Lecture 5

Linear stability of parallel shear flows II

AE209 Hydrodynamic stability
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1. Linear stability analysis of inviscid mixing layer
2. Linearised equation for viscous parallel flows
3. Normal mode solution
4. Squire’s transformation

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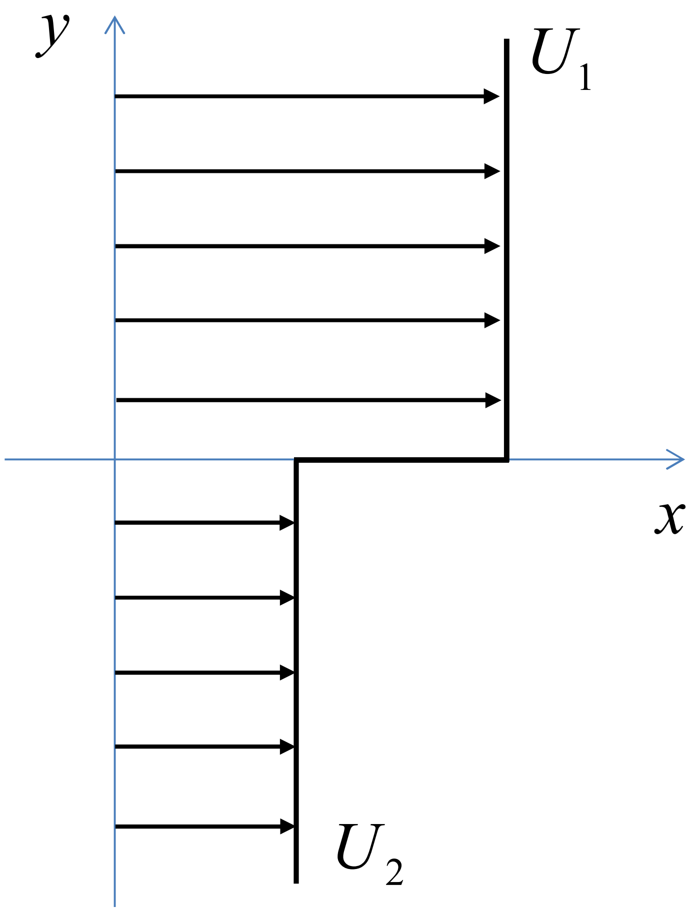
Example 1: Piecewise mixing layer



Base flow profile

$$U(y) = \begin{cases} U_1 & \text{for } y > 0 \\ U_2 & \text{for } y < 0 \end{cases}$$

Velocity ratio

$$R = \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\overline{U}}$$

Jump condition 1: Continuity of displacement

$$\frac{\tilde{v}}{U-c} \quad \text{at} \quad y=0$$

Jump condition 2: Continuity of pressure (force)

$$\tilde{p} = \frac{i}{\alpha}\left[DU\,\tilde{v} - (U - c)D\tilde{v}\right] \quad \text{at} \quad y = 0$$

Summary:

Rayleigh equation

$$(U-c)\left(D^2-\alpha^2\right)\tilde{v}-D^2U\,\tilde{v}=0$$

with boundary conditions

$$\tilde{v}(y=\infty)=\tilde{v}(y=-\infty)=0$$

and jump conditions

$(U-c)D\tilde{v}-DU\,\tilde{v}$ and $\dfrac{\tilde{v}}{U-c}$ are continuous at $y=0$

Solution

1) Rayleigh equation and the boundary condition gives

$$\tilde{v}(y) = \begin{cases} Ae^{-\alpha y} & \text{for } y > 0 \\ Be^{\alpha y} & \text{for } y < 0 \end{cases} \qquad \text{with} \quad \alpha > 0$$

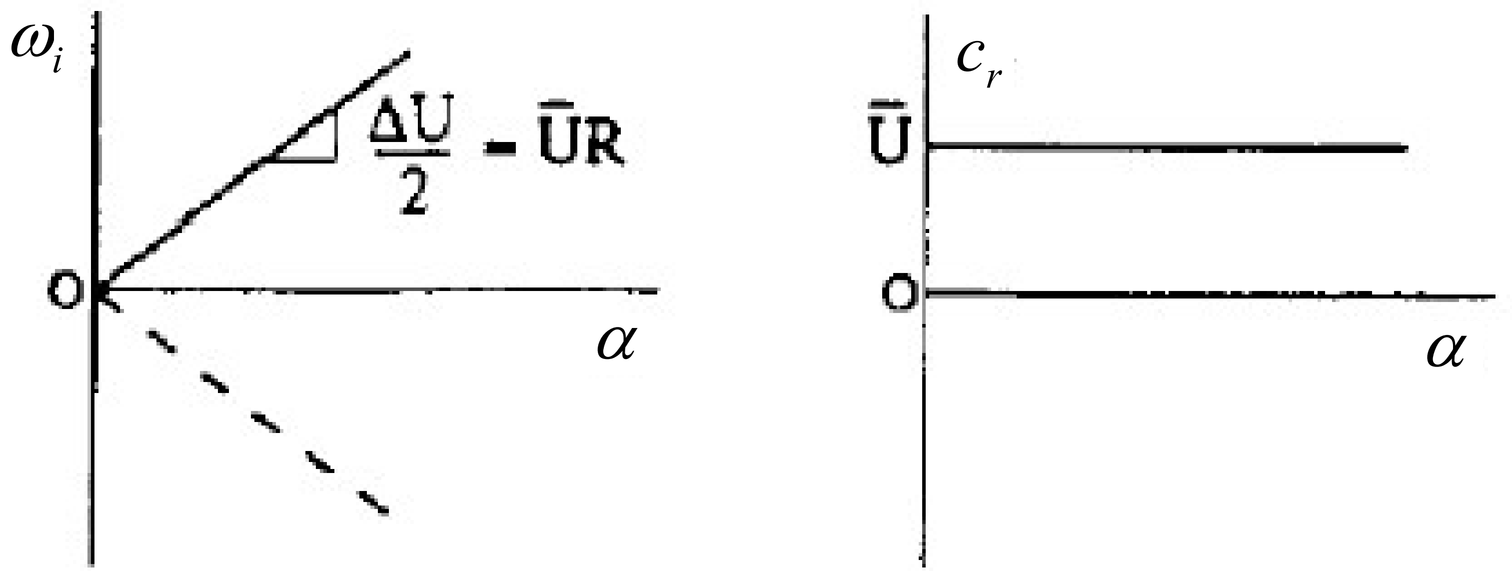
2) Apply the jump conditions

i) $$-\alpha\left(U_1 - c\right)A = \alpha\left(U_2 - c\right)B$$

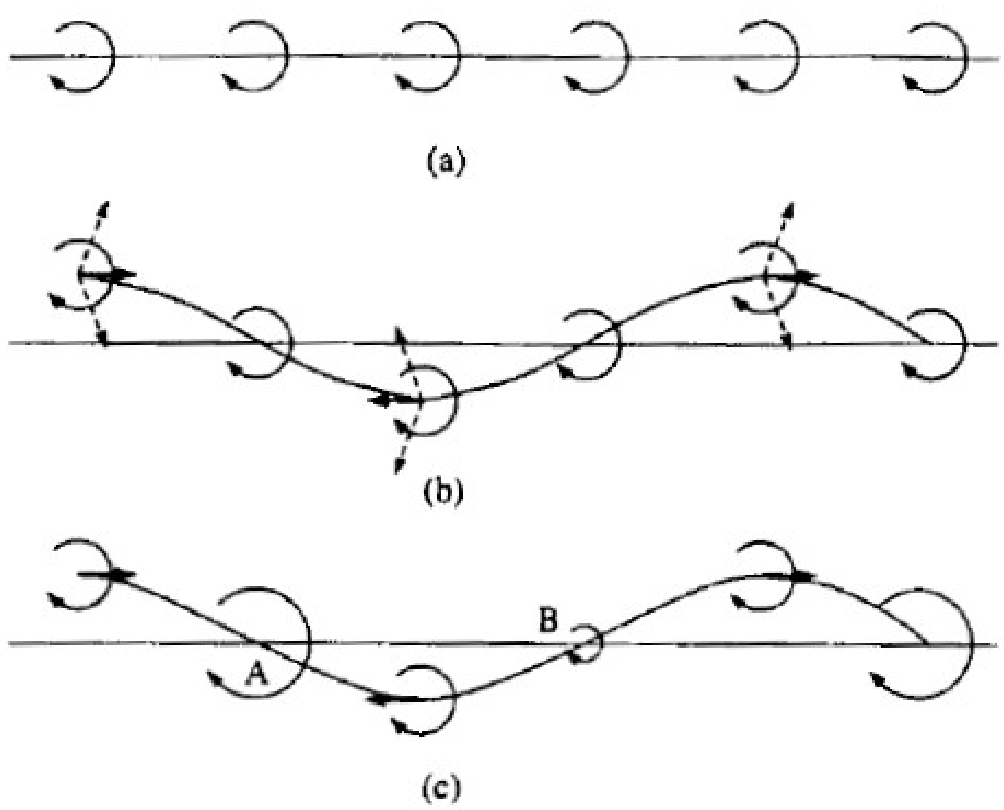
ii) $$\frac{A}{\left(U_1 - c\right)} = \frac{B}{\left(U_2 - c\right)}$$

3) Solve the eigenvalue problem and obtain dispersion relation

$$c = \overline{U} \pm \frac{i\Delta U}{2} \qquad \text{or} \qquad \omega = \alpha\overline{U} \pm i\alpha R\overline{U}$$



Physical mechanism



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Linearised equation for viscous parallel flow

Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

Consider $\mathbf{u}(x, y, z, t) = (U(y), 0, 0) + \varepsilon\, \mathbf{u}'(x, y, z, t),$

$$p(x, y, z, t) = P(x, y) + \varepsilon\, p'(x, y, z, t)$$

and neglect the terms at $O(\varepsilon^2)$. Then,

Linearised equation for viscous parallel flow

Linearised Navier-Stokes equation around parallel base flow

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+v'\frac{dU}{dy}=-\frac{\partial p'}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}+\frac{\partial^2 u'}{\partial z^2}\right)$$

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}=-\frac{\partial p'}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}+\frac{\partial^2 v'}{\partial z^2}\right)$$

$$\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}=-\frac{\partial p'}{\partial z}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2}+\frac{\partial^2 w'}{\partial z^2}\right)$$

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Velocity and vorticity form of linearised Navier-Stokes equation

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^4\right]v'=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^2\right)\eta'+\frac{dU}{dy}\frac{\partial v'}{\partial z}=0$$

with the boundary condition,

$$v'=\frac{\partial v'}{\partial n}=\eta'=0$$ at solid boundary and/or the far field

and the initial condition,

$$v'(x,y,z,t=0)=v_0'(x,y,z)$$
$$\eta'(x,y,z,t=0)=\eta_0'(x,y,z)$$

Lecture outline

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Normal mode solution

Consider full three-dimensional case such that

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^4\right]v'=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^2\right)\eta'+\frac{dU}{dy}\frac{\partial v'}{\partial z}=0$$

Then, the normal mode solution takes the following form:

$$v'(x,y,z,t)=\tilde{v}(y)e^{i(\alpha x+\beta z-\omega t)}+c.c$$

$$\eta'(x,y,z,t)=\tilde{\eta}(y)e^{i(\alpha x+\beta z-\omega t)}+c.c$$

where $\alpha,\beta\in R$ and $\omega\in C$.

Orr-Sommerfeld equation (for wall-normal velocity):

$$\left[\left(-i\omega+i\alpha U\right)\left(D^2-k^2\right)-i\alpha D^2U-\frac{1}{\mathrm{Re}}\left(D^2-k^2\right)^2\right]\tilde{v}=0$$

Squire equation (for wall-normal vorticity):

$$\left[\left(-i\omega+i\alpha U\right)-\frac{1}{\mathrm{Re}}\left(D^2-k^2\right)\right]\tilde{\eta}=-i\beta DU\,\tilde{v}$$

where $k^2=\alpha^2+\beta^2$ with boundary conditions:

$\tilde{v}=D\tilde{v}=\tilde{\eta}=0$ at a solid wall and in the far field.

Remark

If $\alpha,\beta\in R$ are given, then $\omega\in C$ becomes unknown with $\tilde{v}$ and $\tilde{\eta}$
Resulting in an **eigenvalue problem** as in Rayleigh equation.

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Squire transformation

Theorem: Damped Squire modes

The solutions to the Squire equation are always damped, i.e. $\omega_i < 0$ for all α, β and Re.

Remark

Instability comes from Orr-Sommerfeld equation.

Squire transformation

Consider the Orr-Sommerfeld equation for $\beta = 0$ in the following form:

$$\left[(U-c)\left(D^2-\alpha_{2D}^2\right)-D^2U-\frac{1}{i\alpha_{2D}\,\mathrm{Re}_{2D}}\left(D^2-\alpha_{2D}^2\right)^2\right]\tilde{v}=0$$

Now, consider the Orr-Sommerfeld for $\beta \neq 0$ by setting $\mathrm{Re}_{3D} = \dfrac{\alpha}{k}\mathrm{Re}$

$$\left[(U-c)\left(D^2-k^2\right)-D^2U-\frac{1}{ik\,\mathrm{Re}_{3D}}\left(D^2-k^2\right)^2\right]\tilde{v}=0$$

It follows that the critical Reynolds numbers for the onset of an instability should be

$$\mathrm{Re}_{2D,c} = \mathrm{Re}_{3D,c} = \frac{\alpha}{k}\mathrm{Re}_c,$$

indicating that

$$\mathrm{Re}_{2D,c} < \mathrm{Re}_c$$

Theorem: Squire

Given Re_L as the critical Reynolds number for the onset of linear instability for a given α and β, the Reynolds number Re_c below which no exponential instabilities exist for any wave numbers satisfies

$$\mathrm{Re}_c \equiv \min_{\alpha,\beta} \mathrm{Re}_L(\alpha,\beta) = \min_{\alpha} \mathrm{Re}_L(\alpha,0)$$

Remark

The **most unstable linear instability** is always **two dimensional**.

1. **Linear stability analysis of inviscid mixing layer**
2. **Linearised equation for viscous parallel flows**
3. **Orr-Sommefed and Squire equation**
4. **Squire’s transformation and theorem**