A1: The Boussinesq equations

The governing equations for a nonrotating, inviscid, adiabatic, fluid are:

$$\begin{aligned}
\rho\frac{d\mathbf{u}}{dt} &= -\boldsymbol{\nabla} p - g\rho\hat{\mathbf{z}} \\
\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) &= 0 \\
\frac{d\rho}{dt} &= 0\ .
\end{aligned}$$

(Here $\hat{\mathbf{z}}$ is the upward unit normal.) In the *Boussinesq* approximation, which is appropriate for an almost- incompressible fluid, it assumed that variations of density are small, so that in the intertial terms, and in the continuity equation, we may substitute $\rho \to \rho_0$, a constant. However, even weak density variations are important in *buoyancy*, and so we retain variations in ρ in the buoyancy term in the vertical equation of motion. We define the buoyancy as

$$b = g(\rho_0 - \rho)/\rho_0\ ,$$

and also define a reference pressure, $p_0 = -g\rho_0 z + const.$, in hydrostatic balance with the reference density, and then introduce $\tilde{p} = p - p_0(z)$. Then we have

$$-\boldsymbol{\nabla} p - g\rho\hat{\mathbf{z}} = \begin{pmatrix} -\frac{\partial p}{\partial x} \\ -\frac{\partial p}{\partial y} \\ -\frac{\partial p}{\partial z} - g\rho \end{pmatrix} = \begin{pmatrix} -\frac{\partial \tilde{p}}{\partial x} \\ -\frac{\partial \tilde{p}}{\partial y} \\ -\frac{\partial \tilde{p}}{\partial z} + g\rho_0 - g\rho \end{pmatrix} = -\boldsymbol{\nabla}\tilde{p} + \rho_0 b\hat{\mathbf{z}}\ ,$$

Writing $\Phi = \tilde{p}/\rho_0$ (analogous to geopotential in pressure coordinates), the resulting set of Boussinesq equations for a nonrotating system are

$$\begin{aligned}
\frac{d\mathbf{u}}{dt} &= -\boldsymbol{\nabla}\Phi + b\hat{\mathbf{z}} \\
\nabla\cdot\mathbf{u} &= 0 \\
\frac{db}{dt} &= 0
\end{aligned} \tag{1}$$

Note that hydrostatic balance is just

$$\frac{\partial\Phi}{\partial z} = b\ .$$

We will add rotational terms to these equations as the context requires.