Additional Problem Set for Lecture 7: Voting

(1) Show that the Condorcet method and the Borda count yield the same ordering if there are only two options.

(2) Consider the case of three options A, B, and C and construct an example where the Borda ranking differs from the 2-approval ranking, i.e. the ranking that one obtains if each voter approves of two options.

(3) A group of n members has to make a decision on three logically interconnected propositions A, B, and C. All group members accept that A $\vee$ B $\rightarrow$ C, where "$\rightarrow$" is the material conditional (see http://en.wikipedia.org/wiki/Material_conditional). (a) What are the possible situations? (b) Try to construct a discursive dilemma!

(4) A group of n voters has to make a yes-no decision. We assume that n is odd, that is that $n = 2m + 1$. Their verdicts of the voters are independent of each other and the reliability of each voter is $r := P(\mathrm{Vote_Y}|\mathrm{Y}) = P(\mathrm{Vote_N}|\mathrm{N})$ with $r \in (0, 1)$. Let M be the proposition that the majority makes the right decision. One can then show that

$$P(\mathrm{M}) = \sum_{k=m+1}^{n} \binom{n}{k} r^k \, (1 - r)^{n-k} \tag{1}$$

(try to show this!), where the *binomial* is defined as follows:

$$\binom{n}{k} := \frac{n!}{k! \, (n - k)!}$$

with $n! := 1 \cdot 2 \cdot \dots \cdot n$.

Consider the case $n = 3$ and show that $P(\mathrm{M}) > r$ for $r > 1/2$ and $P(\mathrm{M}) < r$ for $r < 1/2$. Note that the Condorcet Jury Theorem follows from eq. (1) and the *Law of Large Numbers*. See http://en.wikipedia.org/wiki/Law_of_large_numbers. For more on the Condorcet Jury Theorem, see the papers by F. Dietrich (http://arno.unimaas.nl/show.cgi?fid=11796) and by C. List and R. Goodin (http://personal.lse.ac.uk/list/PDF-files/listgoodin.pdf).

(5) Reconsider the Pareto axiom. How could it be criticized?

Solutions

(1) There are N voters, a voters prefer A to B, b voters prefer B to A, and c voters are indifferent between A and B. Clearly, $a + b + c = N$. Hence, A is the Condorcet winner if $a > b$. If $a = b$, then there is a tie, and if $b > a$, then B is the Condorcet winner. Next, let us calculate the Borda scores. We obtain: Score(A) $= 1 \cdot a + 0 \cdot b = a$ and Score(B) $= 1 \cdot b + 0 \cdot a = b$. Here we have given 0 points to A and B in the indifference case as both options rank lowest then. Hence, A is the Borda winner if $a > b$, there is a tie if $a = b$, and B is the Borda winner if $a > b$. Hence, the Condorcet method and the Borda count yield the same ordering if there are only two options.

(2) Here is one example. One voter ranks A $\succ$ B $\succ$ C, one voter ranks B $\succ$ C $\succ$ A, and three voters rank C $\succ$ B $\succ$ A. Hence, one voter approves of A, five voters approve of B, and four voters approve of C. Hence, 2-approval voting yields the group ranking B $\succ$ C $\succ$ A. Next we calculate the Broda scores and obtain: $\text{Score(A)} = 2, \text{Score(B)} = 6$ and $\text{Score(C)} = 7$. Hence, the Borda ranking is C $\succ$ B $\succ$ A. There are many more examples. Can you find one, where the orderings are reverse?

(3a) There are five situations for the instantiations of (A, B, C): $S_1 = (1, 1, 1), S_2 = (1, 0, 1), S_3 = (0, 1, 1), S_4 = (0, 0, 1)$, and $S_5 = (0, 0, 0)$.
(3b) Play with it!

(4) For $n = 3$, we obtain from eq. (1):

$$P(\mathrm{M}) = \left(\begin{array}{c} 3 \\ 2 \end{array} \right) r^2 \, (1 - r) + \left(\begin{array}{c} 3 \\ 3 \end{array} \right) r^3 = 3 \, p^2 - 2 \, p^3$$

Next, we calculate

$$\Delta := P(\mathrm{M}) - r = 2 \, r \, (1 - r) \, (r - 1/2)$$

(check this equation). From this equation we see that $\Delta > 0$ for $r > 1/2$ and $\Delta < 0$ for $r < 1/2$, which completes the proof. Try to gernalize this result to n voters! The algebraic calculations can be done by hand, but you can also make your life easier by using powerful and excellent software such as Mathematica, Maple or Matlab.

(5) As a start, read
http://dash.harvard.edu/bitstream/handle/1/3612779/Sen_ImpossibilityParetian.pdf?sequence=4

Also have a look at the following two entries from the Stanford Encyclopedia of Philosophy.
http://plato.stanford.edu/entries/justice-distributive/.
http://plato.stanford.edu/entries/equality/