Introduction to Mathematical Philosophy

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Week 1: Overview

Overview of Lecture 1: Infinity

1.1 Introduction: Infinity has always been an important philosophical topic. But what exactly does 'infinite' mean? And can we understand infinity by finite means?

1.2 Arguments and Paradoxes: Zeno of Elea argued that there is no change in the world by putting forward a famous paradox of infinity. An argument is a sequence of sentences the last one of which is called the 'conclusion'; the statements before the conclusion are called the 'premises'; and the conclusion of an argument is introduced by a term such as 'therefore'. A logically valid argument is one that has the property that if all its premises are true then necessarily also its conclusion is true. Finally, a paradox is a logically valid argument in which each premise is plausible if taken by itself, but where the conclusion of the argument is absurd.

1.3 Zeno's Paradox: Zeno puts forward a paradox in this sense of the word. It can be reconstructed in terms of nine premises P1-P9 and a conclusion. The conclusion is that Achilles will never overtake the tortoise in the hypothetical race that is described by Zeno, which is indeed absurd.

1.4 Calculus to the Rescue: We find that premise P6 in Zeno's paradox might have looked true initially, but when we analyze it by means of the calculus of real numbers, we see that it is not justified. Adding up infinitely many periods of time of positive duration does not necessarily lead to arbitrarily large periods of time. But Zeno's argument only goes through if that had been assumed. Perhaps Zeno might come back to us with a new argument, but his original argument concerning Achilles and the tortoise rests on a premise for which we do not have good reasons to believe it to be true.

1.5 Sets: We still have not clarified what one means by an infinite set. But what is a set anyway? We turn to that question first, and we enumerate some of the most basic properties of sets. In particular, sets satisfy the Principle of Extensionality: sets are identical if and only if they have the same members.

1.6 Comparing Sets in Terms of Size: At next we want to compare sets in terms of size. On the one hand, we can do so by means of the notion of proper subset: a set is a proper subset of another if and only if every member of the former is a member of the latter but not the other way around. On the other hand, we can also try to pair off the members of two sets so that, first of all, each member of the first set is paired off with precisely one member of the second set, and secondly, each member of the second set is paired off with precisely one member of the first set. Intuitively, and as is in fact true for finite sets, if a set is a proper subset of another, then the second set is of greater size than the former; and: if one manages to pair off the members of two sets, then the two sets are of equal size. But if one takes these two, as it seems, plausible statements together and turns them into premises of an argument, one can construct a paradox again. This had been observed by Galilei. The paradox can be reconstructed in terms of four premises and one conclusion.

1.7 Diagnosis of Galileo's Paradox: One way out of the paradox is to distinguish between two kinds or concepts of size. The second one, according to which two sets are of equal size if and only if there is a pairing off between the members of the two sets, turned out to be enormously fruitful in set theory. We consider some of its implications, and we end up defining infinity in terms of it, as follows: a set is infinite if and only if the set is of equal size as at least one of its proper subsets.

1.8 Cantor's Theorem: Cantor proved that there is not just one infinite size but that there are infinities of different sizes. In particular, he proved that the (infinite) set of natural numbers is of smaller size than the (infinite) set of real numbers. We sketch Cantor's proof of his theorem.

1.9 Conclusions: The calculus helped us avoid the absurd conclusion of Zeno's paradox. And the language of set theory gave us the means to define 'infinite' in a simple, natural and fruitful manner; we were able to state this definition using finitely many set-theoretic terms that are perfectly understandable.

Chapter 1

Week 1: Infinity

1.1 Introduction (04:09)

Welcome to our introduction to mathematical philosophy! The goal of our course is to demonstrate that mathematical methods can be as useful in philosophy as they are in the sciences. For some of these mathematical methods we will show you how to apply them to philosophical questions and problems, we will tell you how such applications have helped philosophers to make philosophical progress, and hopefully at the end of the course you will have found that these mathematical methods help you, too, to gain new philosophical insight.

Today we will deal with the perennial topic of infinity. This has become a popular subject, on which there are lots of popular books and films. So you might have heard about the topics that I will be dealing with today: if so even better, for this will enable you to concentrate on the more methodological questions which are typically not raised by the popular expositions but which I will discuss. Whether or not you have heard about the topics of today, they will allow me to introduce some of the concepts that we will need in the other lectures, and which every philosopher should have heard about anyway. So let us jump straight in.

In many ways, we seem to be finite beings, and what surrounds us seems to be finite, too. The house that I am living in has a finite spatial extension: its diameter is a certain number of, say, meters or feet; my life span here, I am afraid, is finite in time: it's some number of years; I have finitely many kids – that is, two of them – and although the number of students of this course is fantastically large, it is still finite. I am only capable of cutting my birthday cake into finitely many pieces, and so on.

No wonder, therefore, that quite a few scholars of the past thought that the infinite would have to remain a mystery to us. In the Western tradition this presumption was strengthened often by associating the infinite with God or God-like abilities: by believing God to have, in contrast with ourselves, an infinitely complex intellect, to be infinitely powerful, to be infinitely good, to be eternal, and the like. And if one holds these beliefs, it becomes tempting to think that understanding the infinite will be just as difficult, or maybe even just as impossible, as understanding God.

Whether or not one believes that there is a god that is infinitely powerful or infinitely good, a philosopher can't help asking: what would such a belief amount to? What exactly does one mean when one ascribes terms such as 'infinitely powerful' or 'infinitely good' to God? Maybe 'infinitely powerful' means something like: there are infinitely many possible ways the world might be, infinitely many possible worlds; and for each of them, God would in principle have the power to make it real or actual, to turn it into the actual world that we are actually living in (see Figure 1.1).

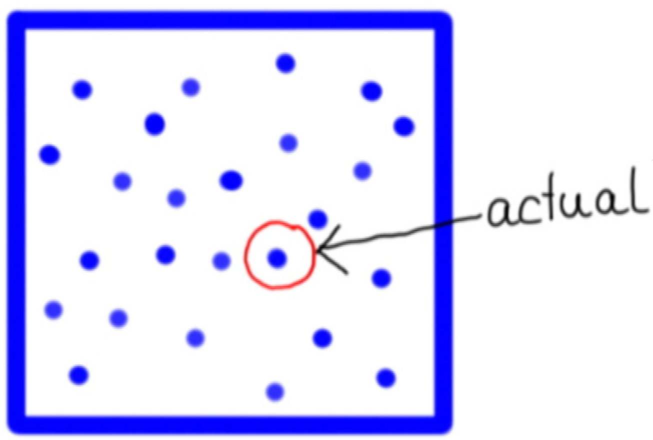

Figure 1.1: Possible worlds

Remark on possible worlds: I will return to the topic of possible worlds in Lecture 3.

Accordingly, maybe, 'infinitely good' means something like: having infinitely many positive properties. I have used the expression 'infinitely many' twice now. But what exactly does that term mean? And since we seem to be finite beings, are we even in a position to answer such a question? Can we reason in finite terms about the infinite without getting tangled up in paradoxes?

Remark on God and infinity: One of the philosophers who thought that God and the infinite were closely related, and who argued that for that reason there would be serious limitations to our understanding of God, was Nicolaus Cusanus in the 15th century. If you want to know more about him and his views, take a look at the corresponding entry in the Stanford Encyclopedia of Philosophy:
http://plato.stanford.edu/entries/cusanus/.
By the way: I will quite regularly refer you to entries in the Stanford Encyclopedia of Philosophy: they are freely accessible; they are written by professional philosophers who specialize in the subject matter of the respective entries; and they are subject to a serious peer-review process. I can very much recommend the Stanford Encyclopedia to you. It is also where philosophy students and philosophy faculty start their searches when they need some philosophical background information.

1.2 Arguments and Paradoxes (06:19)

One of the first to invoke paradoxes of infinity in their philosophical work was the ancient Greek philosopher Zeno of Elea: in line with the philosophical views of his teacher Parmenides, Zeno's aim was to argue against commonsensical statements such as that there is more than one object, or that there is change.

Remark on Zeno and Parmenides: For more on the historical background, see the corresponding entries in the Stanford Encyclopedia of Philosophy:
http://plato.stanford.edu/entries/parmenides/
and
http://plato.stanford.edu/entries/zeno-elea/.
You might also like to check out my LMU colleague Peter Adamson's (freely available) podcast 'History of Philosophy without any gaps' which includes episodes on Parmenides and Zeno:
http://www.historyofphilosophy.net

You may think it's odd to try to argue against such statements since they are obviously right. Well, Zeno was appealing to a reality/appearance distinction here, as many philosophers have done after him, and, for that matter, as many scientists do as well. The idea is: it appears as if there is more than one object, and it appears as if there is change, but really, on a more fundamental level, this might be an illusion. It's like putting a straight stick into water: the stick appears to be bent, but really it isn't. Accordingly, in philosophy we do allow ourselves to go beyond commonsense, but only if there are good reasons for doing so; and in order to demonstrate that there are good reasons, one is required to put forward arguments.

Now what is an argument?

(Slide 1)

> (P1) ...
>
> (P2) ...
>
> ⋮
>
> ---
>
> (C) ...

It is a sequence of statements: the last one of them is called the ‘conclusion’; the statements before the conclusion are called the ‘premises’; and the premises are usually meant to support the conclusion, which is why the conclusion of an argument is sometimes introduced by a term such as ‘therefore’, or the like. Amongst all arguments, the so-called logically valid ones are particularly important in philosophy: a logically valid argument is such that, independently of whether its premises are true, if they are true, then the conclusion must be true as well. For instance:

(Slide 2)

> (P1) Zeno is a philosopher.
>
> (P2) If Zeno is a philosopher, then he puts forward arguments.
>
> ---
>
> (C) Zeno puts forward arguments.

More generally:

(Slide 3)

> (P1) A.
>
> (P2) If A, then B.
>
> ---
>
> (C) B.

If the two premises are true, then the conclusion must be true, too; it could not be false given that the premises are true. In order for this argument to be logically valid, it does not matter really whether its premises are actually true – the point is: if they are true, then the conclusion follows from the premises logically and therefore is true as well.

Remark on the argument 'A. If A then B. Therefore, B.': The logical rule by which one can derive this conclusion from these premises is called 'Modus Ponens' in the philosophical tradition.
Please note that, whenever I can, I will avoid fancy names and funny symbols in the lectures themselves, since these names and symbols are not important really. I will still tell you, from time to time, about some of them in some of my written remarks, but feel free to ignore these remarks on terminology if you are not interested.
Oh, and I should add that we will return to the topic of truth in Lecture 2, and we will deal with 'if-then' in detail in Lecture 4.

Logic, which began with Aristotle and which thus belongs to the oldest disciplines in philosophy, deals with methods by which we can construct arguments that are logically valid, or by which we can distinguish logically valid arguments from those that are not.

Remark on Aristotle and his logic: For more on Aristotle and his logic, see
http://plato.stanford.edu/entries/aristotle/
and
http://plato.stanford.edu/entries/aristotle-logic/.

These logical methods belong to the larger class of formal or mathematical methods that are applied in philosophy. But logic is not the topic of this lecture.

Remark on Logic: It is a real pity that I cannot say more here about logical concepts, such as the logical validity of arguments – concepts that are of fundamental importance for philosophy – but doing logic properly would require a course of its own. Unfortunately, in our present course, we will have to leave things on a very informal (and sometimes pretty sloppy!) level as far as logic is concerned.
If that does not satisfy you: In case you study philosophy at a university, then you should have completed a serious logic course anyway. But most of you won't be students in some philosophy program, so if you still want to learn more about logic, then one option would be to take the Introduction to Logic course that you find in Coursera:
https://www.coursera.org/course/intrologic.
I should add, though, that that course has not been designed specifically for philosophers. Finally, should you speak German, you might also be interested in the lecture notes of my own introductory logic course at LMU which is in fact designed specifically for philosophers – the lecture notes are available freely at
http://www.mcmp.philosophie.uni-muenchen.de/people/faculty/hannes_leitgeb/logik_1_wise20142015/index.html

I only mentioned it because we may understand Zeno to put forward logically valid arguments. Not just any old logically valid argument, though: the premises of his arguments involve statements to the effect that there is more than one object, or that there is change; and these premises appear to be true, at least at first glance, while the conclusions of Zeno's arguments happen to be patently absurd. This is exactly what is called a 'paradox' in the philosophical literature.

(Slide 5):

Paradox:

> a logically valid argument in which each premise is plausible if taken by itself, but where the conclusion of the argument is absurd.

$$\left.\begin{array}{cl} \text{(P1)} & \ldots\ [\text{plausible}] \\ \text{(P2)} & \ldots\ [\text{plausible}] \\ \vdots & \\ \hline \text{(C)} & \ldots\ [\text{absurd}] \end{array}\right\}\ \text{logically valid}$$

Now, when a philosopher manages to state a paradox in this sense of the word, this means the following: since the argument is logically valid, then, if all of the premises are true, the conclusion must be true as well. But the conclusion of a paradoxical argument is absurd and hence overwhelmingly likely to be false. Therefore, at least one of the premises must be false, too, in spite of its initial appearance to be true. So stating a paradox is nothing but a way of arguing against some premises that might initially seem to be innocuous. And that is precisely what Zeno does.

Several such paradoxical arguments are ascribed to Zeno, but we will just deal with the most famous of them which concerns a hypothetical race between Achilles and a tortoise, a turtle – it's a thought experiment, as it is quite typical of philosophical reasoning. The conclusion of the argument is that Achilles will not overtake the tortoise in the race, which is absurd, since of course Achilles is by far the superior runner. The argument will be logically valid, and some of the premises will, at least at first glance, appear to be true in virtue of our apparent experience of motion; it is these premises that Zeno aims to attack by stating the paradox. I won't talk you through the ancient texts on the paradox, all of which we only have in the form of commentaries by other ancient philosophers. What I will do instead is to reconstruct the content of one version of Zeno's argument in my own terms, and to state that content as clearly and precisely as is necessary to evaluate the argument.

Remark on Zeno's paradoxes: More historical background information on Zeno's paradoxes can be found at http://plato.stanford.edu/entries/paradox-zeno/.

Quiz 01:
Do you think the following argument is logically valid, too?

(P1) Zeno puts forward arguments.
(P2) If Zeno is a philosopher, then he puts forward arguments.

(C) Zeno is a philosopher.

Or more generally:

(P1) B.
(P2) If A, then B.

(C) A.

Solution

1.3 Zeno's Paradox (08:16)

Picture the following situation:

(Slide 6)

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

(see Figure 1.2)

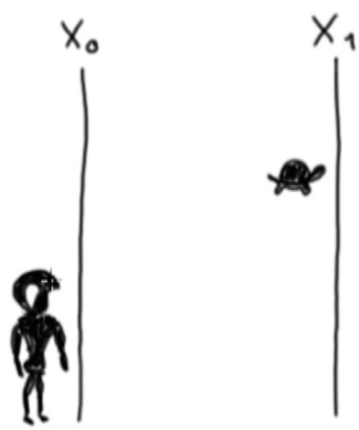

Figure 1.2: Start of the race

So the tortoise has a headstart. From now on we only consider points that are to the right of x_0 when we describe what is happening as time progresses or, in any case, as it seems to progress.

Our second premise is:

(Slide 7)

> (P2) When Achilles is at point x_1, the tortoise is at point x_2, where x_1 is to the left of x_2, and Achilles does not overtake the tortoise at any point between x_0 and x_1.

(see Figure 1.3)

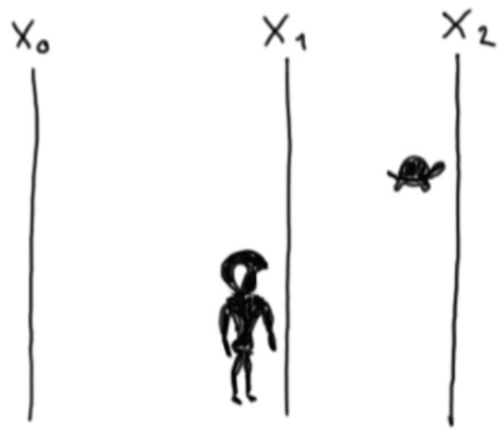

Figure 1.3: Second step

This is certainly plausible: When Achilles reaches x_1, the tortoise will have advanced a little bit in the meantime; and x_2 is the point to which she has advanced. Similarly:

(Slide 8)

> When Achilles is at point x_2, the tortoise is at point x_3, where x_2 is to the left of x_3, and Achilles does not overtake the tortoise at any point between x_1 and x_2.

And so on (see Figure 1.4).

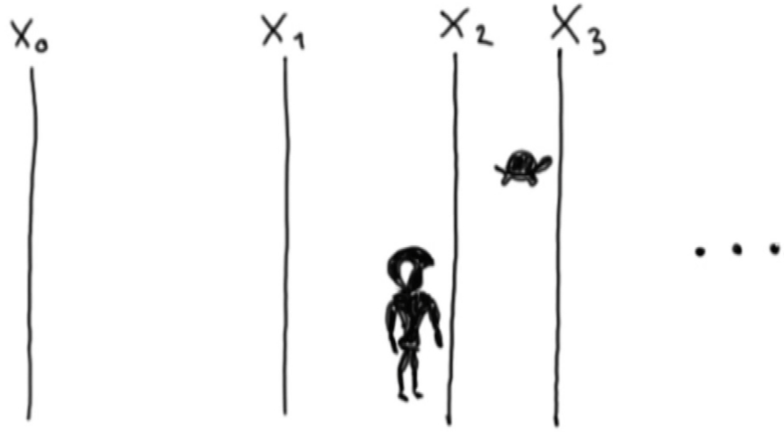

Figure 1.4: Third step (and so on)

Generally:

(Slide 10)

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

(P2) For all $n \geq 1$, when Achilles is at x_n, the tortoise is at x_{n+1}, where x_n is to the left of x_{n+1}, and Achilles does not overtake the tortoise at any point between x_{n-1} and x_n.

Here the variable 'n' denotes any of the numbers $1, 2, 3, ...$

Furthermore, we may assume:

(Slide 11)

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

(P2) For all $n \geq 1$, when Achilles is at x_n, the tortoise is at x_{n+1}, where x_n is to the left of x_{n+1}, and Achilles does not overtake the tortoise at any point between x_{n-1} and x_n.

(P3) If P1 and P2 are the case, then Achilles does not overtake the tortoise at any of x_1, x_2,... nor at any point in between any two of them.

The '...' in the premise simply means: and so on. Premise 3 should be unproblematic, since if premises 1 and 2 are the case, then Achilles does not overtake the tortoise at x_0, nor at any point between x_0 and x_1, nor at x_1, nor at any point between x_1 and x_2, and so on.

So far, so good. The underlying idea of the remaining premises of Zeno's argument is that Achilles will never be able to move beyond all of the points x_0, x_1, x_2,... No matter how long one is going to wait, Achilles will still be to the left of some of these points. And so he will never overtake the tortoise, which is absurd (see Figure 1.5).

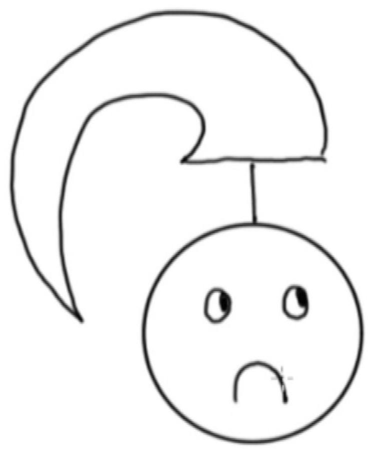

Figure 1.5: Grrr

Let us state this more precisely now. First of all, this should be obvious again:

(Slide 12)

> (P4) It takes Achilles a positive amount T_0 of time to get from x_0 to x_1. It takes Achilles a positive amount T_1 of time to get from x_1 to x_2.
> $\vdots$

Generally:

(Slide 13)

> (P4) For all n, it takes Achilles a positive amount T_n of time to get from x_n to x_{n+1}.

Here 'n' denotes any of the numbers $0, 1, 2, ...$, and 'positive' just means: longer than nothing. Moreover, it certainly appears to be true that the periods of time that it takes

Achilles to pass the various distances run by the tortoise add up to longer periods of time; so we have:

(Slide 14)

> (P5) If P4 is the case, then for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time.

But when one puts together infinitely many positive periods of time, they seem to become arbitrarily large, larger than any given finite amount of time. That is:

(Slide 15)

> (P6) If for all n, it takes Achilles $T_0+\ldots+T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time, then the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time.

'Not bounded by any finite amount of time' means: the periods taken together get arbitrarily large; for any given amount, the sum exceeds that amount at some point along Achilles' way.

Now surely the following is the case, too:

(Slide 16)

> (P7) If the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time, then Achilles never actually passes all of $x_0, x_1, x_2, \ldots$

And by the positions of the points $x_0, x_1, x_2, ...$, we also know:

(Slide 17)

> (P8) If Achilles never actually passes all of $x_0, x_1, x_2, \ldots$, then he is always somewhere to the left of at least one of $x_1, x_2, \ldots$

Again by looking at what the spatial situation is like, the following seems fine as well:

(Slide 18)

> (P9) If Achilles does not overtake the tortoise at any of $x_1, x_2, \ldots$ nor at any point in between either of them, and if he is always somewhere to the left of at least one of $x_1, x_2, \ldots$, then Achilles never overtakes the tortoise.

But now by taking all of our premises together we may logically conclude:

> (Conclusion) Achilles never overtakes the tortoise.

(Slide 19)

> (P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.
>
> $\vdots$
>
> (P9) If Achilles does not overtake the tortoise at any of $x_1, x_2, \ldots$ nor at any point in between either of them, and if he is always somewhere to the left of at least one of $x_1, x_2, \ldots$, then Achilles never overtakes the tortoise.
>
> ---
>
> (C) Achilles never overtakes the tortoise.

This is because premises 1-3 give us that Achilles does not overtake the tortoise at any of x_1, x_2, nor at any point in between any two of them:

(Slide 20)

> (P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.
>
> (P2) For all $n \geq 1$, when Achilles is at x_n, the tortoise is at x_{n+1}, where x_n is to the left of x_{n+1}, and Achilles does not overtake the tortoise at any point between x_{n-1} and x_n.
>
> (P3) If P1 and P2 are the case, then Achilles does not overtake the tortoise at any of x_1, x_2,... nor at any point in between any two of them.

And premises 4-8 give us that Achilles is always somewhere to the left of at least one of $x_1, x_2, \ldots$:

(Slide 21)

(P4) For all n, it takes Achilles a positive amount T_n of time to get from x_n to x_{n+1}.

(P5) If P4 is the case, then for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time.

(P6) If for all n, it takes Achilles $T_0+\ldots+T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time,then the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time.

(P7) If the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time, then Achilles never actually passes all of $x_0, x_1, x_2, \ldots$

(P8) If Achilles never actually passes all of $x_0, x_1, x_2, \ldots$, then he is always somewhere to the left of at least one of $x_1, x_2, \ldots$

Taking these together with premise 9 entails the conclusion:

(Slide 22)

(P9) If Achilles does not overtake the tortoise at any of $x_1, x_2, \ldots$ nor at any point in between either of them, and also he is always somewhere to the left of at least one of $x_1, x_2, \ldots$, then Achilles never overtakes the tortoise.

Therefore:

(C) Achilles never overtakes the tortoise.

Our commonsense assumptions concerning apparent motion, supplemented by a little bit of reasoning, seem to justify the claim that Achilles never overtakes the tortoise. That is an absurd conclusion. In any case, it is in conflict with another of our commonsense assumptions concerning apparent motion, that is, that Achilles should be able to overtake the tortoise in this hypothetical situation.

So something has gone wrong: since the logic of the argument is fine, one or several of the premises must actually be false, and this is what Zeno wants us to conclude. Our apparent experience of change is merely apparent: really there is no change, and hence no movement at all; the premises that are based on this illusion are false, and maybe our whole conception of change is incoherent.

Quiz 02:
Please check for yourself: First of all, did you understand the logic of the argument? Reconsider the structure of the premises, and how the premises jointly entail the conclusion. Secondly: What do you think about the argument? Do you find the conclusion absurd? If so: which of the premises seem fishy to you? When I turn to my assessment of the argument in the next part of the lecture, please compare your own findings with mine.

1.4 Calculus to the Rescue (10:10)

Now back to our own subject matter: where do the mathematical methods enter the picture? How can mathematics help us to improve philosophical argumentation in this case?

Once the mathematical philosopher has stated a philosophical argument clearly enough in order to judge it, and if she has already determined the argument to be logically valid, she subjects its premises to mathematical scrutiny. In the case of Zeno's argument, Premise 6 is the one that she should be suspicious of:

(Slide 23)

> (P6) If for all n, it takes Achilles $T_0+\ldots+T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0,\ldots,T_n$ is a positive amount of time, then the amount of time $T_0+T_1+T_2+\ldots$ in which Achilles is to pass each of $x_0,x_1,x_2,\ldots$ is not bounded by any finite amount of time.

In the original texts on the paradox this premise is not stated explicitly; in fact, hardly any of the premises above are stated explicitly, rather all of them need to be extracted from the very brief story that we are told. But it is difficult to see how Zeno's argument is meant to go through without anything like Premise 6 above.

Let us put this premise to the test now.

Say, T_0 is half a minute: so it takes Achilles half a minute to move from x_0 to x_1; furthermore, assume T_1 to be a quarter of a minute: it takes Achilles a quarter of a minute to move from x_1 to x_2; and so on:

(Slide 24)

$$T_0 = \frac{1}{2}$$

$$T_1 = \frac{1}{4}$$

$$T_2 = \frac{1}{8}$$

$$\vdots$$

Generally: Let T_n be $\frac{1}{2^{n+1}}$ of a minute:

(Slide 25)

$$T_0 = \frac{1}{2}$$

$$T_1 = \frac{1}{4}$$

$$T_2 = \frac{1}{8}$$

$$\vdots$$

$$T_n = \frac{1}{2^{n+1}}$$

If one thinks about it a little bit, this entails that we are assuming Achilles to be twice as fast as the tortoise. A conservative assumption, that is, since presumably Achilles is much faster than just twice as fast as the tortoise; but never mind.

More importantly, you might think: hey – none of these additional assumptions were part of Zeno's story. Why are we allowed to invent them? And aren't we all of a sudden changing the subject matter, because instead of talking about space and time we are talking about numbers now?

But that's not the right way of thinking about it. First of all, although we did use the familiar '+' sign when we formulated Premise 6, all we meant with this initially was that amounts of time are put together somehow. There was no need as yet to interpret the '+' as a sum of numbers. What Zeno does need is a way of getting from the tortoise being ahead of Achilles at each point x_n to the conclusion that the tortoise will be ahead of Achilles forever. And it is hard to see how Zeno could get that without "putting together" periods of time, and in fact infinitely many of them, to the effect that these added up periods would become arbitrarily large. But that means Zeno had to presuppose Premise 6, or something like it, to be plausible.

Now about the numbers: presently we are putting Premise 6 to the test. Our method of doing so is by building a little mathematical model of the situation that Zeno is describing, and in which we can then determine whether Premise 6 is true in it. For that purpose we are now measuring amounts of time by means of numbers; or, since only the numbers will be relevant in our model, we might just as well identify amounts of time with the numbers. Accordingly, the "putting together" of non-overlapping periods of time gets identified with summing up their corresponding numbers. Why are we allowed to do so? Because we are describing a possible way in which the race might have proceeded – Achilles might have been precisely twice as fast as the tortoise – and we are describing it in the same manner that we would find plausible also in real-life races: by measuring times by numbers. If we had good reasons for not believing it to be plausible to measure time like that, our toy model would not be a good one; but that does not seem to be so.

Now back to Premise 6: what our formal toy model tells us is that we do not have good reasons to believe it to be true. One might have thought that adding up infinitely many positive numbers must lead to arbitrarily large sums of such numbers, but that is not necessarily so. For consider the amount of time $T_0 + T_1 + \ldots + T_n$ that it takes Achilles to get from x_0 to x_{n+1} in our little mathematical model: $T_0 + T_1 + \ldots + T_n$ is nothing but $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n+1}}$:

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$$T_0 + T_1 + \ldots + T_n \;=\; \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n+1}}$$

Now, it is easy to prove mathematically that:

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$$T_0 + T_1 + \ldots + T_n \; = \; \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n+1}} \; < \; 1$$

however large the n: whether 1 or 2 or 3 or any other positive integer. So the corresponding sum does not become arbitrarily large but is always bounded by 1. In other words: whatever the n, Achilles reaches x_n in less than one minute.

We do not state the proof here, but geometrically one can see that if we view these numbers as areas of regions in a square with side length 1, then they add up to less than an area of 1, whatever the n (see Figure 1.6).

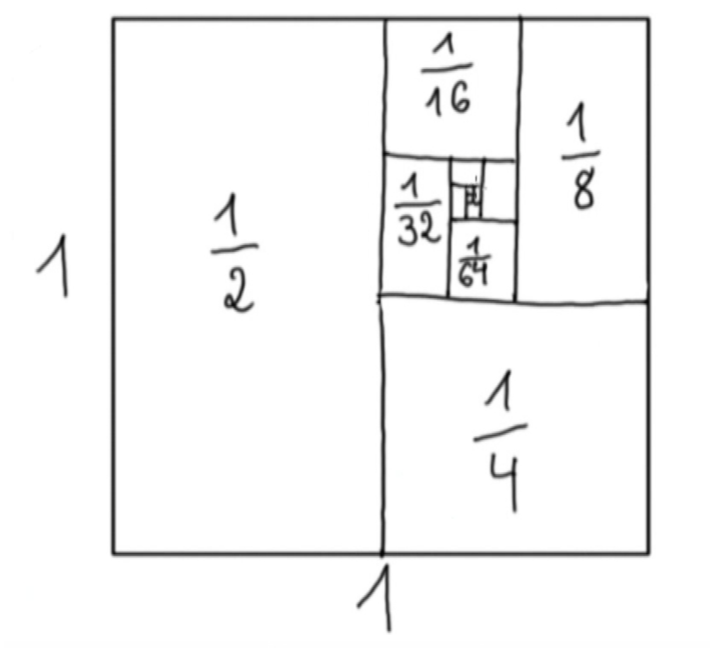

Figure 1.6: Infinite series

By a standard theorem in the calculus of real numbers, this implies that the whole infinite series $T_0 + T_1 + T_2 + \ldots$, that is, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ converges to a real number. In our case, that number can be shown to be exactly 1:

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$$T_0 + T_1 + T_2 + \ldots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$$

This means that in our little mathematical toy model it takes Achilles precisely 1 minute to catch the tortoise. And this is so in spite of the fact that he needs to traverse infinitely many segments of space: the reason is that from one segment to the next the distances that he needs to cross become much smaller, and the time that Achilles needs to pass them becomes much shorter. So much shorter that overall these periods of time add up to a finite number: 1 minute.

All of this can be made perfectly exact: the only instrument that one needs to do so is the calculus, or what is also called real number analysis – one of the classic fields of modern mathematics in which concepts such as infinite sum, convergence, and limit are made precise, and in which various theorems that involve these concepts are proven rigorously.

> Remark on infinite sums: If you want to know more about the formal details of all of that, please consult virtually any web entry or lecture notes or textbook on the calculus – on the area that is often called 'analysis (of real numbers)' in mathematics. The mathematical key terms to look for are: 'infinite sequence'; 'infinite series'; 'convergence' and 'limit'; and 'geometric series' (for our series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is a simple example of a so-called geometric series).

This shows us that, as things stand, Premise 6 of Zeno's argument is not plausible after all.

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> (P6) If for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time, then the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time.

As things stand, Zeno's argument does not force us to question our commonsensical assumptions on movement: instead we ought to reject one of the premises that he seems to have taken for granted, that is, that putting together infinitely many positive amounts of time must lead to arbitrarily large amounts of time. Because that's just not true. Without that premise, Zeno's argument does not go through anymore: the argument from premises 1-5 and 7-9 to the conclusion is not logically valid anymore.

Now, this does not mean that Zeno could not have come back to us with a different argument by which, perhaps, he would have rationally convinced us of the illusion of movement after all – this is not excluded by anything that we said. For instance, he might have replaced Premise 6 above by a couple of new premises, so that the resulting argument would be logically valid again, and where the new premises would be more stably plausible. The new premises might e.g. show that one cannot actually measure amounts of time in terms of numbers, or that one cannot add up amounts of time in the same way as we are adding up numbers, all of which would prohibit us from invoking our toy model from before, or the like. But as things stand we have no indication of any of this. We have done our best to state Zeno's argument in clear and precise terms, and we found that the resulting argument should not convince us of the nonexistence of movement. That's all.

None of this is a criticism of Zeno either: his paradox is wonderfully creative and stimulating, and he did not know about the later mathematical progress on the treatment of infinite sums. But we are still perfectly allowed to criticize Zeno's argument; in fact, being the great philosopher that he was, that's exactly what he would have wanted us to do. It is modern mathematics which prevents us from taking one of the premises of Zeno's argument to be justified; and in this way, mathematics helps us to avoid drawing the wrong philosophical conclusions. That is one reason why it can be very useful to apply mathematics when we do philosophy.

Quiz 03:
(1) Try out the following: Let $t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ Hence $\frac{t}{2} = \frac{1}{4} + \frac{1}{8} + \ldots$. Now subtract $\frac{t}{2}$ from t, and try to determine the value of 't' in this way.
(2) How does it follow in our toy model that Achilles is twice as fast as the tortoise?
Solution

1.5 Sets (04:39)

We have seen that mathematics can help us avoid drawing the wrong philosophical conclusions from Zeno's argument: an argument that invited us to consider infinite sequences of points and amounts of time. More precisely: an argument that invited us to consider points and amounts of time which one could enumerate in terms of the familiar sequence of natural numbers, the non-negative integers:

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Natural numbers: $0, 1, 2, 3, \ldots$

And clearly it is good to avoid such errors. But by this we have not yet gained much insight into the concept of infinity itself: it is one thing not to draw wrong conclusions about certain infinite sequences; it is another thing to understand what we even mean when we call something 'infinite'.

As I said before, the set of all my kids is finite: it has exactly 2 members. The set of God's positive properties is, perhaps, infinite; and the set of natural numbers is definitely infinite. That's all very well: but what do we mean by that? And even prior to that: what do we mean by 'set' here, as in: the 'set' of natural numbers is infinite? Let us turn to that question first.

As a first approximation, think of a set as a list of items. The members of the set are the items on the list, and the set is the list itself. The only difference between actual lists and sets is: lists are linguistic objects, something that we write up, while sets are not; and when we deal with a set it is only meant to be important what is a member of the set and what is not – what is on the list and what is not. For instance,

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$$\{2, 1, 3\} = \{1, 2, 3\}$$

The order in which we list members of a set is irrelevant. It is only relevant that 1, 2, and 3 are members of the set, and nothing else is. As you can see, we use these curly brackets to denote sets; the objects mentioned in between the curly brackets are the members of the set.

Accordingly:

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$$\{1, 2, 2, 3\} = \{1, 2, 3\}$$

It does not matter if we mention a member of a set twice or more often: it is only important what is a member of the set and what is not.

By the way, also

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$$\{\text{Hannes Leitgeb}\}$$

is a set. It is a set with just one member, that is, myself; so we can form sets not just of numbers but of any kind of objects really, and a set does not need to have more than one member either. Furthermore, it holds that, e.g.,

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$$\{\text{Hannes Leitgeb}\} = \{\text{the philosopher who is giving this lecture}\}$$

since the set denoted on the left has precisely the same members as the set denoted on the right.

We may even accept the existence of the empty set:

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$$\{\}$$

Just think of a list without any entries. Accordingly, the empty set is a set without members; and it is the only set without any members, because sets are completely determined by what their members and non-members are, and hence if some sets X and Y both have no members at all, since they share precisely the same members and non-members, they must be identical.

Remark on the empty set: More usually in mathematics, the empty set is denoted by '$\varnothing$', that is, by a crossed 'O'. But we will stick to the more self-explanatory '$\{\}$' in the following.

More generally, identities and differences of sets are meant to be governed by the following principle, which is called the Principle of Extensionality:

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Principle of Extensionality:

> For all sets X, Y: $X = Y$ if and only if for all z it holds: z is a member of X if and only if z is a member of Y.

("Two sets are identical if and only if they have the same members.")

'If and only if' means the same as: is equivalent to. When one states an equivalence like that, one wants to say: if what is stated to the left of the 'if and only if' is true, then what is stated to the right of it is true as well, and the other way around. The left-hand side and the right-hand side are equivalent.

Extensionality is just the more precise way of expressing what I meant before when I said: it is only important for a set what is a member of it and what is not.

Remark on membership: The membership relation for sets is usually denoted by means of a formal symbol that looks like the Greek letter epsilon: '$\in$'. But, as I said before, I will avoid symbols in the lectures wherever I can.

Quiz 04:
(1) Is the set $\{8, 3, 5, 5\}$ identical to the set $\{5, 3, 8, 8, 5\}$?
(2) Is the set $\{1, 2, 3, 4\}$ identical to the set $\{1, 4\}$?
Solution

1.6 Comparing Sets in Terms of Size (07:28)

Next, we want to compare sets in terms of their sizes; for instance, we might want to say ultimately that a finite set is somehow smaller in size than some infinite set. But there are different ways of comparing sets to this effect. For instance: we might say that sets are subsets of some other sets. Like:

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$\{1\}$ is a subset of $\{1, 2\}$.
$\{1, 2\}$ is a subset of $\{1, 2, 3\}$.

The idea is:

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Subsets:

> For all sets X, Y: X is a subset of Y if and only if for all z it holds: if z is a member of X, then z is a member of Y.

("One set is a subset of another if and only if every member of the former set is also a member of the latter set.")

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$\{1, 2\}$ is a subset of $\{1, 2, 3\}$.

Indeed, 1 and 2 are the only members of the set $\{1, 2\}$, and both of them are also members of the set $\{1, 2, 3\}$; therefore, $\{1, 2\}$ is a subset of $\{1, 2, 3\}$.

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$\{1, 2, 3\}$ is not a subset of $\{1, 2\}$.

On the other hand, $\{1, 2, 3\}$ is not a subset of $\{1, 2\}$, since not every member of $\{1, 2, 3\}$ is a member of $\{1, 2\}$: indeed 3 is a member of the set $\{1, 2, 3\}$, but not of the set $\{1, 2\}$.

We can summarize this example by pointing out that

(Slide 40)

$\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$.

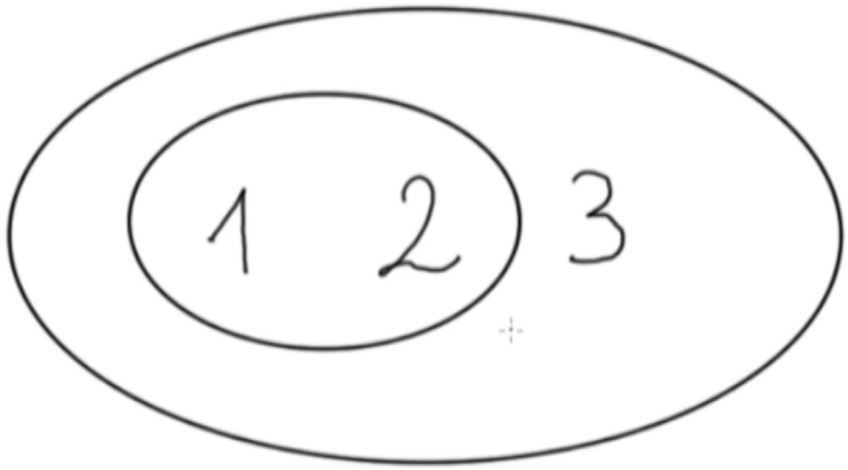

Figure 1.7: Proper subset relation

In general terms:

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Proper Subsets:

> For all sets X, Y: X is a proper subset of Y if and only if X is a subset of Y, but Y is not a subset of X.

("One set is a proper subset of another if and only if the former is a subset of the latter, but not the other way around.")

So it holds, for example, that every set is a subset of itself, but no set is a proper subset of itself. Alright?

Remark on the subset relation and on the proper-subset relation: The usual manner of denoting the proper-subset relation in mathematics is in terms of a symbol that looks like a 'U' rotated to the right by 90 degrees: '$\subset$'. The symbol for the subset relation is the same one except that a horizontal line is added: '$\subseteq$'.

We have just compared sets by means of the proper-subset relation; intuitively, in some sense, if a set X is a proper subset of a set Y, then X is smaller than Y.

Now here is another way of comparing sets in terms of size: by pairing off their members.

Pairing off means here that there is a way of correlating the members of one set with the members of another set so that one does not forget about any member of either set, and each member of the one set is correlated with exactly one member of the other.

Remark on 'pairing off': In order to make this fully precise, one would first have to define set theoretically what an ordered pair is, then what a relation is, then what a function (a mapping) is, after which one would define what a one-to-one function, what an onto function, and finally what a bijective function is. Our "pairing-offs" coincide with bijective functions. More on this can be found on every website, in all lecture notes, and in every text book on basic set theory.

For instance:

There is a pairing off between the set $\{1, 2\}$ and the set {Hannes Leitgeb, Stephan Hartmann} (see Figure 1.8).

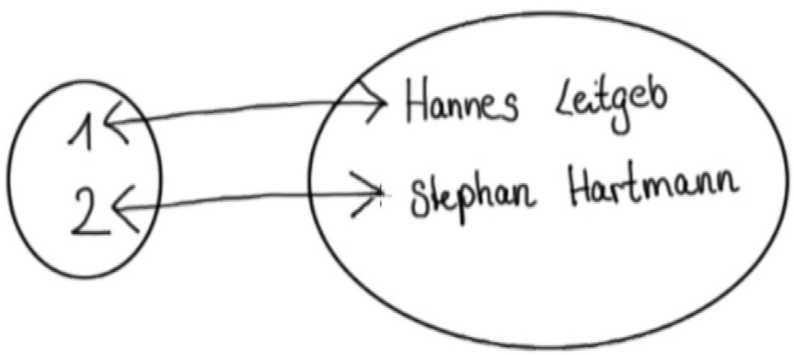

Figure 1.8: Example 1 pairing off

This is a different manner of pairing off the members of these sets (see Figure 1.9).

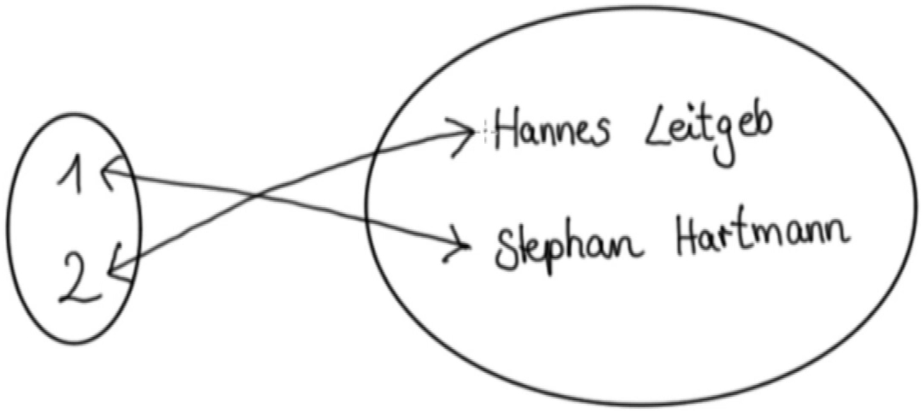

Figure 1.9: Example 2 pairing off

On the other hand, there is no way at all of pairing off the members of these two sets: the set $\{1, 2\}$ and the set {Hannes Leitgeb, Stephan Hartmann, Zeno} (see Figure 1.10).

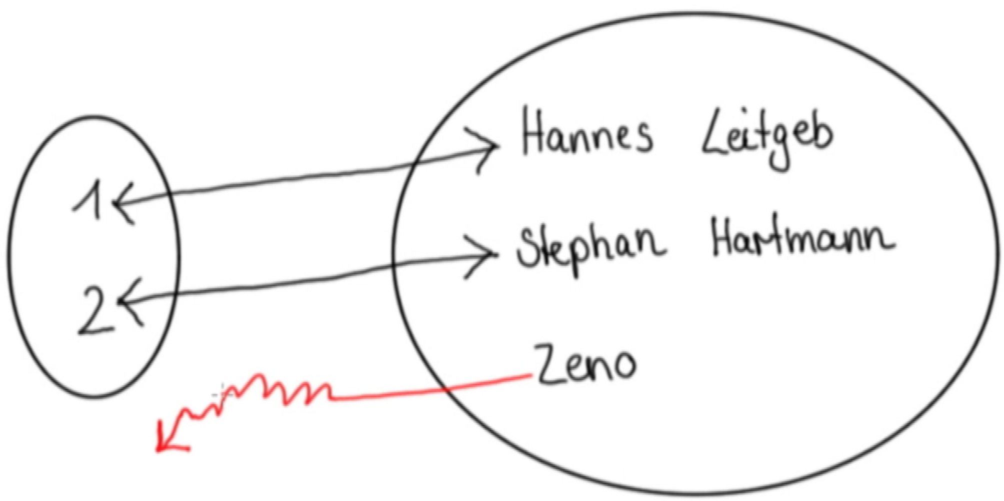

Figure 1.10: Pairing off doesn't work

However one tries to correlate the members of the one set with the members of the other, some member of the smaller set would have to be correlated with more than one member of the other set, which is not what we mean when we say ‘pairing’.

Why is there a pairing between $\{1, 2\}$ and {Hannes Leitgeb, Stephan Hartmann}, while there is none between $\{1, 2\}$ and {Hannes Leitgeb, Stephan Hartmann, Zeno}? Because $\{1, 2\}$ and {Hannes Leitgeb, Stephan Hartmann} are of equal size, while {Hannes Leitgeb, Stephan Hartmann, Zeno} is larger than $\{1, 2\}$: it has more members.

We have introduced two ways of comparing sets with each other: sets can be subsets, and indeed proper subsets, of other sets; and the members of sets can be paired off with members of other sets. We can use either of these concepts in order to get a better grasp of when a set has equally many members as another set and when this is not the case. But we need to be very careful in doing so, or we might end up in paradox again.

The first one who actually realized this was Galileo Galilei. Yes, I mean the famous physicist from the 16th and 17th century that you all know about.

Remark on Galileo: More on Galileo can be found at http://plato.stanford.edu/entries/galileo/.

In a fictitious dialogue, Galileo presented a paradoxical argument about infinite sets. In the following, I present a slight variant of Galileo's original argument:

Remark on Galileo's Paradox: If you want to know more about Galileo's original text, take a look here: http://en.wikipedia.org/wiki/Galileo's_paradox. Note that Galileo takes squares of numbers where we take doubles of numbers.

These are the premises:

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> (P1) If X is a proper subset of Y, then X does not have equally many members as Y.

This seems plausible, if one thinks of our previous example:

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$\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$.

Indeed: $\{1, 2\}$ does not have equally many members as $\{1, 2, 3\}$.

And indeed $\{1, 2\}$ does have less members than 1,2,3; so $\{1, 2\}$ does not have equally members as $\{1, 2, 3\}$.

Also the following premise looks just fine:

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> (P2) If there is a pairing off between the members of X and the members of Y, then X has equally many members as Y.

Just reconsider our previous example, the pairing off between the set $\{1, 2\}$ and the set {Hannes Leitgeb, Stephan Hartmann} (see Figure 1.11).

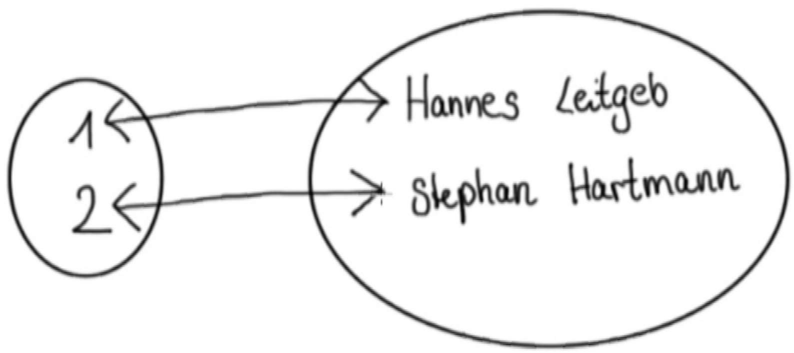

Figure 1.11: Example 1 pairing off

Once again, $\{1, 2\}$ does in fact have equally many members as {Hannes Leitgeb, Stephan Hartmann}.

The next premise is clearly true:

(Slide 46)

> (P3) The set of even natural numbers is a proper subset of the set of positive natural numbers.

$\{2, 4, 6, \ldots\}$ is a proper subset of $\{1, 2, 3, 4, 5, 6, \ldots\}$.

After all, every even natural number is also a positive natural number where by 'positive' we mean here simply that we are excluding zero, and that is just for convenience.

But also the following is the case:

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> (P4) There is a pairing off between the even natural numbers and the positive natural numbers.

This can be seen by correlating each even natural number with its half, or, the other way around, each positive natural number with its double (see Figure 1.12).

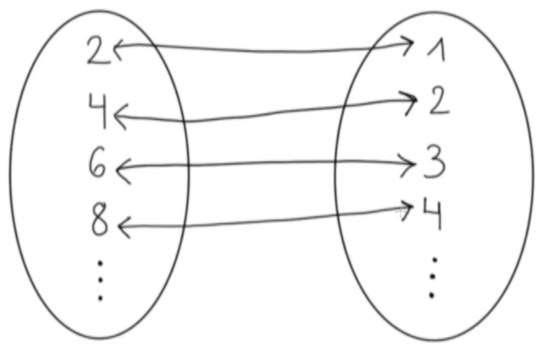

Figure 1.12: Example 3 pairing off

But from all of these premises taken together we may logically conclude (C):

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> (P1) If X is a proper subset of Y, then X does not have equally many members as Y.
>
> (P2) If there is a pairing off between the members of X and the members of Y, then X has equally many members as Y.
>
> (P3) The set of even natural numbers is a proper subset of the set of positive natural numbers.
>
> (P4) There is a pairing off between the even natural numbers and the positive natural numbers.
>
> ---
>
> (C) The set of even natural numbers does not have equally many members as the set of positive natural numbers, and the set of even natural numbers does have equally many members as the set of positive natural numbers.

For by premises 1 and 3, we get that the set of even natural numbers does not have equally many members as the set of natural numbers; and from premises 2 and 4 we can derive that the set of even natural numbers does have equally many members as the set of positive natural numbers.

This is a logically valid argument in which each premise is plausible if taken by itself, but the conclusion of which is absurd; in fact, the conclusion is even a logical contradiction.

Galileo hit upon a paradox again. Just as in the case of Zeno's paradox before, some premise needs to go. But which one, or, which ones?

Quiz 05:
(1) Is $\{1, 3, 5\}$ a subset of $\{8, 1, 5, 3\}$? Is it a proper subset?
(2) Show that there is a pairing off between the set $\{1, 2, 3, 4, ...\}$ of positive integers and the set $\{3, 6, 9, 12, ...\}$.
Solution

1.7 Diagnosis of Galileo's Paradox (09:56)

Galileo's own diagnosis seemed to be that, unlike finite sets, infinite sets just cannot be compared in terms of size. Upon reflection, it simply does not make sense to say of an infinite set that it has equally members as another one, nor that this would not be so. One should say of infinite sets that they are infinite and leave things at that; the concepts of equal or greater size should be restricted to finite sets only.

(Slide 48)

> (P1) If X is a proper subset of Y, then X does not have equally many members as Y.
>
> (P2) If there is a pairing off between the members of X and the members of Y, then X has equally many members as Y.
>
> (P3) The set of even natural numbers is a proper subset of the set of positive natural numbers.
>
> (P4) There is a pairing off between the even natural numbers and the positive natural numbers.
>
> ---
>
> (C) The set of even natural numbers does not have equally many members as the set of positive natural numbers, and the set of even natural numbers does have equally many members as the set of positive natural numbers.

Hence, according to Galileo, the right response to his paradox would be to reject premises 1 and 2 as false or meaningless if applied to infinite sets X and Y.

But that is not the only possible reaction. Here is a different response: questioning the absurdity of the conclusion.

Perhaps, in one sense of the term, the set of even natural numbers does not have equally many members as the set of positive natural numbers, while in a different sense, the set of even natural numbers does have equally many members as the set of positive natural numbers. The two statements might sound as if they could not both be true, but if what is meant by ‘equally many’ in each of them is very different, then maybe they are not in conflict after all. In this case, the argument should really be stated like this:

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> (P1*) If X is a proper subset of Y, then X does not have equally$_1$ many members as Y.
>
> (P2*) If there is a pairing off between the members of X and the members of Y, then X has equally$_2$ many members as Y.
>
> (P3) The set of even natural numbers is a proper subset of the set of positive natural numbers.
>
> (P4) There is a pairing off between the even natural numbers and the positive natural numbers.
>
> ---
>
> (C*) The set of even natural numbers does not have equally$_1$ many members as the set of positive natural numbers, and the set of even natural numbers does have equally$_2$ many members as the set of positive natural numbers.

Focus on the first two premises and the conclusion.

If there are two senses of 'having equally many members as' in play here, then the conclusion is not actually a logical contradiction. It's like saying: 'That building is a bank, but it is not a bank.' This might sound like a contradiction, but if it means 'That building is an institution in which money is kept, but it is not a slope beside the river', there is no contradiction anymore.

If so, the situation really is this: as long as we are dealing with finite sets, we understand very well what it means to count its number of members; and from this we know what it means to say that a finite set has equally many members as another one. But now we want to extrapolate from the finite to the infinite: we want to extend our concept of size also to infinite sets, and it turns out that there is more than one way of achieving this. If we follow one path, the set of even natural numbers turns out not to have equally members as the set of positive natural numbers; that's equality$_1$. But if we follow another path, the set of even natural numbers does have equally many members as the set of positive natural numbers; that's equality$_2$. Rather than ending up in an incoherent situation, as Galileo seemed to think, we end up with two ways of extending our concept of size from the finite to the infinite. As soon as we turn to the infinite, neither of the two ways preserves everything that we think is true about the finite; and Galileo's paradox shows that we could not preserve everything without running into a contradiction. But each of the two ways of extending our concept does preserve something that we held true about finite sets: the one concept preserves Premise 1, which is now supposed to hold for all sets, whether finite or infinite, while the other concept preserves Premise 2, which is now supposed to hold for all sets, whether finite or infinite.

But maybe we can do better than that: maybe we can show that one of the two concepts is in fact "better" than the other one; perhaps one ought to be preferred as extrapolation of the concept of size to the infinite case. Say, one of the concepts is such that a precise, consistent, and systematic theory of size for sets can be built around it; where the theory is enormously fruitful in the sense that lots of theorems about size for infinite sets can be derived from it, which also throw new light also on other topics in mathematics, science, and philosophy; and where the underlying concept of size is still simple and illuminating. And assume that this is all true of the one concept, but it is not true of the other, or at least not to the same extent, then we would have good reasons for preferring the former concept over the latter.

This is exactly what we find to be the case here. First of all, Premise 1^* only tells us something negative about equality$_1$: it only tells us when it is not the case that sets are of equal size in that sense of the word. On the other hand, Premise 2^* has something positive to say about equality$_2$. Accordingly, it is easy to strengthen Premise 2^* into a definition of equal size for sets in general:

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Equal Size (Equinumerosity, Equipollence):

> For all sets X, Y: X has equally$_2$ many members as Y if and only if there is a pairing off between the members of X and those of Y.

Equivalently, one often says in this case: X is equinumerous to Y. Or X and Y are equipollent.

In contrast, it is not clear at all how to state a definition of equality$_1$. For instance, we would not want to define X to have equally$_1$ many members as Y if and only if X is a subset of Y and Y is a subset of X: according to that definition not even $\{1, 2\}$ and {Hannes Leitgeb, Stephan Hartmann} would be equal in size. Indeed, X would have equally$_1$ many members as Y if and only if $X = Y$, which is not right.

One might think: perhaps equality$_2$ is good for telling us what is the case when two sets are of the same size, but it does not tell us much what is the case when two sets are not of the same size. Of course, by our definition of equality$_2$, it is clear that X does not have equally$_2$ many members as Y precisely when there is no pairing off between the members of the two sets. But that does not seem to be very informative: in particular, which of the two, X or Y, is then smaller in size, if any of them? Perhaps there is no suitable notion of smaller size at all that would accompany equality$_2$?

It turns out that we need not worry: just define

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Smaller Size:

> For all sets X, Y: X has less$_2$ members than Y if and only if X has equally$_2$ many members as a proper subset of Y, but X does not have equally$_2$ many members as Y.

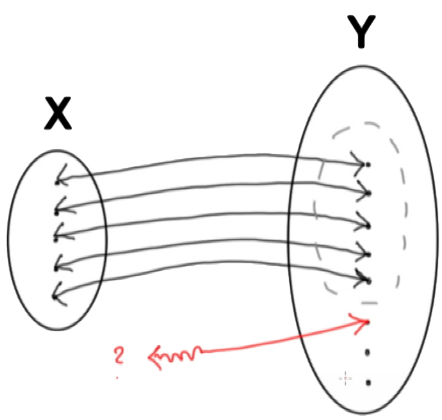

Figure 1.13: X has less$_2$ members than Y

For instance, $\{1, 2\}$ has less$_2$ members than $\{1, 2, 3\}$; as the required proper subset of $\{1, 2, 3\}$ one may take $\{1, 2\}$ itself; on the other hand, once again, the members of $\{1, 2\}$ cannot be paired off with the members of $\{1, 2, 3\}$.

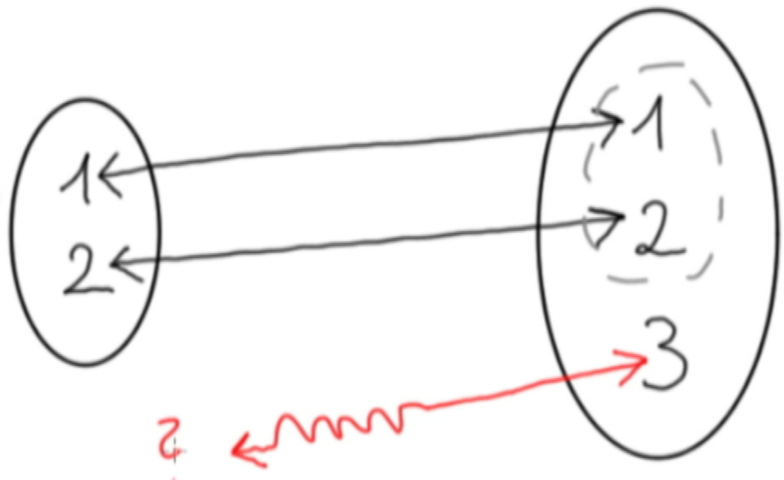

Figure 1.14: $\{1, 2\}$ has less_2 members than $\{1, 2, 3\}$

For analogous reasons, $\{1, 2, 3\}$ turns out to have less_2 members than the set of positive natural numbers, just as it should be the case.

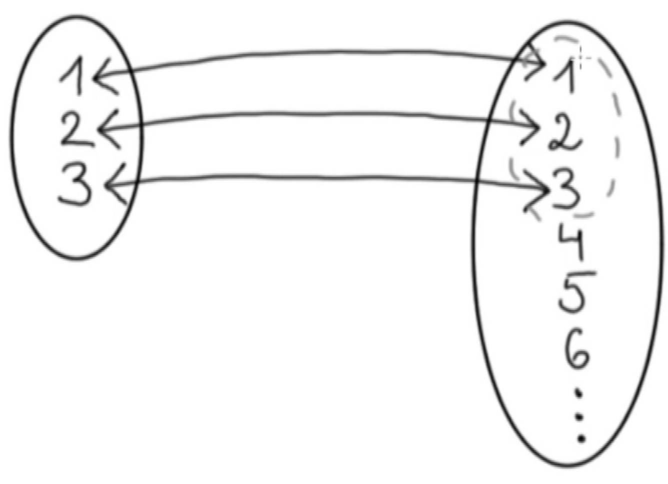

Figure 1.15: $\{1, 2, 3\}$ has less_2 members than the set of positive natural numbers

But there is more: it follows from the principles of modern set theory that the following is the case:

(Slide 52)

For all sets X and Y: Either X has less_2 many members than Y, or X has equally_2 many members as Y, or Y has less_2 many members than X.

This is just like for natural numbers m and n, where we have: either $m<n$, or $m = n$, or $n<m$.

It is not clear at all how anything like that could be derived for equality_1 and a corresponding notion of less_1-many-members on the basis of some reasonably simple definition of these terms.

Finally, it is very easy now to use equality_2 in order to define what it means to call a set infinite:

(Slide 53)

Infinity and Finiteness:

> For all sets X: X is infinite_2 if and only if X has equally_2 many members as a proper subset of X.

And we can add:

(Slide 54)

X is finite_2 if and only if X is not infinite_2.

For instance:

(Slide 55)

$\{1, 2\}$ is finite_2.

There is no pairing off between $\{1, 2\}$ and any of its proper subsets.

On the other hand, the truth of Premise 4 in our argument from above amounts to the set of positive natural numbers being infinite, as intended.

(Slide 56)

$\{1, 2, 3, 4, 5, 6, \ldots\}$ is infinite_2.

By means of this concept of same-size_2 we have finally been able to define infinity for sets in a simple and illuminating manner: infinity amounts to a kind of self-similarity property, much as with fractals in geometry – an infinite set contains something which, in terms of size, looks like itself.

Quiz 06:
(1) Assume that we would define X to have equally$_1$ many members as Y if and only if X is a subset of Y and Y is a subset of X. Use the Principle of Extensionality for sets in order to show that this would entail: X has equally$_1$ many members as Y if and only if $X = Y$.
(2) Say, we would have defined 'less$_2$' in this slightly different manner:
X has less$_2$ members than Y if and only if X has equally$_2$ many members as a subset of Y, but X does not have equally$_2$ many members as Y.
(So we say 'subset' now where in the original definition we had said 'proper subset'.)
Would that change anything regarding which sets are having less$_2$ members than which other sets?
Solution

1.8 Cantor's Theorem (11:52)

One can still say more: As the development of modern set theory in the late 19th century and in the first half of the 20th century shows, a precise, elegant, systematic, and enormously fruitful theory of sizes for infinite sets can be built upon these notions of equal$_2$, less$_2$, and infinite$_2$. Indeed, today these terms are simply taken to determine the right notions of size; no need to attach indices to them in order to signal that alternative notions of size would also be available.

It was one person who more or less single-handedly discovered this theory: the German mathematician Georg Cantor. However, it should be said that the definition of infinity is due to another German mathematician, Richard Dedekind. And both the theory and the definition of infinity are now part of what is called axiomatic set theory, which may be regarded as the foundation of all of modern mathematics, and which, by the way, we have all reasons to believe consistent.

Remark on early set theory: For more on the early history of set theory, see `http://plato.stanford.edu/entries/settheory-early/`.

Now here is something that I hope is bothering you: you might like the definition of infinity; and you might have begun to accept the concept of equal-size on which it is based. But what's the big deal with the concept of 'has less members than' that was defined before?

For you might think to yourself: o.k., the set $\{1, 2\}$ has less members than $\{1, 2, 3\}$ (see

Figure 1.14 again). In turn, $\{1, 2, 3\}$ has less members than the set of positive natural numbers, or the set of all natural numbers, for that matter (see Figure 1.15 again). And the set of positive natural numbers has equally many members as the set of even natural numbers (see Figure 1.12 again), and so on. But that suggests the following simple picture: finite sets can be compared in size just as normal; all the infinite sets have more members than all the finite ones; and all the infinite sets are equal in size to each other. And so we could have just said that instead of bothering to analyze the concept 'has less members than' in much detail.

Well: Wrong!

As Georg Cantor showed by means of various mathematical theorems, the landscape of infinity is much more complex and fascinating than that.

Most importantly: there are infinities of different sizes! Here is an example:

(Slide 57)

Theorem:
The set of natural numbers has less many members than the set of real numbers.

Since both of these sets are infinite, this shows that there exist infinite sets X and Y, such that X has less many members than Y.

By the set of real numbers, I mean here the set of all numbers that one can denote by decimal expansions in the usual manner, such as:

(Slide 58)

The real numbers: $0, 1, \frac{1}{2}, -\frac{1}{3}, -[18233 + 16/17], \sqrt{2}, \pi, \ldots$

$$
\begin{array}{ccrl}
0 & = & & 0.000000000000000\ldots \\
1 & = & & 1.000000000000000\ldots \\
\frac{1}{2} & = & & 0.500000000000000\ldots \\
-\frac{1}{3} & = & - & 0.333333333333333\ldots \\
-[18233 + 16/17] & = & - & 18233.94117647058\ldots \\
\sqrt{2} & = & & 1.414213562373095\ldots \\
\pi & = & & 3.141592653589793\ldots
\end{array}
$$

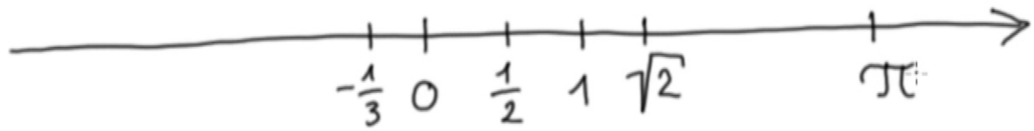

Figure 1.16: Real number line

Or, alternatively, one can also think of the real numbers geometrically as the points on the real number line. Anyway.

Let me now try to convey the main idea behind the proof of Cantor's theorem above; note that I will suppress a couple of details that one needs to fill in if one wants to state the proof in completely precise terms – issues to do, e.g., with the fact that the real number $0.99999\ldots$ is actually identical to the real number $1.000\ldots$, and the like; think about that, if you are interested, but I won't mind for now:

O.k.: by the definition of 'has less members than', we need to show the following in order to prove the theorem:

(Slide 59)

1. The set of natural numbers has equally many members as a proper subset of the set of real numbers,
2. but the set of natural numbers does not have equally many members as the set of real numbers itself.

The first part is obvious, since the set of natural numbers is itself a proper subset of the set of real numbers, so correlate:

(Slide 60)

$$\begin{array}{ccc} 0 & \leftrightarrow & 0.000000000000000\ldots \\ 1 & \leftrightarrow & 1.000000000000000\ldots \\ 2 & \leftrightarrow & 2.000000000000000\ldots \\ \vdots & & \vdots \end{array}$$

We can simply pair off the natural numbers with themselves.

So the tricky part is the second one. The way to prove it is what philosophers traditionally call a reductio ad absurdum, or what mathematicians often call an indirect proof – a method of proof that is actually related to Zeno's way of arguing. Let us assume for reductio that the set of natural numbers does have equally many members as the set of real numbers. From this assumption we will derive a logical contradiction. Hence the assumption cannot be true – so its negation must be true, that is: the set of natural numbers does not have equally many members as the set of real numbers. That is the logical structure of the proof.

Now let's carry this out: Assume for reductio that the set of natural numbers has equally many members as the set of real numbers. Therefore, by definition of 'has equally members as', there must be a pairing off between natural numbers and real numbers:

(Slide 61)

$$\begin{array}{ccc} 0 & \leftrightarrow & [\ldots].\, x_1^0\, x_2^0\, x_3^0 \ldots \\ 1 & \leftrightarrow & [\ldots].\, x_1^1\, x_2^1\, x_3^1 \ldots \\ 2 & \leftrightarrow & [\ldots].\, x_1^2\, x_2^2\, x_3^2 \ldots \\ \vdots & & \vdots \end{array}$$

x_n^m: the n-th decimal place of the real number paired off with m. E.g.: if $0 \quad \leftrightarrow \quad [\ldots].\,3\,3\,3\ldots$

then $x_1^0 = 3$, $x_2^0 = 3$, $x_2^0 = 3, \ldots$

Each natural number is paired off with precisely one real number, and the other way round. By the meaning of 'pairing off', the pairing does not miss any natural number, nor does it miss any real number.

By '[...]' I only mean that I will not actually care about the decimal expansion of our real numbers to the left of the point. Instead I will only care about their decimal expansion to the right of it. By the way, on the slides the point is coloured in red.

Now, when I say 'x_n^m' I mean: the n-th decimal place, that is, the n-th digit to the right of the point of the real number that is paired off with natural number m.

For instance: if 0 got say paired off above with $\frac{1}{3}$, that is, $0.33333\ldots$, then x_1^0 would be 3, x_2^0 would be 3, x_3^0 would be 3, and so on.

Now we have to prove that assuming the existence of a pairing off between natural numbers and real numbers implies a contradiction. This can be seen as follows: one shows that a real number exists that is missing in the pairing that we had assumed to exist. But that's a contradiction, since by definition a pairing does not miss any members of either of the sets involved.

Let me now describe a real number that must be missing in the pairing off above: I choose 0 to be its digit before the point. As the first digit after the point, take any digit that is distinct from x_1^0: since there are 10 available digits – from '0' to '9' – there is such a digit distinct from x_1^0. Let that digit be y_1 (see Figure 1.17). For instance, if x_1^0 is the digit '5', then y_1 could be '4', or '7', or the like – anything other than '5'. Similarly, as the second digit after the point use any digit that is distinct from x_2^1. Call it y_2 (see Figure 1.18). As the third digit after the point choose any digit that is distinct from x_3^2; call it y_3 (see Figure 1.19). And so on. In general terms: for all natural numbers n, let y_n be any digit other than x_n^{n-1}.

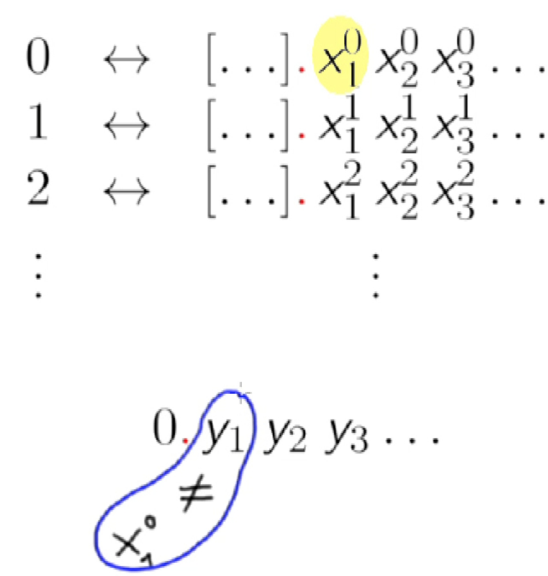

Figure 1.17: Digit y_1

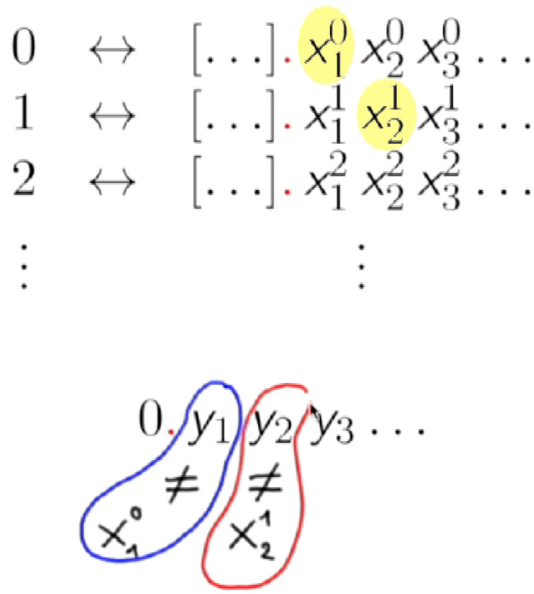

Figure 1.18: Digit y_2

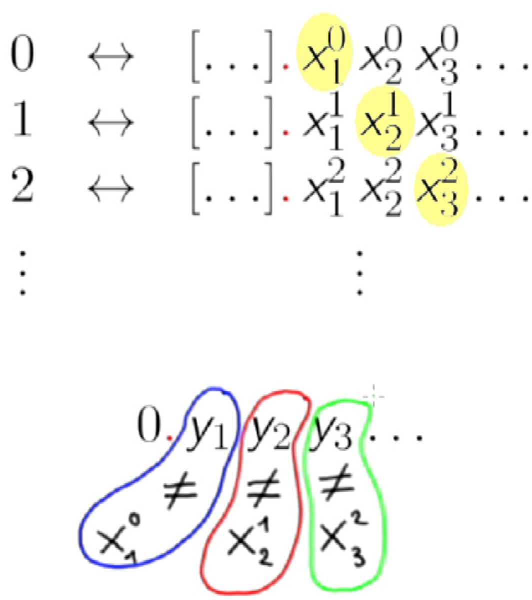

Figure 1.19: Digit y_3

The real number that we describe in this way is given by the decimal expansion:

$$0.\, y_1\, y_2\, y_3 \ldots$$

This is certainly a real number, since it is denoted by a decimal expansion in the standard manner. However, its decimal expansion was chosen so that it denotes a real number that was missing from the pairing which we assumed to exist before:

for $0.\, y_1\, y_2\, y_3 \ldots$ differs from the real number paired with 0 at digit x_1^0 (see Figure 1.20),

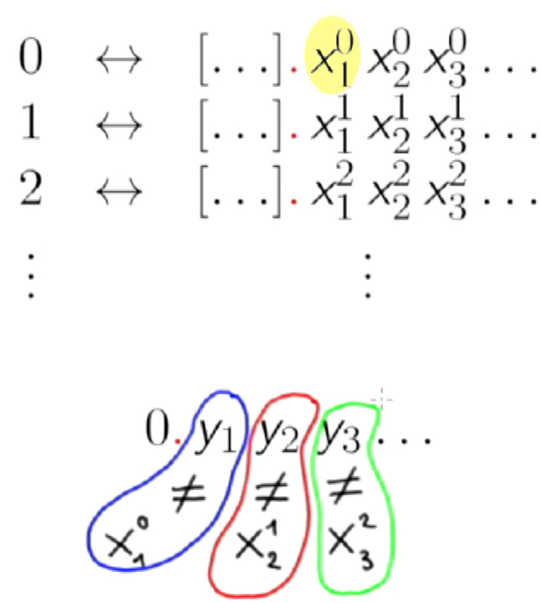

Figure 1.20: Difference 1

$0.\,y_1\,y_2\,y_3\ldots$ differs from the real number paired with 1 at digit x_2^1 (see Figure 1.21),

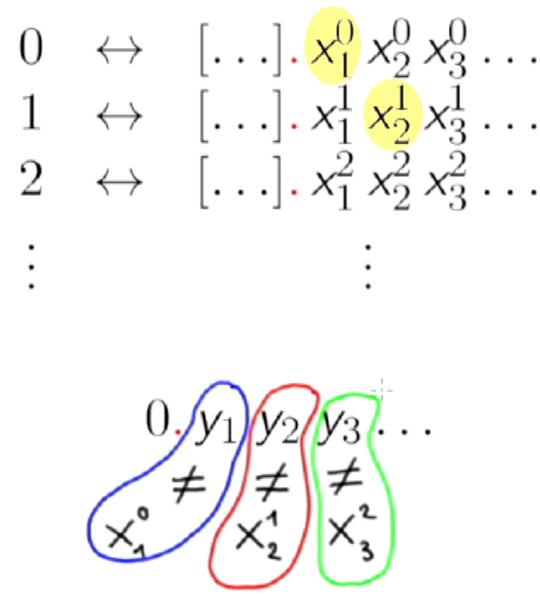

Figure 1.21: Difference 2

$0.\, y_1\, y_2\, y_3 \ldots$ differs from the real number paired with 2 at digit x_3^2 (see Figure 1.22), and so on.

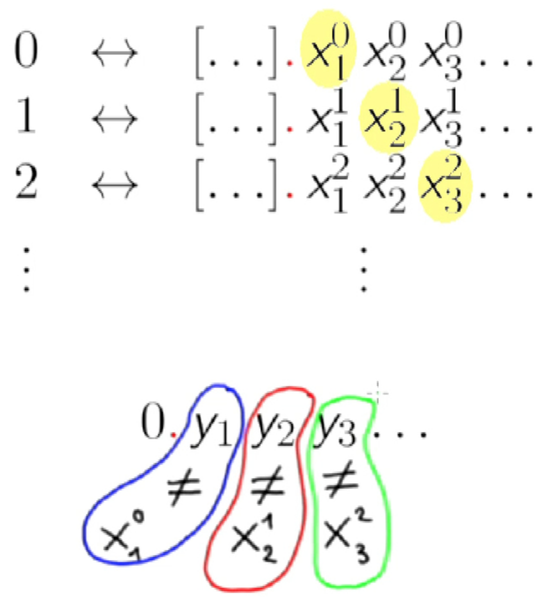

Figure 1.22: Difference 3

By altering the diagonal of the two-dimensional array of digits above, we have determined a real number that was not paired off with any natural number above. But by assumption each real number would have to be paired off with a natural number. That is a contradiction.

So our assumption that the set of natural numbers has equally many members as the set of real numbers leads to a contradiction. Hence, the negation of this assumption must be true: the set of natural numbers does not have equally many members as the set of real numbers. And that is what we meant to prove.

Remark on Cantor's proof: A precise proof of this theorem by Cantor is contained in every introductory textbook in set theory.

But Cantor's methods of proof by diagonalization is really much more general than that. Cantor did not just show that the set of natural numbers is of smaller size than the set of real numbers; he also showed that the set of real numbers is of smaller size than yet another infinite set; and that infinite set is smaller than yet another infinite set; and so on. There are not just two infinite sizes: there are infinitely many of them. More than the size of the set of natural numbers, more than the size of the set of real numbers, and so on.

And Cantor can do even more: he shows that one can measure the sizes of infinite sets in terms of new numbers: the so-called transfinite cardinal numbers. One can calculate with them in many ways, though not all, that are familiar from the natural numbers. And so on and so forth. And all of this is part of modern set theory.

We have not just found an answer to our question from the beginning of this lecture: what is infinity? We discovered a whole universe of infinities. Cantor's paradise, as the famous German mathematician David Hilbert once called it, from which no one shall expel us. It is a mathematical theory – set theory – that gave us the answer, and which improved our understanding of infinity in unexpected ways.

Remark on modern set theory: If you want to learn more about set theory, take a look at http://plato.stanford.edu/entries/set-theory/.

Quiz 07:
(1) I mentioned along the way that the real number $0.99999\ldots$ is identical to the real number 1.00000.
Show this to be so by considering
$x = 0.99999\ldots$,
then multiplying x by 10, and by deriving from this that $x = 1$.
(2) In the proof of Cantor's theorem I say at one point:
"Let me now describe a real number that must be missing in the pairing off above: I choose 0 to be its digit before the point. As the first digit after the point, take any digit that is distinct from x_1^0: since there are 10 available digits – from '0' to '9' – there is such a digit distinct from x_1^0. Let that digit be y_1. For instance, if x_1^0 is the digit '5', then y_1 could be '4', or '7', or the like – anything other than '5'. Similarly, as the second digit after the point use any digit that is distinct from x_2^1. Call it y_2. As the third digit after the point choose any digit that is distinct from x_3^2; call it y_3. And so on. In general terms: for all natural numbers n, let y_n be any digit other than x_n^{n-1}."
Here is the question: When one alters the diagonal of the enumerated
$[\ldots].\, x_1^0\, x_2^0\, x_3^0 \ldots$
$[\ldots].\, x_1^1\, x_2^1\, x_3^1 \ldots$
$[\ldots].\, x_1^2\, x_2^2\, x_3^2 \ldots$
in this way, are there any constraints at all on which digits y_1, y_2, y_3,... to choose and which not?
Solution

1.9 Conclusions (06:15)

The topic of this lecture was the infinite: a traditional philosophical topic. In the first part of the lecture, the calculus helped us to avoid drawing the wrong philosophical conclusions from Zeno's paradox of infinity – a philosophical argument that was meant to undermine the appearance of change. In the second part of the lecture, we found a way of avoiding Galileo's paradox of infinity without restricting the notion of size just to finite sets. Modern set theory supplied us with an exact, fruitful, and simple concept of size that generalizes the concept of size of finite sets; on its basis we were able to define a concept of infinity that is equally simple and illuminating and in terms of which one can prove that there is not just *the* infinite, but there are actually different sizes of infinity, and indeed infinitely many of them.

Let me wrap up this lecture with a couple of observations.

First of all, here's a worry: what we have been doing here isn't really philosophy anymore – this is mathematics. Well, Zeno's paradox certainly is regarded as philosophy; the calculus was only a method of testing the plausibility of certain premises in Zeno's argument. As far as the second part of this lecture is concerned, it is only important to keep in mind that the question "What is the infinite?" is a classical philosophical question. And it is hard to doubt that set theory has led to enormous progress on that question. How one ought to draw the borderline between the foundations of mathematics on the one hand and philosophy on the other is a question that I will simply put to one side here.

Secondly, a different kind of worry: in some parts of the talk I was using mathematics; but how do we justify mathematics in the first place? More particularly: how do we justify it philosophically? For instance: in some parts of the talk I presupposed the existence of the natural numbers. What do we mean by that? On what grounds are we allowed to do so? Should we really believe in the existence of infinitely many numbers? And shouldn't philosophers worry about that? The answer is: absolutely! There is a whole area of philosophy that deals with just these questions: the philosophy of mathematics.

Remark on philosophy of mathematics: If you want to know more about the area of philosophy of mathematics, check out http://plato.stanford.edu/entries/philosophy-mathematics/.

I am very much interested in that area myself. It is just that it was not the topic of today's lecture. Also there is nothing wrong about using mathematical methods in one part of philosophy, and questioning these mathematical methods in another. It's like: there is nothing wrong about using traditional philosophical methods, such as conducting thought experiments, in one part of philosophy, and questioning these philosophical methods in another. I should also emphasize that the definition of infinity that we ended up with did not refer to the natural numbers in any way; it only involved logical or quasi-logical notions. And the definition did not entail the existence of infinitely many objects either, it merely determined a concept – that's all.

Thirdly: what happened to our initial worry that finite beings, such as, maybe, ourselves, are not in the position to understand infinity. To some extent, we found this to be unjustified. We can talk about infinitely many objects – such as the natural numbers – and also sets of infinitely many objects, and we do so in finite terms. We are able to define infinity in finite terms. In set theory we prove the existence of infinite sets of different sizes in finite terms. And so on. We need to be careful, of course, to avoid being stuck with paradoxes. And of course it would be ridiculous to think that we have gained anything like a complete understanding of the infinite, whatever that might be. But we did gain some understanding of some aspects of the infinite, and mathematics was crucial for that. Actually, things might just be the other way around: as much as we are finite beings, it might be easier for us to understand the infinite than the finite. Think of our definition from before: we defined infinity of sets; and then we defined finiteness as non-infinity. So it is finiteness that is defined negatively: as the non-existence of a pairing off with a proper subset. Perhaps we are beings whose understanding of the infinite is prior to their understanding of the finite.

Remark on the definition of infinity: I should add that there are also alternative definitions of infinity – definitions that differ from the one that I used in this lecture (and which originated with Richard Dedekind). But it is fair to say that the definition of infinity by Dedekind is generally regarded as the most important and fruitful one in modern mathematics and philosophy. The different kinds of infinity are also contained in the usual textbooks on set theory.

Fourthly: you might think that we are taking away everything that was mystical about the infinite by approaching it in a precise, systematic, and rational manner. And isn't that a pity? Well, to the extent that we do, that's just the way it is: philosophy does aim to conceptualize and rationally reconstruct what might have seemed incomprehensible beforehand. But there is not much of a worry that a lecture like this would dissolve the mystique completely: for instance, modern set theorists believe with Cantor that if all the transfinite cardinal numbers – all the numbers that measure the sizes of infinite sets – are taken together, then it is so many of them that they do not form a set anymore. It's simply too many of them. As Cantor said: they form an absolute infinity. An infinity that is greater than all transfinite cardinal numbers; an infinity whose size, therefore, cannot be measured anymore in set-theoretic terms. If this absolutely infinite is not mystical, what is?

Here are some books on philosophical paradoxes which also cover the paradoxes that we are going to encounter in these lectures (including Zeno's and Galileo's paradox):

Clark, M., *Paradoxes from A to Z*, second edition, London: Routledge, 2007.
Rescher, N., *Paradoxes. Their Roots, Range, and Resolution*, Chicago: Open Court, 2001.
Sainsbury, R.M., *Paradoxes*, third edition, Cambridge: Cambridge University Press, 2009.
Sorensen, R., *A Brief History of the Paradox*, Oxford: Oxford University Press, 2005.

And here are some relevant books on the infinite and on the corresponding history and philosophy of set theory:

Ferreirs, J., *Labyrinth of Thought. A History of Set Theory and Its Role in Modern Mathematics*, second revised edition, Basel: Birkhuser, 2007.
Lavine, S., *Understanding the Infinite*, Cambridge, Mass.: Harvard University Press, 1998.
Moore, A.W., *The Infinite*, second edition, London: Routledge, 2001.
Oppy, G., *Philosophical Perspectives on Infinity*, Cambridge: Cambridge University Press, 2006.

Finally, this is a philosophical journal article which deals with different conceptions of the infinite and with how they relate to Galileo's Paradox:

Mancosu, P., "Measuring the Size of Infinite Collections of Natural Numbers: Was Cantor's Theory of Infinite Numbers Inevitable?", *The Review of Symbolic Logic* 2/4 (2009), 612-646.

Appendix A

Quiz Solutions Week 1: Infinity

Quiz 01:

Do you think the following argument is logically valid, too?

(P1) Zeno puts forward arguments.

(P2) If Zeno is a philosopher, then he puts forward arguments.

(C) Zeno is a philosopher.

Or more generally:

(P1) B.

(P2) If A, then B.

(C) A.

SOLUTION Quiz 01:

No, that (kind of) argument is not logically valid: if the premises P1 and P2 are true, then this does not imply with necessity that also the conclusion C is true. If the argument were logically true, then just the truth of the premises P1 and P2 by themselves should suffice to guarantee the truth of C. But that is not so: it is perfectly possible that P1 and P2 are true while C is not. Here is one way of seeing why: assume, say, 'Zeno' would denote a scientist who is not also a philosopher: this is not excluded by P1 and P2 being true – and as I said, we are not to suppose tacitly any premise additional to P1 and P2 by which, e.g., we could rule out 'Zeno' denoting a non-philosopher. Given that 'Zeno' denotes a scientist who is not also a philosopher, which describes a perfectly possible situation, premise P1 is true, since not just philosophers put forward arguments, but also scientists do. Premise P2 is still true, since all philosophers put forward arguments, so independently of whether Zeno is a philosopher or not, **if** he is a philosopher, then he puts forward arguments. But the conclusion C is false: for, by assumption, 'Zeno' was meant to denote a non-philosopher. So the joint truth of P1 and P2 taken by itself does not yield the truth of C with necessity.

Accordingly, for the more general version of the argument: Assume the premises (P1) B, (P2) if A then B, to be true. Then there is no guarantee that also A is true: B is true, by premise P1; and A is indeed one possible "reason" why B might be true, as stated by premise P2; but there might well be other "reasons" why B is true: e.g., it might be the case that B is really true because (i) if D then B, and (ii) D are both true. So A does not have to be true just because the premises P1 and P2 are true. Hence, the argument is not logically valid. The truth of P1 and P2 does not logically imply the truth of the conclusion sentence A.

Back to quiz

Quiz 03:

(1) Try out the following: Let $t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ Hence $\frac{t}{2} = \frac{1}{4} + \frac{1}{8} + \ldots$. Now subtract $\frac{t}{2}$ from t, and try to determine the value of 't' in this way.

(2) How does it follow in our toy model that Achilles is twice as fast as the tortoise?

SOLUTION Quiz 03:

Solution (1): We first consider the left-hand sides of these equations, and then their right-hand sides. On the one hand, if one subtracts $\frac{t}{2}$ from t, one ends up with $\frac{t}{2}$. On the other hand, subtracting $\frac{1}{4} + \frac{1}{8} + \ldots$ from $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ leaves $\frac{1}{2}$.

Thus, we have: $\frac{t}{2} = \frac{1}{2}$. That is: $t = 1$. The infinite sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ converges towards 1.

Solution (2): Here is one way of seeing this. The amount of time that it takes the tortoise to move from x_1 to x_2 is precisely the amount of time that it takes Achilles to move from x_0 to x_1, that is, T_0; but it only takes Achilles the amount T_1 of time to move from x_1 to x_2; and, by assumption, T_1 is but half of T_0. So Achilles is twice as fast as the tortoise.

Back to quiz

Quiz 04:

(1) Is the set $\{8, 3, 5, 5\}$ identical to the set $\{5, 3, 8, 8, 5\}$?

(2) Is the set $\{1, 2, 3, 4\}$ identical to the set $\{1, 4\}$?

SOLUTION Quiz 04:

Solution (1): Yes, as they have the same members. The more economical way of denoting that set would be: $\{3, 5, 8\}$.

Solution (2): No. For instance, the number 2 is a member of the set $\{1, 2, 3, 4\}$ while 2 is not a member of $\{1, 4\}$. Therefore, the set $\{1, 2, 3, 4\}$ is not identical to (that is, is distinct from) the set $\{1, 4\}$.)

Back to quiz

Quiz 05:

(1) Is $\{1, 3, 5\}$ a subset of $\{8, 1, 5, 3\}$? Is it a proper subset?

(2) Show that there is a pairing off between the set $\{1, 2, 3, 4, ...\}$ of positive integers and the set $\{3, 6, 9, 12, ...\}$.

SOLUTION Quiz 05:

Solution (1): Yes and yes.

Solution (2): $\{3, 6, 9, 12, ...\}$ is the set of positive multiples of 3. The pairing off between $\{1, 2, 3, 4, ...\}$ and $\{3, 6, 9, 12, ...\}$ is given by mapping each positive integer n to the corresponding number $3n$.)

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Quiz 06:

(1) Assume that we would define X to have equally$_1$ many members as Y if and only if X is a subset of Y and Y is a subset of X. Use the Principle of Extensionality for sets in order to show that this would entail: X has equally$_1$ many members as Y if and only if $X = Y$.

(2) Say, we would have defined 'less$_2$' in this slightly different manner:

X has less$_2$ members than Y if and only if X has equally$_2$ many members as a subset of Y, but X does not have equally$_2$ many members as Y.

(So we say 'subset' now where in the original definition we had said 'proper subset'.)

Would that change anything regarding which sets are having less$_2$ members than which other sets?

SOLUTION Quiz 06:

Solution (1): By definition, X has equally$_1$ many members as Y if and only if X is a subset of Y and Y is a subset of X.

By definition of 'subset of': X is a subset of Y if and only if for all z, if z is a member of X, then z is also a member of Y.

And by the same definition of 'subset of': Y is a subset of X if and only if for all z, if z is a member of Y, then z is also a member of X.

Taking all of these together, we have:

X has equally$_1$ many members as Y if and only if for all z, if z is a member of X, then z is also a member of Y, and for all z, if z is a member of Y, then z is also a member of X.

We can simplify the part to the right of 'if and only if' as follows (using that 'A if and only if B' is equivalent to 'if A, then B, and if B, then A'):

X has equally$_1$ many members as Y if and only if for all z: z is a member of X if and only if z is a member of Y.

But, according to the Principle of Extensionality, the part that is now right to the 'if and only if' can be replaced by an identity statement:

X has equally$_1$ many members as Y if and only if

$X = Y$.

Which is what we wanted to show.

Solution (2): No, it would not change anything. The reason is that the part 'X does not have equally$_2$ many members as Y' already excludes X from having equally$_2$ many members as Y itself (where Y is the only "improper" subset of Y). Therefore,

(i) 'X has equally$_2$ many members as a subset of Y, but X does not have equally$_2$ many members as Y'

is in fact equivalent to

(ii) 'X has equally$_2$ many members as a proper subset of Y, but X does not have equally$_2$ many members as Y'.

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Quiz 07:

(1) I mentioned along the way that the real number $0.99999\ldots$ is identical to the real number 1.00000.

Show this to be so by considering

$x = 0.99999\ldots$,

then multiplying x by 10, and by deriving from this that $x = 1$.

(2) In the proof of Cantor's theorem I say at one point:

"Let me now describe a real number that must be missing in the pairing off above: I choose 0 to be its digit before the point. As the first digit after the point, take any digit that is distinct from x_1^0: since there are 10 available digits – from '0' to '9' – there is such a digit distinct from x_1^0. Let that digit be y_1. For instance, if x_1^0 is the digit '5', then y_1 could be '4', or '7', or the like – anything other than '5'. Similarly, as the second digit after the point use any digit that is distinct from x_2^1. Call it y_2. As the third digit after the point choose any digit that is distinct from x_3^2; call it y_3. And so on. In general terms: for all natural numbers n, let y_n be any digit other than x_n^{n-1}."

Here is the question: When one alters the diagonal of the enumerated

$[\ldots].\, x_1^0\, x_2^0\, x_3^0 \ldots$
$[\ldots].\, x_1^1\, x_2^1\, x_3^1 \ldots$
$[\ldots].\, x_1^2\, x_2^2\, x_3^2 \ldots$

in this way, are there any constraints at all on which digits y_1, y_2, y_3,... to choose and which not?

SOLUTION Quiz 07:

Solution (1): Let $x = 0.99999\ldots$

Hence, $10x = 9.9999\ldots$

Therefore,

$10x - x = 9.9999\ldots - 0.99999\ldots$

which implies that

$9x = 9$

which means

$x = 1$.

Solution (2): There are no constraints except for one: one should avoid choosing 9s and 0s. For one wants to avoid a situation in which, e.g., one constructs the decimal expansion $0.9999999\ldots$ which might well differ from all given decimal expansions in

$[\ldots].\, x_1^0\, x_2^0\, x_3^0 \ldots$
$[\ldots].\, x_1^1\, x_2^1\, x_3^1 \ldots$
$[\ldots].\, x_1^2\, x_2^2\, x_3^2 \ldots$

but where at the same time $1.000000\ldots$ would in fact be amongst the given decimal expansions; because in that case, since $0.99999\ldots = 1.0$, one would not really have constructed a real number that had been missing from the given pairing off between natural numbers and real numbers.

So, in order to stay on the safe side, one might actually define instead, e.g.: let y_1 be '4' if x_1^0 is the digit '5', and let y_1 be '5' if x_1^0 is a digit different from '5'; and so also for all other digits y_n. That is: avoid 9s and 0s by choosing, e.g., 4s and 5s instead.

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