General features of games and game theory

- Interdependent decisions
- Self-interested agents
- Not a theory of teamwork
- Here: non-cooperative game theory

The matrix: games in strategic forms

		Me	
		Heads	Tails
You	Heads		
	Tails		

Pure and mixed strategies

		Me	
		Heads	Tails
You	Heads		
	Tails		

- A mixed strategy is a probability distribution over the set of pure strategies. For instance:
  - (1/2 Heads, 1/2 Tails)
  - (1/3 Heads, 2/3 Tails)
  - ...

Outcomes, payoffs, preferences, other properties

		Me	
		Heads	Tails
You	Heads	(2, -2)	(-2, 2)
	Tails	(-2, 2)	(2, -2)

Prisoners' Dilemma

Bob

Ann		Don't Testify	Testify
	Don't Testify	(-2, -2)	(-10, 0)
	Testify	(0, -10)	(-5, -5)

Pure coordination game

Bob

		Left	Right
Ann	Left	(1, 1)	(0, 0)
	Right	(0, 0)	(1, 1)

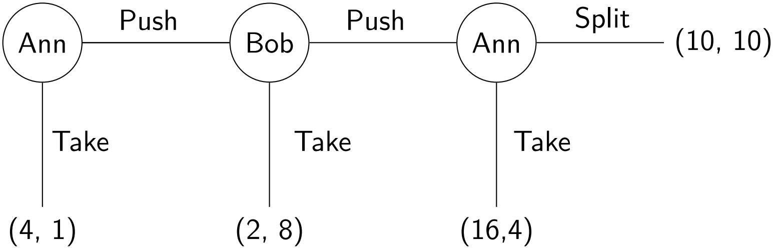
Games in strategic forms

Definition

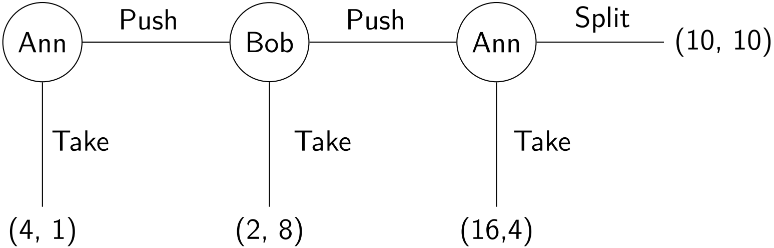
A game in strategic form $\mathbb{G}$ is a tuple $\langle N, S_i, u_i \rangle$ such that :

- N is a finite set of agents.
- S_i is a finite set of *actions* or *strategies* for i. A *strategy profile* $\sigma \in \Pi_{i \in N} S_i$ is a vector of strategies, one for each agent in I. The strategy s_i which i plays in the profile σ is noted σ_i. Similarly for σ_{-i}.
- $u_i : \Pi_{i \in N} S_i \to \mathbb{R}$ is an *utility function* that assigns to every strategy profile $\sigma \in \Pi_{i \in N} S_i$ the utility valuation of that profile for agent i.

Games in extensive form



Actions and strategies



- Bob: Take, Push
- Ann: always Take (T,T), first Take then Split (T,S), first Push then Take (P,T), first Push then Split (P,S)

Information in (extensive) games

- Perfect and imperfect information
- Complete and incomplete information

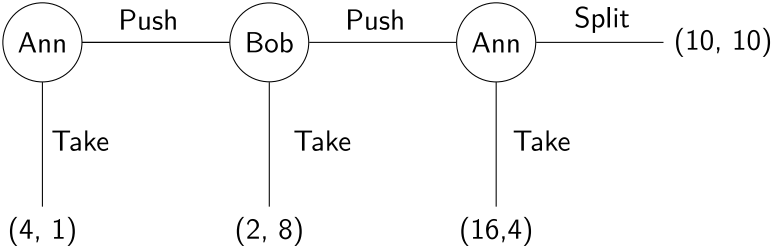
Extensive form games

Definition

A *game in extensive form* $\mathcal{T}$ is a tuple $\langle N, T, \tau, \{U_i\}_{i \in N} \rangle$ such that:

- T is a tree.
- τ : non-terminal nodes $\rightarrow N$ is a *turn function* which assigns to every non-terminal node x the player whose turn it is to play at x.
- U_i : terminal nodes $\rightarrow \mathbb{R}$ is a *payoff function* for player i which assigns i's payoff at each terminal node.

Strategy profiles and paths



- A profile is a combination of strategies, one for each players.
- A strategy profile determines a unique path from the root to an outcome.

		Bob	
		Don't Testify	Testify
Ann	Don't Testify	(-2, -2)	(-10, 0)
	Testify	(0, -10)	(-5, -5)

		Bob	
		Left	Right
Ann	Up	(4, 1)	(7, 0)
	Down	(2, 0)	(2, 1)

		Bob	
		Left	Right
Ann	Left	(1, 1)	(0, 0)
	Right	(0, 0)	(1, 1)

Definition

A strategy profile σ is a *Nash equilibrium* iff for all i and all $s'_i \neq \sigma_i$:

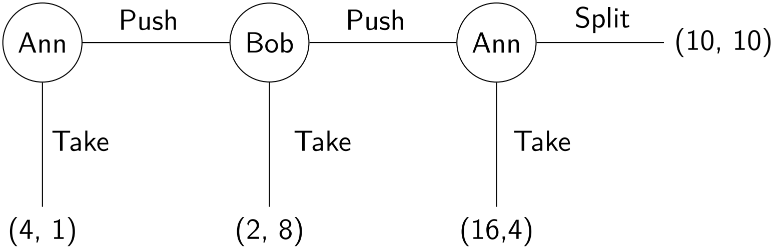
$$u_i(\sigma) \geq u_i(s'_i, \sigma_{-i})$$

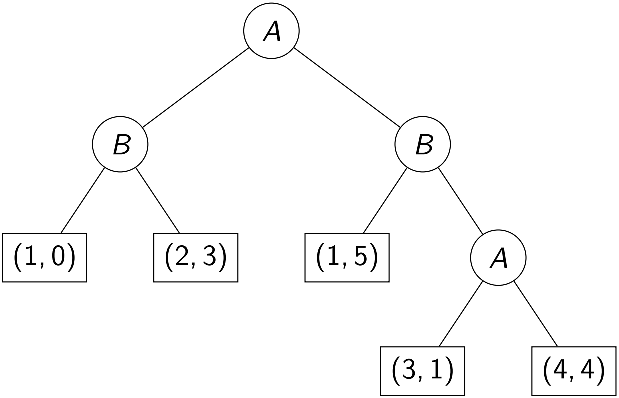
		Me	
		Heads	Tails
You	Heads	(2, -2)	(-2, 2)
	Tails	(-2, 2)	(2, -2)

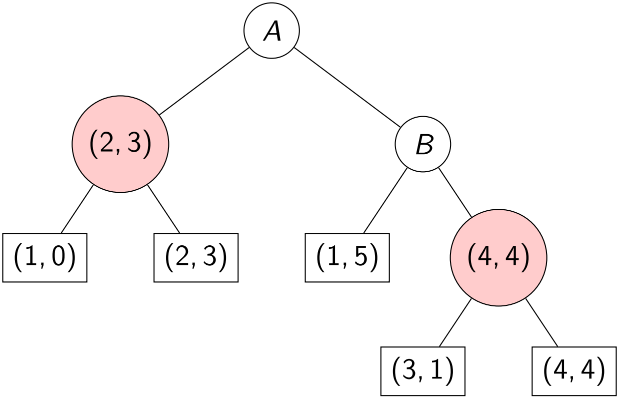
Iterated strict dominance and Nash

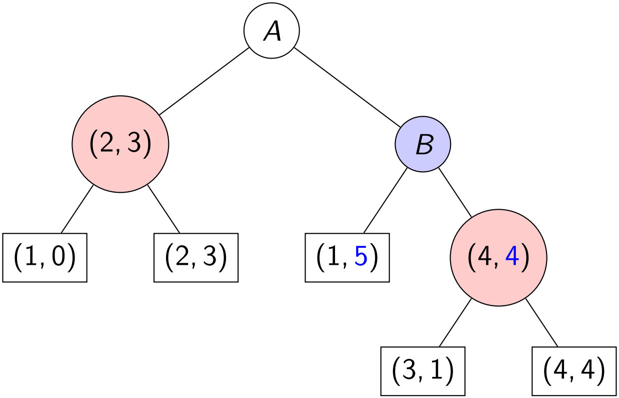
- *All Nash equilibria survive IESDS*. But not all profiles that survive IESDS are Nash equilibria.
- *The algorithm is order independent*: One can eliminate SDS one player at the time, in difference order, or all simultaneously. The end-point of the elimination procedure will always be the same.
- *Weak dominance*: always as bad, sometimes worse.

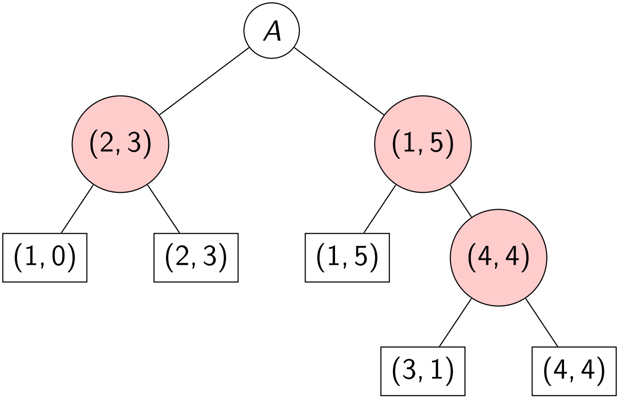
Solution for the tree



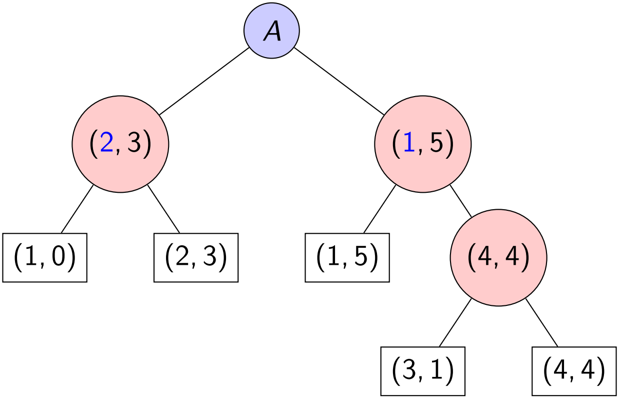

A
B
B
(1, 0)
(2, 3)
(1, 5)
A
(3, 1)
(4, 4)


A
(2, 3)
B
(1, 0)
(2, 3)
(1, 5)
(4, 4)
(3, 1)
(4, 4)

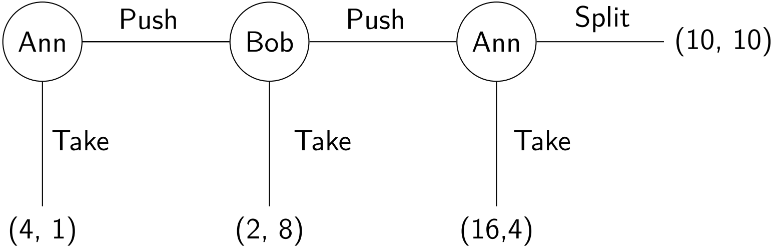

A
(2, 3)
B
(1, 0)
(2, 3)
(1, 5)
(4, 4)
(3, 1)
(4, 4)



A
(2, 3)
(1, 5)
(1, 0)
(2, 3)
(1, 5)
(4, 4)
(3, 1)
(4, 4)



A
$(2, 3)$
$(1, 5)$
$(1, 0)$
$(2, 3)$
$(1, 5)$
$(4, 4)$
$(3, 1)$
$(4, 4)$


Ann
Push
Bob
Push
Ann
Split
(10, 10)
Take
Take
Take
(4, 1)
(2, 8)
(16,4)

Rationality - in games

- The outcome of a rational play should be a Nash equilibrium? Why?
- Rational players should only play strategies that survive iterated elimination by strict dominance? Why?

Decision-theoretic rationality

Choose the action which gives, in your view, the highest chance of achieving a desirable outcome.

$$EU(A) = \sum_{o \in Out} [\text{ subjective prob. of } o \text{ given A}] \times [\text{utility of } o]$$

What should Ann Do?

		Bob	
		Left	Right
Ann	Left	(1, 1)	(0, 0)
	Right	(0, 0)	(1, 1)

- Actions, states of the world, preferences...

What should Ann Do?

		Bob	
		Left	Right
Ann	Left	(1, 1)	(0, 0)
	Right	(0, 0)	(1, 1)

- Actions, states of the world, preferences...and beliefs!

What should Ann do?

Example: Ann believes with degree 0.9 that Bob will drive on the Right.

$$EU(Right) = 0.9(1) + 0.1(0) = 0.9$$

$$EU(Left) = 0.9(0) + 0.1(1) = 0.1$$

She should play Right.

Why maximize?

Ask Ramsey, De Finetti, von Neumann and Morgenstern, Savage.

Coherence!

What should Bob do?

Example: Bob believes with degree 0.8 that Ann will drive on the Left.

$$EU_B(Right) = 0.2(1) + 0.8(0) = 0.2$$

$$EU_B(Left) = 0.2(0) + 0.8(1) = 0.8$$

He should play Left.
Ann believes with degree 0.9 that Bob will drive on the Right.

$$EU_A(Right) = 0.9(1) + 0.1(0) = 0.9$$

$$EU_A(Left) = 0.9(0) + 0.1(1) = 0.1$$

She should play Right.

Bob

		Left	Right
Ann	Left	(1, 1)	(0, 0)
	Right	(0, 0)	(1, 1)

Rational agents need not to play Nash equilibrium.

Fact. A strategy $s_i \in S_i$ is strictly dominated iff there is no belief of i under which s_i maximizes expected utility.

		Bob	
		Don't Testify	Testify
Ann	Don't Testify	(-2, -2)	(-10, 0)
	Testify	(0, -10)	(-5, -5)

Bob

Ann		Left	Right
	Up	(4, 1)	(7, 0)
	Down	(2, 0)	(2, 1)

Different types of uncertainty in games

- Strategic uncertainty
- Imperfect information
- Incomplete information

		Bob	
		Left	Right
Ann	Up	(4, 1)	(7, 0)
	Down	(2, 0)	(2, 1)

Models of contexts

- Kripke models
- Aumann / epistemic-probability models
- Harsanyi type spaces

Models for $\mathcal{G}$: $\langle W, \{prob_i, \pi_i\}_{i \in N}, f \rangle$

W : Set of possible worlds or states

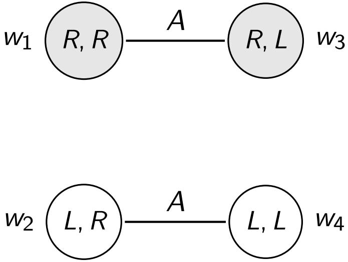
Models for $\mathcal{G}$: $\langle W, \{prob_i, \pi_i\}_{i \in N}, f \rangle$

$$f\colon W \rightarrow \Pi_i S_i$$

Knowledge in partitions models

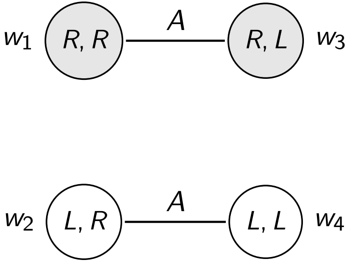
$$K_i(E) \text{ at } w \text{ iff } \pi_i(w) \subseteq E$$

$$K_i(E) \rightarrow E$$



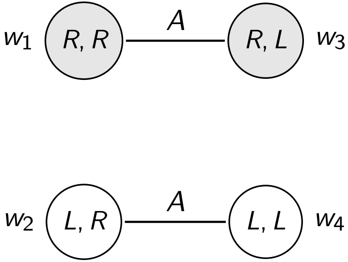
$K_A(Right_B)$ at w_1? No!

$K_A(Right_A)$ at w_1? Yes!



$K_A K_A(Right_A)$ at w_1? Yes!

$$K_i(E) \rightarrow K_i K_i(E)$$

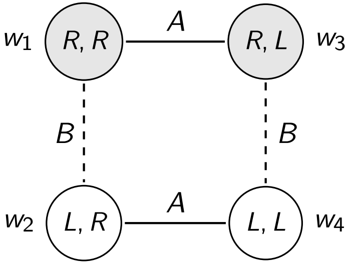


$K_A \neg K_A(Bob_A)$ at w_1? Yes!

$$\neg K_i(E) \to K_i \neg K_i(E)$$

Two assumptions

1. Every partition cell, for each agent, has a positive prior probability.
2. *Ex Interim*: if $v \in \pi_i(w)$ then $(f(v))_i = (f(w))_i$



$K_A \neg K_B(Right_A)$ at w_1

$K_B \neg K_A(Right_B)$ at w_1

$K_A K_B(K_A(Right_A) \vee K_A(Left_A))$ at w_1

Common knowledge:

- What any fool would know in a certain context
- What is agreed upon, established by precedent, assumed, being attended to, salient, or in the conversational record

Shared knowledge - "everybody knows"

E is shared knowledge between Ann and Bob iff

$$K_A(E) \wedge K_B(E)$$

E is shared knowledge (up to level 1) in a context iff

$$\bigwedge_{i \in N} K_i E$$

E is shared knowledge (up to level $n+1$) in a context iff

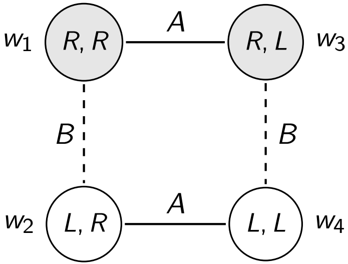
$$\bigwedge_{i \in N} K_i(E \text{ is shared knowledge up to level } n)$$

Common knowledge - iterative def.

E is common knowledge in a context iff

E is shared knowledge up to level n, for all natural numbers n.

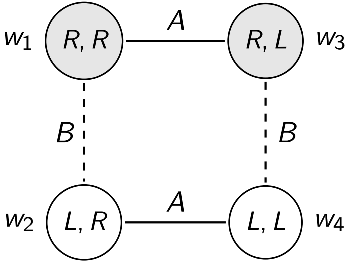
Walking in the model...



Common knowledge - fixed-point def.

E is common knowledge in a context iff

everybody knows that (E, and that E is common knowledge)


w_1
R, R
A
R, L
w_3
B
B
w_2
L, R
A
L, L
w_4

Decision-theoretic rationality

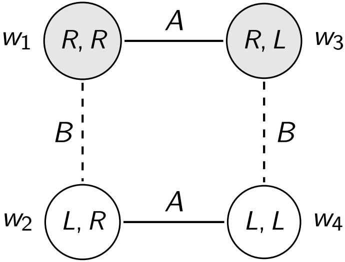
Choose the action which gives, in your view, the highest chance of achieving a desirable outcome.

$$EU(A) = \sum_{o \in Out} [\text{ subjective prob. of } o \text{ given A}] \times [\text{utility of } o]$$

Rationality in a context

Choose the action which gives, in your view, the highest chance of achieving a desirable outcome.

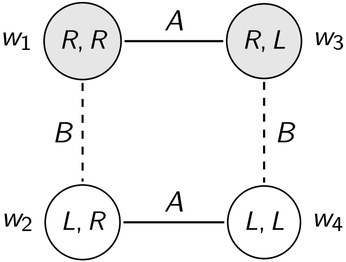
$$EU_i^w(s_i) = \sum_{v \in \pi_i i} [prob_i(v/\pi_i(w)) \times (\text{utility of playing } s_i \text{ at } v)]$$



		Bob	
		Left	Right
Ann	Left	(1, 1)	(0, 0)
	Right	(0, 0)	(1, 1)

$$EU_A^{w_1}(Right_A) = 2/3(1) + 1/3(0)$$

$$EU_A^{w_1}(Left_A) = 2/3(0) + 1/3(1)$$



		Bob	
		Left	Right
Ann	Left	(1, 1)	(0, 0)
	Right	(0, 0)	(1, 1)

Everybody is rational

- $Rational_i$ = the set of states where i is rational
- $Rational = \bigcap_i Rational_i$ = the set of states where all players are rational
- Everybody knows that everyone is rational...
- Rationality is common knowledge...

Bob

		Left	Right
Ann	Up	(4, 1)	(7, 0)
	Down	(2, 0)	(2, 1)

CKR $\rightarrow$ Iterated strict dominance play

Take any state and in a model for a game.

1. No rational agent plays strictly dominated strategies.
2. Suppose everybody knows that (rationality is shared knowledge up to $n-1$).
   - $\rightarrow$ (By IH:) everybody knows that (no one plays strategy that would not survive n rounds).
   - $\rightarrow$ If s_i would be eliminated at round $n+1$, then s_i is not rational given that information.

- IF CKR, THEN iterated strict dominance play.
- Iterated strict dominance play is always consistent with CKR.
  - IF Iterated Dominance Play, THEN $\exists$ a model of that play with CKR.

Bob

		Left	Right
Ann	Left	(1, 1)	(0, 0)
	Right	(0, 0)	(1, 1)

- IF CKR, THEN iterated strict dominance play.
- Iterated strict dominance play is always consistent with CKR.
    - IF Iterated Dominance Play, THEN $\exists$ a model of that play with CKR.

Two is a party, three's...

Suppose:

- Common knowledge of "conjectures"
- Common prior beliefs
- Rational players

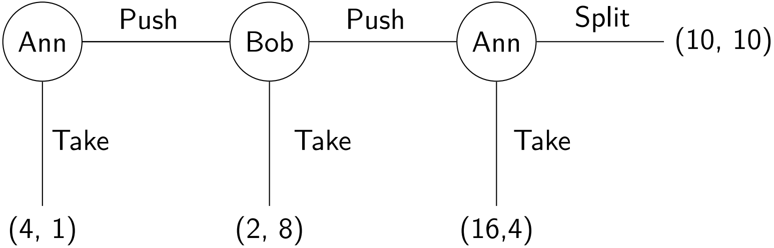
Then these conjectures constitute a Nash equilibrium (possibly in mixed strategies).

$\Rightarrow$ Epistemic interpretation of mixed strategies

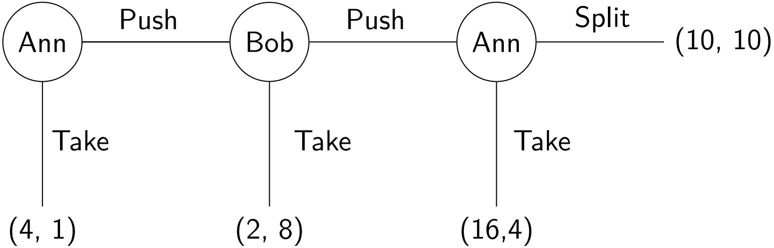
When all is done, and said

- For two players: Rationality and shared knowledge of strategy $\rightarrow$ Nash
- For more: Rationality and common knowledge of conjectures (+CP) "$\rightarrow$" Nash

In both antecedents strategic uncertainty is completely resolved.


Ann
Push
Bob
Push
Ann
Split
(10, 10)
Take
Take
Take
(4, 1)
(2, 8)
(16,4)

Common knowledge of substantive rationality $\rightarrow$ BI



- Rationality in a context, at a node
- Always rational from now on...
- Belief revision!

Epistemic view on games

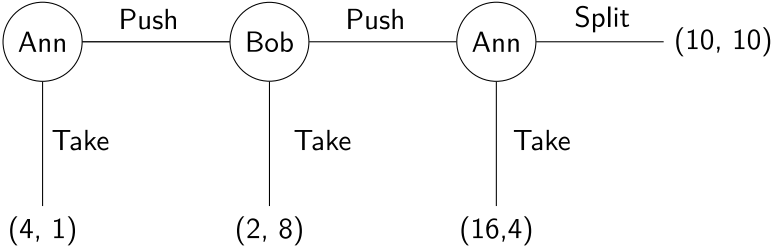
Players should be seen as individual decision makers, choosing what to do on the basis of their own preferences and the information they have in specific contexts.

$\Rightarrow$ Rationality alone does not entail classical solution concepts.

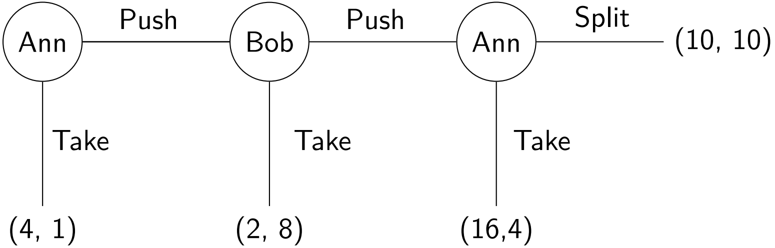
Epistemic characterization results

- CKR $\rightarrow$ Iterated strict dominance (in the matrix)
- Shared knowledge of choice and rationality $\rightarrow$ Nash (2 players, in the matrix)
- CK of substantive rationality $\rightarrow$ backward induction

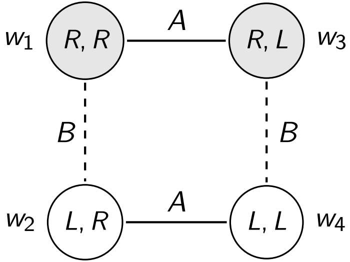
What should Bob think of Ann?



Forward or backward induction?



Better ways to choose?



Ann / Bob

	Left	Right
Left	(1, 1)	(0, 0)
Right	(0, 0)	(1, 1)

Two questions

1. Why play according to the classical solution concept?
2. When will rational players play according to the classical solution concepts?