Propositions and probabilities

- W is the set of possible worlds, $\emptyset$ is the "empty set"
- If X is a proposition, then so is $\neg X$
- If Y is also a proposition then so are $X \wedge Y$ and $X \vee Y$

- $p(W) = 1$
- For all X, $0 \leq p(X) \leq 1$
- If $X \wedge Y = \emptyset$ then $p(X \vee Y) = p(X) + p(Y)$

Conditionalisation and expectation

$$p(H|E) = \frac{p(H \wedge E)}{p(E)}$$

E confirms H iff $p(H|E) > p(H)$

$$EU(A) = \sum_{W} p(w)u(A(w))$$

Sets of probabilities

R	$\frac{1}{3}$
B	$\frac{1}{3}$
Y	$\frac{1}{3}$

Sets of probabilities

R	$\frac{1}{3}$	$\frac{1}{3}$
B	$\frac{1}{3}$	0
Y	$\frac{1}{3}$	$\frac{2}{3}$

Sets of probabilities

R	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
B	$\frac{1}{3}$	0	$\frac{2}{3}$
Y	$\frac{1}{3}$	$\frac{2}{3}$	0

Sets of probabilities

R	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\dots$
B	$\frac{1}{3}$	0	$\frac{2}{3}$	q	$\dots$
Y	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{2}{3} - q$	$\dots$

Sets of probabilities

$$P = \left\{ \begin{array}{llllll} \text{R} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \ldots \\ \text{B} & \frac{1}{3} & 0 & \frac{2}{3} & q & \ldots \\ \text{Y} & \frac{1}{3} & \frac{2}{3} & 0 & \frac{2}{3} - q & \ldots \end{array} \right\}$$

$P(R) = \frac{1}{3}$, $P(B) = \left[0, \frac{2}{3}\right]$
P is a set of functions. p is a function in P.

The Ellsberg choices

G Win the holiday if a red ball is drawn
H Win the holiday if a blue ball is drawn

K Win the holiday if a red or yellow ball is drawn
L Win the holiday if a blue or yellow ball is drawn

Ellsberg expectations I

$u(\text{Holiday}) = 1,\ u(\text{Nothing}) = 0$

$p(\text{Red}) = \frac{1}{3},\ p(\text{Yellow}) = \frac{2}{3},\ p(\text{Blue}) = 0$

G $\frac{1}{3} \times 1 + \frac{2}{3} \times 0 = \frac{1}{3}$

H $0 \times 1 + 1 \times 0 = 0$

Ellsberg expectations I

$$u(\text{Holiday}) = 1,\ u(\text{Nothing}) = 0$$

$$p(\text{Red}) = \tfrac{1}{3},\ p(\text{Yellow}) = \tfrac{2}{3},\ p(\text{Blue}) = 0$$

G $\frac{1}{3} \times 1 + \frac{2}{3} \times 0 = \frac{1}{3}$

H $0 \times 1 + 1 \times 0 = 0$

K $\left(\frac{1}{3} + \frac{2}{3}\right) \times 1 + 0 \times 0 = 1$

L $\left(0 + \frac{2}{3}\right) \times 1 + \frac{1}{3} \times 0 = \frac{2}{3}$

Ellsberg expectations II

$$u(\text{Holiday}) = 1, \; u(\text{Nothing}) = 0$$

$$p(\text{Red}) = \frac{1}{3}, \; p(\text{Yellow}) = 0, \; p(\text{Blue}) = \frac{2}{3}$$

G $\frac{1}{3} \times 1 + \frac{2}{3} \times 0 = \frac{1}{3}$

H $\frac{2}{3} \times 1 + \frac{1}{3} \times 0 = \frac{2}{3}$

K $\left(\frac{1}{3} + 0\right) \times 1 + \frac{2}{3} \times 0 = \frac{1}{3}$

L $\left(0 + \frac{2}{3}\right) \times 1 + \frac{1}{3} \times 0 = \frac{2}{3}$

Summary of Ellsberg expectations

For all p in the credal set P, $p(\text{Red}) = \frac{1}{3}$ And if $p(\text{Blue}) = q$, then $p(\text{Yellow}) = \frac{2}{3} - q$.
$p(\text{Blue}) + p(\text{Yellow}) = \frac{2}{3}$.
$p(\text{Red}) + p(\text{Yellow}) = \frac{1}{3} + \frac{2}{3} - q = 1 - q$.
q ranges over $\left[0, \frac{2}{3}\right]$

- $EU(G) = P(\text{Red}) = \frac{1}{3}$
- $EU(H) = P(\text{Blue}) = \left[0, \frac{2}{3}\right]$
- $EU(K) = P(\text{Red} \vee \text{Yellow}) = \left[\frac{1}{3}, 1\right]$
- $EU(L) = P(\text{Blue} \vee \text{Yellow}) = \frac{2}{3}$

Buying and selling bets

Buy pay qS for the chance to win S if X

Sell gain qS but risk having to pay out S if X

Lower and upper probabilities

$L(X)$ is the largest q such that you would buy a bet on X. $U(X)$ is the smallest q such that you would sell a bet on X.

- $L(W) = 1$
- For all X, $0 \leq L(X) \leq 1$
- For all X, Y, if $X \wedge Y = \emptyset$ then $L(X \vee Y) \geq L(X) + L(Y)$

- $U(W) = 1$
- For all X, $0 \leq U(X) \leq 1$
- For all X, Y if $X \wedge Y = \emptyset$ then $U(X \vee Y) \leq U(X) + U(Y)$

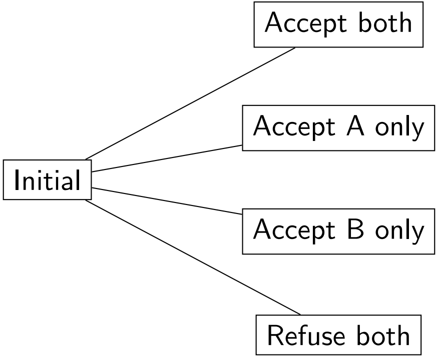
Further, for all X, $L(X) \leq U(X)$ and $U(\neg X) = 1 - L(X)$

A great series of bets

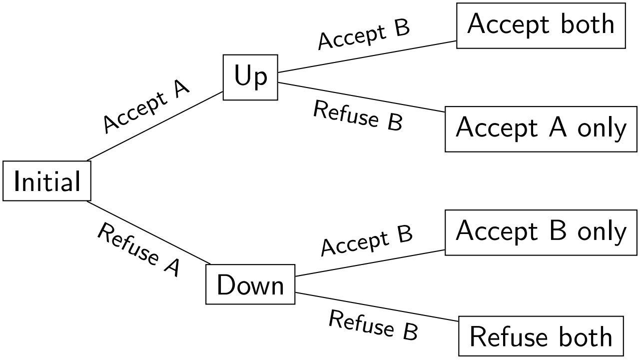
A Win 2 if X but lose 1 if $\neg X$

B Win 2 if $\neg X$ but lose 1 if X

Four-way choice



Sequence of choices



A Win 2 if X but lose 1 if $\neg X$

B Win 2 if $\neg X$ but lose 1 if X

Maximin expectation and mixtures

A Win 10 if X, 0 otherwise

B Win 2 if X, 8 otherwise

C Win 0 if X, 6 otherwise

- $EU(A) = 10 \times q$ which ranges over $[0, 10]$
- $EU(B) = 2 \times q + 8 \times (1 - q)$ which ranges over $[2, 8]$
- $EU\left(A\frac{1}{2}C\right) = \frac{1}{2} \times 10 \times q + \frac{1}{2} \times 6 \times (1 - q) = 3 + 2 \times q$

Maximin expectation and mixtures

A Win 10 if X, 0 otherwise

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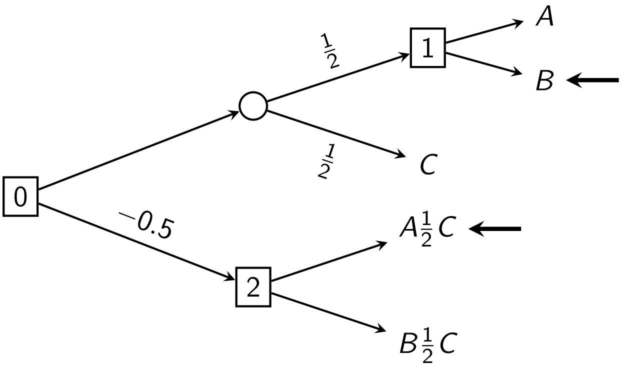
C Win 0 if X, 6 otherwise

- $EU(A) = 10 \times q$ which ranges over $[0, 10]$
- $EU(B) = 2 \times q + 8 \times (1 - q)$ which ranges over $[2, 8]$
- $EU\left(A\frac{1}{2}C\right) = \frac{1}{2} \times 10 \times q + \frac{1}{2} \times 6 \times (1 - q) = 3 + 2 \times q$
- $EU\left(B\frac{1}{2}C\right) = \frac{1}{2}(2 \times q + 8 \times (1 - q)) + \frac{1}{2} \times (6 \times (1 - q)) = 7 - 6 \times q$

Avoiding free information?



Dilation

$$\begin{aligned} P(\text{Black}|X) &= \left\{0, \frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}, 1\right\} \\ &= P(\text{White}|X) = P(\text{Black}|Y) = P(\text{White}|Y) \end{aligned}$$

For every p in the set P:

$$p(\text{Black}|Y) = 1 - p(\text{Black}|X)$$

Dilation

$$
\begin{aligned}
P(\text{Black}|X) &= \left\{0, \frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}, 1\right\} \\
&= P(\text{White}|X) = P(\text{Black}|Y) = P(\text{White}|Y)
\end{aligned}
$$

For every p in the set P:

$$
p(\text{Black}|Y) = 1 - p(\text{Black}|X)
$$

$$
\begin{aligned}
p(\text{Black}) &= p(\text{Black}|X)p(X) + p(\text{Black}|Y)p(Y) \\
&= \frac{1}{2}p(\text{Black}|X) + \frac{1}{2}(1 - p(\text{Black}|X)) \\
&= \frac{1}{2}
\end{aligned}
$$

Dilation decision problem