Lecture 4

Linear stability of parallel flows I

AE209 Hydrodynamic stability

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Lecture outline

1. Concept of parallel flows
2. Linearised equation for inviscid parallel shear flows
3. Normal mode solution
4. When does an instability occur?

Lecture outline

1. **Concept of parallel flows**

Parallel shear flows

The flow configuration, the base flow of which is given by

$$\mathbf{U} = (U(y), 0, 0)$$

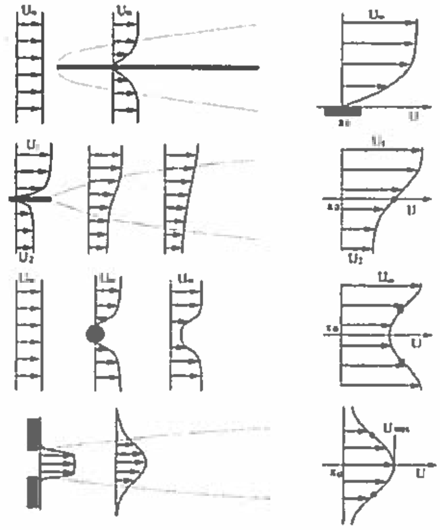
Examples

Plane Couette flow, Poiseulle flow, Pipe flow, and etc.

$U(y)$ $U(y) = 1 - y^2$ $U(r) = 1 - r^2.$

Parallel shear flows

Weakly non-parallel flows (or spatially developing flows)



Boundary layer

Mixing layer

Cylinder wake

Jet

2. Linearised equation for inviscid parallel shear flows

Linearised equation for inviscid parallel flow

Linearised Euler equation around parallel base flow

$$\frac{\partial}{\partial x}\left[\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{dU}{dy} = -\frac{\partial p'}{\partial x}\right]$$

$+$

$$\frac{\partial}{\partial y}\left[\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y}\right]$$

$+.$

$$\frac{\partial}{\partial z}\left[\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z}\right]$$

$$\Rightarrow \quad \nabla^2 p = -2\boxed{\frac{dU}{dy}\frac{\partial v}{\partial x}}$$

Poisson equation for pressure.

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Linearised equation for inviscid parallel flow

Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p$$

$$\nabla \cdot \mathbf{u} = 0$$

Consider $\quad \mathbf{u}(x,y,z,t) = \underbrace{(U(y),0,0)}_{\underline{U}} + \varepsilon\, \mathbf{u}'(x,y,z,t),$

$$p(x,y,z,t) = P(x,y) + \varepsilon\, p'(x,y,z,t)$$

and neglect the terms at $O(\varepsilon^2)$. Then,

$$\frac{\partial \underline{u}'}{\partial t} + (\underline{U} \cdot \nabla)\underline{u}' + (\underline{u} \cdot \nabla)\underline{U} = -\nabla p'$$

$$\nabla \cdot \underline{u}' = 0$$

Linearised equation for inviscid parallel flow

1) Equation for wall-normal velocity

$$\nabla^2 \cdot \left[\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} \right] \qquad \frac{\partial}{\partial y}\left[\nabla^2 p = -2\frac{dU}{dy}\frac{\partial v}{\partial x} \right]$$

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \right)\nabla^2 - \frac{d^2U}{dy^2}\frac{\partial}{\partial x} \right] v' = 0$$

2) Equation for wall-normal vorticity

$$\frac{\partial}{\partial z}\left[\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+v'\frac{dU}{dy}=-\frac{\partial p'}{\partial x}\right]$$

$-$

$$\frac{\partial}{\partial x}\left[\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}=-\frac{\partial p'}{\partial z}\right]$$

Wall-normal vorticity (ω_y)

$$\eta'=\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}$$

$$\underline{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\eta'=-\frac{dU}{dy}\frac{\partial v'}{\partial z}}$$

$\Rightarrow$ Use continuity to retrieve u, w.

Linearised equation for inviscid parallel flow

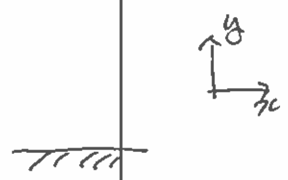
Velocity and vorticity form of linearised Euler equation

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}\right]v'=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\eta'+\frac{dU}{dy}\frac{\partial v'}{\partial z}=0$$

with the boundary condition,

$\underline{v'=\eta'=0}$ at solid boundary and/or the far field



and the initial condition,

$$v'(x,y,z,t=0)=v_0'(x,y,z)$$
$$\eta'(x,y,z,t=0)=\eta_0'(x,y,z)$$

3. Normal mode solution

Normal mode solution

For example,

$v' \to \tilde{v}$.

$$\frac{\partial v'}{\partial t} = \frac{\partial}{\partial t}\left(\tilde{v}\, e^{i\alpha x - i\omega t}\right) = -i\omega\, \tilde{v}\, \cancel{e^{i\alpha x - i\omega t}}.$$

$\frac{\partial}{\partial t} \to -i\omega$

$$\frac{d^2U}{dy^2}\frac{\partial v'}{\partial x} = \frac{d^2U}{dy^2}\frac{\partial}{\partial x}\left[\tilde{v}\, e^{i\alpha x - i\omega t}\right] = i\alpha\frac{d^2U}{dy^2}\tilde{v}\, \cancel{e^{i\alpha x - i\omega t}}$$

$\frac{\partial}{\partial x} \to i\alpha$

$$\nabla^2 v' = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\left(\tilde{v}\, e^{i\alpha x - i\omega t}\right)$$

$\frac{\partial^2}{\partial x^2} \to -\alpha^2$.

$$= \left[-\alpha^2 + \frac{\partial^2}{\partial y^2}\right]\left(\tilde{v}\, \cancel{e^{i\alpha x - i\omega t}}\right)$$

Normal mode solution

Consider two-dimensional case such that

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}\right]v'=0$$

cf

$$\ddot{x}-a^2x=0$$

$$\Rightarrow x=Ce^{\lambda t}$$

Then, the normal mode solution takes the following form:

$$v'(x,y,t)=\tilde{v}(y)e^{i(\alpha x-\omega t)}+c.c$$

Complex

complex conjugate

Assume

where $\alpha\in R$ and $\omega\in C$.

Real. Complex

known. Unknown.

(Prescribed)

Interpretation of normal mode solution

Normal mode solution

$c_r + ic_i$

$$v'(x,y,t) = \tilde{v}(y)e^{i\alpha(x-ct)} + c.c$$
$$= \text{Real}\left\{ \tilde{v}(y) \,|\, e^{i\phi(y)} e^{i\alpha(x-(c_r+ic_i)t)} \right\}$$
$$= |\tilde{v}(y)| e^{\alpha c_i t} \cos[\alpha(x - c_r t) + \phi(y)]$$

$Z = re^{i\theta}$ Phase angle

$r = |Z|$

wall-normal structure of wave

c_r : Phase speed

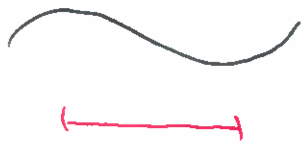
(speed of wave)

$c_i > 0$ Linearly unstable

$c_i = 0$ Marginally stable (or neutral)

$c_i < 0$ Linearly stable

$c_r \rightarrow$



$\lambda_x = \frac{2\pi}{\alpha}$

if $c_i > 0$

Rayleigh equation

Rayleigh equation

Let $\omega = \alpha c$ and $D \equiv d/dy$. Then, we get

$$\left(U - \underbrace{c}_{\text{eigenvalue}}\right)\left(D^2 - \alpha^2\right)\tilde{v} - D^2 U \tilde{v} = 0$$

eigen function

Prescribed by a real number.

with the boundary condition,

$\tilde{v} = 0$ at solid boundary and/or the far field

Remark

If $\alpha \in R$ is given, then $c \in C$ becomes unknown with $\tilde{v}$, resulting in **eigenvalue problem**.

Lecture outline

4. **When does an instability occur?**

Theorem: Rayleigh inflection point criterion (Inverse is NOT true)

There is a linear Instability.

If there exist perturbations with $c_i > 0$, then d^2U/dy^2 must be zero at some $y \in \Omega$ ($\Omega = [a,b]$ is the flow domain in y).

Proof Assume $C_i > 0$ → complex conjugate of $\tilde{v}$.

$$\int_a^b \tilde{v}^* \cancel{(U-c)}(D^2-\alpha^2)\tilde{v} - \frac{D^2U\,\tilde{v}}{(U-c)} = 0 \; dy.$$

$$\int_a^b \tilde{v}^*\left(D^2-\alpha^2\right)\tilde{v}\,dy - \int_a^b \frac{D^2U}{(U-c)}\tilde{v}^*\tilde{v}\,dy = 0$$

$$\int_a^b v^* D^2\tilde{v}\,dy = \cancel{\left[\tilde{v}^* D\tilde{v}\right]_a^b} - \int_a^b D\tilde{v}^* D\tilde{v}\,dy$$

0

integration by part.

$$= -\int_a |D\tilde{v}|^2\,dy$$

Rayleigh inflection point criterion

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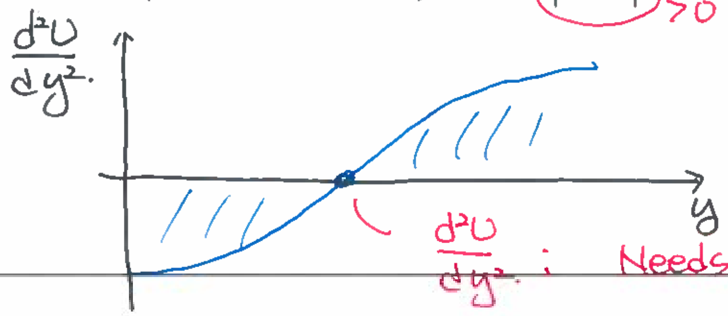
$$\underbrace{\int_a^b |D\tilde{v}|^2 + \alpha^2 |\tilde{v}|^2 \, dy}_{\text{Real Positive}} + \underbrace{\int_a^b \frac{d^2U/dy^2}{(U-c)} |\tilde{v}|^2 \, dy}_{\text{Complex.}} = 0$$

$$\int_a^b \frac{(U-c)^* \, d^2U/dy^2}{(U-c)(U-c)^*} |\tilde{v}|^2 dy$$

$$\Rightarrow \int_a^b \frac{(U-c)^* \, d^2U/dy^2}{|U-c|^2} |\tilde{v}|^2 dy$$

Imaginary part:

$$\mathrm{Im}\left(\int_a^b \frac{d^2U/dy^2}{(U-c)} |\tilde{v}|^2 \, dy \right) = \int_a^b \frac{\overbrace{c_i}^{>0.} \, d^2U/dy^2 \overbrace{|\tilde{v}|^2}^{>0.}}{\underbrace{|U-c|^2}_{>0}} dy = 0$$



$\frac{d^2U}{dy^2}$: Needs sign change at some y.

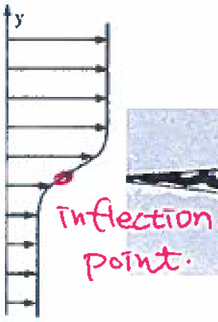
Rayleigh inflection point criterion

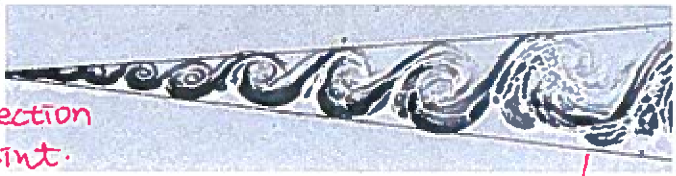
Remark 1

The presence of some **inflection points** is a **necessary condition** for **linear instability**.

Remark 2

The presence of an **infection point** is often a **sign of instability**.





Instability in a mixing layer
(Brown & Roshko 1974)

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3. Normal mode solution
4. Rayleigh thereom