Lecture 8

Non-modal stability analysis II

AE209 Hydrodynamic stability
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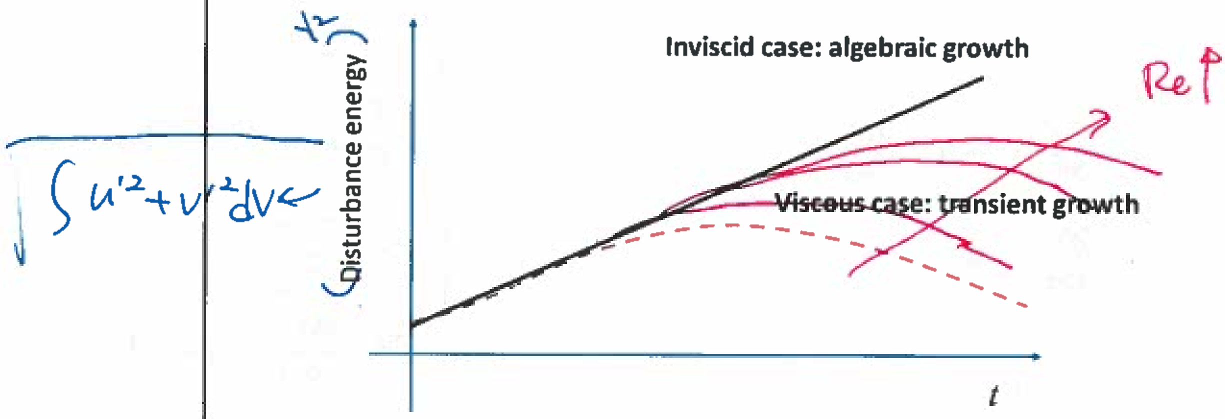
Lecture outline

1. **Non-modal growth in inviscid and viscous flows**
2. **Optimal transient growth**
3. **Orr mechanism and lift-up effect**

Non-normal energy growth in inviscid and viscous flows 4/21

Solution

$$u'(y,t) = u'_0 - v'_0 \frac{dU}{dy} t, \qquad v'(y,t) = v'_0$$



Algebraic instability in inviscid flow

$\frac{\partial}{\partial x} = 0$

Linearised Euler equation with a streamwise uniform disturbance

$$\frac{\partial u'}{\partial t} + U\cancel{\frac{\partial u'}{\partial x}}^{\,0} + v'\frac{dU}{dy} = -\cancel{\frac{\partial p'}{\partial x}}^{\,0} \quad \text{with} \quad u'(t=0,y) = u'_0(y)$$

$$(*) - \quad \frac{\partial v'}{\partial t} + U\cancel{\frac{\partial v'}{\partial x}}^{\,0} = -\cancel{\frac{\partial p'}{\partial y}}^{\,0} \qquad v'(t=0,y) = v'_0(y)$$

① $\frac{\partial u'}{\partial t} = -v'\frac{dU}{dy}$

② $\frac{\partial v'}{\partial t} = 0$

$*$ $\frac{\partial^2 p'}{\partial y^2} = -2\cancel{\frac{\partial U}{\partial y}\frac{\partial v}{\partial x}}^{\,0}$

$\Rightarrow \frac{\partial^2 p'}{\partial y^2} = 0$

$\Rightarrow \frac{\partial p'}{\partial y} = \text{const}$

$\left(\frac{\partial p'}{\partial y} = 0\right.$ at boundary from $\left.(*)\right)$

$\Rightarrow \frac{\partial p'}{\partial y} = 0.$

Solution

$$\underbrace{u'(y,t) = u'_0 - v'_0\frac{dU}{dy}t, \quad v'(y,t) = v'_0}_{\text{Algebraic growth in time}}$$

How to quantify transient growth?

Question

What initial disturbance leads to largest transient growth at a given time?

2. Optimal transient growth

Optimal transient growth

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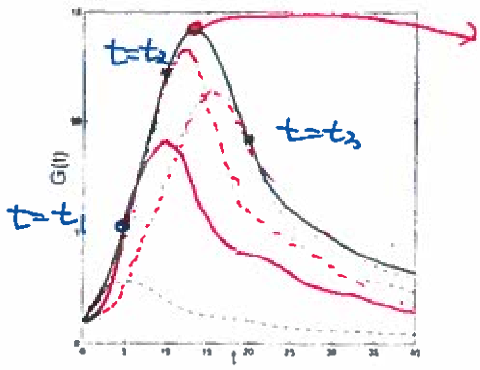
Optimal transient growth for Poiseulle flow

$$G(t) = \max_{\hat{\mathbf{u}}_0} \frac{\|\hat{\mathbf{u}}(t;\alpha,\beta)\|^2}{\|\hat{\mathbf{u}}_0(\alpha,\beta)\|^2}$$



$G_{max} = \max_t G(t).$

FIGURE 4.2 Amplification $G(t)$ for Poiseuille flow with Re = 1000, α = 1 (solid line) and growth curves of selected initial conditions (dashed lines)

Optimal transient growth

Optimal initial disturbance problem

$$\max_{\hat{\mathbf{u}}_0} \frac{\|\hat{\mathbf{u}}(t;\alpha,\beta)\|^2}{\|\hat{\mathbf{u}}_0(\alpha,\beta)\|^2} \quad \text{with} \quad \|\hat{\mathbf{u}}\|^2 = \int_\Omega |\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2 dy$$

subject to

$$\frac{\partial}{\partial t}\begin{bmatrix} k^2 - D^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} + \begin{bmatrix} L_{OS} & 0 \\ i\beta DU & L_{SQ} \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} = 0$$

where $k^2 = \alpha^2 + \beta^2$

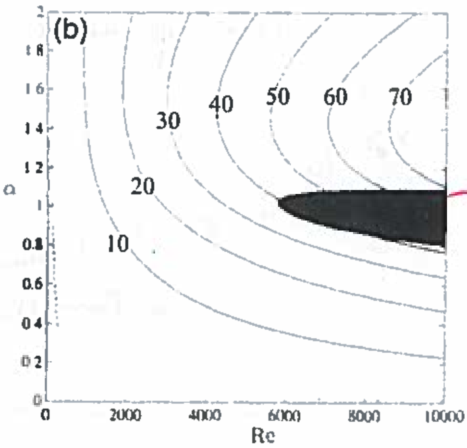
$$L_{OS} = i\alpha U(k^2 - D^2) + i\alpha D^2 U + \frac{1}{\mathrm{Re}}(k^2 - D^2)^2$$

$$L_{SQ} = i\alpha U + \frac{1}{\mathrm{Re}}(k^2 - D^2)$$

Optimal transient growth

Maximum growth with the Reynolds number for Poiseulle flow $(\beta = 0)$

$$G_{\max} = \max_t G(t) = \max_t \max_{\hat{\mathbf{u}}_0} \frac{\|\hat{\mathbf{u}}(t;\alpha,\beta)\|^2}{\|\hat{\mathbf{u}}_0(\alpha,\beta)\|^2}$$



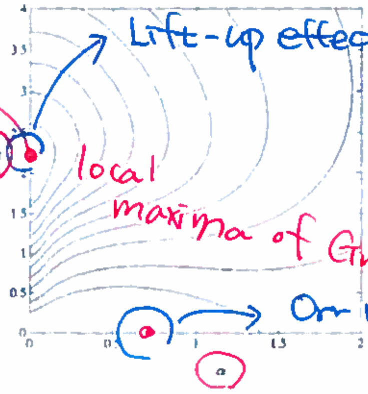
Linearly unstable region.
(i.e. $u(t) \to \infty$)

Optimal transient growth

Maximum growth for Poiseuille flow in the wavenumber plane

$$G_{\max} = \max_{t} G(t) = \max_{t} \max_{\hat{\mathbf{u}}_0} \frac{\|\hat{\mathbf{u}}(t;\alpha,\beta)\|^2}{\|\hat{\mathbf{u}}_0(\alpha,\beta)\|^2}$$

The location where Gmax becomes the global maximum.



Lift-up effect. (3D)

local maxima of Gmax

Orr mechanism. (2D)

FIGURE 4.4 Contours of G_{max} for Poiseuille flow with Re = 1000. The curves from outer to inner correspond to G_{max} = 10, 20, 40, ..., 140, 160, 180. From Reddy & Henningson (1993)

Scaling of optimal transient growth

Scaling with the Reynolds number

Flow configurations	G_{max}	t_{max}	α	β
Couette flow	0.20 Re²	0.076 Re	35/Re	2.04
Poiseulle flow	1.18 Re²	0.117 Re	0	1.6
Pipe flow	0.07 Re²	0.048 Re	0	1
Boundary layer	1.50 Re²	0.778 Re	0	0.65

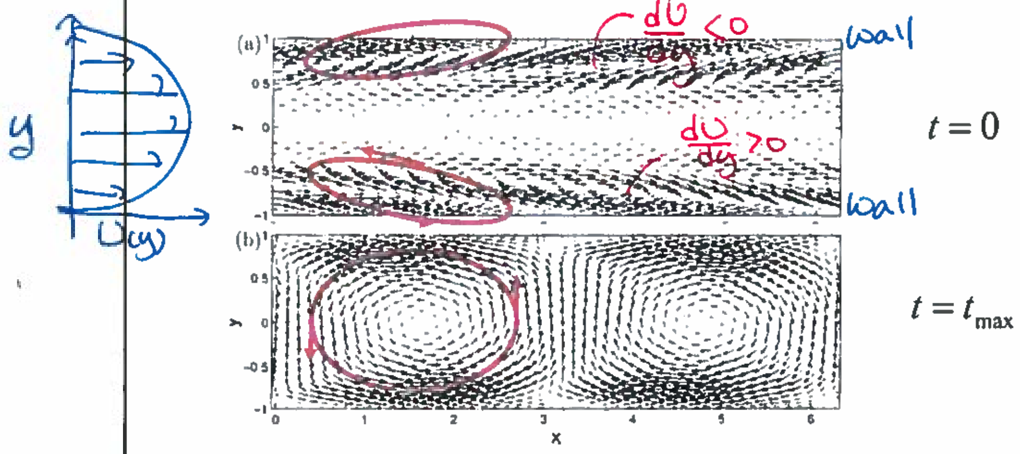
$\underbrace{G_{max} \sim Re^2 \qquad t_{max} \sim Re.}_{\text{Analytical result.}}$

Schmid and Henningson (2001)

3. Orr mechanism and lift-up effect

The structure of optimal disturbance

Optimal disturbance in two dimensional case (Poiseulle flow)



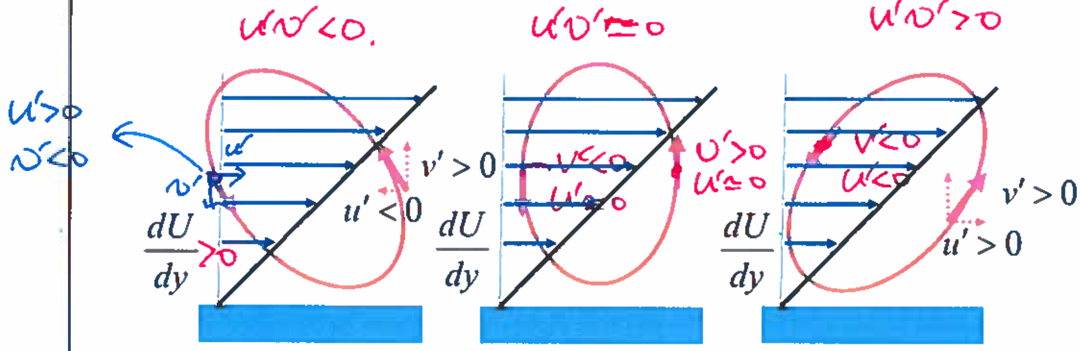
$\alpha = 1, \beta = 0, \mathrm{Re} = 1000$

The Orr mechanism − 1905

The Orr mechanism: tilting by mean shear

$$\frac{dE}{dt} \sim -\int_V u'v' \frac{dU}{dy} dV$$

(+)



$t = 0$ $\Rightarrow \frac{dE}{dt} > 0$

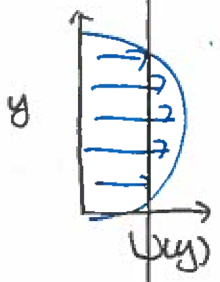
$t \sim t_{max}$ $\Rightarrow \frac{dE}{dt} \simeq 0$

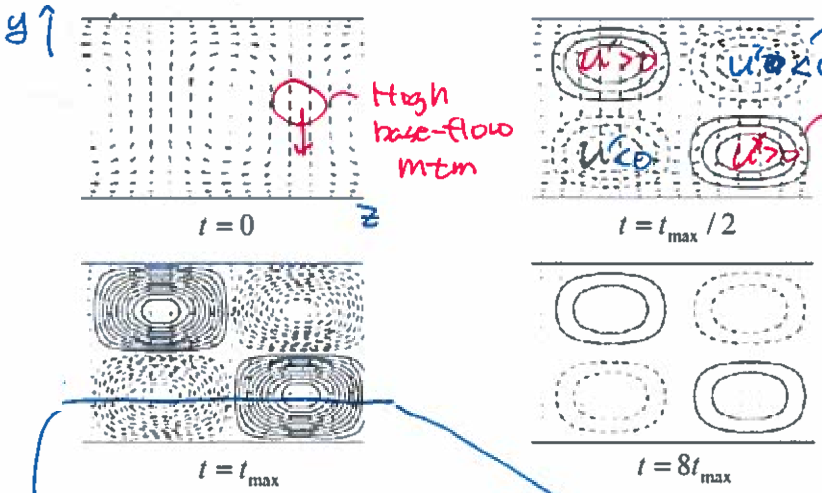
$t > t_{max}$ $\Rightarrow \frac{dE}{dt} < 0.$

Optimal disturbance in three dimensional flow

→ Lift-up effect

Optimal disturbance in three dimensional case (Poiseulle flow)





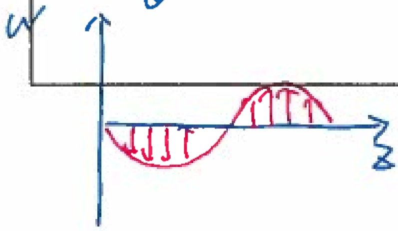
dashed contours $u'<0$

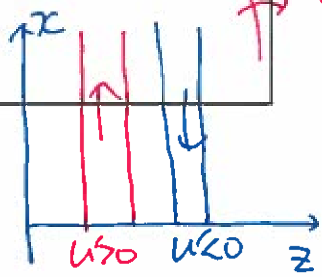
Solid contours $u'>0$

→ Streaks

$\alpha = 0, \beta = 2.044, \mathrm{Re} = 5000$

Bewley & Liu (1997)





Lift-up effect

Lift-up effect: a vortex tilting mechanism for generation of streaks

$$\frac{D\omega_y}{Dt} \sim \omega_x \frac{dU}{dy}$$

$\omega \sim [\omega_x, 0, 0]^T$ → ⇒ $\omega \sim [0, \omega_y, 0]$

$t=0$ $t=t_{max}$

Recall the Orr-Sommefeld-Squire system:

$$\frac{\partial}{\partial t}\begin{bmatrix} k^2 - D^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} + \begin{bmatrix} L_{OS} & 0 \\ i\beta DU & L_{SQ} \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} = 0$$

where $k^2 = \alpha^2 + \beta^2$

$$L_{OS} = i\alpha U(k^2 - D^2) + i\alpha D^2 U + \frac{1}{\mathrm{Re}}(k^2 - D^2)^2$$

1. Optimal transient growth
2. Case study: wall-bounded shear flows