Lecture 8

Non-modal stability analysis II

AE209 Hydrodynamic stability

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1. **Non-modal growth in inviscid and viscous flows**
2. **Optimal transient growth**
3. **Orr mechanism and lift-up effect**

Algebraic instability in inviscid flow

Linearised Euler equation with a streamwise uniform disturbance

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+v'\frac{dU}{dy}=-\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}=-\frac{\partial p'}{\partial y}$$

with

$$u'(t=0,y)=u'_0(y)$$

$$v'(t=0,y)=v'_0(y)$$

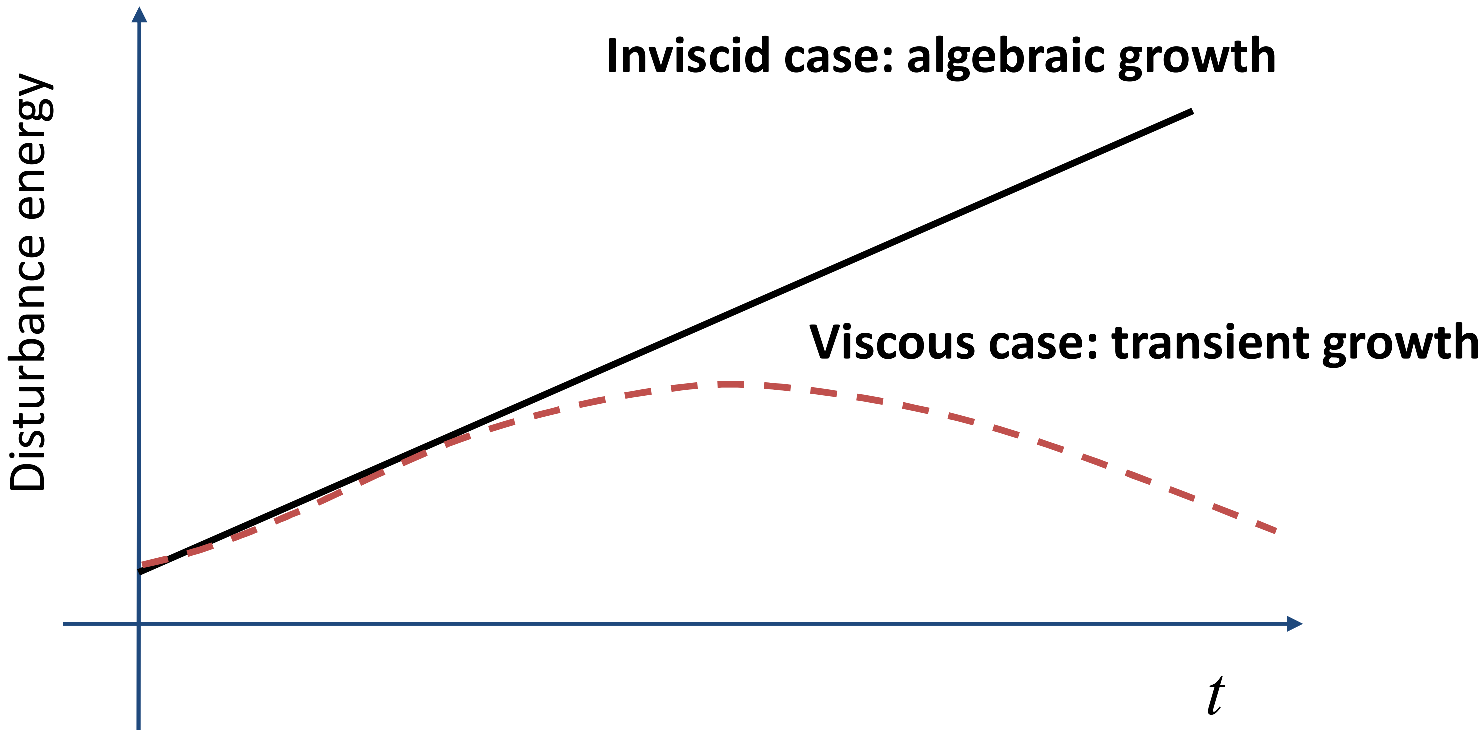
Solution

$$u'(y,t)=u'_0-v'_0\frac{dU}{dy}t,\quad v'(y,t)=v'_0$$

Non-normal energy growth in inviscid and viscous flows

Solution

$$u'(y,t) = u'_0 - v'_0 \frac{dU}{dy} t, \qquad v'(y,t) = v'_0$$



Lecture outline

1. Non-modal growth in inviscid and viscous flows
2. **Optimal transient growth**
3. Orr mechanism and lift-up effect

How to quantify transient growth?

Question

What initial disturbance leads to largest transient growth at a given time?

Optimal initial disturbance problem

$$\max_{\hat{\mathbf{u}}_0} \frac{\left\|\hat{\mathbf{u}}(t;\alpha,\beta)\right\|^2}{\left\|\hat{\mathbf{u}}_0(\alpha,\beta)\right\|^2} \quad \text{with} \quad \left\|\hat{\mathbf{u}}\right\|^2 = \int_\Omega \left|\hat{u}\right|^2 + \left|\hat{v}\right|^2 + \left|\hat{w}\right|^2 dy$$

subject to

$$\frac{\partial}{\partial t}\begin{bmatrix} k^2 - D^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} + \begin{bmatrix} L_{OS} & 0 \\ i\beta DU & L_{SQ} \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} = 0$$

where $k^2 = \alpha^2 + \beta^2$

$$L_{OS} = i\alpha U(k^2 - D^2) + i\alpha D^2 U + \frac{1}{\mathrm{Re}}(k^2 - D^2)^2$$

$$L_{SQ} = i\alpha U + \frac{1}{\mathrm{Re}}(k^2 - D^2)$$

Optimal transient growth for Poiseulle flow

$$G(t) = \max_{\hat{\mathbf{u}}_0} \frac{\left\| \hat{\mathbf{u}}(t;\alpha,\beta) \right\|^2}{\left\| \hat{\mathbf{u}}_0(\alpha,\beta) \right\|^2}$$

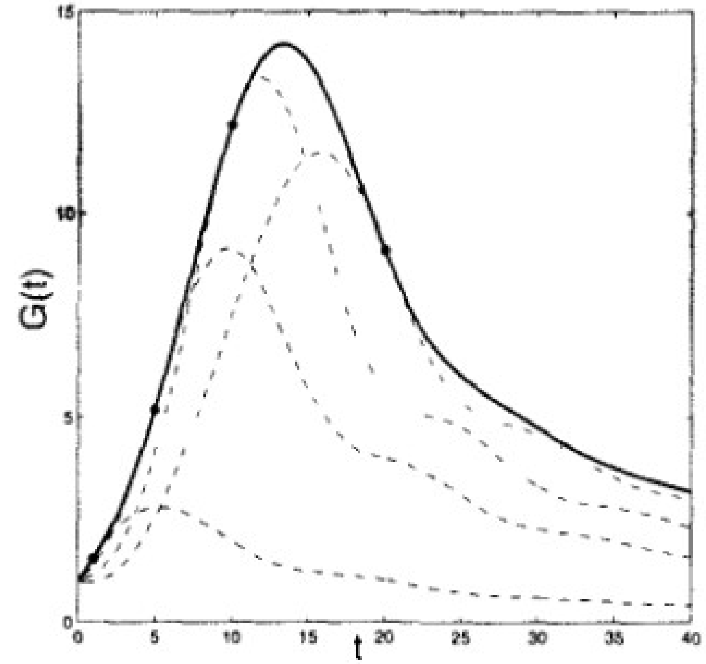


FIGURE 4 2 Amplification $G(t)$ for Poiseuille flow with Re $= 1000$, $\alpha = 1$ (solid line) and growth curves of selected initial conditions (dashed lines)

Maximum growth for Poiseulle flow in the wavenumber plane

$$G_{\max} = \max_t G(t) = \max_t \max_{\hat{\mathbf{u}}_0} \frac{\left\|\hat{\mathbf{u}}(t;\alpha,\beta)\right\|^2}{\left\|\hat{\mathbf{u}}_0(\alpha,\beta)\right\|^2}$$

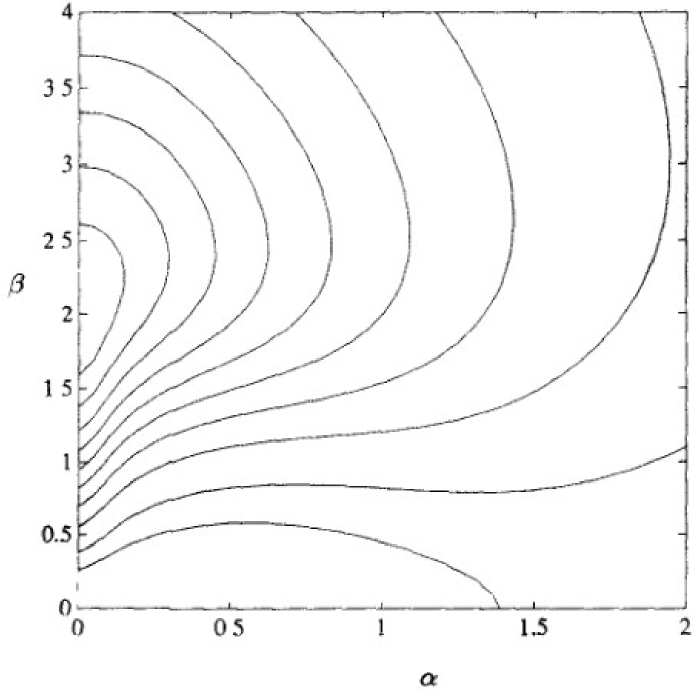
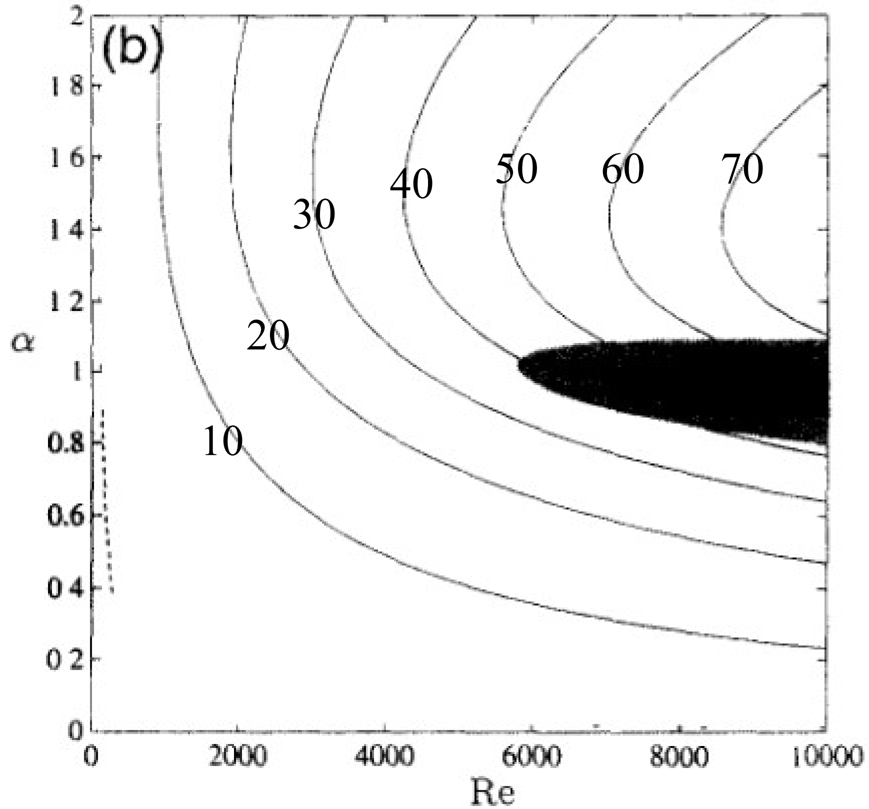


FIGURE 4 4 Contours of G_{max} for Poiseuille flow with Re = 1000 The curves from outer to inner correspond to $G_{max} = 10, 20, 40, \ldots, 140, 160, 180$. From Reddy & Henningson (1993)

Optimal transient growth

Maximum growth with the Reynolds number for Poiseulle flow $(\beta = 0)$

$$G_{\max} = \max_t G(t) = \max_t \max_{\hat{\mathbf{u}}_0} \frac{\left\|\hat{\mathbf{u}}(t;\alpha,\beta)\right\|^2}{\left\|\hat{\mathbf{u}}_0(\alpha,\beta)\right\|^2}$$



Scaling of optimal transient growth

Scaling with the Reynolds number

Flow configurations	G_{max}	t_{max}	α	β
Couette flow	0.20 Re^2	0.076 Re	35/Re	2.04
Poiseulle flow	1.18 Re^2	0.117 Re	0	1.6
Pipe flow	0.07 Re^2	0.048 Re	0	1
Boundary layer	1.50 Re^2	0.778 Re	0	0.65

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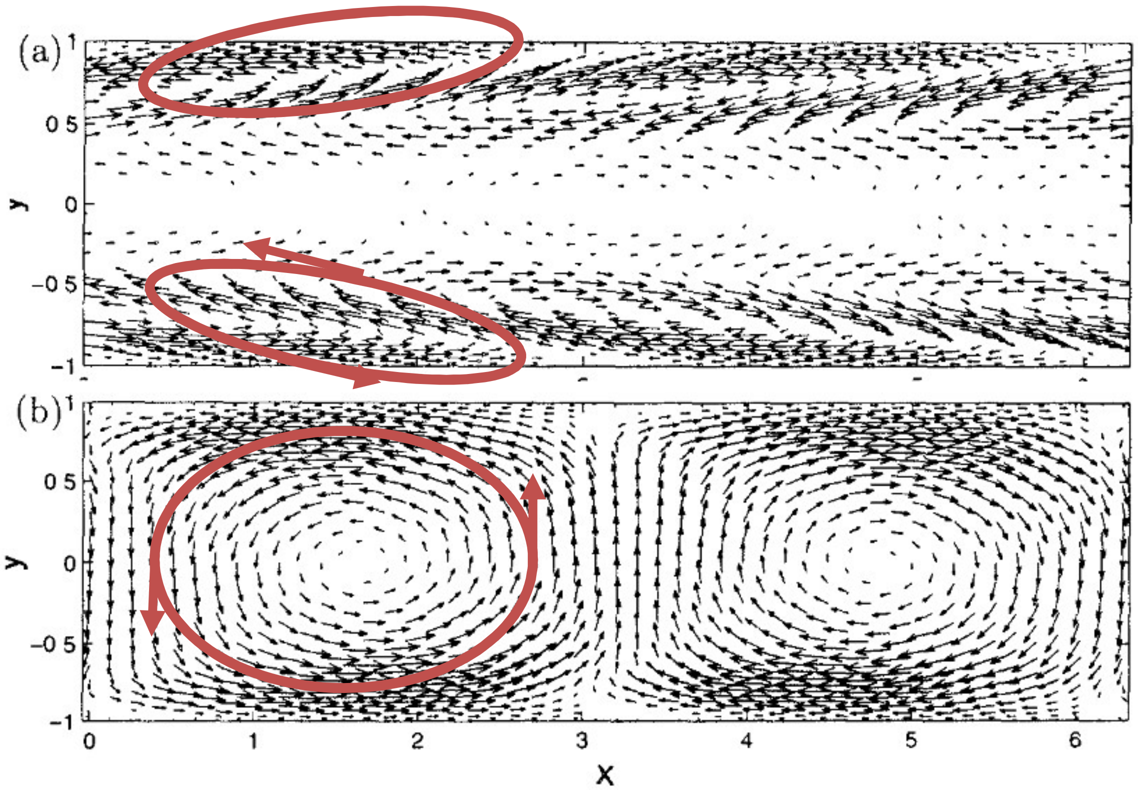
The structure of optimal disturbance

Optimal disturbance in two dimensional case (Poiseulle flow)



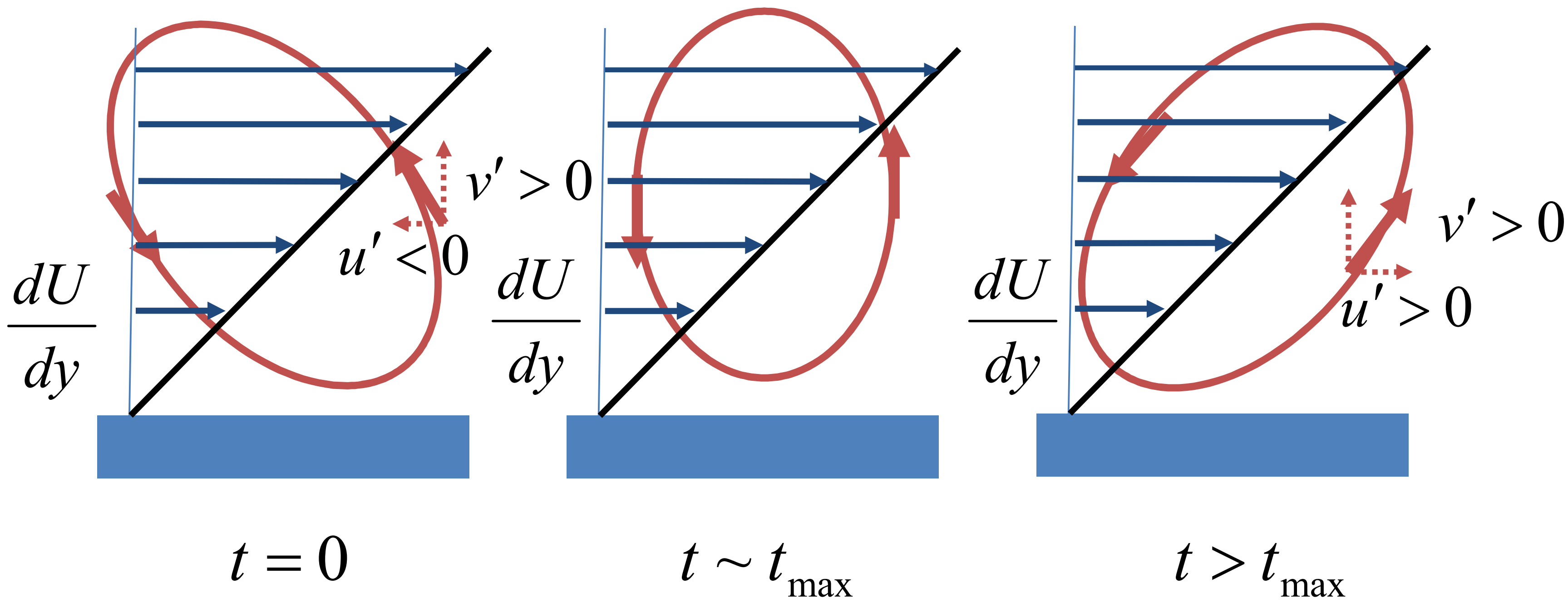
$t = 0$

$t = t_{\max}$

$\alpha = 1, \beta = 0, \mathrm{Re} = 1000$

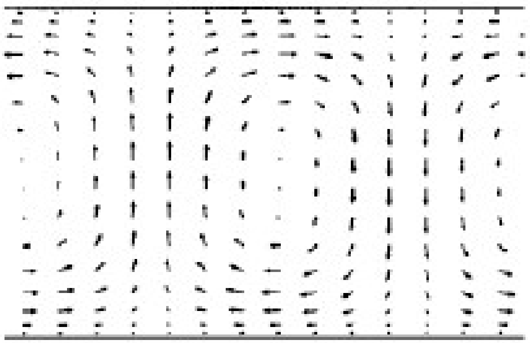
The Orr mechanism: tilting by mean shear

$$\frac{dE}{dt} \sim -\int_V u'v' \frac{dU}{dy} dV$$

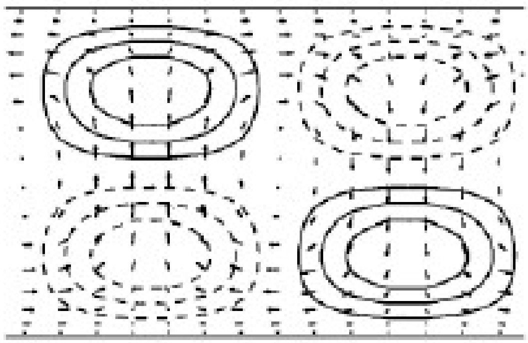


Optimal disturbance in three dimensional flow

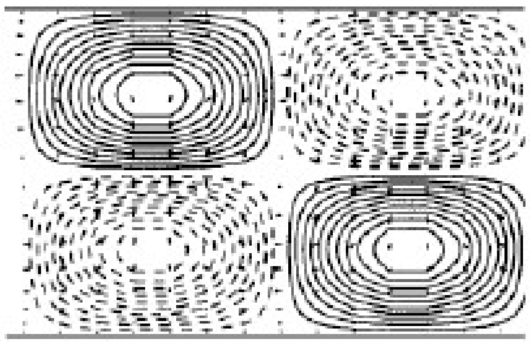
Optimal disturbance in three dimensional case (Poiseulle flow)



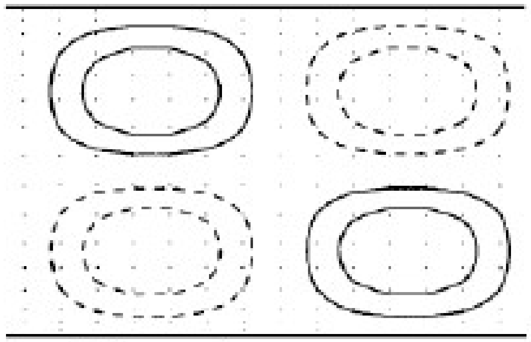
$t = 0$



$t = t_{\max} / 2$



$t = t_{\max}$



$t = 8t_{\max}$

$$\alpha = 0, \beta = 2.044, \mathrm{Re} = 5000$$

Bewley & Liu (1997)

Lift-up effect

Lift-up effect: a vortex tilting mechanism for generation of streaks

$$\frac{D\omega_y}{Dt} \sim \omega_x \frac{dU}{dy}$$

Recall the Orr-Sommefeld-Squire system:

$$\frac{\partial}{\partial t}\begin{bmatrix} k^2 - D^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} + \begin{bmatrix} L_{OS} & 0 \\ i\beta DU & L_{SQ} \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} = 0$$

where $k^2 = \alpha^2 + \beta^2$

$$L_{OS} = i\alpha U(k^2 - D^2) + i\alpha D^2 U + \frac{1}{\mathrm{Re}}(k^2 - D^2)^2$$

1. Optimal transient growth
2. Case study: wall-bounded shear flows