Introduction to Mathematical Philosophy

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Week 8: Overview

Overview of Lecture 8: Quantum Logic

8.1 Introduction: Quantum mechanics is a peculiar scientific theory which comes with a number of paradoxes such as the EPR Paradox and Schr odingers cat. We stress that many of the philosophical conclusions of quantum mechanics are relative to an interpretation, such as the Copenhagen Interpretation of quantum mechanics. Finally, we introduce the concept of a *random variable*.

8.2 Probabilities and Correlations: We introduce the *expectation* of a random variable and the *covariance* of two random variables and show how they can be calculated from a joint probability distribution. Next, we consider the reverse problem: How can we determine the underlying probability distribution from the expectations and covariances which can be extracted from the data?

8.3 Quantum Correlations: We show that quantum mechanics yields correlations that are so strong (or so strange!) that they cannot be obtained from a joint probability distribution. The measured correlations, that can be calculated from quantum mechanics, violate the Clauser-Horne-Shimony-Holt inequality.

8.4 Quantum Logic: We consider the logical structure of propositions reporting quantum measurement outcomes and show that distributivity of *and* ($\wedge$) over *or* ($\vee$) is violated. We discuss the implications of this astonishing result and sketch the program of quantum logic.

8.5 Final Remarks: We discuss the implications of the results of the previous two clips. Logic and probability are intimately connected, and it might be an empirical question whether classical logic and probability theory hold in our world.

Chapter 8

Week 8: Quantum Logic

8.1 Introduction (6:24)

Welcome to Lecture 8 of our Introduction to Mathematical Philosophy! In the last two weeks we have looked at individual and collective decision making. We have tried to find conditions that rational sets of preferences should satisfy, we have formulated conditions that aggregation functions should satisfy, and we have provided an epistemic analysis of various aggregation procedures. These are topics close to the heard of the mathematical philosopher, but they are also topics that social scientists address. Clearly, political scientists are interested in finding the best aggregation procedure, and psychologists want to understand how real people or groups make decisions.

In this lecture we will address philosophical problems related to another science – physics. As we will see, modern physics has something to say about the role and status of probability and perhaps even logic. For example, it turns out that some measured correlations between variables cannot be accounted for by a standard probability distribution, and the algebra of propositions about quantum systems does not seem to be the same as in classical physics or in ordinary life. We will explain in detail what this means below. Interestingly, these two observations are related to each other.

As you can imagine, quantum mechanics is a complicated scientific theory. It is a fundamental theory that accounts for the behavior of electrons, atoms, and many other subatomic particle. It even has implications for macroscopic objects, and it has many strange and counter-intuitive features. You might have heard about the double-slit experiment, the Einstein-Podolski-Rosen Paradox or of Schrödinger's Cat. Quantum mechanics is also mathematically quite demanding, and we cannot go into many technical details here. Fortunately, however, some of the philosophically most troubling aspects of quantum theory can be laid out with the mathematics we encountered so far in this course (plus a little more which I will introduce). I will recommend some more advanced readings at the end for the experts.

Quantum mechanics is not only mathematically quite hard, it is also a conceptually very difficult theory. It is not immediately clear what the formalism means and what it says about the world. There are various interpretations that differ radically in what they say.

(Slide 1)

There are several competing interpretations of quantum mechanics on the market. These interpretations try to make sense out of the theory and explain what the theory tells us about the world.

All interpretations of quantum mechanics are of course compatible with the empirical content of the theory. They do, for example, not make different predictions. But they disagree in the way they make sense of the theory. Interestingly, many philosophical questions that quantum mechanics raises cannot be answered without choosing an interpretation. Questions about determinism are a case in point. Does quantum mechanics really suggest that we live in an indeterministic world?

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Here are two well-known interpretations:

- The Copenhagen Interpretation (defenders: N. Bohr and W. Heisenberg)
- Bohmian Mechanics (defenders: D. Dürr, S. Goldstein and N. Zanghi. Note, however, that the defenders of Bohmian Mechanics do not consider it as an interpretation, but as a theory.)

Yes!, answer the defenders of the Copenhagen Interpretation such as the famous physicists Niels Bohr and Werner Heisenberg, and NO! say the defenders of Bohmian Mechanics such as our colleague Detlef Dürr here at LMU Munich.

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Different interpretations give different answers to various philosophical questions, such as the problem of determinism and the interpretation of probability in quantum mechanics.

We find this interpretation-dependence also when it comes to say what the probabilities that we can calculate with the formalism of quantum mechanics mean. Are they objective, or do they only reflect what a rational agent should believe about a quantum system? The answer depends on the chosen interpretation.

Before we start, let me briefly introduce the concept of a random variable.

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A random variable is a variable that can take on a set of possible values, each with a specific probability.

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Examples:

- the weight of a randomly picked person (this is a **continuous variable** with values in the positive real numbers (on a given scale))
- the spin of an electron (this is a **binary variable** with values $\pm 1/2$)

While weight is represented by a continuous variable, spin is represented by a binary variable, which allows only two values: Given a certain direction along which the spin is measured, there are only two possible outcomes, $+1$ and -1.

In the next clip, we will talk a bit more about probabilities and correlations which will help us to understand why quantum mechanics is so special.

Quiz 54:
More precisely put, a random variable is a function from a sample space S to the real numbers. Let the sample space be the set S = {high, medium, low} and let there be three random variables A, B and C with A(high) = 5, A(medium) = 3 and A(low) = −1, B(high) = 4, B(medium) = 0 and B(low) = −3, C(high) = 3, C(medium) = −1 and C(low) = −5. Which of the following relations is true?
(a) A > B (b) B > C (c) A > C (d) AB > C (e) AC > B (f) BC > A
Solution

8.2 Probabilities and Correlations (11:45)

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Probabilities and Correlations

Let us consider two binary random variables A and B with the values: $a_1, b_1 = +1$ and $a_2, b_2 = -1$.

There is a joint probability distribution P defined over these variables.

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To specify the joint probability distribution P, we have to fix the values of

$$\begin{aligned} P(A=1,B=1),\\ P(A=1,B=-1),\\ P(A=-1,B=1),\\ P(A=-1,B=-1). \end{aligned}$$

which we sometimes abbreviate as follows:

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$$\begin{aligned} P(A=1,B=1) &=: P(1,1),\\ P(A=1,B=-1) &=: P(1,-1),\\ P(A=-1,B=1) &=: P(-1,1),\\ P(A=-1,B=-1) &=: P(-1,-1). \end{aligned}$$

(Note here that as in previous lectures, $P(a,b)$ stands for $P(a \wedge b)$.)

As all four sum to 1, only three of them have to be fixed:

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- $P(1,1) + P(1,-1) + P(-1,1) + P(-1,-1) = 1$

We can now calculate the expectation of A, which is defined as follows:

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The expectation of the random variable A:

$$\begin{aligned} E(A) &= \sum_{i,j=1}^{2} a_i \cdot P(a_i,b_j)\\ &= 1 \cdot P(1,1) + 1 \cdot P(1,-1) + (-1) \cdot P(-1,1)\\ &\quad + (-1) \cdot P(-1,-1)\\ &= P(1,1) + P(1,-1) - P(-1,1) - P(-1,-1) \end{aligned}$$

Similarly, we can calculate the expectation of B:

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The expectation of the random variable B:

$$\begin{aligned}
E(B) &= \sum_{i,j=1}^{2} b_j \cdot P(a_i, b_j) \\
&= 1 \cdot P(1,1) + (-1) \cdot P(1,-1) + 1 \cdot P(-1,1) \\
&\quad + (-1) \cdot P(-1,-1) \\
&= P(1,1) - P(1,-1) + P(-1,1) - P(-1,-1)
\end{aligned}$$

and the expectation value of the product AB:

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The expectation of the random variable AB:

$$\begin{aligned}
E(AB) &= \sum_{i,j=1}^{2} a_i \cdot b_j \cdot P(a_i, b_j) \\
&= 1 \cdot P(1,1) + (-1) \cdot P(1,-1) + (-1) \cdot P(-1,1) \\
&\quad + (-1) \cdot (-1) \cdot P(-1,-1) \\
&= P(1,1) - P(1,-1) - P(-1,1) + P(-1,-1)
\end{aligned}$$

The expectation is the value of the variable in question that one would expect to obtain if one could repeat the data generating process an infinite number of times and then take the average of all the values that one obtained.

Next, we can define the variance. For a random variable A, it is defined as follows:

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The variance of the random variable A:

$$V(A) = E((A - E(A))^2)$$

which can be shown to be

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The variance of the random variable A:

$$\begin{aligned}
V(A) &= E((A - E(A))^2) \\
&= E(A^2) - (E(A))^2
\end{aligned}$$

Finally, we define the covariance of A and B:

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The covariance of the random variables A and B:

$$cov(A, B) = E\left((A - E(A)) \cdot (B - E(B))\right)$$

which can be shown to be

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The covariance of the random variables A and B:

$$\begin{aligned} cov(A, B) &= E\left((A - E(A)) \cdot (B - E(B))\right) \\ &= E(AB) - E(A) \cdot E(B) \end{aligned}$$

The covariance measures how much the two variables are correlated. If A and B are probabilistically independent, then the covariance vanishes. Here is how one shows this:

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If A and B are probabilistically independent, then $P(a_i, b_j) = P(a_i) \cdot P(b_j)$ (remember Lecture 5!).

Hence,

$$\begin{aligned} E(AB) &= \sum_{i,j=1}^{2} a_i \cdot b_j \cdot P(a_i)\, P(b_j) \\ E(AB) &= \sum_{i,j=1}^{2} a_i\, P(a_i) \cdot b_j\, P(b_j) \\ &= (\sum_{i=1}^{2} a_i P(a_i)) \cdot (\sum_{j=1}^{2} b_j P(b_j)) \\ &= E(A) \cdot E(B). \end{aligned}$$

Hence, $cov(A, B) = 0$.

Let us look at a specific example.

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Example

$$\begin{aligned} P(1,1) = P(-1,-1) &= 0.4 \\ P(1,-1) = P(-1,1) &= 0.1 \end{aligned}$$

(See Figure 8.1.)

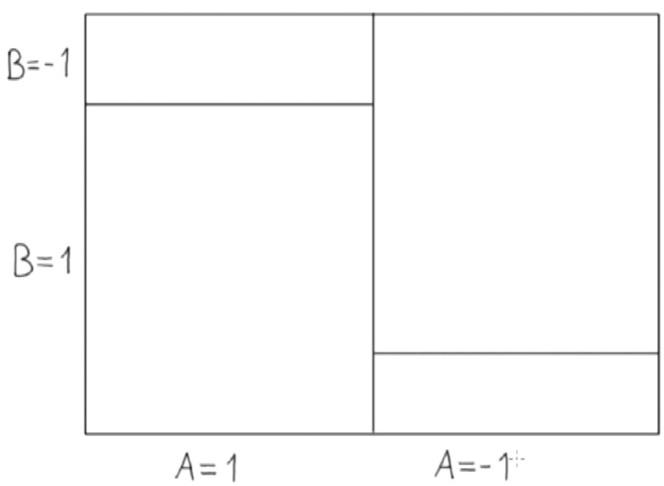

Figure 8.1: Joint probability distribution

From this we see that

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$$P(A=1) = P(1,1) + P(1,-1) = 0.5$$

(See Figure 8.2.)

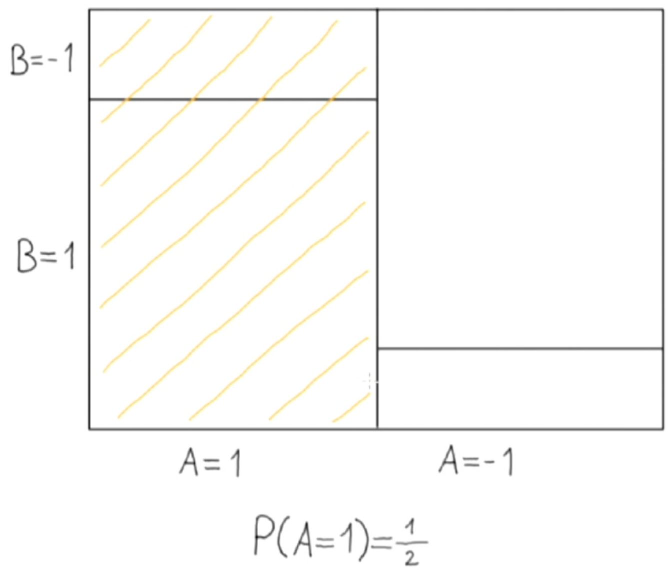

Figure 8.2: $P(A = 1) = P(1, 1) + P(1, -1) = 0.5$

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$$P(B = 1) = P(1, 1) + P(-1, 1) \quad = \quad 0.5$$

(See Figure 8.3.)

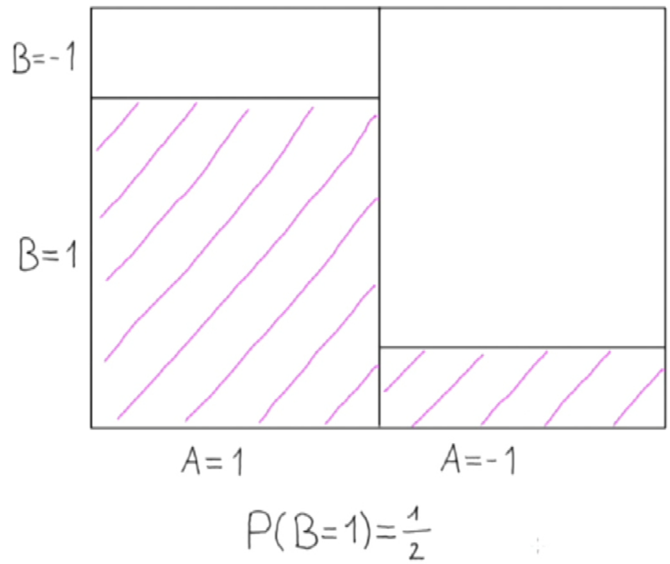

Figure 8.3: $P(B=1) = P(1,1) + P(-1,1) = 0.5$

Accordingly,

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$$P(A=-1) = P(B=-1) = 0.5$$

Quiz 55: Think for yourself about how to represent this graphically.

For the expectation, we then obtain

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$$\begin{aligned} E(A) &= 1 \cdot 0.5 + (-1) \cdot 0.5 = 0 \\ E(B) &= 1 \cdot 0.5 + (-1) \cdot 0.5 = 0 \end{aligned}$$

Similarly, we calculate:

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$$
\begin{aligned}
E(AB) &= \sum_{i,j=1}^{2} a_i\, b_j \cdot P(a_i, b_j) \\
&= 0.4 - 0.1 + 0.4 - 0.1 \\
&= 0.6
\end{aligned}
$$

and we can calculate the covariance of A and B:

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$$
\begin{aligned}
cov(A, B) &= E(AB) - E(A) \cdot E(B) \\
&= E(AB) \\
&= 0.6
\end{aligned}
$$

This indicates that A and B are positively correlated. Learning that the random variable A has the value $+1$ makes it much more likely that the random variable B also has the value 1.

Let us now turn to science. Here one often does not know the underlying joint probability distribution. However, the expectations and covariances can be estimated from the data. And so the problem is to infer the underlying joint probability distribution from this information. In fact, scientists want even more. They want to find out what the causal relations between the variables in question are. It is one thing to know, for example, that smoking and certain heart diseases are statistically correlated. But it is quite another thing, and indeed much more important, to know whether the two variables are also causally related. After all, having yellow fingers is also statistically correlated with certain heart diseases, while none is the cause of the other. There is a lot of interesting research on this important topic, and I will give you a couple of references at the end of this lecture.

To proceed, let us elaborate on how a joint probability distribution can be inferred from statistical data. So let us consider two random variables, A and B, with vanishing expectation, i.e. with

$$E(A) = E(B) = 0,$$

and with

$$cov(A, B) = x$$

where x is some real number in the closed interval $[-1, 1]$. From this we immediately infer that $E(AB) = x$:

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$$E(A) = E(B) = 0, cov(A, B) = x \in [-1, 1] \rightarrow E(AB) = x$$

We can then write down the following four equations that fix the values of the joint probability distribution:

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1. $E(A) = 0$ implies

$$P(1, 1) + P(1, -1) - P(-1, 1) - P(-1, -1) = 0.$$

2. $E(B) = 0$ implies

$$P(1, 1) - P(1, -1) + P(-1, 1) - P(-1, -1) = 0.$$

3. $E(AB) = x$ implies

$$P(1, 1) - P(1, -1) - P(-1, 1) + P(-1, -1) = x.$$

4. Probability Theory implies

$$P(1, 1) + P(1, -1) + P(-1, 1) + P(-1, -1) = 1.$$

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Solving this system of four linear equations leads to:

$$\begin{aligned} P(1, 1) = P(-1, -1) &= (1 + x)/4 \\ P(1, -1) = P(-1, 1) &= (1 - x)/4 \end{aligned}$$

Remember: x is the covariance of A and B.

Note that $(1+x)/4$ and $(1-x)/4$ are always in the closed interval $[0,1]$ whatever value of x we choose from the closed interval $[-1,1]$. And so we conclude that whatever the value of the covariance is, there is always a joint probability distribution that reproduces it. This may sound trivial, but we will learn in the next clip that this is not always the case: There are correlations between variables that are so strange that there is no probability distribution that generates them.

Quiz 56:
Consider two binary variables A and B each of which can take the values 1 and -1. Furthermore, $P(1,1) = 0.2, P(1,-1) = 0.4$, and $P(-1,-1) = 0.3$.
(1): What is $P(-1,1)$?
(2): Are A and B positively correlated?
Solution

8.3 Quantum Correlations (9:27)

Let us now consider four random variables, A, A', B, and B' and let us not worry about what these variables mean. We frame our problem as a purely mathematical problem. What matters is that the expectations $E(AB)$, $E(AB')$, $E(A'B)$ and $E(A'B')$ can be calculated from quantum mechanics. They can also be measured experimentally and the measured values are very close to calculated values, which are

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$$E(AB) = E(AB') = E(A'B) = x, E(A'B') = -x, x = 1/\sqrt{2}$$

Remembering our calculation at the end of the last clip, we can now construct probability distributions over (A,B), (A,B'), (A',B) and (A',B') that reproduce these correlations. These distributions are

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$$\begin{aligned}
P(A=1,B=1)=P(A=-1,B=-1) &= (1+x)/4 \\
P(A=1,B=-1)=P(A=-1,B=1) &= (1-x)/4 \\
\\
P(A=1,B'=1)=P(A=-1,B'=-1) &= (1+x)/4 \\
P(A=1,B'=-1)=P(A=-1,B'=1) &= (1-x)/4 \\
\\
P(A'=1,B=1)=P(A'=-1,B=-1) &= (1+x)/4 \\
P(A'=1,B=-1)=P(A'=-1,B=1) &= (1-x)/4 \\
\\
P(A'=1,B'=1)=P(A'=-1,B'=-1) &= (1-x)/4 \\
P(A'=1,B'=-1)=P(A'=-1,B'=1) &= (1+x)/4
\end{aligned}$$

Interestingly, however, it turns out that there is no joint probability distribution over all four variables A, A', B, and B'. This follows from the following theorem:

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The Clauser-Horne-Shimony-Holt Theorem

There is a joint probability distribution over the random variables A, A', B, and B' if and only if

$$|E(AB) + E(AB') + E(A'B) - E(A'B')| \leq 2.$$

The last inequality is called the Clauser-Horne-Shimony-Holt Inequality, or simply the CHSH Inequality. The vertical bars on the left hand side of the inequality denote the absolute value or modulus of the expression in between them. For example, $|5| = 5$ and $|-5| = 5$.

It is now easy to see that there is no joint probability distribution for the case discussed above. If we insert the four experimentally confirmed expectations, i.e.

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The consequence of the CHSH Theorem

Insert

$$\begin{aligned}
E(AB) &= E(AB') = E(A'B) = 1/\sqrt{2} \\
E(A'B') &= -1/\sqrt{2}
\end{aligned}$$

in the CHSH-inequality:

$$|1/\sqrt{2}+1/\sqrt{2}+1/\sqrt{2}+1/\sqrt{2}| = 2\sqrt{2} \approx 2.828 > 2.$$

Hence, there is no joint probability distribution over A, A', B, and B'.

This is an amazing result. There are correlations which can be observed (and explained by a theory!), but which are so strange that there is no joint probability distribution that accounts for them. Much of the work in the foundations of quantum mechanics is about understanding this and other strange features of quantum mechanics. One would like to understand how the world must be like if these things happen.

Let us take stock. We have shown that there is no joint probability distribution over the variables A, A', B, and B', but that there is a joint probability distribution over the variables (A, B), (A, B'), (A', B) and (A', B'). How can this be? It turns out that if one explicitly constructs the joint probability distribution from the statistical information, i.e. from

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$$\begin{aligned} E(A) = E(A') = E(B) = E(B') &= 0 \\ E(AB) = E(AB') = E(A'B) &= 1/\sqrt{2} \\ E(A'B') &= -1/\sqrt{2} \end{aligned}$$

then one realizes that some "probabilities", such as $P(1, 1, -1, 1)$, turn out to be negative. This can of course not be so because probabilities are always greater than or equal to zero, which in turn explains why there is no joint probability distribution. However, if one uses the resulting "distribution" (including the negative "probabilities") and calculates

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$$\begin{aligned} E(A) = E(A') = E(B) = E(B') &= 0 \\ E(AB) = E(AB') = E(A'B) &= 1/\sqrt{2} \\ E(A'B') &= -1/\sqrt{2} \end{aligned}$$

Some "probabilities", such as $P(1, 1, -1, 1)$, are negative, but, for example,

$$\begin{aligned} P(A = 1, B = 1) &= \sum_{A', B'} P(A = 1, A', B = 1, B') \\ &= \frac{1+\sqrt{2}}{4\sqrt{2}} \geq 0. \end{aligned}$$

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$$\begin{aligned} P(A=1, B=1) &= \sum_{A', B'} P(A=1, A', B=1, B') \\ &= \frac{1+\sqrt{2}}{4\sqrt{2}} \geq 0. \end{aligned}$$

then one gets well-behaved probabilities in the closed interval $[0,1]$ as the negative terms are canceled by corresponding positive terms in the sum. In fact, one gets the value that we stated above.

We have seen that a certain mathematical structure – probabilities defined over a Boolean algebra of variables – cannot always be used to account for statistical correlations that certain – in our case: quantum mechanical – data exhibit. Probability theory is not universally applicable in science. This raises interesting philosophical questions about the relation between mathematics and the world. We will shortly return to these questions below. Before, however, we would like to understand better what is at stake here. Obviously, there is no probability assignment for all instantiations of (A, A', B, B'). Some of these combinations are not even impossible (which would mean that one assigns them a probability of 0). There seems to be a much deeper incompatibility which has to be understood. This incompatibility is reflected in the formalisms of quantum mechanics, which features so-called non-commuting operators, i.e. mathematical objects for which the product XY is not the same as the product YX.

The impossibility of assigning probabilities to all instantiations of (A, A', B, B') also suggests that the underlying Boolean algebra, over which the probabilities are defined, is not the right mathematical structure to use here. A more restricted structure, which excludes some elements of the algebra, should do better. We will explore this route in the next clip.

We will come back to Boolean algebras in 8.4. For now, you might want to read http://en.wikipedia.org/wiki/Boolean_algebra.

Quiz 57:
Consider four binary random variables, A, A', B, and B'. The marginal distributions are as follows:
$P(A=1, B=1) = 0.25, P(A=1, B=-1) = 0.25, P(A=-1, B=1) = 0.25, P(A=-1, B=-1) = 0.25$
$P(A=1, B'=1) = 0.5, P(A=1, B'=-1) = 0, P(A=-1, B'=1) = 0, P(A=-1, B'=-1) = 0.5$
$P(A'=1, B=1) = 0.5, P(A'=1, B=-1) = 0.5, P(A'=-1, B=1) = 0, P(A'=-1, B=-1) = 0$
$P(A'=1, B'=1) = 0.1, P(A'=1, B'=-1) = 0.2, P(A'=-1, B'=1) = 0.4, P(A'=-1, B'=-1) = 0.3$
Is there a joint distribution over all four random variables?
Solution

8.4 Quantum Logic (8:32)

The starting point of the research program of quantum logic is an investigation of the logical structure of propositions reporting quantum measurement outcomes. Measurements typically answer yes-no questions such as:

Does the electron have spin 1/2 in the z-direction?

Doing a number of different measurements on a quantum system, one arrives at a set of propositions. It is interesting to ask whether these propositions obey the laws of standard propositional logic. These propositions form a Boolean algebra, and the following operations are defined.

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Boolean algebra

Operations:

- Conjunction: $\wedge$
- Disjunction: $\vee$
- Negation: $\neg$

We do not need much here to explain the basic ideas of quantum logic. However, if you want to learn more about Boolean algebras, check out the already mentioned wikipedia entry:
http://en.wikipedia.org/wiki/Boolean_algebra
To start with, read also the entries about the mentioned operations:
http://en.wikipedia.org/wiki/Logical_conjunction,
http://en.wikipedia.org/wiki/Logical_disjunction,
and
http://en.wikipedia.org/wiki/Logical_negation.

Moreover, the following laws hold in a Boolean algebra:

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Some laws that hold in a Boolean algebra:

Associativity of $\vee$: $x \vee (y \vee z) = (x \vee y) \vee z$

Associativity of $\wedge$: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

Commutativity of $\vee$: $x \vee y = y \vee x$

Commutativity of $\wedge$: $x \wedge y = y \wedge x$

Distributivity of $\wedge$ over $\vee$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

This list is not complete, but it suffices for our purposes.

Let us now turn to the logic of propositions reporting quantum measurement outcomes. Do they obey these laws? We will show that Distributivity is violated, and we will do so by considering an example.

We mentioned already that electrons have a property called spin. This is a property that has no analogue in classical physics. Tables and chairs, for example, do not have spin.

Here is one surprising feature involving spin.

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A surprising feature involving spin

- Electrons can be prepared in a way that their spin has the value $+1/2$ in the z-direction.

(One always has to choose a direction in which one measures the spin.)

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- That is, if we measure the spin of electrons which are prepared in this way in the z-direction, then we will always obtain the result $+1/2$.
- If we subsequently measure the spin of such electrons in the orthogonal x-direction, then we will find that 50% of the electrons have spin $+1/2$ and 50% of the electrons have spin $-1/2$.

Interestingly, however:

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- If we next consider the electrons that have spin $+1/2$ in the x-direction and measure their spin in the z-direction again, then we will find that 50% of them have spin $+1/2$ and 50% have spin $-1/2$.

This is a very strange observation! After all, the spin in z-direction was measured before and it turned out to be $+1/2$. So how can it now not be $+1/2$ anymore?

It seems that the second measurement, the measurement of the spin in the x-direction, somehow changes the "state" of the system. It disturbs the system, and so a measurement is not just a way to read off the true spin value of the electron. Measurements play a much more active role in the dynamics of quantum systems. They are involved in establishing the fact that the electron has a certain spin in a certain direction. That is, at least, the account given by the Copenhagen interpretation of quantum mechanics. We will use it to illustrate the basic ideas of quantum logic and to show that distributivity of AND over OR is violated here:

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Distributivity is violated!

Let:

- X: The electron has spin $+1/2$ in the z-direction.
- Y: The electron has spin $+1/2$ in the x-direction.
- Z: The electron has spin $-1/2$ in the x-direction.

We note:

- $Y \vee Z$ is true (as $+1/2$ and $-1/2$ are the only possible spin-values).
- Hence $X \wedge (Y \vee Z)$ is true.
- $(X \wedge Y) \vee (X \wedge Z)$ is false.

- Hence, $X \wedge (Y \vee Z) \neq (X \wedge Y) \vee (X \wedge Z)$.

In other words, distributivity is violated in quantum logic.

Let us be a bit more explicit here and recall that once the value of the spin in the x-direction is fixed, i.e. once we know that Y is true or false, the value of the spin in the z-direction is undetermined. It is either 1/2 (with a probability of 0.5) or $-1/2$ (with a probability of 0.5). Hence, it is false that the electron has spin $+1/2$ in the z-direction, which implies that X is false. Hence
$(X \wedge Y) \vee (X \wedge Z)$ is false. (*)
However $(Y \vee Z)$ is true. We know that the spin is either $+1/2$ or $-1/2$. This also holds if we known that the electron has spin 1/2 in the z-direction. Hence,
$X \wedge (Y \vee Z)$ is true. (**)
From (*) and (**) we conclude that
$(X \wedge Y) \vee (X \wedge Z)$ is not equivalent with $X \wedge (Y \vee Z)$ and distributivity is violated in quantum logic.

Our example illustrates that the "logic" of quantum measurement outcomes is non-Boolean. To give a positive account, i.e. to specify the laws that actually hold, one has to carefully study the relevant propositions. This is the business of the research field of quantum logic, which was established by John von Neumann, whom we encountered already in Lecture 6 when we spoke about the von Neumann and Morgenstern axioms and the representation theorem that von Neumann and Morgenstern proved. In 1936, von Neumann and Garrett Birkhoff published a paper entitled "The Logic of Quantum Mechanics" that contains important ideas. Later authors, such as George Mackey and Constantin Piron made important contributions, especially in the 1960ies and 70ies. For example, Mackey formulated a set of axioms for the quantum propositional system as a so-called ortho-complemented lattice, and Piron proved a representation theorem that aims at motivating the Hilbert-space structure of quantum mechanics from quantum logical axioms. Since then much happened in quantum logic, although the field never became mainstream in physics or in the philosophy of physics. In recent years, quantum logic gained momentum due to an interesting relation to the new and thriving research field of quantum computation and pragmatic so-called operational approaches to quantum logic are flourishing.

(Slide 68)

A Short History of Quantum Logic

- 1936: John von Neumann and Garrett Birkhoff published "The Logic of Quantum Mechanics".
- 1960ies and 70ies: George Mackey formulated a set of axioms for the quantum propositional system as a so-called ortho-complemented lattice.

- 1964: Constantin Piron proved a representation theorem that aims at motivating the Hilbert-space structure of quantum mechanics from quantum logical axioms.
- Today many quantum logicians explore the connection to the theory of quantum computation and prefer an operational approach.

Quiz 58:
In a Boolean algebra, Distributivity of $\vee$ over $\wedge$ also holds:
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
Show that
$B \vee (A \wedge (B \vee C))$ is equivalent with $B \vee (A \wedge C)$,
where $\wedge$ and $\vee$ denote the corresponding logical connectives.
Solution

8.5 Final Remarks (3:57)

Let me close with a couple of final remarks. To begin with, we have seen that logic and probability are closely related in quantum mechanics. Recall our discussion of the violation of the CHSH-Inequality. We have seen that there is no probability distribution over the four random variables A, A', B, and B'. However, there are probability distributions over (A, B) etc. Recall further that probability measures are defined over a Boolean algebra of propositions. If one wants to deal with the quantum mechanical case, one has two main options. First, one can replace the Boolean algebra by a suitable non-Boolean algebra and define a probability measure on it. This is what the quantum logic program does. Second, one can stick to the Boolean algebra and explore to what extend it is possible to define a meaningful measure on it that is not a probability measure, but that shares many of the features of a probability measure. This measure should also collapse into a probability measure in cases where probabilities are defined. Both approaches seem to be worth elaborating, but in any case, logic and probability cannot be separated from each other in quantum mechanics. A change in the logic has implications for the probability measure and vice versa.

This brings us to a related topic: the status of logic and probability in quantum mechanics. In the late 1960ies, the philosopher Hilary Putnam famously argued that quantum mechanics provides an empirical case for revising our views about logic. Alluding to the violation of distributivity in quantum mechanics, Putnam argued that classical logic is in difficulties just as Euclidian Geometry gets in trouble when faced with Einstein's General Theory of Relativity and its empirical confirmation. This is a very strong claim that has been the subject of intensive debates. Can a logic really be established empirically? After all, a logic tells us what good reasoning is. And: is quantum logic really a logic? These are interesting questions which you might want to look into. I'll give you some references below.

Finally, let me stress that quantum logic and probability is a wonderful field of research for the mathematical philosopher. It is a philosophically and mathematically challenging field, there are many open problems, and it is interesting to see that there are structural connections to other fields of enquiry. Think, for example, about the representation theorems that show up in decision theory and in quantum logic. There are interesting connections that one might want to further. Perhaps you want to look further into this?

References:
Here are two important books on causal inference:
Pearl, J., Causality: *Models, Reasoning and Inference.* Cambridge: Cambridge University Press, 2009.
Spirtes, P., Glymour, C., and Scheines, R., *Causation, Prediction, and Search.* Cambridge, Mass: MIT Press.
You might also want to have a look at
http://plato.stanford.edu/entries/causation-probabilistic/
For an introduction to quantum mechanics, see
http://plato.stanford.edu/entries/qm/
Here are some more advanced entires on quantum mechanics from the Stanford Encyclopedia of Philosophy:
http://plato.stanford.edu/entries/qm-copenhagen/
http://plato.stanford.edu/entries/qt-epr/
http://plato.stanford.edu/entries/bell-theorem/
http://plato.stanford.edu/entries/qm-bohm/
http://plato.stanford.edu/entries/kochen-specker/
Most relevant for the present topic is the entry
http://plato.stanford.edu/entries/qt-quantlog/
Note, however, that this entry is technically quite demanding.
I also recommend the following two books on quantum mechanics:
Cushing, J., *Philosophical Concepts in Physics.* Cambridge: Cambridge University Press, 1998.
Cushing, J., *Quantum Mechanics: Historical Contingency and the Copenhagen Hegemony.* Chicago: The University of Chicago Press, 1994.
The following book discusses probabilities in physics:
Beisbart, C. and Hartmann, S. (eds.), *Probabilities in Physics.* Oxford: Oxford University Press 2011.
The following book provides the best introduction to quantum logic:
Hughes, R.I.G., *The Structure and Interpretation of Quantum Mechanics.* Cambridge: Harvard University Press 1992.
(If you want to buy one book, then buy this one!)
You might also want to read
http://en.wikipedia.org/wiki/Quantum_logic
Finally, the following book collects many of Patrick Suppes' papers (with Mario Zanotti) on quantum probabilities:
Suppes, P. and Zanotti, M., *Foundations of Probability with Applications: Selected Papers 1974-1995.* Cambridge: Cambridge University Press 2010.
All chapters of this book can also be downloaded at
http://suppes-corpus.stanford.edu.

Appendix A

Quiz Solutions Week 8: Quantum Logic

Quiz 54:

More precisely put, a random variable is a function from a sample space S to the real numbers. Let the sample space be the set S = {high, medium, low} and let there be three random variables A, B and C with A(high) = 5, A(medium) = 3 and A(low) = −1, B(high) = 4, B(medium) = 0 and B(low) = −3, C(high) = 3, C(medium) = −1 and C(low) = −5. Which of the following relations is true?

(a) A > B (b) B > C (c) A > C (d) AB > C (e) AC > B (f) BC > A

SOLUTION Quiz 54:

(a) Yes, because for all three arguments (i.e. high, medium, and low), the value of A is greater than the value of B. (b) Yes. (c) Yes. (d) Yes, because for all three arguments, the value of AB is greater than the value of C, i.e. A(high) B(high) = 20 > C(high) = 3, and so on. (e) No. (f) No.

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Quiz 56:

Consider two binary variables A and B each of which can take the values 1 and -1. Furthermore, $P(1,1) = 0.2, P(1,-1) = 0.4$, and $P(-1,-1) = 0.3$.

(1): What is $P(-1,1)$?

(2): Are A and B positively correlated?

SOLUTION Quiz 56:

Solution (1): $P(-1,1) = 1 - P(1,1) - P(1,-1) - P(-1,-1) = 0.1$

Solution (2): We calculate $E(A) = 0.2+0.4-0.3-0.1 = 0.2, E(B) = 0.2-0.4-0.3+0.1 = -0.4$, and $E(AB) = 0.2-0.4+0.3-0.1 = 0$. Hence, $cov(AB) = E(AB)-E(A)E(B) = 0.08$. A and B are therefore positively correlated.

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Quiz 57:

Consider four binary random variables, A, A', B, and B'. The marginal distributions are as follows:

$P(A = 1, B = 1) = 0.25, P(A = 1, B = -1) = 0.25, P(A = -1, B = 1) = 0.25, P(A = -1, B = -1) = 0.25$

$P(A = 1, B' = 1) = 0.5, P(A = 1, B' = -1) = 0, P(A = -1, B' = 1) = 0, P(A = -1, B' = -1) = 0.5$

$P(A' = 1, B = 1) = 0.5, P(A' = 1, B = -1) = 0.5, P(A' = -1, B = 1) = 0, P(A' = -1, B = -1) = 0$

$P(A' = 1, B' = 1) = 0.1, P(A' = 1, B' = -1) = 0.2, P(A' = -1, B' = 1) = 0.4, P(A' = -1, B' = -1) = 0.3$

Is there a joint distribution over all four random variables?

SOLUTION Quiz 57:

We first calculate the expectations:

$E(AB) = 0.25 - 0.25 - 0.25 + 0.25 = 0$

$E(AB') = 0.5 + 0.5 = 1$

$E(A'B) = 0.5 - 0.5 = 0$

$E(A'B') = 0.1 - 0.2 - 0.4 + 0.3 = -0.2$

Next, we insert these expectations in the CHSH inequality:

$|E(AB) + E(AB') + E(A'B) - E(A'B')| = |0 + 1 + 0 + 0.2| = 1.2 \leq 2$. Hence, there is a joint distribution over A, A', B, and B'. How would you calculate it?

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Quiz 58:

In a Boolean algebra, Distributivity of $\vee$ over $\wedge$ also holds:

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Show that

$B \vee (A \wedge (B \vee C))$ is equivalent with $B \vee (A \wedge C)$,

where $\wedge$ and $\vee$ denote the corresponding logical connectives.

SOLUTION Quiz 58:

We use "=" for "is equivalent with" and make the following transformations:

$B \vee (A \wedge (B \vee C)) = (B \vee A) \wedge (B \vee (B \vee C))$ (Distributivity of $\vee$ over $\wedge$) $= (B \vee A) \wedge ((B \vee B) \vee C)$ (Associativity of $\vee$) $= (B \vee A) \wedge (B \vee C)$ (as $B \vee B = B$) $= B \vee (A \wedge C)$ (Distributivity of $\vee$ over $\wedge$)

You can also use truth tables to convince yourself of the identity. See

http://en.wikipedia.org/wiki/Truth_table

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List of Figures