HYDRODYNAMIC STABILITY

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LECTURE 2

Dynamical system:
$$\begin{cases} \dfrac{d\underline{x}}{dt} = \underline{f}(\underline{x}, t) \\ \underline{x}(t=0) = \underline{x}_0 \end{cases}$$

EXAMPLE: N-S Equations
$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\nabla P + \dfrac{1}{Re}\nabla^2\vec{u} \\ \vec{\nabla}\cdot\vec{u} = 0 \end{cases}$$

let $\underline{x} = [\vec{u}^T \;\; P]^T$, then:

$$\frac{\partial}{\partial t}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \vec{u} \\ P \end{bmatrix} = \begin{bmatrix} 1/Re\,\nabla^2 & -\vec{\nabla} \\ \vec{\nabla} & 0 \end{bmatrix}\begin{bmatrix} \vec{u} \\ P \end{bmatrix} + \begin{bmatrix} -(\vec{u}\cdot\vec{\nabla})\vec{u} \\ 0 \end{bmatrix}$$

Imagining $f(x) \Rightarrow f(x_1), f(x_2), \ldots, f(x_n)$ as $n \to \infty$ we can define this as an infinite dimensional dynamical system.

Note that from a phase portrait it is possible to identify equilibrium points. But how are they defined?

EQUILIBRIUM POINT:

$\vec{\bar{x}}$ is an equilibrium point if $\vec{x}(t) = \vec{\bar{x}}$ is a solution of the dynamical system such that $\vec{f}(\vec{\bar{x}}, t) = 0$

Note that, for a general dynamical system the number of equilibrium points is undefined

JACOBIAN LINEARIZATION

let $\vec{\bar{x}}$ be an equilibrium point such that $\vec{f}(\vec{\bar{x}}) = 0$. Considering a small perturbation, the dynamical system can be approximated.

$$\vec{x} = \vec{\bar{x}} + \underbrace{\varepsilon}_{\varepsilon \ll 1}\delta\vec{x}$$

$$\frac{d\,\delta\vec{x}}{dt} = \left.\frac{\partial \vec{f}}{\partial \vec{x}}\right|_{\vec{x} = \vec{\bar{x}}} \delta\vec{x}$$

$\dfrac{\partial \vec{f}}{\partial \vec{x}}$ is a matrix with constant entries.

How is it found?

$$\frac{d\vec{x}}{dt} = \vec{f}(x) \qquad \vec{x} = \vec{\tilde{x}} + \varepsilon\,\delta\vec{x}$$

$$\frac{d(\tilde{\vec{x}} + \varepsilon\delta x)}{dt} = \vec{f}(\vec{\tilde{x}} + \varepsilon\delta\vec{x}) \longrightarrow \frac{d\,\varepsilon\,\delta\vec{x}}{dt} = \cancelto{0}{f(\vec{\tilde{x}})} + \varepsilon \left.\frac{\partial \vec{f}}{\partial \vec{x}}\right|_{\vec{x}=\vec{\tilde{x}}} \delta\vec{x}$$

ε cancels out and $\dfrac{d\,\delta\vec{x}}{dt} = \left.\dfrac{\partial \vec{f}}{\partial \vec{x}}\right|_{\vec{x}=\vec{\tilde{x}}} \delta\vec{x}$

A linear system is easier to analyse.

EXAMPLE: linearised N-S equations

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}P + \dfrac{1}{Re}\nabla^2\vec{u} \\ \vec{\nabla}\cdot\vec{u} \end{cases}$$

let $\vec{u} = \vec{U} + \varepsilon\vec{u}'$

$P = \bar{P} + \varepsilon P'$

Substituting:

$$\frac{\partial(\cancelto{0}{\vec{U}} + \varepsilon\vec{u}')}{\partial t} + ((\vec{U}+\varepsilon\vec{u}')\cdot\vec{\nabla})(\vec{U}+\varepsilon\vec{u}') = -\vec{\nabla}(\bar{P}+\varepsilon P') + \frac{1}{Re}\nabla^2(\vec{U}+\varepsilon\vec{u}'$$

$$\varepsilon\frac{\partial\vec{u}'}{\partial t} + (\vec{U}\cdot\vec{\nabla} + \varepsilon\vec{u}'\cdot\vec{\nabla})(\vec{U}+\varepsilon\vec{u}') = -\vec{\nabla}\bar{P} - \varepsilon\vec{\nabla}P' + \frac{1}{Re}\nabla^2\vec{U} + \frac{\varepsilon}{Re}\nabla^2\vec{u}'$$

$$\varepsilon\frac{\partial\vec{u}'}{\partial t} + (\vec{U}\cdot\vec{\nabla})\vec{U} + \varepsilon(\vec{U}\cdot\vec{\nabla})\vec{u}' + \varepsilon(\vec{u}'\cdot\vec{\nabla})\vec{U} + \cancelto{0}{\varepsilon^2(\vec{u}'\cdot\vec{\nabla})}\vec{u}' = -\vec{\nabla}\bar{P} - \varepsilon\vec{\nabla}P' + \frac{1}{Re}\nabla^2\vec{U} + \frac{\varepsilon}{Re}\nabla^2\vec{u}'$$

At $O(1)$:

$$\begin{cases} (\vec{U}\cdot\vec{\nabla})\vec{U} = -\vec{\nabla}\bar{P} + \dfrac{1}{Re}\nabla^2\vec{U} \\ \vec{\nabla}\cdot\vec{U} = 0 \end{cases}$$

At $O(\varepsilon)$:

$$\begin{cases} \dfrac{\partial\vec{u}'}{\partial t} + (\vec{U}\cdot\vec{\nabla})\vec{u}' + (\vec{u}'\cdot\vec{\nabla})\vec{U} = -\vec{\nabla}P' + \dfrac{1}{Re}\nabla^2\vec{u}' \rightarrow \underline{\text{LINEAR}}! \\ \vec{\nabla}\cdot\vec{u}' = 0 \end{cases} \quad (2)$$

If a linearised dynamical system has a solution such that:

$\| \delta \vec{x} \| \to \infty$ as $t \to \infty$: linearly unstable

$\| \delta \vec{x} \| \to 0$ as $t \to \infty$

LINEAR STABILITY OF A PLANAR DYNAMICAL SYSTEM

The linearised system is written as mentioned beforehand:

$$\frac{d \delta \vec{x}}{dt} = \left. \frac{\partial \vec{f}}{\partial \vec{x}} \right|_{\vec{x} = \vec{x}^*} \delta \vec{x} = \underline{\underline{A}} \, \delta \vec{x} \quad , \quad \underline{\underline{A}} \;\; 2 \times 2 \text{ matrix}$$

We introduce a normal mode solution: $\delta \underline{x} = e^{\lambda t} \delta \hat{\underline{x}}$

with the eigenvalue problem: $\lambda \delta \hat{\vec{x}} = \underline{\underline{A}} \, \delta \hat{\vec{x}}$

So the solution is:

$$\delta \underline{x}(t) = c_1 e^{\lambda t} \delta \hat{\vec{x}}_1 + c_2 e^{\lambda_2 t} \delta \hat{\vec{x}}_2$$

with: $\begin{cases} \lambda_1, \lambda_2 & \text{eigenvalues} \\ \delta \hat{\vec{x}}_1, \delta \hat{\vec{x}}_2 & \text{eigenvectors} \\ c_1, c_2 & \text{constants to be determined with initial conditions} \end{cases}$

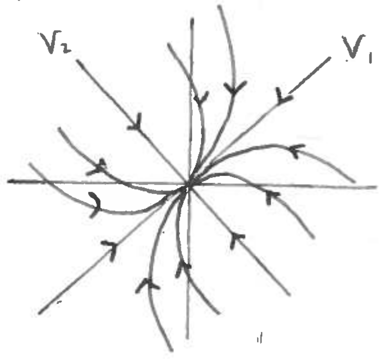
It is important to observe how the system will diverge if any of the eigenvalues is positive. (Real part)

If $Re(\lambda_1)$ or $Re(\lambda_2) > 0 \Rightarrow$ linearly unstable

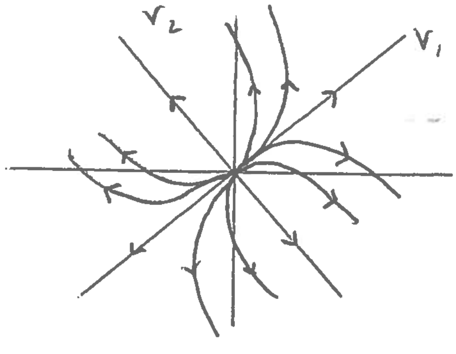
PHASE PORTRAITS

Case 1: $\lambda_1 \lambda_2$ real and $\lambda_1 \neq \lambda_2 \Rightarrow \vec{v}_1, \vec{v}_2$ linearly independent

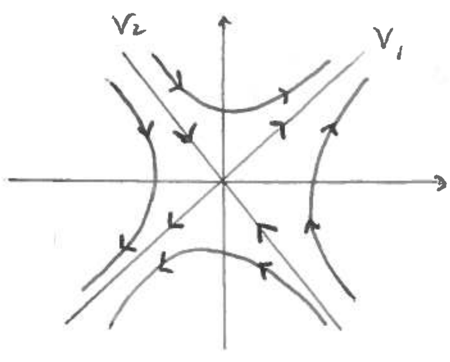
i) $\lambda_1, \lambda_2 < 0 \quad |\lambda_1| > |\lambda_2|$



ii) $\lambda_1, \lambda_2 > 0, \; |\lambda_1| > |\lambda_2|$



iii) $\lambda_2 < 0 < \lambda_1$



Saddle point

CASE 2

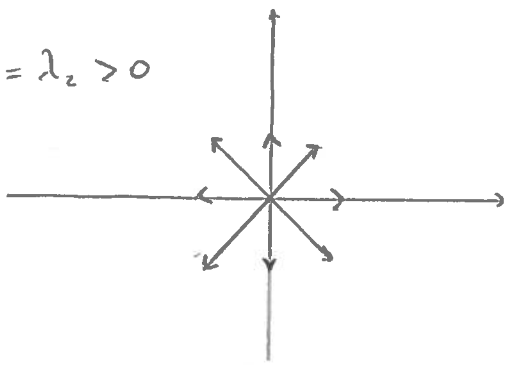
λ_1, λ_2 are both real and $\lambda_1 = \lambda_2$

i) $\text{rank}(A - \lambda I) = 2$

The eigenspace is full rank

$$\begin{cases} x(t) = x_0 e^{\lambda t} \\ y(t) = y_0 e^{\lambda t} \end{cases}$$

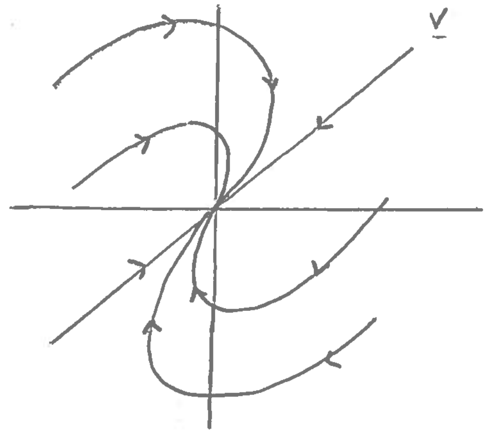
$\lambda_1 = \lambda_2 > 0$



ii) $\text{rank}(A - \lambda I) = 1$

Only one eigenvector exists

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 t + c_2 \\ c_3 t + c_4 \end{bmatrix} e^{\lambda t}$$



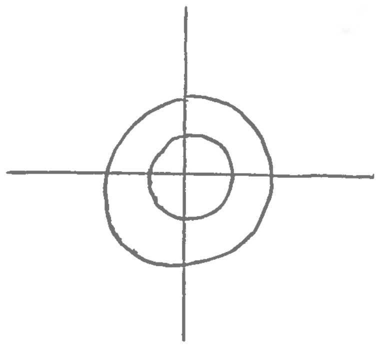
$\lambda < 0$

CASE 3

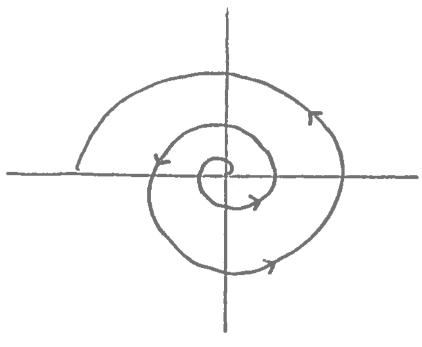
λ_1, λ_2 are complex such that $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

Solution: $e^{\lambda t} = e^{\alpha t}(\cos \beta t + i \sin \beta t)$

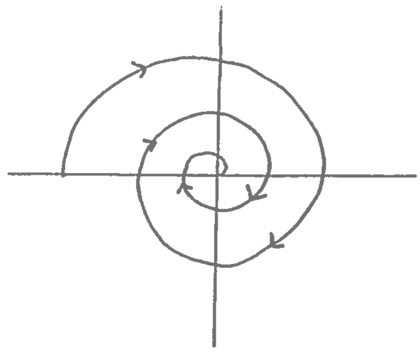
i) $\alpha = 0$



ii) $\alpha > 0$



iii) $\alpha < 0$



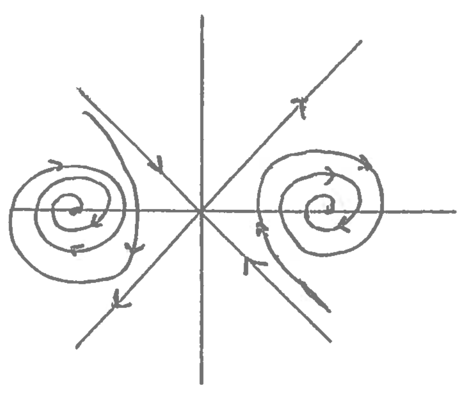
EXAMPLE

: $\ddot{x} + \dot{x} - x + x^3 = 0 \longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - x_1^3 - x_2 \end{cases}$

Equilibrium points: $(0,0)$, $(1,0)$, $(-1,0)$

1) $(0,0) \rightarrow \left.\dfrac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

2) $(\pm 1, 0) \rightarrow \left.\dfrac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$ $\quad \lambda_{1,2}$ Complex, $Re(\lambda_1, \lambda_2) < 0$



LECTURE 3

BIFURCATION

Definition: sudden topological change of given nonlinear dynamical system taking place when a control parameter changes smoothly.
Definition 2: More strictly speaking it is the change in number, or in the qualitative character, of the set of possible steady flows (or unsteady flows in dynamic equilibrium) as Re varies, often linked with the onset of instability

Types of bifurcation here analysed:
1) Transcritical bifurcation
2) Saddle-node bifurcation
3) Pitchfork bifurcation
4) Hopf bifurcation

1) TRANSCRITICAL BIFURCATION

The aim is always plotting the bifurcation diagram, where some variable describing the state is plotted against some parameters.
EXAMPLE: Plot the bifurcation diagram of $\frac{du}{dt} = K(R-R_c)u - \ell u^2$
with K, ℓ constants ($K>0$, $\ell>0$) and R parameter

- Find equilibrium points $\rightarrow f(R) = 0$

$K(R-R_c)u - \ell u^2 = 0 \rightarrow u\left[K(R-R_c) - \ell u\right] = 0$

$u_1 = 0$ (a)

$u_2 = \frac{K}{\ell}(R-R_c)$ (b)

- Examine linear stability of these solutions

(a) $u_0 = 0$ let $u = u_0 + \varepsilon\,\delta u$, $\varepsilon \ll 1$

$$\frac{d\,\delta u}{dt} = \left.\frac{\partial f}{\partial u}\right|_{u=u_0} \delta u = K(R-R_c)\,\delta u - 2\ell u_0\,\delta u$$

Since $u_0 = 0$

$$\left.\frac{\partial f}{\partial u}\right|_{u=u_0} \delta u = k(R-R_c)\,\delta u$$

Since $A = k(R-R_c)\,\delta u$, $\quad \lambda = k(R-R_c)\,\delta u$.

Therefore: $\begin{cases} R-R_c > 0 \longrightarrow R > R_c & \text{linearly unstable} \\ R-R_c < 0 \longrightarrow R < R_c & \text{linearly stable} \end{cases}$

(b) $u_0 = \frac{k}{\ell}(R-R_c)$

let $u = u_0 + \varepsilon\,\delta u$, $\quad \varepsilon \ll 1$

$$\frac{d\,\delta u}{dt} = \left.\frac{\partial f}{\partial u}\right|_{u=u_0} \delta u = \left(k(R-R_c) - 2\ell u_0\right)\delta u$$

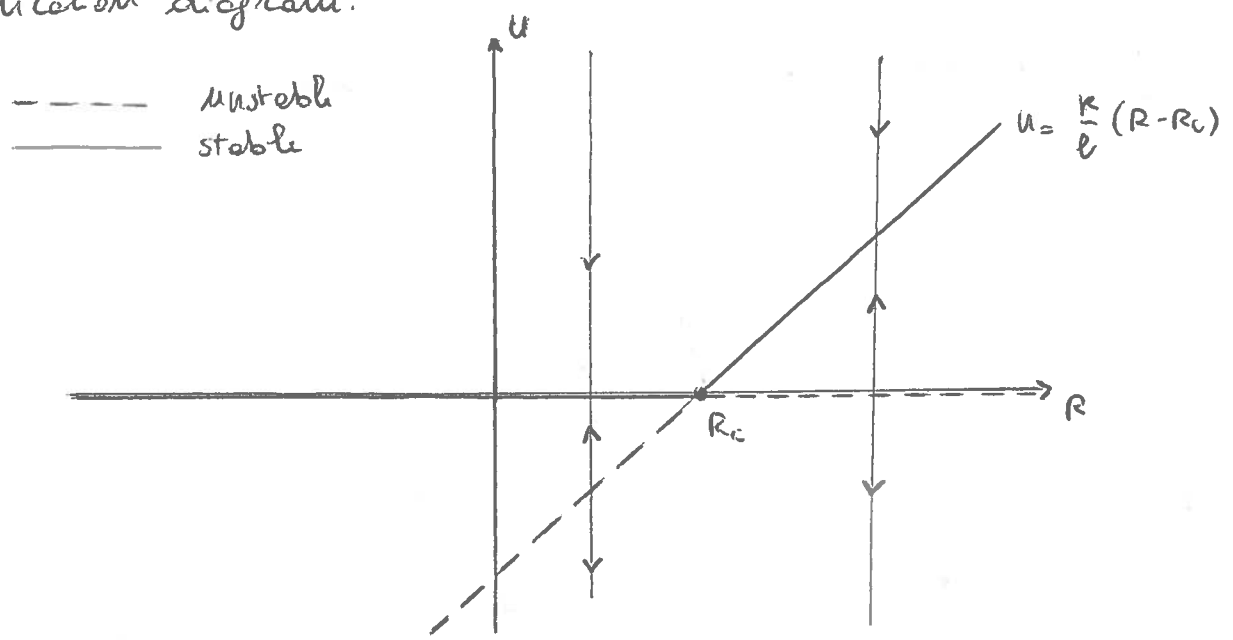
$$\left.\frac{\partial f}{\partial u}\right|_{u=u_0} = A\delta u = k(R-R_c)\,\delta u - 2\ell\cdot\frac{k}{\ell}(R-R_c)\,\delta u$$

$$\delta u\,A = k(R-R_c)\,\delta u - 2k(R-R_c)\,\delta u = -k(R-R_c)\,\delta u$$

Therefore $\lambda = -k(R-R_c) = (R_c - R)k$

$\begin{cases} R_c - R > 0 \longrightarrow R < R_c & \text{linearly unstable} \\ R_c - R < 0 \longrightarrow R > R_c & \text{linearly stable} \end{cases}$

Bifurcation diagram:



2) SADDLE-NODE BIFURCATION

EXAMPLE: Find the bifurcation diagram of $\frac{du}{dt} = K(R-R_c) - \ell u^2$ with $K>0$, $\ell>0$ and R control parameter

- Find equilibrium points

$$f(u) = 0 \rightarrow K(R-R_c) - \ell u^2 = 0 \rightarrow u^2 = \frac{K}{\ell}(R-R_c)$$

So
$$u_1 = \sqrt{\frac{K}{\ell}(R-R_c)} \quad (a)$$
$$u_2 = -\sqrt{\frac{K}{\ell}(R-R_c)} \quad (b)$$

- Examine linear stability of these solutions.

Recap... $u = u_0 + \varepsilon \delta u$, $\varepsilon \ll 1$

$$\frac{du}{dt} = \frac{d(\cancel{u_0} + \varepsilon \delta u)}{dt} = \varepsilon \frac{d\,\delta u}{dt} = f(u_0 + \varepsilon \delta u) = \cancel{f(u_0)} + \left.\frac{\partial f}{\partial u}\right|_{u=u_0} \varepsilon \delta u$$

So
$$\frac{d\,\delta u}{dt} = \underbrace{\left.\frac{\partial f}{\partial u}\right|_{u=u_0}}_{A} \delta u = -2\ell u_0\, \delta u \quad , \quad A = -2\ell u_0$$

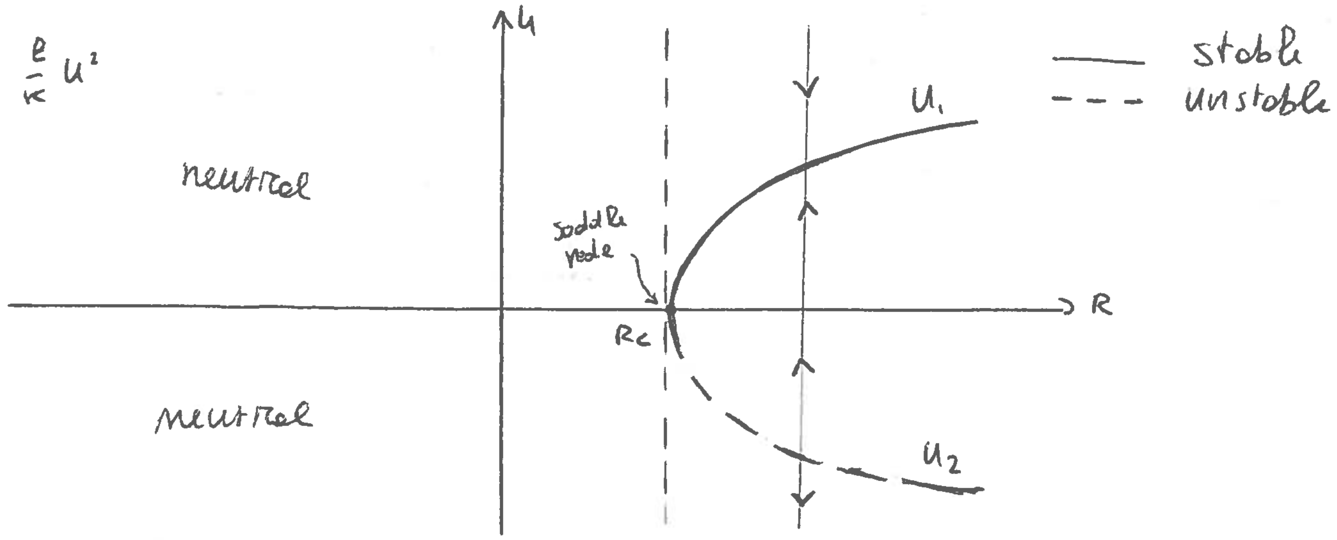
(a) $A = -2\ell\sqrt{\frac{K}{\ell}(R-R_c)} = \lambda$

$$\begin{cases} R-R_c > 0 \longrightarrow R > R_c & \text{linearly stable} \\ R-R_c < 0 \longrightarrow R < R_c & \text{linearly neutral} \end{cases}$$

(b) $A = +2\ell\sqrt{\frac{K}{\ell}(R-R_c)} = \lambda$

$$\begin{cases} R-R_c > 0 \longrightarrow R > R_c & \text{linearly unstable} \\ R-R_c < 0 \longrightarrow R < R_c & \text{linearly neutral} \end{cases}$$

$R = R_c + \frac{\ell}{K}u^2$



3) PITCHFORK BIFURCATION

EXAMPLE: Find the bifurcation diagram of $\frac{du}{dt} = k(R-R_c)u - \ell u^3$

with $k>0$, $\ell>0$ and R parameter

- Find equilibrium points $\quad f(u)=0 \rightarrow k(R-R_c)u - \ell u^3 = 0$

$$u\left[k(R-R_c) - \ell u^2\right] = 0 \qquad \begin{array}{ll} u_1 = 0 & (a) \\ u_2 = \sqrt{\frac{k}{\ell}(R-R_c)} & (b) \\ u_3 = -\sqrt{\frac{k}{\ell}(R-R_c)} & (c) \end{array}$$

- Examine linear stability of these solutions

let $u = u_0 + \varepsilon\, \delta u \quad , \quad \varepsilon \ll 1.$

Then $\frac{d\delta u}{dt} = \underbrace{\frac{\partial f}{\partial u}\Big|_{u=u_0}}_{A} \delta u = \underbrace{\left[k(R-R_c) - 3\ell u_0^2\right]}_{A} \delta u$

(a) $A = k(R-R_c) = \lambda \quad \begin{cases} R-R_c>0 \longrightarrow R>R_c & \text{linearly unstable} \\ R-R_c<0 \longrightarrow R<R_c & \text{linearly stable} \end{cases}$

(b) $A = k(R-R_c) - 3\ell \cdot \frac{k}{\ell}(R-R_c) = -2k(R-R_c) = 2k(R_c-R)$

$\begin{cases} R_c - R>0 \longrightarrow R<R_c & \text{linearly unstable} \\ R_c - R<0 \longrightarrow R>R_c & \text{linearly stable} \end{cases}$

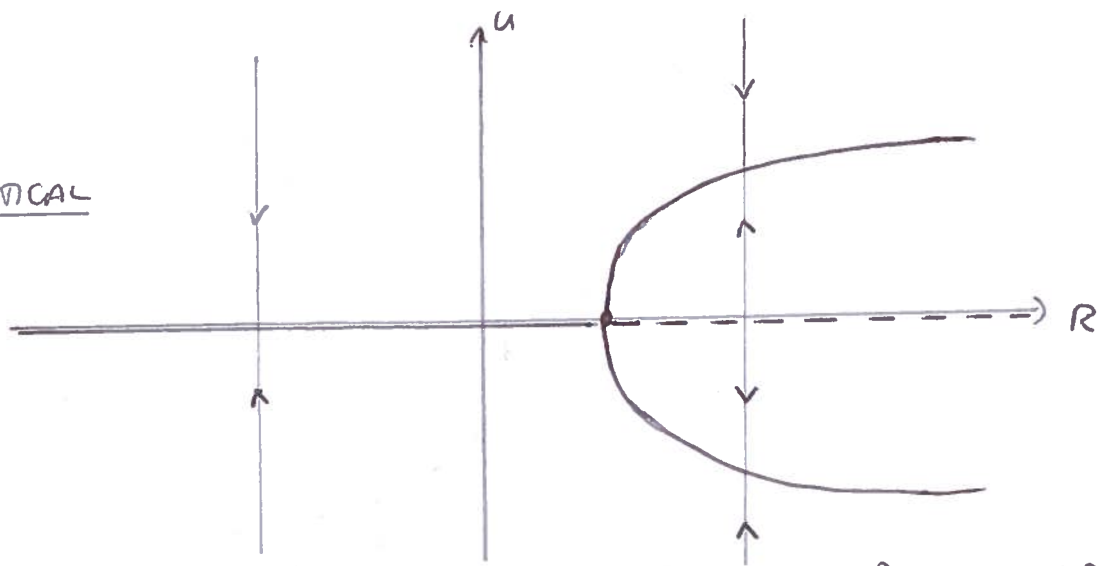
(c) $A = k(R-R_c) - 3\ell\,\frac{k}{\ell}(R-R_c) = 2k(R_c-R)$ same as (b)

$\begin{cases} R_c - R>0 \longrightarrow R<R_c & \text{linearly unstable} \\ R_c - R<0 \longrightarrow R>R_c & \text{linearly stable} \end{cases}$

$\ell > 0$

↓

SUPERCRITICAL CASE



two stable solutions for R greater than R_c for linear stability in addition to the unstable solution $u_1 = 0$.

What if $\ell < 0$? $\rightarrow$ SUBCRITICAL CASE

the equilibrium points are found in the same way but

$$u^2 = \frac{k}{\ell}(R - R_c) \rightarrow R = R_c + \underbrace{\frac{\ell}{k}}_{<0} u^2$$

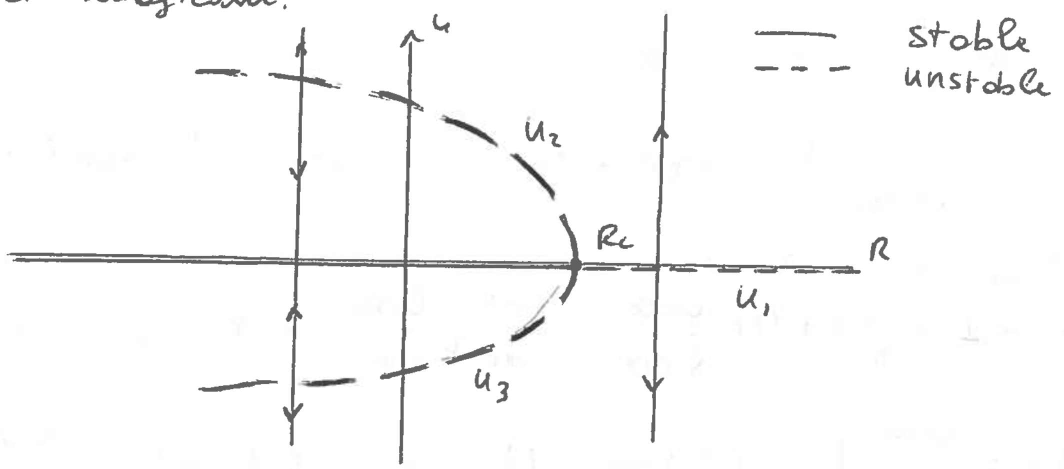
- Examine linear stability

(a) $k(R-R_c) = A$ $\begin{cases} R - R_c > 0 \rightarrow & R > R_c \text{ linearly unstable} \\ R - R_c < 0 \rightarrow & R < R_c \text{ linearly stable} \end{cases}$

(b) $A = \underbrace{-2\frac{k}{\ell}}_{>0}(R - R_c)$ $\begin{cases} R - R_c > 0 \rightarrow & R > R_c \text{ linearly unstable} \\ R - R_c < 0 \rightarrow & R < R_c \text{ linearly stable} \end{cases}$

(c) same as (b).

Bifurcation diagram:



An example of pitchfork bifurcation is the flow wake behind a sphere. $Re_c \cong 210$

$Re_D = 100 \rightarrow$ steady axisymmetric

$Re_D = 250 \rightarrow$ steady plane symmetric

4) HOPF BIFURCATION

Find the bifurcation diagram a model given by:

$$\frac{dx}{dt} = -y + (a - x^2 - y^2)x$$

$$\frac{dy}{dt} = x + (a - x^2 - y^2)y$$

with $a = k(R - R_c)$, $k > 0$

let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$x: \quad \frac{d(r\cos\theta)}{dt} = -r\sin\theta + (a - r^2)\, r\cos\theta$$

$$\frac{dr}{dt}\cos\theta + r\frac{d\theta}{dt}(-\sin\theta) = -r\sin\theta + (a-r^2)\, r\cos\theta$$

$$y: \quad \frac{d(r\sin\theta)}{dt} = r\cos\theta + (a-r^2)\, r\sin\theta$$

$$\frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}(\cos\theta) = +r\cos\theta + (a-r^2)\, r\sin\theta$$

Comparing left-hand and right-hand sides:

$$\begin{cases} \frac{dr}{dt}\cos\theta = (a-r^2)\, r\cos\theta \\ -r\frac{d\theta}{dt}\sin\theta = -r\sin\theta \end{cases}$$

This leads to

$$\begin{cases} \frac{dr}{dt} = r(a-r^2) \quad r > 0 & (Eq. 1) \\ \frac{d\theta}{dt} = 1 & (Eq. 2) \end{cases}$$

Eq. 1

- Equilibrium

$f(R) = 0 \rightarrow r(a-r^2) = 0$

$r_1 = 0$ (a)

$r_2 = \sqrt{a} = \sqrt{K(R-R_c)}$ (b)

$\theta(t) = \text{Const} + t$

- Linearization $\left.\frac{\partial f}{\partial r}\right|_{r=r_2} \delta r = (a - 3r_2^2)\, \delta r$

(b)

$A = K(R-R_c) - 3K(R-R_c) = -2K(R-R_c)$

$r_2 \rightarrow \begin{cases} -R + R_c > 0 \longrightarrow R < R_c & \text{linearly unstable} \\ -R + R_c < 0 \longrightarrow R > R_c & \text{linearly stable} \end{cases}$

(a) $\left.\frac{\partial f}{\partial r}\right|_{r=r_1} \delta r = (a - 3r_1^2)\,\delta r = a\,\delta r \qquad A = a$

$A = K(R - R_c) \quad \begin{cases} R - R_c > 0 \longrightarrow R > R_c & \text{linearly unstable} \\ R - R_c < 0 \longrightarrow R < R_c & \text{linearly stable} \end{cases}$

So, Solution 1: (d)

$\begin{cases} r_1 = 0 \\ \theta = t + \text{Const} \end{cases} \Longrightarrow \begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases}$ stable for $R < R_c$, unstable for $R > R_c$

Solution 2: (b)

$\begin{cases} r_2 = \sqrt{K(R-R_c)} \\ \theta = t + \text{Const} \end{cases} \Longrightarrow \begin{cases} x_2 = \sqrt{a}\cos(t + \text{Const}) \\ y_2 = \sqrt{a}\sin(t + \text{Const}) \end{cases}$ stable for $R > R_c$, unstable for $R < R_c$

Two situations arise:

i) $\alpha > 0$

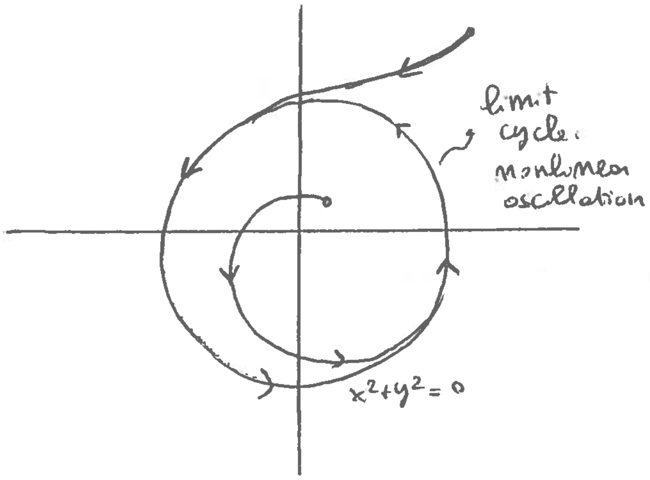
Solution 1 $\rightarrow$ unstable

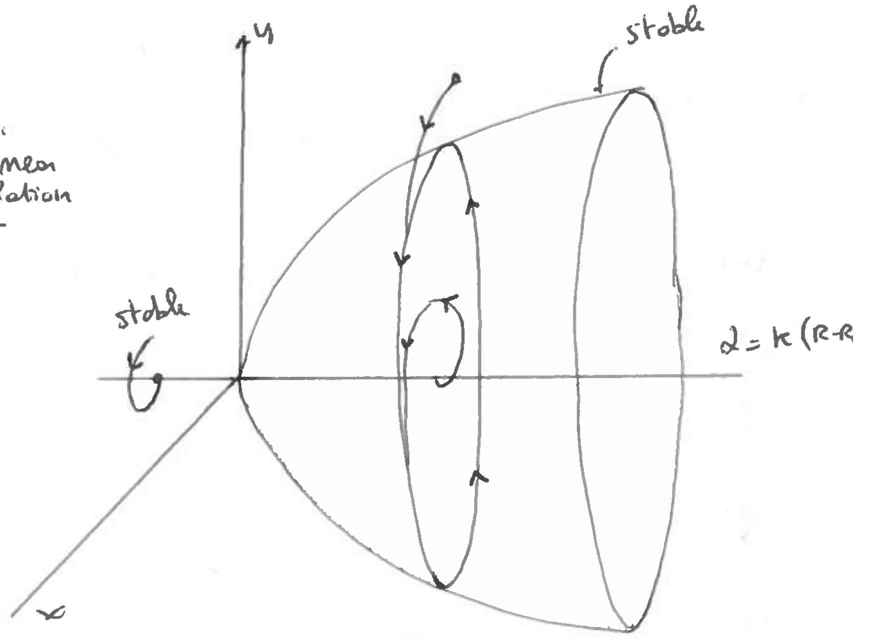
Solution 2 $\rightarrow$ stable

ii) $\alpha < 0$

$\downarrow$

<u>not significant</u>.





Supercritical Hopf bifurcation $\longleftarrow$

$\alpha < 0 \rightarrow$ Sol. 1

$\alpha > 0 \rightarrow$ Sol. 2.

An example is flow wake behind a cylinder. $Re_D = \frac{U_\infty D}{\nu}$ $\quad Re_{D_{cr}} \cong 47$

$Re_D = 27 \longrightarrow$ steady symmetric

$\leq Re_D \leq 188$ ($Re_D = 140$) $\longrightarrow$ unsteady time periodic (Von Karman Vortex)

LECTURE 4

LINEAR STABILITY OF PARALLEL FLOWS

Parallel shear flows have a base flow configuration given by:

$$\vec{U}(y) = (U(y), 0, 0)$$

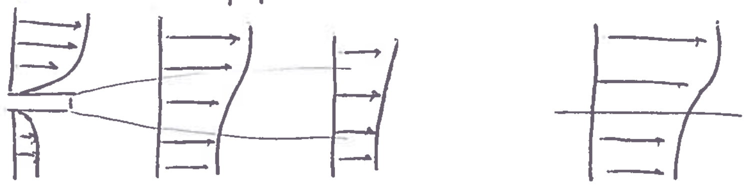
EXAMPLES:

- Plane Couette flow: $U(Y) = Y$
- Poiseuille flow : $U(Y) = 1 - Y^2$
- Pipe flow : $U(r) = 1 - r^2$

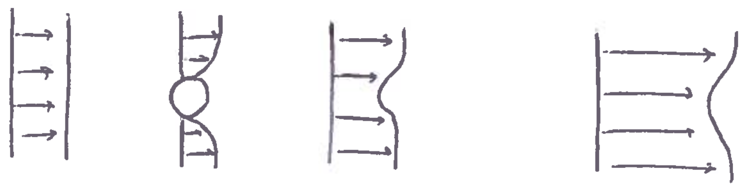
This type of study finds useful application in some weakly non-parallel flows:

- Boundary layer

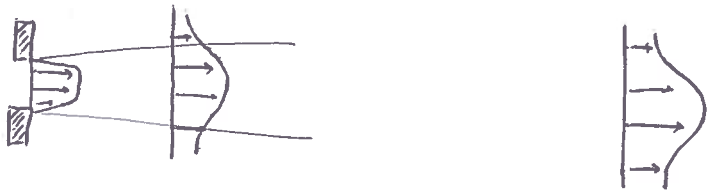
- Mixing layer



- Cylinder wake



- Jet



LINEARISED EQUATION FOR INVISCID PARALLEL FLOW

Euler equation :

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\vec{\nabla} P \\ \vec{\nabla} \cdot \vec{u} = 0 \end{cases}$$

Consider

$$\vec{u}(x,y,z,t) = \overbrace{(U(Y), 0, 0)}^{\vec{U}} + \varepsilon \vec{u}'(x,y,z,t)$$

$$P(x,y,z,t) = \bar{P}(x,y) + \varepsilon \; p'(x,y,z,t)$$

Substituting:

$$\begin{cases} \dfrac{\partial(\vec{U}+\varepsilon\vec{u}')}{\partial t} + ((\vec{U}+\varepsilon\vec{u}')\cdot\vec{\nabla})(\vec{U}+\varepsilon\vec{u}') = -\vec{\nabla}(\bar{P}+\varepsilon P') \\ \vec{\nabla}\cdot(U+\varepsilon u') = 0 \end{cases}$$

momentum equation:

$$\cancel{\frac{\partial\vec{U}}{\partial t}}^{\,0} + \varepsilon\frac{\partial\vec{u}'}{\partial t} + (\vec{U}\cdot\vec{\nabla})\vec{U} + \varepsilon(\vec{U}\cdot\vec{\nabla})\vec{u}' + \varepsilon(\vec{u}'\cdot\vec{\nabla})\vec{U} +$$
$$+ \cancel{\varepsilon^2(\vec{u}'\cdot\vec{\nabla})\vec{u}'}^{\,0} = -\vec{\nabla}\bar{P} - \vec{\nabla}\varepsilon P'$$

Consider $O(\varepsilon)$:

$$\begin{cases} \dfrac{\partial\vec{u}'}{\partial t} + (\vec{U}\cdot\vec{\nabla})\vec{u}' + (\vec{u}'\cdot\vec{\nabla})\vec{U} = -\vec{\nabla}P' \\ \vec{\nabla}\cdot\vec{u}' = 0 \end{cases} \qquad \boxed{\vec{U} = U\hat{x}}$$

Pressure represents the issue to be eliminated. Expanding the equation, the nonlinear term is evidenced:

$$\begin{cases} (\vec{U}\cdot\vec{\nabla})\vec{u}' = (U(y)\hat{x}\cdot\vec{\nabla})\vec{u}' = U(y)\dfrac{\partial\vec{u}'}{\partial x}\hat{x} \\ (\vec{u}'\cdot\vec{\nabla})\vec{U} = \left(u'\dfrac{\partial}{\partial x} + v'\dfrac{\partial}{\partial y} + w'\dfrac{\partial}{\partial z}\right)U(y)\hat{x} = v'\dfrac{\partial U(y)}{\partial y}\hat{x} \end{cases}$$

Euler equations:

$$\begin{cases} \dfrac{\partial u'}{\partial t} + U\dfrac{\partial u'}{\partial x} + v'\dfrac{\partial U}{\partial y} = -\dfrac{\partial P'}{\partial x} \\ \dfrac{\partial v'}{\partial t} + U\dfrac{\partial v'}{\partial x} = -\dfrac{\partial P'}{\partial y} \\ \dfrac{\partial w'}{\partial t} + U\dfrac{\partial w'}{\partial x} = -\dfrac{\partial P'}{\partial z} \\ \dfrac{\partial u'}{\partial x} + \dfrac{\partial v'}{\partial y} + \dfrac{\partial w'}{\partial z} = 0 \end{cases}$$

We now take partial derivatives of the three equations.

$$\frac{\partial}{\partial x}\left[\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = -\frac{\partial P'}{\partial x}\right] \rightarrow$$

$$U\frac{\partial^2 u'}{\partial x^2} + \frac{\partial v'}{\partial x}\frac{\partial U}{\partial y} = -\frac{\partial^2 P'}{\partial x^2} \quad (1) \qquad \left(\text{with } \frac{\partial}{\partial t}\frac{\partial u'}{\partial x} \text{ on the left}\right)$$

$$\frac{\partial}{\partial y}\left[\cancel{\frac{\partial v'}{\partial t}}^{\,0} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y}\right] \longrightarrow \overset{\frac{\partial}{\partial t}\frac{\partial v}{\partial y}}{\vee}\; U\frac{\partial}{\partial x}\left(\frac{\partial v'}{\partial y}\right)\overset{+\frac{dU}{dy}\frac{\partial v}{\partial x}}{\vee} = -\frac{\partial^2 p'}{\partial y^2} \qquad (2)$$

$$\frac{\partial}{\partial z}\left[\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z}\right] \longrightarrow \overset{\frac{\partial}{\partial t}\frac{\partial w'}{\partial z}}{\vee}\; U\frac{\partial}{\partial x}\left(\frac{\partial w'}{\partial z}\right) = -\frac{\partial^2 p}{\partial z^2} \qquad (3)$$

(1) + (2) + (3) ;

$$U\frac{\partial^2 u'}{\partial x^2} + \frac{dU}{dy}\frac{\partial v'}{\partial x} + \frac{dU}{dy}\frac{\partial v'}{\partial x} + U\frac{\partial}{\partial x}\left(\frac{\partial v'}{\partial y}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial w'}{\partial z}\right) = -\frac{\partial^2 p'}{\partial x^2} - \frac{\partial p'}{\partial y^2} - \frac{\partial p'}{\partial z^2}$$

$$U\frac{\partial}{\partial x}\left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right] + 2\frac{\partial U}{\partial y}\frac{\partial v'}{\partial x} = -\nabla^2 p'$$

Incompressibility: $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \rightarrow \frac{\partial}{\partial t}\left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right] = 0$

Therefore: $\boxed{\nabla^2 p' = -2\frac{dU}{dy}\frac{\partial v'}{\partial x}}$ POISSON EQUATION FOR PRESSURE

1)
Now consider the y-component of the momentum equation and the equation just derived:

$$\begin{cases} \frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} \\ \nabla^2 p' = -2\frac{dU}{dy}\frac{\partial v'}{\partial x} \end{cases}$$

We take the laplacian of the first equation and $\frac{\partial}{\partial y}$ in the secon

$$\begin{cases} \nabla^2\left(\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y}\right) \longrightarrow \nabla^2\left(\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x}\right) = -\nabla^2\left(\frac{\partial p'}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(-2\frac{dU}{dy}\frac{\partial v'}{\partial x} = \nabla^2 p'\right) \longrightarrow \frac{\partial}{\partial y}\left(+2\frac{dU}{dy}\frac{\partial v'}{\partial x}\right) = -\nabla^2\left(\frac{\partial p'}{\partial y}\right) \end{cases}$$

This means that:

$$\nabla^2\left(\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x}\right) = \frac{\partial}{\partial y}\left(2\frac{dU}{dy}\frac{\partial v'}{\partial x}\right)$$

$$\boxed{\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - \frac{d^2U}{dy^2}\frac{\partial}{\partial x}\right]v' = 0}$$

Pressure is gone

Further explanation

$$\nabla^2\left(\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}\right)=\frac{\partial}{\partial y}\left(2\frac{dU}{dy}\frac{\partial v'}{\partial x}\right)$$

Consider the left-hand side and expand:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}\right)=\frac{\partial}{\partial y}\left(2\frac{dU}{dy}\frac{\partial v'}{\partial x}\right)$$

$$\nabla^2\frac{\partial v'}{\partial t}+U\frac{\partial^2}{\partial x^2}\left(\frac{\partial v'}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{dU}{dy}\frac{\partial v'}{\partial x}+U\frac{\partial^2 v'}{\partial x\partial y}\right)+U\frac{\partial^2}{\partial z^2}\left(\frac{\partial v'}{\partial x}\right)=$$

$$=2\frac{d^2U}{dy^2}\frac{\partial v'}{\partial x}+2\frac{dU}{dy}\frac{\partial^2 v'}{\partial x\partial y}$$

$$\nabla^2\frac{\partial v'}{\partial t}+U\frac{\partial}{\partial x}\left(\frac{\partial v'}{\partial x^2}\right)+\frac{d^2U}{dy}\frac{\partial v'}{\partial x}+\cancel{\frac{dU}{dy}\frac{\partial^2 v'}{\partial y\partial x}}+\cancel{\frac{dU}{dy}\frac{\partial^2 v'}{\partial x\partial y}}+$$

$$+U\frac{\partial}{\partial x}\left(\frac{\partial^2 v'}{\partial y^2}\right)+U\frac{\partial}{\partial x}\left(\frac{\partial^2 v'}{\partial z^2}\right)=2\frac{d^2U}{dy^2}\frac{\partial v'}{\partial x}+\cancel{2\frac{dU}{dy}\frac{\partial^2 v'}{\partial x\partial y}}$$

$$\nabla^2\frac{\partial v'}{\partial t}+U\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial v'}{\partial x^2}+\frac{\partial v'}{\partial y^2}+\frac{\partial v'}{\partial z^2}\right)}_{\nabla^2 v'}=\frac{d^2U}{dy^2}\frac{\partial v'}{\partial x}$$

Therefore,

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}\right]v'=0$$

2) A second equation without pressure is now sought after.
Having before used y-component of the momentum equation, now x and z components are employed.

$$\begin{cases} \dfrac{\partial u'}{\partial t} + U \dfrac{\partial u'}{\partial x} + v' \dfrac{dU}{dy} = -\dfrac{\partial p'}{\partial x} \\ \dfrac{\partial w'}{\partial t} + U \dfrac{\partial w'}{\partial x} = -\dfrac{\partial p'}{\partial x} \end{cases}$$

We take partial derivatives of these two equations

$$(1) \quad \frac{\partial}{\partial z}\left[\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{dU}{dy} = -\frac{\partial p'}{\partial x} \right] \Rightarrow \frac{\partial}{\partial t}\frac{\partial u'}{\partial z} + U \frac{\partial}{\partial z}\frac{\partial u'}{\partial x} + \frac{\partial}{\partial z}\left(v' \frac{dU}{dy}\right) = -\frac{\partial}{\partial z}\frac{\partial p'}{\partial x}$$

$$(2) \quad \frac{\partial}{\partial x}\left[\frac{\partial w'}{\partial t} + U \frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z} \right] \Rightarrow \frac{\partial}{\partial t}\frac{\partial w'}{\partial z} + U \frac{\partial}{\partial x}\frac{\partial w'}{\partial} = -\frac{\partial}{\partial x}\frac{\partial p'}{\partial z}$$

(1) - (2):

$$\frac{\partial}{\partial t}\left[\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial z} \right] + U \frac{\partial}{\partial x}\left[\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} \right] + \frac{\partial}{\partial z}\left(v' \frac{dU}{dy}\right) = \cancel{-\frac{\partial}{\partial z}\frac{\partial p'}{\partial x}} + \cancel{\frac{\partial}{\partial z}\frac{\partial p'}{\partial x}}$$

Introducing <u>wall-normal vorticity</u> as $\eta' = \dfrac{\partial u'}{\partial z} - \dfrac{\partial w'}{\partial x}$

The equation can be rewritten as:

$$\boxed{\left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right] \eta' = -\frac{\partial}{\partial z}\left(v' \frac{dU}{dy}\right)}$$

We now dispose of two equations that do not contain the pressure term.

So, summing up: there are two equations <u>(PARABOLIC)</u> and conditions

$$\begin{cases} \left[\left(\dfrac{\partial}{\partial t} + U \dfrac{\partial}{\partial x} \right) \nabla^2 - \dfrac{d^2 U}{dy^2} \dfrac{\partial}{\partial x} \right] v' = 0 \\ \left[\dfrac{\partial}{\partial t} + U \dfrac{\partial}{\partial x} \right] \eta' + \dfrac{\partial}{\partial z}\left(v' \dfrac{dU}{dy} \right) = 0 \\ \left.\begin{array}{l} v' = 0 \\ \eta' = 0 \end{array}\right\} \text{B.C.} \\ \left.\begin{array}{l} v'(x,y,z,0) = v'_0(x,y,z) \\ \eta'(x,y,z,0) = \eta'_0(x,y,z) \end{array}\right\} \text{I.C.} \end{cases}$$

NORMAL MODE SOLUTION LEADING TO RAILEIGH's EQUATION

Consider a two-dimensional case and pick the first equation:

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - \frac{d^2U}{dy^2}\frac{\partial}{\partial x}\right] v' = 0$$

The incompressibility condition is: $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial x} = 0$

In a problem of the kind $\ddot{x} - a^2 x = 0$, we make the assumption that $x = c e^{\lambda t}$ $\longrightarrow$ normal mode solution

In particular, we assume the solution is:

$$v'(x,y,t) = \underbrace{\tilde{v}(y)}_{\text{complex}} e^{i(\alpha x - \omega t)} + \underbrace{c.c.}_{\text{complex conjugate of}}$$

WAVE NUMBER

with $\begin{cases} \alpha \in \mathbb{R} \longrightarrow \underline{\text{known}} \\ \omega \in \mathbb{C} \longrightarrow \underline{\text{unknown}} \end{cases}$

NOTE: Normal mode solution couldn't have been assumed in y direction, as $a = a(y)$ and it would have had to be constant with respect to y.

Substituting v' in the equation at the top yields:

- $\frac{\partial v'}{\partial t} = -\tilde{v}(y)\,\omega i\, e^{i(\alpha x - \omega t)} \longrightarrow \frac{\partial}{\partial t} \longrightarrow -i\omega$
- $\frac{\partial v'}{\partial x} = \tilde{v}(y)\, i\alpha\, e^{i(\alpha x - \omega t)} \longrightarrow \frac{\partial}{\partial x} \longrightarrow i\alpha$
- $\frac{\partial^2 v'}{\partial x^2} = -\tilde{v}(y)\,\alpha^2 e^{i(\alpha x - \omega t)} \longrightarrow \frac{\partial^2}{\partial x^2} \longrightarrow -\alpha^2$

$v' \longrightarrow \tilde{v}$

Plugging these into the equation, letting $\boxed{\frac{d}{dy} = D}$ and $\boxed{\omega = \alpha c}$

$$\Big[(\underbrace{-i\omega}_{\frac{\partial}{\partial t}} + \underbrace{iU\alpha}_{U\frac{\partial}{\partial x}})\underbrace{(D^2 - \alpha^2)}_{\text{laplacian term } \nabla^2} - \underbrace{D^2U\,\alpha}_{\frac{d^2U}{dy^2}\frac{\partial}{\partial x}}\Big]\tilde{v} = 0$$

ω complex $\Rightarrow$ c complex

useful in the next page

knowing that $\omega = \alpha c$:

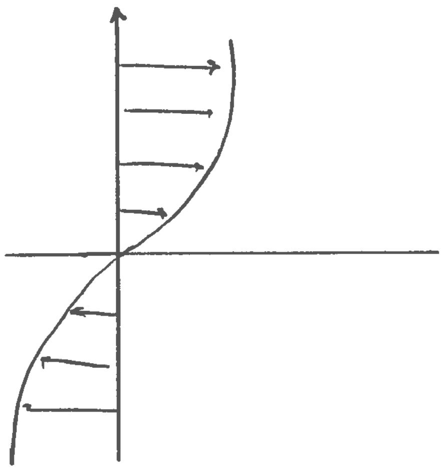
$$\left[(-i\alpha c + iU\alpha)(D^2 - \alpha^2) - D^2U\,i\alpha\right]\tilde{v} = 0$$

FURTHER EXPLANATION: WHAT IS $\tilde{v}$?

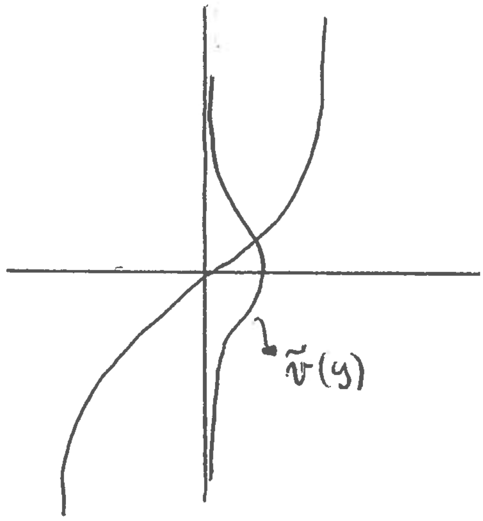
$\tilde{v}$ (eigenfunction) describes the structure of the instability along y.

This is why it is expressed as a function of y.

For example, in a mixing layer in the following form



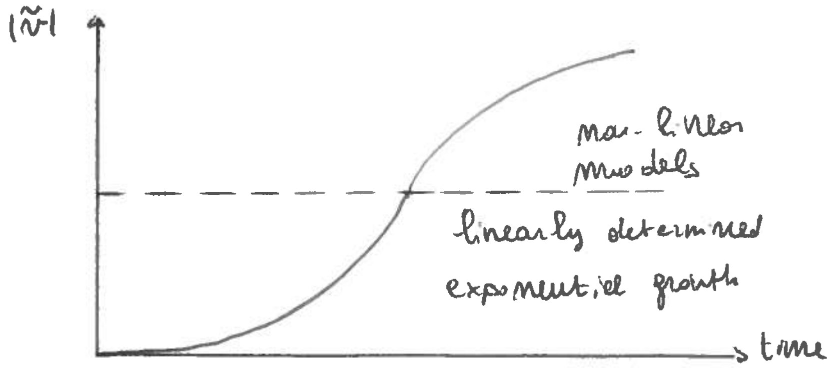
The instability distribution could be seen as:



Also, to be kept in mind is that the module of the exponential $e^{i\alpha(x-ct)}$ is always 1.

Each point in the x-direction will have the same structure.

The exponential growth that we describe with linear theory only gets to a certain point, after which the linear hypothesis is no longer valid.



We may consider only the term in parenthesis.

$$(-i\alpha c + iU\alpha)(D^2-\alpha^2) - D^2 U i\alpha = 0$$

Observe how $i\alpha$ is common to all terms. It is canceled, and:

Finally, Rayleigh equation is:

$$\begin{cases} (U-c)(D^2-\alpha^2)\tilde{v} - D^2 U \tilde{v} = 0 \\ \tilde{v} = 0 \quad \text{at solid boundary / far field} \end{cases}$$ $\rightarrow$ II order ODE for $\tilde{v}$

We notice how writing $\omega = \alpha c$ relates time and space through the scaling parameter c.

The unknowns of the problem are:

$\begin{cases} c \\ \tilde{v} \end{cases}$ with α known because it is prescribed as an input.

This results in an <u>eigenvalue</u> problem.

INTERPRETATION

Each physical quantity is represented by the real part of its complex expression, so:

$$v'(x,y,t) = \tilde{v}(y)\, e^{i\alpha(x-ct)} + c.c.$$

it makes the real part of c imaginary and the imaginary part real

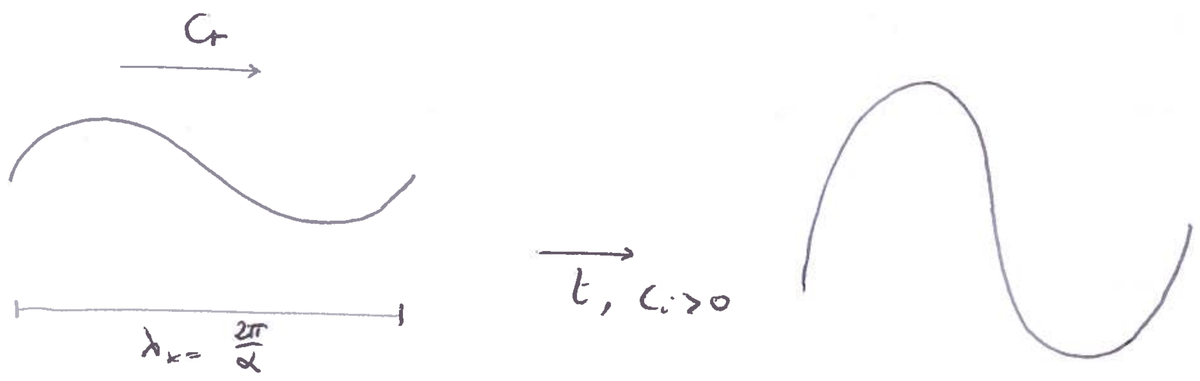
$$= \text{Real}\left\{ \underbrace{|\tilde{v}(y)|}_{\text{amplitude}}\, e^{\overset{\text{phase}}{i\phi(y)}}\, e^{i\alpha(x-(c_r+ic_i)t)} \right\}$$

$$= |\tilde{v}(y)|\, e^{\alpha c_i t} \cos[\alpha(x-c_r t) + \phi(y)]$$

The wave is <u>WALL-NORMAL</u>. Having set $\begin{cases} c_r = Re(c) \\ c_i = Im(c) \end{cases}$ the wave travels with phase velocity c_r in the x (or y?) direction (known), while it decays or grows in time like $e^{\alpha c_i t}$. This means:

- $c_i > 0$ linearly unstable
- $c_i = 0$ marginally stable (or neutral)
- $c_i < 0$ linearly stable

This concept can also be viewed graphically.



THEOREM: RAYLEIGH INFLECTION POINT CRITERION

If there exists perturbations with $c_i > 0$, then $\frac{d^2U}{dy}$ must be zero at some $y \in \Omega$ ($\Omega = [a, b]$ is the flow domain in y). In other words, the occurrence of an inflection point is a <u>necessary</u> condition for instability. The inverse is not true.

PROOF

- Assume $c_i > 0$

Consider Rayleigh's Equation: $(U-c)(D^2-\alpha^2)\tilde{v} - D^2U\tilde{v} = 0$

We multiply it by the complex conjugate $\tilde{v}^*$ of $\tilde{v}$ and divide by $U-c$.

$$\tilde{v}^* \left[\frac{(U-c)(D^2-\alpha^2)\tilde{v} - D^2U\tilde{v}}{U-c} \right] = 0$$

FURTHER EXPLANATION

$$\text{Im}\left(\frac{1}{U-c}\right) = \text{Im}\left(\frac{(U-c)^*}{(U-c)(U-c)^*}\right) = \text{Im}\left(\frac{(U-c)^*}{|U-c|^2}\right)$$

$$(U-c)^* = (U - c_R - i c_i)^* = (U - c_R + i c_i)$$

So $\text{Im}\left(\frac{1}{U-c}\right) = \frac{c_i}{|U-c|^2}$

We then integrate from a to b:

$$\int_a^b \tilde{v}^* \left[\frac{(U-c)(D^2-\alpha^2)\tilde{v} - D^2U\tilde{v}}{U-c} \right] dy = 0$$

$$\int_a^b \tilde{v}^* (D^2-\alpha^2)\tilde{v}\, dy - \int_a^b \tilde{v}^* \frac{D^2U}{U-c} \tilde{v}\, dy = 0$$

Consider just $\downarrow$. Integrating by parts yields:

$$\int_a^b \tilde{v}^* D^2 \tilde{v} = \left[\cancel{\tilde{v}^* D^2 \tilde{v}}\right]_a^b{}^{\,0} - \int_a^b D\tilde{v}^* D\tilde{v}\, dy$$

it cancels out because of the <u>boundary conditions</u>

Since we are just interested in the real positive part, we write $D\tilde{v}^* D\tilde{v}$ as : $|D\tilde{v}|^2$, since $|D\tilde{v}^* D\tilde{v}| = |D\tilde{v}^*||D\tilde{v}|$
$= |D\tilde{v}|^2$

Remembering that, with $\tilde{v}^*$ complex conjugate of $\tilde{v}$, $\tilde{v}\tilde{v}^* = |\tilde{v}|^2$

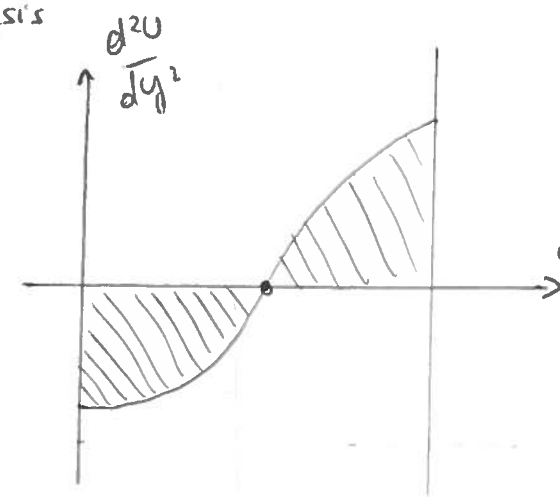
$$\int_a^b \left[|D\tilde{v}|^2 + \alpha^2 |\tilde{v}|^2 \right] dy + \int_a^b \frac{d^2U}{dy^2} \frac{1}{U-c} |\tilde{v}|^2 \, dy = 0$$

We then observe the <u>imaginary part</u> of the equation: The first element is <u>real positive</u>, so it is not present in Im( ).

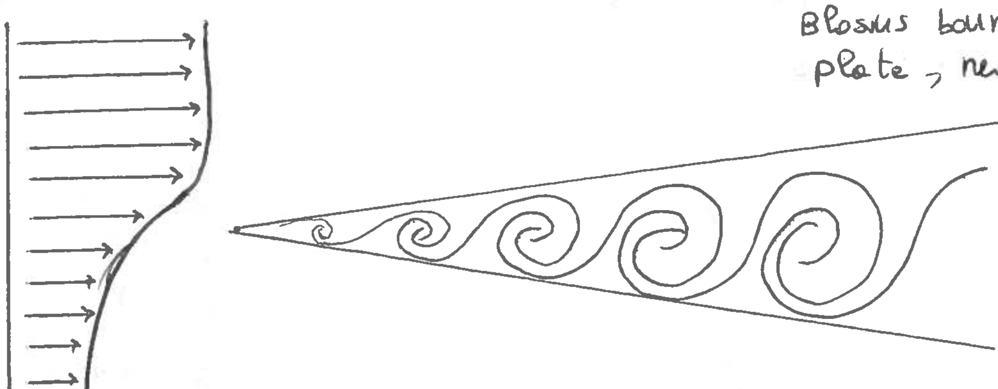
$$\mathrm{Im}\left(\int_a^b \frac{d^2U/dy^2}{(U-c)} |\tilde{v}|^2 dy \right) = \int_a^b \frac{c_i \, d^2U/dy^2 \, |\tilde{v}|^2}{|U-c|^2} = 0$$

We know that :
$$\begin{cases} c_i > 0 & \text{hypothesis} \\ |\tilde{v}|^2 > 0 \\ |U-c|^2 > 0 \end{cases}$$

This means that, for the integral to be zero, the term d^2U/dy^2 must, at some point in the domain, change sign and therefore there must be an <u>INFLECTION POINT</u>



- It has been therefore proven that the presence of an inflection point is a <u>necessary condition for linear instability</u>
- REMARK: The presence of an inflection point is often a <u>sign of instability</u>

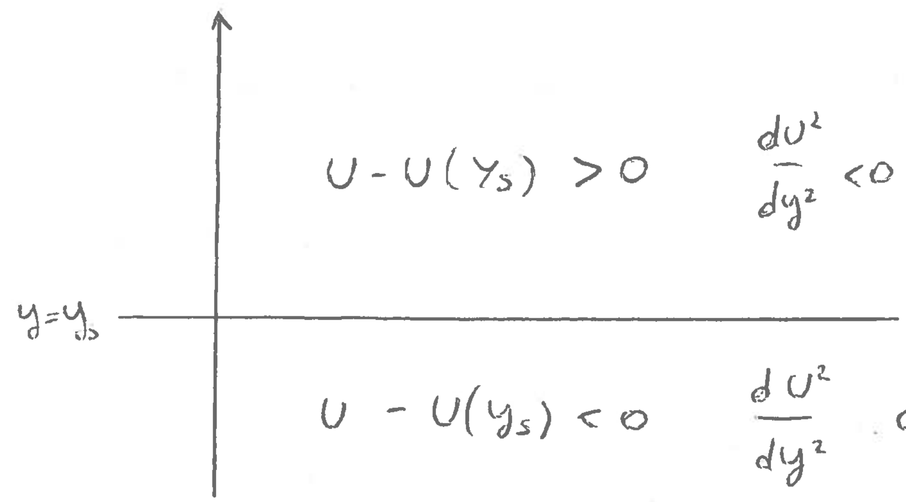


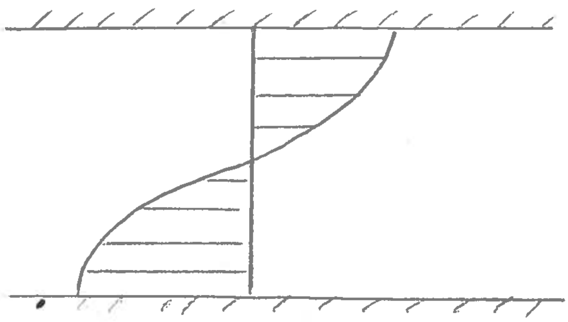
[Note: Plane Poiseuille flow and Blasius boundary layer on a flat plate, neither of which has an inflection point in the velocity profile, are linearly stable as long as the effects of viscosity on the perturbations are ignored]

THEOREM: FJØRTOFT's CRITERION

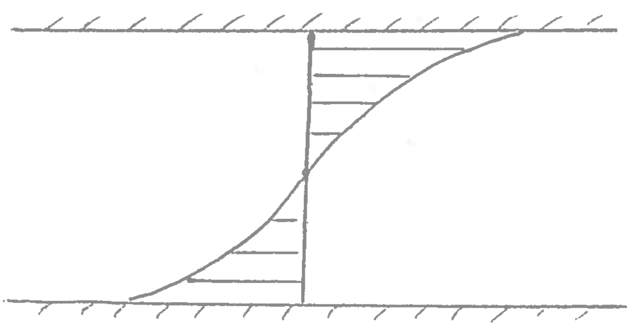
This theorem is not very used now. it is:

Given a monotonic mean velocity profile $U(y)$, a necessary condition for instability is $\frac{d^2U}{dy^2}\left(U - U_s(y)\right) < 0$, where y_s is the inflection point.





Unstable



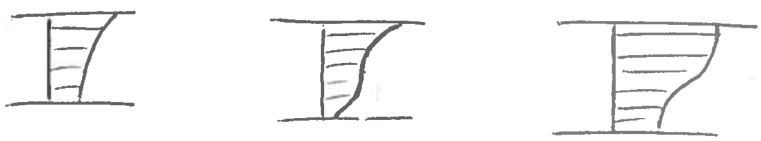
Stable

REMARK

Around the inflection point, the spanwise vorticity should be maximum for instability.

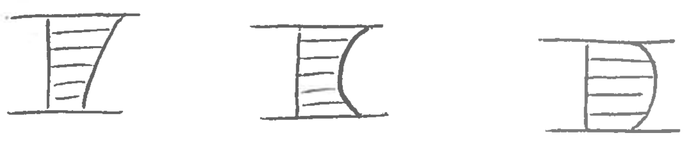
Examples:

- Rayleigh



a) stable
b) ?
c) ?

- Fjørtoft



a) stable
b) stable
c) ?

LECTURE 5

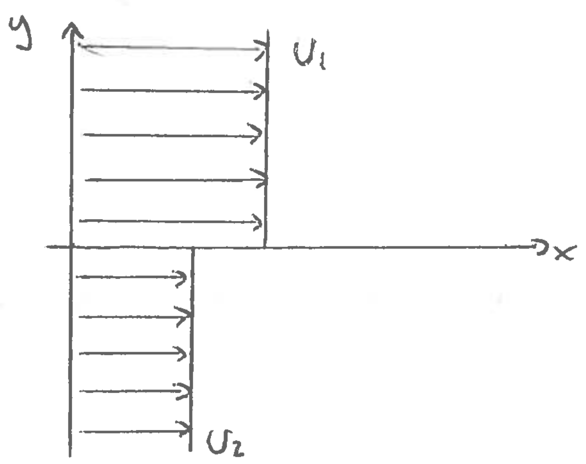
LINEAR STABILITY ANALYSIS OF INVISCID MIXING LAYER

KELVIN - HELMOLTZ INSTABILITY

When facing a practical example, some common features can be discerned. (reference: DRAZIN)

1) Identification of the physical mechanism of instability of a given flow and modelling of the instability by choice of an appropriate system of equations and boundary conditions.
2) Choice of a solution satisfying the system to represent the basic flow.
3) Linearization of the system for small perturbations of the chosen basic flow
4) Use the method of Normal modes
5) Application of the results to understand or control the observed instability.

The problem we're about to analyse was studied theoretically by Helmoltz (1868) and later by Kelvin (1871) for the purpose of explaining, in particular, the formation of ocean waves by the wind. This topic was further investigated by T.B. Benjamin (1957) and J.W. Miles (1959) in famous papers.
Even though K-H modeling is too simplistic to describe the model cited above, it proved to be generic instability of shear flows at large Reynolds numbers (viscosity negligible)



Base flow profile:

$$U(y) = \begin{cases} U_1 & y > 0 \\ U_2 & y < 0 \end{cases}$$

We need to introduce proper jump conditions.

Defining $\Delta U = U_1 - U_2$, the velocity ratio is:

$$R = \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$$

with $\bar{U} = \frac{U_1 + U_2}{2}$ $\longrightarrow$ average of the two velocities

- JUMP CONDITIONS (highly empirical, not to be derived)

The two conditions come from the linearized N-S equations. If available, proof will be attached.

- Jump condition 1: Continuity of pressure

$$\tilde{p} = \frac{i}{\alpha}\left[DU\tilde{v} - (U-c)D\tilde{v} \right] \quad \text{at } y=0$$

- Jump condition 2: Continuity of velocity

$$\frac{\tilde{v}}{U-c} \quad \text{at } y=0$$

So, the problem is:

$$\begin{cases} (U-c)(D^2-\alpha^2)\tilde{v} - D^2U\tilde{v} = 0 & \text{Rayleigh equation, } \alpha>0 \\ \tilde{v}(y=\infty) = \tilde{v}(y=-\infty) = 0 & \text{B.C. at infinity, \underline{no perturbati}} \\ \left.\begin{array}{l} (U-c)D\tilde{v} - DU\tilde{v} \\ \dfrac{\tilde{v}}{U-c} \end{array}\right\} & \text{Jump conditions, continuous at } y=0 \end{cases}$$

We know that the velocity profile for $y \neq 0$ is constant. Therefore:

$U = U_1 \quad y>0 \longrightarrow D^2U = 0$

$U = U_2 \quad y<0 \longrightarrow D^2U = 0$

This means that from Rayleigh equation:

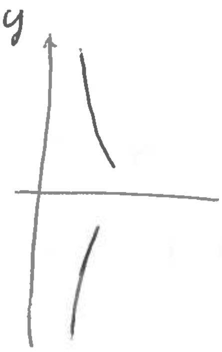
$$(D^2-\alpha^2)\tilde{v} = 0$$

The solution of the problem (normal mode solution) is in the form:

$$\tilde{v} = c_1 e^{-\alpha y} + c_2 e^{\alpha y}$$

We want the solution to tend to zero as $y \to \infty$, so:

$y > 0$: $\lim_{y \to \infty} \tilde{v} = \underbrace{c_1 e^{-\alpha\infty}}_{\text{solution for } y>0} + c_2 e^{\alpha\infty}$



$y < 0$: $\lim_{y \to -\infty} \tilde{v} = c_1 e^{\alpha\infty} + \underbrace{c_2 e^{-\alpha\infty}}_{\text{solution for } y<0}$

The jump conditions are then used to find c_1, c_2.
This means that:

$$\tilde{v}(y) = \begin{cases} A e^{-\alpha y} & y > 0 \\ B e^{\alpha y} & y < 0 \end{cases} \qquad \text{with } \alpha > 0.$$

- Pressure jump condition

$(U-c) D\tilde{v} - DU\tilde{v}$ continuous.

$DU = 0$ both for $y > 0$ and $y < 0$, since $U = \text{const}$. So:

$$\left[(U_1 - c) D (A e^{-\alpha y}) \right] \Big|_{y=0} = \left[(U_2 - c) D (B e^{\alpha y}) \right] \Big|_{y=0}$$

$$-\alpha (U_1 - c)(A) e^{-\alpha y} \Big|_{y=0} = \alpha (U_2 - c)(B) e^{+\alpha y} \Big|_{y=0}$$

Therefore:

$$-\alpha (U_1 - c) A = \alpha (U_2 - c) B \longrightarrow (U_1 - c) A + (U_2 - c) B = 0$$

- Velocity jump condition

$\dfrac{\tilde{v}}{U-c}$ continuous

(next page)

$$\left.\frac{A e^{-\alpha y}}{U_1 - c}\right|_{y=0} = \left.\frac{B e^{\alpha y}}{U_2 - c}\right|_{y=0} \rightarrow \frac{A}{U_1 - c} = \frac{B}{U_2 - c}$$

The two parameters A and B are therefore determined by solving the following system:

$$\begin{cases} (U_1 - c) A + (U_2 - c) B = 0 \\ (U_2 - c) A - (U_1 - c) B = 0 \end{cases} \longrightarrow$$

It has solutions different from the trivial one if

Written in system form, $\underline{\underline{L}}\, \underline{x} = 0$

$$\begin{bmatrix} U_1 - c & U_2 - c \\ U_2 - c & -(U_1 - c) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

The solution is: $\det(L) = 0$

$$\begin{vmatrix} U_1 - c & U_2 - c \\ U_2 - c & -(U_1 - c) \end{vmatrix} = -(U_1 - c)^2 - (U_2 - c)^2 = 0$$

This gives the <u>DISPERSION RELATION</u>

$$-U_1^2 + 2U_1 c - c^2 - U_2^2 - c^2 + 2U_2 c = 0$$

$$-2c^2 + 2(U_1 + U_2)c - U_1^2 - U_2^2 = 0 \rightarrow c^2 - c(U_1 + U_2) + \frac{U_1^2 + U_2^2}{2}$$

$$c = \frac{(U_1 + U_2) \pm \sqrt{U_1^2 + U_2^2 + 2U_1 U_2 - 2U_1^2 - 2U_2^2}}{2}$$

$$c = \frac{1}{2}(U_1 + U_2) \pm \frac{1}{2}\sqrt{-U_1^2 - U_2^2 + 2U_1 U_2}$$

$$c = \frac{1}{2}(U_1 + U_2) \pm \frac{1}{2}\sqrt{-(U_1 - U_2)^2}$$

$$c = \frac{1}{2}(U_1 + U_2) \pm i\frac{1}{2}(U_1 - U_2)$$

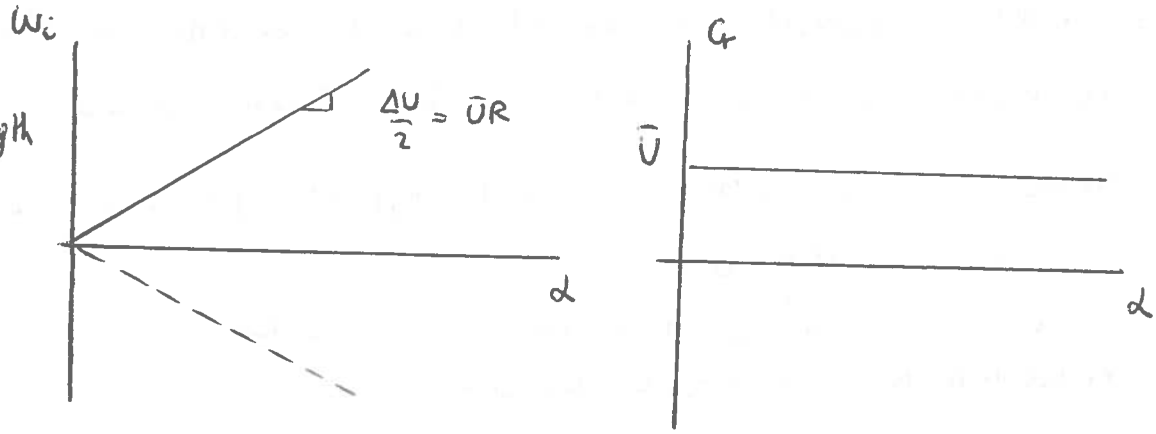
with: $\begin{cases} \bar{U} = \dfrac{U_1 + U_2}{2} \\ \Delta U = U_1 - U_2 \end{cases}$

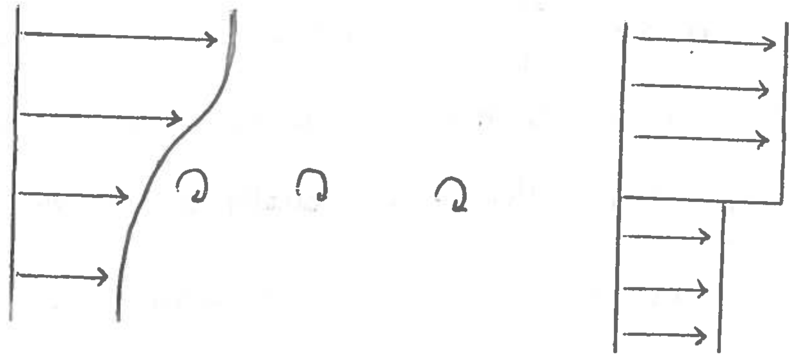
So: $\boxed{c = \bar{U} \pm \dfrac{i \Delta U}{2}}$ → propagation speed is average of the two speeds, since $c_r = \bar{U}$. The time growth rate is instead proportional to the difference of the two speeds.

Recalling that $\omega = c\alpha$, and multiplying the previous relation with α, we obtain:

$$\boxed{\omega = \alpha \bar{U} \pm i \alpha R \bar{U}}$$ DISPERSION RELATION

no length scale





No length
(0 length)

Completing the explanation... (Reference: Charru)

In order to have a non-trivial solution, it has been found that, as a consequence of $\det(L) = 0$, there exist two modes corresponding to two complex-conjugate eigenvalues.

From the viewpoint of the temporal stability of a perturbation of a real wave number, the speed c_r and the temporal growth rate $\omega_i = \alpha c_i$ of these modes are:

$$c_r = \bar{U}$$

$$\omega_i = \pm \frac{\alpha \Delta U}{2} \longrightarrow 1 \text{ mode is always positive!}$$

Therefore, both modes propagate at the same speed equal to the average speed $\bar{U}$ (waves are not dispersive) and because the mode with positive growth rate the flow is <u>unstable</u> for any velocity difference, no matter how small.

Moreover, it is unstable to any perturbation, no matter what its wave number α, with growth rate increasing linearly with α.

The unphysical conclusion that the growth rate is unbounded at large wave numbers (small wavelengths) is a consequence of the fact that all effects of viscous diffusion have been ignored.

In fact, inertial effects dominate viscous effects for wave numbers such that $\rho \alpha \Delta U^2 \gg \mu \alpha^2 \Delta U$

$\rho \alpha \Delta U^2 \rightarrow$ inertial term, N-S. $\alpha \rightarrow$ space derivative

$\mu \alpha^2 \Delta U \rightarrow$ viscous term, NS. $\alpha^2 \rightarrow$ laplacian

This corresponds to $\alpha \ll \frac{\Delta U}{\nu}$. Viscous effects cannot be ignored for $\alpha \gtrsim \frac{\Delta U}{\nu}$. This number therefore determines the wave number scale beyond which the above analysis is no longer valid.

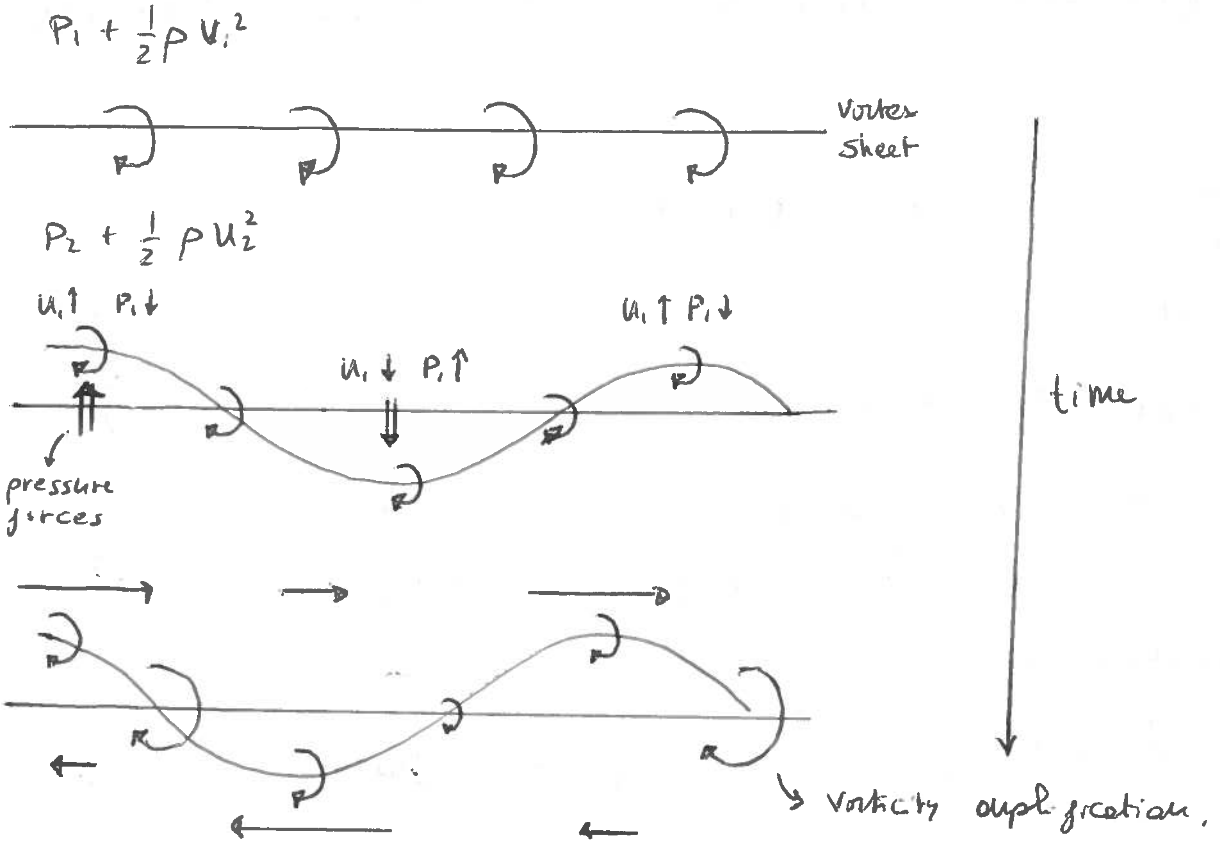
The Kelvin-Helmoltz effect can be explained as a sort of "Bernoull effect". Consider for simplicity a vortex sheet where the speeds of the fluids are $U_1 = -\Delta U$ and $U_2 = \Delta U$. Consider then an initial disturbance which slightly displaces the sheet so that its elevation is sinusoidal.

Assuming that the flow outside of the shear layer is nice and smooth, Bernoulli equation can be reasonably applied.

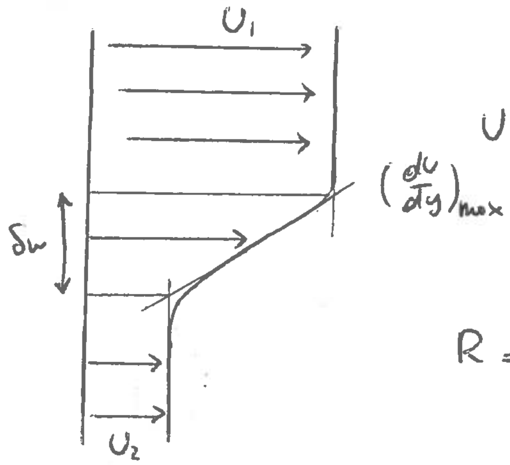
Where the cross-sectional area perpendicular to the flow is decreased, speed increases and pressure decreases.

The instability mechanism can be explained thinking of pressure forces amplifying the perturbation.

But, more importantly, bigger vorticity is generated; an increase in vorticity produces an amplification of the wave in time.



HYPERBOLIC TANGENT MIXING LAYER



Base flow profile

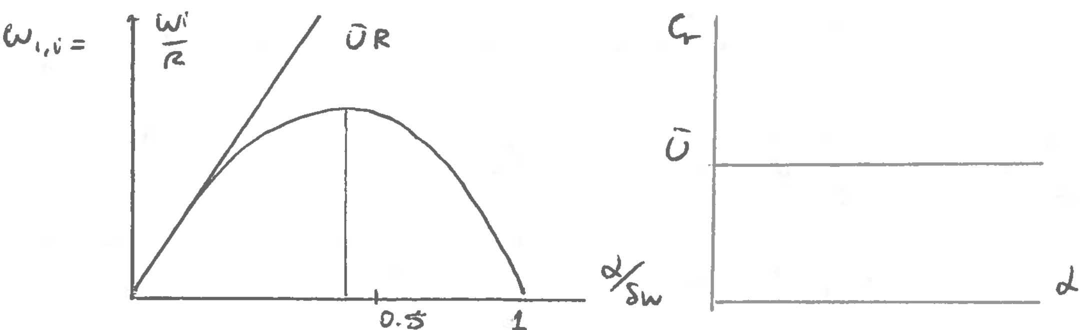
$$U(y) = \bar{U} + \frac{\Delta U}{2} \tanh\left(\frac{2y}{\delta_w}\right)$$

Setting $\Delta U = U_1 - U_2$;

$$R = \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$$

$$\delta_w = \frac{\Delta U}{(dU/dy)_{max}}$$

(There is no analytical solution.) Dispersion relation of the most unstable eigenmode: $\omega(\alpha, R) = \alpha\bar{U} + i R \bar{U} \omega_1(\alpha)$



(Michalke, 1964)

There is length scale for instability

$\alpha \delta_w \simeq 0.5$

$\downarrow$

this is the lengthscale of the instability.

LINEARISED EQUATIONS FOR VISCOUS PARALLEL FLOW

N-S equations:

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}P + \dfrac{1}{Re}\nabla^2\vec{u} \\ \vec{\nabla}\cdot\vec{u} = 0 \end{cases}$$

Consider the following perturbed flow:

$$\vec{u}(x,y,z,t) = (U(y),0,0) + \varepsilon\vec{u}'(x,y,z,t)$$

$$P(x,y,z,t) = \bar{P}(x,y) + \varepsilon P'(x,y,z,t)$$

Substituting:

$$\frac{\partial(\cancel{\vec{U}} + \varepsilon\vec{u}')}{\partial t} + \left((\vec{U}+\varepsilon\vec{u}')\cdot\vec{\nabla}\right)(\vec{U}+\varepsilon\vec{u}') =$$

$$= -\vec{\nabla}(\bar{P}+\varepsilon P') + \frac{1}{Re}\nabla^2(\vec{U}+\varepsilon\vec{u}')$$

$$\varepsilon\frac{\partial\vec{u}'}{\partial t} + (\vec{U}\cdot\vec{\nabla})\vec{U} + \varepsilon(\vec{u}'\cdot\vec{\nabla})\vec{U} + \varepsilon(\vec{U}\cdot\vec{\nabla})\vec{u}' + \cancel{\varepsilon^2(\vec{u}'\cdot\vec{\nabla})\vec{u}'} =$$

$$= -\vec{\nabla}\bar{P} - \varepsilon\vec{\nabla}P' + \frac{1}{Re}\nabla^2\vec{U} + \frac{\varepsilon}{Re}\nabla^2\vec{u}'$$

Incompressibility:

$$\vec{\nabla}\cdot(\vec{U}+\varepsilon\vec{u}') = 0 \longrightarrow \vec{\nabla}\cdot\vec{U} + \varepsilon\vec{\nabla}\vec{u}' = 0$$

$O(\varepsilon)$:

$$\begin{cases} \dfrac{\partial\vec{u}'}{\partial t} + (\vec{u}'\cdot\vec{\nabla})\vec{U} + (\vec{U}\cdot\vec{\nabla})\vec{u}' = -\vec{\nabla}P' + \dfrac{1}{Re}\nabla^2 u' \\ \vec{\nabla}\cdot\vec{u}' = 0 \end{cases}$$

We now want to expand these equations. Let's investigate the advection term first:

- $(\vec{u}'\cdot\vec{\nabla})\vec{U} = \left(u'\dfrac{\partial}{\partial x} + v'\dfrac{\partial}{\partial y} + w'\dfrac{\partial}{\partial z}\right)U(y)\,\hat{x} = v'\dfrac{\partial U}{\partial y}\hat{x}$
- $(\vec{U}\cdot\vec{\nabla})\vec{u}' = \left(U\dfrac{\partial}{\partial x}\right)(u'\hat{x} + v'\hat{y} + w'\hat{z}) = U\dfrac{\partial u'}{\partial x}\hat{x} + U\dfrac{\partial v'}{\partial x}\hat{y} + U\dfrac{\partial w'}{\partial x}\hat{z}$

The various components are:

$$\begin{cases} \dfrac{\partial u'}{\partial t} + v'\dfrac{dU}{dy} + U\dfrac{\partial u'}{\partial x} = -\dfrac{\partial p'}{\partial x} + \dfrac{1}{Re}\left(\dfrac{\partial^2 u'}{\partial x^2} + \dfrac{\partial^2 u'}{\partial y^2} + \dfrac{\partial^2 u'}{\partial z^2}\right) \\ \dfrac{\partial v'}{\partial t} + U\dfrac{\partial v'}{\partial x} = -\dfrac{\partial p'}{\partial y} + \dfrac{1}{Re}\left(\dfrac{\partial^2 v'}{\partial x^2} + \dfrac{\partial^2 v'}{\partial y^2} + \dfrac{\partial^2 v'}{\partial z^2}\right) \\ \dfrac{\partial w'}{\partial t} + U\dfrac{\partial w'}{\partial x} = -\dfrac{\partial p'}{\partial z} + \dfrac{1}{Re}\left(\dfrac{\partial^2 w'}{\partial x^2} + \dfrac{\partial^2 w'}{\partial y^2} + \dfrac{\partial^2 w'}{\partial z^2}\right) \\ \dfrac{\partial u'}{\partial x} + \dfrac{\partial v'}{\partial y} + \dfrac{\partial w'}{\partial z} = 0 \end{cases}$$

We now follow the same procedure as for the inviscid flows.
First, we take partial derivatives of the momentum equations.

$$(1)\quad \frac{\partial}{\partial x}\left[\frac{\partial u'}{\partial t} + v'\frac{dU}{dy} + U\frac{\partial u'}{\partial x} = -\frac{\partial p'}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right)\right]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x}\right) + \frac{\partial}{\partial x}\left(v'\frac{dU}{dy}\right) + \frac{\partial}{\partial x}\left(U\frac{\partial u'}{\partial x}\right) = -\frac{\partial^2 p'}{\partial x^2} + \frac{1}{Re}\frac{\partial}{\partial x}\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right)$$

$$(2)\quad \frac{\partial}{\partial y}\left[\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right)\right]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial y}\right) + \frac{\partial}{\partial y}\left(U\frac{\partial v'}{\partial x}\right) = -\frac{\partial^2 p'}{\partial y^2} + \frac{1}{Re}\frac{\partial}{\partial y}\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2}\right)$$

$$(3)\quad \frac{\partial}{\partial z}\left[\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right)\right.$$

$$\frac{\partial}{\partial t}\left(\frac{\partial w'}{\partial z}\right) + \frac{\partial}{\partial z}\left(U\frac{\partial w'}{\partial x}\right) = -\frac{\partial^2 p'}{\partial z^2} + \frac{1}{Re}\frac{\partial}{\partial z}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right)$$

Summing (1) + (2) + (3) and collecting terms:

$$\cancel{\frac{\partial}{\partial t}\left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right]} + \frac{\partial}{\partial x}\left(v'\frac{dU}{dy}\right) + \frac{dU}{dy}\frac{\partial v'}{\partial x} + \cancel{\frac{\partial}{\partial x}\left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right]} =$$

$$= -\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 p'}{\partial z^2}\right) + \frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}\cancel{\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)} +\right.$$

$$\left. + \frac{\partial^2}{\partial y^2}\cancel{\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)} + \frac{\partial^2}{\partial z^2}\cancel{\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)}\right], \quad \left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0\right]$$

We're left with:

$$\boxed{\nabla^2 P = -2 \frac{dU}{dy} \frac{\partial v'}{\partial x}}$$ POISSON EQUATION

Now consider the y-component of the momentum equation.

$$\begin{cases} \dfrac{\partial v'}{\partial t} + U \dfrac{\partial v'}{\partial x} = -\dfrac{\partial P'}{\partial y} + \dfrac{1}{Re} \nabla^2 v' \\ \nabla^2 P = -2 \dfrac{dU}{dy} \dfrac{\partial v'}{\partial x} \end{cases}$$

We take ∇^2 of the first equation and $\frac{\partial}{\partial y}$ of the second.

$$\begin{cases} \nabla^2 \left(\dfrac{\partial v'}{\partial t} + U \dfrac{\partial v'}{\partial x} = -\dfrac{\partial P'}{\partial y} + \dfrac{1}{Re} \nabla^2 v' \right) \\ \dfrac{\partial}{\partial y} \left(\nabla^2 P = -2 \dfrac{dU}{dy} \dfrac{\partial v'}{\partial x} \right. \end{cases}$$

$$\nabla^2 \left(\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} \right) = -\frac{\partial}{\partial y} \nabla^2 P' + \frac{1}{Re} \nabla^4 v'$$

$$\downarrow$$

$$\boxed{-\frac{\partial}{\partial y} \nabla^2 P} = \nabla^2 \left(\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} \right) - \frac{1}{Re} \nabla^4 v'$$

$$\boxed{-\frac{\partial}{\partial y} \nabla^2 P} = 2 \frac{\partial}{\partial y} \left(\frac{dU}{dy} \frac{\partial v'}{\partial x} \right)$$

So, in the same way as for inviscid parallel flow but with an extra term:

$$\underbrace{\nabla^2 \left(\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} \right)}_{\downarrow} - \frac{1}{Re} \nabla^4 v' = 2 \frac{\partial}{\partial y} \left(\frac{dU}{dy} \frac{\partial v'}{\partial x} \right)$$

Consider . Expanding the laplacian operator:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} \right)$$

Consider only the second term:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) U \frac{\partial v'}{\partial x} = U \frac{\partial^2}{\partial x^2} \left(\frac{\partial v'}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{dU}{dy} \frac{\partial v'}{\partial x} + U \frac{\partial^2 v'}{\partial x \partial y} \right) + \frac{\partial^2}{\partial z^2} \frac{\partial v'}{\partial x}$$

$$\nabla^2\left(U\frac{\partial v'}{\partial x}\right)=U\frac{\partial}{\partial x}\left(\frac{\partial^2 v'}{\partial x}\right)+\frac{d^2U}{dy^2}\frac{\partial v'}{\partial x}+\frac{dU}{dy}\frac{\partial^2 v'}{\partial x\partial y}+$$

$$+\frac{dU}{dy}\frac{\partial^2 v'}{\partial x\partial y}+U\frac{\partial}{\partial x}\left(\frac{\partial^2 v'}{\partial y^2}\right)+U\frac{\partial}{\partial x}\left(\frac{\partial^2 v'}{\partial z}\right)$$

$$=U\frac{\partial}{\partial x}\left(\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}+\frac{\partial^2 v'}{\partial z^2}\right)+\frac{d^2U}{dy^2}\frac{\partial v'}{\partial x}+2\frac{dU}{dy}\frac{\partial^2 v'}{\partial x\partial y}$$

Plugging this back into the former equation yelds:

$$\nabla^2\frac{\partial v'}{\partial t}+U\frac{\partial}{\partial x}(\nabla^2 v')+\frac{d^2U}{dy^2}\frac{\partial v'}{\partial x}+\cancel{2\frac{dU}{dy}\frac{\partial^2 v'}{\partial x\partial y}}-\frac{1}{Re}\nabla^4 v'=$$

$$=2\frac{d^2U}{dy^2}\frac{\partial v'}{\partial x}+\cancel{2\frac{dU}{dy}\frac{\partial^2 v'}{\partial x\partial y}}$$

$$\boxed{\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}-\frac{1}{Re}\nabla^4\right]v'=0}$$

The second equation where pressure disappears comes from the x-component and z-component of the N-S. equations.

$$\begin{cases}\dfrac{\partial u'}{\partial t}+v'\dfrac{dU}{dy}+U\dfrac{\partial u'}{\partial x}=-\dfrac{\partial p'}{\partial x}+\dfrac{1}{Re}\nabla^2 u'\\[2ex]\dfrac{\partial w'}{\partial t}+\qquad U\dfrac{\partial w'}{\partial x}=-\dfrac{\partial p}{\partial z}+\dfrac{1}{Re}\nabla^2 w'\end{cases}$$

We take $\frac{\partial}{\partial z}$ from the first equation and $\frac{\partial}{\partial x}$ from the second

$$(1)\quad\frac{\partial}{\partial z}[\quad]\Rightarrow\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial z}\right)+\frac{\partial v'}{\partial z}\frac{dU}{dy}+U\frac{\partial^2 u'}{\partial x\partial z}=-\frac{\partial^2 p'}{\partial x\partial z}+\frac{1}{Re}\frac{\partial}{\partial z}(\nabla^2 u')$$

$$(2)\quad\frac{\partial}{\partial x}[\quad]\Rightarrow\frac{\partial}{\partial t}\left(\frac{\partial w'}{\partial x}\right)+U\frac{\partial^2 w'}{\partial x^2}=-\frac{\partial^2 p'}{\partial x\partial z}+\frac{1}{Re}\frac{\partial}{\partial x}(\nabla^2 w')$$

Wanting to make (once more) pressure disappear, we take (1) - (2)

$$\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}\right)+\frac{dU}{dy}\frac{\partial v'}{\partial z}+U\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}\right)=\frac{1}{Re}\frac{\partial}{\partial z}\left(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}+\frac{\partial^2 u'}{\partial z^2}\right)+$$

$$-\frac{1}{Re}\frac{\partial}{\partial x}\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2}+\frac{\partial^2 w'}{\partial z^2}\right)$$

Introducing WALL-NORMAL VORTICITY: $\eta' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}$

The equation then becomes:

$$\frac{\partial \eta'}{\partial t} + \frac{dU}{dy}\frac{\partial v'}{\partial z} + U\frac{\partial \eta'}{\partial x} = \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right) + \right.$$
$$\left. + \frac{\partial^2}{\partial z^2}\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right)\right)$$

And:

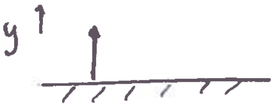
$$\frac{\partial \eta'}{\partial t} + \frac{dU}{dy}\frac{\partial v'}{\partial x} + U\frac{\partial \eta'}{\partial x} - \frac{1}{Re}\nabla^2 \eta' = 0$$

Collecting η':

$$\boxed{\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{Re}\nabla^2\right]\eta' + \frac{dU}{dy}\frac{\partial v'}{\partial z} = 0}$$

We can finally build the problem. How many conditions do we need? 2 initial conditions for sure. Then, 4th order derivative in velocity equation requires 4 boundary conditions, whereas 2nd order derivative in vorticity requires 2 BC for η'.

At the solid boundary:
$$\begin{cases} u' = w' = 0 & \longrightarrow \text{no-slip condition} \\ v' = 0 & \longrightarrow \text{non-penetration} \end{cases}$$



From the incompressibility equation:
$$\left.\cancel{\frac{\partial u'}{\partial x}} + \frac{\partial v'}{\partial y} + \cancel{\frac{\partial w'}{\partial z}}\right|_{wall} = 0 \longrightarrow \frac{\partial v'}{\partial y} = 0$$

The problem is:

$$\begin{cases} \left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - \frac{d^2U}{dy^2}\frac{\partial}{\partial x} - \frac{1}{Re}\nabla^4\right]v' = 0 & (1) \\ \left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{Re}\nabla^2\right]\eta' + \frac{dU}{dy}\frac{\partial v'}{\partial z} = 0 & (2) \\ \left.\begin{array}{l} u' = w' = 0 \\ v' = 0 \\ \frac{\partial v'}{\partial y} = 0 \end{array}\right\} & \text{at solid boundary and/or far field} \\ \left.\begin{array}{l} v'(x,y,z,0) = v_0(x,y,z) \\ \eta'(x,y,z,0) = \eta'(x,y,z) \end{array}\right\} & \text{I.C.} \end{cases}$$

NORMAL MODE SOLUTION

The case under examination is three-dimensional, so the solution must be adequate. Its form is:

$$(1) \quad v'(x,y,z,t) = \tilde{v}(y)\, e^{i(\alpha x + \beta z - \omega t)} + c.c.$$

$$(2) \quad \eta'(x,y,z,t) = \tilde{\eta}(y)\, e^{i(\alpha x + \beta z - \omega t)} + c.c.$$

where: $\alpha \in \mathbb{R}$ streamwise wave number

$\beta \in \mathbb{R}$ spanwise wave number

$\omega \in \mathbb{C}$ frequency : $\omega = \alpha c$

We now need to analyse the derivatives.

$$\frac{\partial v'}{\partial t} = -i\omega \tilde{v}(y)\, e^{i(\alpha x+\beta z-\omega t)} \implies \frac{\partial}{\partial t} \to -i\omega$$

$$\frac{\partial v'}{\partial x} = i\alpha \tilde{v}(y)\, e^{i(\alpha x+\beta z-\omega t)} \implies \frac{\partial}{\partial x} \to i\alpha$$

$$\frac{\partial v'}{\partial z} = i\beta \tilde{v}(y)\, e^{i(\alpha x+\beta z-\omega t)} \implies \frac{\partial}{\partial z} \to i\beta$$

$$\frac{\partial^2 v'}{\partial x^2} = -\alpha^2 \tilde{v}(y)\, e^{i(\alpha x+\beta z-\omega t)} \implies \frac{\partial^2}{\partial x^2} \to -\alpha^2$$

$$\frac{\partial^2 v'}{\partial z^2} = -\beta^2 \tilde{v}(y)\, e^{i(\alpha x+\beta z-\omega t)} \implies \frac{\partial^2}{\partial z^2} \to -\beta^2$$

$v' \to \tilde{v}$, $\eta' \to \tilde{\eta}$

Again, we set $D = \dfrac{d}{dy}$ and $\omega = \alpha c$.

$$(1) \to \left[\left(-i\omega + U i\alpha\right)\left(D^2 - \alpha^2 - \beta^2\right) - D^2 U i\alpha - \frac{1}{Re}\left(D^2 - \alpha^2 - \beta^2\right)^2\right]\tilde{v} = 0$$

Introducing $k^2 = \alpha^2 + \beta^2$; ORR-SOMMERFELD EQUATION

$$\boxed{\left[\left(-i\omega + \alpha U i\right)\left(D^2 - k^2\right) - D^2 U i\alpha - \frac{1}{Re}\left(D^2 - k^2\right)^2\right]\tilde{v} = 0}$$

Assumptions: basic flow plane parallel steady and exact sol. of N-S, so strictly plane Couette - Poiseuille flow.

This equation was first derived in 1907.

We now apply the second approach to the vorticity equation.

$$(2) \longrightarrow \left[(-i\omega + U i \alpha) - \frac{1}{Re}\left(D^2 - \alpha^2 - \beta^2\right)\right]\tilde{\eta} = -i DU\beta\tilde{v}$$

Again using the parameter $K^2 = \alpha^2 + \beta^2$:

<u>SQUIRE EQUATION</u>

$$\boxed{\left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - K^2)\right]\tilde{\eta} = -i DU\beta\tilde{v}}$$

Remark: if $\alpha, \beta \in \mathbb{R}$ are given, then $\omega \in \mathbb{C}$ becomes unknown with $\tilde{v}$ and $\tilde{\eta}$. This results in an eigenvalue problem as in Rayleigh equation.

The problem is set up as:

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix} = \underline{0}$$

B.C.:

$\tilde{v} = D\tilde{v} = \tilde{\eta} = 0$

at solid wall and in the far field

Eigenvalues are from A and C.

They don't come from B, linked to Squire Equation.

Therefore, we study Orr-Sommerfeld equation

<u>SQUIRE'S TRANSFORMATION</u>

<u>THEOREM: DAMPED SQUIRE MODES</u>

The solutions to the Squire equation are always damped, i.e. $\omega_i < 0$ for all α, β, and Reynolds number.

REMARK: instability comes from Orr-Sommerfeld equation.

We consider Orr-Sommerfeld equation for both the 2-D and the 3-D case.

- <u>2D</u> Let $\beta = 0$

  We know that $K^2 = \alpha^2 + \beta^2 \longrightarrow K^2 = \alpha_{2D}^2$, $Re_{cr} = Re_{2D,c}$

  On the onset of instability:

$$\left[(-i\alpha_{2D} c + U i \alpha_{2D})(D^2 - \alpha_{2D}^2) - D^2 U i \alpha_{2D} - \frac{1}{Re_{2D,c}}(D^2 - \alpha_{2D}^2)^2\right]\tilde{v} = 0$$

FURTHER EXPLANATION: WHY DOES SQUIRE EQUATION NOT CONTRIBUTE TO INSTABILITY?

Strategy: we need to prove that ω_i is always negative, when associated with squire equation.

From the beginning:

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix} = \underline{0}$$

B won't contribute to the eigenvalues. we need to show that C does not produce instability.

We proceed in the same way as for Rayleigh's inflection point criterion.

$$\left[(-i\omega + U i\alpha) - \frac{1}{Re}(D^2 - k^2)\right]\tilde{\eta} = 0$$

$$\tilde{\eta}^*\left[(-i\omega + U i\alpha) - \frac{1}{Re}(D^2 - k^2)\right]\tilde{\eta} = 0$$

$$\int_V \tilde{\eta}^*\left[(-i\omega + U i\alpha) - \frac{1}{Re}(D^2 - k^2)\right]\tilde{\eta} = 0$$

$$-i\omega \underbrace{\int_V \tilde{\eta}^*\tilde{\eta}}_{\text{real}} + i\alpha \underbrace{\int_V U\tilde{\eta}^*\tilde{\eta}}_{\text{real}} - \frac{1}{Re}\underbrace{\int \tilde{\eta}^*(D^2 - k^2)\tilde{\eta}\, dy}_{\text{real}} = 0$$

Pick the guy that will have an imaginary part and integrate by parts:

$$\int_V \tilde{\eta}^* D^2 \tilde{\eta}\, dy = \underbrace{\cancel{\left[\tilde{\eta}^* D^2\tilde{\eta}\right]_V}}_{\text{boundary conditions!}}^{0} - \int_V D\tilde{\eta}^* D\tilde{\eta}\, dy$$

$$D\tilde{\eta}^* D\tilde{\eta} = |D\tilde{\eta}|$$

So:

$$-i\omega \int_V \tilde{\eta}^*\tilde{\eta} + i\alpha \int_V U\tilde{\eta}^*\tilde{\eta} + \frac{1}{Re}\int_V |D\tilde{\eta}^*|^2 dV = 0$$

The whole equation is <u>real</u>. $-i\omega$ must be negative (real) for the equation to hold.

$$-i\omega < 0 \rightarrow -i(\omega_r + i\omega_i) < 0 \qquad -i\omega_r + \omega_i < 0 \qquad \boxed{\omega_i < 0}$$

Dividing by $i\alpha$:

$$\left[(U - c)(D^2 - \alpha_{2D}^2) - D^2 U - \frac{1}{i\alpha_{2D} Re_{2D,c}} (D^2 - \alpha_{2D}^2)^2 \right] \tilde{v} = 0$$

- **3D |** $\beta \neq 0$

We know that $k^2 = \alpha^2 + \beta^2$ $\left(\longrightarrow k^2 = \alpha_{3D}^2 + \beta_{3D}^2 \right)$, $Re_{cr} = Re_{3D,c}$

On the onset of instability: (dividing by $i\alpha$)

$$\left[(U - c)(D^2 - k^2) - D^2 U - \frac{1}{i\alpha_{3D} Re_{cr}} (D^2 - k^2)^2 \right] \tilde{v} = 0$$

We set $Re_{3D_c} = \frac{\alpha}{k} Re_c$, So: $Re_c = \frac{k}{\alpha} Re_{3D_c}$ this is "fake"

$$\left[(U - c)(D^2 - k^2) - D^2 U - \frac{1}{ik Re_{3D_c}} (D^2 - k^2)^2 \right] \tilde{v} = 0$$

It follows that the critical Reynolds number of an instability should be:

$$Re_{2D,c} = Re_{3D,c} = \frac{\alpha}{k} Re_c \quad , \quad k > \alpha.$$

this indicates that $\boxed{Re_c > Re_{2D,c}}$

SQUIRE'S THEOREM

Given Re_L as the critical Reynolds number for the onset of linear instability for a given α and β, the Reynolds number Re_c below which no exponential instabilities exist for any wave numbers satisfies:

$$Re_c \equiv \min_{\alpha,\beta} Re_L(\alpha, \beta) = \min_{\alpha} Re_L(\alpha, 0)$$

Remark:

The <u>most unstable</u> <u>linear</u> instability is <u>always</u> <u>two dimensional</u>

LECTURE 6

EIGENSPECTRA AND EIGEN FUNCTIONS

ORR-SOMMERFELD equation (wall-normal velocity):

$$\left[(-i\omega + i\alpha U)(D^2 - K^2) - i\alpha D^2 U - \frac{1}{Re}(D^2-K^2) \right] \tilde{v} = 0$$

SQUIRE EQUATION (wall-normal vorticity)

$$\left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - K^2) \right] \tilde{\eta} = -i\beta DU \tilde{v}$$

There are infinitely many sets of ω_n with the corresponding $\tilde{v}_n$ and $\tilde{\eta}_n$.

We search for the most unstable eigenvalue and the corresponding eigenfunctions. The most unstable is the one with the largest ω_i (or c_i).

POISEULLE FLOW

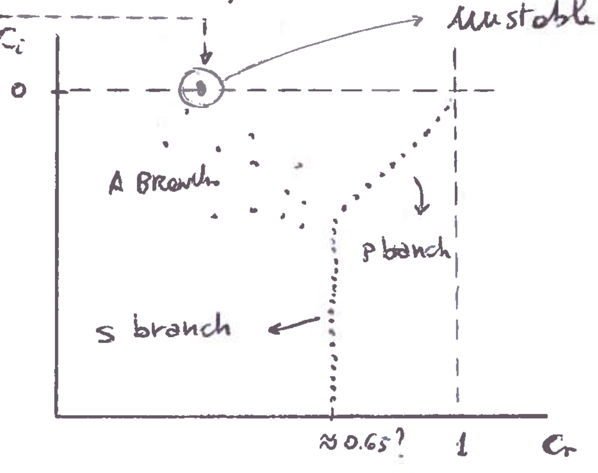
$U(y) = 1 - y^2$

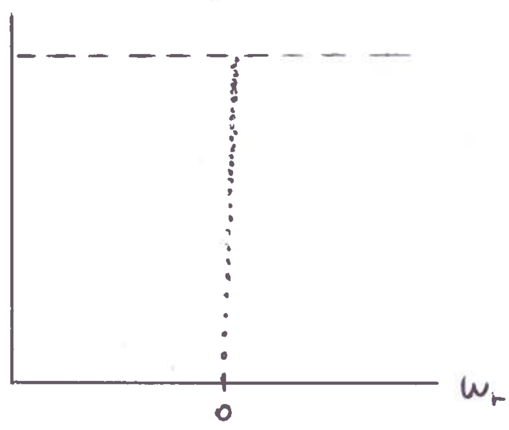
Raileigh criterion would state that there is no inviscid instability. But we will see that introducing the viscous term produces instability. This goes against common sense.

Viscosity-driven instability is called TOLLMIEN-SCHLICHTING INSTABILITY, and it traditionally appears in boundary layers.

Now set $Re = 10000$. EIGENSPECTRA OF PLANE POISEULLE FLOW

(a) let $\alpha = 1$, $\beta = 0$ (b) let $\alpha = 0$, $\beta = 1$

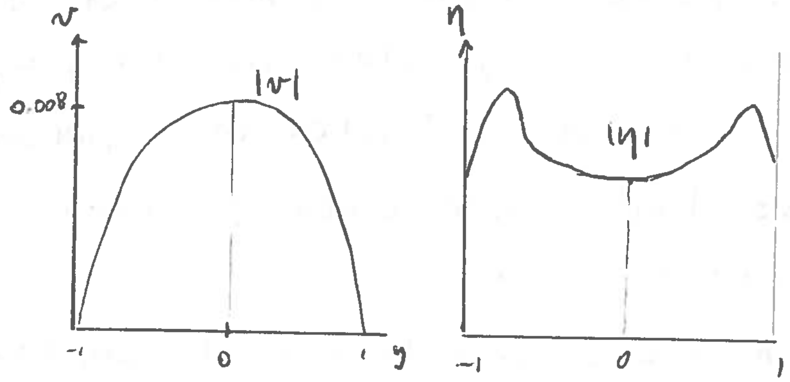




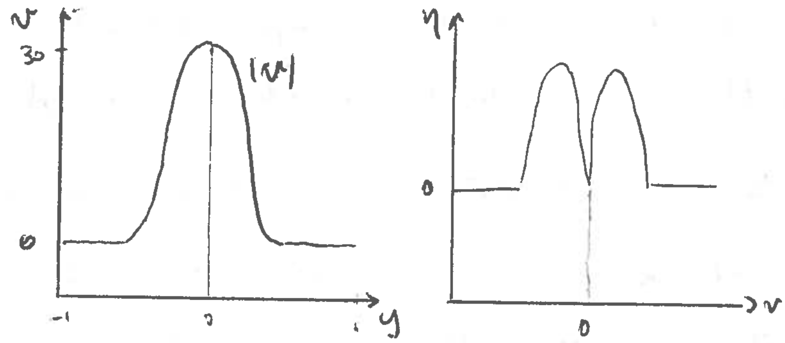
EIGENFUNCTIONS OF PLANE POISEULLE FLOW

let $\alpha = 1$, $\beta = 1$, $Re = 5000$

- A branch
- small C_r → propagation speed
- maximum of v at $y \simeq 0$.
- T-S instability



- P branch
- large C_r
- no instability



- S branch
- strongly damped

BOUNDARY LAYER INSTABILITY

The instability of a laminar boundary layer is a convective instability with a mechanism very similar to that of plane Poiseulle flow. However, there is a difference in that the former develops from an inhomogeneous (translationally noninvariant) flow in the flow direction.

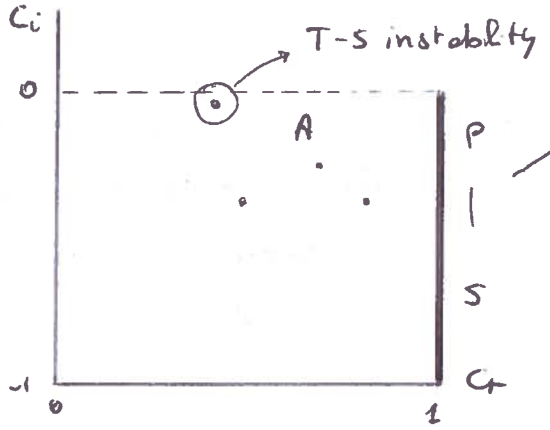
Theoretical study of boundary layer instability is complicated by the fact that since the boundary layer gets thicker downstream, the flow is not strictly parallel, especially near the leading edge where the Reynolds number is not very high.

This problem can be evaded by assuming that the x-gradient of the base flow is sufficently small compared to the wave number of the perturbation $\left(\underbrace{\frac{\partial u}{\partial x}}_{\text{(small)}} + \frac{\partial v}{\partial y} = 0\right)$, and that the characteristics of the perturbation (amplitude, wave number, growth rate) adapt rapidly to the new local conditions encountered as a result of its advection downstream. This hypothesis of rapid relaxation of "fully developed perturbation" makes it possible

to use a local stability analysis, where the x-gradients of the base flow are assumed to be zero.
The problem then becomes that of the Orr-Sommerfeld equation for a parallel velocity profile $U(x,y)$, where the x-coordinate is treated as a parameter. The growth rates, wavelenghts, and eigen functions calculated are then parametric functions of x.
The spatial evolution of the amplitude of an eigenmode of a given frequency is affected by by the nonuniformity of the base velocity profile in the flow direction.

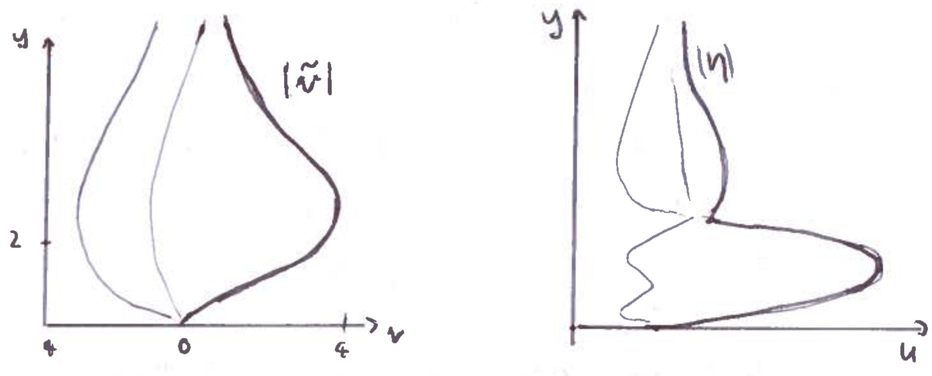
EIGENSPECTRA OF BLASIUS BOUNDARY LAYER

let $\alpha = 0.2$, $\beta = 0$, $Re = 500$
The scatter plot of eigenvalues is:



Continuous set of eigenvalues ($c_r \approx 1$).
This is due to unbounded domain in the y direction.

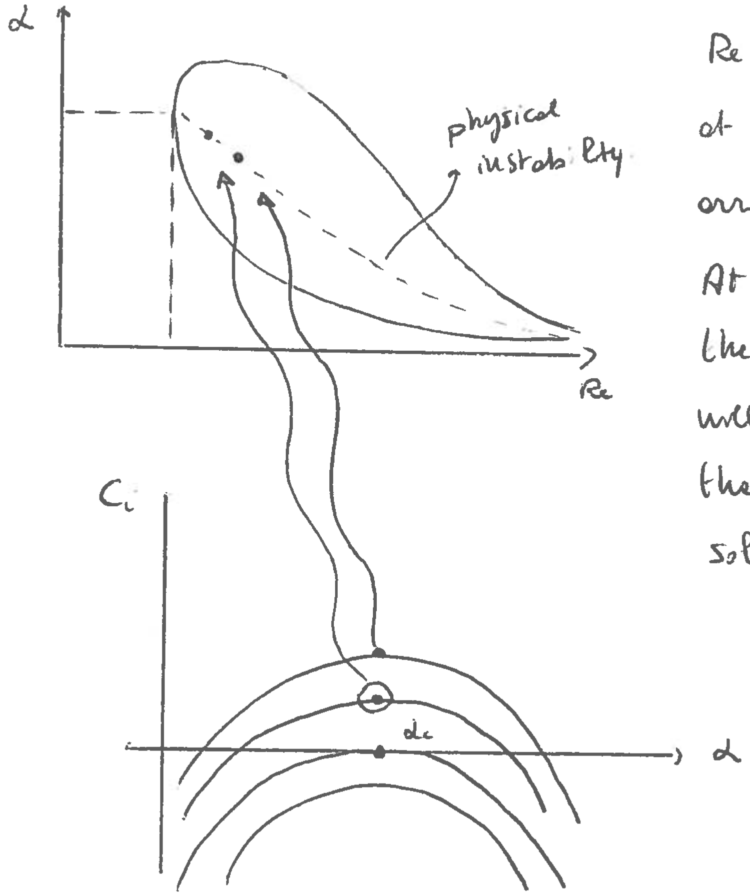
EIGENFUNCTIONS → T-S instability



large inside of the boundary layer.

These graphs are obtained with local analysis. $\alpha = 0.2$, $\beta = 0$, $Re = 500$.
The thick line represents the absolute value; the thin lines represent the real and imaginary part.

FURTHER EXPLANATION ON INSTABILITY MECHANISM



Re starts growing, and at some point it will arrive at Re_{cr}

At every Reynolds number, the most unstable eigenvalue will be the solution and, therefore the most unstable solution will take place.

So, for every Re, the C_i max can be found and it will correspond to some α. Plotting this α in the contour graph α-Re will give the curve instability follows.

NEUTRAL STABILITY CURVE

We now want to investigate every α, β. First let $\beta = 0$. With every α, that is every wavelength, we compute the <u>most unstable</u> eigenvalue.

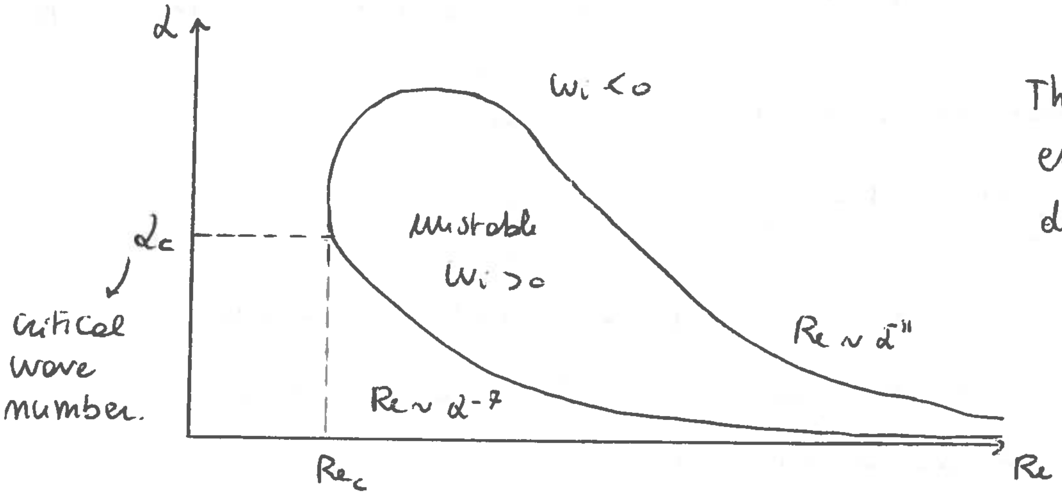
$\alpha \to 0 \qquad \lambda = \dfrac{2\pi}{\alpha} \to \infty$

$\alpha \to \infty \qquad \lambda = \dfrac{2\pi}{\alpha} \to 0$

POISEUILLE FLOW

$\omega_i(\alpha, \beta = 0, Re) = 0$ $\qquad$ $\beta = 0 \to$ thanks to Squire theorem

The aim is to identify the area where $\omega_i > 0$, that is a region of instability.



↳ Critical Reynolds number.

Theoretically, the end part is diminishing as $Re \to \infty$

Contours of c_i and c_r can be traced, on graphs relating α and Re. The same thing can be done with Blasius boundary layer; in both cases, a shaded area will point out where is the region in the parameter space where unstable solution exists.

Comparing data that relate different flow configurations and information coming from linear stability analysis, we notice there is something odd.

Flow	Critical Re (linear stability)	Transition Re	Critical wavenumber	Critical phase speed
Couette flow	∞	350 - 400	–	–
Poiseuille flow	5772.2	1000 - 2000	1.02	0.2639
Pipe flow	∞	2000 - 2500	–	–
Boundary layer	519.4	Depends on dist. env.	0.303	0.3935

Remark: linear stability analysis does not provide a full explanation for the onset of transition.
In fact, both Couette flow and Pipe flow never produce instability, with linear stability analysis. It is not enough.

SPATIAL STABILITY ANALYSIS / VIBRATING RIBBON PROBLEM

Revisit the normal mode solution (2D case).

$$v' = \tilde{v}\, e^{i\alpha x - i\omega t} + c.c.$$

So far we have carried out temporal stability analysis, assuming α known and $\omega \in \mathbb{C}$ unknown. This would result in:

$$\begin{cases} \omega_i > 0 \longrightarrow \text{linearly unstable} \\ \omega_i < 0 \longrightarrow \text{linearly stable} \end{cases}$$

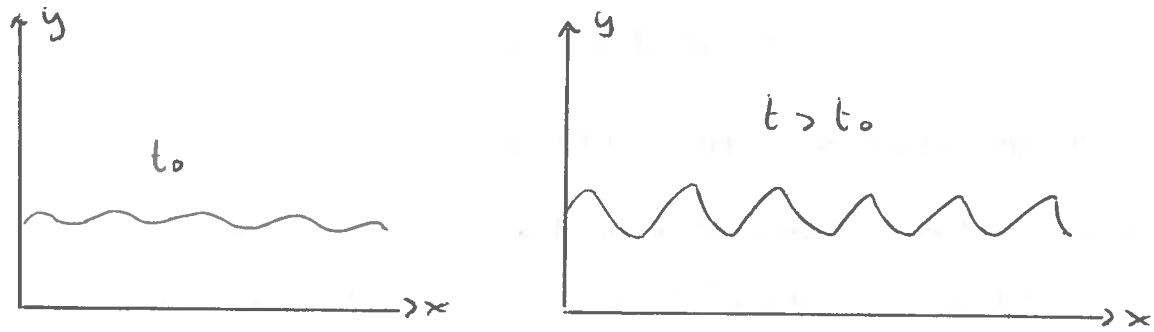
Now, consider $\underline{\omega \in \mathbb{R} \text{ is given}}$ and $\underline{\alpha \in \mathbb{C} \text{ unknown}}$: we perform spatial stability analysis.
We do the other way around, and find:

$$\begin{cases} \alpha_i < 0 \quad \text{linearly unstable} \\ \alpha_i > 0 \quad \text{linearly stable} \end{cases}$$

We are fixed in space and look at the disturbance evolve in space, as it goes to $x \longrightarrow \infty$.

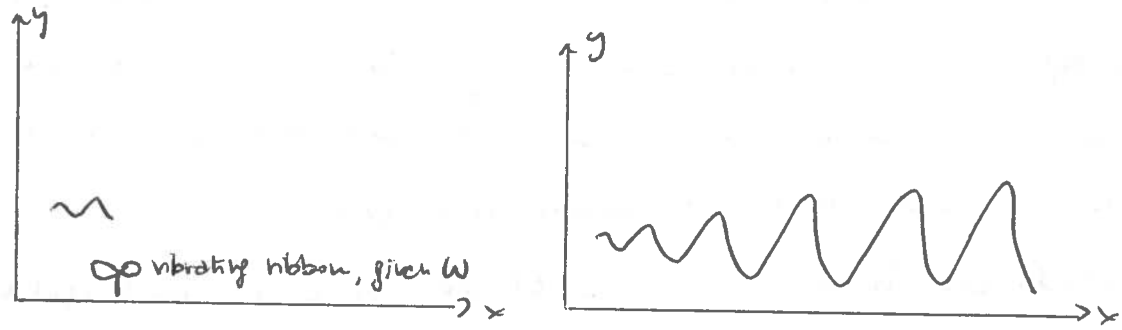
A graphical representation will surely help.

TEMPORAL



this is impossible to analyse experimentally.

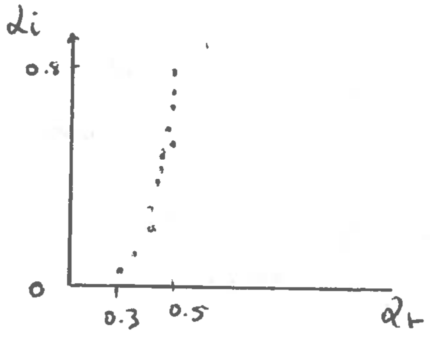
SPATIAL



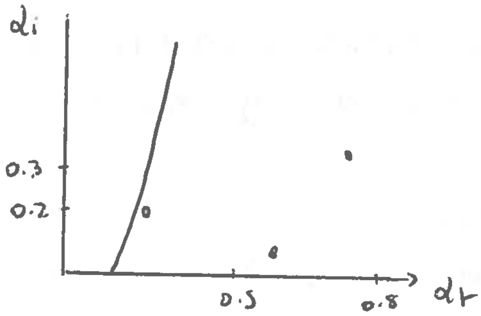
the wave grows in space: it is difficult to solve analytically, but it can be more easily proved experimentally

EIGENSPECTRA

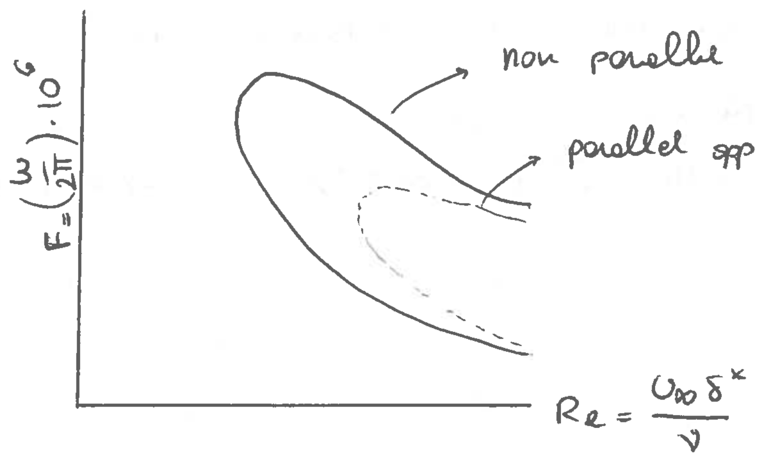
POISEUILLE, $\omega = 0.3$, $Re = 1000$



BLASIUS B.L, $\omega = 0.26$, $Re = 1000$.



NEUTRAL STABILITY CURVE - BLASIUS B.L.



LECTURE 7

NON-MODAL STABILITY ANALYSIS (1980s)

REDDY AND HENNINGSON (MIT, 1992)

Linear stability analysis and energy methods are two standard tools for studying the stability of viscous channel flows.

Linear stability analysis involves examining the evolutions of small perturbations by linearizing the N-S equations and yields the Orr Sommerfeld equation. Stability is then determined by examining the O-S eigenvalues. If there is an eigenvalue in the upper-half plane then is an exponentially growing mode and the flow is said to be linearly unstable.

Energy methods are based on a variational approach and yield conditions for no energy growth for perturbations of arbitrary amplitude.

These results do not agree with experimental studies. In recent years several nonlinear theories, including the secondary instability theory, have been developed, giving better agreement with experiments.

Thus, linear stability analysis gives conditions for exponential instability and energy methods give conditions for no energy growth.

In intermediate cases with $R_g < R < R_c$ the energy of a small perturbation decays to zero as $t \to \infty$ but there may be transient energy growth before the decay. For example, for two-dimensional perturbations to Poiseuille flow, transient growth by a factor as large as 5_0 can occur (Farrell 1988).

Flow configuration	Critical Re for energy growth	Critical Re (linear stability)	Transition Re (experiments
Couette flow	20.7	∞	350 - 400
Poiseuille flow	49.6	5772	1000 - 2000
Pipe flow	81.5	∞	2000 - 2500

We need to try and understand why there is such discrepancy.

Remark: nonlinear terms in the form of perturbed N-S equation play no role in the mechanism of disturbance growth. Thus, linearization is not what is causing the discrepancy: it has to do with the hypothesis of normal mode solution.

NON-LINEAR EQUATIONS:

$$\frac{\partial u_i'}{\partial t} = -u_j' \frac{\partial U_i}{\partial x_j} - U_j \frac{\partial u_i'}{\partial x_j} - u_j' \frac{\partial u_i'}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 u_i'}{\partial x_j \partial x_j} - \frac{\partial P}{\partial x_i}$$

Integrating over the volume V and multiplying by u_i', Reynolds-Orr equation is obtained:

$$\int_V \frac{1}{2} \frac{\partial u_i u_i}{\partial t} dV = -\int_V u_i u_j \frac{\partial U_i}{\partial x_j} dV - \frac{1}{Re} \int_V \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dV$$

Confirming what is explained above

INITIAL VALUE PROBLEM OF LINEARISED EQUATION

We try to solve all linearised equations without making use of normal mode solution.

Consider the full solution of the linearised N-S equations.

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} = L\vec{u} \\ \vec{u}(t=0) = \vec{u}_0 \end{cases} \qquad L \text{ is a } \underline{\text{matrix}}$$

- Assumption: L is a diagonalisable matrix, with the eigenvectors given by $\{\tilde{\vec{u}}_1, \tilde{\vec{u}}_2, \tilde{\vec{u}}_3, \dots, \tilde{\vec{u}}_n\}$
  This assumption corresponds to stating that the eigenspace is <u>full rank</u>.
- Assumption: $n < \infty$, so $\{\tilde{\vec{u}}_1, \tilde{\vec{u}}_2, \dots, \tilde{\vec{u}}_n\}$ is a finite-dimensional vector.

The solution of the equation above is obtained by the eigenfunction expansion technique:

$$\vec{u}(t) = a_1(t)\,\tilde{\vec{u}}_1 + a_2(t)\,\tilde{\vec{u}}_2 + a_3(t)\,\tilde{\vec{u}}_3 + \dots + a_n(t)\,\tilde{\vec{u}}_n .$$

We are now solving considering all eigenvalues <u>COMBINED</u>, not taking separately the solution each one of them would produce. This allows to discover interactions between them.

Let:

$$U = \begin{bmatrix} | & | & & | \\ \tilde{\vec{u}}_1 & \tilde{\vec{u}}_2 & \cdots & \tilde{\vec{u}}_n \\ | & | & & | \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{bmatrix} = \underline{a}(t)$$

This makes it: $\quad \underline{u}(t) = \underset{\substack{\downarrow \\ \text{eigenvectors matrix}}}{U}\,\underline{a}(t)$

Plugging it back:

$$\frac{dU\underline{a}}{dt} = LU\underline{a}$$

$L U = U \Lambda$ general form of the eigenvalue problem.

in fact $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$, λ_i eigenvalues

So L can be rewritten as: $L = U \Lambda U^{-1}$. Plugging it in:

$$\frac{d U \underline{a}}{dt} = U \Lambda U^{-1} U \underline{a} \longrightarrow \frac{d U \underline{a}}{dt} = U \Lambda \underline{a}$$

So the problem is rewritten:

$$\begin{cases} \dfrac{d\underline{a}}{dt} = \Lambda \underline{a} \\ \underline{a}(t_0) = [a_{1,0}, a_{2,0}, a_{3,0}, \dots, a_{n,0}]^T \end{cases}$$

The unknown of the problem is the time dependent vector $\boxed{\underline{a}(t)}$

Expanding the system yields:

$$\frac{d}{dt}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda_1 a_1 \\ \lambda_2 a_2 \\ \lambda_3 a_3 \\ \vdots \\ \lambda_n a_n \end{bmatrix} \Longrightarrow \begin{aligned} a_1(t) &= a_{1,0}\, e^{\lambda_1 t} \\ a_2(t) &= a_{2,0}\, e^{\lambda_2 t} \\ a_3(t) &= a_{3,0}\, e^{\lambda_3 t} \\ &\vdots \\ a_n(t) &= a_{n,0}\, e^{\lambda_n t} \end{aligned}$$

$\downarrow$

initial conditions

We have projected the infinite dimensional space of N-S equations onto a finite-dimensional one, because there exists no technique to analyse infinite modes.

The full solution of the linear system is then given by:

$$\vec{u}(t) = a_{1,0}\, e^{\lambda_1 t}\tilde{u}_1 + a_{2,0}\, e^{\lambda_2 t}\tilde{u}_2 + a_{3,0}\, e^{\lambda_3 t}\tilde{u}_3 + \dots + a_{n,0}\, e^{\lambda_n t}\tilde{u}_n$$

It is now clear how normal mode analysis fails to capture the interaction of eigenmodes in the full solution.

We want to understand what possible kind of interaction could be established.

TRANSIENT GROWTH AND NON-NORMAL LINEAR OPERATOR

Consider velocity and vorticity form of linearised N-S equations:

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - \frac{d^2U}{dy^2}\frac{\partial}{\partial x} - \frac{1}{Re}\nabla^4\right]v' = 0$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{Re}\nabla^2\right)\eta' + \frac{dU}{dy}\frac{\partial v'}{\partial z} = 0$$

Consider an initial value problem of the following wave by setting:

$$\begin{cases} v'(x,y,z,t) = \tilde{v}(t;\alpha,\beta)\, e^{i\alpha x + i\beta z} \\ \eta'(x,y,z,t) = \tilde{\eta}(t;\alpha,\beta)\, e^{i\alpha x + i\beta z}. \end{cases}$$

We no longer assume normal mode with time.

We can then write a matrix form of initial value problem for Orr-Sommerfeld-Squire system.

$$\frac{\partial}{\partial t}\begin{bmatrix} (D^2-K^2) & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix} + \begin{bmatrix} +Ui\alpha(D^2-K^2) - D^2Ui\alpha - \frac{1}{Re}(D^2-K^2)^2 & 0 \\ iDU\beta & Ui\alpha - \frac{1}{Re}(D^2-K^2) \end{bmatrix}\begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix}$$

Changing sign in the first equation

$$\frac{\partial}{\partial t}\begin{bmatrix} K^2-D^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix} + \begin{bmatrix} L_{os} & 0 \\ iDU\beta & L_{sq} \end{bmatrix}\begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix} = \underline{0}$$

with: $$L_{os} = Ui\alpha(K^2-D^2) + D^2U i\alpha + \frac{1}{Re}(D^2-K^2)^2$$

$$L_{sq} = Ui\alpha + \frac{1}{Re}(K^2-D^2)$$

We can see how as $Re \to \infty$ the off-diagonal term becomes more relevant, as the diffusion terms lose one term each.

$\downarrow$

L_{os}, L_{sq}

The off-diagonal term represents a coupling term to be further investigated.

We try to understand what happens with a <u>MODEL PROBLEM</u>.

Consider :

the (-) sign makes it a stable operator.

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} -1/Re & 0 \\ 1 & -2/Re \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix} , \quad \begin{bmatrix} v \\ \eta \end{bmatrix}_{t=0} = \begin{bmatrix} v_0 \\ \eta_0 \end{bmatrix}$$

We now repeat the same procedure already followed on page 44.

$$L = \begin{bmatrix} -1/Re & 0 \\ 1 & -2/Re \end{bmatrix} , \quad \vec{v} = \begin{bmatrix} v \\ \eta \end{bmatrix}$$

Eigenfunctions: $V = \begin{bmatrix} | & | \\ \tilde{\vec{v}}_1 & \tilde{\vec{v}}_2 \\ | & | \end{bmatrix} = \{ \tilde{\vec{v}}_1 \;\; \tilde{\vec{v}}_2 \}$ known

since $L V = V \Lambda$ where $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

So $L = V \Lambda V^{-1}$

The problem could be expressed as $\dfrac{d\vec{v}}{dt} = L\vec{v}$ $(*)$

Eigenfunction expansion technique: $\vec{v} = a_1(t)\,\tilde{\vec{v}}_1 + a_2(t)\,\tilde{\vec{v}}_2$ (Δ)

$\vec{v} = V\vec{a}(t)$

Plugging everything in $(*)$:

$$\frac{d\, V\vec{a}(t)}{dt} = V\Lambda V^{-1} V \vec{a}(t) \longrightarrow \frac{d\,V\vec{a}(t)}{dt} = V\Lambda\vec{a}(t)$$

Multiplying on the left by V^{-1}:

$$\frac{d\vec{a}}{dt} = \Lambda\vec{a}(t) \longrightarrow \begin{cases} a_1(t) = a_{1,0}\, e^{\lambda_1 t} \\ a_2(t) = a_{2,0}\, e^{\lambda_2 t} \end{cases}$$

(Δ): $$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = a_{1,0}\, e^{\lambda_1 t}\begin{bmatrix} \tilde{v}_1 \\ \tilde{\eta}_1 \end{bmatrix} + a_{2,0}\, e^{\lambda_2 t}\begin{bmatrix} \tilde{v}_2 \\ \tilde{\eta}_2 \end{bmatrix}$$

with λ_1, λ_2 eigenvalues

$\tilde{\vec{v}}_1, \tilde{\vec{v}}_2$ eigenfunctions

We now want to find eigenvalues and eigenfunctions.

$$L \tilde{\vec{v}} = \lambda \tilde{\vec{v}} \quad \rightarrow \quad (L - \lambda \mathbb{I}) \tilde{\vec{v}} = 0$$

$$(L - \lambda I) = 0 \quad \longrightarrow \quad \det (L - \lambda I) = 0$$

$$\begin{vmatrix} -\frac{1}{Re} - \lambda & 0 \\ 1 & -\frac{2}{Re} - \lambda \end{vmatrix} = \left(+ \frac{1}{Re} + \lambda \right) \left(+ \frac{2}{Re} + \lambda \right) = 0 \qquad \begin{cases} \lambda_1 = -\frac{1}{Re} \\ \lambda_2 = -\frac{2}{Re} \end{cases}$$

- $\lambda_1 = -\frac{1}{Re} \rightarrow$ eigenfunctions?

$$\begin{bmatrix} 0 & 0 \\ 1 & -\frac{2}{Re} + \frac{1}{Re} \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{\eta}_1 \end{bmatrix} = 0 \longrightarrow \tilde{v} - \frac{1}{Re} \tilde{\eta}_1 = 0$$

choosing $\tilde{v}_1 = \dfrac{1}{\sqrt{1 + Re^2}} \longrightarrow \tilde{\eta}_1 = Re\, \tilde{v} = \dfrac{Re}{\sqrt{1 + Re^2}}$

$$\text{So: } \begin{bmatrix} \tilde{v}_1 \\ \tilde{\eta}_1 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{1+Re^2}}}_{\text{normalization}} \begin{bmatrix} 1 \\ Re \end{bmatrix} = \tilde{\vec{v}}_1$$

- $\lambda_2 = -\frac{2}{Re}$

$$\begin{bmatrix} -\frac{1}{Re} + \frac{2}{Re} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix} = 0 \longrightarrow \tilde{v} = 0$$

$$\text{So } \begin{bmatrix} \tilde{v}_2 \\ \tilde{\eta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \tilde{\vec{v}}_2$$

These eigenvectors are <u>not orthogonal</u>!

What happens to the initial conditions?

$$\begin{bmatrix} v_0 \\ \eta_0 \end{bmatrix} = a_{1,0} \begin{bmatrix} 1 \\ Re \end{bmatrix} + a_{2,0} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{cases} a_{1,0} & \text{unknown} \\ a_{2,0} & \text{unknown} \end{cases}$$

$$\begin{cases} v_0 = a_{1,0} & \longrightarrow \quad a_{1,0} = v_0 \\ \eta_0 = a_{1,0} \cdot Re + a_{2,0} & \longrightarrow \quad a_{2,0} = \eta_0 - Re\, v_0 \end{cases}$$

We therefore have a complete form of the solution found with the eigenfunction expansion technique:

$$\boxed{\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = v_0\, e^{-t/Re} \begin{bmatrix} 1 \\ Re \end{bmatrix} + (\eta_0 - Re\, v_0)\, e^{-\frac{2t}{Re}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

This can be re-written in system form:

$$\begin{cases} v(t) = v_0\, e^{-t/Re} \\ \eta(t) = Re\, v_0 \left(e^{-t/Re} - e^{-\frac{2t}{Re}} \right) + \eta_0\, e^{-\frac{2t}{Re}} \end{cases}$$

The non-orthogonality has already been pointed out. What can we infer from it?

$$\tilde{\vec{v}}_1 \cdot \tilde{\vec{v}}_2 = \underbrace{\|\tilde{\vec{v}}_1\|}_{1}\ \underbrace{\|\tilde{\vec{v}}_2\|}_{1} \cos\theta \quad \longrightarrow \text{ both have normalized dimension}$$

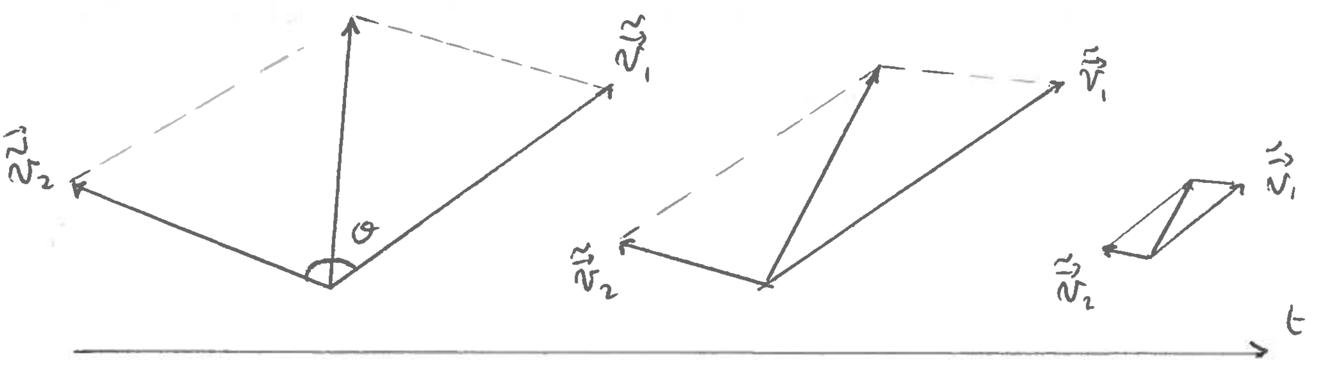
$$\cos\theta = \tilde{\vec{v}}_1 \cdot \tilde{\vec{v}}_2 = \frac{1}{\sqrt{1+Re^2}} \begin{bmatrix} 1 \\ Re \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{Re}{\sqrt{1+Re^2}}$$

- If $Re \to \infty$, $\cos\theta \to 1 \implies \theta = 0°$ or $\theta = 180°$

  so if $Re \to \infty$ $\tilde{\vec{v}}_1 \parallel \tilde{\vec{v}}_2$

- If $Re \to 0$, $\cos\theta \to 0 \implies \theta = 90°$

  so if $Re \to 0$ $\tilde{\vec{v}}_1 \perp \tilde{\vec{v}}_2$



At sufficiently large Re, the eigenvectors are not orthogonal to each other. Therefore, their interaction could lead to any kind of growth, but the solution then decays to zero.

This allows <u>SHORT TERM GROWTH MECHANISM IN TIME</u>

The two extremes are:

1) $Re \to 0$: the eigenvectors are orthogonal to each other, yielding a <u>monotonically decaying</u> solution

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} v_0 e^{-t/Re}, \\ \eta_0 e^{-2t/Re} \end{bmatrix}$$

2) $Re \to \infty$: $\lambda_1 = \lambda_2 = 0$ and the two eigenvectors become the same, yielding the following algebraically growing solution:

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = (\eta_0 + v_0 t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + v_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \|u\|^2 \sim t^2$$

3) $0 < Re < \infty$ There will be some sort of growth mechanism.

Remarks:

- The non-orthogonal superposition of exponentially decaying solutions can give rise to short-term transient growth.
- Eigenvalues alone only describe the asymptotic fate of the disturbance, but fail to capture transient effects.
- The "source" of the transient amplification of the initial condition lies in the nonorthogonality of the eigenfunctions.
- The non-normal eigenfunctions are the typical nature of the non-normal linear operator.

DEF: NON-NORMAL OPERATOR

Linear operators, the eigenfunctions (or eigenvectors) of which are non-orthogonal to one another with respect to the given inner product, is called non-normal.

Remark: linearised N-S equations with non-zero advection term is a non-normal linear operator. In fact, the problem arose from the coupling term $iDU\beta$ in O-S-S. system and it is there because of advection.

A fundamental implication of the non-normality is that there can be substantial transient growth in the energy of small perturbations even if the Reynolds number is less than the critical value. This growth occurs in the absence of nonlinearities.

We now want to compare the viscous situation of transient growth with the phenomenon taking place in the case of an inviscid formulation.

ALGEBRAIC INSTABILITY IN INVISCID FLOW

Linearised Euler equation with a streamwise uniform disturbance

We take $Re \to \infty$ and $\frac{\partial}{\partial x} = 0$

We are in 2-D.

$$\begin{cases} \frac{\partial \vec{u}'}{\partial t} + (\vec{u}' \cdot \vec{\nabla})\vec{U} + (\vec{U} \cdot \vec{\nabla})\vec{u}' = -\vec{\nabla} p' \\ \vec{\nabla} \cdot \vec{u}' = 0 \end{cases} \qquad \vec{\nabla} = \left(\frac{\partial}{\partial y} \hat{y} \right)$$

$$\begin{cases} \frac{\partial u'}{\partial t} + v' \frac{dU}{dy} + U \cancelto{0}{\frac{\partial u'}{\partial x}} = -\cancelto{0}{\frac{\partial p'}{\partial x}} \\ \frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} = -\frac{\partial p}{\partial y} \\ \frac{\partial v'}{\partial y} + \cancelto{0}{\frac{\partial u'}{\partial x}} = 0 \quad \longrightarrow \quad v' = v'(t) \end{cases}$$

↳ because there is no x dependance

Now consider the boundary conditions.

$a \rightarrow v'(y=a)=0$

$b \rightarrow v'(y=b)=0$

We know that the Poisson equation for pressure is:

$$\nabla^2 p = -2 \frac{dU}{dy} \frac{\partial v'}{\partial x}$$

Having set as hypothesis that there is no variation in x-direction, $\frac{\partial v'}{\partial x} = 0$. What follows is: $\nabla^2 p = 0$

$$\downarrow$$

$$\cancel{\frac{\partial^2 p'}{\partial x^2}} + \frac{\partial^2 p'}{\partial y^2} = 0$$

This tells us that $\frac{\partial^2 p'}{\partial y^2} = 0 \rightarrow \frac{\partial p'}{\partial y} = \text{Const.}$

From boundary conditions, we set $\left.\frac{\partial p'}{\partial y}\right|_{wall} = 0$

The equations we obtain are:

$$\begin{cases} \dfrac{\partial u'}{\partial t} = -v' \dfrac{dU}{dy} & u' = u'(y,t) \\ \dfrac{\partial v'}{\partial t} = 0 & v' = v'(t) \end{cases}$$

From the second, $v' = v_0'$

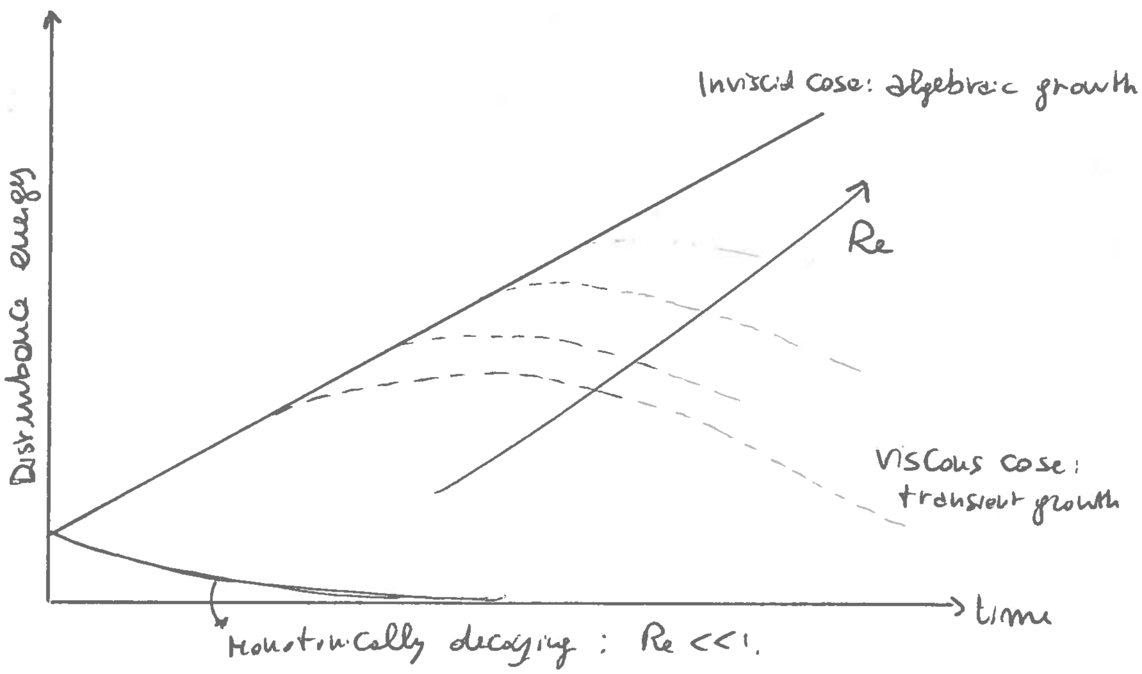
From the first, having find out that v' is <u>constant</u>:

$$u'(t) = u_0' - v' \frac{dU}{dy} t.$$

So the solution is:

$$\boxed{\begin{aligned} u'(y,t) &= u_0 - v_0 \frac{dU}{dy} t \\ v'(t) &= v_0 \end{aligned}}$$

We can finally compare the two different models:

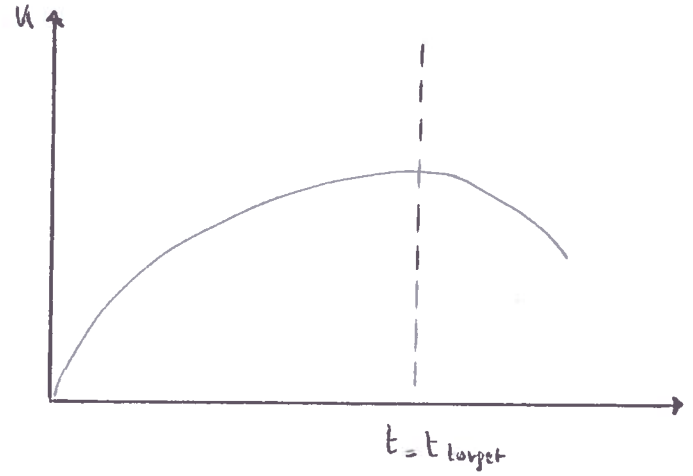


LECTURE 8

OPTIMAL TRANSIENT GROWTH

Aim: we want to find the initial disturbance (initial condition) that leads to the largest transient growth at a given time.

We need to solve an optimization problem, with different targets fixed.



A physically relevant quantity for measuring growth is the energy norm:

$$\|\tilde{\vec{u}}\|^2 = \int_\Omega |\tilde{u}|^2 + |\tilde{v}|^2 + |\tilde{w}|^2 \, dy$$

The total energy of the perturbation is obtained by first dividing $\|\tilde{\vec{u}}\|$ by $2k^2$ and then integrating the resulting quantity over all α and β. (Gustavsson 1986).

To measure the greatest possible growth in energy of an initial perturbation at time t, we introduce the growth function

$$G(\alpha, \beta, Re, t) = \max_{\hat{u}_0} \frac{\|\tilde{\vec{u}}(t; \alpha, \beta)\|^2}{\|\tilde{\vec{u}}_0(\alpha, \beta)\|^2}$$

That fetches energy growth results for $G^{max} > 1$; and it is for definition such that $G^{max} \geq 1$.

From what I understand, fixed Re, α and β come as a consequence (see page 39). Also, t_{target} is fixed, and we look for the initial conditions (disturbance) that make this happen.

they correspond to the most unstable instability

more likely, they

The problem is subject to:

$$\frac{\partial}{\partial t}\begin{bmatrix} k^2-D^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix} + \begin{bmatrix} L_{os} & 0 \\ i\beta DU & L_{sq} \end{bmatrix}\begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix} = 0$$

where $k^2 = \alpha^2 + \beta^2$

$$L_{os} = i\alpha U(k^2 - D^2) + i\alpha D^2 U + \frac{1}{Re}(k^2 - D^2)^2$$

$$L_{sq} = i\alpha U + \frac{1}{Re}(k^2 - D^2)$$

What could a theoretical procedure be?

- Fix Re
- For all α, β ... solve the system. Find the most unstable solution and use it for working on the growth function.

MODEL PROBLEM

We get rid of the presence of α and β in a model problem to better understand what is happening.

$$\max_{\vec{u}_0} \frac{\| \vec{u}(t) \|^2}{\| \vec{u}_0 \|^2} \qquad \text{with} \qquad \| \vec{u} \| = u^T u$$

Subject to:

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} -1/Re & 0 \\ 1 & -2/Re \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix}$$

The technique we employ is based on a paper by Butler and Farrell (1992, Physics of Fluids)

We now repeat the same procedure presented to find the solution of this differential problem.

STEP 1 : Project the optimization problem in the eigenspace.

1) Find eigenvalues and eigenfunctions of the linear operator

$$\lambda \underbrace{\begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix}}_{\vec{u}} = \begin{bmatrix} -1/Re & 0 \\ 1 & -2/Re \end{bmatrix}\begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix}$$

$$\lambda_1 = -\frac{1}{Re} \qquad \begin{bmatrix} \tilde{v}_1 \\ \tilde{\eta}_1 \end{bmatrix} = \frac{1}{\sqrt{1+Re^2}} \begin{bmatrix} 1 \\ Re \end{bmatrix}$$

$$\lambda_2 = -\frac{2}{Re} \qquad \begin{bmatrix} \tilde{v}_2 \\ \tilde{\eta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For full solution, see page 48.

2) Construct the solution of the problem using the eigenfunction expansion technique

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = a_1(t) \begin{bmatrix} \tilde{v}_1 \\ \tilde{\eta}_1 \end{bmatrix} + a_2(t) \begin{bmatrix} \tilde{v}_2 \\ \tilde{\eta}_2 \end{bmatrix}$$

We had: $\frac{d\vec{u}}{dt} = L\vec{u}$, $\underset{\substack{\downarrow \\ \text{matrix of} \\ \text{eigenfunctions}}}{LV} = V\Lambda \rightarrow L = V\Lambda V^{-1}$

$\vec{u} = V\vec{a}$

So:

$$\frac{dV\vec{a}}{dt} = V\Lambda V^{-1}V\vec{a} \rightarrow \frac{d\vec{a}}{dt} = \Lambda\vec{a} \rightarrow \vec{a} = \vec{a}_0 e^{\overset{\substack{\text{diagonal} \\ \downarrow}}{\Lambda} t}$$

So:

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = a_{1,0} e^{\lambda_1 t} \begin{bmatrix} \tilde{v}_1 \\ \tilde{\eta}_1 \end{bmatrix} + a_{2,0} e^{\lambda_2 t} \begin{bmatrix} \tilde{v}_2 \\ \tilde{\eta}_2 \end{bmatrix}$$

We need to find the optimal initial coefficients a_{10} and a_{20}, and if needed report them back to $\tilde{v}_0$ and $\tilde{\eta}_0$.
It can be convenient to change the solution in the form:

$$\vec{u}(t) = \underbrace{\begin{bmatrix} | & | \\ \tilde{\vec{u}}_1 & \tilde{\vec{u}}_2 \\ | & | \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}}_{e^{\Lambda t}} \underbrace{\begin{bmatrix} a_{1,0} \\ a_{2,0} \end{bmatrix}}_{\vec{a}_0} = U e^{\Lambda t} \vec{a}_0$$

Using this solution, we remove the governing equations.

Remember that $\vec{u}_0 = U\vec{a}_0$.

STEP 2 : Solve the optimization problem

It is the case to open a big parenthesis and look at how an optimization problem is implemented.

1) We restate the optimization problem in the projected space.

We had $\max_{\vec{u}_0} \frac{\|\vec{u}(t)\|^2}{\|\vec{u}_0\|^2}$

- Numerator $\rightarrow (U e^{\Lambda t} \vec{a}_0)^T (U e^{\Lambda t} \vec{a}_0) \Longleftarrow \|\vec{u}(t)\|^2$

- Denominator $\rightarrow \vec{u}_0^T \vec{u}_0 = \|\vec{u}_0\| = 1$

The problem is subject to: $\underbrace{(U\vec{a}_0)^T}_{\vec{u}_0^T} \underbrace{(U\vec{a}_0)}_{\vec{u}_0} = 1$

We can then rewrite the problem, with:

$$\max_{\vec{a}_0} \vec{a}_0^T (e^{\Lambda t})^T U^T U e^{\Lambda t} \vec{a}_0$$

$$\text{s.t.} \quad \vec{a}_0^T \underbrace{U^T U}_{\hookrightarrow \text{ non-orthogonal}} \vec{a}_0 = 1$$

We define a few quantities that will come in handy:

$A = (e^{\Lambda t})^T U^T U e^{\Lambda t}$

$\vec{x} = \vec{a}_0$

$Q = U^T U$

And we obtain

$$\max_{\vec{x}} \vec{x}^T A \vec{x} \quad , \text{ subject to } \quad \vec{x}^T Q \vec{x} = 1$$

THEORY OF OPTIMIZATION

It may come in handy to have a brief summary of how this thing is done.

It is just a "big" parenthesis on optimization

Problem:

$$\min_{x \in \mathbb{R}^n} L(x) \quad \text{s.t.} \quad \underbrace{f(x) = 0}_{\text{constraint}} \quad , \quad f : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad (n \geq m)$$

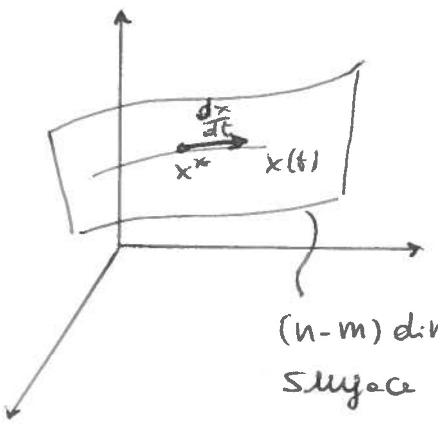
Note: if $n < m$ there are more constraints than degrees of freedom. More likely, there is no solution!

GENERAL FORMULATION (Lagrange multipliers)

The problem is the one stated just above:

$$\min_{x \in \mathbb{R}^n} L(x) \quad \text{s.t.} \quad f(x) = 0$$

$$S \triangleq \{ x \in \mathbb{R}^n \mid f(x) = 0 \} \qquad f : \mathbb{R}^n \to \mathbb{R}^m$$



(n-m) dimensional surface

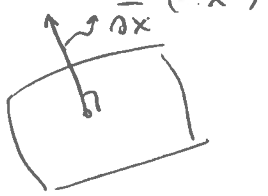
let $x^* =$ local minimizer

$x(t) =$ curve on S s.t. $x(0) = x^*$

i) $$\left.\frac{dL}{dt}\right|_{t=0} = \left.\frac{\partial L}{\partial x}\right|_{t=0} \left.\frac{dx}{dt}\right|_{t=0} = 0$$

$$\underbrace{\frac{\partial L}{\partial x}(x^*)}_{} \cdot \dot{x}(0) = 0$$

↳ normal to S; $S \perp \frac{\partial L}{\partial x}(x^*)$



$$\longrightarrow \frac{\partial L}{\partial x}(x^*) \perp \dot{x}(0)$$

ii) $f(x) = 0$

$$\frac{df}{dt} = \left.\frac{\partial f}{\partial x} \frac{dx}{dt}\right|_{x = x^*} = 0$$

↳ matrix $m \times n$

$$\dot{x}(0) \in N\left(\frac{\partial f}{\partial x}\right) \cup R\left(\frac{\partial f^T}{\partial x}(x^*)\right) = \mathbb{R}^N$$

$$\longrightarrow \frac{\partial L}{\partial x} \in R\left(\frac{\partial f^T}{\partial x}(x^*)\right)$$

i.e.

$$\frac{\partial L}{\partial x}(x^*)^T = \frac{\partial f^T}{\partial x}(x^*)\, c \qquad c \in \mathbb{R}^m$$

$$= -\lambda \in \mathbb{R}^m$$

$$\longrightarrow \frac{\partial L}{\partial x}(x^*) + \lambda^T \frac{\partial f}{\partial x}(x^*) = 0$$

FIRST ORDER NECESSARY CONDITION

<u>Theorem:</u> let x^* be a local extremum point of L subject to the constraints $f(x) = 0$

Assume x^* is a regular point.

$\Updownarrow$

dim of (tangent plane of f at this) is equal to dim f

Then

Necessary $\longrightarrow$ $\frac{\partial L}{\partial x}(x^*) + \lambda^T \frac{\partial f}{\partial x}(x^*) = 0$

<u>Application</u>

let $H(x,\lambda) = L(x) + \lambda^T f(x)$ $\longrightarrow$ $\underbrace{\min_{x,\lambda} H(x,\lambda)}_{\text{unconstrained Problem}}$

If x^* is relative minimum

$$\longrightarrow \frac{\partial H(x,\lambda)}{\partial (x,\lambda)} = 0$$

Note:

$$\frac{\partial H}{\partial x} = \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x}$$

$$\frac{\partial H}{\partial x} = f(x) = 0$$

EXAMPLE 1

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T x \quad \text{s.t.} \quad Ax = b$$

Solution:

$$H(x,\lambda) = \frac{1}{2} x^T x + \lambda^T (Ax - b)$$

$$H(x,\lambda)_x = x^T + \lambda^T A = 0$$

$$H(x,\lambda)_\lambda = Ax - b = 0$$

$$x + A^T \lambda = 0$$

$$Ax + AA^T \lambda = 0$$

$$b + AA^T \lambda = 0$$

$$\lambda = -(AA^T)^{-1} b \qquad (\text{assuming } (AA^T)^{-1} \text{ exists})$$

$$x^* = A^T (AA^{-1}) b$$

EXAMPLE 2

$$\max_{x \in \mathbb{R}^n} x^T A^T Q A x \quad \text{s.t.} \quad x^T Q x = 1$$

Solution:

$$H(x,\lambda) = x^T A^T Q A x + \underbrace{\lambda^T (x^T Q x - 1)}_{\text{scalar}}$$

$$\frac{\partial H}{\partial x} = x^T A^T Q A + \lambda^T x^T Q = 0 \longrightarrow Q^{-1} A^T Q A x = \lambda x$$

$$\frac{\partial H}{\partial \lambda} = x^T Q x - 1 = 0$$

λ: eigenvalue of $(Q^{-1} A^T Q A)$

Premultiply (1) by $x^T Q$; $\Rightarrow$ $x^T A^T Q A x = \lambda$

$$\therefore \quad \|A\|_Q^2 = \lambda_{max} (Q^{-1} A^T Q A)$$

Lagrange multiplier $\Longleftrightarrow$ eigenvalue

$\downarrow$

sensitivity of $\|Ax\|$

Returning to transient growth optimization:

2) Apply the optimality condition to the problem:

$$\max_{\vec{x}} \ \vec{x}^T A \vec{x} \qquad \text{s.t.} \quad \vec{x}^T Q \vec{x} = 1$$

We introduce the Lagrangian:

$$\mathcal{L} \triangleq \vec{x}^T A \vec{x} + \mu (1 - \vec{x}^T Q \vec{x})$$

μ: lagrange multiplier.
↓
scalar

Optimality condition:

$$\frac{\partial \mathcal{L}}{\partial \vec{x}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = 0$$

We are in matrix form, so $\frac{df}{d\vec{x}} = 2\vec{x}$, $\frac{\partial f}{\partial \vec{x}} = \vec{\nabla} f = \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \right]$

$$\frac{\partial \mathcal{L}}{\partial \vec{x}} = 2 \vec{x}^T A - 2 \mu \vec{x}^T Q = 0$$ → (gradient of a vector)

$$\frac{\partial \mathcal{L}}{\partial \mu} = 1 - \vec{x}^T Q \vec{x} = 0$$ → Normalisation of eigenvector

We get to...

$$A^T \vec{x} = \mu Q^T \vec{x}$$ Eigenvalue problem!

$$A = A^T \qquad Q = Q^T \longrightarrow A\vec{x} = \mu Q \vec{x}$$

3) The solution becomes:

$$\underbrace{\underbrace{\vec{a}_0^T (e^{\Lambda t})^T U^T U e^{\Lambda t} \vec{a}_0}_{A\vec{x} = \mu Q \vec{x}}}_{\| u(t) \|^2} = \mu \vec{a}_0^T \underbrace{Q \vec{x}}_{} = \mu \underbrace{\vec{a}_0^T U^T U \vec{a}_0}_{\vec{u}_0^T \vec{u}_0 = \| \vec{u}_0 \|^2}$$

This is the crucial step. take the first and the last elements, and divide the equation by $\| \vec{u}_0 \|^2$.

We obtain that:

$$\mu_{max} = \max \frac{\| \vec{u}(t) \|^2}{\| \vec{u}_0 \|^2}$$

$\downarrow$

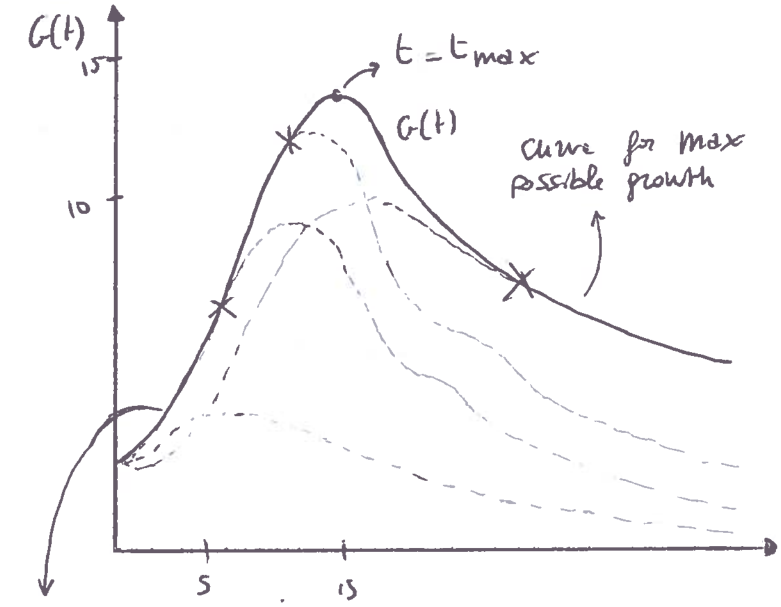
this is the biggest eigenvalue

OPTIMAL INITIAL CONDITION:

$$\vec{u}_0 = U \vec{a}_0 \qquad (\underline{x}_0 \text{ eigenvector})$$

WALL - BOUNDED SHEAR FLOWS

$$G(t) = \max_{\tilde{\vec{u}}_0} \frac{\| \tilde{\vec{u}}(t, \alpha, \beta) \|^2}{\| \tilde{\vec{u}}_0(\alpha, \beta) \|^2}$$



" x_i " are the target times.

Out of all the different initial conditions that can be imposed, there is one that gives maximum transient growth, and it is $G(t)$. This curve is here represented for Poiseuille flow with $Re = 1000$, $\alpha = 1$.

The dashed lines, instead, represent the growth curves of selected initial conditions.

Now we will proceed in the following way:

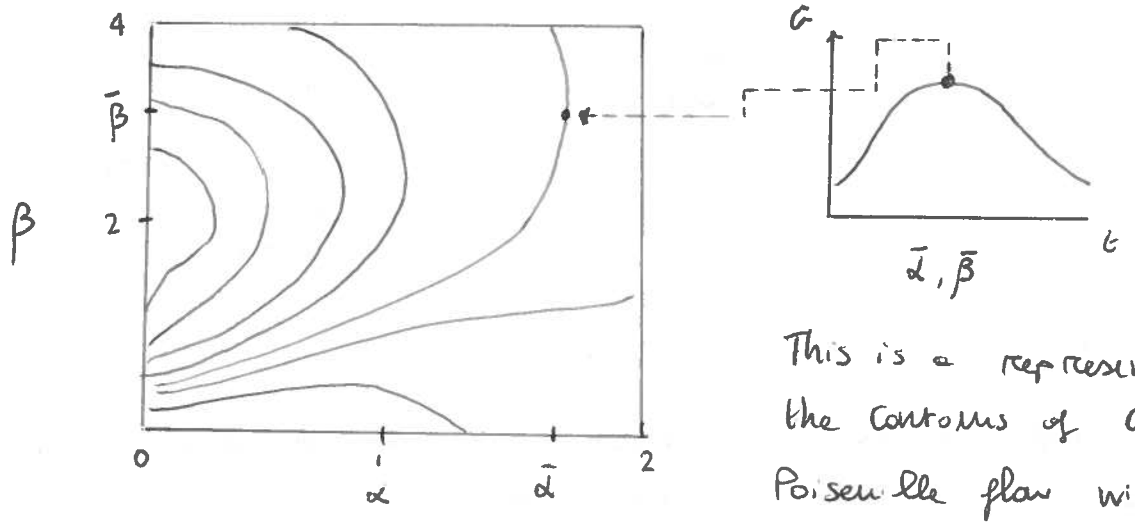
- For every α, β in the range of interest, we calculate $G(t)$ that is we select the curve associated with the initial conditions that produce the maximum transient growth for that set of α, β.
- We select the maximum of the curve $G(t)$ and we create a contour plot of G_{max}, for the specified set of α, β.

In equations, we are selecting:

$$\boxed{G_{max} = \max_t G(t) = \max_t \max_{\hat{u}_0} \frac{\|\hat{\vec{u}}(t,\alpha,\beta)\|^2}{\|\hat{\vec{u}}_0(\alpha,\beta)\|^2}}$$

Every point in the contour plot will be related to a different instant in time:

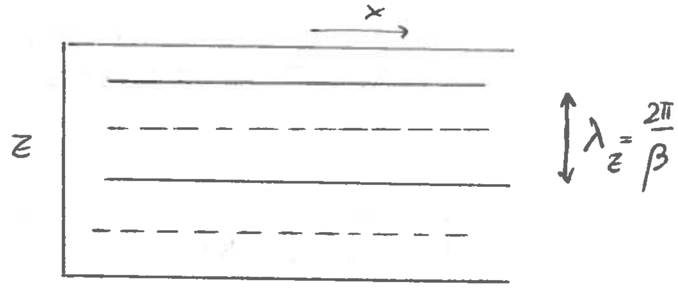
WAVENUMBER PLANE - POISEUILLE FLOW



This is a representation of the contours of G_{max} for Poiseuille flow with $Re = 1000$.

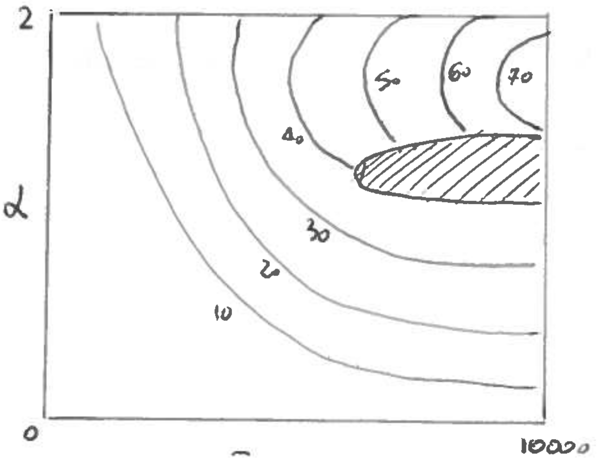
The curves from outer to inner correspond to 10 ... 180.

We can observe how a long streamwise structure is most amplified: $\lambda_x = \frac{2\pi}{\alpha}$. This, in the graph, corresponds to the region where α is close to zero and β close to two.



The picture on the right represents the wavelength of the spanwise instability, as seen from above.

The same kind of plot can be produced fixing $\beta = 0$ and varying Re



Again, G_{max} is plotted.

The dark region can't be analysed with this tool, since it is linearly unstable.

Scaling with the Reynolds number

Flow configurations	G_{max}	t_{max}	α	β
Couette flow	$0.20\ Re^2$	$0.076\ Re$	$35/Re$	2.04
Poiseuille flow	$1.18\ Re^2$	$0.117\ Re$	0	1.6
Pipe flow	$0.07\ Re^2$	$0.048\ Re$	0	1
Boundary layer	$1.50\ Re^2$	$0.778\ Re$	0	0.65

$\downarrow$

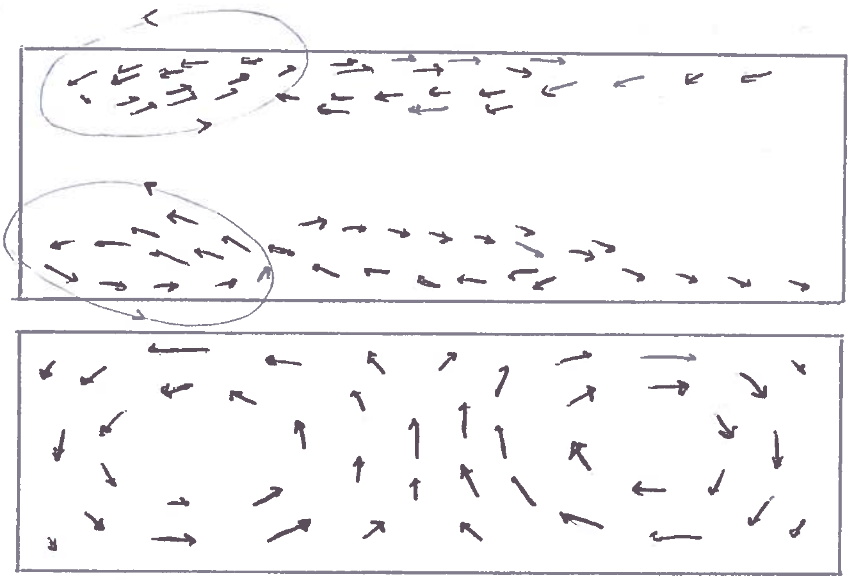
$G_{max} \sim Re^2$

$\downarrow$

long streamwise structure appears with $O(1)$ spanwise width

THE OPTIMAL DISTURBANCE (POISEUILLE FLOW) – 2D

Consider the figure at pag. 62. What is the initial disturbance that induces the maximum transient growth?
Remember that the parameters of the problem were: $\begin{cases} \alpha = 1 \\ \beta = 0 \\ Re = 1000 \end{cases}$



$t = 0$: initial condition

$t = t_{max}$
$\downarrow$
most amplified case

How can we understand the mechanism behind this temporal evolution?

It is linked to the evolution of vorticity.

THE ORR MECHANISM

The energy growth and decay is somehow proportional to:

$$\frac{dE}{dt} \sim - \int_V u'v' \frac{dU}{dY} \, dV$$

$\frac{dU}{dY}$ is given by the base flow in Poiseuille flow it is positive and will remain the same as time passes.

What rules growth and decay is the combination $u'v'$.

Initially, vorticity is tilted upstream. (fluctuations).

The large velocity on top makes this vorticity tilt downstream.

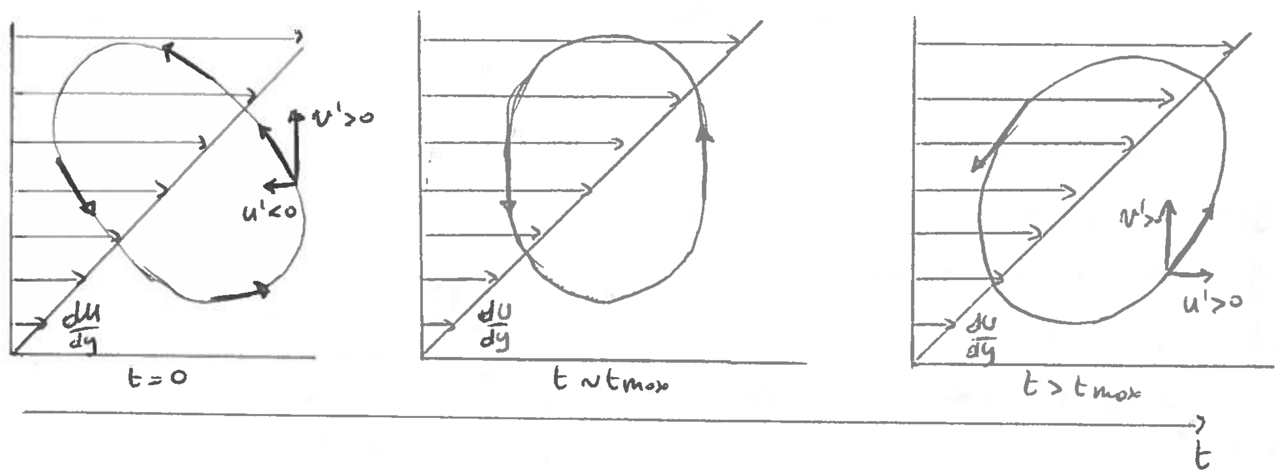
The steps are the following:

1) $\left.\begin{array}{l} u'<0 \text{ with } v'>0 \\ u'>0 \text{ with } v'<0 \end{array}\right\} \rightarrow u'v'<0 \rightarrow \frac{dE}{dt}>0$ <u>GROWTH</u>

2) $u'v' = 0 \rightarrow \frac{dE}{dt} = 0 \quad t \sim t_{max}$ maximum growth achieved

3) $\left.\begin{array}{l} u'>0 \text{ with } v'>0 \\ u'<0 \text{ with } v'<0 \end{array}\right\} \rightarrow u'v'>0 \rightarrow \frac{dE}{dt}<0$ <u>DECAY</u>

Again, the tilting is due to the large velocity on top.



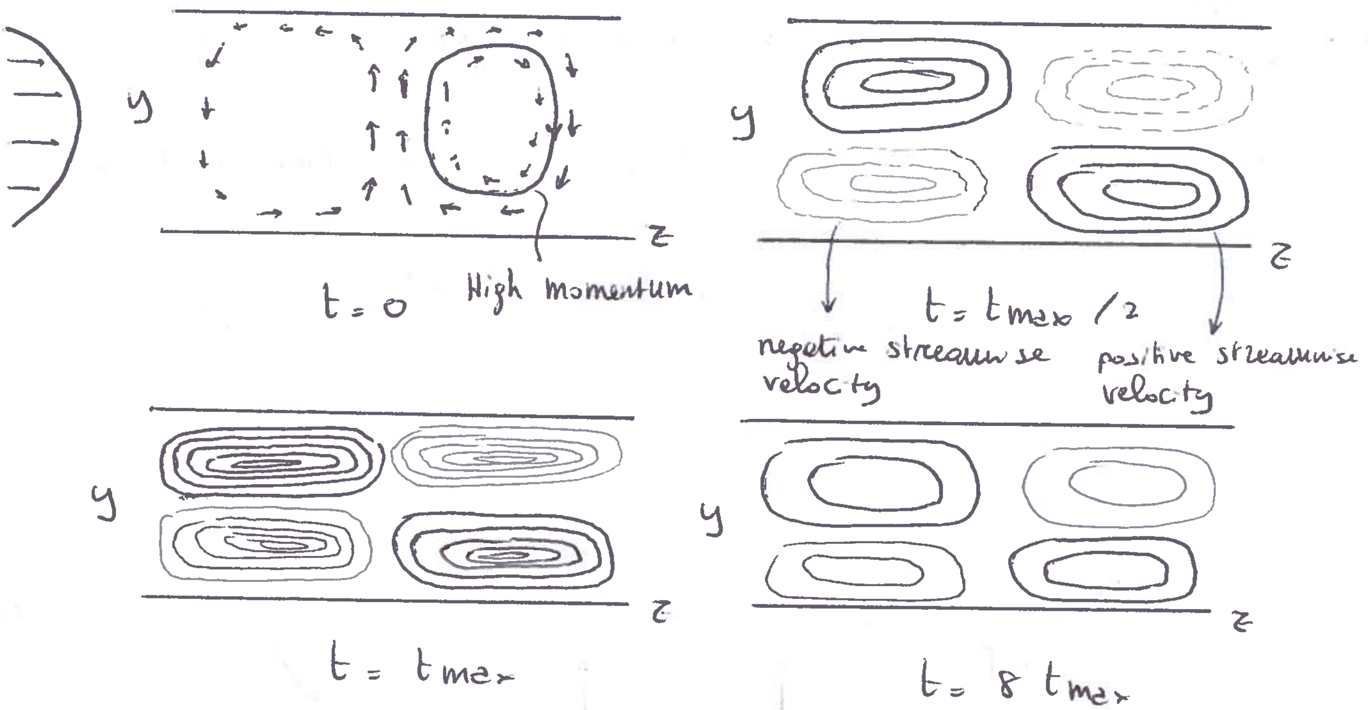
In brief: optimal initial conditions are narrow structures inclined against the mean shear. As time evolves, they tilt into the mean shear direction, thus "releasing" their energy.

Viscous diffusion limits their growth, as they are further stretched by the mean flow.

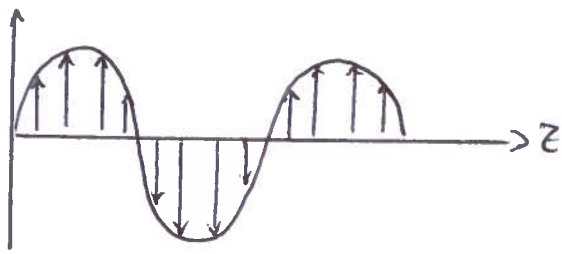
THE OPTIMAL DISTURBANCE (POISEUILLE FLOW) – 3D

The structure that leads to maximum amplification is different from the 2-D case. (REF. PETER SCHMID)
Three dimensional optimal disturbances resemble streamwise vortices which transport low-energy fluid close to the walls into regions of higher mean flow velocities, thus creating a streamwise velocity defect called streaks. The process associated with the formation of streaks from streamwise vortices is called lift-up mechanism.

Poiseuille flow with $\alpha = 0$, $\beta =$, $Re = 5000$
(Bewley & Liu, 1997)



Streaks are the alternating pattern of streamwise velocity. Since the system is stable they decay to zero. It's wall-normal vorticity



LIFT-UP EFFECT

It is a vortex tilting mechanism for the generation of streaks.

$$\underbrace{\frac{D\omega_y}{Dt}}_{\text{material derivative}} \sim \underbrace{\omega_x \frac{dU}{dy}}_{\text{initial condition}}$$

→ Initial condition ω_x (streamwise vortex) generates wall-normal vorticity → streaks.
It is $\omega_x \frac{dU}{dy}$, main source of transient growth, that causes vortex tilting.

LECTURE 9

SPATIO-TEMPORAL EVOLUTION OF INSTABILITIES

GINZBURG-LANDAU EQUATION

We introduce a toy model of the linearised N-S equations.
It is called <u>complex linear</u> Ginzburg-Landau equation.

$$\underbrace{\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x}}_{\text{advection}} - \mu u - (\underbrace{1}_{\text{diffusion}} - i\underbrace{\overset{\text{real}}{c_d}}_{\text{dispersion}})\frac{\partial^2 u}{\partial x^2} = 0$$

μ is a control parameter that can be mentally associated with Re. It plays a role in instabilities. It is generally complex.

The boundary conditions we impose are the following:

BC: $\quad u(x = \pm\infty) = 0$

IC: $\quad u(x, t=0) = u_0(x)$

This equation was originally derived with reference to superconductors and superfluids.

We don't derive it from scratch.

We proceed guessing a normal mode solution:

$$u = \underset{\substack{\downarrow \\ \text{constant, complex and with no dependence on } y.}}{A}\, e^{i(kx-\omega t)}$$

$$\frac{\partial u}{\partial t} = A(-i\omega)\, e^{i(kx-\omega t)} \longrightarrow \frac{\partial}{\partial t} \Rightarrow -i\omega$$

$$\frac{\partial u}{\partial x} = A(ik)\, e^{i(kx-\omega t)} \longrightarrow \frac{\partial}{\partial x} \Rightarrow ik$$

$$\frac{\partial^2 u}{\partial x^2} = -Ak^2\, e^{i(kx-\omega t)} \longrightarrow \frac{\partial^2}{\partial x^2} \Rightarrow -k^2$$

We now compute the dispersion relation. So, we plug these new derivatives back in the G-L equation.

$$-i\omega u + U i k u - \mu u - (1 - i C_d)\left(-k^2 u\right) = 0$$

$$-i\omega + U i k - \mu + (1 - i C_d) k^2 = 0$$

multiply by i and obtain...

$$\omega - Uk - \mu i + (i + C_d) k^2 = 0$$

Collect i and obtain the dispersion relation:

$$D(\omega, k) = \omega - Uk + C_d k^2 - i(\mu - k^2) = 0$$

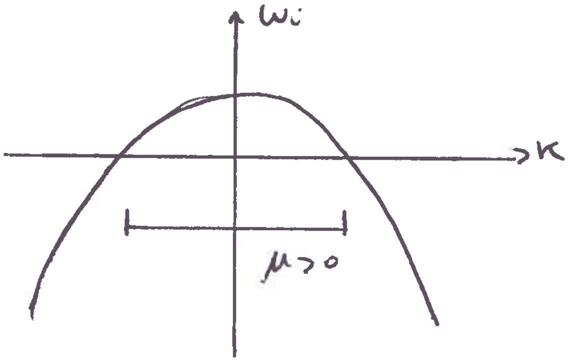
This is an algebraic equation that can be solved by hand.
There are two possibilities of further analysis.

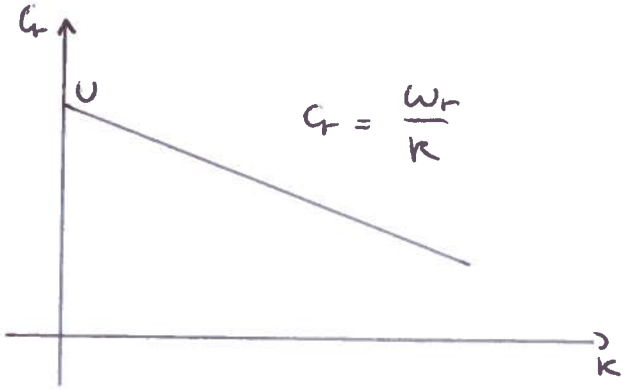
- Temporal stability

$$\omega(k) = Uk - C_d k^2 + i(\mu - k^2)$$

$$\begin{cases} \omega_r = Uk - C_d k^2 \\ \omega_i = \mu - k^2 \end{cases}$$

We observe that if $\mu > 0$, ω_i is positive for some k, therefore linearly unstable.





It can be shown that $\omega_i > 0$ when $-\mu^{1/2} \leq k \leq \mu^{1/2}$.

- Spatial stability

The idea here is to prescribe ω and try to find k. There will be two solutions.

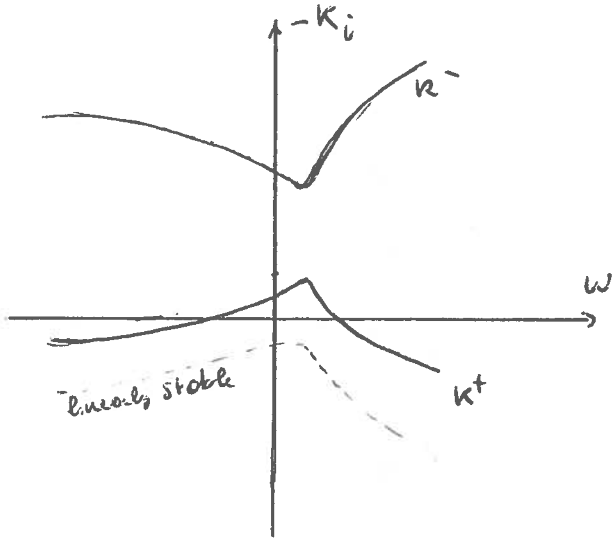
$$k^2(C_d + i) - Uk + \omega - i\mu = 0$$

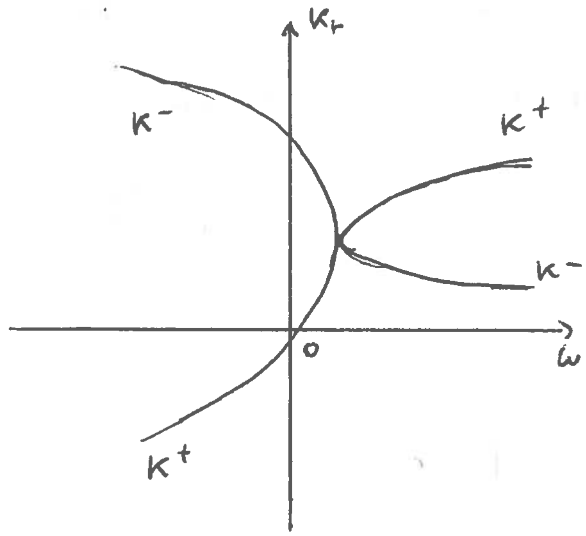
$$k^{\pm}(\omega) = \frac{U}{2(C_d + i)} \pm \frac{1}{2(C_d + i)} \sqrt{U^2 - 4(C_d + i)(\omega - i\mu)}$$

So the two solutions are the following:

$$k^{\pm}(\omega) = \frac{U}{2(C_d+i)} \pm \left(\frac{-1}{C_d+i}\right)^{\frac{1}{2}}\left[\omega - \frac{C_d U^2}{4(1+C_d^2)} - i\left\{\mu - \frac{U^2}{4(1+C_d^2)}\right\}\right]^{\frac{1}{2}}$$

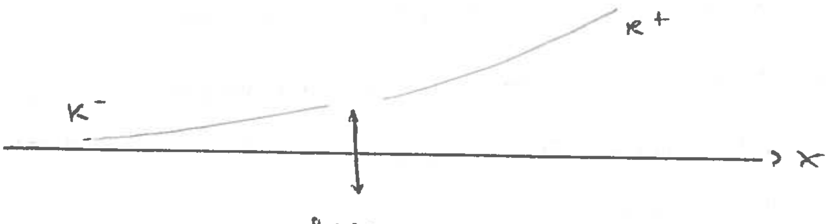
Graphical representation:





If $-k_i > 0 \rightarrow$ spatially unstable, from $e^{ikx - i\omega t}$.

Considering the first of the two graphs, we can say the following.
Suppose we have a disturbance, at some point of the flow.

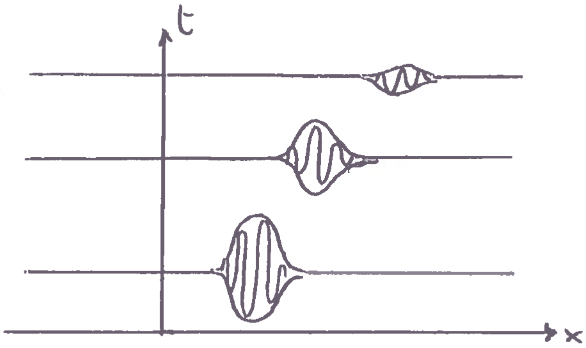


Information from k^- is upstream from disturbance, so we don't worry about it.
Instead, the downstream disturbance is driven by k^+.

ABSOLUTE AND CONVECTIVE INSTABILITIES IN PARALLEL FLOW

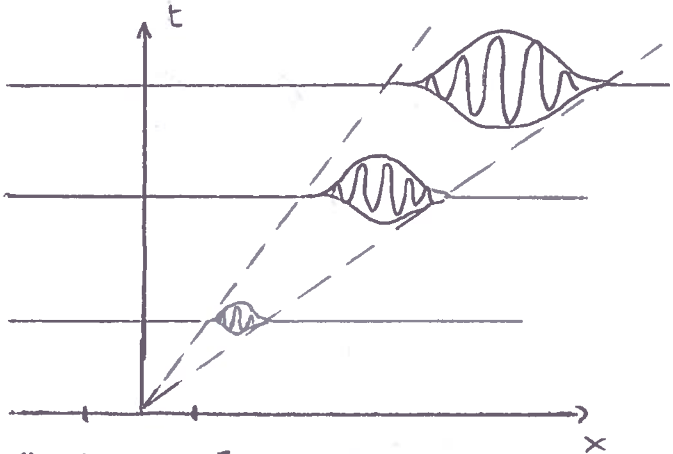
The spatio-temporal evolution of a wave packet (i.e. impulse) in a parallel flow can be of three different kinds.

- LINEARLY STABLE

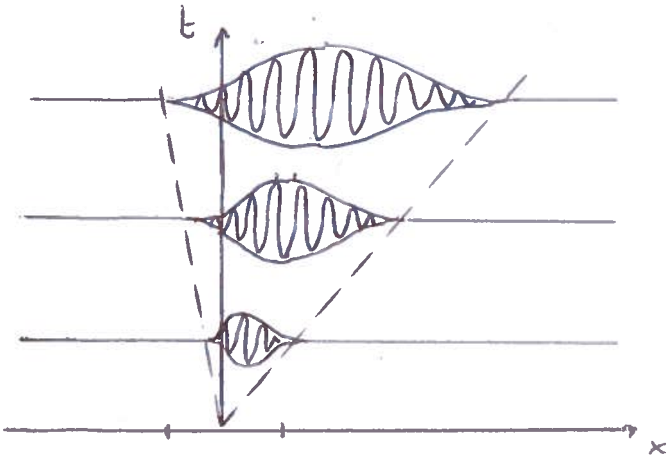


All the perturbations decay both in space and time

- LINEARLY UNSTABLE



CONVECTIVELY UNSTABLE



ABSOLUTELY UNSTABLE

In both cases the perturbation grows as $t \to \infty$.
The qualitative difference is the following: selected a region of interest around the origin, a convectively unstable behaviour will result in a stable situation after a while; instead, the absolutely unstable behaviour will cause the region to be contaminated by the instability thereafter.

We can more rigorously define these differences considering the impulse response (<u>GREEN'S FUNCTION</u>) of the G-L equation.

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \mu - (1 - ic_d)\frac{\partial^2}{\partial x^2}\right] \underbrace{G(x,t)}_{\text{Green's function}} = \underbrace{\delta(x)\,\delta(t)}_{\text{impulse}}$$

- LINEARLY STABLE:

$$\lim_{t \to \infty} G(x,t) = 0 \quad \text{for all rays} \quad \frac{x}{t} = \text{const} \qquad (t = cx)$$

- LINEARLY UNSTABLE

$$\lim_{t \to \infty} G(x,t) = \infty \quad \text{for at least one ray} \quad \frac{x}{t} = \text{const.}$$

So how do we distinguish among different kinds of unstable wavepackets?
The answer is: we check the ray $\frac{x}{t} = 0$.

- IF $G(x,t) \to 0$ along $\frac{x}{t} = 0 \Rightarrow$ CONVECTIVELY UNSTABLE
- IF $G(x,t) \to \infty$ along $\frac{x}{t} = 0 \Rightarrow$ ABSOLUTELY UNSTABLE

CRITERION FOR ABSOLUTE INSTABILITY.

Consider the impulse response of Ginzburg - Landau equation.

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \mu - (1 - i c_d)\frac{\partial^2}{\partial x^2}\right] G(x,t) = \delta(x)\,\delta(t)$$

The criterion is found through three steps.

1) Perform Fourier transform in x and Laplace transform in t:

$$\tilde{G}(k,\omega) = \int_0^\infty \int_{-\infty}^\infty G(x,t)\, e^{-i(kx - \omega t)}\, dx\, dt$$

2) Construct solution in the wavenumber space
3) Invert the Fourier - Laplace transform

$$G(x,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^\infty \int_{-\infty}^\infty \tilde{G}(k,\omega)\, e^{i(kx - \omega t)}\, dx\, dt.$$

This last part is very technical and it will be therefore skipped.
Only the final solution is provided.

STEP 1) We perform a Fourier transform in x and Laplace transform in t.

i) Fourier transform in x.

$$\hat{G}(k,t) = \int_{-\infty}^{\infty} G(x,t)\, e^{-ikx}\, dx$$

- $$\int_{-\infty}^{+\infty} \frac{\partial G(x,t)}{\partial t} e^{-ikx} dx = \frac{\partial}{\partial t}\int_{-\infty}^{+\infty} G(x,t)\, e^{-ikx} dx = \frac{\partial \hat{G}(k,t)}{\partial t}$$

the integral does not depend on t, so

- $$\int_{-\infty}^{\infty} \frac{\partial G(x,t)}{\partial x} e^{-ikx} dx = \Big[\underbrace{\cancel{G(x,t)\, e^{-ikx}}}_{0\ (\because BC)}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} G(x,t) \frac{\partial e^{-ikx}}{\partial x} dx$$

$$= ik \int_{-\infty}^{\infty} G(x,t)\, e^{-ikx} dx = ik\, \hat{G}(k,t)$$

- $$\int_{-\infty}^{+\infty} \frac{\partial^2 G(x,t)}{\partial x^2} e^{-ikx} = \Big[\underbrace{\cancel{\frac{\partial G(x,t)}{\partial x} e^{-ikx}}}_{0\ (\because BC)}\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{\partial G(x,t)}{\partial x} \frac{\partial e^{-ikx}}{\partial x} dx$$

$$= ik \int_{-\infty}^{+\infty} \frac{\partial G(x,t)}{\partial x} e^{-ikx} dx$$

$$= ik \left\{ \Big[\underbrace{\cancel{G(x,t)\, e^{-ikx}}}_{0\ (\because BC)}\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} G(x,t) \frac{\partial e^{-ikx}}{\partial x} dx \right.$$

$$= -ik(-ik) \int_{-\infty}^{+\infty} G(x,t)\, e^{-ikx} dx$$

$$= -k^2\, \hat{G}(k,t)$$

- $$\int_{-\infty}^{+\infty} \delta(x)\, \delta(t)\, e^{-ikx} dx = \delta(t)$$

ii) Laplace transform in t

$$\tilde{G}(k,\omega) = \int_0^\infty \hat{G}(k,t)\, e^{i\omega t}\, d\omega$$

- $$\int_0^\infty \frac{\partial \hat{G}(k,t)}{\partial t} e^{i\omega t}\, dt = \left[\hat{G}(k,t)\, e^{i\omega t} \right]_0^\infty - \int_0^\infty \hat{G}(k,t) \frac{\partial e^{i\omega t}}{\partial t}\, dt$$
$$= -i\omega \int_0^\infty \hat{G}(k,t)\, e^{i\omega t} = -i\omega\, \tilde{G}(k,\omega)$$

- $$\int_0^\infty \delta(t)\, e^{i\omega t}\, dt = 1$$

STEP 2) We can now construct the solution in the wavenumber space. Using what is listed in step 1, Ginzburg-Landau equation becomes:

$$-i\omega\, \tilde{G}(k,\omega) + U ik\, \tilde{G}(k,\omega) - \mu\, \tilde{G}(k,\omega) + (1 - ic_d)\, k^2\, \tilde{G}(k,\omega) = 1$$

Collecting $\tilde{G}(k,\omega)$, we obtain the following:

$$\boxed{D(k,\omega)\, \tilde{G}(k,\omega) = 1}$$

G-L equation with impulse forcing in k and ω space

<u>DISPERSION RELATION</u>:

$$\boxed{D(k,\omega) = -i\omega + U ik - \mu + (1 - ic_d)\, k^2}$$

The solution in the wavenumber space is then easily found, just inverting an algebraic equation.

$$\tilde{G}(k,\omega) = \frac{1}{D(k,\omega)}$$

STEP 3) Invert the Fourier-Laplace transform.

$$G(x,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{D(k,\omega)}\, e^{i(kx - \omega t)}\, dx\, dt$$

Starting from the inverse laplace-Fourier transform, Huerre developed an asymptotic solution throug the method of steepest descent (2000)

In fact:

$$G(x,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{\frac{1}{D(k,\omega)}}_{\tilde{G}(k,\omega)} e^{i(kx-\omega t)} \, dx\, dt \sim$$

$$\sim \frac{e^{i(k_0 x - \omega_0 t)}}{\frac{\partial D}{\partial \omega}(k_0, \omega_0) \left[\frac{\partial^2 \omega}{\partial k^2}(k_0)\, t \right]^{1/2}} \quad \text{as } t \to \infty$$

$e^{i(k_0 x - \omega_0 t)}$ is the dominant term.

we have that:

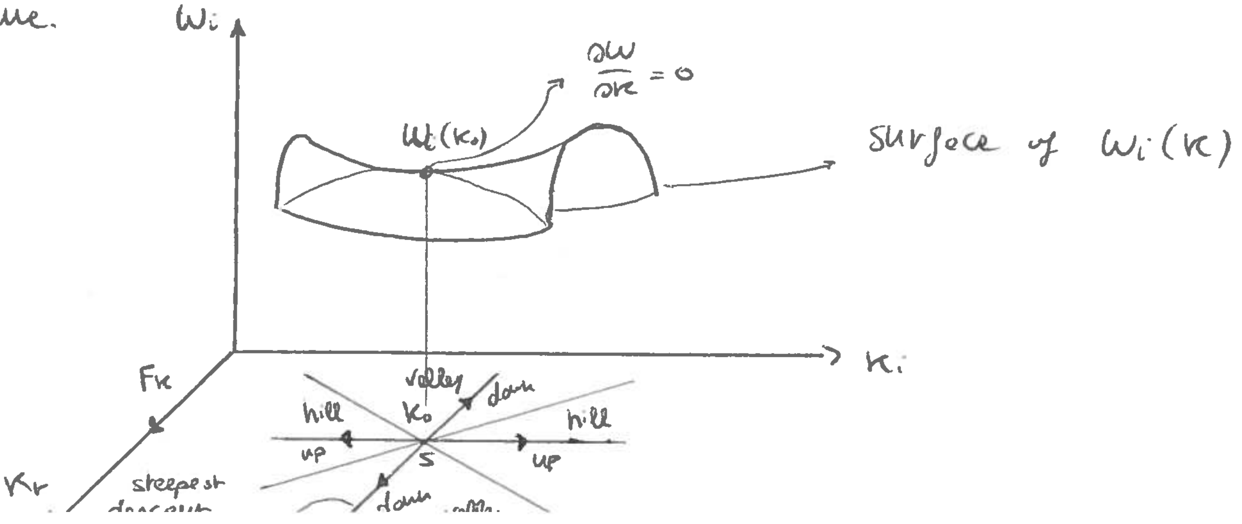
$$\underbrace{\frac{\partial \omega(k)}{\partial k} = 0}_{\text{Complex group velocity}} \quad \text{at} \quad \underbrace{k = k_0}_{\text{complex absolute wavenumber}} \quad \text{and} \quad \underbrace{\omega_0 = \omega(k_0)}_{\text{complex absolute frequency}}$$

Complex group velocity → Roughly average velocity of the impulse response (wave packet)

complex absolute frequency → if $\omega_{0,i} > 0 \Rightarrow$ ABSOLUTELY UNSTABLE

VERY IMPORTANT: both ω and k are complex.

REMARK: The point $\omega_0 = \omega_0(k_0)$ forms a saddle point a er complex k plane.



CRITERION (ABSOLUTE INSTABILITY)

$$G(x,t) \sim e^{i(k_0 x - \omega_0 t)}$$

Procedure: Assumption of normal mode solution $\rightarrow$ apply criterion $\rightarrow$ find k_0 $\rightarrow$ find ω_0.

From the definition of absolute instability, the growth rate along $\frac{x}{t} = 0$ is given by ω_{0i}, called absolute growth rate. In general, denoting with k_{max} the wave number that gives the maximum growth rate:

- $\omega_i(k_{max}) < 0$: LINEARLY STABLE
- $\omega_i(k_{max}) > 0$ and $\omega_{0,i} < 0$: CONVECTIVELY UNSTABLE
- $\omega_i(k_{max}) > 0$ and $\omega_{0,i} > 0$: ABSOLUTELY UNSTABLE

Remember, from the dispersion relation $\rightarrow$ $D(k,\omega) = 0$ $\rightarrow$ $\omega(k)$

LECTURE 10

APPLICATION TO G-L EQUATION

We remark that the equation of interest is complex and linear.
Starting from the dispersion relation, we have: (normal mode solution)

$$D(k,\omega)=0 \longrightarrow \omega(k) = Uk - C_d k^2 + i(\mu - k^2)$$

- LINEAR STABILITY. (temporal)

We calculate the maximum growth rate by checking all real k. We observe that:

$$k_{max} = 0 \Longrightarrow \omega(k_{max}) = i\,\omega_{i,max} = i\mu$$

$\downarrow$
wavenumber for maximum growth

This means that $\omega_i > 0$ if $\mu > 0$

- ABSOLUTE INSTABILITY

We calculate the absolute growth rate. With reference to page 74, we perform the calculation:

$$\frac{\partial \omega}{\partial k} = 0 \;, \quad \omega(k) = Uk - C_d k^2 + i(\mu - k^2)$$

$\downarrow$

$$\frac{\partial \omega}{\partial k} = U - 2C_d k_0 - 2ik_0 = 0 \longrightarrow k_0 = \frac{U}{2(C_d+i)}$$

$\downarrow$
complex absolute wavenumber

We can then find the complex absolute frequency by substituting.

$$\omega_0 = \omega(k_0) = U\cdot\frac{U}{2(C_d+i)} - C_d\frac{U^2}{4(C_d+i)^2} + i\left(\mu - \frac{U^2}{4(C_d+i)^2}\right)$$

$$\omega_0 = \frac{U^2}{2(C_d+i)}\cdot\frac{(C_d-i)}{(C_d-i)} - \frac{C_dU^2}{4(C_d+i)^2}\frac{(C_d-i)^2}{(C_d-i)^2} + i\left(\mu - \frac{U^2}{4(C_d+i)^2}\frac{(C_d-i)^2}{(C_d-i)^2}\right)$$

$$w_0 = \frac{U^2(C_d - i)}{2(C_d^2+1)} - \frac{C_d U^2 (C_d - i)^2}{4(C_d^2+1)^2} + i\left(\mu - \frac{U^2(C_d-i)^2}{4(C_d^2+1)^2}\right)$$

$$w_0 = \frac{U^2(C_d - i)}{2(C_d^2+1)} - \frac{U^2(C_d - i)^2 (C_d + i)}{4(C_d^2+1)^2} + i\mu$$

$$w_0 = \frac{U^2(C_d - i)}{2(C_d^2+1)} - \frac{U^2(C_d - i)\cancel{(C_d^2+1)}}{4(C_d^2+1)^{\cancel{2}}} + i\mu$$

$$w_0 = \frac{U^2(C_d - i)}{2(C_d^2+1)} - \frac{U^2(C_d - i)}{4(C_d^2+1)} + i\mu$$

$$w_0 = \frac{U^2(C_d - i)}{4(C_d^2+1)} + i\mu$$

$$w_0 = \frac{C_d U^2}{4(C_d^2+1)} + i\left(\mu - \frac{U^2}{4(C_d^2+1)}\right)$$

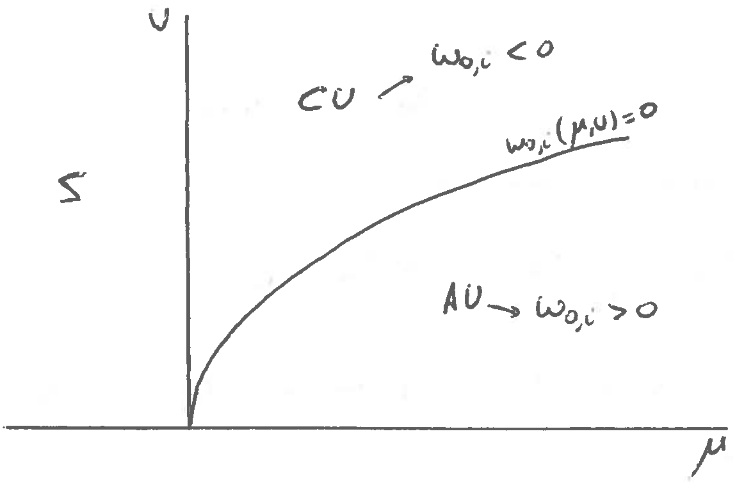
The theorem says that we have absolute instability for $w_{0,i} > 0$.

$$w_{0,i} = \mu - \frac{U^2}{4(C_d^2+1)} > 0$$

The value of the control parameter μ that gives absolute instability is larger than the previous case, and precisely:

$$\mu > \frac{U^2}{4(C_d^2+1)}$$

We can therefore draw the absolute and convective instabilities in the parametric case:



$$w_{0,i} = w_{0,i}(\mu, U) = \mu - \frac{U^2}{4(C_d^2+1)}$$

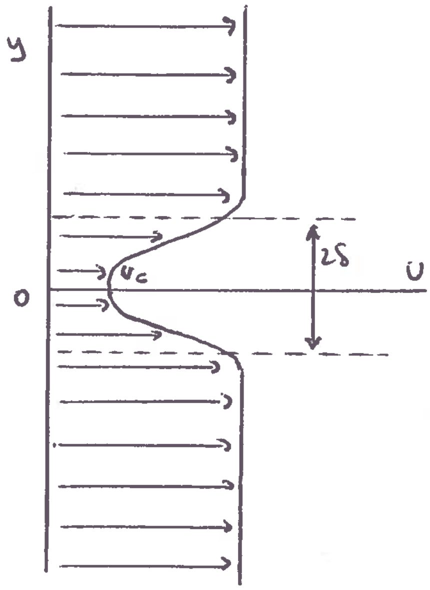
AU vs CU: compensation between instability and advection.

As U increases, the propagation tends to go downstream, therefore advection dominates (with the control parameter μ fixed). Instead, if the instability is stronger than advection, for example for low U, instability dominates and the propagation is also upstream.

APPLICATION TO BLUFF-BODY WAKE

PARALLEL WAKE

We consider the following family of wake profiles:



$$U(y) = U_\infty + (U_\infty - U_c)\, U_1\left(\frac{y}{\delta}; N\right)$$

where: $\xi = \dfrac{y}{\delta}$

$$U_1(\xi; N) = \left[1 + \sinh^{2N}\left\{\xi \sinh^{-1}(1)\right\}\right]^{-1}$$

with $Re = \dfrac{\bar{U}\delta}{\nu}$, $\bar{U} = \dfrac{U^\infty + U_c}{2}$

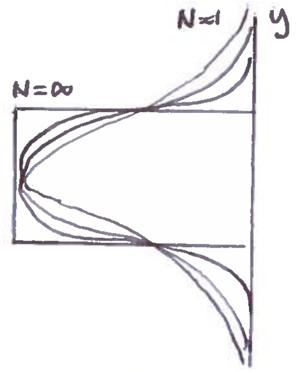
N : stiffness.

This type of analysis can be performed on a varied number of situations. In fact, we define a VELOCITY RATIO R:

$$R = \frac{U_c - U_\infty}{U_c + U_\infty}$$

We therefore generate all possible profiles, combining R and N:

$-1 < R < 0$ $\qquad$ $R = 0$ $\qquad$ $R < -1$



N: Stiffness

We then consider the Orr-Sommerfeld equation:

$$\left[(-i\omega + ikU)(D^2 - k^2) - ikD^2U - \frac{1}{Re}(D^2 - k^2)^2\right]\tilde{v} = 0$$

Solving this numerically we obtain the complex dispersion relation

$D(k, \omega) = 0 \quad \longrightarrow \quad \omega(k) = 0.$

$\downarrow \downarrow$

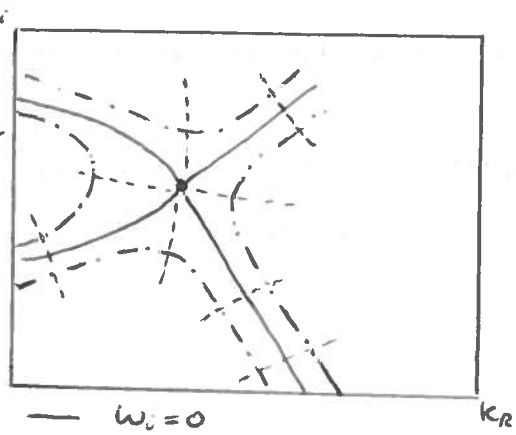
Complex

This is subject to exponential - decay boundary at $y = \pm\infty$.

We then search numerically, for each combination of R, N, Re, the saddle point, that analytically would correspond to $\frac{\partial \omega}{\partial k} = 0$.
The parameter k_0 that satisfies $\frac{\partial \omega}{\partial k} = 0$ allows us to find $\omega(k_0)$, and therefore analyze the behaviour of the flow.

For example fixing $R = -1$, $N = 2$, $Re = 11.3$, Monkewitz (1988) obtained:

$$\omega - \omega_0 \sim \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2}(k_0)(k-k_0)^2$$



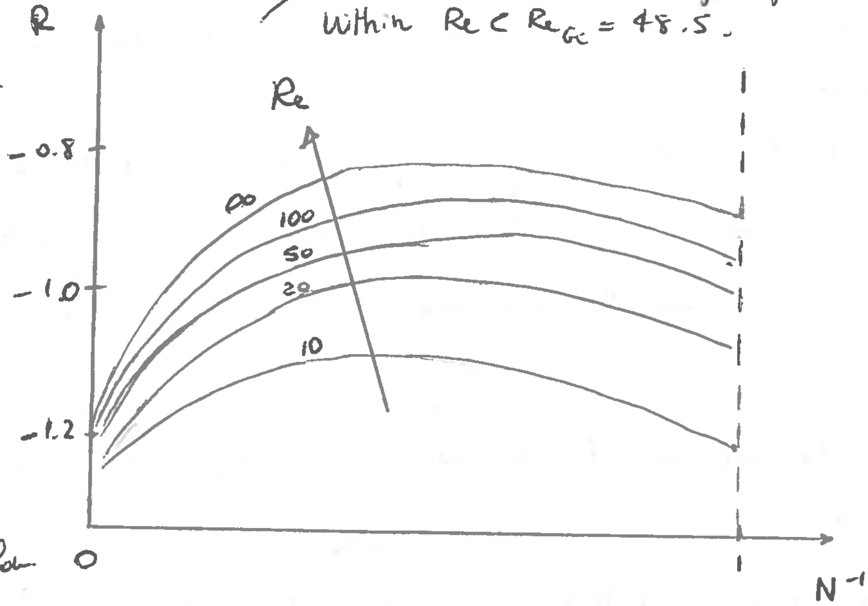
$\omega_0 = 1.008 + 0i$

It is possible to graphically appreciate the saddle point.

By varying the three parameters above, we can get families of curves.
Qualitatively, the effect of velocity ratio, stiffness and Reynolds number is the following:

NOTE: In the inviscid limit ($Re = \infty$) the onset of instability first takes place at $N \sim 2$ for a coflowing wake $R = -0.85$. Thus, an appropriately shaped velocity profile may undergo a transition to absolute instability even in the absence of counterflow. Viscous effects delay the onset.

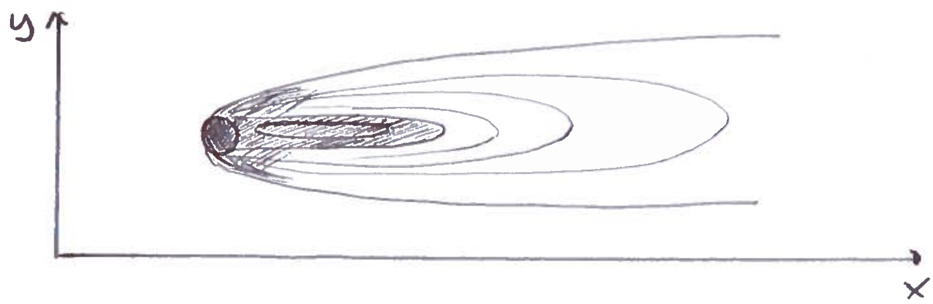
This representation is faithful to experiments within $Re < Re_{Gc} = 48.5$.



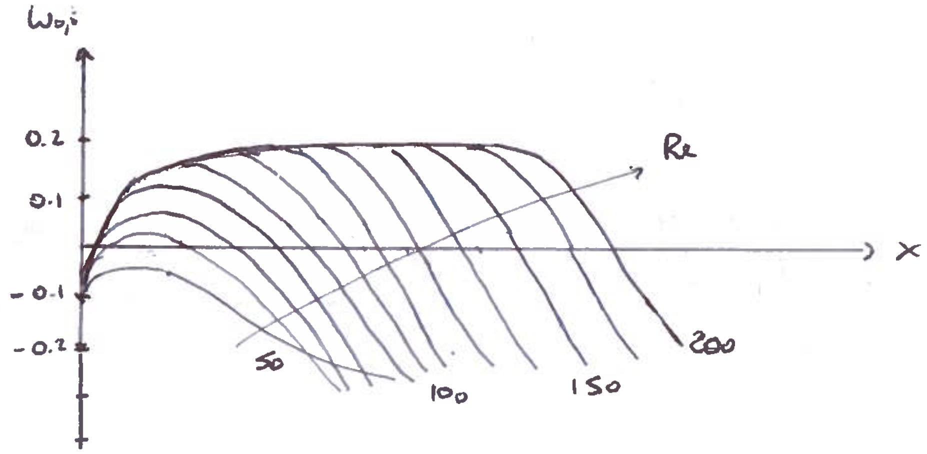
Monkewitz (1988)

NON-PARALLEL CYLINDER WAKE

We picture a base profile at $Re = 100$, $Re = \frac{U_\infty d}{\nu}$



We then perform the analysis at each streamwise location starting from the base flow.
It is then possible to plot a family of curves representing $\omega_{0,i}$ as a function of x for every Re number of interest. The result looks like this:



Instability is <u>local</u>, as the curves become negative as x grows.

From this graph it is possible to distinguish between absolute and convective instability.
We observe that AU appears in the near-wake region at $Re \cong 25$;
Instead, vortex shedding appears at $Re = 47$. This because the analysis we have performed assumes parallel flow, which explains the discrepancy.
Vortex shedding corresponds to global instability, that is temporally growing instability of non-parallel flow.
There is a paper extending this type of analysis to non-parallel flow. Local AU is a necessary condition for global instability situation. Without non-parallel flow, the wake would have to have vortex shedding at $Re = 25$

The emergence of vortex s

- $5 < Re < 25$ Same region
- $25 < Re < 47$ Same region
  ↓
  recircul
- $Re \approx 47$ Strong local
  in the form

REMARK:

local absolute instability

global instability of d

↓

(in this case, vortex sh

The streamwise station most susceptible to absolute instability is located one diameter downstream of the cylinder axis, at the point of maximum counter-flow velocity, in the same cross-stream plane as the steady vortex centres within the recirculation bubble.

- In the interval $5 < Re < 25$ corresponding to $2 < Re < 10$, the sinuous mode becomes locally convectively unstable in a gradually increasing streamwise domain around $x = d$.
- In the interval $25 < Re < 48.5$ corresponding to $10 < Re < 16$, a pocket of absolute local instability is nucleated around $x = d$, its streamwise extent increasing with Re to cover a larger and larger portion of the recirculation bubble.

- local instability point of view: 1) locally stable everywhere 2) local convective instability; 3) local absolute instability embedded within a convectively unstable domain
- global instability point of view: single transition to oscillatory regime at $Re = Re_{cr}$ via supercritical Hopf bifurcation

PHYSICAL IMPLICATIONS: OSCILLATOR VS. AMPLIFIER FLOWS

REMARK 1: CONVECTIVE INSTABILITY

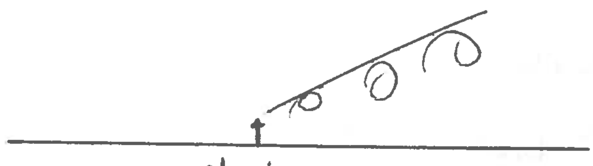
- The reference control volume returns to the original state after the impulse moves away downstream. This happens in a fashion similar to transient growth.

  ↓

  Convection term $U \frac{d}{dx}$ is the source of non-normality of LNS equations.

- Spatial stability analysis becomes meaningful in this situation.
- Instability dynamics is driven by upstream noise $\Leftrightarrow$ IF there no noise, there is no downstream instability.

This kind of flow is defined as a <u>noise - amplifier</u>



at $t = 0$

$t = \Delta t$

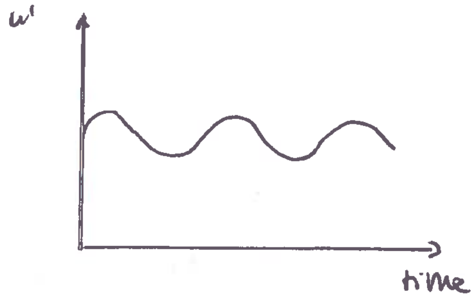
$\vdots$

$t = n\Delta t$

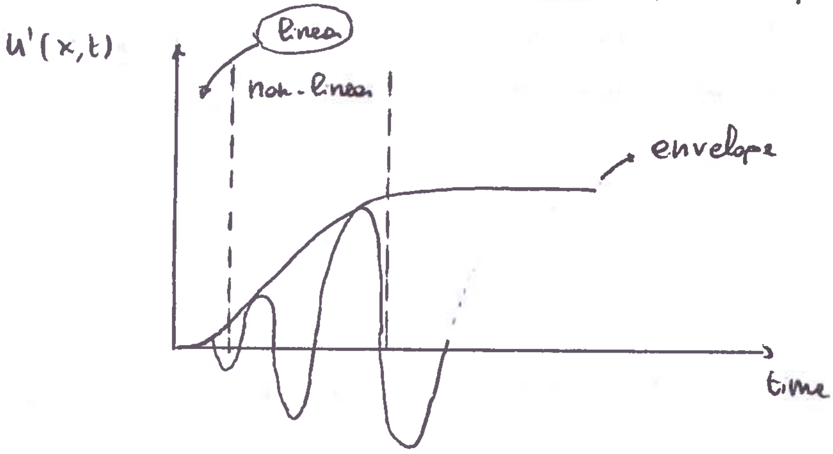
EXAMPLES: Boundary layer, cold jet, co-flowing mixing layer.

REMARK 2: ABSOLUTE INSTABILITY

- The reference control volume never returns to the original state: any perturbation contaminates the region of interest
- Spatial stability analysis becomes meaningless in this situation: there is no reference point for comparison of instability. Switch-on transient contaminates the whole flow.
- Instability dynamics is <u>intrinsically</u> driven by the given system and often results in a nonlinear oscillation with a distinct frequency $\Rightarrow$ <u>OSCILLATOR</u>

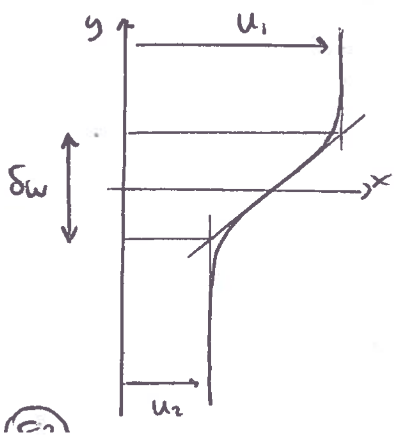


observe that nonlinearity will produce a damping mechanism:



EXAMPLES: wake, hot jet, counter-flowing mixing layer.

EXAMPLE: <u>MIXING LAYER</u>



Base flow profile:

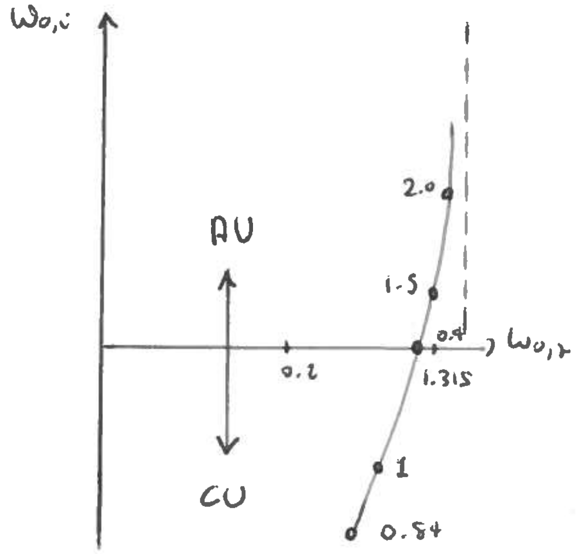
$$U(y) = \bar{U} + \frac{\Delta U}{2} \tanh\left(\frac{2y}{\delta_w}\right)$$

velocity ratio: $R = \dfrac{U_1 - U_2}{U_1 + U_2} = \dfrac{\Delta U}{2\bar{U}}$

$$\delta_w = \frac{\Delta U}{(dU/dy)_{max}}$$

Note: Absolute and convective instabilities are introduced in a paper by Huerre & Monkewitz (1985), published on JFM.

• **THEORY**: TRANSITION FROM CONVECTIVE TO ABSOLUTE INSTABILITY WITH R.

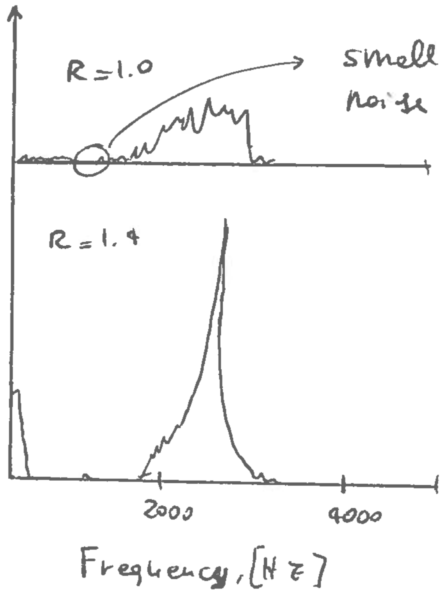


$$R = \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$$

The transition between absolute and convective happens at $\underline{R = 1.315}$

• **EXPERIMENT**: SHIFT FROM NOISE AMPLIFIER TO OSCILLATOR WITH R.

BROAD-BAND SIGNAL (EXPERIMENTAL NOISE)



Velocity power spectra (linear scale) measured in the jet shear layer drawn for different values of the velocity ratio R.

Square of saturation amplitude at fixed spatial position x/d = 0.25 versus velocity ratio R, where d is the jet diameter at the exit plane

--- transition point to oscillator.