Lecture 5

Linear stability of parallel shear flows II

AE209 Hydrodynamic stability
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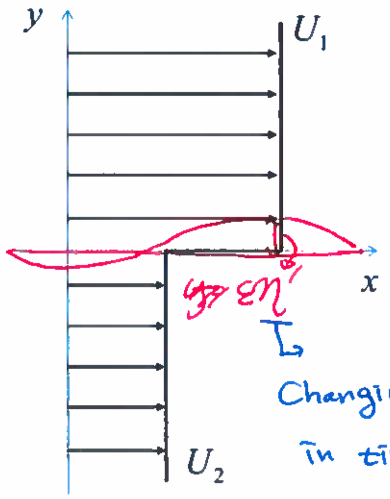
Lecture outline

1. Linear stability analysis of inviscid mixing layer
2. Linearised equation for viscous parallel flows
3. Normal mode solution
4. Squire's transformation

1. **Linear stability analysis of inviscid mixing layer**

Example 1: Piecewise mixing layer



Changing in time and space.

Base flow profile

$$U(y)=\begin{cases} U_1 & \text{for } y>0 \\ U_2 & \text{for } y<0 \end{cases}$$

Velocity ratio

$$R=\frac{U_1-U_2}{U_1+U_2}=\frac{\Delta U}{2\bar{U}}$$

ΔU → difference between U_1 and U_2

$\bar{U}$ → Average velocity of U_1 and U_2.

Jump condition 1: Continuity of displacement $\Rightarrow$ η' must be the same at $y = 0$.

$$\frac{\tilde{v}}{U-c} \quad \text{at} \quad y = 0$$

$$\underbrace{\varepsilon v'(x,y,t)\big|_{y=0}}_{} = \underbrace{\left[\frac{\partial}{\partial t} + (U + \underbrace{\varepsilon u'(x,y,t)}_{O(\varepsilon^2)})\frac{\partial}{\partial x}\right]\varepsilon\eta'}_{\varepsilon \frac{D\eta'}{Dt}}$$

At $O(\varepsilon)$

$$v'(x,y,t) = \left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]\eta'$$

Now, $\eta' = \hat{\eta} e^{i(\alpha x - \omega t)}$ $\quad v = \hat{v} e^{i(\alpha x - \omega t)}$

$$\Rightarrow \quad \hat{v} = [-i\omega + i\alpha U]\hat{\eta} \quad \text{and} \quad \omega = \alpha c$$

$$\Rightarrow \quad \hat{\eta} = \left[\frac{\hat{v}}{i\alpha(U-c)}\right]$$

Jump condition 2: Continuity of pressure (force)

$$\tilde{p} = \frac{i}{\alpha}\left[\overbrace{DU}^{\frac{d}{dy}}\,\tilde{v} - (U-c)D\tilde{v}\right] \quad \text{at} \quad y=0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

$$\Rightarrow i\alpha\tilde{u} + D\tilde{v} = 0.$$

$$\Rightarrow \tilde{u} = -\frac{iD\tilde{v}}{\alpha}.$$

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{dU}{dy} = -\frac{\partial p'}{\partial x} \rightarrow$$ Linearizes N-S eq in x-dir.

$$\Rightarrow -i\omega\tilde{u} + i\alpha U\tilde{u} + \tilde{v}\frac{dU}{dy} = -i\alpha\tilde{p}.$$

$$\Rightarrow \tilde{p} = \frac{i}{\alpha}\left[DU\tilde{v} - (U-c)D\tilde{v}\right] \rightarrow$$ must be continuous at $y=0$

Summary:

Rayleigh equation

$$(U-c)(D^2-\alpha^2)\tilde{v} - \underbrace{D^2U}_{=0 \text{ except for } y=0}\tilde{v} = 0 \quad \Rightarrow \quad (D^2-\alpha^2)\tilde{v} = 0. \quad (\text{for } y \neq 0)$$

$(\alpha > 0)$.

$$\tilde{v} = \underbrace{C_1 e^{-\alpha y}}_{\text{① Solution for } y>0} + \underbrace{C_2 e^{\alpha y}}_{\text{② Solution for } y<0}$$

with boundary conditions

$$\underbrace{\tilde{v}(y=\infty)}_{①} = \underbrace{\tilde{v}(y=-\infty)}_{②} = 0$$

and jump conditions

$(U-c)D\tilde{v} - DU\,\tilde{v}$ and $\dfrac{\tilde{v}}{U-c}$ are continuous at $y=0$

Use to find C_1 and C_2.

$(U-c)\,D\tilde{v} - DU\,\tilde{v}$

$\dfrac{\tilde{v}}{U-c}$

Solution

1) Rayleigh equation and the boundary condition gives

$$\tilde{v}(y) = \begin{cases} Ae^{-\alpha y} & \text{for } y > 0 \\ Be^{\alpha y} & \text{for } y < 0 \end{cases} \qquad \text{with } \alpha > 0$$

2) Apply the jump conditions

i) $-\alpha(U_1 - c)A = \alpha(U_2 - c)B$

ii) $\dfrac{A}{(U_1 - c)} = \dfrac{B}{(U_2 - c)}$

i) $(U_1-c)A + (U_2-c)B = 0$

ii) $(U_2-c)A - (U_1-c)B = 0$

$$\Rightarrow \underbrace{\begin{bmatrix} U_1-c & U_2-c \\ U_2-c & -(U_1-c) \end{bmatrix}}_{L} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

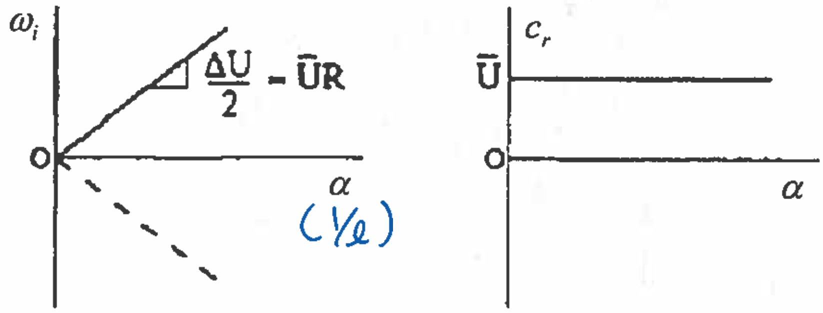
$\Rightarrow \det L = 0$

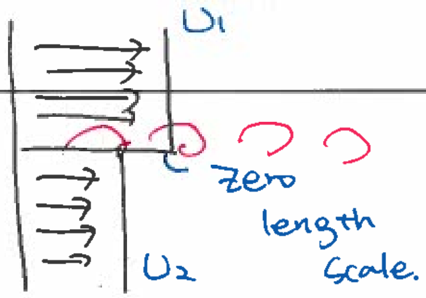
$\Rightarrow \quad c = \dfrac{U_1+U_2}{2} \pm i\,\dfrac{U_1-U_2}{2}$

Piecewise mixing layer

3) Solve the eigenvalue problem and obtain dispersion relation

$$c = \overline{U} \pm \frac{i\Delta U}{2} \qquad \text{or} \qquad \omega = \alpha\overline{U} \pm i\alpha R\overline{U}$$



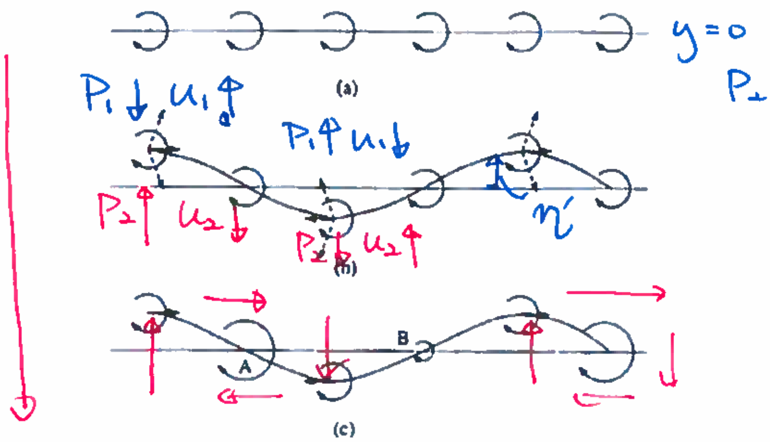


Physical mechanism

$P_1 + \frac{1}{2}\rho u_1^2 = 0$

$P_2 + \frac{1}{2}\rho u_2^2 = 0$

Displacement of the interface grows in time.



* Kelvin-Helmholtz Instability

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2. **Linearised equation for viscous parallel flows**

Linearised equation for viscous parallel flow

Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}$$

$$\nabla\cdot\mathbf{u} = 0$$

Consider $\mathbf{u}(x,y,z,t) = (U(y),0,0) + \varepsilon\,\mathbf{u}'(x,y,z,t),$

$p(x,y,z,t) = P(x,y) + \varepsilon\, p'(x,y,z,t)$

and neglect the terms at $O(\varepsilon^2)$. Then,

$$\frac{\partial \underline{u}'}{\partial t} + (\underline{U}\cdot\nabla)\underline{u}' + (\underline{u}'\cdot\nabla)\underline{U} = -\nabla p' + \frac{1}{\mathrm{Re}}\nabla^2\underline{u}'$$

Linearised equation for viscous parallel flow

Linearised Navier-Stokes equation around parallel base flow

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+v'\frac{dU}{dy}=-\frac{\partial p'}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}+\frac{\partial^2 u'}{\partial z^2}\right)$$

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}=-\frac{\partial p'}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}+\frac{\partial^2 v'}{\partial z^2}\right)$$

$$\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}=-\frac{\partial p'}{\partial z}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2}+\frac{\partial^2 w'}{\partial z^2}\right)$$

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Linearised equation for viscous parallel flow

Velocity and vorticity form of linearised Navier-Stokes equation

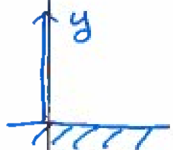
$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}-\boxed{\frac{1}{\text{Re}}\nabla^4}\right]v'=0$$

$\nabla^4 = \nabla^2\nabla^2$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\frac{1}{\text{Re}}\nabla^2\right)\eta'+\frac{dU}{dy}\frac{\partial v'}{\partial z}=0$$

with the boundary condition, (no-slip)

$$v'=\frac{\partial v'}{\partial n}=\eta'=0 \quad \text{at solid boundary and/or the far field}$$



$$\cancel{\frac{\partial u'}{\partial x}}^{\,0}+\frac{\partial v'}{\partial y}+\cancel{\frac{\partial w'}{\partial z}}^{\,0}=0$$

and the initial condition,

$$v'(x,y,z,t=0)=v_0'(x,y,z)$$
$$\eta'(x,y,z,t=0)=\eta_0'(x,y,z)$$

3. Normal mode solution

Normal mode solution

Consider full three-dimensional case such that

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^4\right]v'=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^2\right)\eta'+\frac{dU}{dy}\frac{\partial v'}{\partial z}=0$$

Then, the normal mode solution takes the following form:

$$v'(x,y,z,t)=\widetilde{v}(y)e^{i(\alpha x+\beta z-\omega t)}+c.c$$

$$\eta'(x,y,z,t)=\widetilde{\eta}(y)e^{i(\alpha x+\beta z-\omega t)}+c.c$$

where $\alpha,\beta\in R$ and $\omega\in C$.

Orr-Sommerfeld and Squire equations

Orr-Sommerfeld equation (for wall-normal velocity):

$$\left[\underbrace{(-i\omega+i\alpha U)(D^2-k^2)-i\alpha D^2U}_{\text{Rayleigh eqn.}}-\frac{1}{\mathrm{Re}}(D^2-k^2)^2\right]\tilde{v}=0$$

Squire equation (for wall-normal vorticity):

$$\left[(-i\omega+i\alpha U)-\frac{1}{\mathrm{Re}}(D^2-k^2)\right]\tilde{\eta}=-i\beta DU\,\tilde{v}$$

where $k^2=\alpha^2+\beta^2$ with boundary conditions:

$\tilde{v}=D\tilde{v}=\tilde{\eta}=0$ at a solid wall and in the far field.

Remark

If $\alpha,\beta\in R$ are given, then $\omega\in C$ becomes unknown with $\tilde{v}$ and $\tilde{\eta}$
Resulting in an **eigenvalue problem** as in Rayleigh equation.

4. Squire's transformation

Squire transformation

Theorem: Damped Squire modes

The solutions to the Squire equation are always damped, i.e. $\omega_i < 0$ for all α, β and Re.

(Andrea's lecture note)

Remark

Instability comes from Orr-Sommerfeld equation.

Consider the Orr-Sommerfeld equation for $\beta = 0$ in the following form:

$$\left[(U-c)(D^2-\alpha_{2D}^2)-D^2U-\frac{1}{i\alpha_{2D}\,\mathrm{Re}_{2D}}(D^2-\alpha_{2D}^2)^2\right]\tilde{v}=0$$ $\Rightarrow$ $\mathrm{Re}_{2D,c}$

Critical Re for instability ($c_i=0$)

Now, consider the Orr-Sommerfeld for $\beta \neq 0$ by setting $\mathrm{Re}_{3D}=\dfrac{\alpha}{k}\mathrm{Re}$

$$\left[(U-c)(D^2-\underbrace{k^2}_{=\alpha^2+\beta^2})-D^2U-\frac{1}{\underbrace{ik\,\mathrm{Re}_{3D}}_{\Rightarrow i\alpha\mathrm{Re}}}(D^2-k^2)^2\right]\tilde{v}=0$$

It follows that the critical Reynolds numbers for the onset of an instability should be

$$\mathrm{Re}_{2D,c}=\mathrm{Re}_{3D,c}=\frac{\alpha}{k}\mathrm{Re}_c$$ $\rightarrow$ Critical Re for instability with $\beta\neq 0$

indicating that

$$\mathrm{Re}_{2D,c}<\mathrm{Re}_c$$

$$\left[(U-c)(D^2-k^2)-D^2U-\frac{1}{i\alpha\mathrm{Re}}(D^2-k^2)\right]\tilde{v}=0$$

Squire's theorem

Theorem: Squire

Given Re_L as the critical Reynolds number for the onset of linear instability for a given α and β, the Reynolds number Re_c below which no exponential instabilities exist for any wave numbers satisfies

$$\mathrm{Re}_c \equiv \min_{\alpha,\beta} \mathrm{Re}_L(\alpha,\beta) = \min_{\alpha} \mathrm{Re}_L(\alpha,0)$$

Remark

The **most unstable linear instability** is always **two dimensional.** $(\beta = 0)$

Summary

1. **Linear stability analysis of inviscid mixing layer**
2. **Linearised equation for viscous parallel flows**
3. **Orr-Sommefed and Squire equation**
4. **Squire's transformation and theorem**