Additional Problem Set for Lecture 8: Quantum Logic and Probability

(1) In the lecture, we considered the case of two random variables, A and B which can take the values ± 1, with $E(\mathrm{A}) = E(\mathrm{B}) = 0$ and $cov(\mathrm{A}, \mathrm{B}) = x \in [-1, 1]$. Show that $P(1, 1) = P(-1, -1) = (1 + x)/4$ and $P(1, -1) = P(-1, 1) = (1 - x)/4$.

(2) Consider three binary random variables A, B, and C which can take the values ± 1. Furthermore, $E(\mathrm{A}) = E(\mathrm{B}) = E(\mathrm{C}) = 0$ and $E(\mathrm{AB}) = E(\mathrm{BC}) = -1$. Show that a joint probability distribution over the three variables exists if and only if $E(\mathrm{AC}) = 1$.

(3) Consider four random variables A, A', B and B' which can take the values ± 1 and let there be a probability distribution $P(A, A', B, B')$ defined over these variables. Many calculations make the so-called *symmetry assumption*, i.e. they assume that $p_{1111} = p_{0000}, p_{1010} = p_{0101}, p_{0001} = p_{1110}$ etc., where we use the shorthand-notation $p_{1010} := P(A = 1, A' = -1, B = 1, B' = -1)$, etc. Show that the symmetry assumption implies that $E(\mathrm{A}) = E(\mathrm{A}') = E(\mathrm{B}) = E(\mathrm{B}') = 0$.

(4) Read about the Kochen-Specker theorem in the Stanford Encyclopedia of Philosophy: http://plato.stanford.edu/entries/kochen-specker/ Does the theorem establish that all observables defined for a quantum mechanical system have definite values at all times?

Solutions

(1) Since $E(\mathrm{A}) = E(\mathrm{B}) = 0$, we have $cov(\mathrm{A}, \mathrm{B}) = E(\mathrm{AB}) = x$. We have seen in the lecture that $E(\mathrm{A}) = 0$ implies that

$$P(1,1) + P(1,-1) - P(-1,1) - P(-1,-1) = 0. \tag{1}$$

$E(\mathrm{B}) = 0$ implies that

$$P(1,1) - P(1,-1) + P(-1,1) - P(-1,-1) = 0. \tag{2}$$

$E(\mathrm{AB}) = x$ implies

$$P(1,1) - P(1,-1) - P(-1,1) + P(-1,-1) = x. \tag{3}$$

Finally, Probability Theory implies

$$P(1,1) + P(1,-1) + P(-1,1) + P(-1,-1) = 1 \tag{4}$$

To solve this system of four linear equations, we first add eq. (4) to eq. (1) and obtain:

$$P(1,1) + P(1,-1) = 1 \tag{5}$$

Next, we add eq. (4) to eq. (2) and obtain:

$$P(1,1) + P(-1,1) = 1 \tag{6}$$

Comparing eq. (5) and eq. (6) we conclude that

$$P(1,-1) = P(-1,1). \tag{7}$$

From eq. (7) and eq. (2) we conclude that

$$P(1,1) = P(-1,-1). \tag{8}$$

Next, we add eq. (3) and eq. (4) and use eq. (8) to obtain:

$$P(1,1) = P(-1,-1) = \frac{1+x}{4} \tag{9}$$

Inserting eq. (9) into eq. (6) and remember eq. (7), we obtain

$$P(1,-1) = P(-1,1) = \frac{1-x}{4}, \tag{10}$$

which completes the proof.

(2) We introduce the shorthand-notation $P(A=1, B=1, C=1) =: p_{111}, P(A=1, B=-1, C=-1) =: p_{100}$ etc. and write $p_{11\cdot}$ for $p_{111}+p_{110}$ etc. Here the dot indicates that we sum over the two possible values it can take. With this, it is easy to see that $E(\mathrm{AB}) = -1$ implies

$$p_{11\cdot} - p_{10\cdot} - p_{01\cdot} + p_{00\cdot} = -1. \tag{11}$$

Similarly, $E(\mathrm{BC}) = -1$ implies

$$p_{\cdot 11} - p_{\cdot 10} - p_{\cdot 01} + p_{\cdot 00} = -1. \tag{12}$$

As all probabilities sum up to 1, we also have

$$\begin{aligned} p_{11\cdot} + p_{10\cdot} + p_{01\cdot} + p_{00\cdot} &= 1, & (13) \\ p_{\cdot 11} + p_{\cdot 10} + p_{\cdot 01} + p_{\cdot 00} &= 1. & (14) \end{aligned}$$

We add eq. (13) to eq. (11) and (14) to eq. (12) and obtain:

$$\begin{aligned} p_{11\cdot} + p_{00\cdot} &= 0, & (15) \\ p_{\cdot 11} + p_{\cdot 00} &= 0. & (16) \end{aligned}$$

Hence,

$$p_{111} = p_{110} = p_{100} = p_{011} = p_{001} = p_{000} = 0. \tag{17}$$

Inserting eqs. (17) into eq. (11), we obtain:

$$p_{101} + p_{010} = 1 \tag{18}$$

Let us now look at the implications of $E(\mathrm{A}) = E(\mathrm{B}) = E(\mathrm{C}) = 0$, using eqs. (17):

$$\begin{aligned} E(\mathrm{A}) &= p_{1\cdot\cdot} - p_{0\cdot\cdot} = p_{101} - p_{010} = 0 & (19) \\ E(\mathrm{B}) &= p_{\cdot 1\cdot} - p_{\cdot 0\cdot} = p_{010} - p_{101} = 0 & (20) \\ E(\mathrm{C}) &= p_{\cdot\cdot 1} - p_{\cdot\cdot 0} = p_{101} - p_{010} = 0 & (21) \\ & & (22) \end{aligned}$$

Hence,

$$p_{101} = p_{010}, \tag{23}$$

and with eq. (18), we obtain

$$p_{101} = p_{010} = 1/2. \tag{24}$$

Eq. (17) and (24) fix the probability distribution uniquely. We can now calculate E(AC) and obtain:

$$E(\mathrm{AC}) = p_{1\cdot 1} - p_{1\cdot 0} - p_{0\cdot 1} + p_{0\cdot 0} = p_{101} + p_{010} = 1. \tag{25}$$

Hence, a probability distribution only exists if $E(\mathrm{AC}) = 1$. For a generalization of the result of this problem, see http://suppes-corpus.stanford.edu/article.html?id=215.

(3) We calculate

$$\begin{aligned} E(\mathrm{A}) &= p_{1\cdots} - p_{0\cdots} \\ &= (p_{1111} - p_{0000}) + (p_{1110} - p_{0001}) + (p_{1101} - p_{0010}) + (p_{1100} - p_{0011}) \\ &\quad +(p_{1011} - p_{0100}) + (p_{1010} - p_{0101}) + (p_{1001} - p_{0110}) + (p_{1000} - p_{0111}), \end{aligned} \tag{26}$$

where we grouped corresponding terms together. Symmetry then implies that $E(\mathrm{A}) = 0$. Similarly for $E(\mathrm{A}'), E(\mathrm{B})$ and $E(\mathrm{B}')$.

(4) No, one can also deny the conditions (NC) and (O) discussed in the entry.