Lecture 1

Question 1. What must hold for (G, *) to be a group?

Answer. (i) Identity exists
(ii) Inverse exists
(iii) (x*y)*z = x*(y*z)

Question 2. Sketch a proof for the uniqueness of inverses

Answer. Suppose y,z are inverses of x. y = y*e = y*(x*z) and use associativity to show y = z.

Motivation: think about this fact is contingent on - if associativity didn't hold, it'd make sense for there to exist a left inverse and a right inverse. Therefore, we want to try and sneak associativity into this. The only way we can do this is by introducing three terms.

Question 3. Show that the identity element is unique

Answer. Formally state what it means for e and $\hat{e}$ to be identities and show that this implies they are equal.

Lecture 2

Question 4. When is a group H cyclic?

Answer. If there exists an element h in H such that each element of H is a power of h.

Question 5. Prove the only subset of Z under addition is nZ

Answer. If $H = \{0\} = 0\mathbb{Z}$

Otherwise, choose $0 < n \in H$ with n minimal. $n\mathbb{Z} \subseteq H$ by closure and inverses. We show $H = n\mathbb{Z}$. Suppose $\exists h \in H \setminus n\mathbb{Z}$, can write $h = nk + h'$, with $h' \in \{1, 2, \dots, n-1\}$

But $h' = h - nk \in H$, contradicting minimality of n. Thus $H = n\mathbb{Z}$

Question 6. When is the function

$$\theta : G \to H$$

homomorphism from $(G, *_G)$ to $(H, *_H)$?

Answer.

$$\theta(x *_G y) = \theta(x) *_H \theta(y)$$

$\forall x, y \in G$

Question 7. Let $\theta : G \to H$ be a homomorphism. What is the image of θ?

Answer. $\theta(G) = \{\theta(g) : g \in G\}$

Question 8. Prove the image of $\theta(G) : G \to H$ is a subgroup of H

Answer. Consider axioms

Lecture 3

Question 9. Prove the composition of two homomorphisms is a homomorphism.

Answer. Do it.

Question 10. Let $o(g) = m$, and $n \in \mathbb{N}$. Prove that

$$g^n = e \iff m|n$$

Answer. $\Leftarrow$ trivial verification
$\Rightarrow$ Suppose $g^n = e$ and write $n = qm + r$, $0 \leq r < m$.

Lecture 4

Question 11. How do we algebraically write D_{2n}?

Answer.

$$D_{2n} =< r, t | r^n = e, t^2 = e, trt = r^{-1} >$$

Question 12. Sketch a geometric proof that the group D_{2n} is $\{e, r, r^2, \ldots, r^{n-1}, t, rt, \ldots, r^{n-1}t\}$ under composition.

Answer. Let f be a symmetry of P f(1) maps to another vertex of P (say k). Let g be the rotation of f such that f(1) = 1. The vertex adjascent to 1 is either fixed or reflected. If it is fixed, g is the identity, meaning f is a rotation. Otherwise, g is a reflection then a rotation.

Lecture 5

Question 13. How do we denote the set of all permutations of X?

Answer. Sym(X).

Question 14. What is the symmetric group of degree n?

Answer.

$$S_n = Sym(\{1, 2, \ldots, n\})$$

the set of all permutations of $\{1, 2, \ldots, n\}$.

Question 15. Prove Sym(X) is a group under composition.

Answer. *closure - composition of bijections forms a bijection
*identity, define $c(x) = x$ for all x in X
*inverse - $f \in Sym(X)$, since f is a bijection, f^{-1} exists and is a bijection, satisfying $f^{-1} \circ f = e = f^{-1} \circ f$
*associativity - composition of functions is associative

Question 16. Define (informally) a k-cycle

Answer. $\sigma = (a_1, a_2, \ldots, a_k)$, mapping a_1 to a_2, a_2 to $a_3, \ldots, a_k$ to a_1. It leaves all other elements of S_n fixed.

Question 17. Define (formally) a k-cycle

Answer. Let $a_1, \ldots a_k$ be distinct integers in $\{1, \ldots n\}$. Suppose $\sigma \in S_n$ and

$$\sigma(a_i) = a_{i+1}$$

$$\sigma(a_k) = a_1$$

for $1 \leq i \leq k-1$, and $\sigma(x) = x, \;\; \forall x \in \{1, \ldots, n\}/\{a_1, \ldots, a_n\}$.

Question 18. Show that disjoint cycles σ, τ commute

Answer. Consider logically what happens and write it algebraically

Lecture 6

Question 19. How do we write a permutation σ as a product of disjoint cycles?

Answer. $a_1 \in \{1, 2, \ldots, n\} = X$
Consider $a_1, \sigma(a_1), \sigma^2(a_1), \ldots$
Since X is finite, $\exists$ minimal j s.t. $\sigma^j(a_i) \in \{a_1, \sigma(a_1), \ldots, \sigma^{j-1}(a_1)\}$

Claim: $\sigma^j(a_1) = a_1$.
Since if not,
$\sigma^j(a_1) = \sigma^i(a_i)$ for $j > i \geq 1$
$\sigma^{j-i}(a_1) = a_1$ contradicting minimality of j.

Question 20. Show a k-cycle can be written as a product of transpositions

Answer. $(a_1, a_2, \ldots, a_n) = (a_1, a_2)(a_2, a_3), \ldots (a_{k-2}, a_{k-1})(a_{k-1}, a_k)$.

Question 21. Let σ, τ be disjoint cycles in S_n. Prove $o(\sigma\tau) = \text{lcm}(o(\sigma), o(\tau))$

Answer. ($\Leftarrow$)verify, recall σ and τ commute because disjoint
($\Rightarrow$) Suppose $o(\tau\sigma) = n$,

$$(\sigma\tau)^n = e$$

$$\sigma^n \tau^n = e$$

But σ, τ move different elements of X,
So $\sigma^n = e$, $\tau^n = e$
So $o(\sigma)|n$ and $o(\tau)|n$
So $k = \text{lcm}(o(\sigma), o(\tau))$

Question 22. What does it mean that $sgn(S_n)$ is well defined?

Answer. If

$$\sigma = \tau_1 \dots \tau_a$$
$$= \tau_1' \dots \tau_b'$$

then $(-1)^a = (-1)^b$.

Question 23. Sketch a proof that the function
sgn: $S_n \to \{\pm 1\}$ $\sigma \to \text{sgn}(\sigma)$
is well-defined.

Answer. Given $\tau = (a, b)$ is a transposition, we show $\sigma\tau \mod 2 \cong \sigma + 1$ by considering the two cases: a and b are in the same cycle, and a and b are in distinct cycles.
In both cases we can rewrite the product of cycles such that the claim holds.
We know c(identity) = n, so assuming σ has two decompositions, we can show a = b mod 2, from which claim holds.

Question 24. Suppose (k,l) are in the same cycle within σ. How do we "factor" $\sigma(k, l)$ into disjoint cycles?

Answer.

$$(k, a_1, \dots, a_r, l, b_1, \dots, b_s)(k, l)$$
$$(k, b_1, \dots, b_s)(l, a_1, \dots, a_r)$$

Question 25. Suppose (k,l) are in different cycles of σ. How do we "factor" $\sigma(k, l)$ into disjoint cycles?

Answer.

$$(k, a_1, \dots, a_r)(l, b_1, \dots, b_s)(k, l)$$
$$= (k, b_1, b_2, \dots, b_s, l_1, a_2, \dots, a_r)$$

Lecture 7

Question 26. Sketch a proof of the claim that the map

$$\text{sgn}(S_n, \circ) \to (\{\pm 1\}, \times)$$

is a well-defined and non-trivial homomorphism. (assuming sgn well defined)

Answer. - Well defined since $sgn(S_n)$ is well defined. - $sgn((1,2)) = -1$, so non-trivial. (not just equal to identity)

Question 27. Sketch a proof of the claim that the map

$$\text{sgn}(S_n, \circ) \to (\{\pm 1\}, \times)$$

is a homomorphism (assuming well-defined and non-trivial)

Answer. Suppose $a, b \in S_n$ Writing out a,b as transpositions it follows that sgn(ab) = sgn(a)sgn(b).

Question 28. Let $H \leq G$, $a, b \in G$. Prove
$a^{-1}b \in H \Rightarrow aH = bH$

Answer. Suppose $a^{-1}b = k \in H$
$b = ak \in aH$
also $b \in bH$
So $aH = bH$.

Question 29. What is a coset of G?

Answer. Let $H \leq G$, and $g \in G$. The left coset gH is defined to be

$$\{gh : h \in H\}$$

, similar for right coset.

Question 30. What is the index of H in G denoted $|G : H|$?

Answer. It is the number of left cosets of H in G.

Question 31. What is $|G : H|$ for finite G,H in terms of the order of G and H?

Answer. $|G : H| = |G|/|H|$.

Question 32. Prove that if $aH \cap bH$ isn't empty, then $aH = bH$.

Answer. Let c be an element in the intersection. $c = ak$ for some k in H, by writing set definition of cH we see that cH is a improper subset of aH.

Similarly, $a = ck^{-1}$, which is an element of cH
So aH is a improper subset of cH.
So $aH = cH$. Similarly $cH = bH$ so $aH = bH$.

Question 33. Let $H \leq G$, $a, b \in G$. Prove $aH = bH \to a^{-1}b \in H$

Answer. $b \in bH = aH$ $b = ah$ for some $h\ inH$ $a^{-1}b = h \in H$

Question 34. What does Lagrange's Theorem state?

Answer. Let H be a subgroup of the finite group G. Then the order of H divides the order of G.

Question 35. Prove Lagrange's theorem.

Answer. G is partitioned into k distinct cosets of H. Each coset is the same size. So $|G| = |H|$ k.

Question 36. Sketch a proof of Lagranges corollary.

Answer. Apply Lagranges theorem to the subgroup $\rangle g \langle$ of G.

Question 37. State Lagranges corollary

Answer. G is a finite group and $g \in G$. Then $o(g) \big| |G|$. In particular, $g^{|G|} = e$.

Lecture 8

Question 38. When is a subgroup K called normal?

Answer. if $gK = Kg$ for all $g \in G$.

Question 39. State the Fermat-Euler Theorem

Answer. Let $a \in \mathbb{N}, n \in \mathbb{Z}, (a, n) = 1$.

$$a^{\phi(n)} \equiv 1 \mod n$$

Question 40. Prove that the inverse of $a \in R_n^*$ is in R_n^*

Answer. Claim follows from Bezout on (a,n)=1 and rearranging.

Lecture 9

Question 41. What is the first isomorphism theorem?

Answer. Let G, H be groups and

$$\theta : G \to H$$

a group homomorphism. Then $\text{Im}\theta \leq H$, $\text{Ker}\theta \trianglelefteq G$
and $G \backslash \text{Ker}\theta \cong \text{Im}\theta$

Question 42. (Sketch) proof that $K \trianglelefteq G$ means $G \backslash K$ exists

Answer. Check coset multiplication is well defined, ie. that two cosets multiplied always give the same result (this $\iff$ normal). Verify closure, identity and inverse.

Question 43. Prove the 1st isomorphism theorem

Answer. Construct an isomorphism ϕ

$$G\backslash\mathrm{Ker}\theta \to \mathrm{Im}\theta$$

Let $K = \mathrm{Ker}\theta$,

$$gK \to \theta(g)$$

Prove ϕ is well-defined, homomorphism and bijection.

Lecture 10

Question 44. What is a simple group?

Answer. A group with no non-trivial normal subgroups.

Question 45. Prove that a homomorphism $\theta : G \to H$ is injective iff

$$\mathrm{Ker}\theta = \{e\}$$

Answer. Consider both directions to show it is injective.

Question 46. (i) Let $N \trianglelefteq G$, $H \leq G$. Prove that then $NH \leq G$.
(ii) Let $N \trianglelefteq G$, $M \trianglelefteq G$, prove then $NM \trianglelefteq G$.

Answer. diy

Lecture 11

Question 47. Let G be a group with subgroups H and K.

Answer. Consider both directions to show it is injective.

Question 48. Prove that if,
(i) each elem of G can be written as hk ($h \in H$, $k \in K$)
(ii) $H \cap K = \{e\}$
(iii) $hk = kh \forall k \in K, h \in H$,
then
$G \cong H \times K$

Answer. Show the map

$$\theta : H \times K \to G$$

$$\theta(h, k) \to hk$$

is an isomorphism.

Question 49. There are alternate, equivalent definitions of internal direct product.
Show that
(i) each elem of G can be written as hk ($h \in H$, $k \in K$)

(ii) $H \cap K = \{e\}$
(iii) $hk = kh \forall k \in K, h \in H$,
is equivalent to (i)' $H \trianglelefteq G, K \trianglelefteq G$ (ii)' $H \cap K = \{e\}$
(iii) HK = G.

Answer. $\Rightarrow$ use (iii) to show (i)' and (i) implies (iii)'
$\Leftarrow$ Use (i)' to show (iii), (i) immediate from (iii)

Lecture 12

Question 50. Classify groups of order 6 up to isomorphism.

Answer. Consider order of elements with lagrange. if $o(g) = 6$, then $G \cong C_6$.
Can't have all elems order 2, since $|G|$ isn't a power of two.
We always have an element a of order 3. (g^2 if o(g) = 6).
Consider $b \in G \backslash \langle a \rangle$; $b^2 \in \langle a \rangle$
$bab^{-1} \in \langle a \rangle$ and consider cases.

Question 51. Classify groups of order 8 up to isomorphism

Answer. If all elems order 2, then we have $C_2 \times C_2 \times C_2$. Prove by quotienting out
If an elem has order 8, then C_8.
Let a have order 4. Consider $b \in G \backslash \langle a \rangle$.
$b^2 \in \langle a \rangle$
If $b^2 = a$ or a^3, then $o(b) = 8$, so C_8.
Consider $bab^{-1} = a^i$, and evaluate $b^2 a b^{-2}$ in two ways to show $i = \pm 1$.
Consider the four cases in i and b^2 to get groups.

Lecture 13

Question 52. Define a group action ϕ.

Answer. G acts on X with the map
$\phi : G \times X \to X$
$(g, x) \to \phi(g, x) = g(x)$
such that
(1) closure (implied by notation) (2) gh(x) = g(h(x)) (3) identity acting on x is sent to x

Lecture 14

Question 53. Define a group action in terms of Sym(X)

Answer. If G is a group and X is a set such that

$$\Phi : G \to \text{Sym}(X)$$

is a group homomorphism, then

$$\phi : G \times X \to X$$

$$(g, x) \to \phi_g(x)$$

where $\Phi(g) = \phi_g$, is a group action.

Question 54. What is a faithful action?
Answer. $\text{Ker}\Phi = \{e\}$; the only x acted to the identity is the identity.

Question 55. What does Cayley's theorem state?
Answer. Every group is isomorphic to a subgroup of Sym(X) for some X.

Question 56. Prove Cayley's theorem
Answer. Consider the left regular action.

$$\Phi : G \times G \to G$$

$$(g, h) \to gh$$

This action is faithful (show), so we have an injective homomorphism

$$\Phi : G \to \text{Sym}(G)$$

So $G \cong \text{Im}\Phi \leq \text{Sym}(G)$ as required.

Question 57. Define the orbit of $x \in X$
Answer.

$$\text{Orb}_G(x) = \{g(x) : g \in G\} \subseteq X$$

The orbit of x is the set of points in X that it can map to.

Lecture 15

Question 58. Define a transitive action
Answer. G acts transitively on X if for any $x \in X$, $\text{Orb}_G(x) = X$.

Question 59. Prove the distinct G-orbits form a partition of X.
Answer. Suppose $z \in \text{Orb}_G(x) \cap \text{Orb}_G(y)$.
Show $\text{Orb}_G(x) = \text{Orb}_G(z) = \text{Orb}_G(y)$.

Question 60. Show that $\text{Orb}_G(x)$ is invariant, i.e.

$$g(\text{Orb}_G(x)) \subseteq \text{Orb}_G(x).$$

Answer. just do it

Question 61. Show that G is transitive on $\mathrm{Orb}_G(x)$

Answer. this means any element in $\mathrm{Orb}_G(x)$ can go to any other element in $\mathrm{Orb}_G(x)$. construct this

Question 62. Define the stabilizer of x in G

Answer. $\mathrm{Stab}_G(x) = \{g \in G : g(x) = x\}$

Question 63. Prove $\mathrm{Stab}_G(x)$ is a subgroup of G.

Answer. just do it

Question 64. Prove the orbit-stabilizer theorem.

Answer. We prove that the index of $\mathrm{Stab}_G(x)$ is $|\mathrm{Orb}_G(x)|$.
Consider the map

$$\theta : \mathrm{Orb}_G(x) \to (G : \mathrm{Stab}_G(x))$$

$$g(x) \to g\mathrm{Stab}_G(x)$$

Now prove this is a well-defined bijection.

Lecture 16

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Lecture 17

Question 65. State Cauchy's theorem.

Answer. Let G be a finite group and p a prime that divides $|G|$. Then there exists an element in G of order p.

Question 66. Prove Cauchy's theorem.

Answer.

$$X = \{(x_1, x_2, \dots, x_p) : x_1 x_2 \cdots x_p = e, x_i \in G\}$$

Let $H = \{h : h^p = e\} \cong C_p$ which acts on X as follows

$$H \times X \to X$$

$$(h, (x_1, \dots, x_p)) \to (x_2, x_3, \dots, x_p, x_1)$$

In general

$$(h^i, (x_1, x_2, \dots, x_p)) \to (x_{1+i}, x_{2+i}, \dots, x_{p+i})$$

With suffix k taken $(k-1 \mod p)+1$.
Check this is a group action. Using the fact $|\text{Orb}_H(\bar{x})| = 1$ or p (by O/S), that

$$p \,\Big|\, \sum_{\text{distinct orbits}} \text{Orb}_H(\bar{x}),$$

and that

$$\text{Orb}_H(e, e, \dots e)) = 1$$

to show that $\bar{x} \neq e$ exists such that

$$\text{Orb}_H(\bar{x}) = 1,\ \bar{x} = (x, x, \dots x)$$

meaning $x^n = e$.

Question 67. What is the conjugacy action?
Answer.

$$G \times G \to G$$
$$(g, h) \to ghg^{-1}$$

Question 68. Given $z \in Z(G)$, what is $\text{ccl}_G(z)$?
Answer. 1

Question 69. Write $Z(G)$ in terms of the centralizers in G.
Answer. $Z(G) = \bigcap_{h \in G} C_G(h)$

Question 70. How can we relate the order of two elements in the same conjugacy class
Answer. They are equal

Question 71. What is the orbit and stabilizer of the conjugacy action?
Answer. The orbit are conjugacy classes and the stabilizers are called centralizers.

Question 72. Prove that if $|G| = p^n$ for some prime p, then $Z(G)$ is non-trivial.
Answer.

$$G = \bigcup_{\text{distinct conj classes}} \text{ccl}_G(x)$$

By Orbit-Stabilizer theorem, $\text{ccl}_G(x) \,\big|\, |G| = p^n$
So $|\text{ccl}_G(x)| = 1$ or $p \,\big|\, |\text{ccl}_G(x)|$
But

$$|G| = \sum_{x \in Z(G)} \text{ccl}_G(x) + \sum_{\text{distinct classes w/}p \big| \text{ccl}_G(x)} \text{ccl}_G(x)$$

So

$$p \mid \sum_{x \in Z(G)} \mathrm{ccl}_G(x) = Z(G)$$

so $|Z(G)| > 1$.

Question 73. Prove that if $G \backslash Z(G)$ is cyclic, then G is abelian.

Answer. just do it lolololol

Lecture 18

Question 74. Prove that if $G \backslash Z(G)$ is cyclic, then G is abelian.

Answer. just do it lolololol

Question 75. Let $\sigma \in S_n$. Define the cycle type of σ.

Answer. σ can be written as a product of disjoint cycles including 1-cycles. Then the cycle type of σ is $(n_1, n_2, \ldots, n_k)$ where $n_1 \leq n_2 \cdots \leq n_k$ and the ith cycle of σ has length n_i

Question 76. Prove that π and σ are conjugate in S_n iff they have the same cycle type.

Answer. σ can be written as a product of disjoint cycles including 1-cycles. Then the cycle type of σ is $(n_1, n_2, \ldots, n_k)$ where $n_1 \leq n_2 \cdots \leq n_k$ and the ith cycle of σ has length n_i.
Consider τ conjugating σ, and consider the (general!) element $\sigma(a_{ij})$ and unwrap notation until you can write a product of disjoint cycles for the conjugation.

To show the converse, write π similarly to σ but with b_{ij} and then show, with $\tau(a_{ij}) = b_{ij}$, that $\pi = \tau\sigma\tau^{-1}$.

Question 77. Let $x \in A_n$. If $\mathrm{C}_{A_n}(x) = \mathrm{C}_{S_n}(x)$, how can we relate the size of $\mathrm{cl}_{A_n}(x)$ and $\mathrm{ccl}_{S_n}(x)$?

Answer.

$$|\mathrm{ccl}_{A_n}(x)| = \frac{|\mathrm{ccl}_{S_n}(x)|}{2}$$

Question 78. Let $x \in A_n$. If $\mathrm{C}_{A_n}(x) \nleq \mathrm{C}_{S_n}(x)$, how can we relate the size of $\mathrm{cl}_{A_n}(x)$ and $\mathrm{ccl}_{S_n}(x)$?

Answer.

$$|\mathrm{ccl}_{A_n}(x)| = |\mathrm{ccl}_{S_n}(x)|$$

Lecture 19

Question 79. What are the number of elements in S_n with k_L cycles of length L?

Answer.

$$\frac{n!}{\prod_L k_L! L^{k_L}}$$

Question 80. Prove A_5 is a simple group.

Answer. Suppose $N \trianglelefteq A_5$. Then N is a union of conjugacy classes.
$\Rightarrow |N| = 1 + 15a + 20b + 12c$ where $a, b \in \{0, 1\}, c \in \{0, 1, 2\}$.
But by Lagrange, $|N| \big| |A_5| = 60$.
Only possibility is $|N| = 1$ or 60.

Question 81. Prove $\mathrm{GL}_n(\mathbb{R})$ is a group under matrix multiplication.

Answer. closure, identity, inverse: don't be noob associative: use index notation

Question 82. Prove

$$\mathrm{Det} : \mathrm{GL}_n(\mathbb{R}) \to (\mathbb{R}\backslash\{0\}, \times)$$

$$A \to \det A$$

is a surjective homomorphism

Answer. do it

Question 83. What is $\mathbb{F}_p$?

Answer. It is the finite field

$$\mathbb{F}_p = (\{0, 1, \ldots, p-1\}, +_p, \times_p).$$

Lecture 20

Question 84. What is $|GL_3(\mathbb{F}_p)|$?

Answer. Consider no. choices for each column.
Column 1: $p^3 - 1$
Column 2: $p^3 - p$
Column 3: $p^3 - p^2$

Question 85. What is $O(\mathbb{R})$

Answer. Orthonormal group.

$$\mathrm{O}_n(\mathbb{R}) = \{A \in M_n : AA^T = I\}$$

Lecture 21

Question 86. Prove $O(\mathbb{R})$ is a group
Answer. :3

Question 87. Let $A \in \mathrm{O}_n(\mathbb{R})$ and $\mathbf{x}, \mathbf{y} \in M_n$, then
(i) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$
(ii) $|Ax| = |x|$ So A is an isometry.
Answer. Start from givens and unravel definitions.

Question 88. Find all 2x2 orthogonal matrices.
Answer. Use $AA^T = I$ definition to find conditions on the entries, then consider cases.

Question 89. Let $A \in \mathrm{SO}_3(\mathbb{R})$. Show that then A is conjugate to a matrix of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $\theta \in [0, 2\pi)$.
Answer. Can show matrix is conjugate to a 3x3 matrix that fixes e_3. Then we just need to consider the 2x2 matrix case, from which it immediately follows 2x2 case.

Question 90. Write $\mathrm{O}_3(\mathbb{R})$ in terms of $\mathrm{SO}_3(\mathbb{R})$
Answer. $\mathrm{O}_3(\mathbb{R}) = \mathrm{SO}_3(\mathbb{R}) \cup \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \mathrm{SO}_3(\mathbb{R})$

Lecture 22

Question 91. What is a Mobius transform?
Answer. A Mobius transform is a function of a complex variable that can be written

$$f(z) = \frac{az + b}{cz + d}$$

for some $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

Question 92. Prove any element of $O_3(\mathbb{R})$ is a product of at most 3 reflections
Answer. Can consider fixing an axis at at time by adding a new reflection for each.

Question 93. Prove the set $\mathcal{M}$ of all Mobius maps on $\mathbb{C}$ is a group under composition.

Answer. Composition of maps is associative.
$I(z) = z \in \mathcal{M}$.
Closure can be verified manually by considering cases and bash.

Question 94. Prove
$$\mathrm{GL}_2(\mathbb{C})\backslash Z \cong \mathcal{M}$$
where $Z = \{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) : \lambda \in \mathbb{C}/\{0\}\}$.
Answer. Construct surjective homomorphism
$$\begin{aligned} \Phi : \quad \mathrm{GL}_2(\mathbb{C}) &\to \mathcal{M} \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\to \frac{az+b}{cz+d} \end{aligned}$$
and apply first isomorphism theorem.

Lecture 23

Question 95. Prove
$$\mathrm{SL}_2(\mathbb{C})\backslash\{\pm I\} \cong \mathcal{M}$$
Answer. Construct surjective homomorphism
$$\begin{aligned} \Phi : \quad \mathrm{SL}_2(\mathbb{C}) &\to \mathcal{M} \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\to \frac{az+b}{cz+d} \end{aligned}$$
and apply first isomorphism theorem.

Question 96. Prove every Mobius map can be written as a composiiton of maps of the following forms:
(i) $z \to az$, $a \neq 0$
(ii) $z \to z + b$
(iii) $z \to \frac{1}{z}$
Answer. Let $f(z) = \frac{az+b}{cz+d}$ Consider $c = 0$ and $c \neq 0$ cases and construct f by composition.

Question 97. What does it mean for a group G to act triply transitively on a set X?
Answer. Given distinct $x_1, x_2, x_3 \in X$ and distinct $y_1, y_2, y_3 \in X$, then there exists a $g \in G$ such that $g(x_i) = y_i$.

Question 98. Prove the action of $\mathcal{M}$ on $\mathbb{C}_\infty$ is sharply triply transitive.

Answer. Label first triple $\{z_0, z_1, z_\infty\}$ and second triple $\{w_0, w_1, w_\infty\}$.
We construct $g \in \mathcal{M}$ such that

$$\begin{aligned} g: \quad & z_0 \to 0 \\ & z_1 \to 1 \\ & z_\infty \to \infty \end{aligned}$$

First suppose $z_1, z_2, z_3 \neq \infty$. Then

$$g(z) = \frac{(z - z_0)(z_1 - z_\infty)}{(z - z_\infty)(z_1 - z_0)}$$

Check: "$ad - bc'' = (z_0 - z_\infty)(z_1 - z_\infty)(z_1 - z_0) \neq 0$.
If $z_\infty = \infty$, then $g(z) = \frac{z - z_0}{z_1 - z_0}$
If $z_1 = \infty$, then $g(z) = \frac{z - z_0}{z - z_\infty}$
If $z_\infty = \infty$, then $g(z) = \frac{z_1 - z_\infty}{z - z_\infty}$.

Similarly find h such that

$$\begin{aligned} h: \quad & z_0 \to 0 \\ & z_1 \to 1 \\ & z_\infty \to \infty \end{aligned}$$

Then $f = h^1 g : z_i \to w_i$ as required.

Now to prove uniqueness
Suppose $f' : z_i \to w_i$.
Then $f^{-1} f' : z_i \to z_i$
Let g be as above. Can show $g f^{-1} f' g^{-1} = \text{id}$,
so $f = f'$.

Lecture 24

Question 99. Any non-identity Mobius map is conjugate to one of
(i) $z \to \omega z$
(ii) $z \to z + 1$.

Answer. Consider jordan normal forms

Question 100. Prove that a non-identity Mobius map has either 1 or 2 fixed points

Answer. Note fixed points are preserved under conjugation. Hence consider number of fixed points of ωz and $z + 1$

Question 101. If $f \in \mathcal{M}$ and C is a circle, prove that $f(C)$ is a circle.

Answer. It suffices to consider the transforms which constitute a mobius transform and show that these individually preserve circles. Consider the complex equation of a circle to verify each case.

Question 102. Define the cross-ratio of $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$.

Answer.

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_2)}$$

If we have $\omega_i = \infty$, just apply the dubious simplifiation of $\frac{\infty}{\infty} = 1$.

Question 103. Given $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ and $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{C}_\infty$, then $\exists f \in \mathcal{M}$ such that $f(z_i) = \omega_i$ iff $[z_1, z_2, z_3, z_4] = [\omega_1, \omega_2, \omega_3, \omega_4]$

Answer. ($\Rightarrow$) Suppose $f(z_j) = \omega_j$. Find $\omega_j - \omega_k$ in terms of function and show cross-ratios are the same.

($\Leftarrow$) Consider g mapping z_i to $0, 1, \infty$ and h mapping ω_i to $0, 1, \infty$, and using $[0, 1, x, \infty] = x$, show that $h^{-1}g$ works.

Question 104. z_1, z_2, z_3, z_4 lie in some circle in $\mathbb{C}_\infty$ iff $[z_1, z_2, z_3, z_4] \in \mathbb{R}$.

Answer. Construct a function

$$: C \to \mathbb{R} \cup \{\infty\}$$

and construct circle out of z_1, z_2, z_4. Map $g : z_i$ to $0, 1, \infty$ and apply cross-ratios to show that $z_3 \in C \iff [z_1, z_2, z_3, z_4] \in \mathbb{R}$.