# Lecture 6

## Linear stability of parallel shear flows III

AE209 Hydrodynamic stability
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## Lecture outline



1. Eigenspectra and eigenfunctions
2. Neutral stability curve
3. Spatial stability analysis and vibrating ribbon problem

1. **Eigenspectra and eigenfunctions**

## Review of Orr-Sommerfeld and Squire equations



**Orr-Sommerfeld equation** (for wall-normal velocity):

$$\left[(-i\omega+i\alpha U)(D^2-k^2)-i\alpha D^2U-\frac{1}{\mathrm{Re}}(D^2-k^2)^2\right]\tilde{v}=0$$

**Squire equation** (for wall-normal vorticity):

$$\left[(-i\omega+i\alpha U)-\frac{1}{\mathrm{Re}}(D^2-k^2)\right]\tilde{\eta}=-i\beta DU\,\tilde{v}$$

where $k^2=\alpha^2+\beta^2$ with boundary conditions:

**Infinitely many sets of** $\omega_n$ with the corresponding $\tilde{v}_n$ and $\tilde{\eta}_n$

Search for the most unstable eigenvalue and eigenfunction.

## Eigenspectra and eigenfunctions



### Eigenspectra of plane Poiseulle flow

$$U(y) = 1 - y^2$$

unstable eigenvalue ($c_i > 0$)

Viscosity-driven Instability

↳ Tollmien-Schlichting Instability

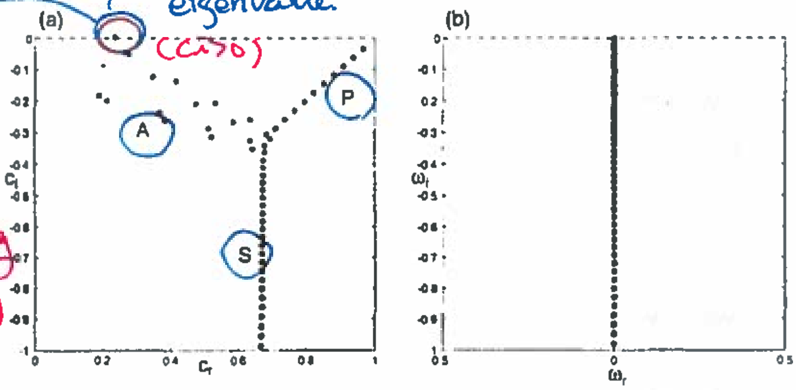



FIGURE 3.1 Orr-Sommerfeld spectrum of plane Poiseuille flow for Re = 10000. (a) wave numbers $\alpha = 1, \beta = 0$. (b) wave numbers $\alpha = 0, \beta = 1$

Schmid & Henningson (2001)

## Eigenspectra and eigenfunctions



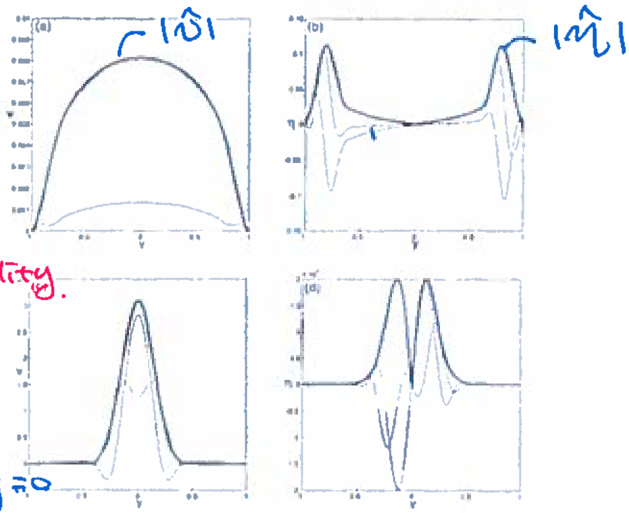
Eigenfunctions of plane Poiseulle flow $(\alpha = 1, \beta = 1, \text{and } \mathrm{Re} = 5000)$

**A branch**

- Small $C_r$.
- maximum of $v$ at $y=0$
- T-S instability.

**P branch**

- large $C_r$.
- Centered around $y=0$
- No instability.




Schmid & Henningson (2001)

S branch

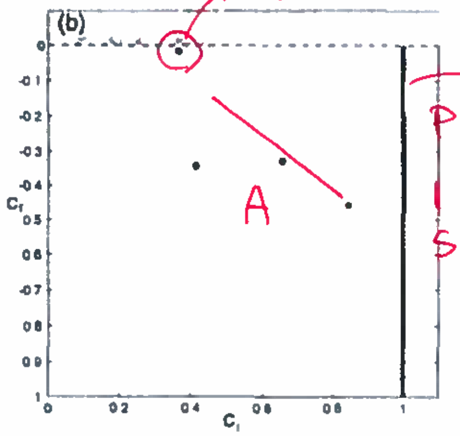
: strongly damped.

## Eigenspectra and eigenfunctions



**Eigenspectra of Blasius boundary layer** $(\alpha = 0.2, \beta = 0, \text{and } \mathrm{Re} = 500)$

T-S mode




continuous spectra (due to unbounded domain)

Always damped.

Schmid & Henningson (2001)

## Eigenspectra and eigenfunctions

**Eigenfunction of Blasius boundary layer** ($\alpha = 0.2, \beta = 0$, and $\mathrm{Re} = 500$)

T-S mode.

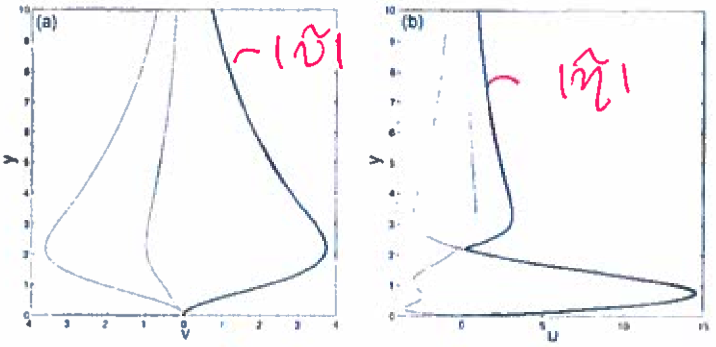



FIGURE 3.5 Eigenfunctions for Blasius boundary layer flow. (a,b) Eigenfunction of the discrete spectrum, vertical (a) and streamwise (b) velocity component for $\alpha = 0.2$, $\mathrm{Re} = 500$. The thick line represents the absolute value of $v$ or $u$, the thin lines represent the real and imaginary part.

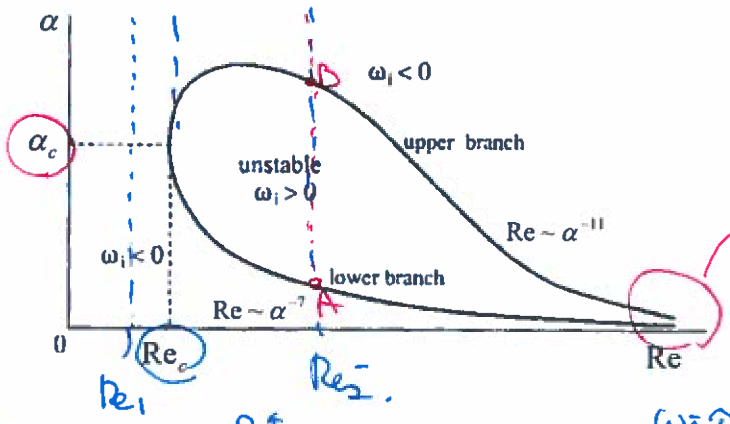
2. **Neutral stability curve**

## Neutral stability curve

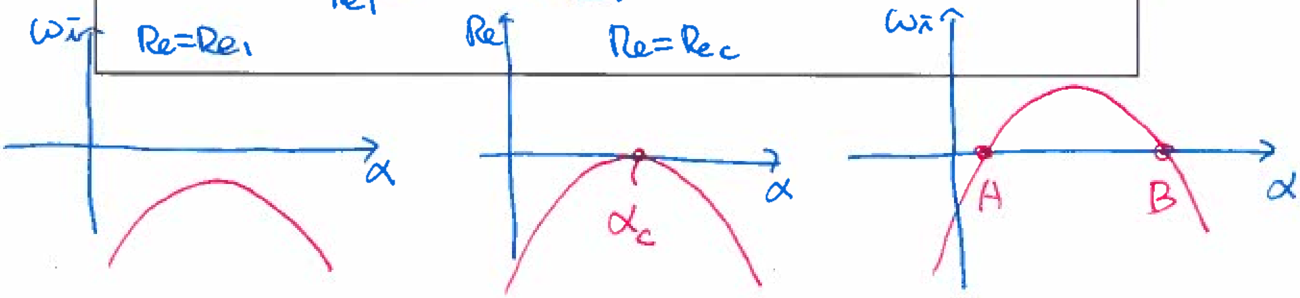


**Schematic structure of neutral stability curve for Poiseulle flow**

$$\omega_i(\alpha, \beta = 0, \mathrm{Re}) = 0$$




Should disappear as $Re \to \infty$.



### Neutral stability curve of Poiseulle flow

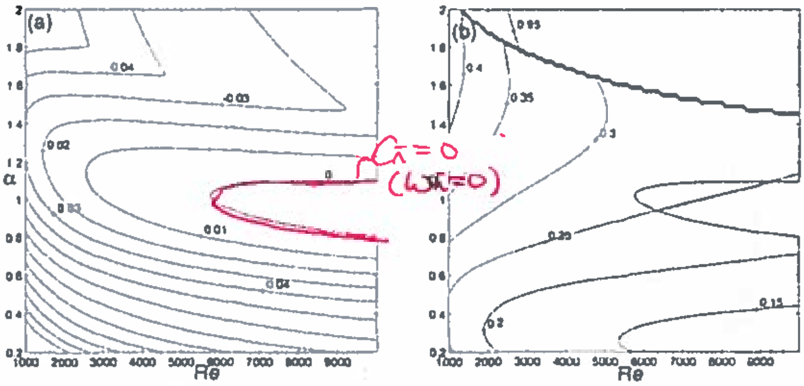



FIGURE 3.8 Neutral curve for plane Poiseuille flow (a) contours of constant growth rate $c_i$, (b) contours of constant phase velocity $c_r$. The shaded area represents the region of parameter space where unstable solutions exist

## Neutral stability curve

### Neutral stability curve of Blasius boundary layer

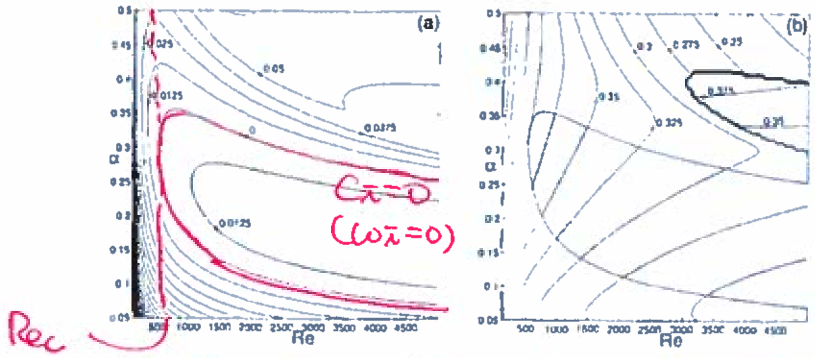



FIGURE 3.9 Neutral curve for Blasius boundary layer flow. (a) contours of constant growth rate $c_i$; (b) contours of constant phase velocity $c_r$. The shaded area represents the region in parameter space where unstable solutions exist.

## Neutral stability curve



**Critical Reynolds numbers and streamwise wavenumbers**

Linear stability analysis

| Flow configurations | Critical Re (Linear stability) | Transition Re | Critical wavenumber | Critical phase speed |
|---|---|---|---|---|
| Couette flow | ∞ | 350-400 | - | - |
| Poiseulle flow | 5772.2 | 1000-2000 | 1.02 | 0.2639 |
| Pipe flow | ∞ | 2000-2500 | - | - |
| Boundary layer | 519.4 | Depends on dist. env. | 0.303 | 0.3935 |

**Remark**

**Linear stability analysis** does **not provide a full explanation** for the onset of **transition.**

3. Spatial stability analysis and vibrating ribbon problem

## Spatial stability analysis



**Normal mode solution (2D case) revisited**

$$v'(x,y,t)=\tilde{v}(y)e^{i\alpha x-i\omega t}+c.c$$

So far, $\alpha \in R$ is given and $\omega \in C$ unknown

$\omega_i > 0$ Linearly unstable $\quad$ $\omega_i < 0$ Linearly stable

Temporal Stability analysis

Now, consider $\omega \in R$ is given and $\alpha \in C$ unknown

$\alpha_i < 0$ Linearly unstable $\quad$ $\alpha_i > 0$ Linearly stable

perturbation grows as $x \to \infty$

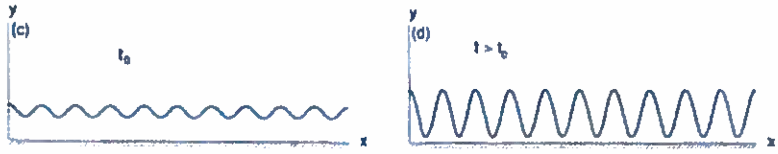
Perturbation decays as $x \to \infty$

Spatial Stability analysis

## Temporal vs spatial stability analysis

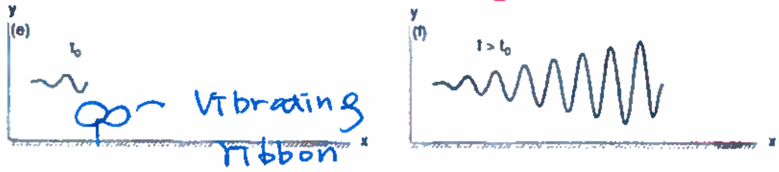


### Temporal stability analysis




⇒ very difficult to implement experimentally

### Spatial stability analysis (Vibrating Ribbon Problem)

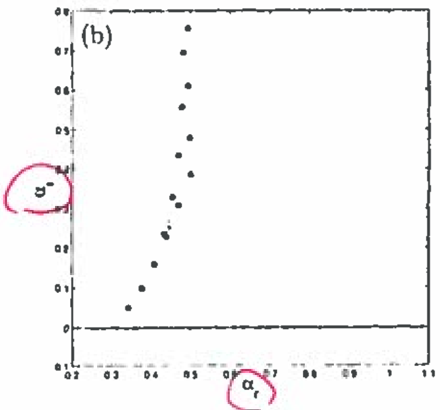



Vibrating ribbon with a given frequency.

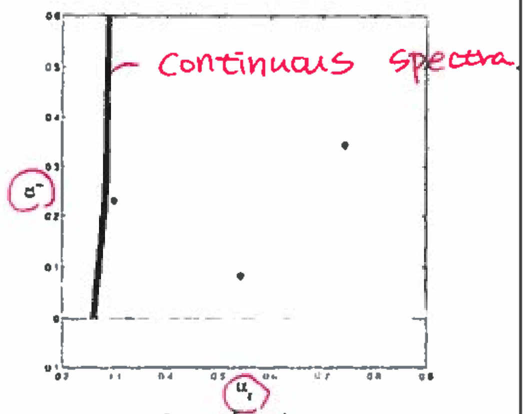
⇒ Easy to do experiment. ↔ difficult to solve.

⇒ Nonlinear eigenvalue problem.

### Eigenspectra of Poiseulle flow and Blasius boundary layer




Poiseulle flow
($\omega = 0.3$ and $\mathrm{Re} = 1000$)



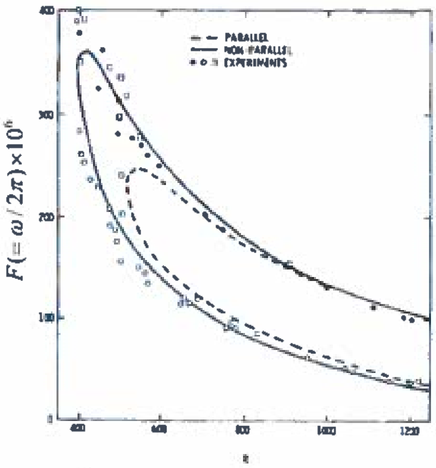

Boundary layer
($\omega = 0.26$ and $\mathrm{Re} = 1000$)

Schmid & Henningson (2001)

## Neutral stability curve



### Neutra stability curve of Blasius boundary layer




$$\mathrm{Re}\left(=\frac{U_\infty \delta^*}{\nu}\right)$$

FIG. 1. Comparison between the neutral stability curves based on parallel and nonparallel stability theories and experimental data—o, data of Schubauer and Skramstad, □, ●, data of Ross *et al*.

Saric & Nayfeh (1975)

## Summary



1. Eigenspectra and eigenfunctions
2. Neutral stability curve
3. Spatial stability analysis and vibrating ribbon problem