Lecture 4

Linear stability of parallel flows I

AE209 Hydrodynamic stability

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Lecture outline

1. **Concept of parallel flows**
2. **Linearised equation for inviscid parallel shear flows**
3. **Normal mode solution**
4. **When does an instability occur?**

Lecture outline

Parallel shear flows

The flow configuration, the base flow of which is given by

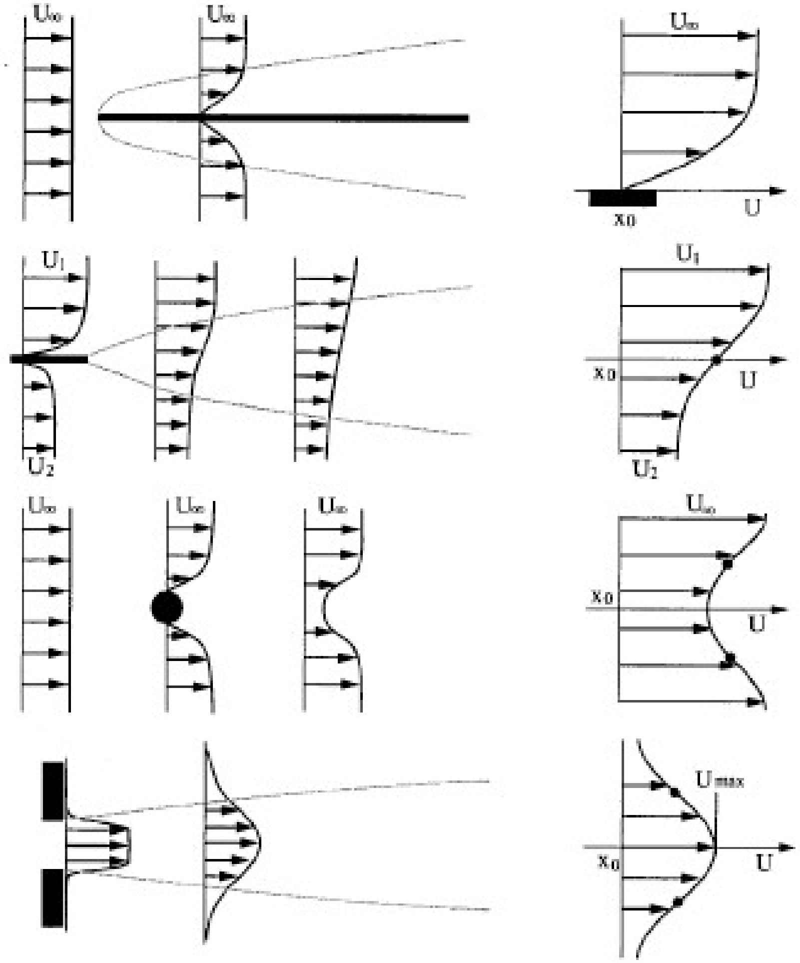
$$\mathbf{U} = (U(y), 0, 0)$$

Examples

Plane Couette flow, Poiseulle flow, Pipe flow, and etc.

Parallel shear flows

Weakly non-parallel flows (or spatially developing flows)



Boundary layer

Mixing layer

Cylinder wake

Jet

1. Concept of parallel flows
2. **Linearised equation for inviscid parallel shear flows**
3. Normal mode solution
4. When does an instability occur?

Linearised equation for inviscid parallel flow

Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p$$

$$\nabla \cdot \mathbf{u} = 0$$

Consider $\mathbf{u}(x, y, z, t) = (U(y), 0, 0) + \varepsilon\, \mathbf{u}'(x, y, z, t),$

$$p(x, y, z, t) = P(x, y) + \varepsilon\, p'(x, y, z, t)$$

and neglect the terms at $O(\varepsilon^2)$. Then,

Linearised equation for inviscid parallel flow

Linearised Euler equation around parallel base flow

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{dU}{dy} = -\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} \quad \Rightarrow \quad \nabla^2 p = -2\frac{dU}{dy}\frac{\partial v}{\partial x}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Linearised equation for inviscid parallel flow

1) Equation for wall-normal velocity

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} \qquad \nabla^2 p = -2\frac{dU}{dy}\frac{\partial v}{\partial x}$$

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - \frac{d^2U}{dy^2}\frac{\partial}{\partial x}\right]v' = 0$$

Linearised equation for inviscid parallel flow

2) Equation for wall-normal vorticity

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{dU}{dy} = -\frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z}$$

Wall-normal vorticity

$$\eta' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta' = -\frac{dU}{dy}\frac{\partial v'}{\partial z}$$

Velocity and vorticity form of linearised Euler equation

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}\right]v'=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\eta'+\frac{dU}{dy}\frac{\partial v'}{\partial z}=0$$

with the boundary condition,

$v'=\eta'=0$ at solid boundary and/or the far field

and the initial condition,

$$v'(x,y,z,t=0)=v_0'(x,y,z)$$
$$\eta'(x,y,z,t=0)=\eta_0'(x,y,z)$$

1. Concept of parallel flows
2. Linearised equation for inviscid parallel shear flows
3. **Normal mode solution**
4. When does an instability occur?

Normal mode solution

Consider two-dimensional case such that

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}\right]v'=0$$

Then, the normal mode solution takes the following form:

$$v'(x,y,t)=\widetilde{v}(y)e^{i(\alpha x-\omega t)}+c.c$$

where $\alpha\in R$ and $\omega\in C$.

Normal mode solution

For example,

$$\frac{\partial v'}{\partial t} =$$

$$\frac{d^2 U}{dy^2}\frac{\partial v'}{\partial x} =$$

$$\nabla^2 v' =$$

Rayleigh equation

Let $\omega = \alpha c$ and $D \equiv d/dy$. Then, we get

$$(U-c)\left(D^2-\alpha^2\right)\tilde{v} - D^2U\,\tilde{v} = 0$$

with the boundary condition,

$\tilde{v} = 0$ at solid boundary and/or the far field

Remark

If $\alpha \in R$ is given, then $c \in C$ becomes unknown with $\tilde{v}$, resulting in **eigenvalue problem**.

Interpretation of normal mode solution

Normal mode solution

$$v'(x,y,t) = \tilde{v}(y)e^{i\alpha(x-ct)} + c.c$$
$$= \text{Real}\left\{|\tilde{v}(y)|\, e^{i\phi(y)} e^{i\alpha(x-(c_r+ic_i)t)}\right\}$$
$$= |\tilde{v}(y)|\, e^{\alpha c_i t} \cos[\alpha(x - c_r t) + \phi(y)]$$

c_r : Phase speed

$c_i > 0$ Linearly unstable

$c_i = 0$ Marginally stable (or neutral)

$c_i < 0$ Linearly stable

When does an instability occur?

Theorem: Rayleigh inflection point criterion

If there exist perturbations with $c_i > 0$, then d^2U/dy^2 must be zero at some $y \in \Omega$ ($\Omega = [a,b]$ is the flow domain in y).

Proof

$$(U-c)\left(D^2-\alpha^2\right)\tilde{v} - D^2U\,\tilde{v} = 0$$

$$\int_a^b \tilde{v}^*\left(D^2-\alpha^2\right)\tilde{v}\,dy - \int_a^b \frac{D^2U}{(U-c)}\tilde{v}^*\tilde{v}\,dy = 0$$

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$$\int_a^b \left|D\tilde{v}\right|^2 + \alpha^2 \left|\tilde{v}\right|^2 dy + \int_a^b \frac{d^2U/dy^2}{(U-c)} \left|\tilde{v}\right|^2 dy = 0$$

Imaginary part:

$$\mathrm{Im}\left(\int_a^b \frac{d^2U/dy^2}{(U-c)} \left|\tilde{v}\right|^2 dy \right) = \int_a^b \frac{c_i d^2U/dy^2 \left|\tilde{v}\right|^2}{\left|U-c\right|^2} dy = 0$$

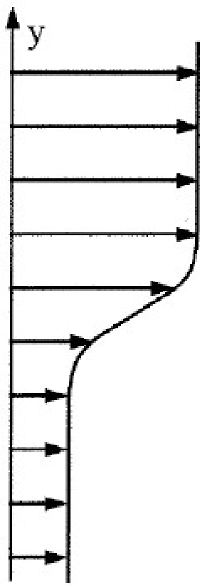
Rayleigh inflection point criterion

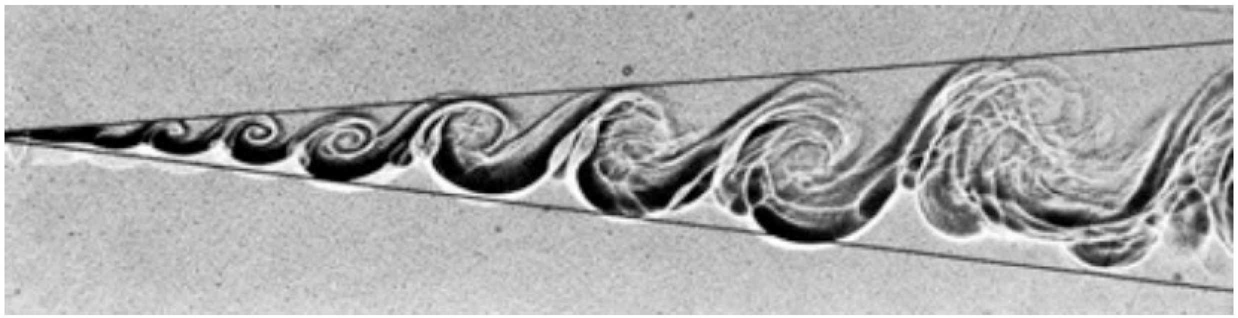
Remark 1

The presence of some **inflection points** is a **necessary condition** for **linear instability**.

Remark 2

The presence of an **infection point** is often **a sign of instability**.





Instability in a mixing layer
(Brown & Roshko 1974)

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4. Rayleigh thereom