

Capturing Dependency of Data using Graphical Models

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Objectives

- Understand **equivalence of conditional independence of R.Vs and factorizations** of their probability distribution over a graph.
- Simple **undirected graphical models**:
 - Gaussian Markov Network
 - Logistic Model

Example: Scores of Units

- Imagine a table of unit scores.

Name	SPS	Math	Python	Mach. Learn.
Song	80	70	50	60
Harry	50	40	70	80
Ron	50	50	...	45
Hermione	90	100	...	100
...

Dependency of Datasets and Its Graphical Representation

- Scores of units are **dependent!**
 - Student with **high** Math, Python score is likely to receive **high** SPS score.
 - Student with **high** SPS score is likely to receive a **high** Mach. Learn. score.

Problem Formulation

- Given a dataset $\{\mathbf{x}_i\}_{i=1}^n$,
 - $\mathbf{x}_i = [x_i^{(1)}, x_i^{(2)} \dots x_i^{(d)}] \in R^d$
 - \mathbf{x}_i is a vector of a student i 's scores.
 - e.g., $x^{(1)}$ is SPS, $x^{(2)}$ is Math...
- **What does $p(x^{(1)}, x^{(2)} \dots x^{(d)})$ look like?**

Independence of R.V.s

- Let's look at how independence between R.V.s are **expressed in probability**:
- R.V. X is **independent** of Y :
 - $X \perp Y$
 - $\Leftrightarrow p(X, Y) = p(X)p(Y)$
 - Factorization
 - $\Leftrightarrow p(X|Y) = p(X) \Leftrightarrow p(Y|X) = p(Y)$
 - No Information flows between X and Y .

Example: Likelihood with Independent Datapoints:

- Likelihood over the dataset
 - Factorizes into product over each x_i
 - $p(x_1, x_2, \dots, x_n; \theta) = \prod_{i=1}^n p(x_i; \theta)$
 - We can do so as $x_1 \dots x_n$ are independent.
- Maximum Likelihood Estimation
 - $\max_{\theta} \prod_{i=1}^n p(x_i; \theta)$
 - **Lab sheet 4.1**

Conditional Independence of R.V.s

- R.V. X is independent of Y **given** Z
 - $X \perp Y|Z$
 - $\Leftrightarrow p(X, Y|Z) = p(X|Z)p(Y|Z)$
 - $\Leftrightarrow p(X, Y, Z) \propto g_1(X, Z) \cdot g_2(Y, Z)$
 - Factorization
 - $\Leftrightarrow p(X|Y, Z) = p(X|Z)$
 - Information flow: Y does not give any additional info which changes the prob. of X given Z .
 - $\Leftrightarrow p(Y|X, Z) = p(Y|Z)$

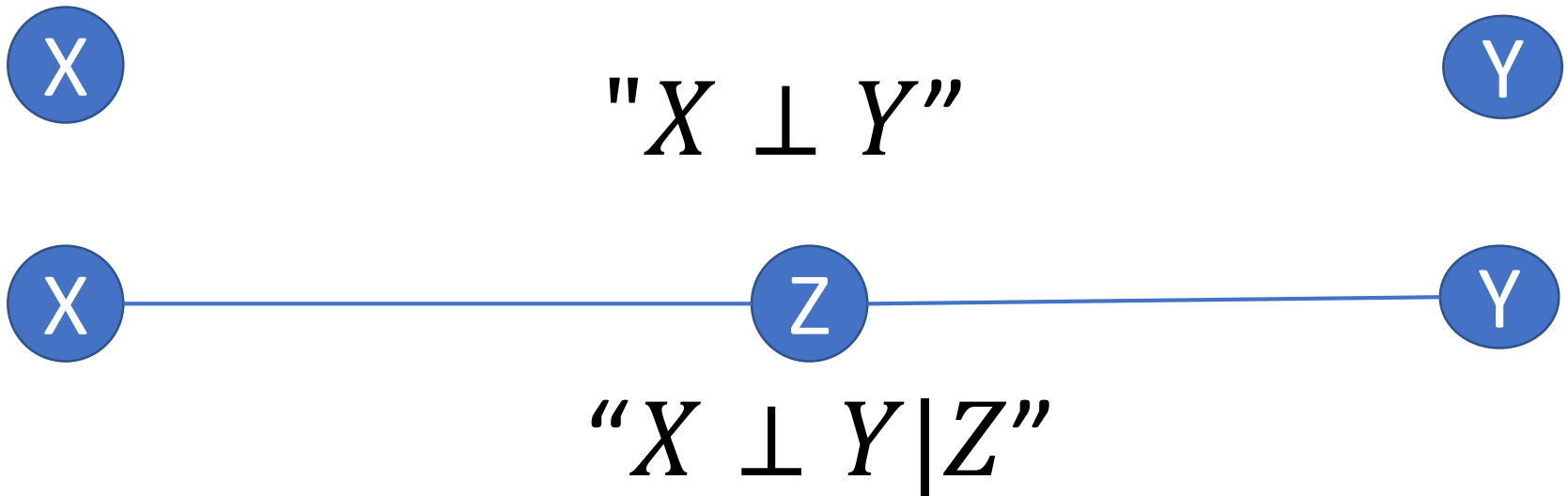
(Conditional) Independence and Information Flow

- (Conditional) Independence tells how information **flows** between R.V.s
 - $X \perp Y \Leftrightarrow$ no information flows in-between X and Y .
 - $X \perp Y | Z \Leftrightarrow$ information **flows between** X and Y **via** Z .



Representing (Conditional) Independence by Graph

- Given many R.Vs, listing all (cond.) independence can be cumbersome.
- A **graphical representation** is helpful:



Representing Conditional Independence by Graph

- Given a graph $G = \langle E, V \rangle$, and three random variables $X, Y, Z \subseteq V$
 - if X and Y are completely “**blocked**” by Z , we say $X \perp Y | Z$ is represented by G .

Example: Encoding (cond.) indep. by graph

$\text{Math} \perp \text{ML} \mid \text{SPS}$

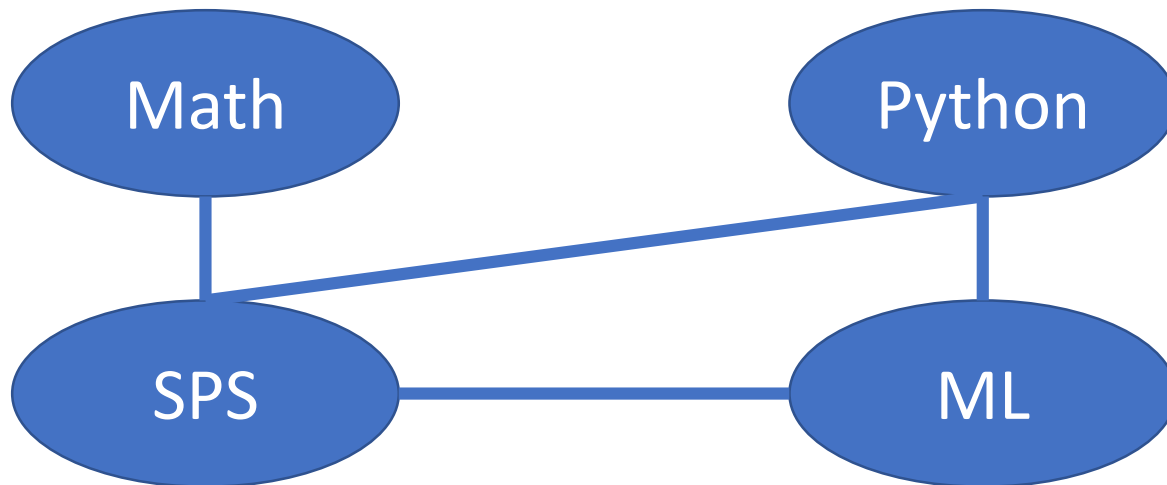
$\text{Math} \perp \text{Python} \mid \text{SPS}$

$\text{Math} \perp \text{ML} \mid \text{SPS}, \text{Python}$

$\text{Math} \perp \text{Python}, \text{ML} \mid \text{SPS}$

$\text{Math} \perp \text{Python} \mid \text{SPS}, \text{ML}$

List of
conditional
independen
ce encoded
by Graph!





Representing Prob. Distribution Factorization by Graph

- Factorizing a probability dist. greatly reduces complexity of modelling and computation of a probability dist.
 - Think about that Maximum Likelihood example you did in Lab!

Representing Prob. Distribution Factorization by Graph

- Writing the factorization of a probability distribution of many factors can be cumbersome.
- Can we also use graph to help??

 $P(X, Y) = P(X)P(Y)$ 



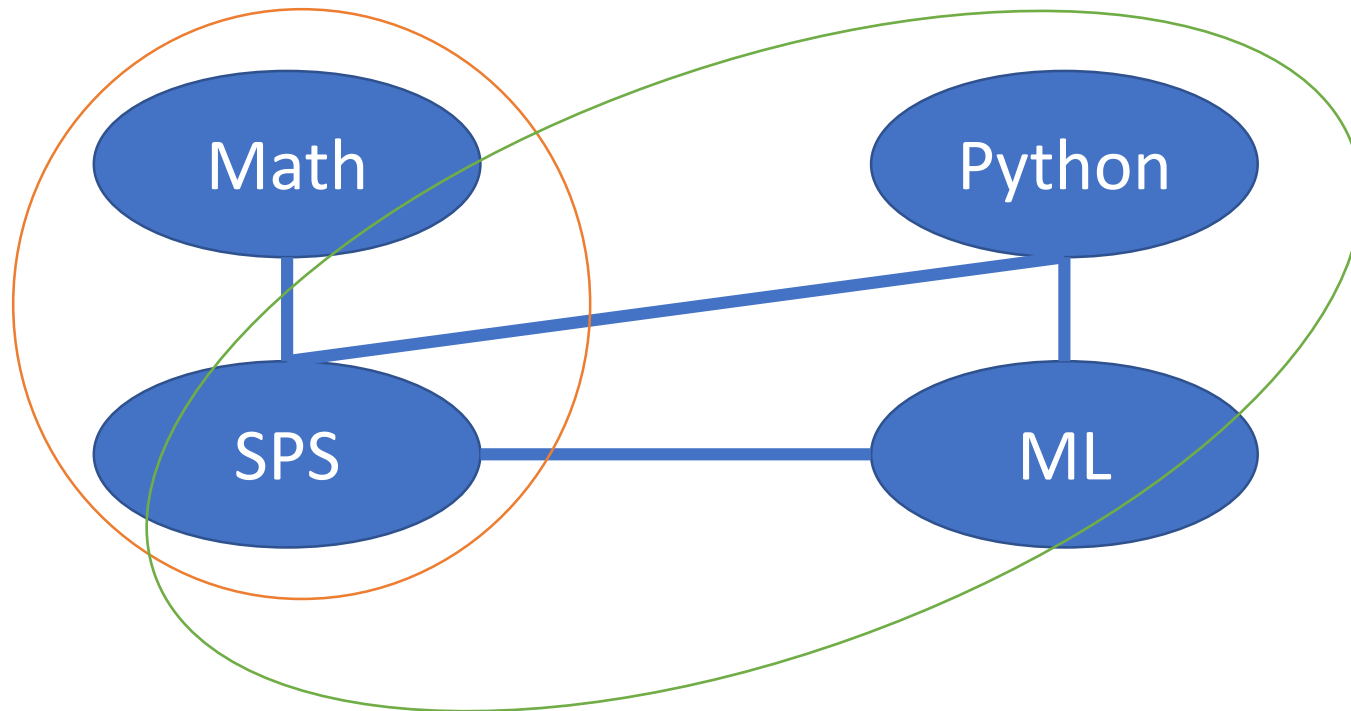
$P(X, Y, Z) \propto g_1(X, Z)g_2(Y, Z)$

Representing Prob. Distribution Factorization by Graph

- Given a graph $G = \langle E, V \rangle$,
- We say $p(X)$ factorizes over G :
- If $p(X) \propto \prod_{c \in \mathcal{C}} g_c(X^{(c)})$
 - where \mathcal{C} is set of all **cliques** in G .
 - Clique: fully connected subgraph.
 - g_c is a function defined on $X^{(c)}$, which is the subset of X **restricted on c** .

Example

$$p(Ma, SPS, Py, ML) \\ \propto g_1(Ma, SPS) \cdot g_2(Py, ML, SPS).$$



Equivalency between Factorization and Conditional Independence over G

- Using graph represent a factorization of a probability distribution
- Using graph represent a list of conditional independence
- Remarkably, these two seemingly irrelevant notions are **equivalent!**

Equivalency between Factorization and Conditional Independence over G

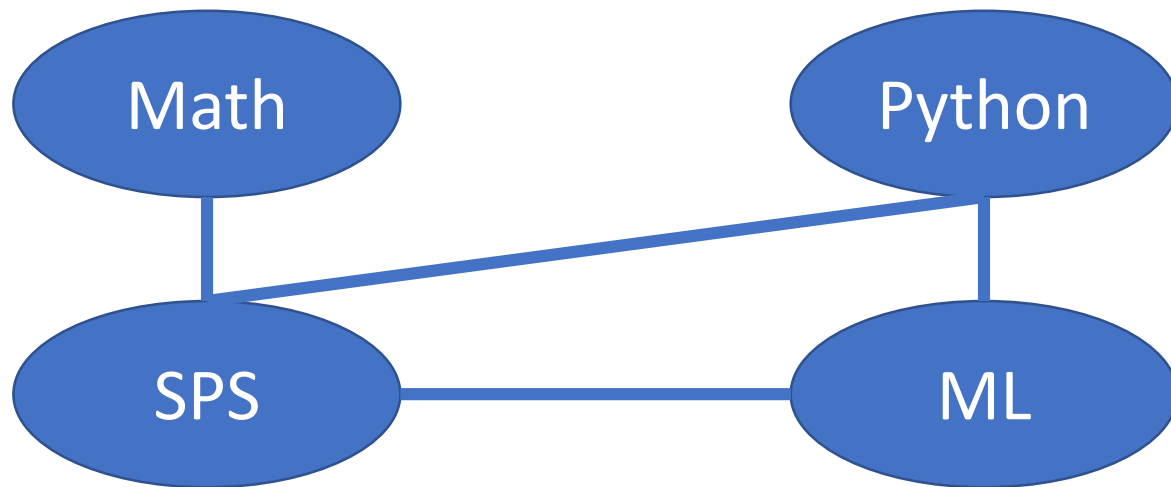
- If p factorizes over G , p satisfies all conditional independence represented by G .
- If p satisfies all conditional independence represented by G , then p factorizes over G .

Equivalency between Factorization and Conditional Independence over G

- Verify this on Scores of Units example!
- Live demonstration.

Example

$$p(Ma, SPS, Py, ML) \\ \propto g_1(Ma, SPS) \cdot g_2(Py, ML, SPS).$$



$$\text{Hint: } X \perp Y|Z \Leftrightarrow p(X, Y, Z) \propto g_1(X, Z) \cdot g_2(Y, Z) \\ X \perp Y, W|Z \Rightarrow X \perp Y|Z$$

Example

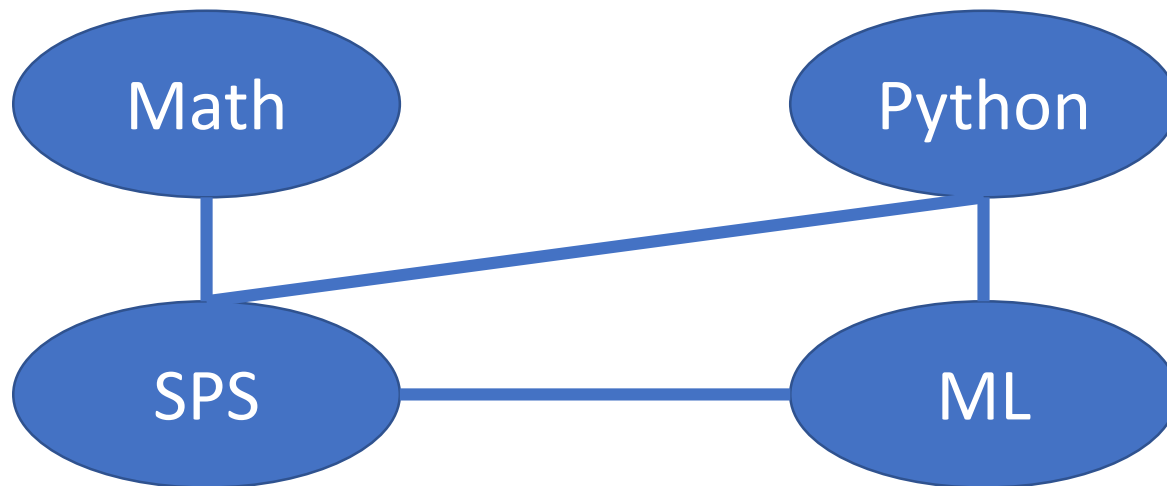
Math \perp ML | SPS

Math \perp Python | SPS

Math \perp ML | SPS, Python

Math \perp Python, ML | SPS

Math \perp Python | SPS, ML



Hint: $X \perp Y | Z \Leftrightarrow p(X, Y, Z) \propto g_1(X, Z) \cdot g_2(Y, Z)$

Markov Network

- A probability distribution $p(X)$ which uses undirected graph representing its conditional independence, is called an **undirected graphical model**, or a **Markov network**.

Gaussian Markov Network

- Multivariate Gaussian distribution:

- $\mathbf{x} \in R^d, \mathbf{x} \sim N(\mathbf{0}, \Sigma)$

- $p(\mathbf{x}) \propto \exp \left[-\frac{\mathbf{x}(\Sigma)^{-1} \mathbf{x}^T}{2} \right]$ Let $\Theta = (\Sigma)^{-1}$.

$$\propto \exp \left[-\frac{\sum_{u,v} \Theta^{(u,v)} x^{(u)} x^{(v)}}{2} \right]$$
$$\propto \prod_{u,v; \Theta^{(u,v)} \neq 0} \exp(-\Theta^{(u,v)} x^{(u)} x^{(v)})$$

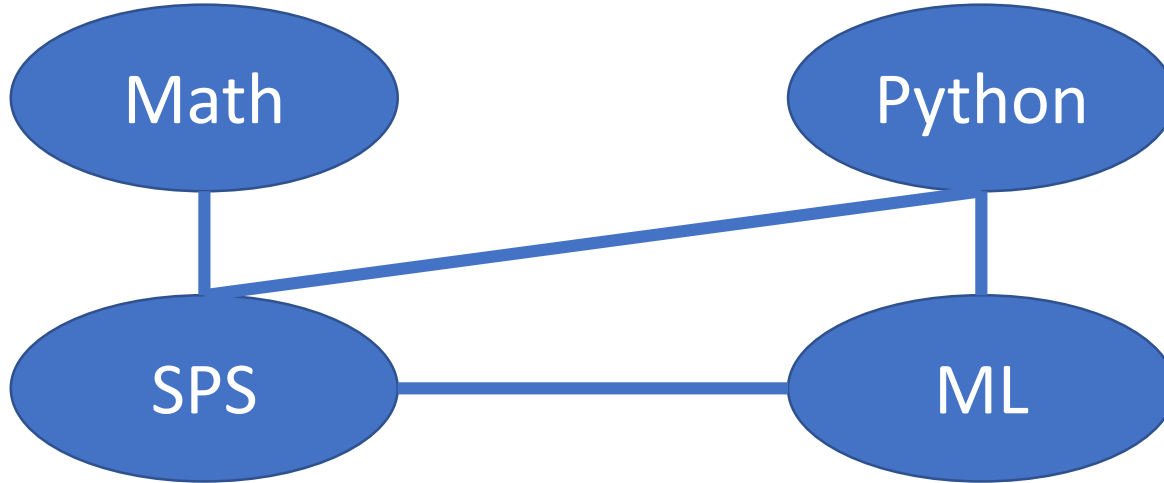
Gaussian Markov Network

- $p(\mathbf{x}) \propto \prod_{u,v; \Theta(u,v) \neq 0} g_{u,v}(x^{(u)}, x^{(v)})$
- $p(\mathbf{x})$ **factorizes over G !**
 - G defined by the adjacency matrix
$$A^{(u,v)} = \begin{cases} 0, & \Theta(u,v) == 0 \\ 1, & \Theta(u,v) \neq 0 \end{cases}$$
 - G must be an undirected graph (why?)
 - \Leftrightarrow satisfies the conditional independence encoded in G .

Gaussian Markov Network

- Knowing a graph G that encodes all conditional independence of your dataset, I can use its adjacency matrix G to construct Θ !
 - Use sparsity of the adjacency matrix
 - NOT its actual values!
 - Θ must be positive definite!!

Example



• $x^{(1)}:\text{Math}; x^{(2)}:\text{Py}; x^{(3)}:\text{SPS}; x^{(4)}:\text{ML}$

$$\bullet \Theta = \begin{bmatrix} \Theta_{11} & 0 & \Theta_{13} & 0 \\ 0 & \Theta_{22} & \Theta_{23} & \Theta_{24} \\ \Theta_{13} & \Theta_{23} & \Theta_{33} & \Theta_{34} \\ 0 & \Theta_{24} & \Theta_{34} & \Theta_{44} \end{bmatrix}$$

Quiz

- Suppose graph G encodes all cond. indep. in your probability dist. G contains **three edges, five nodes**. How many **non-zero elements** are there in Θ , the parameter to your Gaussian Markov Net?
- A.3
- B.8
- C.6
- D.10
- E.11

<https://bit.ly/2uIFZUu>

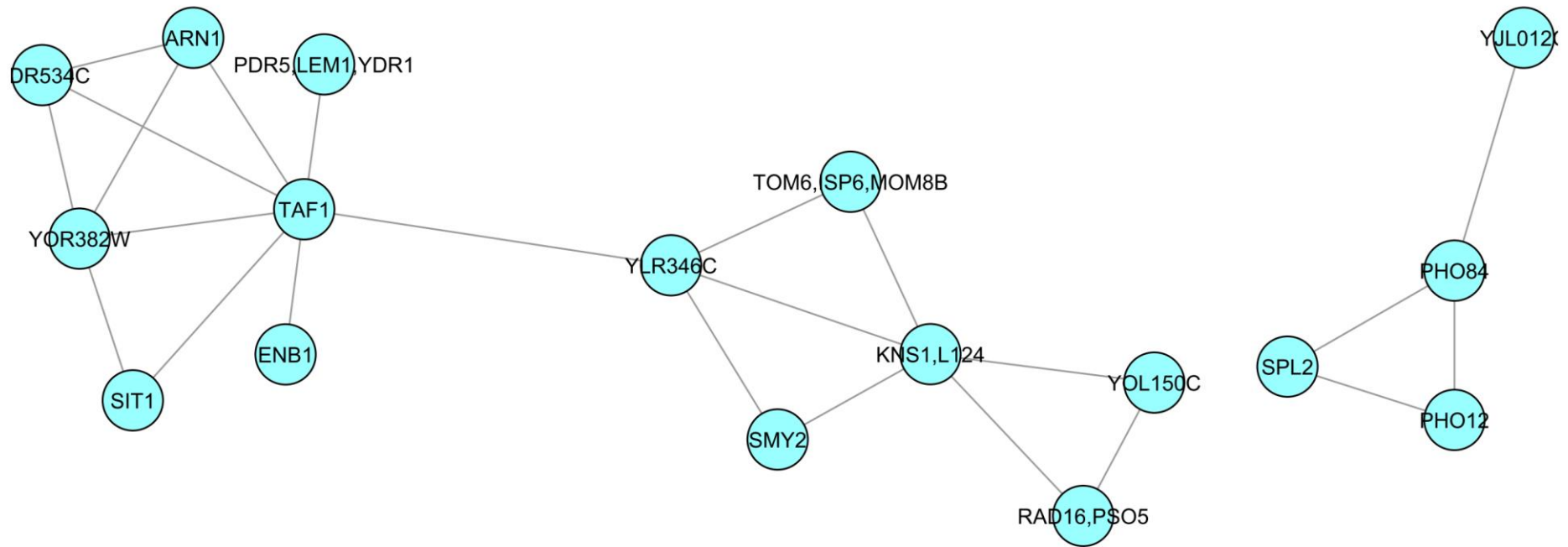
Gaussian Markov Network

- **Not knowing** G encodes all cond. independence of $p(\mathbf{x})$. Given dataset D , we can fit a sparse $\hat{\Theta}$.
 - Using MLE: $\hat{\Theta} = \underset{\Theta}{\operatorname{argmax}} \log p(D; \Theta)$
 - The sparsity of $\hat{\Theta}$ gives a graphical representation of $p(\mathbf{x})$!
 - Such representation reveals how random variables “interacts” with each other!

Example: Gene Expression Data

Time stamp	Gene1	Gene2	Gene3	Gene4
t1	.1	.2	.5	.2
t2	.5	.4	.7	.8
t3	.5	.545
t4	.9	.201
...

Gene Network (Banerjee et al., 2008)



Exponential Family Distribution

- Gaussian Markov network belongs to a wider **family** of distributions, which are defined using a generic form:

- $$p(\mathbf{x}; \boldsymbol{\theta}) := \frac{\exp(\langle \boldsymbol{\theta}, \mathbf{f}(\mathbf{x}) \rangle)}{Z(\boldsymbol{\theta})}$$

- $\mathbf{f}(\mathbf{x})$ is a feature transform on \mathbf{x} .

- $$Z(\boldsymbol{\theta}) := \int \exp(\langle \boldsymbol{\theta}, \mathbf{f}(\mathbf{x}) \rangle) d\mathbf{x}$$

- PC: show when \mathbf{f} is 2nd degree poly. transform with pairwise terms, $p(\mathbf{x}; \boldsymbol{\theta})$ is a multivariate Gaussian distribution.

Conditional Markov Network

- In many tasks, the conditional distribution is the key interest.
 - $p(Y|X)$ measures the randomness on Y given X and help us make a prediction.
 - Both regression and classification requires a **conditional** model.
- How to factorize a conditional distribution over G ?

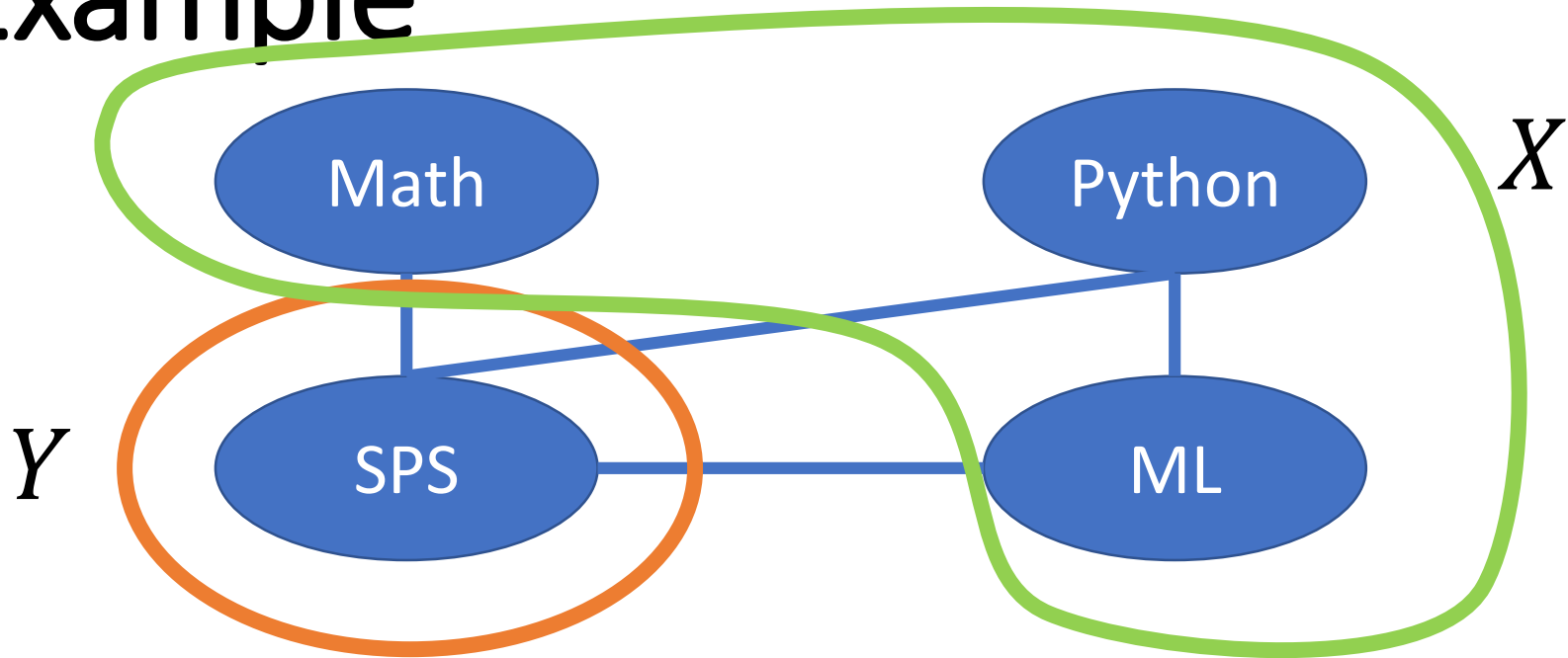
Conditional Markov Network

- We say a conditional probability distribution $P(Y|X)$ factorizes over G whose nodes $V = X \cup Y$, if
- $p(Y|X) = \frac{1}{N(X)} \prod_{c \in \mathcal{C}} g_c(Z), Z \subseteq X \cup Y$
- $N(X) := \int \prod_{c \in \mathcal{C}} g_c(Z) dY$
- Normalizing constant:
 - It normalizes the distribution to 1 over the domain of the random variable (Y).

Conditional Markov Network

- PC: show $Z \not\subseteq X$
 - $p(Y|X)$ does not include factors on conditioning variable X !

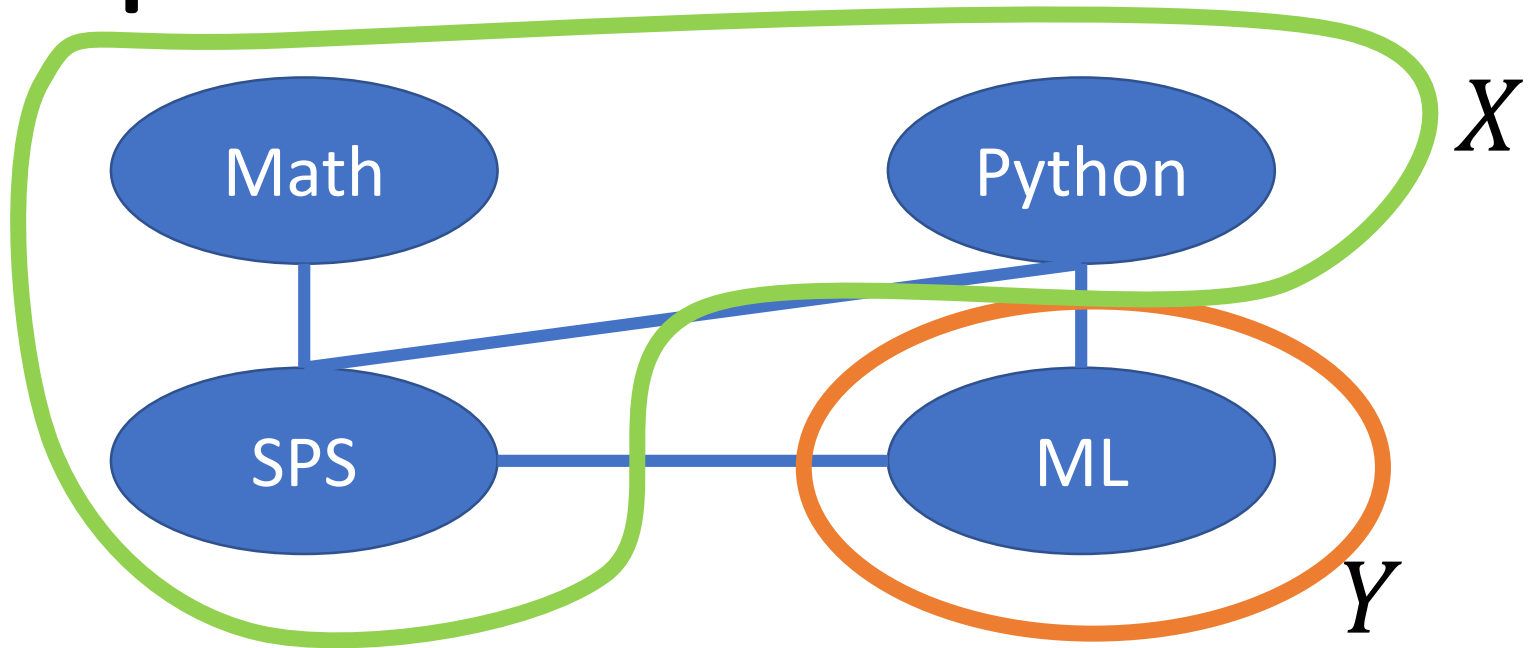
Example



$$\begin{aligned} & \bullet p(SPS | Ma, Py, ML) \\ &= \frac{1}{Z(Ma, Py, ML)} g_1(SPS, Py, ML) g_2(SPS, Ma) \end{aligned}$$

$$\begin{aligned} & \bullet Z(Ma, Py, ML) = \\ & \int g_1(SPS, Py, ML) g_2(SPS, Ma) dSPS \end{aligned}$$

Example



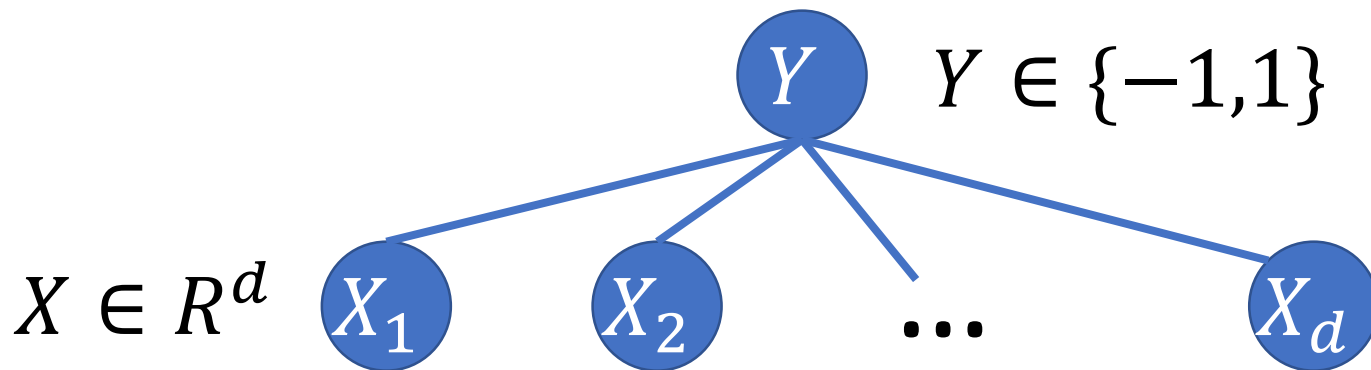
- $p_1(ML|Ma, Py, SPS)$
$$= \frac{1}{Z(Ma, Py, SPS)} g_1(SPS, Py, ML)$$

- $Z(Ma, Py, SPS) = \int g_1(SPS, Py, ML) dML$

- g_2 is gone!

Logistic Regression

- The way of constructing a conditional P.D. gives us a simple classification tool: Logistic Regression.
- Consider a simple Markov Net

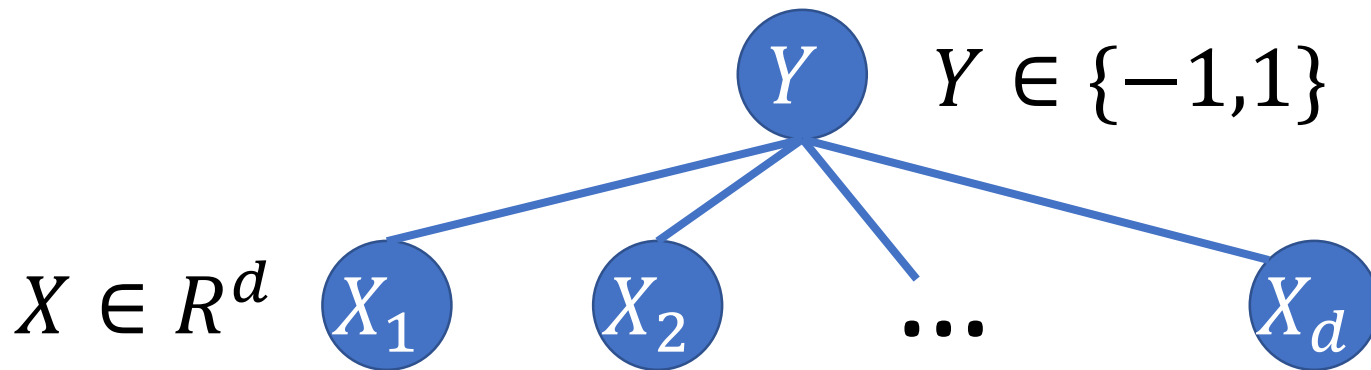


Logistic Model

- Using the factorization rule above,

- $p(Y|X) = \frac{1}{N(X)} \prod_i g_i(Y, X^{(i)})$

- $N(X) = \sum_{c \in \{-1, 1\}} \prod_i g_i(Y, X^{(i)})$



Logistic Model

- Let us construct a model of $p(Y|X)$!

- By setting

$$g_i(Y = y, X_i = x^{(i)}; \beta_i) := \exp(\beta_i \cdot y x^{(i)})$$

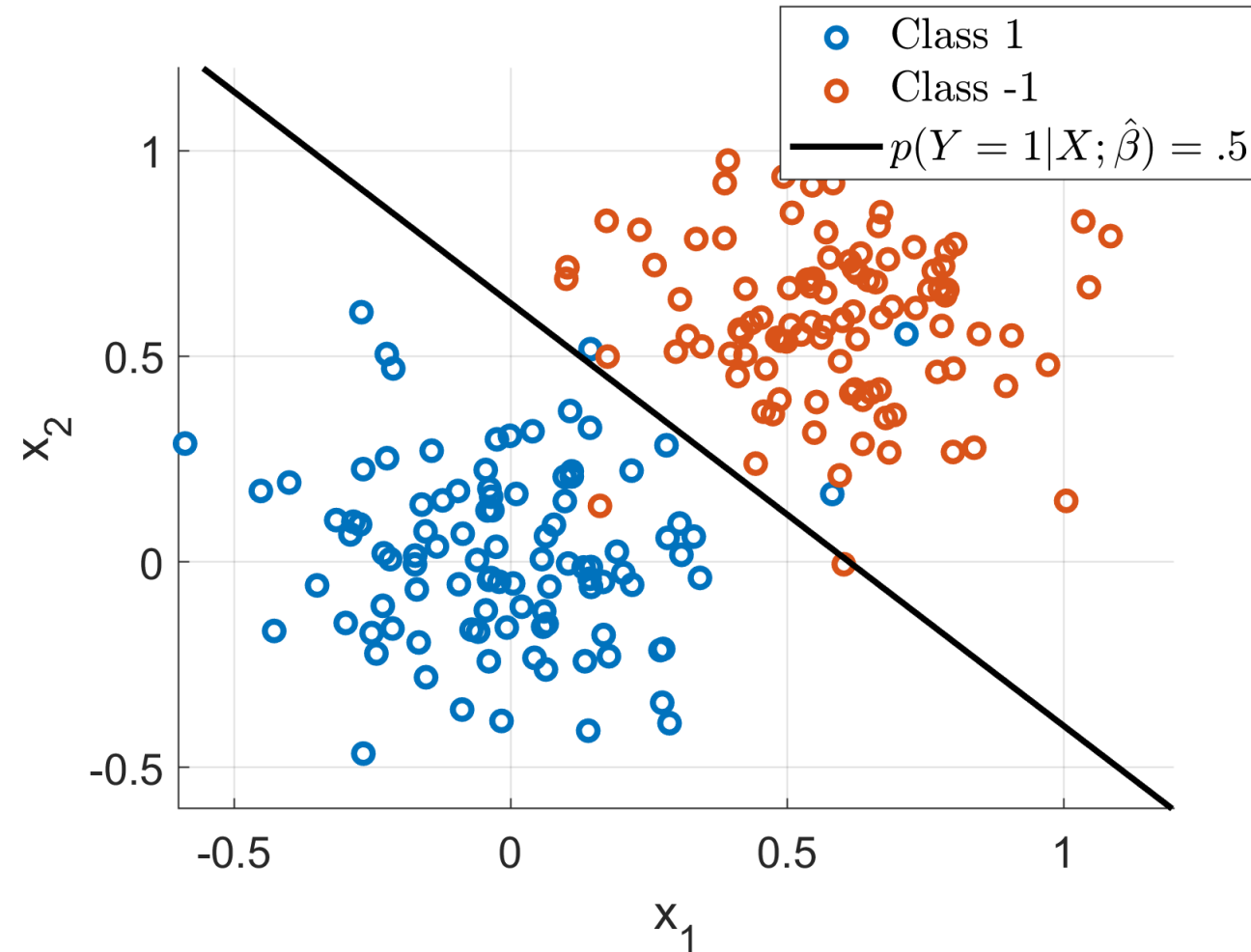
- $$p(y|\mathbf{x}; \boldsymbol{\beta}) = \frac{1}{N(X)} \exp\left(\sum_i \beta^{(i)} \cdot y x^{(i)}\right)$$
$$= \frac{1}{N(X)} \exp(\langle \boldsymbol{\beta}, \mathbf{x} \rangle y).$$

- $$N(X; \boldsymbol{\beta}) = \sum_{y \in \{1, -1\}} \exp(\langle \boldsymbol{\beta}, \mathbf{x} \rangle y)$$

Logistic Regression

- Logistic model:
- $p(y|\mathbf{x}; \boldsymbol{\beta}) = \frac{1}{N(\mathbf{x})} \exp(\langle \boldsymbol{\beta}, \mathbf{x} \rangle y)$
- $N(\mathbf{x}) = \exp(\langle \boldsymbol{\beta}, \mathbf{x} \rangle) + \exp(-\langle \boldsymbol{\beta}, \mathbf{x} \rangle)$
- $\boldsymbol{\beta}$ can be fitted using MLE.
 - $\hat{\boldsymbol{\beta}} = \arg \max_{\boldsymbol{\beta}} \sum_{i=1}^n \log p(y_i | \mathbf{x}_i; \boldsymbol{\beta})$
 - The process of fitting $\boldsymbol{\beta}$ using MLE is called Logistic Regression.
 - `sklearn.linear_model.LogisticRegression`

Example



- Unlike least squares classifier, logistic classifier is a probabilistic classifier, which outputs $p(Y|X; \hat{\beta})$, which is more interpretable!

Conclusion

- Markov network uses a graph to represent its conditional independencies.
 - It visualizes interactions of R.V.s in a P.D.
- Two examples of Markov network
 - Gaussian Markov network factorizes over the graph defined by its **inverse covariance**.
 - Logistic model is a conditional P.D. model factorizes over a classification network.