There are three holiday destinations:

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B: Barbados

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- Family member 1: A $\succ$ B $\succ$ C
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There are three holiday destinations:

A: Aruba

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- Family member 1: $A \succ B \succ C$
- Family member 2: $B \succ C \succ A$
- Family member 3: $C \succ A \succ B$

The individual rankings of the family members can be summarized in a table:

no. 1	no. 2	no. 3
A	B	C
B	C	A
C	A	B

Question: What, then, is the will of the family?

An aggregation procedure is a function that maps the individual preference orderings into a group (or social) ordering.

Method 1: Pairwise Comparisons (“the Condorcet Method”)

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- Note that there is a cycle and that transitivity is therefore violated (show this!). Hence, the proposed method does not yield a group ordering in this case.

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- ...
- Assign $n - 1$ points to the highest ranked option (where n is the number of group members).

- Then add up the points of all group members.

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Return to our family-holiday example and calculate the Borda scores of options A, B and C:

- Score(A) $= 2 + 0 + 1 = 3$

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Hence, $A \sim B \sim C$.

Condorcet and Borda may yield different results!

Consider this example:

5	1	4
A	A	B
B	C	C
C	B	A

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1. The Condorcet Method

- A beats B 6:4.
- A beats C 6:4.
- B beats C 9:1.

A is the Condorcet winner, as A beats all other alternatives in a pairwise comparison. C is the Condorcet loser, as it loses against all other alternatives in a pairwise comparison.

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$\rightarrow$ The group ordering is $A \succ_M B \succ_M C$.

5	1	4
A	A	B
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2. The Borda Count

- $\text{Score(A)} = 5 \times 2 + 1 \times 2 = 12$
- $\text{Score(B)} = 5 \times 1 + 4 \times 2 = 13$
- $\text{Score(C)} = 1 \times 1 + 4 \times 1 = 5$

$\rightarrow$ The group ordering is B $\succ_B$ A $\succ_B$ C, where the subscript B refers to the Borda count.

5	1	4
A	A	B
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C	B	A

3. Majority Voting

A is the majority winner as

- A is ranked first by six group members,

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A is the majority winner as

- A is ranked first by six group members,
- B is ranked first by four group members,
- C is ranked first by no group member.

5	1	4
A	A	B
B	C	C
C	B	A

4. Approval Voting

Each group member submits those options that she approves of. Let us assume that each candidate approves of her first two highest ranked options. Then

- A is approved by six group members,

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Each group member submits those options that she approves of. Let us assume that each candidate approves of her first two highest ranked options. Then

- A is approved by six group members,
- B is approved by nine group members,

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Each group member submits those options that she approves of. Let us assume that each candidate approves of her first two highest ranked options. Then

- A is approved by six group members,
- B is approved by nine group members,
- C is approved by five group members.

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Each group member submits those options that she approves of. Let us assume that each candidate approves of her first two highest ranked options. Then

- A is approved by six group members,
- B is approved by nine group members,
- C is approved by five group members.

Hence, the ranking according to the 2-approval ranking agrees (in this example) with the Borda ranking, i.e. B $\succ$ A $\succ$ C.

Anonymity: Changing the preference orderings of two group members does not affect the outcome of the aggregation.

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May's Theorem (1952): An aggregation method for choosing between two options satisfies (i) Anonymity, (ii) Neutrality and (iii) Positive Responsiveness if and only if it is the majority rule.

Universal Domain: No preference ordering over the options can be ignored by an aggregation method.

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Arrow's Impossibility Theorem: For a finite number of group members (> 1) and for at least three distinct options, there is no aggregation method that satisfies (i) Universal Domain, (ii) Pareto, (iii) Independence of Irrelevant Alternatives, and (iv) Non-Dictatorship.

Borda violates Independence of Irrelevant Alternatives. Consider this ranking:

1	2	2
A	A	B
B	C	C
C	B	A

Here Score(A) $= 6$ and Score(B) $= 5$, hence A $\succ_B$ B.

Next, one option (D) is added, but the relative rankings of A, B and C are oft unchanged:

1	2	2
D	A	B
A	C	C
B	B	D
C	D	A

Now, Score(A) $= 8$ and Score(B) $= 9$, hence B $\succ_B$ A.

Judgment Aggregation

- A city council has to make a decision on whether to build a new harbor site (= proposition C).
- It is consensus that this project should be approved of if and only if there is sufficient demand for new sites that cannot be met by existing sites (= proposition A) and
- the consequences for a nearby natural reserve are supportable (= proposition B).

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Member	A	B	C
1,2,3	Yes	Yes	Yes
4,5	Yes	No	No
6,7	No	Yes	No

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Member	A	B	C
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6,7	No	Yes	No

Note that all council members accept $(A \wedge B) \leftrightarrow C$. They are individually rational.

The Discursive Dilemma

Member	A	B	C
1,2,3	Yes	Yes	Yes
4,5	Yes	No	No
6,7	No	Yes	No
Majority	Yes	Yes	No

Way Out 1: The Premise-Based Procedure

Member	A	B	C
1,2,3	Yes	Yes	Yes
4,5	Yes	No	No
6,7	No	Yes	No
Majority	**Yes**	**Yes**	**Yes**

The group verdict on C is inferred from the group verdicts on A (= Yes) and B (= Yes) using $(A \wedge B) \leftrightarrow C$.

Upshot: The harbor site will be built.

Way Out 2: The Conclusion-Based Procedure

Member	A	B	C
1,2,3	Yes	Yes	**Yes**
4,5	Yes	No	**No**
6,7	No	Yes	**No**
Majority	–	–	**No**

Upshot: The harbor site will not be built.

The Condorcet Jury Theorem

Consider a group of n voters has to make a yes-no decision on a proposition H. We make two assumptions:

1. **The Independence Assumption:** Given the truth or falsity of H, the verdict of one voter does not depend on the verdict of any other voter.

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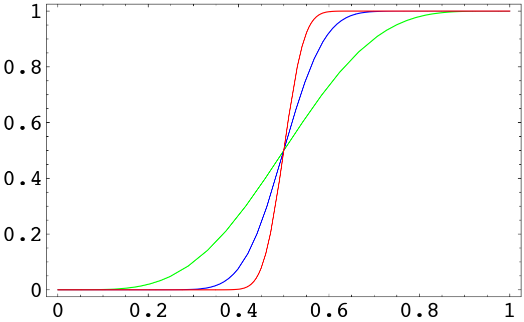
Then: The probability that the majority makes the right decision (i) increases monotonically and (ii) goes to 1 as $n \to \infty$.

The Condorcet Jury Theorem

We plot the probability that voting tracks the truth as a function of the reliability r for 9 (in green), 49 (in blue), and 199 (in red) voters.



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- All members accept $(\mathrm{A} \wedge \mathrm{B}) \leftrightarrow \mathrm{C}$.
- Simplifying notation: We represent a Yes-vote by 1 and a No-vote by 0.
- There are four admissible situations for (A, B, C):

$$
\begin{array}{ccc}
S_1 = (1,1,1) & , & S_2 = (1,0,0) \\
S_3 = (0,1,0) & , & S_4 = (0,0,0)
\end{array}
$$

- Note that only these four situations satisfy $(\mathrm{A} \wedge \mathrm{B}) \leftrightarrow \mathrm{C}$.
- One of these four situations is the true one. We just do not know which.

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- If S_2 is the true situation, then a council member will vote Yes on proposition A with probability r, and No on proposition B with probability r.
- Etc.

Prior Probabilities

- The prior probability that S_1 is the true situation is q_1.
- The prior probability that S_2 is the true situation is q_2.
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Principle of Indifference: $q_1 = q_2 = q_3 = q_4 = 1/4$

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- Use the Premise-Based Procedure, the Conclusion-Based Procedure and the Bayesian Procedure (see below) to aggregate the individual verdicts.
- Do this many times and count how often situation 1 results from the respective aggregation procedure. Works out the corresponding fraction.
- Do the same for situations 2, 3 and 4.
- Weigh the four fractions with the prior probabilities and obtain the wanted probability.
- Compare the three procedures.

Two Main Results

1. The Premise-Based Procedure performs much better than the Conclusion-Based Procedure.
2. The Premise-Based Procedure performs almost as good as the (optimal) Bayesian Procedure.

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- Update the priors using Bayes' Theorem taking the evidence into account.
- Select the situation with the highest posterior probability.

The Bayesian Procedure

This is the optimal aggregation procedure.

- Start with the priors $q_1, \dots, q_4$.
- Take the judgments of the (partially reliable) council members as evidence.
- Update the priors using Bayes' Theorem taking the evidence into account.
- Select the situation with the highest posterior probability.

Note that this method takes all available information into account (including the voting margins).