Lecture 2

Basic dynamical systems theory I

AEM-ADV12 Hydrodynamic stability
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Lecture outline

1. Phase portrait and equilibria
2. Linear stability analysis

Lecture outline

1. **Phase portrait and equilibria**

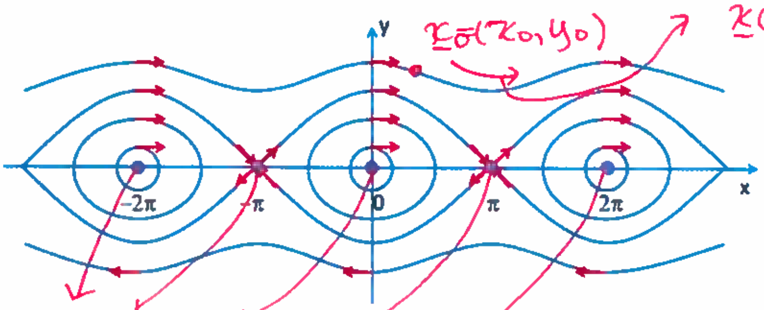
A set of state (solution) trajectory in time.

Phase portrait for a planar (2D) dynamical system

Example: Nonlinear pendulum

$$\dot{x} = y \quad \text{and} \quad \dot{y} = -\sin x$$

$\underline{x}_0 = (x_0, y_0)$ $\quad$ $\underline{x}(t) = (x(t), y(t))$



Equilibrium points
(or fixed points) $\Leftrightarrow$ $\underline{f}(\underline{x}) = \underline{0}$

Equilibrium point (or fixed point)

Definition: Equilibrium point

$\overline{\mathbf{x}}$ is an equilibrium point if $\mathbf{x}(t) = \overline{\mathbf{x}}$ is a solution of the given dynamical system such that

$$\mathbf{f}(\overline{\mathbf{x}}) = \mathbf{0}$$

Steady

Solution.

Equilibrium point (or fixed point)

Example 1: Nonlinear pendulum

$$\dot{x} = y \quad \text{and} \quad \dot{y} = \sin x$$

Let $\dot{x} = \dot{y} = 0$. Then $y = 0$, $\sin x = 0$.

$\therefore$ $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Therefore.

Equilibrium points

$\Rightarrow$ $(0,0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$ $\ldots$

Equilibrium point (or fixed point)

Example 2: Plane Couette flow

$$(\mathbf{U}\cdot\nabla)\mathbf{U} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{U}$$

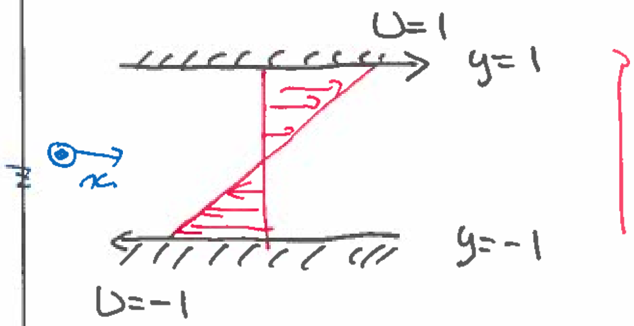
$$\nabla\cdot\mathbf{U} = 0$$

$\underline{u} = [u \quad v \quad w]$

$\frac{\partial}{\partial x} = 0$

$\frac{\partial}{\partial z} = 0.$

$v = w = 0$



$\underline{U(y) = y}$

An equilibrium point.

Lecture outline

2. **Linear stability analysis**

Jacobian linearisation

$\varepsilon \ll 1$, $\delta x \sim O(1)$.

Let $\overline{\mathbf{x}}$ be an equilibrium point such that $\mathbf{f}(\overline{\mathbf{x}}) = \mathbf{0}$. Consider a small perturbation $\boldsymbol{\delta}\mathbf{x}$, i.e. $\mathbf{x} = \overline{\mathbf{x}} + \varepsilon\boldsymbol{\delta}\mathbf{x}$, then the given nonlinear system is approximated by the following linear dynamical system:

$$\frac{d\boldsymbol{\delta}\mathbf{x}}{dt} = \underbrace{\left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\overline{\mathbf{x}}}}_{\text{Jacobian.}} \boldsymbol{\delta}\mathbf{x} \rightarrow \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$\Rightarrow$ matrix with constant entry.

Remark

Linear dynamical system is much easier to analyse.

Let $\underline{x} = \overline{\underline{x}} + \varepsilon \delta \underline{x}$ and $\frac{d\underline{x}}{dt} = f(\underline{x})$

$$\Rightarrow \underbrace{\frac{d(\overline{\underline{x}} + \varepsilon\delta\underline{x})}{dt}}_{\varepsilon \frac{d\delta\underline{x}}{dt}} = f(\overline{\underline{x}} + \varepsilon\delta\underline{x})$$

$$= \underbrace{f(\overline{\underline{x}})}_{\Rightarrow 0.} + \varepsilon \left.\frac{\partial f}{\partial \underline{x}}\right|_{\underline{x}=\overline{\underline{x}}} \delta\underline{x}. + O(\varepsilon^2)$$

$$\Rightarrow \frac{d\delta\underline{x}}{dt} = \left.\frac{\partial f}{\partial \underline{x}}\right|_{\underline{x}=\overline{\underline{x}}} \delta\underline{x}$$

Example 1

Find the linearised system around the equilibrium point.

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}-\begin{bmatrix}0\\ x^2 y\end{bmatrix}=\begin{bmatrix}-y\\ x-x^2y\end{bmatrix}$$

$\to f_1(x,y)$

$\to f_2(x,y).$

Let $x=\bar{x}+\varepsilon\delta x,\quad y=\bar{y}+\varepsilon\delta y \quad\Rightarrow\quad \bar{x}=\bar{y}=0$

$$\Rightarrow \begin{bmatrix}\delta\dot{x}\\ \delta\dot{y}\end{bmatrix}=\left.\begin{bmatrix}\underbrace{\frac{\partial f_1}{\partial x}}_{=0} & \underbrace{\frac{\partial f_1}{\partial y}}_{=-1.}\\ \underbrace{\frac{\partial f_2}{\partial x}}_{=1} & \underbrace{\frac{\partial f_2}{\partial y}}_{=0.}\end{bmatrix}\right|_{(0,0)}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix}$$

Example 2: Linearised Navier-Stokes equation

Find the linearised equation around an equilibrium point (basic state) given by $\mathbf{U}$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}$$

$$\nabla\cdot\mathbf{u} = 0$$

Let $\underline{u} = \underline{U} + \varepsilon \underline{u}'$ and $p = P + \varepsilon p'$

Equilibrium point

perturbation

① $\frac{\partial \underline{u}}{\partial t} = \cancel{\frac{\partial \underline{U}}{\partial t}}^{0} + \varepsilon \frac{\partial \underline{u}'}{\partial t}$.

② $(\underline{u}\cdot\nabla)\underline{u} = \left[(\underline{U}+\varepsilon\underline{u}')\cdot\nabla\right](\underline{U}+\varepsilon\underline{u}')$.

$= (\underline{U}\cdot\nabla)\underline{U} + \varepsilon(\underline{U}\cdot\nabla)\underline{u}'$

$+ \varepsilon(\underline{u}'\cdot\nabla)\underline{U} + \cancel{\varepsilon^2(\underline{u}'\cdot\nabla)\underline{u}'}$

③ $-\nabla p = -\nabla(P+\varepsilon p') = -\nabla P - \varepsilon\nabla p'$

④ $\frac{1}{Re}\nabla^2\underline{u} = \frac{1}{Re}\nabla^2(\underline{U}+\varepsilon\underline{u}') = \frac{1}{Re}\nabla^2\underline{U} + \varepsilon\nabla^2\underline{u}'$

$\Rightarrow$ if $\varepsilon = 0$ $\Rightarrow$

$(\underline{U}\cdot\nabla)\underline{U} = -\nabla P + \frac{1}{Re}\nabla^2\underline{U}$

At $O(\varepsilon)$

$$\boxed{\frac{\partial \underline{u}'}{\partial t} + (\underline{U}\cdot\nabla)\underline{u}' + (\underline{u}'\cdot\nabla)\underline{U} = -\nabla p' + \frac{1}{Re}\nabla^2\underline{u}'}$$

Linearised N-S equations.

Definition: Linear instability ~~or stability~~

If the linearised dynamical system around the given basic state $\overline{\mathbf{x}}$ has a solution such that $\|\delta\mathbf{x}\| \rightarrow \infty$ as $t \rightarrow \infty$, the basic state is called **linearly unstable**.

equilibrium point. (state).

two dimensional.

Linear stability of a planar dynamical system

Let the linearised system around the basic state $\overline{\mathbf{x}}$ be

$$\frac{d\delta\mathbf{x}}{dt} = \mathbf{A}\,\delta\mathbf{x} \qquad \text{where} \qquad \mathbf{A} = \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{\mathbf{x}=\overline{\mathbf{x}}}$$

Normal mode solution : $\delta\underline{x} = e^{\lambda t}\,\delta\hat{\underline{x}}$

$$\underline{\lambda\,\delta\hat{\underline{x}} = A\,\delta\hat{\underline{x}}}.$$ Eigenvalue problem for λ.

λ_1, λ_2 and $\delta\hat{x}_1, \delta\hat{x}_2$.

eigenvalues.

$$\Rightarrow \delta\underline{x}(t) = \underline{C_1}\, e^{\lambda_1 t}\,\underline{\delta\hat{x}_1} + \underline{C_2}\, e^{\lambda_2 t}\,\underline{\delta\hat{x}_2}.$$

from initial condition.

eigenvectors

If $Re(\lambda_1)$ or $Re(\lambda_2) > 0$., $\|\delta\underline{x}\| \to \infty$. as $t \to \infty$

$\Rightarrow$ Linearly unstable.

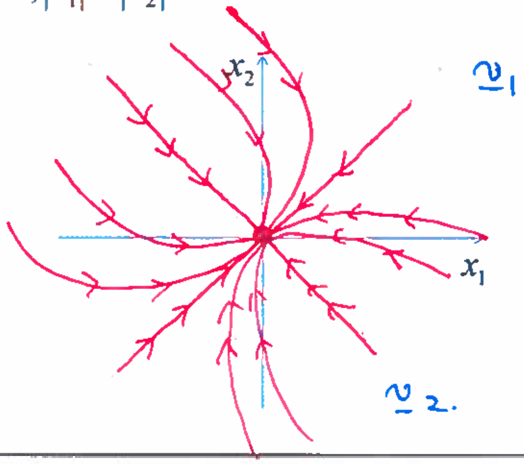
Phase portrait of 2D linear system

Case I

λ_1, λ_2 are both real, and $\lambda_1 \neq \lambda_2$. The two corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are then linearly independent.

$i)\ \lambda_1, \lambda_2 < 0, |\lambda_1| > |\lambda_2|$

This type of Equilibrium point : Stable "Node"

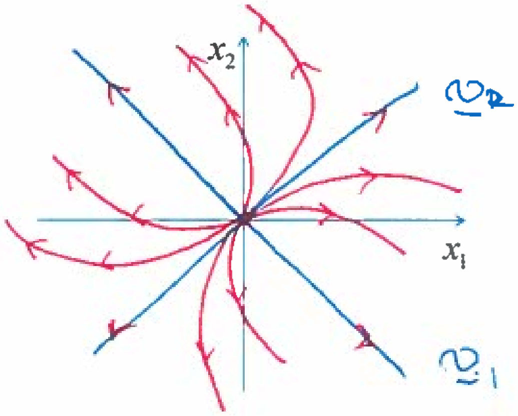


Case I

λ_1, λ_2 are both real, and $\lambda_1 \neq \lambda_2$. The two corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are then linearly independent.

$$ii)\ \lambda_1, \lambda_2 > 0, |\lambda_1| > |\lambda_2|$$

Unstable Node.



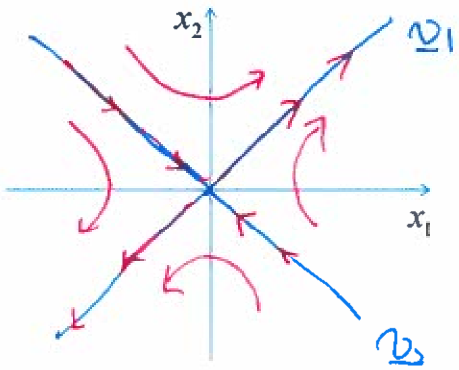
Phase portrait of 2D linear system

Case I

λ_1, λ_2 are both real, and $\lambda_1 \neq \lambda_2$. The two corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are then linearly independent.

iii) $\lambda_2 < 0 < \lambda_1$

Saddle.

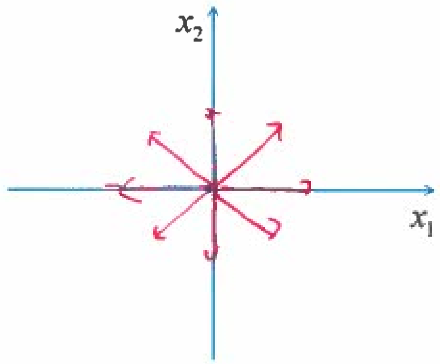


Case II

λ_1, λ_2 are both real, and $\lambda_1 = \lambda_2$.

$i)\ \mathrm{rank}(\mathbf{A} - \lambda \mathbf{I}) = 0$ → eigenvector space is full rank.

2-dimensional



$x(t) = x_0 e^{\lambda t}$

$y(t) = y_0 e^{\lambda t}$.

if $\lambda_1, \lambda_2 > 0$.

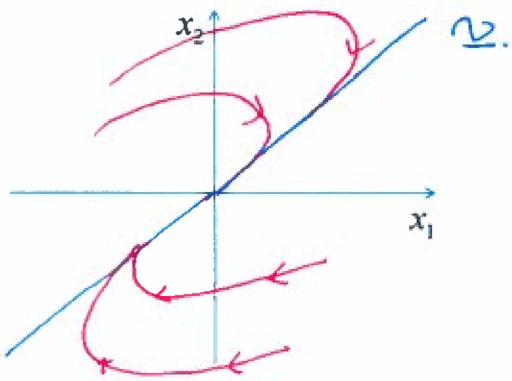
Case II

λ_1, λ_2 are both real, and $\lambda_1 = \lambda_2$.

ii) $\underline{\text{rank}(\mathbf{A} - \lambda \mathbf{I}) = 1}$

one eigenvector exists.

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_1 t + C_2 \\ C_3 t + C_4 \end{bmatrix} e^{\lambda t}.$$



if $\lambda_1 = \lambda_2 < 0$

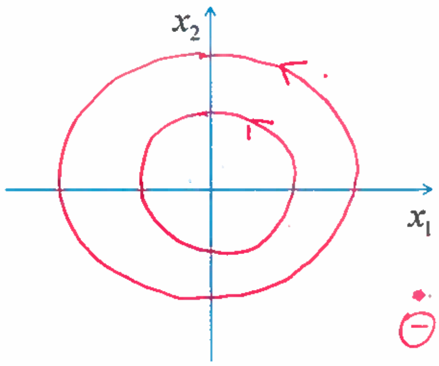
Phase portrait of 2D linear system

Case III

λ_1, λ_2 are both complex such that $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

$$e^{\lambda t} = e^{\alpha t}(\cos \beta t + i \sin \beta t)$$

$i)\ \alpha = 0$



$\dot{\Theta}$

where

$\Theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$.

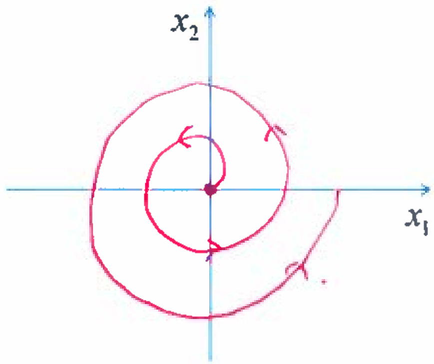
$\dot{\Theta} > 0.$

Case III

λ_1, λ_2 are both complex such that $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

$ii)\, \alpha > 0$

Unstable focus.

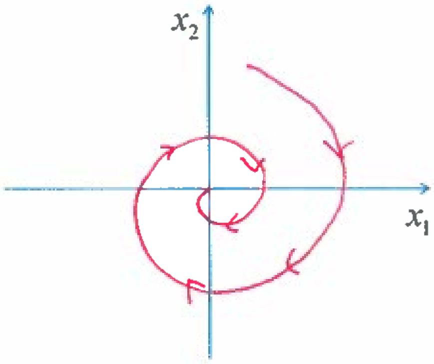


Case III

λ_1, λ_2 are both complex such that $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

$iii)\ \alpha < 0$

Stable focus.



Example: Unforced duffing equation

$$\ddot{x}+\dot{x}-x+x^3=0$$

$(x.=)\dot{x}_1 = x_2$

$\dot{x}_2 = x_1 - x_1^3 - x_2.$

i) Equilibria $\Rightarrow$ $(0,0)$, $(1,0)$, $(-1,0)$

① $\bar{x}=(0,0)$

$$\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \begin{aligned} \lambda_1 &= -\frac{1}{2}+\frac{\sqrt{5}}{2} > 0 \\ \lambda_2 &= -\frac{1}{2}-\frac{\sqrt{5}}{2} < 0 \end{aligned}$$

Saddle.

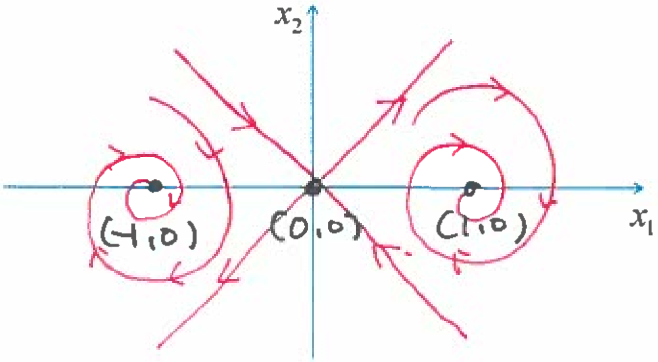
② $\bar{x}=(\pm 1, 0)$

$$\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$$

λ_1, λ_2 are complex

$\mathrm{Re}(\lambda_1, \lambda_2) < 0$ / Stable focus

Phase portrait of 2D linear system



Summary

1. Phase portrait and equilibria
2. Linear stability analysis