Lecture 9

Spatio-temporal evolution of instabilities I

AE209 Hydrodynamic stability
Dr Yongyun Hwang

Lecture outline

1. Ginzburg-Landau equation
2. Absolute and convective instabilities in parallel flow
3. Criterion for absolute instability

1. **Ginzburg-Landau equation**
2. Absolute and convective instabilities in parallel flow
3. Criterion for absolute instability

A toy model of linearised Navier-Stokes equation

Complex linear Ginzburg-Landau equation

u: complex
1D field
u(x,t).

diffusion / dispersion

$$\underbrace{\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}}_{\text{Advection with } U}\boxed{-\mu u}+(1-ic_d)\frac{\partial^2 u}{\partial x^2}=0$$

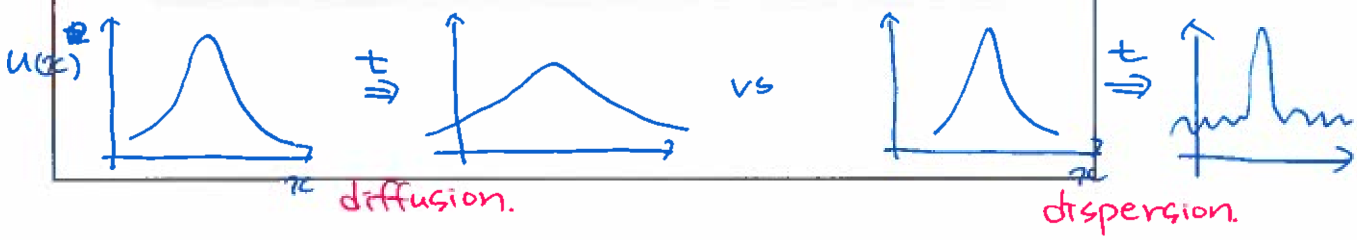
instability driving term.
with control parameter
μ.

with boundary condition

$$u(x=\pm\infty)=0$$

and initial condition

$$u(x,t=0)=u_0(x)$$



Temporal stability : k is given and real, find ω.

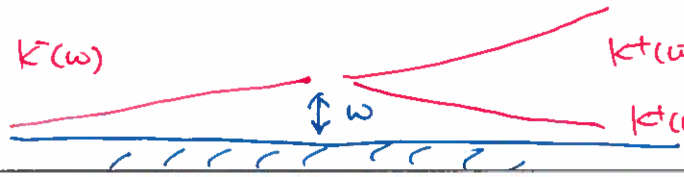
$$\omega(k) = \underbrace{Uk - c_d k^2}_{\omega_r} + i\underbrace{(\mu - k^2)}_{\omega_i}$$

⇒ if $\mu > 0$, $\omega_i > 0$ for some real k.

⇒ Linearly unstable

Spatial stability

$$k^{\pm}(\omega) = \frac{U}{2(c_d + i)} \pm \left(\frac{-1}{c_d + i}\right)^{1/2} \left[\omega - \frac{c_d U^2}{4(1 + c_d^2)} - i\left\{\mu - \frac{U^2}{4(1 + c_d^2)}\right\}\right]$$



$k^+(\omega)$: linearly unstable

$k^+(\omega)$: linearly stable

Linear stability analysis

Complex linear Ginzburg-Landau equation

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} - \mu u - (1 - ic_d)\frac{\partial^2 u}{\partial x^2} = 0$$

Normal mode solution

$$u = Ae^{i(kx-\omega t)}$$

↳ complex constant.

Dispersion relation

$\frac{\partial}{\partial t} \Rightarrow -i\omega$ $\frac{\partial}{\partial x} \Rightarrow ik$ $u \Rightarrow A.$

$$\Rightarrow \underline{\omega - Uk + c_d k^2 - i(\mu - k^2) = 0}$$

Lecture outline

1. [illegible]
2. **Absolute and convective instabilities in parallel flow**
3. Criterion for absolute instability

Linear stability analysis

Temporal

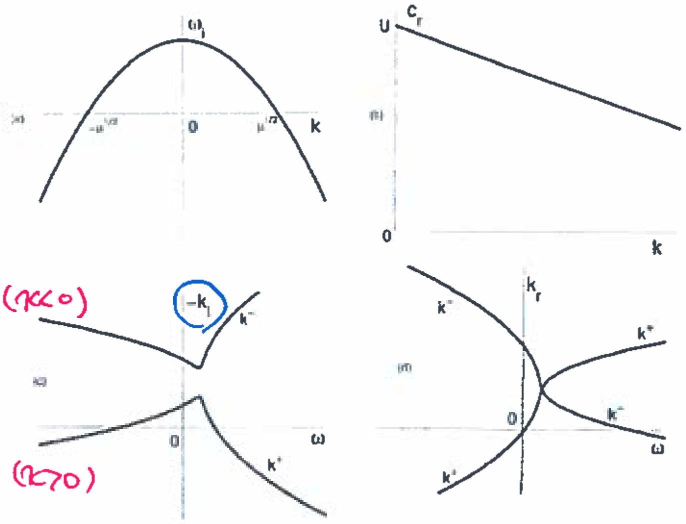
$(\mu > 0)$

e^{ik_ix}

$= e^{-k_ix}$ Spatial

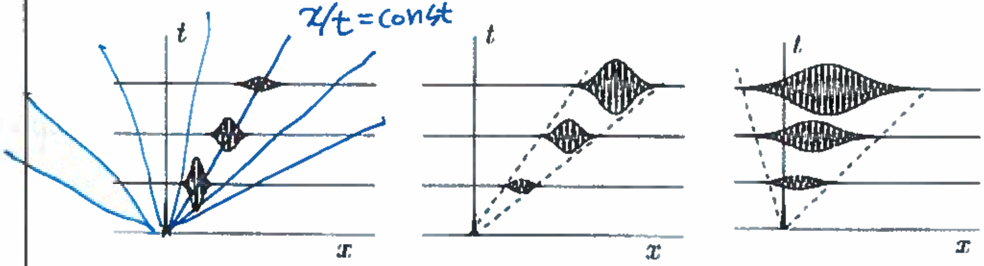
$(x<0)$

$(x>0)$



Evolution of localised disturbances

Spatio-temporal evolution of a wave packet (i.e. impulse) in parallel flow



Linearly Stable

Linearly unstable

Convective unstable

Absolutely unstable

Linear stability revisited

Impulse response of Ginzburg-Landau equation

$$\left[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\mu-(1-ic_d)\frac{\partial^2}{\partial x^2}\right]\underline{G(x,t)}=\delta(x)\delta(t)$$

Green's function : impulse response.

Linearly stable:

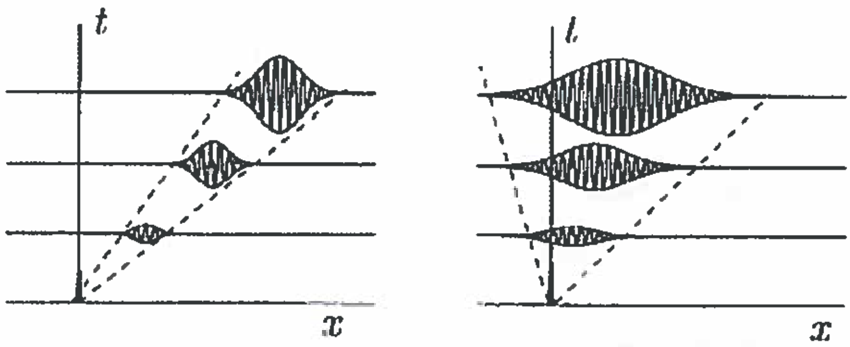
$$\lim_{t\to\infty} G(x,t)=0 \qquad \text{for all rays} \quad x/t=const$$

Linearly unstable:

$$\lim_{t\to\infty} G(x,t)=\infty \qquad \text{for at least one ray} \quad x/t=const$$

Convective and absolute instabilities

Spatio-temporal evolution of **unstable wavepackets** in parallel flow



Convectively unstable

$\lim_{t\to\infty} G(x,t) \to 0$

along $x/t = 0$

Absolutely unstable

$\lim_{t\to\infty} G(x,t) \to \infty$

along $x/t = 0$

Convective and absolute instabilities

Impulse response of Ginzburg-Landau equation

$$\left[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\mu-(1-ic_d)\frac{\partial^2}{\partial x^2}\right]G(x,t)=\delta(x)\delta(t)$$

Convectively unstable

$\lim_{t\to\infty} G(x,t)=0$ along the ray of $x/t=0$

Absolutely unstable

$\lim_{t\to\infty} G(x,t)=\infty$ along the ray of $x/t=0$

1. Ginzburg-Landau equation
2. Absolute and convective instabilities in parallel flow
3. **Criterion for absolute instability**

Impulse response of parallel flow

Impulse response of Ginzburg-Landau equation

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \mu - (1 - ic_d)\frac{\partial^2}{\partial x^2}\right]G(x,t) = \delta(x)\delta(t)$$

Solution)

Step 1) Perform Fourier transform in x and Laplace transform in t : i.e.

$$\tilde{G}(k,\omega) == \int_0^\infty \int_{-\infty}^\infty G(x,t)e^{-i(kx-\omega t)}dxdt$$

Step 2) Construct solution in the wavenumber space

Step 3) Invert the Fourier-Laplace transform

$$G(x,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^\infty \int_{-\infty}^\infty \tilde{G}(k,\omega)e^{i(kx-\omega t)}dkd\omega$$

Impulse response of parallel flow

Step 1) Perform Fourier transform in x and Laplace transform in t

i) Fourier transform in x

$$\hat{G}(k,t) = \int_{-\infty}^{\infty} G(x,t)e^{-ikx}dx$$

Examples:

$$\int_{-\infty}^{\infty} \frac{\partial G(x,t)}{\partial t} e^{-ikx}dx = \frac{\partial \hat{G}(k,t)}{\partial t}$$

$$\frac{\partial}{\partial t}\left[\underbrace{\int_{-\infty}^{\infty} G(x,t)e^{-ikx}dx}_{\hat{G}(k,t)}\right]$$

$$\int_{-\infty}^{\infty} \underbrace{\frac{\partial G(x,t)}{\partial x}}_{u'} \underbrace{e^{-ikx}}_{v}dx = \underbrace{\cancel{\left[\underbrace{G(x,t)}_{u}\underbrace{e^{-ikx}}_{v}\right]_{-\infty}^{\infty}}}_{0\ (B.C)} - \int_{-\infty}^{\infty} \underbrace{G(x,t)}_{u}\underbrace{\frac{\partial e^{-ikx}}{\partial x}}_{v'}dx$$

$ik\, e^{-ikx}.$

$$= ik\hat{G}(k,t)$$

$$\int_{-\infty}^{\infty} \delta(x)\delta(t)e^{-ikx}dx = \delta(t)$$

Impulse response of parallel flow

Step 1) Perform Fourier transform in x and Laplace transform in t

ii) Laplace transform in t

$$\tilde{G}(k,\omega)=\int_0^\infty \hat{G}(k,t)e^{i\omega t}d\omega$$

Examples:

$\hat{G}=0$ at $t=0$

0 ($\hat{G}=0$ at $t=\infty$

$$\int_0^\infty \frac{\partial \hat{G}(k,t)}{\partial t}e^{i\omega t}dt=\cancel{[\hat{G}(k,t)e^{i\omega t}]_0^\infty}-\int_0^\infty \hat{G}(k,t)\frac{\partial e^{i\omega t}}{\partial t}dt$$

$$=-i\omega\tilde{G}(k,\omega)$$

$$\int_0^\infty \delta(t)e^{i\omega t}dt=1$$

Step 2) Construct solution in the wavenumber space

$$D(k,\omega)\widetilde{G}(k,\omega) = 1$$

where $D(k,\omega) = -i\omega + iUk - ic_d k^2 - (\mu - k^2)$

$\frac{\partial}{\partial t} \to -i\omega$; $\frac{\partial}{\partial x} \to ik$;

$G(x,t) \to \widetilde{G}(k,\omega)$

$\delta(x)\,\delta(t) \to 1.$

$\widetilde{G}(k,\omega)$

$$= \frac{1}{D(k,\omega)}$$

Step 3) Construct solution in the wavenumber space

$\tilde{G}(k,\omega)$ * Method of Steepest descent.

$$G(x,t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{D(k,\omega)}e^{i(kx-\omega t)}dk\,d\omega$$

$$\sim\frac{\boxed{e^{i(k_0x-\omega_0 t)}}}{\frac{\partial D}{\partial\omega}(k_0,\omega_0)\left[\frac{\partial^2\omega}{\partial k^2}(k_0)t\right]^{1/2}}\quad\text{as } t\to\infty$$

dominant term as $t\to\infty$.

where

dispresion relation.

Complex

$$\frac{\partial\omega(k)}{\partial k}=0 \quad\text{at}\quad \underline{k=k_0}\quad\text{and}\quad \underline{\omega_0=\omega(k_0)}$$

Complex absolute wavenumber

Complex Absolute frequency

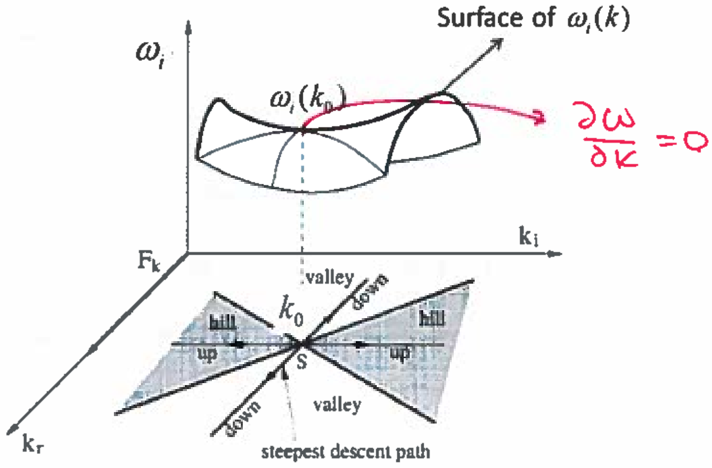
$\Rightarrow$ Along $x/t=0$.

* if $\omega_{0,i}>0$,
$G(x,t)\to\infty$
along $x/t=0$.

Impulse response of parallel flow

Remark

The point $\omega_0 = \omega(k_0)$ forms a saddle point over complex k plane



Impulse response of parallel flow

Criterion of absolute instability

$$G(x,t) \sim e^{i(k_0 x - \omega_0 t)}$$

From the definition of absolute instability, the growth rate along $x/t = 0$ is given by $\omega_{0,i}$ called absolute growth rate. In general,

$\omega_i(k_{\max}) < 0$ **: Linearly stable**

$\omega_i(k_{\max}) > 0$ and $\omega_{0,i} < 0$ **: Convectively unstable**

$\omega_i(k_{\max}) > 0$ and $\omega_{0,i} > 0$ **: Absolutely unstable**

Lecture outline

1. Ginzburg-Landau equation: a toy model of NS equation
2. Absolute and convective instabilities in parallel flow
3. Criterion for absolute instability