Lecture 2

Basic dynamical systems theory I

AEM-ADV12 Hydrodynamic stability

Dr Yongyun Hwang

Lecture outline

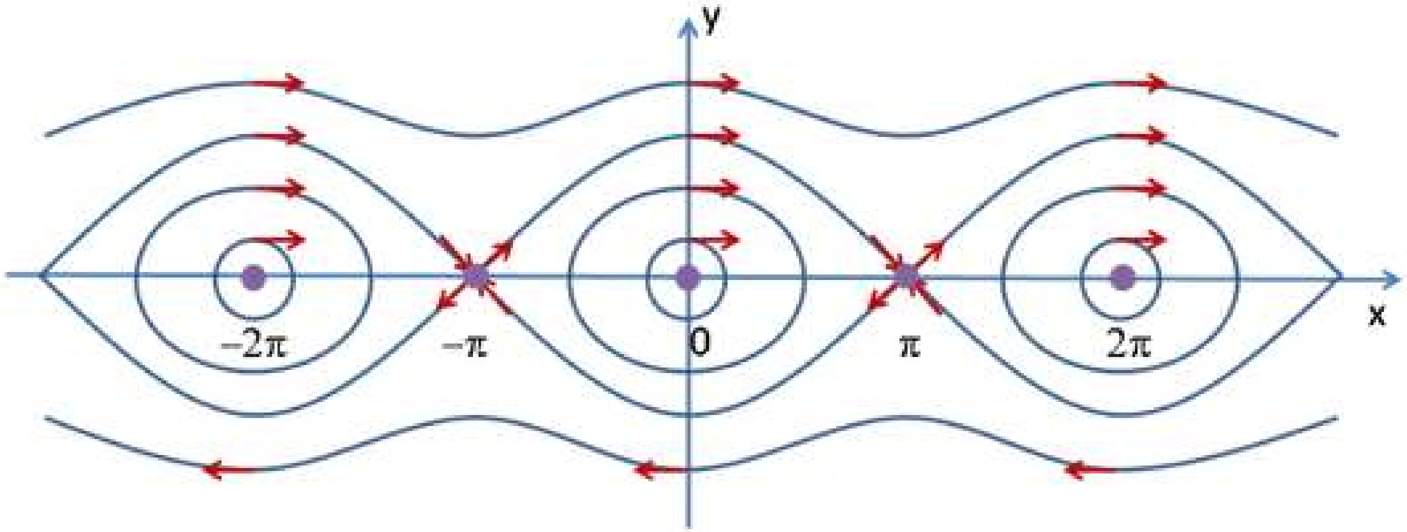
1. Phase portrait and equilibria
2. Linear stability analysis

1. **Phase portrait and equilibria**
2. Linear stability analysis

Phase portrait for a planar (2D) dynamical system

Example: Nonlinear pendulum

$$x = y \quad \text{and} \quad y = -\sin x$$



Equilibrium point (or fixed point)

Definition: Equilibrium point

$\overline{\mathbf{x}}$ is an equilibrium point if $\mathbf{x}(t) = \overline{\mathbf{x}}$ is a solution of the given dynamical system such that

$$\mathbf{f}(\overline{\mathbf{x}}) = \mathbf{0}$$

Equilibrium point (or fixed point)

Example 1: Nonlinear pendulum

$$x = y \quad \text{and} \quad y = \sin x$$

Equilibrium point (or fixed point)

Example 2: Plane Couette flow

$$(\mathbf{U} \cdot \nabla)\mathbf{U} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{U}$$

$$\nabla \cdot \mathbf{U} = 0$$

Lecture outline

1. Phase portrait and equilibria
2. **Linear stability analysis**

Jacobian linearisation

Let $\bar{\mathbf{x}}$ be an equilibrium point such that $\mathbf{f}(\bar{\mathbf{x}}) = \mathbf{0}$. Consider a small perturbation $\boldsymbol{\delta}\mathbf{x}$, i.e. $\mathbf{x} = \bar{\mathbf{x}} + \varepsilon\boldsymbol{\delta}\mathbf{x}$, then the given nonlinear system is approximated by the following linear dynamical system:

$$\frac{d\boldsymbol{\delta}\mathbf{x}}{dt} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\bar{\mathbf{x}}} \boldsymbol{\delta}\mathbf{x}$$

Remark

Linear dynamical system is much easier to analyse.

Example 1

Find the linearised system around the equilibrium point.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ x^2 y \end{bmatrix}$$

Example 2: Linearised Navier-Stokes equation

Find the linearised equation around an equilibrium point (basic state) given by $\mathbf{U}$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

Linear instability

Definition: Linear instability or stability

If the linearised dynamical system around the given basic state $\overline{\mathbf{x}}$ has a solution such that $\left\|\boldsymbol{\delta}\mathbf{x}\right\| \to \infty$ as $t \to \infty$, the basic state is called **linearly unstable**.

Linear stability of a planar dynamical system

Let the linearised system around the basic state $\bar{\mathbf{x}}$ be

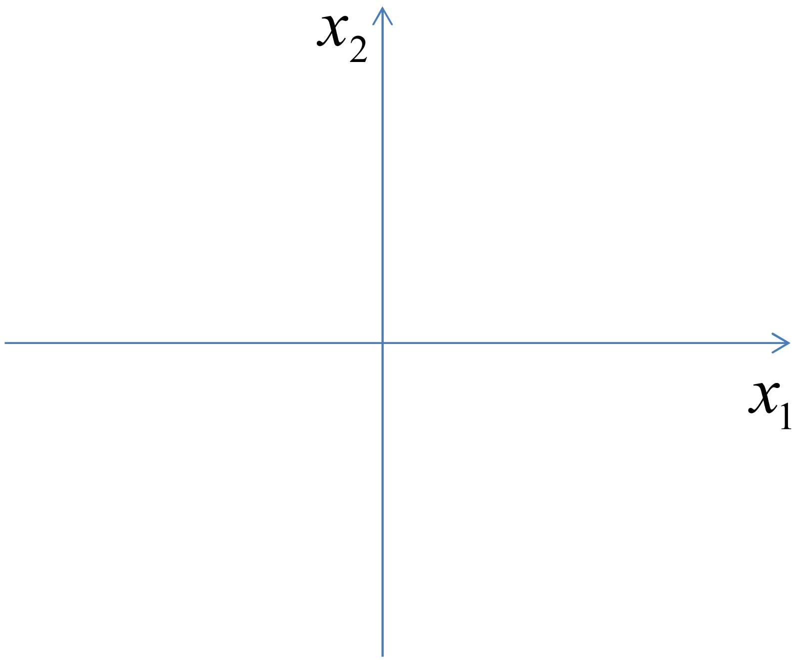
$$\frac{d\boldsymbol{\delta}\mathbf{x}}{dt} = \mathbf{A}\,\boldsymbol{\delta}\mathbf{x} \qquad \text{where} \qquad \mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\bar{\mathbf{x}}}$$

Phase portrait of 2D linear system

Case I

λ_1, λ_2 are both real, and $\lambda_1 \neq \lambda_2$. The two corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are then linearly independent.

$$i)\ \lambda_1, \lambda_2 < 0, |\lambda_1| > |\lambda_2|$$

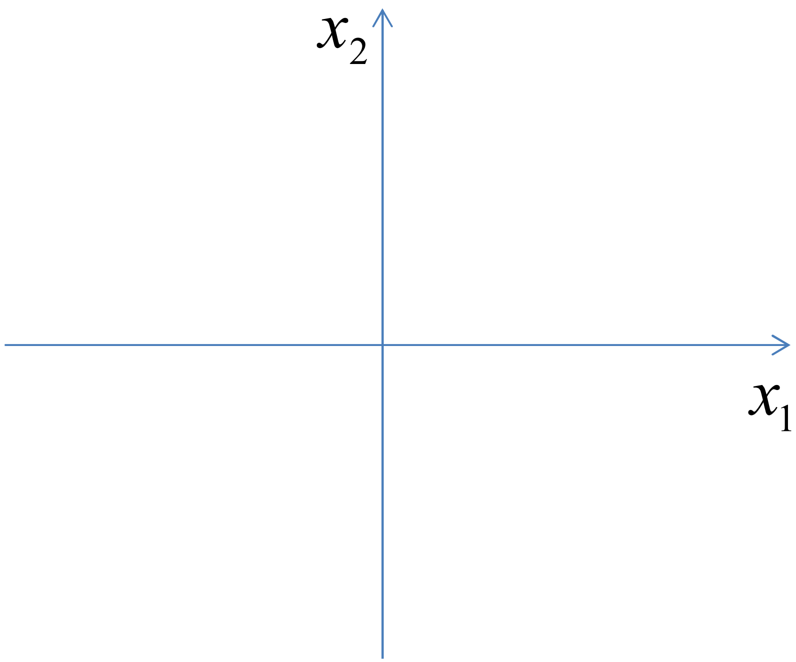


Phase portrait of 2D linear system

Case I

λ_1, λ_2 are both real, and $\lambda_1 \neq \lambda_2$. The two corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are then linearly independent.

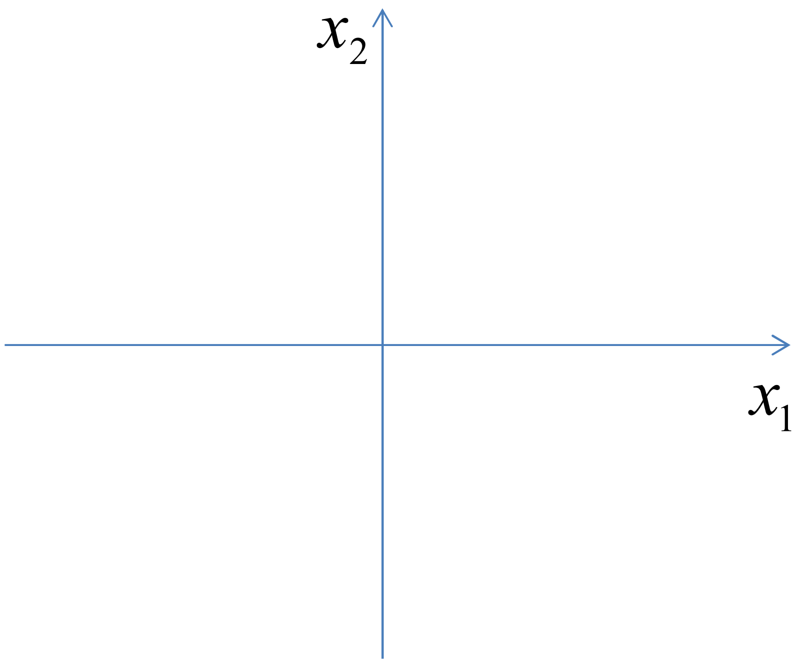
$$ii)\ \lambda_1, \lambda_2 > 0, |\lambda_1| > |\lambda_2|$$



Case I

λ_1, λ_2 are both real, and $\lambda_1 \neq \lambda_2$. The two corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are then linearly independent.

$$iii)\ \lambda_2 < 0 < \lambda_1$$

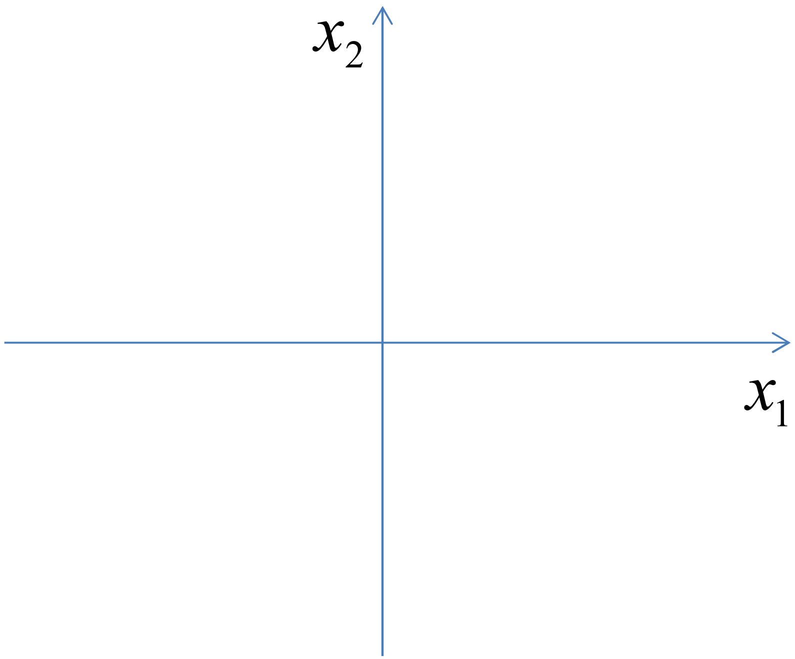


Phase portrait of 2D linear system

Case II

λ_1, λ_2 are both real, and $\lambda_1 = \lambda_2$.

$i)\,\text{rank}(\mathbf{A} - \lambda\mathbf{I}) = 0$

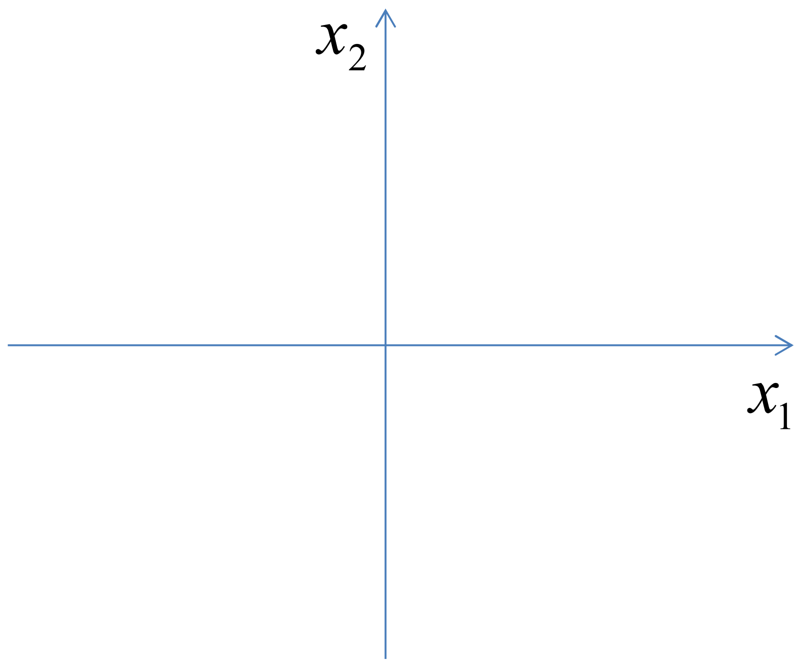


Phase portrait of 2D linear system

Case II

λ_1, λ_2 are both real, and $\lambda_1 = \lambda_2$.

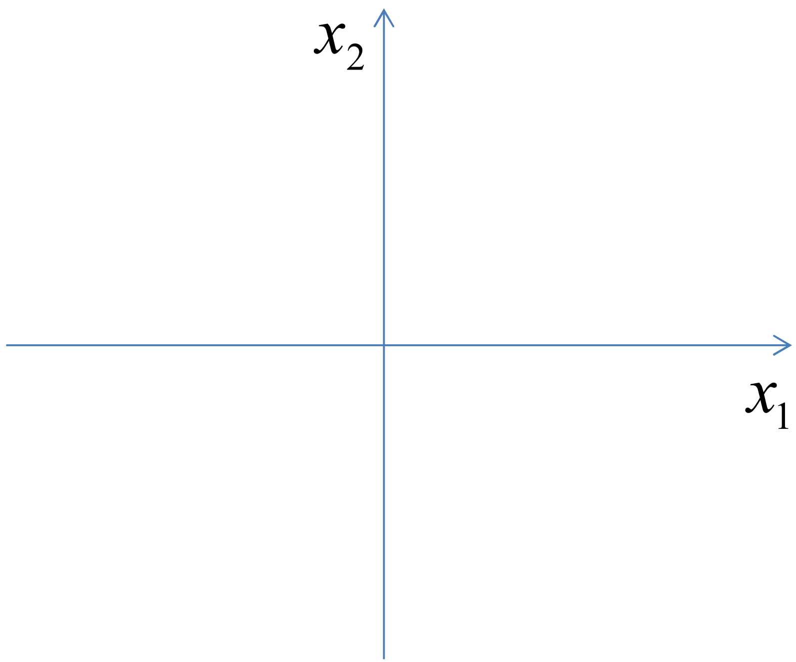
$ii)$ rank$(\mathbf{A} - \lambda\mathbf{I}) = 1$



Case III

λ_1, λ_2 are both complex such that $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

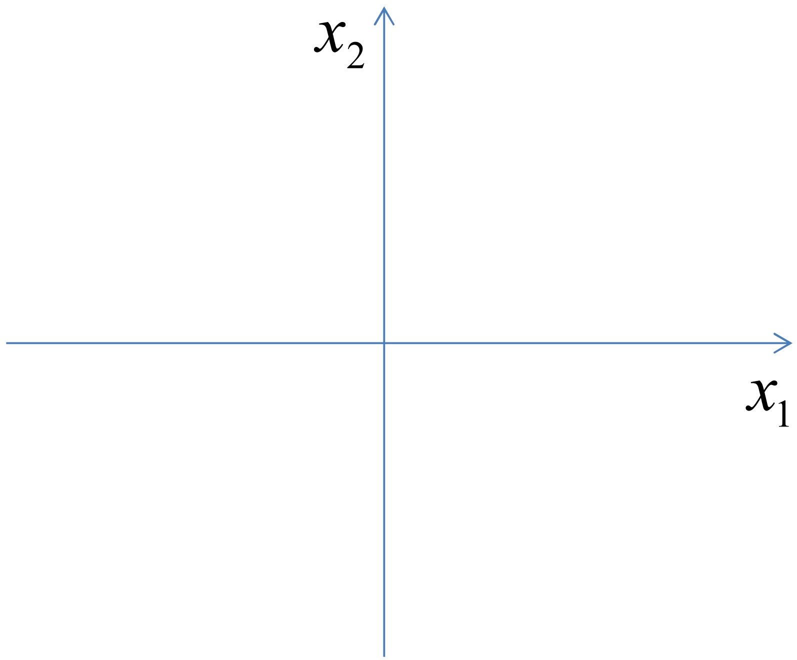
$i)\, \alpha = 0$



Case III

λ_1, λ_2 are both complex such that $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

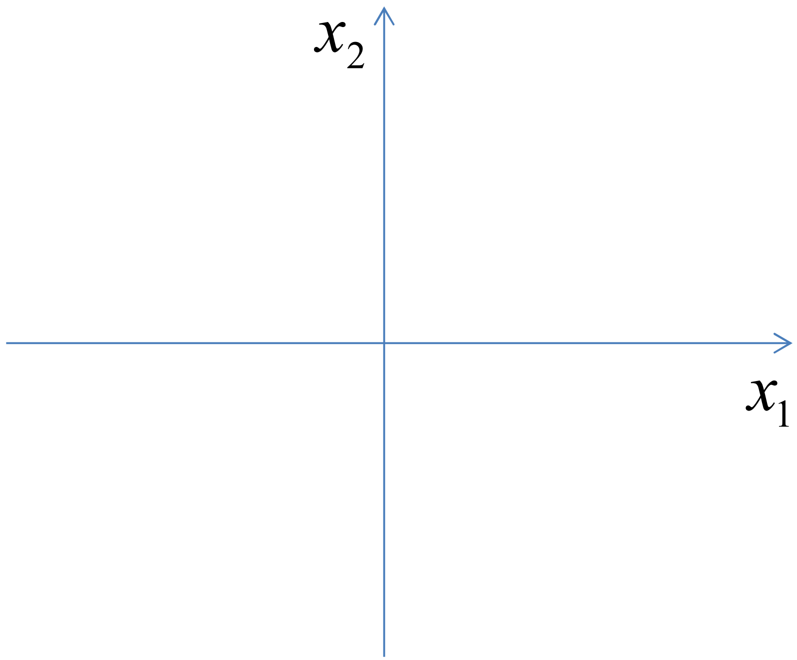
$ii)\, \alpha > 0$



Case III

λ_1, λ_2 are both complex such that $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

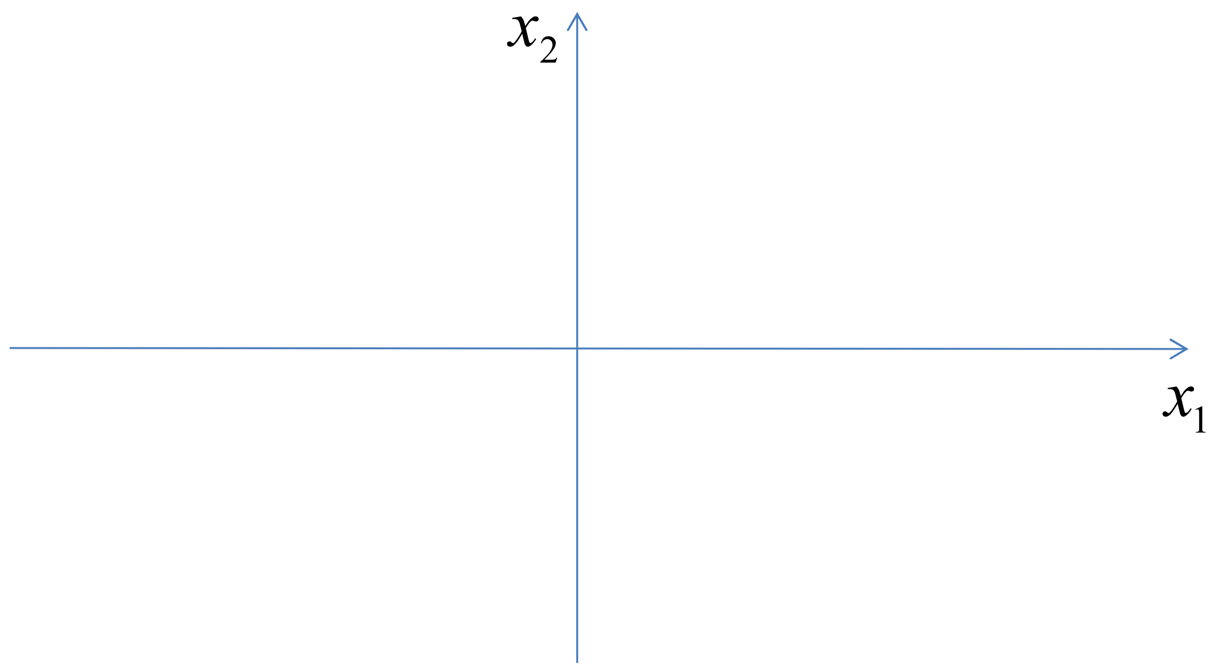
$iii)\ \alpha < 0$



Example: Unforced duffing equation

$$x + x - x + x^3 = 0$$

Phase portrait of 2D linear system



1. **Phase portrait and equilibria**
2. **Linear stability analysis**