Additional Problem Set for Lecture 3: Belief

(1) Take our principles ($\text{Id}_{\text{Worlds}}$) and ($\text{Id}_{\text{Prop}}$) from the lecture as given. Assume that w is a possible world that is distinct from the possible world w'. Can we conclude that there is a proposition that is true at one of these worlds but not at the other?

(2) Picture the set W of all possible worlds in terms of a square again; draw two distinct but intersecting circles in the square–one representing the proposition X, the other one representing the proposition Y; finally, determine (and color) graphically (i) the proposition $W \setminus (X \cap Y)$, and also (ii) the proposition $(W \setminus X) \cup (W \setminus Y)$. What does the graphical representation in (i) look like in comparison with that in (ii)?

(3) Just as in the lecture, let $W = \{w_1, ..., w_8\}$, $X = \{w_1, w_2, w_4, w_5\}$, $Y = \{w_2, w_3, w_5, w_6\}$, $Z = \{w_4, w_5, w_6, w_7\}$. Determine $W \setminus ((X \cup Y) \cup (X \cup Z))$.

(4) As in the lecture, let $W = \{w_1, ..., w_8\}$ and take $B_W = \{w_2, w_5\}$ to be the least proposition again that is believed by an inferentially perfectly rational person p. Please determine whether the following proposition is believed by p, or whether it is disbelieved by p (that is, its negation is believed), or whether p suspends judgement on it (that is, p neither believes it nor disbelieves it):

$$W \setminus w_5$$

(5) Assume our Rational Degree of Belief postulates 1-3 to be the case (for a given set W of possible worlds). Let P be the degree of belief function of an inferentially perfectly rational person.
Show that for all propositions X, Y: if $P(X) = 1$ and $P(Y) = 1$, then $P(X \cap Y) = 1$.

(6) Just as in the lecture, assume that an inferentially perfectly rational person's degree of belief function P is determined by the following function B (in the manner explained by the corresponding theorem in the lecture):
$B(w_1) = 1/15$
$B(w_2) = 1/3$
$B(w_3) = 1/15$
$B(w_4) = 1/15$
$B(w_5) = 1/3$
$B(w_6) = 1/15$
$B(w_7) = 1/15$
$B(w_8) = 0$
Determine the following degree of belief: $P(\{w_1, w_3, w_5, w_7\})$.

Solutions:

(1) Yes, this follows from ($\mathrm{Id}_{\mathrm{Worlds}}$). Some proposition must be true at one of the worlds but not at the other.

(2) They look the same. The proposition $W \setminus (X \cap Y)$ is identical to the proposition $(W \setminus X) \cup (W \setminus Y)$. This is the set-theoretic version of the second of the two so-called De Morgan laws for complement/intersection/union (or negation/conjunction/disjunction).

(3) The result is $\{w_8\}$.

(4) p suspends judgement on $W \setminus w_5$: neither the proposition nor its negation is believed.

(5) Assume that $P(X) = 1$ and $P(Y) = 1$.

We know already from the lecture that $P(\neg X) = 1 - P(X)$, which means in the present case, by the assumption that $P(X) = 1$: $P(\neg X) = 1 - 1 = 0$.

We also know from the lecture that for all X,Y: if X is a subset of Y, then $P(X) \leq P(Y)$. In the present case, we can apply this to $\neg X \cap Y$, which is a subset of $\neg X$, and conclude: $P(\neg X \cap Y) \leq P(\neg X)$, which entails with what we have shown before: $P(\neg X \cap Y) = 0$.

Finally, we also proved in the lecture that for all X, Y: $P(Y) = P(X \cap Y) + P(\neg X \cap Y)$. In the present case, this means: $P(Y) = P(X \cap Y) + P(\neg X \cap Y) =$, using what we have shown before, $= P(X \cap Y)$. But since $P(Y) = 1$ by the assumption from above, it follows that $P(X \cap Y) = 1$.

(6) $P(\{w_1, w_3, w_5, w_7\}) = 1/15 + 1/15 + 1/3 + 1/15 = 3/15 + 1/3 = 3/15 + 5/15 = 8/15$.