Lecture 3

Basic dynamical system theory II

Bifurcation theory.

AEM-ADV12 Hydrodynamic stability
Dr Yongyun Hwang

Lecture outline

1. Transcritical bifurcation
2. Saddle-node bifurcation
3. Pitchfork bifurcation
4. Hopf bifurcation

Lecture outline

1. Transcritical bifurcation

Bifurcation

Definition: Bifurcation

Bifurcation refers to a **sudden topological change** of given nonlinear dynamical system taking place **when a control parameter changes smoothly.**

Re. Angle of attack.

Ra.

Example: Transcritical bifurcation → Reynolds number.

Find the bifurcation diagram of a model given by

$$\frac{du}{dt} = k(R - R_c)u - lu^2$$

→ critical R_c.

where k, l are constants and R is the control parameter.

Step 1) Find equilibrium points ⇒ Set $\frac{du}{dt} = 0$.

$$0 = k(R-R_c)u_0 - lu_0^2$$

$$= [k(R-R_c) - lu_0]u_0.$$

$$\Rightarrow \quad u_0 = 0 \quad \text{and} \quad u_0 = \frac{k}{l}(R-R_c)$$

Step 2) Examine linear stability of the equlibrium points

① $u_0 = 0$ Let $u = u_0 + \varepsilon \delta u$ with $\varepsilon \ll 1$

Basic State

$\delta u(t) = C_1 e^{k(R-R_c)t}$

$$\frac{d\delta u}{dt} = \left.\frac{\partial f}{\partial u}\right|_{u=u_0} \delta u. = (k(R-R_c) - \cancel{2\ell u_0})\,\delta u.$$

$$= k(R-R_c)\,\delta u. \quad (\text{assume } k>0)$$

$R > R_c$: unstable.

$R < R_c$: stable.

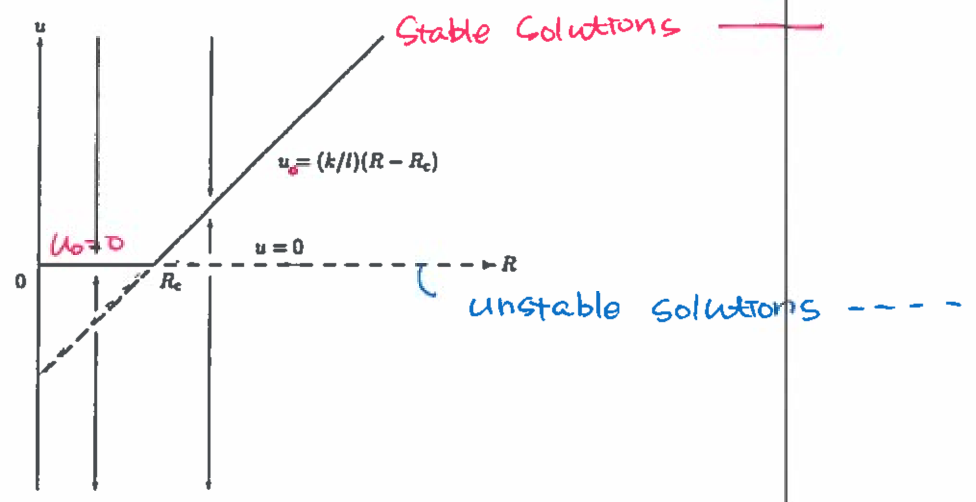
② $u_0 = \frac{k}{\ell}(R-R_c)$

$$\frac{d\delta u}{dt} = -k(R-R_c)\,\delta u$$

$R > R_c$: stable

$R < R_c$: unstable

Transcritical bifurcation



Bifurcation diagram of transcritical bifurcation

At $R = R_c$, two equilibrium points interchange their stability

2. Saddle-node bifurcation $\mapsto$ Couette flow
Pipe flow.
Poiseulle flow

Example: Saddle-node bifurcation

Find the bifurcation diagram of a model of given by

$$\frac{du}{dt} = k(R - R_c) - lu^2$$

where k, l are real constants and R is the control parameter. $(k>0,\ l>0)$

Step 1) Find equilibrium points $u \in \mathbb{R}$.

$$0 = k(R-R_c) - l u_0^2$$

$$u_0 = \pm\sqrt{\frac{k}{l}(R-R_c)}$$

: Equilibrium points exist only for $R > R_c$

Step 2) Examine linear stability of the equlibrium points

Stable Node.

① $u_0 = \sqrt{\frac{k}{\ell}(R-R_c)}$ Let $u = u_0 + \varepsilon\, \delta u$.

$$\frac{d\delta u}{dt} = -2\ell u_0\, \delta u. = \underline{-2\ell\sqrt{\frac{k}{\ell}(R-R_c)}}\, \delta u.$$

$R>R_c$ Stable.

$R<R_c$: does not exist.

Saddle.

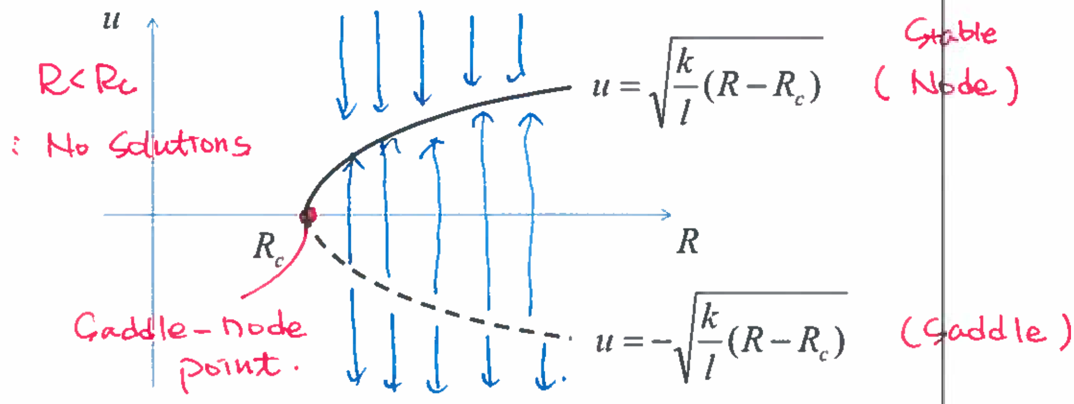
② $u_0 = -\sqrt{\frac{k}{\ell}(R-R_c)}$

$$\frac{d\delta u}{dt} = -2\ell u_0\, \delta u = \underline{2\ell\sqrt{\frac{k}{\ell}(R-R_c)}}\, \delta u$$

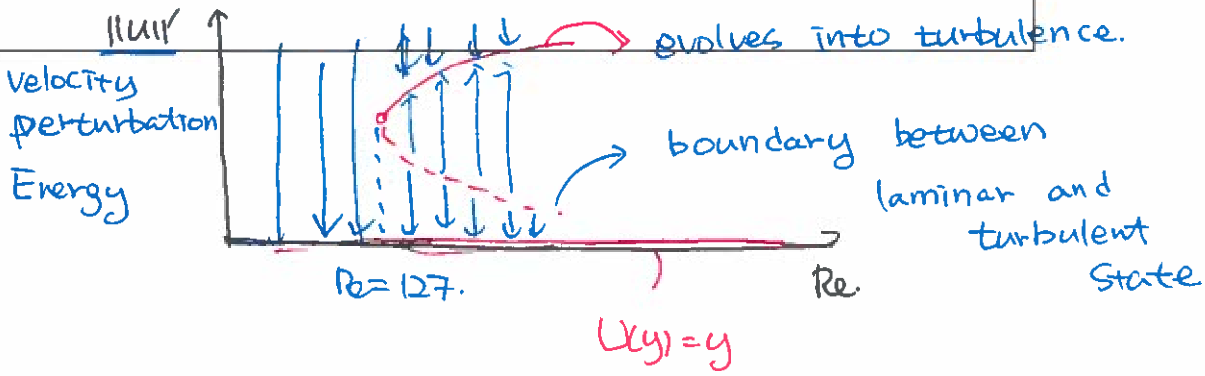
$R>R_c$: unstable.

Saddle-node bifurcation

Example: Saddle-node bifurcation



Bifurcation diagram of Saddle-node bifurcation



3. Pitchfork bifurcation

Example: Pitchfork bifurcation

Find the bifurcation diagram of a model of given by

$$\frac{du}{dt} = k(R - R_c)u - lu^3$$

where k, l are real constants and R is the control parameter. $(k>0,\ l>0)$

Step 1) Find equilibrium points

$$0 = k(R-R_c)u_0 - l u_0^3$$

$$= u_0\left(k(R-R_c) - l u_0^2\right)$$

$u_0 = 0$ and $u_0 = \pm\sqrt{\frac{k}{l}(R-R_c)}$

exist if $R > R_c$

Step 2) Examine linear stability of the equlibrium points

① $u_0 = 0$. and $u = u_0 + \varepsilon \delta u$.

$$\frac{d\delta u}{dt} = \underline{k(R-R_c)}\,\delta u.$$

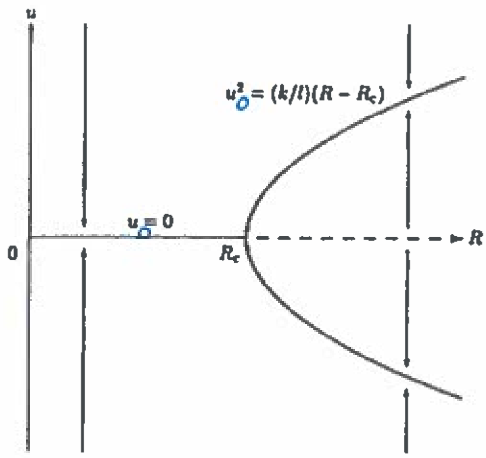
$R>R_c$: unstable
$R<R_c$: Stable.

② $u_0 = \pm\sqrt{\frac{k}{l}(R-R_c)}$

$$\frac{d\delta u}{dt} = \underline{-2k\,(R-R_c)}\,\delta u.$$

$R>R_c$ stable
$R<R_c$ does not exist.

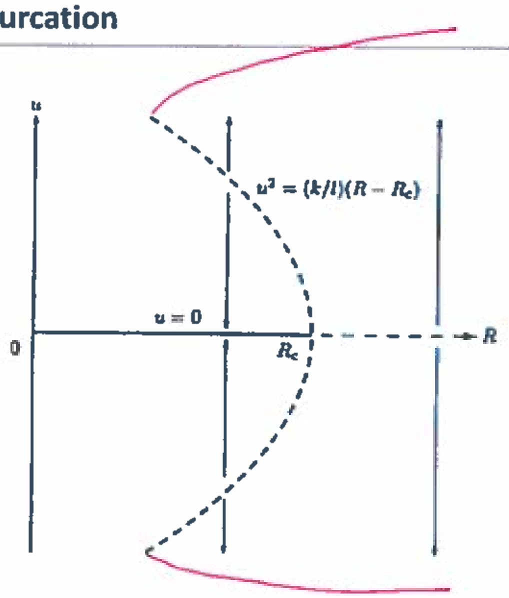
Pitchfork bifurcation



3D Bluff-body wake

Bifurcation diagram of Pitchfork bifurcation
(supercritical case, i.e. $l > 0$)

Pitchfork bifurcation



: Plane Poiseulle flow

Bifurcation diagram of Pitchfork bifurcation
(subcritical case, i.e. $l < 0$)

Pitchfork bifurcation

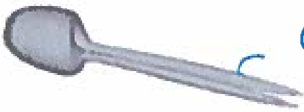
Flow example: Wake behind a sphere

$$\mathrm{Re}_D = \frac{U_\infty D}{\nu}$$

$\mathrm{Re}_{D,critical} \approx 210$

$\mathrm{Re}_D = 100$ → $\mathrm{Re}_D = 250$

(a) 

(b) 

like axial vorticity.

Steady axisymmetric **Steady planar symmetric**

Kim & Choi (2001)

4. Hopf bifurcation → Wake behind a 2D bluff body.

Example 1: Hopf bifurcation

Find the bifurcation diagram a model given by

① — $\frac{dx}{dt} = -y + (a - x^2 - y^2)x$, $\frac{dy}{dt} = x + (a - x^2 - y^2)y$, ②

where $a = k(R - R_c)$ and $k > 0$.

Let $x = r\cos\theta$ $y = r\sin\theta$. where $r(t)$ and $\theta(t)$

$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}$, $\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$.

At ①: $\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt} = -r\sin\theta + (a - r^2)\cos\theta$.

$\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt} = r\cos\theta + (a - r^2)\sin\theta$.

$\Rightarrow \frac{dr}{dt} = r(a - r^2)$, $\frac{d\theta}{dt} = 1$

$(r > 0)$.

$$\frac{dr}{dt} = r(a-r^2)$$

$r_0 = 0$ — Stable for $R<R_c$; unstable for $R>R_c$.

$r_0 = \sqrt{a}$ → Stable for $R>R_c$.

$$\frac{d\theta}{dt} = 1$$

$\theta(t) = t + \text{const.}$

① $r_0 = 0$, $\theta(t) = t+\text{const.}$

$\Rightarrow$ $x_0 = 0$ $y_0 = 0$.

Stable for $R<R_c$

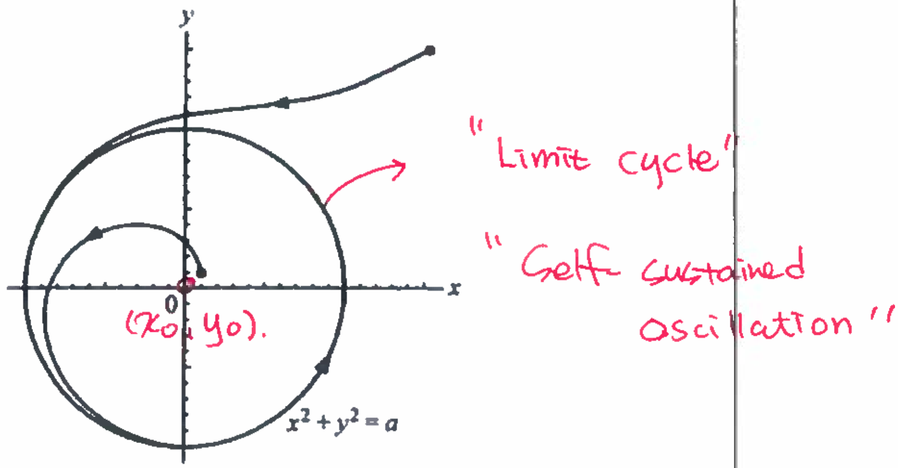
unstable for $R>R_c$.

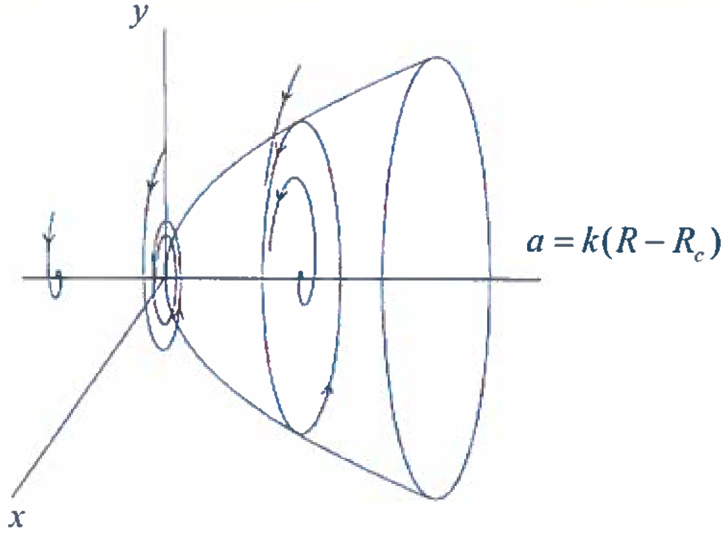
② $r_0 = \sqrt{a}$ $\theta(t) = t + \text{const.}$

$x_0(t) = \sqrt{a}\cos(t+\text{const})$ $y_0(t) = \sqrt{a}\sin(t+\text{const.})$

Time periodic solution. Stable if $R>R_c$

Phase portrait for $a > 0$ $(R > R_c)$





Bifurcation diagram of
supercritical Hopf bifurcation

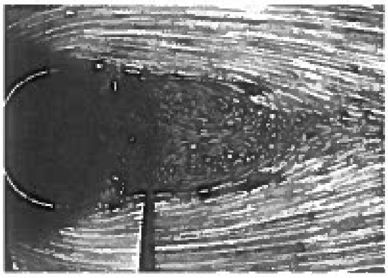
Hopf bifurcation

Flow example: Wake behind a circular cylinder

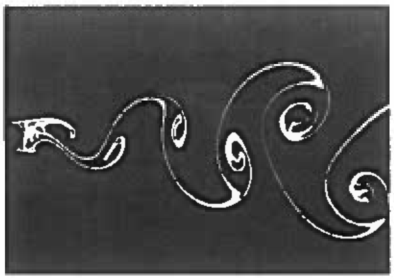
$$\mathrm{Re}_D = \frac{U_\infty D}{\nu}$$

$$\mathrm{Re}_{D,critical} \approx 47$$

$$\mathrm{Re}_D = 27 \qquad \mathrm{Re}_D = 140$$



Steady symmetric
Coutanceau & Bouard (1977)



Unsteady time periodic
Taneda (1982)

1. Transcitical bifurcation
2. Saddle-node bifurcation
3. Pitchfork bifurcation
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