Lecture 9

Spatio-temporal evolution of instabilities I

AE209 Hydrodynamic stability

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Lecture outline

1. Ginzburg-Landau equation
2. Absolute and convective instabilities in parallel flow
3. Criterion for absolute instability

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1. **Ginzburg-Landau equation**
2. Absolute and convective instabilities in parallel flow
3. Criterion for absolute instability

A toy model of linearised Navier-Stokes equation

Complex linear Ginzburg-Landau equation

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} - \mu u - (1 - ic_d)\frac{\partial^2 u}{\partial x^2} = 0$$

with boundary condition

$$u(x = \pm\infty) = 0$$

and initial condition

$$u(x, t = 0) = u_0(x)$$

Linear stability analysis

Complex linear Ginzburg-Landau equation

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}-\mu u-(1-ic_d)\frac{\partial^2 u}{\partial x^2}=0$$

Normal mode solution

$$u=Ae^{i(kx-\omega t)}$$

Dispersion relation

$$\omega-Uk+c_d k^2-i(\mu-k^2)=0$$

Linear stability analysis

Temporal stability

$$\omega(k) = Uk - c_d k^2 + i(\mu - k^2)$$

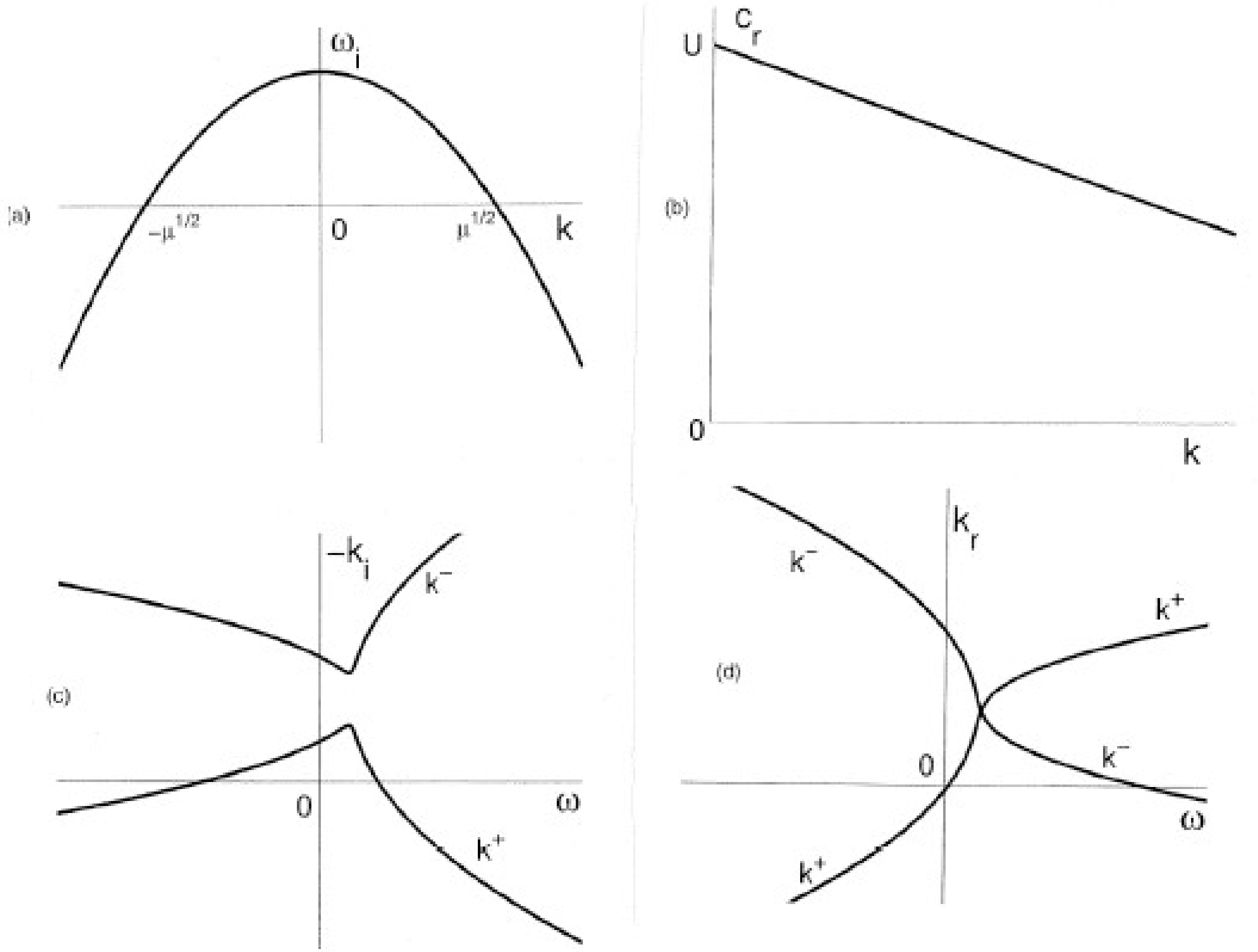
Spatial stability

$$k^{\pm}(\omega) =$$

$$\frac{U}{2(c_d + i)} \pm \left(\frac{-1}{c_d + i} \right)^{1/2} \left[\omega - \frac{c_d U^2}{4(1 + c_d^2)} - i \left\{ \mu - \frac{U^2}{4(1 + c_d^2)} \right\} \right]$$

Temporal

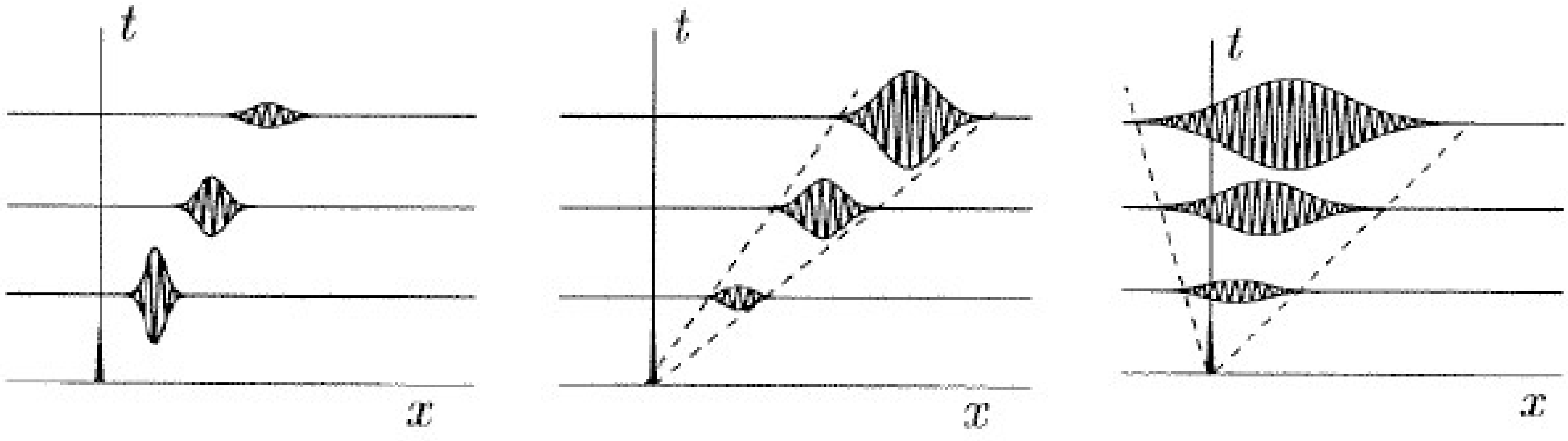
Spatial



Lecture outline

Evolution of localised disturbances

Spatio-temporal evolution of a wave packet (i.e. impulse) in parallel flow



Linearly Stable

Linearly unstable

Linear stability revisited

Impulse response of Ginzburg-Landau equation

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \mu - (1 - ic_d)\frac{\partial^2}{\partial x^2}\right] G(x,t) = \delta(x)\delta(t)$$

Linearly stable:

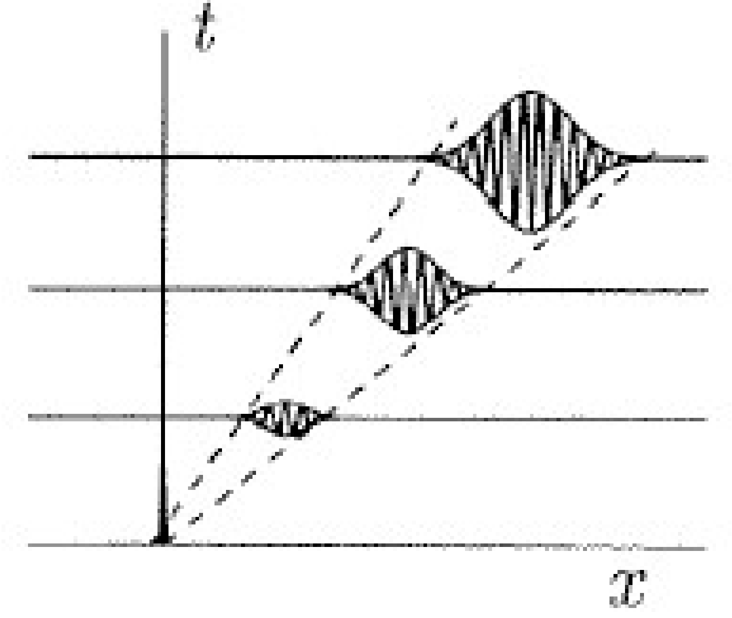
$$\lim_{t\to\infty} G(x,t) = 0 \qquad \text{for all rays} \quad x/t = const$$

Linearly unstable:

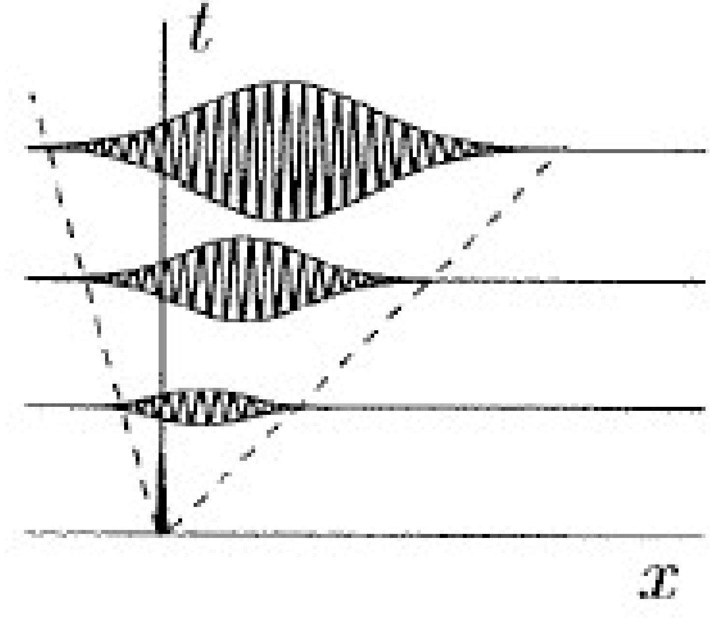
$$\lim_{t\to\infty} G(x,t) = \infty \qquad \text{for at least one ray} \quad x/t = const$$

Convective and absolute instabilities

Spatio-temporal evolution of **unstable wavepackets** in parallel flow



Convectively unstable



Absolutely unstable

Impulse response of Ginzburg-Landau equation

$$\left[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\mu-(1-ic_d)\frac{\partial^2}{\partial x^2}\right]G(x,t)=\delta(x)\delta(t)$$

Convectively unstable

$$\lim_{t\to\infty}G(x,t)=0 \qquad \text{along the ray of} \quad x/t=0$$

Absolutely unstable

$$\lim_{t\to\infty}G(x,t)=\infty \qquad \text{along the ray of} \quad x/t=0$$

1. Ginzburg-Landau equation
2. Absolute and convective instabilities in parallel flow
3. **Criterion for absolute instability**

Impulse response of parallel flow

Impulse response of Ginzburg-Landau equation

$$\left[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\mu-(1-ic_d)\frac{\partial^2}{\partial x^2}\right]G(x,t)=\delta(x)\delta(t)$$

Solution)

Step 1) Perform Fourier transform in x and Laplace transform in t : i.e.

$$\widetilde{G}(k,\omega)==\int_0^{\infty}\int_{-\infty}^{\infty}G(x,t)e^{-i(kx-\omega t)}dxdt$$

Step 2) Construct solution in the wavenumber space

Step 3) Invert the Fourier-Laplace transform

$$G(x,t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{G}(k,\omega)e^{i(kx-\omega t)}dkd\omega$$

Step 1) Perform Fourier transform in x and Laplace transform in t

i) Fourier transform in x

$$\hat{G}(k,t) = \int_{-\infty}^{\infty} G(x,t) e^{-ikx} dx$$

Examples:

$$\int_{-\infty}^{\infty} \frac{\partial G(x,t)}{\partial t} e^{-ikx} dx = \frac{\partial \hat{G}(k,t)}{\partial t}$$

$$\int_{-\infty}^{\infty} \frac{\partial G(x,t)}{\partial x} e^{-ikx} dx = \left[G(x,t) e^{-ikx} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} G(x,t) \frac{\partial e^{-ikx}}{\partial x} dx$$

$$= ik\hat{G}(k,t)$$

$$\int_{-\infty}^{\infty} \delta(x)\delta(t) e^{-ikx} dx = \delta(t)$$

Impulse response of parallel flow

Step 1) Perform Fourier transform in x and Laplace transform in t

ii) Laplace transform in t

$$\widetilde{G}(k,\omega)=\int_0^\infty \hat{G}(k,t)e^{i\omega t}d\omega$$

Examples:

$$\int_0^\infty \frac{\partial \hat{G}(k,t)}{\partial t}e^{i\omega t}dt=[\hat{G}(k,t)e^{i\omega t}]_0^\infty-\int_0^\infty \hat{G}(k,t)\frac{\partial e^{i\omega t}}{\partial t}dt$$

$$=-i\omega\widetilde{G}(k,\omega)$$

$$\int_0^\infty \delta(t)e^{i\omega t}dt=1$$

Impulse response of parallel flow

Step 2) Construct solution in the wavenumber space

$$D(k,\omega)\widetilde{G}(k,\omega) = 1$$

where $D(k,\omega) = -i\omega + iUk - ic_d k^2 - (\mu - k^2)$

Step 3) Construct solution in the wavenumber space

$$G(x,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{D(k,\omega)} e^{i(kx-\omega t)} dk d\omega$$

$$\sim \frac{e^{i(k_0 x - \omega_0 t)}}{\frac{\partial D}{\partial \omega}(k_0, \omega_0) \left[\frac{\partial^2 \omega}{\partial k^2}(k_0) t \right]^{1/2}} \quad \text{as} \quad t \to \infty$$

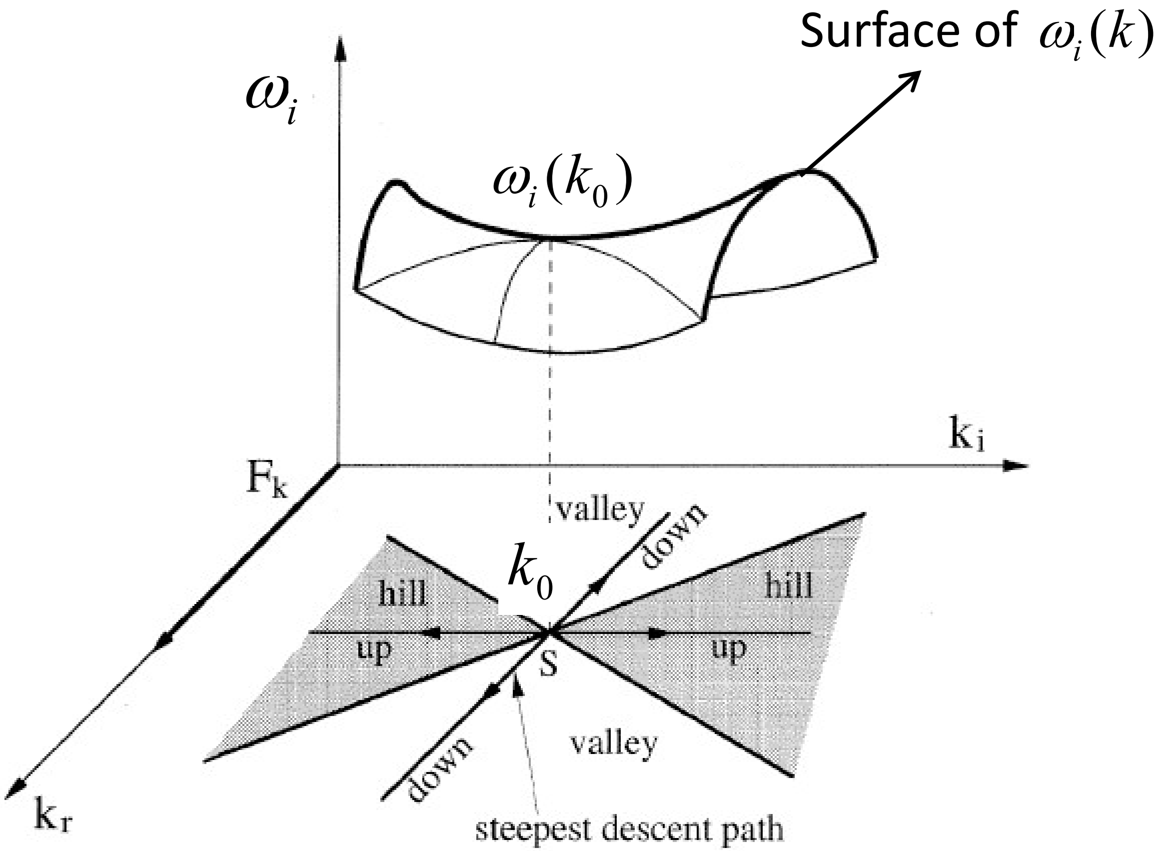
where

$$\frac{\partial \omega(k)}{\partial k} = 0 \quad \text{at} \quad k = k_0 \qquad \text{and} \qquad \omega_0 = \omega(k_0)$$

Impulse response of parallel flow

Remark

The point $\omega_0 = \omega(k_0)$ forms a saddle point over complex k plane



Impulse response of parallel flow

Criterion of absolute instability

$$G(x,t) \sim e^{i(k_0 x - \omega_0 t)}$$

From the definition of absolute instability, the growth rate along $x/t=0$ is given by $\omega_{0,i}$ called absolute growth rate. In general,

$\omega_i(k_{\max}) < 0$: **Linearly stable**

$\omega_i(k_{\max}) > 0$ and $\omega_{0,i} < 0$: **Convectively unstable**

$\omega_i(k_{\max}) > 0$ and $\omega_{0,i} > 0$: **Absolutely unstable**

1. Ginzburg-Landau equation: a toy model of NS equation
2. Absolute and convective instabilities in parallel flow
3. Criterion for absolute instability