1. Consider the following two-dimensional dynamical system: Lecture note ~~3~~ 4

$$\frac{dx}{dt} = -y + (\mu - x^2 - y^2)x,$$

$$\frac{dy}{dt} = x + (\mu - x^2 - y^2)y, \tag{1}$$

where μ is the real-valued bifurcation control parameter.

(a) Using $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$, show that the system (1) can be transformed as follows: [40%]

$$\frac{dr}{dt} = r(\mu - r^2), \quad \frac{d\theta}{dt} = 1,$$

where $r > 1$.

(b) The system (1) has an equilibrium solution and a time periodic solution. Find these two solutions and their domain of existence with respect to μ. [30%]

(c) Determine linear stability of the two solutions obtained in (b) for all μ. [20%]

(d) Using the answers from (b) and (c), draw the bifurcation diagram of the system (1) and classify the type of the bifurcation. [10%]

2. Consider the linearized Navier-Stokes equation around a parallel base flow $\mathbf{U} = (U(y), 0, 0)$. If the perturbation velocity is assumed to be $\mathbf{u}(x, y, z, t) = \hat{\mathbf{u}}(y)e^{i(\alpha x+\beta z)}$ with the streamwise wavenumber α and the spanwise wavenumber β, the following form of the linearized Navier-Stokes equation is obtained with the transverse velocity $\hat{v}$ and the transverse vorticity $\hat{\eta}$:

$$[(\frac{\partial}{\partial t} + i\alpha U)(\mathcal{D}^2 - k^2) - i\alpha\frac{d^2U}{dy^2} - \frac{1}{Re}(\mathcal{D}^2 - k^2)^2]\hat{v} = 0, \tag{2}$$

$$[(\frac{\partial}{\partial t} + i\alpha U) - \frac{1}{Re}(\mathcal{D}^2 - k^2)]\hat{\eta} + i\beta\frac{dU}{dy}\hat{v} = 0, \quad \leftarrow \times \tag{3}$$

where $k^2 = \alpha^2 + \beta^2$, $\mathcal{D} = \partial/\partial y$, and Re is the Reynolds number.

(a) First, consider a special case where fluid flow is two-dimensional and inviscid (i.e. $\hat{\eta} = 0$, $\beta = 0$ and $Re \to \infty$). For a given α, we consider the normal-mode solution of (2), such that $\hat{v}(y, t) = \tilde{v}(y)e^{-i\alpha ct}$ where c is the unknown complex phase velocity. Demonstrate that the base flow contains some points where $d^2U/dy^2 = 0$, if (2) is linearly unstable (i.e. $c_i > 0$ where c_i is the imaginary part of c). Rayleigh inflexion point theorem (Lecture 4) [40%]

(b) Now, we return to the case of three-dimensional and viscous fluid flow. In this case, (3) is known not to generate any instability. Assuming that (2) causes an instability, show that the lowest critical Reynolds number for the onset of the instability is always given when $\beta = 0$. Squire's theorem (Lecture 5) [30%]

(c) Lastly, we consider a case where there is no instability in (2) and (3) even at sufficiently high Reynolds numbers. If we assume that (2) and (3) have only discrete non-degenerate eigenvalues, their general solution is given by

$$\hat{\mathbf{x}}(t) = a_1 e^{-i\alpha c_1 t}\tilde{\mathbf{x}}_1 + a_2 e^{-i\alpha c_2 t}\tilde{\mathbf{x}}_2 + ... + a_n e^{-i\alpha c_n t}\tilde{\mathbf{x}}_n + ..., \tag{4}$$

for $n = 1, 2, 3, ...$, where $\hat{\mathbf{x}}(t) = [\hat{v}(t)\ \hat{\eta}(t)]^T$, a_n are the constant coefficients from the given initial condition, c_n the eigenvalues of the system (complex phase velocity), and $\tilde{\mathbf{x}}_n = [\tilde{v}_n\ \tilde{\eta}_n]^T$ the corresponding eigenfunctions, respectively. Using this form of the solution, explain a possible short-term transient growth mechanism of the perturbation velocity field in relation to the non-orthogonal nature of the eigenfunctions of (2) and (3). Lecture note 7 [30%]

3. Consider the equation: Lecture note 9–10. $\varphi(x,t) = \varepsilon \varphi'(x,t)$

$$\frac{\partial \psi(x,t)}{\partial t} - \frac{\partial \psi(x,t)}{\partial x} = \mu\psi(x,t) + \frac{\partial^2 \psi(x,t)}{\partial x^2} + |\psi(x,t)|^2\psi(x,t) - |\psi(x,t)|^4\psi(x,t), \quad (5)$$

$O(\varepsilon)$ $O(\varepsilon)$ $O(\varepsilon)$ $O(\varepsilon)$ $O(\varepsilon^3)$ $O(\varepsilon^5)$

where $\psi(x,t)$ is complex and μ is the real-valued bifurcation control parameter.

(a) Equation (5) has an equilibrium solution given by $\psi_0(x,t) = 0$. Consider a perturbation to the equilibrium solution such that $\psi(x,t) = \psi_0(x,t) + \epsilon\psi'(x,t) + O(\epsilon^2)$. Then, show that the linearised version of (5) is given as follows: [30%]

$$\frac{\partial \psi'(x,t)}{\partial t} - \frac{\partial \psi'(x,t)}{\partial x} = \mu\psi'(x,t) + \frac{\partial^2 \psi'(x,t)}{\partial x^2}. \quad (6)$$

(b) Now, consider the normal-mode solution of (6), such that $\psi'(x,t) = Ae^{i(kx-\omega t)}$ where A is a constant, k the spatial wavenumber and ω the complex angular frequency. Find the dispersion relation of (6). $\frac{\partial}{\partial t} \to -i\omega$, $\frac{\partial}{\partial x} \to ik$, $\varphi \to A$ [30%]

(c) Plot the neutral stability curve of the basic state $\psi(x,t) = 0$ with respect to k and μ by performing a temporal stability analysis. [20%]

(d) Find the condition for the linearised equation obtained in (a) to be absolutely unstable in terms of μ. [20%]