There we asked: which conditions should the preferences of an individual decision/maker satisfy? We identified four of these conditions – the so-called von Neumann and Morgenstern axioms – and showed that our preferences can be represented by an ordinal utility function if and only if these conditions hold.

In this lecture, we consider the decision-making of a group. We assume that the group members are individually rational, i.e we assume for example that their preferences are transitive and that the goal of the group is to make a fair decision. Such a fair decision clearly excludes that the preferences of the group are always the preferences of a single group member. There should not be a dictator.

We assume that each group member submits a linear preference ordering. Consider the case of a family that wants to go on vacation. After some deliberation, there are three destinations left:

(Slide 1)

There are three holiday destinations:

A: Aruba

B: Barbados

C: Cape Cod

Family member 1 prefers A to B to C.

(Slide 2)

- Family member 1: A $\succ$ B $\succ$ C

(See Figure 7.1.)

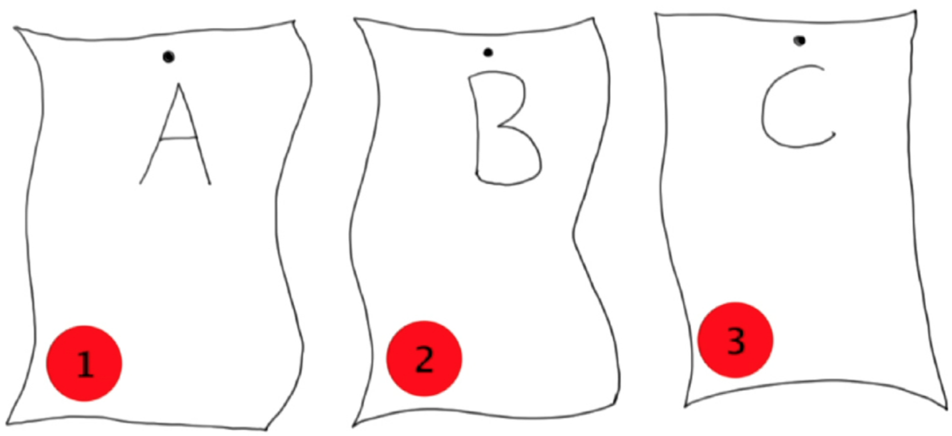


Figure 7.1: Family member 1: A $\succ$ B $\succ$ C

Family member 2 prefers B to C to A.

(Slide 3)

- Family member 2: B $\succ$ C $\succ$ A

(See Figure 7.2.)

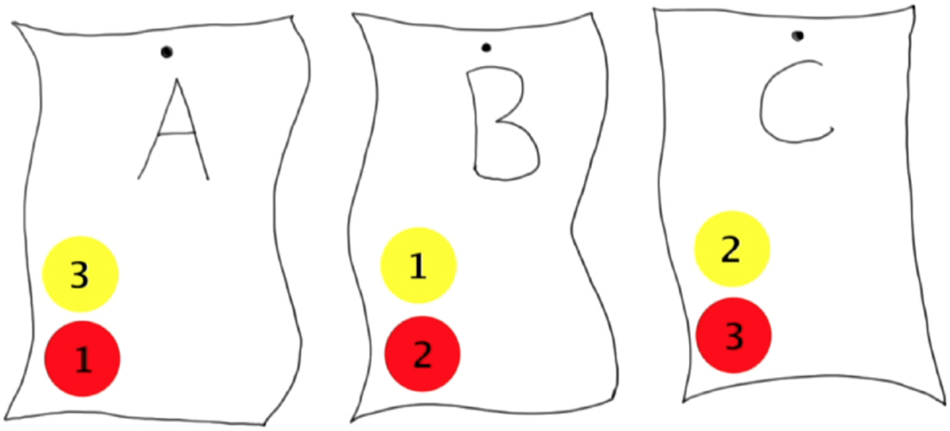


Figure 7.2: Family member 2: B $\succ$ C $\succ$ A

And family member 3 prefers C to A to B.

(Slide 4)

- Family member 3: C $\succ$ A $\succ$ B

(See Figure 7.3.)

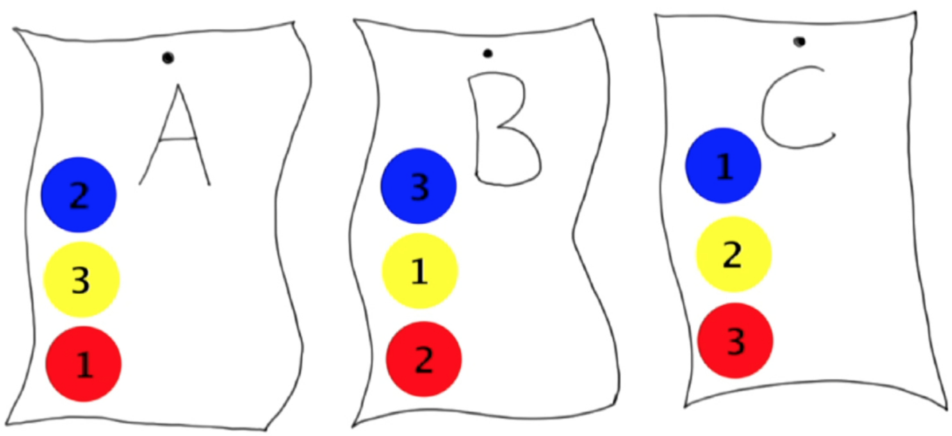


Figure 7.3: Family member 3: C $\succ$ A $\succ$ B

(Slide 5)

The individual rankings of the family members can be summarized in a table:

| no. 1 | no. 2 | no. 3 |
|---|---|---|
| A | B | C |
| B | C | A |
| C | A | B |

Question: What, then, is the will of the family?

Where shall they go on vacation? We will see that this depends, in general, on the aggregation procedure that is used.

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An aggregation procedure is a function that maps the individual preference orderings into a group (or social) ordering.

Here is one plausible procedure. We compare the different destinations pairwise.

(Slide 7)

**Method 1: Pairwise Comparisons ("the Condorcet Method")**

- Two family members prefer A to B. One family member prefers B to A. $\rightarrow$ The majority prefers A to B.

Similarly, we see:

(Slide 8)

- Two family members prefer B to C. One family member prefers C to B. $\rightarrow$ The majority prefers B to C.

Interestingly, there is also a majority that prefers C to A.

(Slide 9)

- Two family members prefer C to A. One family member prefers A to C. $\rightarrow$ The majority prefers C to A.

Hence, we arrive at the following preference ordering of the majority:

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- Hence, A $\succ_M$ B $\succ_M$ C $\succ_M$ A, where $\succ_M$ denotes the preference of the majority.

Here the subscript $M$ refers to the majority ordering.

You have probably noticed already that we arrived at a cycle, which implies that there is no preference ordering of the group.

(Slide 11)

- Note that there is a cycle and that transitivity is therefore violated (show this!). Hence, the proposed method does not yield a group ordering in this case.

We could ask, for instance, which is the highest ranked option, or which are the highest ranked options? It cannot be A, because C is ranked higher. It also cannot be B, because A is ranked higher. And it cannot be C, because B is ranked higher. Applying the majority rule to compare the different options pairwise leads, in this case, to a cycle, and therefore no group ordering exists if we apply this method.

The method just described goes back to the Marquis de Condorcet (1743-1794).

It is a very attractive method because it selects as a winner the option that wins against all other options in a pairwise comparison (unless there is a tie in which case there are two or more Condorcet winners). The second ranked option wins against all remaining options (unless there is a tie). And so on. Unfortunately this methods does not always yield a social ordering as the above example shows. Our example is an instance of the so-called Condorcet Paradox, which is one of many challenging paradoxes of voting theory.

The question, then, is what the will of the group is in the considered case. It seems quite natural to conclude that the three voting profiles, i.e. $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C}$, $\mathrm{B} \succ \mathrm{C} \succ \mathrm{A}$, and $\mathrm{C} \succ \mathrm{A} \succ \mathrm{B}$, suggest that the group is indifferent between the three options, i.e. that

$\mathrm{A} \sim \mathrm{B} \sim \mathrm{C}$.

This is the result that one obtains if one uses a scoring method to make a group decision. Let us assume that we want to rank $n$ alternatives and that each voter submits her ranking.

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**Method 2: The Borda Count**

- Assign 0 points for the lowest ranked option.
- Assign 1 point for the second-lowest option.
- ...
- Assign $n - 1$ points to the highest ranked option (where $n$ is the number of group members).

- Then add up the points of all group members.

This is the so-called Borda count, named after Jean-Charles de Borda (1733-1799) who was, besides the Marquis de Condorcet, one of the great voting theorists in the 18th century.

We apply the Borda count to our example.

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**Method 2: The Borda Count**

Return to our family-holiday example and calculate the Borda scores of options A, B and C:

- Score(A) $= 2 + 0 + 1 = 3$

(See Figure 7.4.)

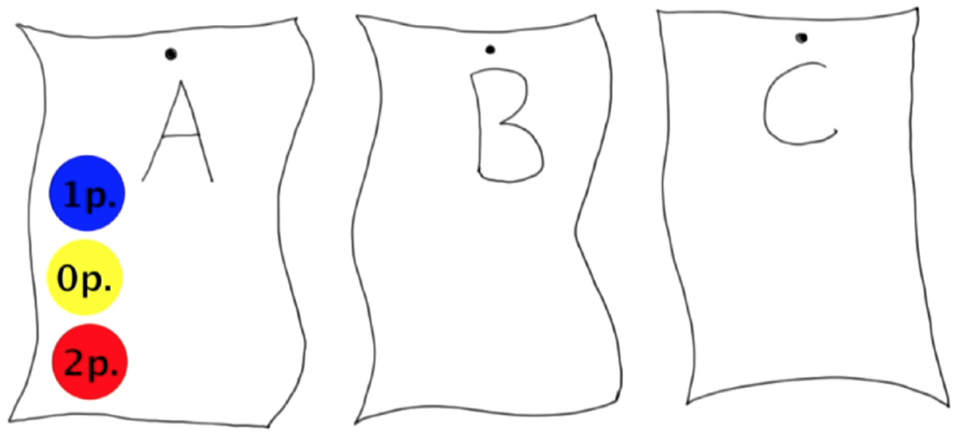


Figure 7.4: Score(A) = 2 + 0 + 1 = 3

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- Score(B) = 1 + 2 + 0 = 3

(See Figure 7.5.)

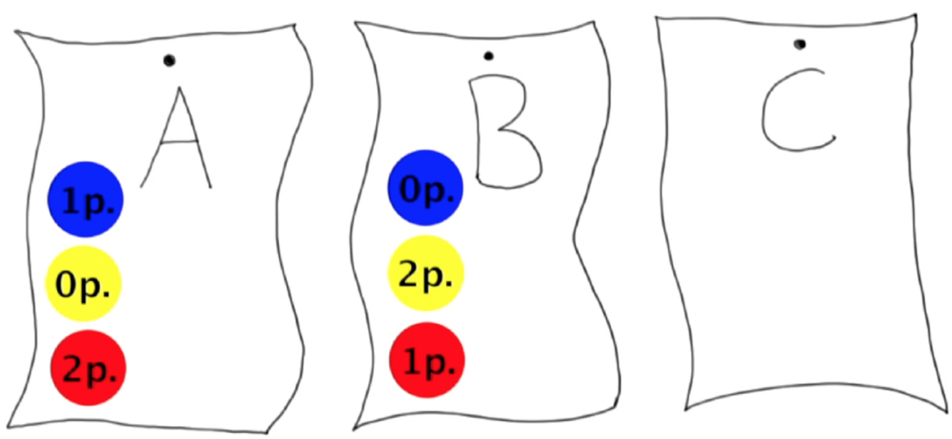


Figure 7.5: Score(B) = 1 + 2 + 0 = 3

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- Score(C) = 0 + 1 + 2 = 3

(See Figure 7.6.)

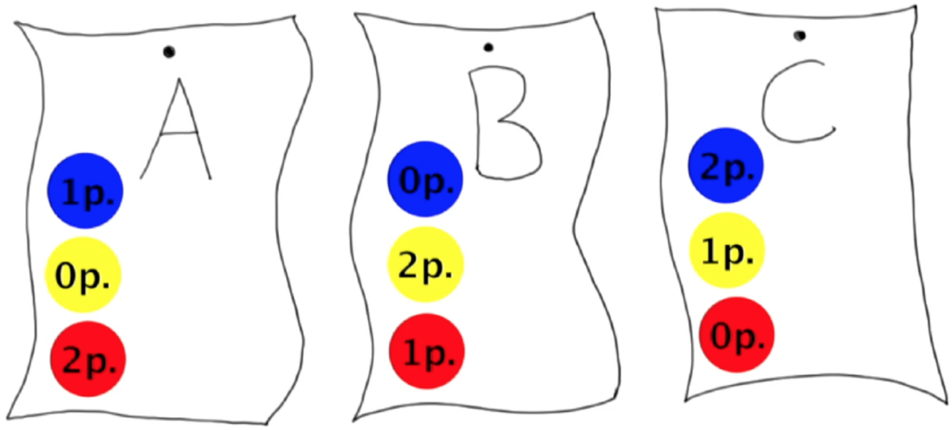


Figure 7.6: Score(C) = 0 + 1 + 2 = 3

Hence, all three options get the same score, and so the group is indifferent between them.

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Hence, A $\sim$ B $\sim$ C.

It is an altogether different question how the family should decide in this case. If all three options are equally well preferred by the family, it seems natural to apply a chance mechanism (such as throwing a dice) to select one option – at least if we assume that each family member prefers a vacation to no vacation even if she ranked the chosen destination last.

Where do we stand? We have introduced two methods of group decision-making based on the preference orderings submitted by the group members. The Condorcet method of pairwise comparisons and the ranking method associated with the name of Borda. Unlike the Borda count, the Condorcet method does not always yield a group order, but we may ask whether the orderings that result from both methods yield the same result in the cases where both methods do yield to a group ordering. The answer to this question is no, as the following example shows.

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**Condorcet and Borda may yield different results!**

Consider this example:

| 5 | 1 | 4 |
|---|---|---|
| A | A | B |
| B | C | C |
| C | B | A |

Here 5 group members rank A over B over C. 1 group member ranks A over C over B, and 4 group members rank B over C over A.

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| 5 | 1 | 4 |
|---|---|---|
| A | A | B |
| B | C | C |
| C | B | A |

**1. The Condorcet Method**

- A beats B 6:4.
- A beats C 6:4.
- B beats C 9:1.

A is the Condorcet winner, as A beats all other alternatives in a pairwise comparison. C is the Condorcet loser, as it loses against all other alternatives in a pairwise comparison.

→ The group ordering is $A \succ_M B \succ_M C$.

We see that A is the Condorcet winner (A beats B and C 6:4). We also see that B ranks second as it beats C 9:1. Hence, the Condorcet methods yields the group ordering A $\succ$ B $\succ$ C. The Borda count, however, leads to a different group ordering.

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| 5 | 1 | 4 |
|---|---|---|
| A | A | B |
| B | C | C |
| C | B | A |

**2. The Borda Count**

- Score(A) $= 5 \times 2 + 1 \times 2 = 12$
- Score(B) $= 5 \times 1 + 4 \times 2 = 13$
- Score(C) $= 1 \times 1 + 4 \times 1 = 5$

$\rightarrow$ The group ordering is B $\succ_B$ A $\succ_B$ C, where the subscript $B$ refers to the Borda count.

Note that in this example, the Condorcet winner is also the one who wins the majority:

(Slides 26-28)

| 5 | 1 | 4 |
|---|---|---|
| A | A | B |
| B | C | C |
| C | B | A |

**3. Majority Voting**

A is the majority winner as

- A is ranked first by six group members,
- B is ranked first by four group members,
- C is ranked first by no group member.

That is the majority ranking is A $\succ$ B $\succ$ C, i.e. the order that results from the Condorcet method. Interestingly, however, the majority rule sometimes selects the Condorcet loser, i.e. the option that does worst in pairwise comparisons.

Quiz 48:
Give an example for the case that the majority winner is the Condorcet loser.
Solution

Another procedure to arrive at a group ordering is to ask the group members to only submit those options that they approve, i.e. those that they consider to be above a certain threshold:

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| 5 | 1 | 4 |
|---|---|---|
| A | A | B |
| B | C | C |
| C | B | A |

**4. Approval Voting**

Each group member submits those options that she approves of. Let us assume that each candidate approves of her first two highest ranked options. Then

- A is approved by six group members,
- B is approved by nine group members,
- C is approved by five group members.

Hence, the ranking according to the 2-approval ranking agrees (in this example) with the Borda ranking, i.e. B $\succ$ A $\succ$ C.

Quiz 49:
Give an example for the case that the two-approval ranking (i.e. all voters approve of two options) is the same as the Condorcet ranking.
Solution

Let us take stock. We have introduced and discussed a number of voting rules. And there are many more. The picture that emerges is that for any group ordering you like, there is a prima facie plausible voting rule that yields that ordering, given the rankings of the individual group members. This is an unfortunate situation, and so the question is this: How shall we evaluate these different voting procedures or aggregation rules? Which one is the best?

To address this question, there are two principled ways to proceed. First, one can follow the methodology developed in the last lecture and come up with axioms that a "social welfare function" should satisfy – "social welfare function" is the expression that social choice theorists use for a function that maps the voters' linear preference orderings to a single social preference ordering. If a certain procedure satisfies these axioms, then it is acceptable, otherwise not. This is a very popular approach and it has triggered much research in social choice theory since the publication of Ken Arrow's book *Social Choice and Individual Values* in 1951. We will discuss the axiomatic approach in the next clip.

Second, we can ask which aggregation procedure is best given a certain goal. Consider a jury in the court room. Here each jury member has only one goal: she wants to make the right decision. However, for various reasons, our abilities to make the right decision are limited. And yet what we want is that the group of all jurors makes the best decision possible. We do not want that an innocent person is convicted, and we also do not want that a guilty person is not convicted. That is, we want to minimize the probability that the group makes a mistake. We ask: Is it possible that the group decision is better than the decision of an individual group member? And: which aggregation procedure maximizes the probability that the group makes the right decision? We will come back to these questions below.

## 7.2 Arrow (7:37)

Let us now consider the axiomatic approach to voting theory. Here are three axioms that were proposed in the literature.

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**Anonymity:** Changing the preference orderings of two group members does not affect the outcome of the aggregation.

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**Neutrality:** The names of the options do not matter, that is, if two options are exchanged in every individual ordering, then the outcome of the aggregation changes accordingly.

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**Positive Responsiveness:** If option A is tied for the win and moves up in one of the individual orderings, then option A is the unique winner.

May then proved in 1952 the following theorem:

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May's Theorem (1952): An aggregation method for choosing between two options satisfies (i) Anonymity, (ii) Neutrality and (iii) Positive Responsiveness if and only if it is the majority rule.

This is an interesting result as it shows that the majority rule is unique: It is the only aggregation function that satisfies these three reasonable conditions. At the same time, we know that the majority rule is not without problems. It may leave a large number of group members unhappy, and it may choose the Condorcet loser, as we have seen above.

Quiz 50:
There are two options, A and B. According to the (simple) majority rule, the option which gets most votes wins. According to the absolute majority rule, the option which gets more than 1/2 of the votes wins. If none of the options gets the absolute majority, then the group is indifferent between the two options. Construct an example which shows that the absolute majority rule violates Positive Responsiveness. Note that for two options A and B, a voter either prefers A to B, or she prefers B to A, or she is indifferent between A and B.
Solution

So far, we have assumed that there are only two distinct options and it turned out that there is an aggregation procedure that satisfies a number of reasonable requirements: majority voting. Does this also hold for more than two options? To address this question, we assume that there are at least three distinct options to choose from. We also assume that the number of group members is finite and greater than 0. With these two assumptions, we can now turn to Arrow's Impossibility Theorem.

Here are some more axioms.

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**Universal Domain:** No preference ordering over the options can be ignored by an aggregation method.

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**Pareto:** If all group members prefer option A to option B, then the group prefers option A to option B.

This axiom seems to be perfectly plausible. Why should the group rank B higher than A if all group members prefer A to B?

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**Independence of Irrelevant Alternatives:** The social ordering of two options A and B depends only on the relative orderings of A and B for each group member.

This condition seems plausible. Assume that there are two possible prizes – an expensive car and cheap car – that a group has to rank. Using a certain aggregation method, the group prefers (perhaps not too surprisingly!) the expensive car to the cheap car. Next, a third alternative – a goat – is introduced while leaving the relative individual orderings of the expensive car and the cheap car unaffected. Now, however, the group prefers the cheap car to the expensive car. This sounds strange! The additional option should be irrelevant for the relative ranking of the first two options. This is exactly what this condition demands.

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**Non-Dictatorship:** The group ordering cannot simply mimic the preferences of a single group member.

We are now in the position to state Arrow's famous Impossibility Theorem:

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Arrow's Impossibility Theorem: For a finite number of group members ($> 1$) and for at least three distinct options, there is no aggregation method that satisfies (i) Universal Domain, (ii) Pareto, (iii) Independence of Irrelevant Alternatives, and (iv) Non-Dictatorship.

To prove this remarkable theorem, Arrow showed that assuming the first three conditions, one has to accept that there is a dictator.

What are the implications of this theorem? First, let us examine how the Condorcet method and the Borda count relate to the theorem. Clearly, the Condorcet method violates the universal domain assumption as we have seen that there is not always a group ordering. The Borda count, on the other hand, violates the Independence of Irrelevant Alternatives condition as the following example shows.

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**Borda violates Independence of Irrelevant Alternatives.** Consider this ranking:

| 1 | 2 | 2 |
|---|---|---|
| A | A | B |
| B | C | C |
| C | B | A |

Here Score(A) $= 6$ and Score(B) $= 5$, hence A $\succ_B$ B.

Next, one option (D) is added, but the relative rankings of A, B and C are oft unchanged:

| 1 | 2 | 2 |
|---|---|---|
| D | A | B |
| A | C | C |
| B | B | D |
| C | D | A |

Now, Score(A) $= 8$ and Score(B) $= 9$, hence B $\succ_B$ A.

We see that without the additional option D, the group prefers A to B, as A has a higher Borda count than B. If option D is included, then the group prefers B to A, in violation of the Independence of Irrelevant Alternatives condition. The ordering of A and B does depend on the irrelevant alternative D. Hence, if we consider the Borda count to be the right aggregation method, than we have to reject the Independence of Irrelevant Alternatives condition. If you are interested in this, have a look at the writings of Donald Saari mentioned at the end of this lecture.

Second and more generally, if we want to arrive at a possibility result, i.e. at an aggregation method that is in accordance with a number of reasonable conditions, then we have to relax one of the conditions in Arrow's Theorem. There is a tremendous literature on this, which we cannot discuss in this lecture. I would like to mention, however, that besides the Independence of Irrelevant Alternatives condition, a natural condition to relax is the Universal Domain condition. Sen's Paradox of the Paretian Liberal suggests, at least to some authors, such as Robert Nozick, that individuals have preferences about options which should not be included in the social preference ordering.

Quiz 51:
Consider a decision situation with only two options A and B and a finite number of group members ($> 0$). Show that the (simple) majority rule satisfies all four axioms mentioned in Arrow's Theorem.
Solution

## 7.3 Judgement Aggregation (6:39)

Sometimes, groups have to make decisions involving logically interrelated propositions. Here is an example:

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**Judgment Aggregation**

- A city council has to make a decision on whether to build a new harbor site (= proposition C).
- It is consensus that this project should be approved of if and only if there is sufficient demand for new sites that cannot be met by existing sites (= proposition A) and
- the consequences for a nearby natural reserve are supportable (= proposition B).

The three relevant propositions are thus:

A: There is sufficient demand for a new site.

B: The site will not endanger the natural reserve.

C: The site should be built.

The council comprises 7 members, and they express their judgments as follows.

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**Judgment Aggregation**

A city council has to make a decision on whether to build a new harbor site (C). It is consensus that this project should be approved of if and only if there is sufficient demand for new sites that cannot be met by existing sites (A) and the consequences for a nearby natural reserve are supportable (B).

| Member | A | B | C |
|---|---|---|---|
| 1,2,3 | Yes | Yes | Yes |
| 4,5 | Yes | No | No |
| 6,7 | No | Yes | No |

Note that all members respect the logical connection that holds between the three propositions:

(Slide 45)

Note that all council members accept $(A \wedge B) \leftrightarrow C$. They are individually rational.

The question now is how the group should decide. It turns out that this is not at all clear.

(Slide 46)

**The Discursive Dilemma**

| Member | A | B | C |
|---|---|---|---|
| 1,2,3 | Yes | Yes | Yes |
| 4,5 | Yes | No | No |
| 6,7 | No | Yes | No |
| Majority | Yes | Yes | No |

Note that there is a majority for A and a majority for B, but no majority for C, in contradiction with the condition just stated. The majority judgments on A, B and C violate the condition

$(A \wedge B) \leftrightarrow C$,

and hence proposition-wise majority voting leads in this case to a contradiction.

What can be done to solve this problem? Here are two ways out of this so-called discursive dilemma. The first way out is the Premise-Based Procedure, and it goes as follows:

One applies the method of proposition-wise majority voting to all premises, and then infers the group judgment on the conclusion by using the rule

$(A \wedge B) \leftrightarrow C$.

This procedure rests on the assumption that the reasons for a decision matter. The council should support the reasons for the decision. Once we are clear about them, then the judgment on the conclusion follows directly from the judgments on the premises. In this case, the group decides that the harbor site will be built.

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**Way Out 1: The Premise-Based Procedure**

| Member | A | B | C |
|---|---|---|---|
| 1,2,3 | Yes | Yes | Yes |
| 4,5 | Yes | No | No |
| 6,7 | No | Yes | No |
| Majority | **Yes** | **Yes** | **Yes** |

The group verdict on C is inferred from the group verdicts on A (= Yes) and B (= Yes) using $(A \wedge B) \leftrightarrow C$.

Upshot: The harbor site will be built.

The second way out of the discursive dilemma is the Conclusion-Based Procedure. The idea here is that the council should support the decision as to whether the harbor site is built or not. Applying this procedure, the council decides that the harbor site will not be built.

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**Way Out 2: The Conclusion-Based Procedure**

| Member | A | B | C |
|---|---|---|---|
| 1,2,3 | Yes | Yes | **Yes** |
| 4,5 | Yes | No | **No** |
| 6,7 | No | Yes | **No** |
| Majority | – | – | **No** |

Upshot: The harbor site will not be built.

We sees that both procedures make opposing recommendations. Which decision is taken depends on the chosen aggregation method. And so we are in a similar situation as in the case of preference aggregation discussed in the last two clips. Note that both procedures – the Premise-Based Procedure and the Conclusion-Based Procedure – are not ideal. The Conclusion-Based Procedure neglects the reasons for the decision, and the Premise-Based Procedure can lead to a group decision that almost no one supports. Both procedures have the disadvantage that they throw away what seems to be relevant information: The voting margin does not matter in either of them although it is plausible that it should. A 9 to 0 vote seems more convincing than a 5 to 4 vote...

But perhaps there is a better procedure? A procedure that satisfies a number of reasonable axioms, perhaps inspired by the axioms discussed before. It turns out that no such method exists. In fact, the theory of Judgment Aggregation can be developed in analogy to the theory of preference aggregation, and so we do again encounter various impossibility theorems.

In the next clip, we will discuss an alternative to the axiomatic approach to preference and judgment aggregation by bringing epistemological considerations to the fore. We will ask which aggregation procedure performs best when the goal is to make the right decision. The idea is this: There might be an aggregation procedure that does not satisfy certain desirable axioms. And yet, the procedure might well be a good guide to the truth. It might maximize the probability of making the right decision.

Quiz 52:
Construct a discursive dilemma for seven voters with the condition $(A \vee B) \leftrightarrow C$ where at least three of the four possible situations get at least one vote. One way to do this is to start from the example at the beginning of this clip and try to adapt it recalling de Morgan's Laws.
Solution

## 7.4 Epistemic Considerations (10:51)

At least sometimes, groups have to make decisions about facts of the matter. For example, whether we should reduce the emission of greenhouse gases is not merely a matter of preference. Most people would be in favor of doing so because they believe it is the right strategy to solve an important and urgent environmental problem. Whether the reduction of greenhouse gases will reduce global warming is a factual question. And so, at least sometimes, as a group we want to make the right decision not (only?) in the sense that it conforms with what people like, but also in the sense that it captures the truth. In any case, this is the group decision situation that we will consider now.

The starting point of this strand of research is the Condorcet Jury Theorem, which goes back to the Marquis de Condorcet whom we already encountered in this lecture. Here a group of $n$ voters has to make a yes-no decision on a proposition H. The theorem makes two assumptions:

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**The Condorcet Jury Theorem**

Consider a group of $n$ voters has to make a yes-no decision on a proposition H. We make two assumptions:

1. **The Independence Assumption:** Given the truth or falsity of H, the verdict of one voter does not depend on the verdict of any other voter.

2. **The Reliability Assumption:** Each voter has a certain reliability $r := P(\text{Vote}_\text{Y}|\text{Y}) = P(\text{Vote}_\text{N}|\text{N}) > .5$ to make the right decision.

Then: The probability that the majority makes the right decision (i) increases monotonically and (ii) goes to 1 as $n \to \infty$.

As an illustration, consider this picture:

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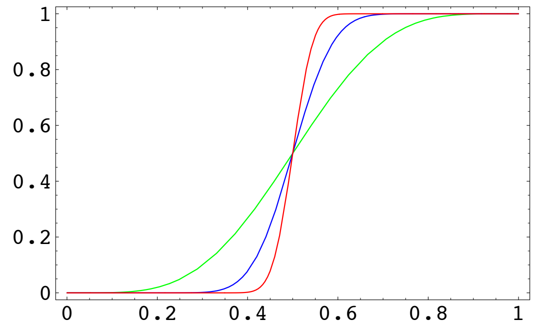
**The Condorcet Jury Theorem**

We plot the probability that voting tracks the truth as a function of the reliability $r$ for 9 (in green), 49 (in blue), and 199 (in red) voters.

It shows the probability that voting tracks the truth as a function of the reliability $r$ for 9 voters (in green), 49 voters (in blue), and 199 voters (in red).




Hence, as long as the group is large enough and as long as the individual group members are sufficiently reliable, the group will track the truth if majority voting is applied. It actually turns out to be enough that the average reliability of all voters is greater than .5. Hence, voting is (given the above conditions) truth-conducive, and we have a nice argument in support of the usual democratic decision-making process. If only enough of us fallible voters vote, the group as a whole will make the right decision.

The conclusion of the Condorcet Jury Theorem is of course only as strong as the assumptions it makes. You might want to question, for example, the independence assumption: After all, we are all reading the same newspapers, we talk to each other etc. So how can our judgments be independent? Besides, it is not clear that the Reliability Assumption holds. Are we, perhaps, too optimistic about our judgmental capacities? Putting these worries aside, the Condorcet Jury Theorem is an important result.

I will now show that the modeling ideas that underly the Condorcet Jury Theorem can be extended to the case of Judgment Aggregation.

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**Setting the Stage**

- A council has $n$ members with a reliability $r$ to make the right judgment on propositions A and B.
- The judgments on A and B are probabilistically independent.
- All members accept (A $\wedge$ B) $\leftrightarrow$ C.
- Simplifying notation: We represent a Yes-vote by 1 and a No-vote by 0.
- There are four admissible situations for (A, B, C):

$$S_1 = (1,1,1) \quad , \quad S_2 = (1,0,0)$$
$$S_3 = (0,1,0) \quad , \quad S_4 = (0,0,0)$$

- Note that only these four situations satisfy (A $\wedge$ B) $\leftrightarrow$ C.
- One of these four situations is the true one. We just do not know which.

However,

(Slide 60)

**Reliability**

There is a certain reliability $r$ that a council member makes the right decision. (Note: we assume that all council members have the same reliability.) Hence, the following holds:

- If $S_1$ is the true situation, then a council member will vote Yes on proposition A with probability $r$, and Yes on proposition B with probability $r$.
- If $S_2$ is the true situation, then a council member will vote Yes on proposition A with probability $r$, and No on proposition B with probability $r$.
- Etc.

Next, we assume that there is a prior probability $q_1$ that $S_1$ is the true situation, a prior probability $q_2$ that $S_2$ is the true situation, a prior probability $q_3$ that $S_3$ is the true situation, and a prior probability $q_4$ that $S_4$ is the true situation.

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**Prior Probabilities**

- The prior probability that $S_1$ is the true situation is $q_1$.
- The prior probability that $S_2$ is the true situation is $q_2$.
- The prior probability that $S_3$ is the true situation is $q_3$.
- The prior probability that $S_4$ is the true situation is $q_4$.

Constraint: $q_1 + q_2 + q_3 + q_4 = 1$

If we do not know anything, we apply the Principle of Indifference and assign a prior probability of $1/4$ to each situation:

(Slide 63)

Principle of Indifference: $q_1 = q_2 = q_3 = q_4 = 1/4$

With these assumptions, we can start running computer simulations:

(Slide 69)

**Running Simulations**

- Fix the number of council members (i.e. $n$), the reliability parameter (i.e. $r$), and the priors (i.e. $q_1, \ldots, q_4$).
- First assume that situation 1 is the true one. Then each council member picks a situation (given her reliability $r$) by a chance process as described above.
- Use the Premise-Based Procedure, the Conclusion-Based Procedure and the Bayesian Procedure (see below) to aggregate the individual verdicts.
- Do this many times and count how often situation 1 results from the respective aggregation procedure. Works out the corresponding fraction.
- Do the same for situations 2, 3 and 4.
- Weigh the four fractions with the prior probabilities and obtain the wanted probability.
- Compare the three procedures.

Interestingly, it turns out that for a wide range of parameters the Premise-Based Procedure outperforms the Conclusion-Based Procedure. What is more, the Premise-Based Procedure is also very close to the result that the optimal Bayesian aggregation method yields.

(Slide 70)

**Two Main Results**

1. The Premise-Based Procedure performs much better than the Conclusion-Based Procedure.
2. The Premise-Based Procedure performs almost as good as the (optimal) Bayesian Procedure.

The Bayesian Procedure works as follows:

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**The Bayesian Procedure**

This is the optimal aggregation procedure.

- Start with the priors $q_1, \ldots, q_4$.
- Take the judgments of the (partially reliable) council members as evidence.
- Update the priors using Bayes' Theorem taking the evidence into account.
- Select the situation with the highest posterior probability.

Note that this method takes all available information into account (including the voting margins).

Given that the Premise-Based Procedure disregards the voting margin, it is surprising that it performs so well. And so we have good reasons to use the Premise-Based Procedure, which is a very simple aggregation rule indeed.

For details about this study, see
Hartmann, S. and Sprenger, J., "Judgment Aggregation and the Problem of Tracking the Truth", *Synthese* 187: 209-221 (2012).
The paper can be downloaded at
http://link.springer.com/article/10.1007/s11229-011-0031-5

Quiz 53:
A voter has a rather low reliability of $P(VoteY|Y) = P(VoteN|N) = 0.1$. She votes "$N$". From this you conclude that you can be pretty confident that $Y$ is the right answer. Is this a correct way of reasoning?
Solution

## 7.5 Final Remarks (2:43)

While we have covered many topics in this lecture, we have also skipped several important issues. In closing, I would like to shortly discuss two of them.

First, we have assumed that the group members simply submit their verdicts which are then aggregated in a certain way. Typically, however, people talk to each other, especially in smaller groups. They try to convince each other, they learn from each other, and they might, if all goes well, end up with a consensual decision that all group members support. Clearly, such situations are much harder to model. But this is how the world is like, and so it would be interesting to explore, for example, whether decision-making by deliberation also has the epistemic virtues that voting has. Is there something like a Condorcet Jury Theorem for deliberation? And if so, which of the two procedures is the better truth-tracker, straight forward voting or deliberation? One hypothesis is that deliberation is better in inhomogeneous groups, i.e. in groups where different group members have different reliabilities. They can then learn from each other, and the hope would be that the less reliable group members follow the more reliable ones.

Second, we disregarded any strategic considerations. We assumed, for example, that the voters sincerely report their preferences. However, if I really want a certain option (and do not care too much about my second and third choice), then I might be tempted to report an ordering that differs from my real ordering. Clearly, these maneuvers cannot be prevented, but it is an interesting question to ask which aggregation procedures are especially susceptible for these considerations.

So there is much more to do in this exciting and practically important research field, where mathematical philosophers join economists, social scientists and even computer scientists.

The following entry from the Stanford Encyclopedia of Philosophy provides an excellent survey of various voting methods:
http://plato.stanford.edu/entries/voting-methods/
Here is a great book-length introduction to social choice theory:
Gaertner, W., 2006, *A Primer in Social Choice Theory*, Oxford: Oxford University Press.
The following books, mentioned already in the last lecture, also have relevant introductory chapters:
Peterson, M., *An Introduction to Decision Theory*, Cambridge: Cambridge University Press, 2009.
Resnik, M., Choices: *An Introduction to Decision Theory*, Minneapolis: University of Minnesota Press, 1987.
Here are some more advanced texts that you will enjoy reading:
Arrow, K., *Social Choice and Individual Values*, 2nd edition, New Haven: Yale University Press, 1963. (A classic text, written by one of the founders of modern social choice theory.)
Brams, S., *Mathematics and Democracy*, Princeton: Princeton University Press, 2008.
List, C. and Pettit, P., Group Agency: *The Possibility, Design, and Status of Corporate Agents*, Oxford: Oxford University Press 2011.
Luce, R.D. and Raiffa, H., *Games and Decisions: Introduction and Critical Survey*, Dover Publications, 1989. (This is a classic textbook. It first appeared in 1957 and is still worth reading.)
Nurmi, H., *Comparing Voting Systems*, Dordrecht: D. Reidel, 1987.
Nurmi, H., *Voting Paradoxes and How to Deal with Them*, Berlin: Springer, 1999.
Regenwetter, M., Grofman, B., Marley, A., and Tsetlin, I., *Behavioral Social Choice: Probabilistic Models, Statistical Inference, and Applications*, Cambridge: Cambridge University Press, 2006.
Saari, D., *Basic Geometry of Voting*, Berlin: Springer, 1995.
Saari, D., *Decisions and Elections: Explaining the Unexpected*, Cambridge: Cambridge University Press, 2001.
Saari, D., *Disposing Dictators, Demystifying Voting Paradoxes: Social Choice Analysis*, Cambridge: Cambridge University Press, 2008.
Sen, A., *Collective Choice and Social Welfare*, San Francisco: Holden Day, Inc., 1970. (This is another classic text.)
Here are some articles:
List, C., "The Discursive Dilemma and Public Reason", *Ethics*, 116(2): 362?402, 2006.
List, C. and R. Goodin, "Epistemic Democracy: Generalizing the Condorcet Jury Theorem", *Journal of Political Philosophy*, 9(3): 277?306, 2001.
Saari, D., "A Dictionary of Voting Paradoxes", *Journal of Economic Theory*, 48(2): 443?475, 1989.
You might also be interested in listening to a lecture Donald Saari gave:
http://www.youtube.com/watch?v=oUcXIeuGYs8

# Appendix A

# Quiz Solutions Week 7: Voting

Quiz 48:

Give an example for the case that the majority winner is the Condorcet loser.

SOLUTION Quiz 48:

Three group members: A $\succ$ B $\succ$ C, two group members: B $\succ$ C $\succ$ A, and two group members: C $\succ$ B $\succ$ A. Then, A wins the majority; but in pairwise comparison, A loses against B as well as against C 3:4. Hence, A is the Condorcet loser.

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Quiz 49:

Give an example for the case that the two-approval ranking (i.e. all voters approve of two options) is the same as the Condorcet ranking.

SOLUTION Quiz 49:

Again, three group members: A $\succ$ B $\succ$ C, two group members: B $\succ$ C $\succ$ A, and two group members: C $\succ$ B $\succ$ A. Then, the Condorcet ranking is B $\succ$ C $\succ$ A. A is approved of by three group members, B is approved of by seven group members, and C is approved of by four group members. Hence, the two-approval ranking is also B $\succ$ C $\succ$ A.

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Quiz 50:

There are two options, A and B. According to the (simple) majority rule, the option which gets most votes wins. According to the absolute majority rule, the option which gets more than 1/2 of the votes wins. If none of the options gets the absolute majority, then the group is indifferent between the two options. Construct an example which shows that the absolute majority rule violates Positive Responsiveness. Note that for two options A and B, a voter either prefers A to B, or she prefers B to A, or she is indifferent between A and B.

SOLUTION Quiz 50:

Let us assume that three voter prefers A to B, three voters prefers B to A, and four voters are indifferent between A and B. According to the (simple) majority rule, the group is indifferent between A and B as this option gets most votes. Let us now assume that one of the four indifferent voters changes her mind. She now prefers A to B. Then four voters prefer A to B, three voter prefer B to A, and three voters are indifferent between A and B. According to the (simple) majority rule, the group now prefers A to B which is in accordance with Positive Responsiveness. However, if we consider the absolute majority rule, then we see that there is no absolute majority for A (as 4 is smaller than $10/2 = 5$). Hence, according to the absolute majority rule, the group remains indifferent between A and B (in violation of Positive Responsiveness).

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Quiz 51:

Consider a decision situation with only two options A and B and a finite number of group members ($> 0$). Show that the (simple) majority rule satisfies all four axioms mentioned in Arrow's Theorem.

SOLUTION Quiz 51:

Universal Domain: None of the three possible group orderings – A $\succ$ B, B $\succ$ A, and A $\sim$ B is excluded by the majority rule.

Pareto: If all group members prefer A to B, then clearly the group prefers A to B according to the (simple) majority rule (which is exactly what Pareto demands).

Independence of Irrelevant Alternatives: As there are only two options, this axiom is trivially satisfied.

Non-Dictatorship: The group decides for the option that gets the most votes, which is not automatically the option that one group member prefers.

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Quiz 52:

Construct a discursive dilemma for seven voters with the condition (A $\vee$ B) $\leftrightarrow$ C where at least three of the four possible situations get at least one vote. One way to do this is to start from the example at the beginning of this clip and try to adapt it recalling de Morgan's Laws.

SOLUTION Quiz 52:

The four situations are S1' := (1,1,1), S2' := (1,0,1), S3' := (0,1,1), and S4' := (0,0,0). Recalling de Morgan's Laws, we see that (A $\wedge$ B) $\leftrightarrow$ C is logically equivalent with ($\neg$ A $\vee$ $\neg$ B) $\leftrightarrow$ $\neg C$. Hence, we can modify the example given in the clip and obtain that 0 x S1', 2 x S2', 2 x S3', and 3 x S4' leads to a discursive dilemma. To confirm this, note that then, according to the Premise-Based Procedure, A = 0, B = 0, hence (because (A $\vee$ B) $\leftrightarrow$ C) $C = 0$. According to the Conclusion-Based Procedure, we obtain C = 1. Hence, the two procedures lead to different group judgments on C.

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Quiz 53:

A voter has a rather low reliability of $P(VoteY|Y) = P(VoteN|N) = 0.1$. She votes "$N$". From this you conclude that you can be pretty confident that $Y$ is the right answer. Is this a correct way of reasoning?

SOLUTION Quiz 53:

We know that $P(VoteY|Y) = P(VoteN|N) = .1$. Hence, $P(VoteN|Y) = P(VoteY|N) = .9$. (We assume here that the voter has to vote.) That is, a voter with a low reliability is much more likely to vote $Y$ if $N$ is true and $N$ if $Y$ is true. What we are interested in, however, is $P(Y|VoteN)$. To calculate it, we apply Bayes Rule and obtain:

$P(Y|VoteN) = \frac{P(VoteN|Y)P(Y)}{(P(VoteN|Y)P(Y)+P(VoteN|N)P(N))} = \frac{.9P(Y)}{(.9P(Y)+.1(1-P(Y)))} = \frac{9P(Y)}{(9P(Y)+1-P(Y))} = frac9P(Y)(1+8P(Y))$.

To evaluate this expression, we plug in few numbers:

$P(Y) = 0 : P(Y|VoteN) = 0$ $P(Y) = 1/8 : P(Y|VoteN) = 9/16 \approx 0.56$ $P(Y) = 1/4 : P(Y|VoteN) = 3/4 = 0.75$ $P(Y) = 1/2 : P(Y|VoteN) = 0.9$ $P(Y) = 3/4 : P(Y|VoteN) = 27/28 \approx 0.96$ $P(Y) = 1 : P(Y|VoteN) = 1$

We see that we get the (perhaps) putative value $P(Y|VoteN) = 0.9$ only for $P(Y) = 1/2$, i.e. if we have no idea what the voter will vote. For lower base-rates $P(Y)$, $P(Y|VoteN)$ is lower than 0.9. In general, the result depends on the base-rate $P(Y)$, which we encountered already in Lecture 5. Neglecting it means to commit the base-rate fallacy.

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# List of Figures