There are two possible *acts*:

- Act 1: Switch
- Act 2: Not-Switch

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- Act 1: Switch
- Act 2: Not-Switch

There are two possible *outcomes*:

- Outcome 1: You get a goat.
- Outcome 2: You get the car.

We calculate the expected utilities of both acts:

$$\begin{aligned} \mathsf{EU}(\text{Switch}) &= (2/3) \cdot \mathsf{U} + (1/3) \cdot \mathsf{u} \\ \mathsf{EU}(\text{Not} - \text{Switch}) &= (1/3) \cdot \mathsf{U} + (2/3) \cdot \mathsf{u} \end{aligned}$$

We calculate the expected utilities of both acts:

$$\begin{aligned} EU(\text{Switch}) &= (2/3)\cdot U + (1/3)\cdot u \\ EU(\text{Not}-\text{Switch}) &= (1/3)\cdot U + (2/3)\cdot u \end{aligned}$$

As $U > u$, we see that

$$EU(\text{Switch}) - EU(\text{Not}-\text{Switch}) = (1/3)\cdot(U-u) > 0.$$

The decision problem

1. There are a number of acts $A_1, A_2, \ldots, A_n$ of which we have to choose one.
2. There are a number of mutually exclusive and exhaustive outcomes $O_1, O_2, \ldots, O_m$, i.e. one of these outcomes occurs.
3. If we choose an act, outcome i occurs with probability $P(O_i)$ and has a utility $U(O_i)$.

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The expected utility of an act A is then given by

$$EU(A) = \sum_{i=1}^{m} P(O_i)U(O_i).$$

There are two possible acts.

- Act 1: We insure the car.
- Act 2: We do not insure the car.

There are two possible *acts*.

- Act 1: We insure the car.
- Act 2: We do not insure the car.

There are two possible *states of the world*.

- State 1: The car breaks in an accident.
- State 2: The car does not break in an accident.

The expected monetary values:

$$
\begin{aligned}
EV(A_1) &= p\,(-m\$) + (1-p)\,(-m\$) = -m\$ \\
EV(A_2) &= p\,(-M\$) + (1-p)\,(0\$) = -p\,M\$
\end{aligned}
$$

There is hence an expected monetary gain to buying insurance whenever $m \leq p\,M$.

The expected utilities:

$$
\begin{aligned}
EU(A_1) &= p\,U(-m\$) + (1-p)\,U(-m\$) = U(-m\$) \\
EU(A_2) &= p\,U(-M\$) + (1-p)\,U(0\$) = p\,U(-M\$)
\end{aligned}
$$

Here we have set $U(0\$) = 0$.

Dominance: An act A dominates an act B if and only if the outcome of A will be as good as the outcome of B no matter which state of the world happens to be the true one, and strictly better under at least one state of the world.

Utility functions are only unique up to a positive linear transformation:

$$u_i \to u_i' = a \cdot u_i + b \text{ with } a > 0.$$

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EU is the utility function under u, and EU' is the utility function under u'. We then obtain:

$$\begin{aligned} EU'(A) &= \sum p_i\, u_i' = \sum p_i\,(a \cdot u_i + b) \\ &= a \cdot (\sum p_i\, u_i) + b \\ &= a \cdot EU(A) + b \end{aligned}$$

Hence, $EU'(A) > EU'(B)$ if and only if $EU(A) > EU(B)$.

A set of basic prizes $X = \{A, B, C, \ldots\}$.

Each basic prize is a lottery which you may win.

Choose a lottery in which you win A with probability p and B with a probability $1 - p$. That is, if A and B are lotteries, then

$$p\,A + (1 - p)\,B$$

is also a lottery.

We denote the latter lottery by ApB.

- $A \succ B$: you strictly prefer lottery A to lottery B.
- $A \prec B$: you strictly prefer lottery B to lottery A.
- $A \sim B$: you are indifferent between lotteries A and B.

The von Neumann Morgenstern Axioms

Consider a set $\mathcal{L}$ of lotteries.

1. Completeness: For all lotteries A, B in $\mathcal{L}$, $A \succ B$ or $A \prec B$ or $A \sim B$.

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4. **Independence:** For all lotteries A, B, C in $\mathcal{L}$, $A \succ B$ if and only if $ApC \succ BpC$.

The Neumann and Morgenstern representation theorem.

A preference relation $\succ$ satisfies the axioms 1 to 4, if and only if there exists a utility function u such that

(i) if $A \succ B$, then $u(A) > u(B)$,

(ii) $u(ApB) = p\,u(A) + (1-p)\,u(B)$,

(iii) for every other function u' that satisfies (i) and (ii), there are numbers $a > 0$ and b such that $u' = a\,u + b$.

$$
\begin{aligned}
u(A) - u(B) &= u(C) - u(D) \\
&\Leftrightarrow \\
u'(A) - u'(B) &= u'(C) - u'(D)
\end{aligned}
$$

O: optimal element in the lottery
W: worst element in the lottery

We set:

$$\begin{aligned} u(O) &= 1 \\ u(W) &= 0 \end{aligned}$$

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If $OpW \sim A$, then $u(A) = p$.

The Allais Paradox

Choice 1:

- A: a cheap car for sure
- B: nothing (1%) or an expensive car (10%) or a cheap car (89%)

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- A: a cheap car for sure
- B: nothing (1%) or an expensive car (10%) or a cheap car (89%)

Choice 2:

- E: a cheap car (11%) or nothing (89%)
- F: an expensive car (10%) or nothing (90%)

We set:

$$u(\text{nothing}) = 0$$
$$u(\text{expensive car}) = 1$$
$$u(\text{cheap car}) = x$$

We set:

$$u(\text{nothing}) = 0,$$
$$u(\text{expensive car}) = 1$$
$$u(\text{cheap car}) = x$$

Then

$$\begin{aligned} u(A) - u(B) &= x - (.1 + .89\,x) = .11\,x - .1 \\ u(E) - u(F) &= .11\,x - .1 \\ &= u(A) - u(B) \end{aligned}$$

Hence, if there is a utility function u, then $A \succ B \Leftrightarrow E \succ F$.

The Ellsberg Paradox

An urn contains contains 90 balls. 30 of these balls are red. The remaining 60 balls are either blue or yellow.

Choice 1:

G: three nights in a luxury hotel in St. Petersburg if a red ball is drawn

H: three nights in a luxury hotel in St. Petersburg if a blue ball is drawn

The Ellsberg Paradox

An urn contains contains 90 balls. 30 of these balls are red. The remaining 60 balls are either blue or yellow.

Choice 1:

G: three nights in a luxury hotel in St. Petersburg if a red ball is drawn

H: three nights in a luxury hotel in St. Petersburg if a blue ball is drawn

Choice 2:

K: three nights in a luxury hotel in St. Petersburg if a red or yellow ball is drawn

L: three nights in a luxury hotel in St. Petersburg if a blue or yellow ball is drawn