Lecture 7

Non-modal stability analysis I

AE209 Hydrodynamic stability

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Lecture outline

1. **Motivation**
2. **Initial value problem of linearised equation**
3. **Transient growth and non-normal linear operator**

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2. Initial value problem of linearised equation
3. Transient growth and non-normal linear operator
4. Non-normal energy growth in inviscid and viscous flows

Limitation of linear stability analysis

Critical Reynolds numbers from linear stability analysis

Flow configurations	Critical Re for energy growth	Critical Re (Linear stability)	Transition Re
Couette flow	20.7	∞	350-400
Poiseulle flow	49.6	5772.2	1000-2000
Pipe flow	81.5	∞	2000-2500

Remark

Nonlinear terms in the form of perturbed Navier-Stokes equation play **no role in the mechanism of disturbance growth.**

Limited role of nonlinear terms

Reynolds-Orr equation

$$\int_V \frac{1}{2}\frac{\partial u_i u_i}{\partial t}\,dV = -\int_V u_i u_j \frac{\partial U_i}{\partial x_j}\,dV - \frac{1}{\mathrm{Re}}\int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\,dV$$

$$\frac{\partial u_i}{\partial t} = -u_j\frac{\partial U_i}{\partial x_j} - U_j\frac{\partial u_i}{\partial x_j} - u_j\frac{\partial u_i}{\partial x_j} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u_i}{\partial x_j \partial x_j}, \qquad \frac{\partial u_j}{\partial x_j} = 0$$

1. Motivation
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What is missing in the classical linear stability analysis?

Full solution of a linearised equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{L}\mathbf{u} \quad \text{with} \quad \mathbf{u}(t=0) = \mathbf{u}_0$$

Assume that $\mathbf{L}$ is a diagonalisable matrix with the eigenvectors given by $\{\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2, \tilde{\mathbf{u}}_3, \tilde{\mathbf{u}}_4 ...., \tilde{\mathbf{u}}_n\}$. Then, the solution of the equation above is obtained by the eigenfunction expansion technique:

$$\mathbf{u}(t) = a_1(t)\tilde{\mathbf{u}}_1 + a_2(t)\tilde{\mathbf{u}}_2 + a_3(t)\tilde{\mathbf{u}}_3 + ... + a_n(t)\tilde{\mathbf{u}}_n$$

Then,

$$\frac{d\mathbf{Ua}}{dt} = \mathbf{LUa}$$

$$\frac{d\mathbf{a}}{dt} = \mathbf{\Lambda a} \quad \text{with} \quad \mathbf{a}(t=0) = \left[a_{1,0} \;\; a_{2,0} \;\; a_{3,0} \;\; a_{4,0} \;\; .... \;\; a_{n,0}\right]^T$$

Full solution of the linear system is then given by

$$\mathbf{u}(t) = a_{1,0}e^{\lambda_1 t}\widetilde{\mathbf{u}}_1 + a_{2,0}e^{\lambda_2 t}\widetilde{\mathbf{u}}_2 + a_{3,0}e^{\lambda_3 t}\widetilde{\mathbf{u}}_3 + \ldots\ldots + a_{n,0}e^{\lambda_n t}\widetilde{\mathbf{u}}_n$$

Eigenvalue problem

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The structure of linearised Navier-Stokes equation

Velocity and vorticity form of linearised Navier-Stokes equation

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^4\right]v'=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^2\right)\eta'+\frac{dU}{dy}\frac{\partial v'}{\partial z}=0$$

Consider an initial value problem of the following wave by setting

$$v'(x,y,z,t)=\hat{v}(t;\alpha,\beta)e^{i\alpha x+i\beta z}$$

$$\eta'(x,y,z,t)=\hat{\eta}(t;\alpha,\beta)e^{i\alpha x+i\beta z}$$

The structure of linearised Navier-Stokes equation

Matrix form of initial value problem for Orr-Sommerfeld-Squire system:

$$\frac{\partial}{\partial t}\begin{bmatrix} k^2 - D^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} + \begin{bmatrix} L_{OS} & 0 \\ i\beta DU & L_{SQ} \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} = 0$$

where $k^2 = \alpha^2 + \beta^2$

$$L_{OS} = i\alpha U(k^2 - D^2) + i\alpha D^2 U + \frac{1}{\mathrm{Re}}(k^2 - D^2)^2$$

$$L_{SQ} = i\alpha U + \frac{1}{\mathrm{Re}}(k^2 - D^2)$$

Remark

As $\mathrm{Re} \to \infty$, the role of the off-diagonal term becomes more and more important.

Transient growth in a linear model

A model problem

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} -1/\mathrm{Re} & 0 \\ 1 & -2/\mathrm{Re} \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix}$$

with the initial condition $\begin{bmatrix} v & \eta \end{bmatrix}_{t=0} = \begin{bmatrix} v_0 & \eta_0 \end{bmatrix}$

Solution) from page 9

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = a_{1,0} e^{\lambda_1 t} \begin{bmatrix} \hat{v}_1 \\ \hat{\eta}_1 \end{bmatrix} + a_{2,0} e^{\lambda_2 t} \begin{bmatrix} \hat{v}_2 \\ \hat{\eta}_2 \end{bmatrix}$$

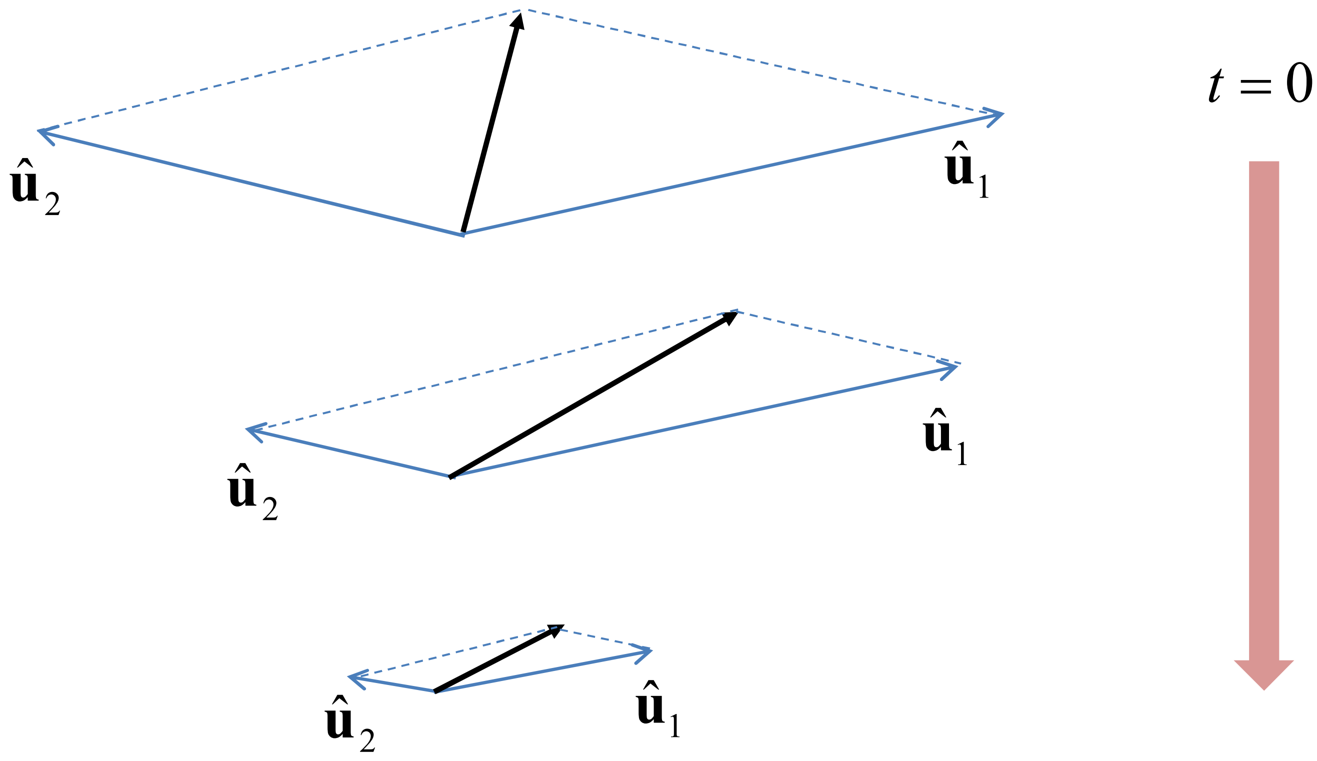
with $\lambda_1 = -1/\mathrm{Re}$ and $\lambda_2 = -2/\mathrm{Re}$

$$\begin{bmatrix} \hat{v}_1 \\ \hat{\eta}_1 \end{bmatrix} = \frac{1}{\sqrt{1+\mathrm{Re}^2}}\begin{bmatrix} 1 \\ \mathrm{Re} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{v}_2 \\ \hat{\eta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Transient growth in a linear model

Schematic diagram of temporal evolution of the solution

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = v_0 e^{-t/\mathrm{Re}} \begin{bmatrix} 1 \\ \mathrm{Re} \end{bmatrix} + \left(\eta_0 - v_0\,\mathrm{Re}\right) e^{-2t/\mathrm{Re}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Remark 1

1) **In the limit of** $\mathrm{Re} \to 0$, the two eigenvectors are orthogonal to each other, yielding a **monotonically decaying solution in time**: i.e.

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} v_0 e^{-1/\mathrm{Re}\, t} \\ \eta_0 e^{-2/\mathrm{Re}\, t} \end{bmatrix}$$

2) **In the limit of** $\mathrm{Re} \to \infty$, $\lambda_1 = \lambda_2 = 0$, and two eigenvectors become the same, yielding the following **algebraically growing solution**: i.e.

$$\begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = (\eta_0 + v_0 t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + v_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Transient growth in a linear model

Remark 2

1) The non-orthogonal superposition of exponentially decaying solutions can give rise to short-term transient growth.

2) **Eigenvalues alone** only describe **the asymptotic fate of the disturbance**, but **fail to capture transient** effects.

3) The **“source” of the transient amplification** of the initial condition lies in the **nonorthogonality of the eigenfunction basis**.

4) The **non-orthogonal eigenfunctions** are the typical nature of the **non-normal linear operator**.

Non-normal linear operator

Definition: Non-normal operator

Linear operators, the eigenfunctions (or eigenvectors) of which are **non-orthogonal** to one another with respect to the given inner product, is cal led **non-normal**.

Remark

Linearised Navier-Stokes equation with non-zero advection term is **a non -normal** linear operator

Summary

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