Lecture 7

Non-modal stability analysis I

AE209 Hydrodynamic stability
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Lecture outline

1. Motivation
2. Initial value problem of linearised equation
3. Transient growth and non-normal linear operator

Lecture outline

1. **Motivation**

Limitation of linear stability analysis

Critical Reynolds numbers from linear stability analysis

Flow configurations	Critical Re for energy growth	Critical Re (Linear stability)	Transition Re
Couette flow	20.7	∞	350-400
Poiseulle flow	49.6	5772.2	1000-2000
Pipe flow	81.5	∞	2000-2500

Remark

Nonlinear terms in the form of perturbed Navier-Stokes equation play **no role in the mechanism of disturbance growth.**

Reynolds-Orr equation

$$\frac{1}{2}\frac{d(KE)}{dt} \leftarrow \int_V \frac{1}{2}\frac{\partial u_i u_i}{\partial t} dV = -\int_V u_i u_j \frac{\partial U_i}{\partial x_j} dV - \frac{1}{Re}\int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} dV < 0.$$

$$\int_V u_i \left[\frac{\partial u_i}{\partial t} = -u_j \frac{\partial U_i}{\partial x_j} - U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial p}{\partial x_i}\right] dV, \quad \frac{\partial u_j}{\partial x_j} = 0$$

Production term. of instability

$= 0$ $= 0$ $= 0$

dissipation.

$$u_i \frac{\partial u_i}{\partial t} = \frac{1}{2}\left(u_i \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial t}\right) = \frac{1}{2}\frac{\partial (u_i u_i)}{\partial t}$$

$$-\int_V u_i u_j \frac{\partial u_i}{\partial x_j} dV$$

$$\int u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j} dV = \cancel{\left[u_i \frac{\partial u_i}{\partial x_j}\right]_{\partial V_j}}^{\,0} - \int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} dV$$

$$\int_V u_i U_j \frac{\partial u_i}{\partial x_j} dV = \int_V \underbrace{U_j}_{a} \underbrace{\frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i u_i\right)}_{b'} dV$$

$$= \underbrace{\cancel{\left[\frac{1}{2}U_j u_i u_i\right]_{\partial V_j}}^{\,0}}_{ab} - \int_V \frac{1}{2}\underbrace{\cancel{\frac{\partial U_j}{\partial x_j}}}_{0} \underbrace{u_i u_i}_{b} dV$$

$$\doteq 0$$

2. Initial value problem of linearised equation

What is missing in the classical linear stability analysis? 7/21

Full solution of a linearised equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{L}\mathbf{u} \quad \text{with} \quad \mathbf{u}(t=0) = \mathbf{u}_0$$

Assume that $\mathbf{L}$ is a diagonalisable matrix with the eigenvectors given by $\{\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2, \tilde{\mathbf{u}}_3, \tilde{\mathbf{u}}_4, \ldots, \tilde{\mathbf{u}}_n\}$. Then, the solution of the equation above is obtained by the eigenfunction expansion technique:

$$\mathbf{u}(t) = a_1(t)\tilde{\mathbf{u}}_1 + a_2(t)\tilde{\mathbf{u}}_2 + a_3(t)\tilde{\mathbf{u}}_3 + \ldots + a_n(t)\tilde{\mathbf{u}}_n$$

$$\underline{L} = \underline{U}\,\underline{\Lambda}\,\underline{U}^{-1}$$

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & & \\ 0 & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

eigenvalues

$$\underline{U} = \begin{bmatrix} | & | & & | \\ \tilde{\underline{u}}_1 & \tilde{\underline{u}}_2 & \cdots & \tilde{\underline{u}}_n \\ | & | & & | \end{bmatrix} \quad \text{and} \quad \underline{a}(t) = \begin{bmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{bmatrix}$$

$$\underline{u}(t) = \underline{U}\,\underline{a}(t).$$

What is missing in the classical linear stability analysis? 8/21

Then,

$$\frac{d\mathbf{Ua}}{dt} = \underline{\mathbf{L}}\mathbf{Ua}$$

$$= \underline{U}\Delta$$

$\underline{L} = \underline{U}\,\underline{\Lambda}\,\underline{U}^{-1}$ and $\underline{u} = \underline{U}\,\underline{a}$

U^{-1}

$$\frac{d\mathbf{a}}{dt} = \mathbf{\Lambda a} \quad \text{with} \quad \mathbf{a}(t=0) = \left[a_{1,0}\ \ a_{2,0}\ \ a_{3,0}\ \ a_{4,0}\ \ \cdots\ \ a_{n,0}\right]^T$$

$$\frac{d}{dt}\begin{bmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 a_1 \\ \lambda_2 a_2 \\ \vdots \\ \lambda_n a_n \end{bmatrix} \Rightarrow \begin{aligned} a_1(t) &= a_{1,0}\, e^{\lambda_1 t} \\ a_2(t) &= a_{2,0}\, e^{\lambda_2 t} \\ &\vdots \\ a_n(t) &= a_{n,0}\, e^{\lambda_n t}. \end{aligned}$$

What is missing in the classical linear stability analysis? 9/21

Full solution of the linear system is then given by

$\mathrm{Re}(\lambda_1) > \mathrm{Re}(\lambda_2) > \ldots$

$$\mathbf{u}(t) = \underbrace{a_{1,0}e^{\lambda_1 t}\tilde{\mathbf{u}}_1}_{\text{Normal mode solution.}} + \underbrace{a_{2,0}e^{\lambda_2 t}\tilde{\mathbf{u}}_2 + a_{3,0}e^{\lambda_3 t}\tilde{\mathbf{u}}_3 + \ldots + a_{n,0}e^{\lambda_n t}\tilde{\mathbf{u}}_n}_{\times}$$

Eigenvalue problem

Normal mode solution.

→ Examines only the evolution of the most unstable eigenmode.

⇒ Fails to capture the interaction between all the eigenmodes

Lecture outline

3. **Transient growth and non-normal linear operator**

The structure of linearised Navier-Stokes equation

Velocity and vorticity form of linearised Navier-Stokes equation

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^4\right]v'=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^2\right)\eta'+\frac{dU}{dy}\frac{\partial v'}{\partial z}=0$$

Consider an initial value problem of the following wave by setting

$$v'(x,y,z,t)=\hat{v}(t;\alpha,\beta)e^{i\alpha x+i\beta z}$$

$$\eta'(x,y,z,t)=\hat{\eta}(t;\alpha,\beta)e^{i\alpha x+i\beta z}$$

The structure of linearised Navier-Stokes equation

Matrix form of initial value problem for Orr-Sommerfeld-Squire system:

$$\frac{\partial}{\partial t}\begin{bmatrix} \underbrace{k^2-D^2}_{\text{Const.}} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix}+\begin{bmatrix} L_{OS} & 0 \\ \underbrace{i\beta DU}_{\text{Const.}} & L_{SQ} \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix}=0$$

where $k^2=\alpha^2+\beta^2$ (assume $\alpha=0$)

$$L_{OS}=\cancel{i\alpha U(k^2-D^2)+i\alpha D^2 U}+\frac{1}{\text{Re}}(k^2-D^2)^2$$

$$L_{SQ}=\cancel{i\alpha U}+\frac{1}{\text{Re}}(k^2-D^2)$$

$L_{OS} \sim \frac{1}{\text{Re}}$

$L_{SQ} \sim \frac{1}{\text{Re}}$

Remark

As $\text{Re}\to\infty$, the role of the off-diagonal term becomes more and more important.

A model problem

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} -1/\mathrm{Re} & 0 \\ 1 & -2/\mathrm{Re} \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix}$$

with the initial condition $\begin{bmatrix} v & \eta \end{bmatrix}_{t=0} = \begin{bmatrix} v_0 & \eta_0 \end{bmatrix}$

Solution) from page 9

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = a_{1,0} e^{\lambda_1 t}\begin{bmatrix} \hat{v}_1 \\ \hat{\eta}_1 \end{bmatrix} + a_{2,0} e^{\lambda_2 t}\begin{bmatrix} \hat{v}_2 \\ \hat{\eta}_2 \end{bmatrix}$$

with $\lambda_1 = -1/\mathrm{Re}$ and $\lambda_2 = -2/\mathrm{Re}$

$$\begin{bmatrix} \hat{v}_1 \\ \hat{\eta}_1 \end{bmatrix} = \frac{1}{\sqrt{1+\mathrm{Re}^2}}\begin{bmatrix} 1 \\ \mathrm{Re} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{v}_2 \\ \hat{\eta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

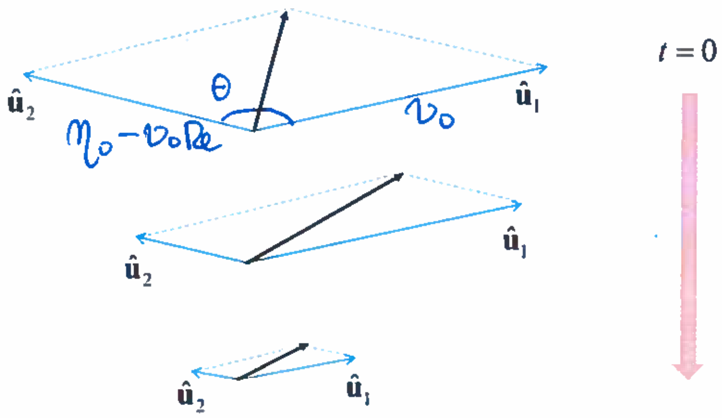
Transient growth in a linear model

Schematic diagram of temporal evolution of the solution

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = v_0 e^{-t/\mathrm{Re}} \begin{bmatrix} 1 \\ \mathrm{Re} \end{bmatrix} + (\eta_0 - v_0\,\mathrm{Re}) e^{-2t/\mathrm{Re}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\tilde{\underline{u}}_1 \cdot \tilde{\underline{u}}_2 = \|\tilde{\underline{u}}_1\| \|\tilde{\underline{u}}_2\| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\mathrm{Re}}{\sqrt{1+\mathrm{Re}^2}}$$



Short term growth mechanism in time due to non-orthogonal nature of the two eigenvectors

If $\mathrm{Re} \to 0$ $\quad \cos\theta = 1 \quad \theta = 0°$ or $180°$

$\mathrm{Re} \to \infty \quad \cos\theta = 0 \quad \theta = 90°$

Remark 1

1) In the limit of $\mathrm{Re} \to 0$, the two eigenvectors are orthogonal to each other, yielding a **monotonically decaying solution in time**: i.e.

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} v_0 e^{-1/\mathrm{Re}\,t} \\ \eta_0 e^{-2/\mathrm{Re}\,t} \end{bmatrix}$$

$e^{\lambda t}$, $te^{\lambda t}$

2) In the limit of $\mathrm{Re} \to \infty$, $\lambda_1 = \lambda_2 = 0$, and two eigenvectors become the same, yielding the following **algebraically growing solution**: i.e.

growth of energy.

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = \underline{(\eta_0 + v_0 t)}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + v_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

t

$\|\underline{u}\|^2 \sim t^2$

Short term growth.

Transient growth in a linear model

Remark 2

1) The non-orthogonal superposition of exponentially decaying soluti ons can give rise to short-term transient growth.

2) **Eigenvalues alone** only describe **the asymptotic fate of the disturb ance**, but **fail to capture transient** effects.

3) The **"source" of the transient amplification** of the initial condition lies in the **nonorthogonality of the eigenfunction basis.**

4) The **non-orthogonal eigenfunctions** are the typical nature of the **non-normal linear operator**.

Non-normal linear operator

Definition: Non-normal operator

Linear operators, the eigenfunctions (or eigenvectors) of which are **non-orthogonal** to one another with respect to the given inner product, is cal led **non-normal**.

Remark

Linearised Navier-Stokes equation with non-zero advection term is a non **-normal** linear operator

Summary

1. **Motivation**
2. **Initial value problem of linearised equation**
3. **Transient growth and non-normal linear operator**