There are several competing interpretations of quantum mechanics on the market. These interpretations try to make sense out of the theory and explain what the theory tells us about the world.

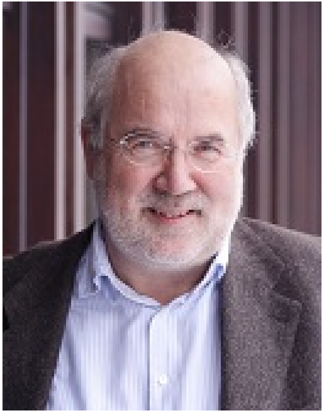
There are several competing interpretations of quantum mechanics on the market. These interpretations try to make sense out of the theory and explain what the theory tells us about the world.

Here are two well-known interpretations:

- The Copenhagen Interpretation (defenders: N. Bohr and W. Heisenberg)
- Bohmian Mechanics (defenders: D. Dürr, S. Goldstein and N. Zanghi. Note, however, that the defenders of Bohmian Mechanics do not consider it as an interpretation, but as a theory.)

(Niels Bohr
Image in the public domain)

(Werner Heisenberg
Source: Bundesarchiv, Bild183-R57262 / CC-BY-SA)



(Prof. Dethlef Dürr
used by permission)

There are several competing interpretations of quantum mechanics on the market. These interpretations try to make sense out of the theory and explain what the theory tells us about the world.

Here are two well-known interpretations:

- The Copenhagen Interpretation (defenders: N. Bohr and W. Heisenberg)
- Bohmian Mechanics (defenders: D. Dürr, S. Goldstein and N. Zanghi. Note, however, that the defenders of Bohmian Mechanics do not consider it as an interpretation, but as a theory.)

Different interpretations give different answers to various philosophical questions, such as the problem of determinism and the interpretation of probability in quantum mechanics.

A random variable is a variable that can take on a set of possible values, each with a specific probability.

A random variable is a variable that can take on a set of possible values, each with a specific probability.

Examples:

- the weight of a randomly picked person
- the spin of an electron

A random variable is a variable that can take on a set of possible values, each with a specific probability.

Examples:

- the weight of a randomly picked person (this is a **continuous variable** with values in the positive real numbers (on a given scale))
- the spin of an electron (this is a **binary variable** with values $\pm 1/2$)

Probabilities and Correlations

Let us consider two binary random variables A and B with the values: $a_1, b_1 = +1$ and $a_2, b_2 = -1$.

Probabilities and Correlations

Let us consider two binary random variables A and B with the values: $a_1, b_1 = +1$ and $a_2, b_2 = -1$.

To specify the joint probability distribution P, we have to fix the values of

$$
\begin{aligned}
P(A = 1, B = 1), \\
P(A = 1, B = -1), \\
P(A = -1, B = 1), \\
P(A = -1, B = -1).
\end{aligned}
$$

Probabilities and Correlations

Let us consider two binary random variables A and B with the values: $a_1, b_1 = +1$ and $a_2, b_2 = -1$.

To specify the joint probability distribution P, we have to fix the values of

$$
\begin{aligned}
P(A = 1, B = 1) &=: P(1, 1),\\
P(A = 1, B = -1) &=: P(1, -1),\\
P(A = -1, B = 1) &=: P(-1, 1),\\
P(A = -1, B = -1) &=: P(-1, -1).
\end{aligned}
$$

Probabilities and Correlations

Let us consider two binary random variables A and B with the values: $a_1, b_1 = +1$ and $a_2, b_2 = -1$.

To specify the joint probability distribution P, we have to fix the values of

$$\begin{aligned} P(A = 1, B = 1) &=: P(1, 1), \\ P(A = 1, B = -1) &=: P(1, -1), \\ P(A = -1, B = 1) &=: P(-1, 1), \\ P(A = -1, B = -1) &=: P(-1, -1). \end{aligned}$$

Please note:

- $P(a, b) \equiv P(a \wedge b)$

Probabilities and Correlations

Let us consider two binary random variables A and B with the values: $a_1, b_1 = +1$ and $a_2, b_2 = -1$.

To specify the joint probability distribution P, we have to fix the values of

$$\begin{aligned} P(A=1, B=1) &=: P(1,1), \\ P(A=1, B=-1) &=: P(1,-1), \\ P(A=-1, B=1) &=: P(-1,1), \\ P(A=-1, B=-1) &=: P(-1,-1). \end{aligned}$$

Please note:

- $P(a, b) \equiv P(a \wedge b)$
- $P(1,1) + P(1,-1) + P(-1,1) + P(-1,-1) = 1$

The expectation of the random variable A:

$$\begin{aligned} E(A) &= \sum_{i,j=1}^{2} a_i \cdot P(a_i, b_j) \\ &= 1 \cdot P(1,1) + 1 \cdot P(1,-1) + (-1) \cdot P(-1,1) \\ &\quad + (-1) \cdot P(-1,-1) \\ &= P(1,1) + P(1,-1) - P(-1,1) - P(-1,-1) \end{aligned}$$

The expectation of the random variable A:

$$\begin{aligned} E(A) &= \sum_{i,j=1}^{2} a_i \cdot P(a_i, b_j) \\ &= 1 \cdot P(1,1) + 1 \cdot P(1,-1) + (-1) \cdot P(-1,1) \\ &\quad + (-1) \cdot P(-1,-1) \\ &= P(1,1) + P(1,-1) - P(-1,1) - P(-1,-1) \end{aligned}$$

The expectation of the random variable B:

$$\begin{aligned} E(B) &= \sum_{i,j=1}^{2} b_j \cdot P(a_i, b_j) \\ &= 1 \cdot P(1,1) + (-1) \cdot P(1,-1) + 1 \cdot P(-1,1) \\ &\quad + (-1) \cdot P(-1,-1) \\ &= P(1,1) - P(1,-1) + P(-1,1) - P(-1,-1) \end{aligned}$$

The expectation of the random variable AB:

$$\begin{aligned} E(AB) &= \sum_{i,j=1}^{2} a_i \cdot b_j \cdot P(a_i, b_j) \\ &= 1 \cdot P(1,1) + (-1) \cdot P(1,-1) + (-1) \cdot P(-1,1) \\ &\quad + (-1) \cdot (-1) \cdot P(-1,-1) \\ &= P(1,1) - P(1,-1) - P(-1,1) + P(-1,-1) \end{aligned}$$

The variance of the random variable A:

$$V(A) = E((A - E(A))^2)$$

The variance of the random variable A:

$$
\begin{aligned}
V(A) &= E((A - E(A))^2) \\
&= E(A^2) - (E(A))^2
\end{aligned}
$$

The covariance of the random variables A and B:

$$cov(A, B) = E\left((A - E(A)) \cdot (B - E(B))\right)$$

The covariance of the random variables A and B:

$$\begin{aligned} cov(A, B) &= E\left((A - E(A)) \cdot (B - E(B))\right) \\ &= E(AB) - E(A) \cdot E(B) \end{aligned}$$

If A and B are probabilistically independent, then $P(a_i, b_j) = P(a_i) \cdot P(b_j)$ (remember Lecture 5!).

If A and B are probabilistically independent, then $P(a_i, b_j) = P(a_i) \cdot P(b_j)$ (remember Lecture 5!).

Hence,

$$
\begin{aligned}
E(AB) &= \sum_{i,j=1}^{2} a_i \cdot b_j \cdot P(a_i)\, P(b_j) \\
E(AB) &= \sum_{i,j=1}^{2} a_i\, P(a_i) \cdot b_j\, P(b_j) \\
&= (\sum_{i=1}^{2} a_i P(a_i)) \cdot (\sum_{j=1}^{2} b_j P(b_j)) \\
&= E(A) \cdot E(B).
\end{aligned}
$$

If A and B are probabilistically independent, then $P(a_i, b_j) = P(a_i) \cdot P(b_j)$ (remember Lecture 5!).

Hence,

$$
\begin{aligned}
E(AB) &= \sum_{i,j=1}^{2} a_i \cdot b_j \cdot P(a_i)\, P(b_j) \\
E(AB) &= \sum_{i,j=1}^{2} a_i\, P(a_i) \cdot b_j\, P(b_j) \\
&= (\sum_{i=1}^{2} a_i P(a_i)) \cdot (\sum_{j=1}^{2} b_j P(b_j)) \\
&= E(A) \cdot E(B).
\end{aligned}
$$

Hence, $cov(A, B) = 0$.

Example

$$
\begin{aligned}
P(1,1) = P(-1,-1) &= 0.4 \\
P(1,-1) = P(-1,1) &= 0.1
\end{aligned}
$$

An Example

$$\begin{aligned} P(1,1) = P(-1,-1) &= 0.4 \\ P(1,-1) = P(-1,1) &= 0.1 \end{aligned}$$

$$P(A=1) = P(1,1) + P(1,-1) = 0.5$$

An Example

$$\begin{aligned} P(1,1) = P(-1,-1) &= 0.4 \\ P(1,-1) = P(-1,1) &= 0.1 \end{aligned}$$

$$\begin{aligned} P(A=1) = P(1,1) + P(1,-1) &= 0.5 \\ P(B=1) = P(1,1) + P(-1,1) &= 0.5 \end{aligned}$$

An Example

$$
\begin{aligned}
P(1,1) = P(-1,-1) &= 0.4 \\
P(1,-1) = P(-1,1) &= 0.1
\end{aligned}
$$

$$
\begin{aligned}
P(A=1) = P(1,1) + P(1,-1) &= 0.5 \\
P(B=1) = P(1,1) + P(-1,1) &= 0.5
\end{aligned}
$$

$$
P(A=-1) = P(B=-1) = 0.5
$$

An Example

$$\begin{aligned} P(1,1) = P(-1,-1) &= 0.4 \\ P(1,-1) = P(-1,1) &= 0.1 \end{aligned}$$

$$\begin{aligned} P(A=1) = P(1,1) + P(1,-1) &= 0.5 \\ P(B=1) = P(1,1) + P(-1,1) &= 0.5 \end{aligned}$$

$$P(A=-1) = P(B=-1) = 0.5$$

$$\begin{aligned} E(A) &= 1 \cdot 0.5 + (-1) \cdot 0.5 = 0 \\ E(B) &= 1 \cdot 0.5 + (-1) \cdot 0.5 = 0 \end{aligned}$$

An Example

$$
\begin{aligned}
E(AB) &= \sum_{i,j=1}^{2} a_i \, b_j \cdot P(a_i, b_j) \\
&= 0.4 - 0.1 + 0.4 - 0.1 \\
&= 0.6
\end{aligned}
$$

An Example

$$
\begin{aligned}
E(AB) &= \sum_{i,j=1}^{2} a_i\, b_j \cdot P(a_i, b_j) \\
&= 0.4 - 0.1 + 0.4 - 0.1 \\
&= 0.6
\end{aligned}
$$

$$
\begin{aligned}
cov(A, B) &= E(AB) - E(A) \cdot E(B) \\
&= E(AB) \\
&= 0.6
\end{aligned}
$$

$$E(A) = E(B) = 0, cov(A, B) = x \in [-1, 1] \rightarrow E(AB) = x$$

$$E(A) = E(B) = 0, cov(A, B) = x \in [-1, 1] \to E(AB) = x$$

1. $E(A) = 0$ implies

$$P(1, 1) + P(1, -1) - P(-1, 1) - P(-1, -1) = 0.$$

$$E(A) = E(B) = 0, cov(A, B) = x \in [-1, 1] \rightarrow E(AB) = x$$

1. $E(A) = 0$ implies

$$P(1, 1) + P(1, -1) - P(-1, 1) - P(-1, -1) = 0.$$

2. $E(B) = 0$ implies

$$P(1, 1) - P(1, -1) + P(-1, 1) - P(-1, -1) = 0.$$

$$E(A) = E(B) = 0, cov(A, B) = x \in [-1, 1] \to E(AB) = x$$

1. $E(A) = 0$ implies
$$P(1,1) + P(1,-1) - P(-1,1) - P(-1,-1) = 0.$$
2. $E(B) = 0$ implies
$$P(1,1) - P(1,-1) + P(-1,1) - P(-1,-1) = 0.$$
3. $E(AB) = x$ implies
$$P(1,1) - P(1,-1) - P(-1,1) + P(-1,-1) = x.$$

$$E(A) = E(B) = 0, cov(A, B) = x \in [-1, 1] \to E(AB) = x$$

1. $E(A) = 0$ implies
$$P(1,1) + P(1,-1) - P(-1,1) - P(-1,-1) = 0.$$
2. $E(B) = 0$ implies
$$P(1,1) - P(1,-1) + P(-1,1) - P(-1,-1) = 0.$$
3. $E(AB) = x$ implies
$$P(1,1) - P(1,-1) - P(-1,1) + P(-1,-1) = x.$$
4. Probability Theory implies
$$P(1,1) + P(1,-1) + P(-1,1) + P(-1,-1) = 1.$$

Solving this system of four linear equations leads to:

$$\begin{aligned} P(1,1) = P(-1,-1) &= (1+x)/4 \\ P(1,-1) = P(-1,1) &= (1-x)/4 \end{aligned}$$

Remember: x is the covariance of A and B.

$$E(AB) = E(AB') = E(A'B) = x, E(A'B') = -x, x = 1/\sqrt{2}$$

$$E(AB) = E(AB') = E(A'B) = x, E(A'B') = -x, x = 1/\sqrt{2}$$

$$\begin{aligned}
P(A = 1, B = 1) = P(A = -1, B = -1) &= (1 + x)/4 \\
P(A = 1, B = -1) = P(A = -1, B = 1) &= (1 - x)/4
\end{aligned}$$

$$\begin{aligned}
P(A = 1, B' = 1) = P(A = -1, B' = -1) &= (1 + x)/4 \\
P(A = 1, B' = -1) = P(A = -1, B' = 1) &= (1 - x)/4
\end{aligned}$$

$$\begin{aligned}
P(A' = 1, B = 1) = P(A' = -1, B = -1) &= (1 + x)/4 \\
P(A' = 1, B = -1) = P(A' = -1, B = 1) &= (1 - x)/4
\end{aligned}$$

$$E(AB) = E(AB') = E(A'B) = x, E(A'B') = -x, x = 1/\sqrt{2}$$

$$\begin{aligned}
P(A=1, B=1) = P(A=-1, B=-1) &= (1+x)/4 \\
P(A=1, B=-1) = P(A=-1, B=1) &= (1-x)/4
\end{aligned}$$

$$\begin{aligned}
P(A=1, B'=1) = P(A=-1, B'=-1) &= (1+x)/4 \\
P(A=1, B'=-1) = P(A=-1, B'=1) &= (1-x)/4
\end{aligned}$$

$$\begin{aligned}
P(A'=1, B=1) = P(A'=-1, B=-1) &= (1+x)/4 \\
P(A'=1, B=-1) = P(A'=-1, B=1) &= (1-x)/4
\end{aligned}$$

$$\begin{aligned}
P(A'=1, B'=1) = P(A'=-1, B'=-1) &= (1-x)/4 \\
P(A'=1, B'=-1) = P(A'=-1, B'=1) &= (1+x)/4
\end{aligned}$$

The Clauser-Horne-Shimony-Holt Theorem

There is a joint probability distribution over the random variables A, A', B, and B' if and only if

$$|E(AB) + E(AB') + E(A'B) - E(A'B')| \leq 2.$$

The Clauser-Horne-Shimony-Holt Theorem

There is a joint probability distribution over the random variables A, A', B, and B' if and only if

$$|E(AB) + E(AB') + E(A'B) - E(A'B')| \leq 2.$$

Note: Here are two examples for the modulus or absolute value of a number:

- $|5| = 5$
- $|-5| = 5$

The consequence of the CHSH Theorem

Insert

$$\begin{aligned} E(AB) &= E(AB') = E(A'B) = 1/\sqrt{2} \\ E(A'B') &= -1/\sqrt{2} \end{aligned}$$

The consequence of the CHSH Theorem

Insert

$$\begin{aligned} E(AB) &= E(AB') = E(A'B) = 1/\sqrt{2} \\ E(A'B') &= -1/\sqrt{2} \end{aligned}$$

in the CHSH-inequality:

$$|1/\sqrt{2} + 1/\sqrt{2} + 1/\sqrt{2} + 1/\sqrt{2}| = 2\sqrt{2} \approx 2.828 > 2.$$

The consequence of the CHSH Theorem

Insert

$$\begin{aligned} E(AB) &= E(AB') = E(A'B) = 1/\sqrt{2} \\ E(A'B') &= -1/\sqrt{2} \end{aligned}$$

in the CHSH-inequality:

$$|1/\sqrt{2} + 1/\sqrt{2} + 1/\sqrt{2} + 1/\sqrt{2}| = 2\sqrt{2} \approx 2.828 > 2.$$

Hence, there is no joint probability distribution over A, A', B, and B'.

$$
\begin{aligned}
E(A) = E(A') = E(B) = E(B') &= 0 \\
E(AB) = E(AB') = E(A'B) &= 1/\sqrt{2} \\
E(A'B') &= -1/\sqrt{2}
\end{aligned}
$$

$$\begin{aligned} E(A) = E(A') = E(B) = E(B') &= 0 \\ E(AB) = E(AB') = E(A'B) &= 1/\sqrt{2} \\ E(A'B') &= -1/\sqrt{2} \end{aligned}$$

Some "probabilities", such as $P(1, 1, -1, 1)$, are negative, but, for example,

$$\begin{aligned} P(A = 1, B = 1) &= \sum_{A', B'} P(A = 1, A', B = 1, B') \\ &= \frac{1 + \sqrt{2}}{4\sqrt{2}} \geq 0. \end{aligned}$$

Boolean algebra

Operations:

- Conjunction: $\wedge$
- Disjunction: $\vee$
- Negation: $\neg$

Some laws that hold in a Boolean algebra:

Associativity of $\vee$: $x \vee (y \vee z) = (x \vee y) \vee z$

Some laws that hold in a Boolean algebra:

Associativity of $\vee$: $x \vee (y \vee z) = (x \vee y) \vee z$

Associativity of $\wedge$: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

Some laws that hold in a Boolean algebra:

Associativity of $\vee$: $x \vee (y \vee z) = (x \vee y) \vee z$

Associativity of $\wedge$: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

Commutativity of $\vee$: $x \vee y = y \vee x$

Some laws that hold in a Boolean algebra:

Associativity of $\vee$: $x \vee (y \vee z) = (x \vee y) \vee z$

Associativity of $\wedge$: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

Commutativity of $\vee$: $x \vee y = y \vee x$

Commutativity of $\wedge$: $x \wedge y = y \wedge x$

Some laws that hold in a Boolean algebra:

Associativity of $\vee$: $x \vee (y \vee z) = (x \vee y) \vee z$

Associativity of $\wedge$: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

Commutativity of $\vee$: $x \vee y = y \vee x$

Commutativity of $\wedge$: $x \wedge y = y \wedge x$

Distributivity of $\wedge$ over $\vee$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

A surprising feature involving spin

- Electrons can be prepared in a way that their spin has the value $+1/2$ in the z-direction.

A surprising feature involving spin

- Electrons can be prepared in a way that their spin has the value $+1/2$ in the z-direction.
- That is, if we measure the spin of electrons which are prepared in this way in the z-direction, then we will always obtain the result $+1/2$.

A surprising feature involving spin

- Electrons can be prepared in a way that their spin has the value $+1/2$ in the z-direction.
- That is, if we measure the spin of electrons which are prepared in this way in the z-direction, then we will always obtain the result $+1/2$.
- If we subsequently measure the spin of such electrons in the orthogonal x-direction, then we will find that 50% of the electrons have spin $+1/2$ and 50% of the electrons have spin $-1/2$.

A surprising feature involving spin

- Electrons can be prepared in a way that their spin has the value $+1/2$ in the z-direction.
- That is, if we measure the spin of electrons which are prepared in this way in the z-direction, then we will always obtain the result $+1/2$.
- If we subsequently measure the spin of such electrons in the orthogonal x-direction, then we will find that 50% of the electrons have spin $+1/2$ and 50% of the electrons have spin $-1/2$.
- If we next consider the electrons that have spin $+1/2$ in the x-direction and measure their spin in the z-direction again, then we will find that 50% of them have spin $+1/2$ and 50% have spin $-1/2$.

Distributivity is violated!

Let:

- X: The electron has spin $+1/2$ in the z-direction.

Distributivity is violated!

Let:

- X: The electron has spin $+1/2$ in the z-direction.
- Y: The electron has spin $+1/2$ in the x-direction.
- Z: The electron has spin $-1/2$ in the x-direction.

Distributivity is violated!

Let:

- X: The electron has spin $+1/2$ in the z-direction.
- Y: The electron has spin $+1/2$ in the x-direction.
- Z: The electron has spin $-1/2$ in the x-direction.

We note:

- $Y \vee Z$ is true (as $+1/2$ and $-1/2$ are the only possible spin-values).

Distributivity is violated!

Let:

- X: The electron has spin $+1/2$ in the z-direction.
- Y: The electron has spin $+1/2$ in the x-direction.
- Z: The electron has spin $-1/2$ in the x-direction.

We note:

- $Y \vee Z$ is true (as $+1/2$ and $-1/2$ are the only possible spin-values).
- Hence $X \wedge (Y \vee Z)$ is true.

Distributivity is violated!

Let:

- X: The electron has spin $+1/2$ in the z-direction.
- Y: The electron has spin $+1/2$ in the x-direction.
- Z: The electron has spin $-1/2$ in the x-direction.

We note:

- $Y \vee Z$ is true (as $+1/2$ and $-1/2$ are the only possible spin-values).
- Hence $X \wedge (Y \vee Z)$ is true.
- $(X \wedge Y) \vee (X \wedge Z)$ is false.

Distributivity is violated!

Let:

- X: The electron has spin $+1/2$ in the z-direction.
- Y: The electron has spin $+1/2$ in the x-direction.
- Z: The electron has spin $-1/2$ in the x-direction.

We note:

- $Y \vee Z$ is true (as $+1/2$ and $-1/2$ are the only possible spin-values).
- Hence $X \wedge (Y \vee Z)$ is true.
- $(X \wedge Y) \vee (X \wedge Z)$ is false.
- Hence, $X \wedge (Y \vee Z) \neq (X \wedge Y) \vee (X \wedge Z)$.

(John von Neumann
Source: The United States Department of Energy, image in the public domain)

A Short History of Quantum Logic

- 1936: John von Neumann and Garrett Birkhoff published "The Logic of Quantum Mechanics".

A Short History of Quantum Logic

- 1936: John von Neumann and Garrett Birkhoff published “The Logic of Quantum Mechanics”.
- 1960ies and 70ies: George Mackey formulated a set of axioms for the quantum propositional system as a so-called ortho-complemented lattice.

A Short History of Quantum Logic

- 1936: John von Neumann and Garrett Birkhoff published "The Logic of Quantum Mechanics".
- 1960ies and 70ies: George Mackey formulated a set of axioms for the quantum propositional system as a so-called ortho-complemented lattice.
- 1964: Constantin Piron proved a representation theorem that aims at motivating the Hilbert-space structure of quantum mechanics from quantum logical axioms.

A Short History of Quantum Logic

- 1936: John von Neumann and Garrett Birkhoff published “The Logic of Quantum Mechanics”.
- 1960ies and 70ies: George Mackey formulated a set of axioms for the quantum propositional system as a so-called ortho-complemented lattice.
- 1964: Constantin Piron proved a representation theorem that aims at motivating the Hilbert-space structure of quantum mechanics from quantum logical axioms.
- Today many quantum logicians explore the connection to the theory of quantum computation and prefer an operational approach.