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EIGENVALUES OF THE ORR-SOMMERFELD EQUATION

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Abstract. A very simple proof is presented of the fact that the Orr-Sommerfeld equation for flows that approach exponentially to a constant at infinity has at most finitely many eigenvalues. A completely elementary argument shows that the Orr-Sommerfeld equation for such flows has no eigenvalues when the product of the Reynolds number and the wave number is small enough.

1. Introduction. Numerical studies have shown [2, 5] that the Orr-Sommerfeld equation [3] for the Blasius flow may have only finitely many eigenvalues (whose eigenvectors decay at infinity). This has led to numerous studies of the spectrum of the Orr-Sommerfeld equation; for some references see [7, 10]. In [7] I proved that the generalized Orr-Sommerfeld equation has only finitely many eigenvalues if the flow is approaching a constant exponentially, which is true for the Blasius flow, and in [8] it has been shown that it is the generalized Orr-Sommerfeld equation that has to be studied in connection with nonlinear problems. Nevertheless, [8] also shows that for stability purposes the relevant properties of the generalized Orr-Sommerfeld equation may be determined by the eigenvalues of the (usual) Orr-Sommerfeld equation which is much simpler. The proof given in [7] is based on rather technical details which are perhaps inaccessible to persons who are not well trained in spectral theory; see for example [9]. In this paper, I shall present an elementary proof that the (usual) Orr-Sommerfeld equation has at most finitely many eigenvalues when the flow approaches a constant exponentially and; in addition I prove a new stability result for low Reynolds numbers. To do this the eigenvalue problem is transformed, by completely elementary operations, into an integral equation and then the result follows immediately from one well-known and often-quoted result from functional analysis which is for convenience of the reader stated in the Appendix as Lemma 4. Generalization of this proof so that it would be applicable to flows that approach a constant algebraically does not seem possible; the counterexample presented in [6] suggests that beside the rate of approach several other properties would have to be considered.

From the integral formulation it is very easy to deduce that when $|\alpha|R$ (α is the wave number in x-direction, R is the Reynolds number) is small then there are no eigenvalues. This is a new result. It says that at any fixed Reynolds number we do not need to worry about possible instability at small wave numbers (in realistic

cases (α, β real) the continuous spectrum by itself would cause exponential decay of solutions [6]—except when $\alpha = \beta = 0$, in which case it is easy to show dissipation directly; see [6]). This result, combined with other estimates in [6, 7] does also give $R_1 > 0$ such that when $R < R_1$ we have stability for all possible (real) wave numbers. A similar result can be obtained when the flow approaches a constant algebraically [6].

To obtain these results one has to assume some decay of the eigenvector ϕ. To be consistent with [6, 7] one can assume that $\phi(x)$, $\phi'(x)$, $\phi''(x)$, $\phi'''(x)$ converge to 0 as $x \to \infty$. However, this behaviour of ϕ is obtained even if somewhat weaker conditions are initially imposed. An example is presented in the theorem below; however, to preserve clarity of the presentation this has not been pursued further.

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2. Results.

Theorem. *Suppose*

   i) $U \in C^2([0,\infty))$, $\lim_{x\to\infty} U(x) = U(\infty)$

   ii) *there exist $M < \infty$ and $\epsilon > 0$ such that $|U'(x)| \le Me^{-\epsilon x}$ for $x \ge 0$*

   iii) $R > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $\lambda^2 = \alpha^2 + \beta^2$, $\operatorname{Re}(\lambda) \ge 0$.

Then there exist at most finitely many (eigenvalues) $\zeta \in \mathbb{C}$ for which there exists $\phi \in C^4([0,\infty))$ with the following properties:

   a) $i\alpha R U''\phi + (\zeta R - \lambda^2 - i\alpha R U)(\phi'' - \lambda^2\phi) + \phi'''' - \lambda^2\phi'' = 0$ *on* $[0,\infty)$

   b) $\phi(0) = \phi'(0) = 0$, $\phi \not\equiv 0$

   c) $\phi(x)$, $\phi'(x)$, $\phi''(x)$ *and* $e^{-\lambda x}\phi'''(x)$ *converge to 0 as* $x \to \infty$.

Moreover, there exists $R_c > 0$, which depends only on α, β and U, such that if $R < R_c$ then there are no such eigenvalues ζ. If, in addition,

$$\sup_x |U''(x)|e^{3\epsilon x/4} < \infty$$

then there exists $c > 0$, which depends only on U, such that there are no eigenvalues when $|\alpha|R < c$.

Proof: Let ζ, ϕ be as above. Define $V(x) = i\alpha R(U(x) - U(\infty))$ and pick $z \in \mathbb{C}$ such that $\operatorname{Re}(z) \ge 0$ and $z^2 = \lambda^2 - \zeta R + i\alpha R U(\infty)$. Then

$$\phi'''' - \lambda^2\phi'' - z^2(\phi'' - \lambda^2\phi) = V(\phi'' - \lambda^2\phi) - V''\phi \quad \text{on } [0,\infty) \tag{1}$$

and note that

$$|V(x)| \le |\alpha|R \int_x^\infty |U'(t)|\,dt \le |\alpha|RMe^{-\epsilon x}/\epsilon \quad \text{for } x \ge 0.$$

Define

$$h(x) = e^{-\mu x}, \quad \mu = \epsilon/4 \tag{2}$$

$$f_1 = \lambda\phi + \phi', \quad f = hf_1 \in L^2(0,\infty) \tag{3}$$

$$f_2 = zf_1 + f_1' \quad (= z\lambda\phi + (z+\lambda)\phi' + \phi'') \tag{4}$$

$$f_3 = zf_2 - f_2' \quad (= z^2\lambda\phi + z^2\phi' - \lambda\phi'' - \phi'''). \tag{5}$$

We shall transform (1) into an integral equation for f. The notation considerably simplifies if the following operators are introduced. For $\nu \in \mathbb{C}$, $\mathrm{Re}\,(\nu) > 0$, $g \in L^2(0,\infty)$ define functions $F(\nu)g$, $G(\nu)g$ in $L^2(0,\infty)$ by

$$(F(\nu)g)(x) = \int_0^x e^{\nu(t-x)} g(t)\, dt$$

$$(G(\nu)g)(x) = \int_x^\infty e^{\nu(x-t)} g(t)\, dt.$$

$F(\nu)$, $G(\nu)$ are bounded operators on $L^2(0,\infty)$ and are analysed in the Appendix. Note that (5) implies

$$\lambda f_3 - f_3' = \phi'''' - \lambda^2 \phi'' - z^2(\phi'' - \lambda^2 \phi) = V(\phi'' - \lambda^2 \phi) - V''\phi$$

and hence

$$\frac{d}{dx}\left(e^{-\lambda x}\left(f_3 + V\phi' - (V' - \lambda V)\phi\right)\right) = 2\lambda e^{-\lambda x} V'\phi.$$

Integration from x to L and letting $L \to \infty$ gives that

$$
\begin{aligned}
f_3 &= -V\phi' + (V' - \lambda V)\phi - 2\lambda \int_x^\infty e^{\lambda(x-t)} V'(t)\phi(t)\, dt \\
&= -V\phi' + (V' - \lambda V)\phi - 2\lambda h^2 G(\lambda + 2\mu)V'h^{-2}\phi \\
&= -Vh^{-1}f + V'\phi - 2\lambda h^2 G(\lambda + 2\mu)V'h^{-2}\phi.
\end{aligned}
\tag{6}
$$

Using (3) and the fact that $\phi(0) = 0$ we get

$$\phi(x) = \int_0^x e^{\lambda(t-x)} f_1(t)\, dt$$

$$\phi = h^{-1}F(\lambda + \mu)f; \tag{7}$$

inserting this into (6) gives

$$f_3 = h^2 Bf, \tag{8}$$

where B is a bounded operator on $L^2(0,\infty)$ defined by

$$Bg = -Vh^{-3}g + V'h^{-3}F(\lambda + \mu)g - 2\lambda G(\lambda + 2\mu)V'h^{-3}F(\lambda + \mu)g \tag{9}$$

for $g \in L^2(0,\infty)$. Note that B depends only on V, λ and μ. Equation (5) gives that

$$e^{-zL}f_2(L) - e^{-zx}f_2(x) = -\int_x^L e^{-zt} f_3(t)\, dt;$$

this, equation (8) and the fact that $f_2(L) \to 0$ as $L \to \infty$ give

$$f_2 = hG(z + \mu)hBf. \tag{10}$$

Equation (4) and the fact that $f_1(0) = 0$ (from (3)) give

$$f_1(x) = \int_0^x e^{z(t-x)} f_2(t)\, dt$$

$$f_1 = h^{-1}F(z+\mu)hf_2$$

this and equation (10) give an integral formulation of (1)

$$f = F(z+\mu)h^2 G(z+\mu)hBf. \tag{11}$$

For $\nu \in \mathbb{C}$, $\text{Re}\,(\nu) > -\mu$ define $K(\nu)$, a bounded operator on $L^2(0,\infty)$, by

$$K(\nu)g = F(\nu+\mu)h^2 G(\nu+\mu)hBg, \quad g \in L^2(0,\infty).$$

Lemma 2 of Appendix gives that

$$\|K(\nu)\| \leq \frac{2\|B\|}{\mu|\nu|} \quad \text{for } \text{Re}\,(\nu) \geq 0, \ \nu \neq 0. \tag{12}$$

Hence (11), the integral formulation of (1), can be written as

$$f = K(z)f \tag{13}$$

and by (7) we have that $f \not\equiv 0$. If $z \neq 0$ then (12) and (13) imply that

$$0 \neq \|f\| = \|K(z)f\| \leq 2\|B\|\,\|f\|/(\mu|z|)$$

and therefore
$$\text{Re}\,(z) \geq 0, \quad |z| \leq 2\|B\|/\mu. \tag{14}$$

If $\text{Re}\,(\nu) > -\mu$ then $h^2 G(\nu+\mu)h$ is a compact operator on $L^2(0,\infty)$ (Lemma 3) and therefore $K(\nu)$ is compact for $\text{Re}\,(\nu) > -\mu$.

Since $F(\nu)$, $G(\nu)$ are holomorphic for $\text{Re}\,(\nu) > 0$ (Lemma 1) we have that $K(\nu)$ is a holomorphic family of compact operators for $\text{Re}\,(\nu) > -\mu$. (12) implies that $\|K(\nu)\| < 1$ for ν large and therefore by a well-known theorem (which is stated for convenience of the reader as Lemma 4) we have that in each compact subset of the half plane $\text{Re}\,(\nu) > -\mu$ there are at most finitely many z for which equation (13) has a nontrivial solution. Because z has to lie in the compact set given by (14), there are at most finitely many z (and hence at most finitely many eigenvalues $\zeta = (\lambda^2 - z^2 + i\alpha RU(\infty))/R$) for which (13) has a nontrivial solution.

To see that no eigenvalues ζ exist for small enough R let us estimate norm of $K(z)$ by simply using Lemma 1. Thus, if (13) has a nontrivial solution then

$$0 \neq \|f\| = \|K(z)f\| \leq \frac{\|B\|\,\|f\|}{(\mu + \text{Re}\,(z))^2} \leq \frac{\|B\|\,\|f\|}{\mu^2}$$

$$\mu^2 \leq \|B\|.$$

Therefore if
$$\|B\| < \mu^2 \tag{15}$$

then (13) has no nontrivial solutions. From (9) and from the definition of V we see that $\|B\|$ is proportional to R and therefore (15) will be true for R small enough (fixed U, α, β).

From here on assume in addition that

$$\sup_x |U''(x)|e^{3\mu x} < \infty.$$

If g_1 is absolutely continuous on $[0, x]$ for all $x > 0$ and if $g_1, g_1' \in L^2(0, \infty)$ then an integration by parts gives

$$g_1 = G(\lambda + 2\mu)((\lambda + 2\mu)g_1 - g_1').$$

Applying this identity to $g_1 = V'h^{-3}F(\lambda + \mu)g$, $g \in L^2(0, \infty)$, gives

$$(2\lambda + 3\mu)G(\lambda + 2\mu)V'h^{-3}F(\lambda + \mu)g$$

$$= V'h^{-3}F(\lambda + \mu)g + G(\lambda + 2\mu)V'h^{-3}g + G(\lambda + 2\mu)(V'h^{-3})'F(\lambda + \mu)g,$$

and if this is used to replace the last term in (9) then

$$Bg = -Vh^{-3}g + \frac{3\mu}{2\lambda + 3\mu}V'h^{-3}F(\lambda + \mu)g - \frac{2\lambda}{2\lambda + 3\mu}G(\lambda + 2\mu)V'h^{-3}g$$

$$- \frac{2\lambda}{2\lambda + 3\mu}G(\lambda + 2\mu)(V'h^{-3})'F(\lambda + \mu)g$$

for all $g \in L^2(0, \infty)$. Since $\|F(\lambda + \mu)\| \le 1/\mu$, $\|G(\lambda + 2\mu)\| \le 1/(2\mu)$ by Lemma 1, we have that

$$\|B\| \le |\alpha|R \left(c_1 + c_2/\mu + c_2/(2\mu) + c_3/(2\mu^2) \right) \tag{16}$$

where

$$c_1 = \sup_x |U(x) - U(\infty)|e^{3\mu x}$$

$$c_2 = \sup_x |U'(x)|e^{3\mu x}$$

$$c_3 = \sup_x |U''(x) + 3\mu U'(x)|e^{3\mu x}.$$

Therefore, if

$$|\alpha|R < c \equiv \mu^2/ \left(c_1 + 3c_2/(2\mu) + c_3/(2\mu^2) \right)$$

then (15) and (16) imply that there are no eigenvalues.

Appendix.

Lemma 1. *If $\nu \in \mathbb{C}$ and $\mathrm{Re}\,(\nu) > 0$ then the operators $F(\nu)$, $G(\nu)$ are bounded linear operators on $L^2(0, \infty)$; moreover,*

$$\|F(\nu)\| \le 1/\mathrm{Re}\,(\nu), \quad \|G(\nu)\| \le 1/\mathrm{Re}\,(\nu).$$

Furthermore, $F(\nu)$, $G(\nu)$ are holomorphic for $\mathrm{Re}\,(\nu) > 0$.

Proof: Pick $\nu \in \mathbb{C}$, $\kappa \equiv \mathrm{Re}\,(\nu) > 0$ and $g \in L^2(0, \infty)$. For $x \ge 0$ define

$$h_1(x) = \int_0^x e^{\kappa(t-x)}|g(t)|\,dt$$

$$h_2(x) = \int_x^\infty e^{\kappa(x-t)}|g(t)|\,dt.$$

Note that

$$h_1' + \kappa h_1 = |g| = \kappa h_2 - h_2' \quad \text{a.e.}$$

$$(h_1')^2 + (\kappa h_1)^2 + \kappa(h_1^2)' = |g|^2 = (\kappa h_2)^2 + (h_2')^2 - \kappa(h_2^2)' \quad \text{a.e.}$$

$$\kappa^2 \int_0^x h_1^2 \leq \int_0^x |g|^2 \geq -\kappa h_2^2(x) + \kappa^2 \int_0^x h_2^2 \,.$$

Therefore $\kappa\|h_1\| \leq \|g\|$; the Schwarz inequality implies $h_2(x) \to 0$ as $x \to \infty$ and hence $\kappa\|h_2\| \leq \|g\|$. Observe that $|F(\nu)g| \leq h_1$ and $|G(\nu)g| \leq h_2$.

If $\text{Re}\,(\nu) > 0$, $\text{Re}\,(\eta) > 0$ then Fubini's theorem implies that

$$F(\nu) - F(\eta) = (\eta - \nu)F(\eta)F(\nu);$$

thus if $\nu \neq \eta$ then

$$\left\| \frac{1}{\nu - \eta}(F(\nu) - F(\eta)) + F(\eta)F(\eta) \right\| \leq \frac{|\nu - \eta|}{\text{Re}\,(\nu)(\text{Re}\,(\eta))^2} \,.$$

Therefore, F is a holomorphic operator-valued function and the same is true for G by exactly the same argument.

Lemma 2. $\|K(\nu)\| \leq \frac{2\|B\|}{|\nu|(\mu + \text{Re}\,(\nu))} \leq \frac{2\|B\|}{|\nu|\mu}$ whenever $\text{Re}\,(\nu) \geq 0$, $\nu \neq 0$.

Proof: Pick $g \in L^2(0,\infty)$, $\text{Re}\,(\nu) \geq 0$, $\nu \neq 0$ and let $g_1 = hBg$,

$$g_2 = F(\nu + \mu)h^2 G(\nu + \mu)g_1.$$

Fubini's theorem implies

$$g_2(x) = \int_0^\infty \frac{e^{2\nu \min\{t,x\}} - 1}{2\nu} e^{-(\nu+\mu)x - (\nu+\mu)t} g_1(t)\, dt$$

and since $\text{Re}\,(\nu) \geq 0$ we have

$$|\nu g_2(x)| \leq \int_0^x e^{(\text{Re}\,(\nu)+\mu)(t-x)}|g_1(t)|\, dt + \int_x^\infty e^{(\text{Re}\,(\nu)+\mu)(x-t)}|g_1(t)|\, dt$$

and therefore by Lemma 1

$$\|g_2\| \leq \frac{2\|g_1\|}{|\nu|(\mu + \text{Re}\,(\nu))} \leq \frac{2\|B\|\,\|g\|}{|\nu|(\mu + \text{Re}\,(\nu))} \,.$$

Lemma 3. $h^2 G(\nu + \mu)h$ is a compact operator on $L^2(0,\infty)$ for $\text{Re}\,(\nu) > -\mu$.

Proof: If $g \in L^2(0,\infty)$ then

$$(h^2 G(\nu + \mu)hg)(x) = \int_0^\infty k(x,y)g(y)\, dy$$

$$|k(x,y)| \leq e^{-2\mu x - \mu y}.$$

Therefore

$$\int_0^\infty \int_0^\infty |k(x,y)|^2 dx\, dy < \infty$$

and this completes the proof [1, p. 518, 4, p. 264].

Lemma 4 [4, p. 370, 1, p. 592]. *Let $\mathcal{D}$ be an open connected subset of $\mathbb{C}$. Suppose that $T(\xi)$, for $\xi \in \mathcal{D}$, is a compact operator on a Banach space X and that the mapping $\xi \to T(\xi)$ is holomorphic for $\xi \in \mathcal{D}$ (i.e., T is differentiable in operator norm). If $\|T(\xi_0)\| < 1$ for some $\xi_0 \in \mathcal{D}$ then in each compact subset of $\mathcal{D}$ there exist at most finitely many ξ for which there exists $g \in X$, $g \neq 0$, such that $g = T(\xi)g$.*

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