COMS21202: Symbols, Patterns and Signals
Data Acquisition and Data Characteristics

[based on Dima Damen lecture notes]

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Agenda

Today we are going to talk about

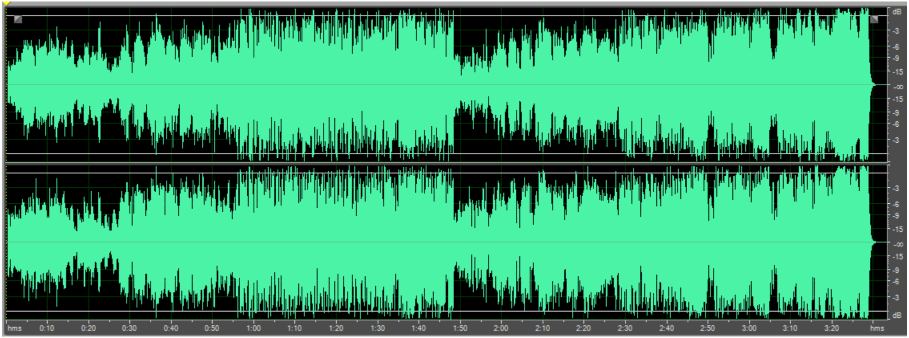
1. Data acquisition
2. Data characteristics: distance measures
3. Data characteristics: summary statistics [reminder]

Data Acquisition - Analogue to Digital Conversion

Analogue to Digital conversion involves

1. Sampling
2. Quantisation

e.g. Audio Signal - 1D [low zoom]



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Analogue to Digital conversion involves

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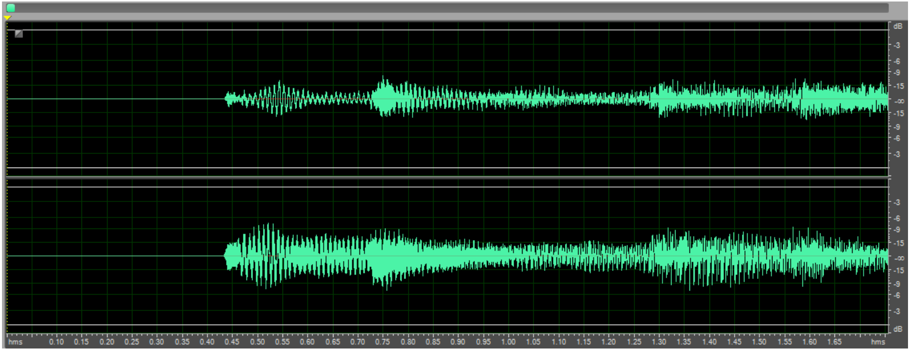
e.g. Audio Signal - 1D [medium zoom]



Data Acquisition - Analogue to Digital Conversion

Analogue to Digital conversion involves

1. Sampling
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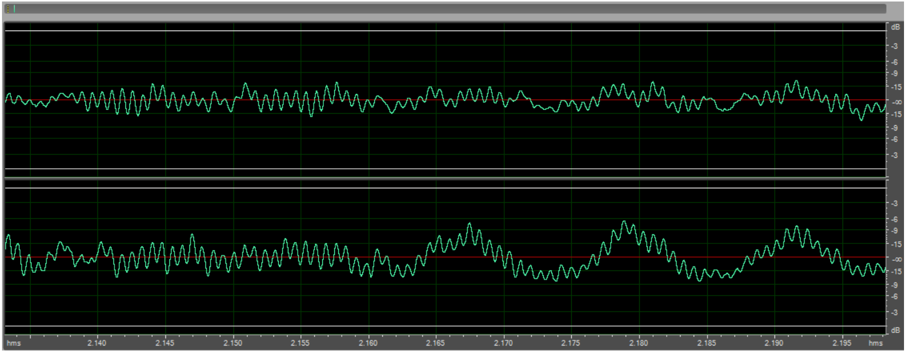
e.g. Audio Signal - 1D [high zoom] : How do you represent data digitally?

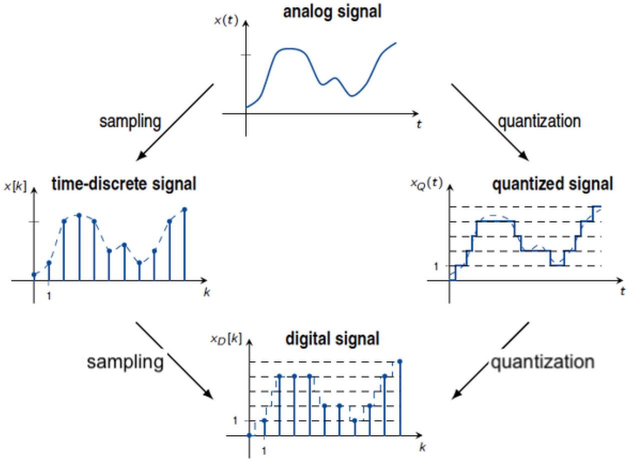


Data Acquisition - Analogue to Digital Conversion

You need to:

1. Sample
2. Quantise

example from dsp-nbsphinx.readthedocs.io (chapter 5.1)



Data Acquisition - Analogue to Digital Conversion

Theorem

Nyquist-Shannon sampling theorem:
If a function $x(t)$ contains no frequencies higher than f_{max} hertz, it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2f_{max}}$ seconds apart.

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In other words,

- Suppose the highest frequency for a given analog signal is f_{max},
- According to the Theorem:
  sampling period, $T_s \leq \frac{1}{2f_{max}}$
  which is equivalent to *sampling rate*, $f_s \geq 2f_{max}$

Data Acquisition - Analogue to Digital Conversion

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Note: Higher sampling/quantisation achieves better signal quality, but also larger memory/storage.

Data Acquisition - Analogue to Digital Conversion

Images - Multi-Dimensional

- Sampling: Resolution in digital photography
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- 8 Mega Pixel Camera - 3264x2448 pixels
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- Colour images: 3 channels: Red, Green, Blue

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- Binary Images: Black/White 1 bit per pixel

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2. **Data characteristics: distance measures**
3. Data characteristics: summary statistics [reminder]

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- Distance is important as it:
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  - enables calculating similarity and dissimilarity
- Without defining a distance measure, almost all statistical and machine learning algorithms will not be able to function.

Distance

A valid distance measure $D(a, b)$ between two components a and b has the following properties

- non-negative: $D(a, b) \geq 0$
- reflexive: $D(a, b) = 0 \iff a = b$
- symmetric: $D(a, b) = D(b, a)$
- satisfies triangular inequality: $D(a, b) + D(b, c) \geq D(a, c)$

Distance (Numerical)

Distances between numerical data points in Euclidean space $\mathbb{R}^n$, for a point $x = (x_1, x_2, .., x_n)$ and a point $y = (y_1, y_2, .., y_n)$, the Minkowski distance of order p (p-norm distance) is defined as:

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$$\begin{aligned} D(x, y) &= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \\ &= \|\mathbf{x} - \mathbf{y}\| \\ &= \sqrt{(\mathbf{x} - \mathbf{y})^T (\mathbf{x} - \mathbf{y})} \end{aligned} \tag{1}$$

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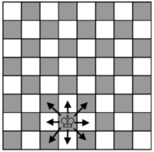

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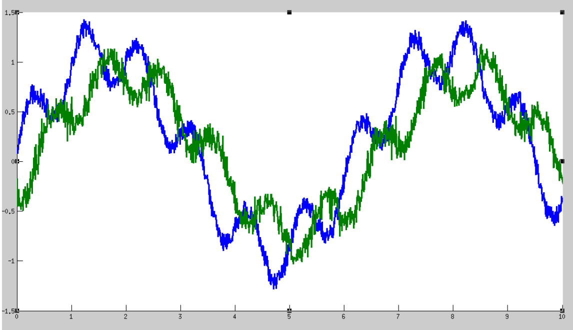

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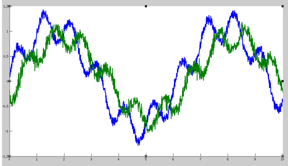

P-Norm distances can only

- Compare time series of the same length
- very sensitive to signal transformations:
  - shifting
  - uniform amplitude scaling
  - non-uniform amplitude scaling
  - uniform time scaling

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Adv. distance: Dynamic Time Warping (Berndt and Clifford, 1994)

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- Dynamic Time Warping (DTW) can be defined recursively as
  For two time series $\mathbf{X} = (x_0, .., x_n)$ and $\mathbf{Y} = (y_0, .., y_m)$

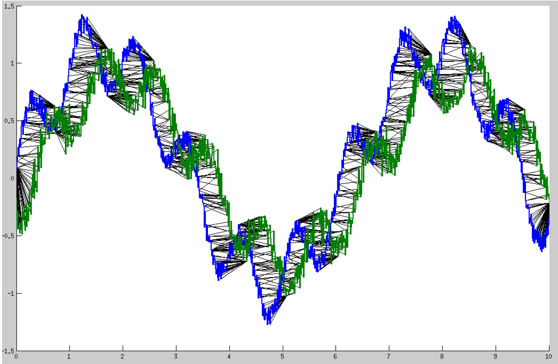
$$DTW(\mathbf{X}, \mathbf{Y}) = D(x_0, y_0) + min\{DTW(\mathbf{X}, REST(\mathbf{Y})), DTW(REST(\mathbf{X}), \mathbf{Y}), DTW(REST(\mathbf{X}), REST(\mathbf{Y}))\}$$

where $REST(\mathbf{X}) = (x_1, .., x_n)$

Distance (Numerical Time Series)

Adv. distance: Dynamic Time Warping (Berndt and Clifford, 1994)

- Can be used for aligning sequences



Distance (Symbolic)

- Distance is not always between numerical data
- Distance between symbolic data is less well-defined, but gaining interest (e.g. text data)
- Distance in text could be:
  - syntactic
  - semantic

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Syntactic - e.g. Hamming Distance

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- For binary strings, hamming distance equals L_1

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Syntactic - e.g. Edit Distance

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- Measures the *minimum* number of ‘operations’ required to transform one sequence of characters into another
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- used in spelling correction, DNA string comparisons

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Distance (Symbolic)

Semantic - e.g. WUP Relatedness Measure

- Built on top of a hierarchy of word semantics
- Most commonly used is WordNet (Princeton) `http://wordnet.princeton.edu/`
- WordNet contains more than 117,000 synsets (synset: set of one or more synonyms that are interchangeable in some context)

Distance (Symbolic)

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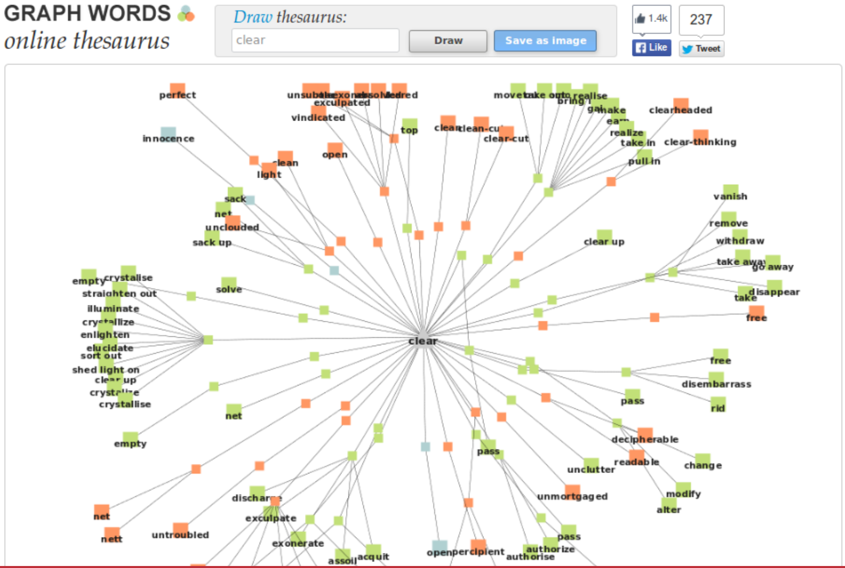

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  - antonymy (strong contract) e.g. wet $\leftrightarrow$ dry

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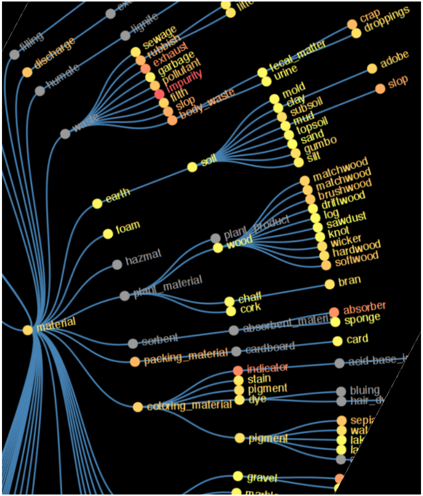
Semantic - e.g. hyponymy



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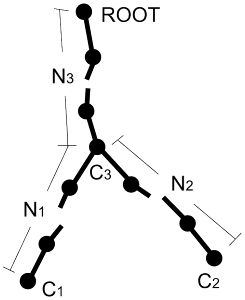

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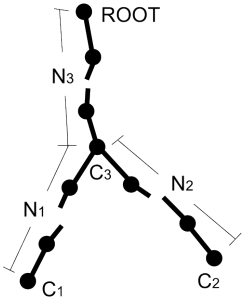

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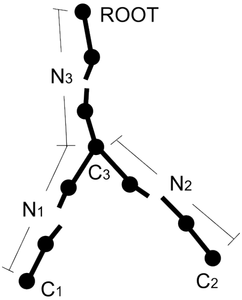

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- or online: `http://ws4jdemo.appspot.com/`

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Semantic - e.g. WUP Measure

- **HOWEVER** WUP is a similarity measure, not a distance measure
- It is effectively the inverse of a distance measure, taking higher values for similar data points.
- WUP(w1, w1) = 1

Distance (Symbolic)

Semantic - e.g. WUP Measure

- **HOWEVER** WUP is a similarity measure, not a distance measure
- It is effectively the inverse of a distance measure, taking higher values for similar data points.
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- Similarity measures can be converted to distance measures, depending on the values they take:

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- WUP(w1, w1) = 1
- Similarity measures can be converted to distance measures, depending on the values they take:

$$D_{WUP} = 1 - WUP$$

Distance - Conclusion

- Once you define a distance measure on your data, you can perform numeric operations
- Different distance measures will enable you to use the same data for various goals

Questions?

Questions?

Which are numeric distances? [Menti.com]

Agenda

Today we are going to talk about

1. Data acquisition
2. Data characteristics: distance measures
3. **Data characteristics: summary statistics [reminder]**

Mean and Variance (Reminder)

For one-dimensional data $\{x_1, .., x_n\}$,
Mean: [average]

$$\mu = \frac{1}{N} \sum_i x_i$$

Variance: [spread]

$$\sigma^2 = \frac{1}{N-1} \sum_i (x_i - \mu)^2$$

Standard Deviation:

$$\sigma = \sqrt{\frac{1}{N-1} \sum_i (x_i - \mu)^2}$$

Mean and Covariance

For multi-dimensional data $\{\mathbf{x}_1, .., \mathbf{x}_n\}$ where $\mathbf{x}_i$ is an m-dimensional vector,
Mean: calculated independently for each dimension

$$\boldsymbol{\mu} = \frac{1}{N}\sum_i \mathbf{x}_i$$

Variance can still be computed along each dimension

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Covariance Matrix: spread and correlation

$$\begin{aligned} \Sigma &= \frac{1}{N-1} \sum_i (\mathbf{x}_i - \boldsymbol{\mu})^2 \\ &= \frac{1}{N-1} \sum_i (\mathbf{x}_i - \boldsymbol{\mu})^T (\mathbf{x}_i - \boldsymbol{\mu}) \end{aligned}$$

WARNING: Σ is the capital letter of σ, not the summation sign!

Covariance Matrix

In two dimensions,

$$\Sigma = \frac{1}{N-1}\sum_i \begin{bmatrix} (v_{i1}-\mu_1)^2 & (v_{i1}-\mu_1)(v_{i2}-\mu_2) \\ (v_{i1}-\mu_1)(v_{i2}-\mu_2) & (v_{i2}-\mu_2)^2 \end{bmatrix}$$

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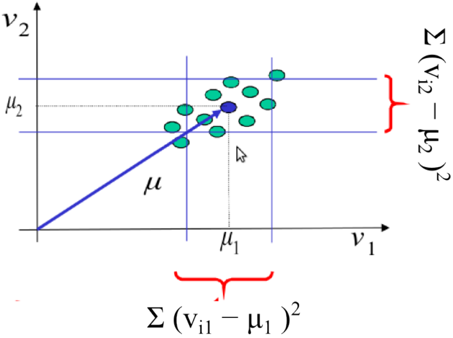

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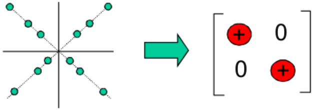
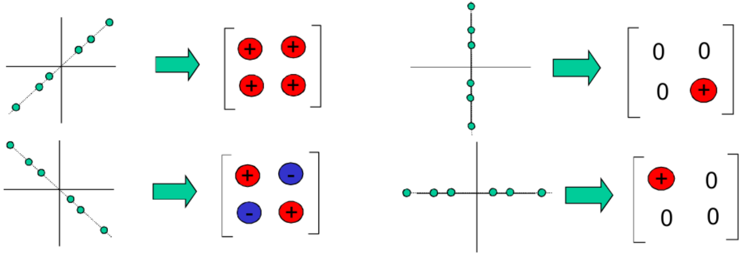
$$\Sigma = \frac{1}{N-1} \sum_i \begin{bmatrix} (v_{i1} - \mu_1)^2 & (v_{i1} - \mu_1)(v_{i2} - \mu_2) \\ (v_{i1} - \mu_1)(v_{i2} - \mu_2) & (v_{i2} - \mu_2)^2 \end{bmatrix}$$

- In addition to the variances along each dimension, the covariance matrix measures the correlation between components
- A positive covariance between two components means a proportional relationship between the variables.
- A negative covariance value indicates and inverse proportional relationship.

Covariance Matrix

$$C = \frac{1}{N-1} \sum_i \begin{bmatrix} (v_{i1} - \mu_1)^2 & (v_{i1} - \mu_1)(v_{i2} - \mu_2) \\ (v_{i1} - \mu_1)(v_{i2} - \mu_2) & (v_{i2} - \mu_2)^2 \end{bmatrix}$$





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COMS21202: Data Acquisition

Covariance and correlation: Demo

geogebra.org/m/wrSFAnkh

Covariance Matrix

In three dimensions,

$$\Sigma = \frac{1}{N-1}\sum_i \begin{bmatrix} (v_{i1}-\mu_1)^2 & (v_{i1}-\mu_1)(v_{i2}-\mu_2) & (v_{i1}-\mu_1)(v_{i3}-\mu_3) \\ (v_{i1}-\mu_1)(v_{i2}-\mu_2) & (v_{i2}-\mu_2)^2 & (v_{i2}-\mu_2)(v_{i3}-\mu_3) \\ (v_{i1}-\mu_1)(v_{i3}-\mu_3) & (v_{i2}-\mu_2)(v_{i3}-\mu_3) & (v_{i3}-\mu_3)^2 \end{bmatrix}$$

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Covariance matrix is always

- square and symmetric

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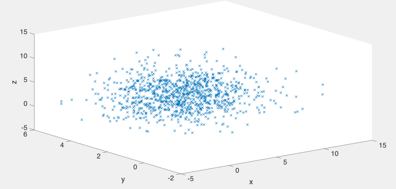
Covariance matrix is always

- square and symmetric
- variances on the diagonal
- covariance between each pair of dimensions is included in non-diagonal elements

Covariance Matrix - e.g.

For the covariance matrix,

$$\Sigma = \begin{bmatrix} 5 & -2 & 2 \\ -2 & 1 & 0 \\ 2 & 0 & 7 \end{bmatrix}$$



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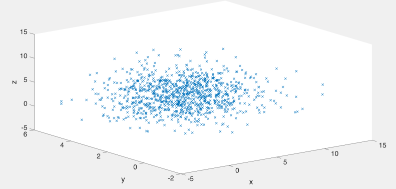

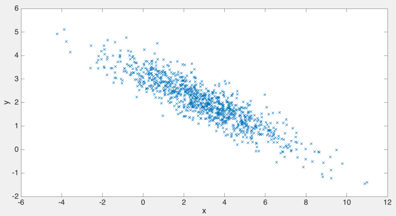

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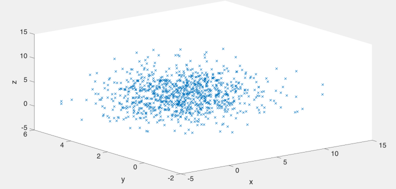

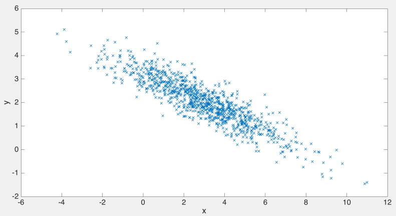

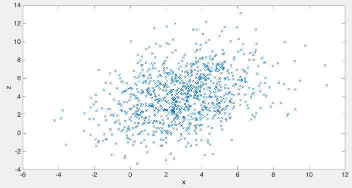

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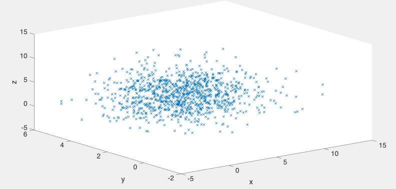

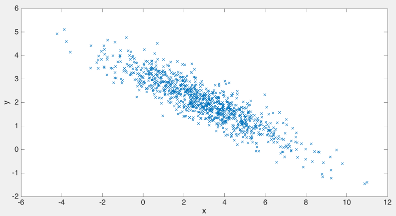

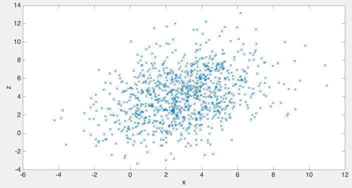

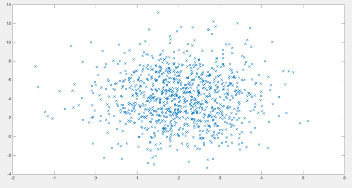

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Covariance Matrix: Eigen analysis

- Eigenvectors and eigenvalues define **principal axes** and spread of points along directions
- Commonly used to reduce data dimensionality (e.g. Principal component analysis [PCA])

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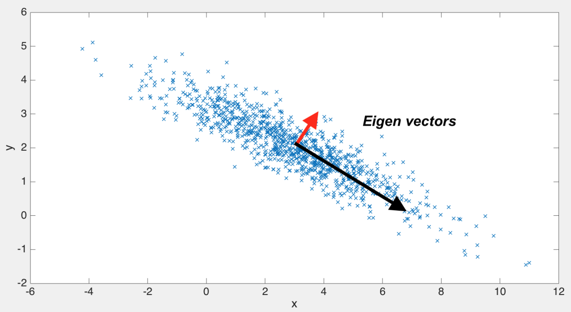

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Covariance Matrix: Eigen analysis

Definition

For a square matrix A,
if there exists a non-zero column vector v where

$$Av = \lambda v$$

then,
$v \rightarrow$ eigenvector of matrix A
$\lambda \rightarrow$ is eigenvalue of matrix A

e.g.

$$A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Covariance Matrix: Eigen analysis

- To calculate eigenvectors of a square matrix, solve $|A - \lambda\mathbf{I}| = 0$ where
  - $\mathbf{I}$ is the identity matrix
  - |A| is the determinant of the matrix
- For 2×2 matrices, two eigenvalues are found λ_1, λ_2

e.g.

$$A - \lambda\mathbf{I} = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & -1 \\ 2 & 3 - \lambda \end{bmatrix}$$

$$|A - \lambda\mathbf{I}| = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

$$\lambda_1 = 1,\ \lambda_2 = 2$$

Covariance Matrix: Eigen analysis

- After the eigenvalues are found, the eigenvectors can be calculated

For $\lambda_1 = 1$

$$\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \tag{2}$$

- This simplified to:

$$\begin{bmatrix} -v_{12} \\ 2v_{11} + 3v_{12} \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \tag{3}$$

- Its set $v_{12} = 1$ then we get the eigenvector[1]:

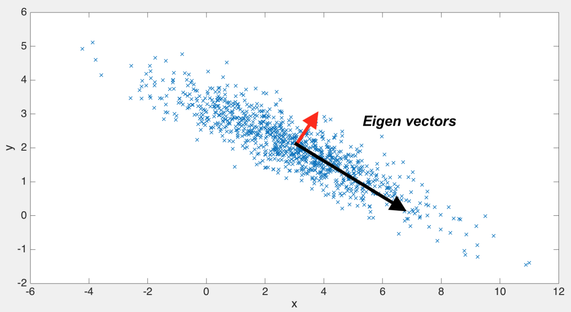
$$\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4}$$

- Verify that this is indeed a valid eigenvector by calculating $Av = \lambda v$

[1] Note that there many eigenvectors that work for a particular eigenvalue, but they all have the same direction. We could consider the eigenvector with $\|v_1\| = 1$, $v_1 = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$.

Covariance Matrix: Eigen analysis

- **Major axis** - eigenvector corresponding to larger eigenvalue (i.e. larger variance)
- **Minor axis** - eigenvector corresponding to smaller eigenvalue (i.e. smaller variance)
- These can be represented using major and minor axes of ellipses



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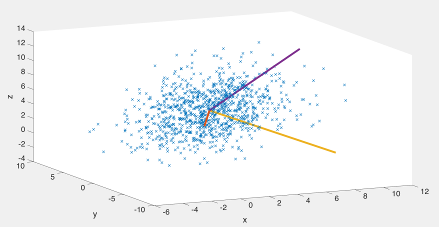
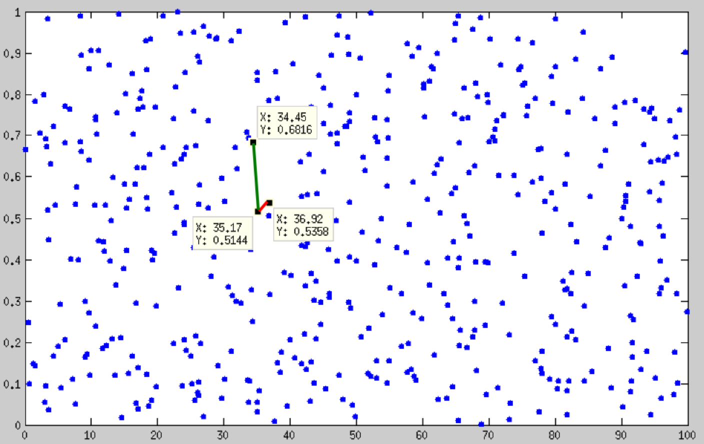
Covariance Matrix: another example



- $\lambda_1 = 0.08 \qquad \lambda_2 = 4.52 \qquad \lambda_3 = 8.40$
- $v_1 = \begin{bmatrix} -0.42 \\ -0.90 \\ 0.12 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0.71 \\ -0.40 \\ -0.57 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0.57 \\ -0.15 \\ 0.81 \end{bmatrix}$
- Principal/Major axis is v_3 (corresponding to largest eigenvalue)

Data Characteristic - Data Normalisation

- Multi-dimensional data may need to be normalised before distance is calculated [2]



[2] note the difference in magnitude between the two dimensions below!

Data Characteristic - Data Normalisation

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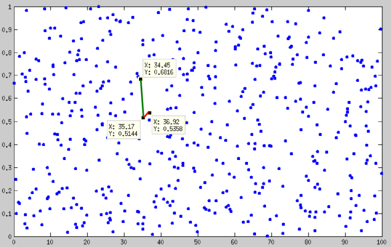
- Methods for normalisation:



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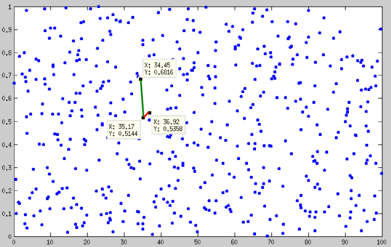
- Methods for normalisation:
  1. Rescaling

$$x' = \frac{x - min(x)}{max(x) - min(x)}$$



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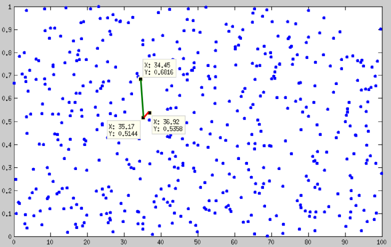
$$x' = \frac{x - \mu}{\sigma}$$



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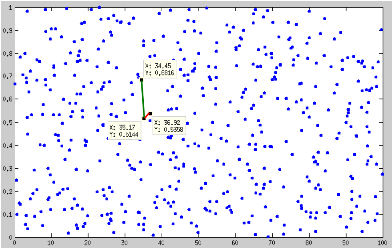
  3. Scaling to unit length

$$x' = \frac{x}{\|x\|}$$



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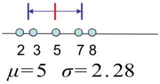

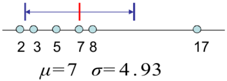

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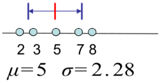
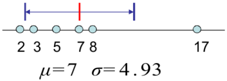
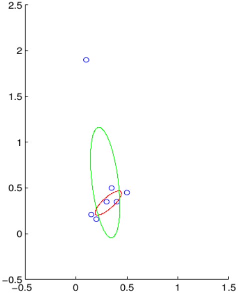
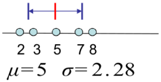


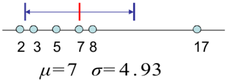


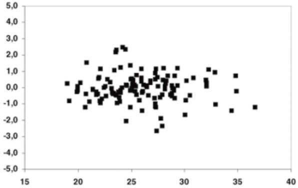


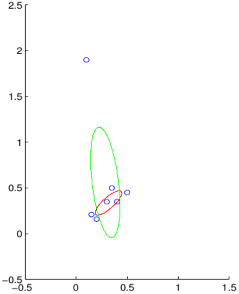
Data Characteristic - Outliers

- Mean, variance and covariance can provide concise description of ‘average’ and ‘spread’
  - but not when outliers are present in the data
  - **outliers**: small number of points with values significantly different from that other points
  - usually due to fault in measurement
  - not always easy to remove









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Mean vs. Median

- An alternative to arithmetic mean is the **median value**
- But median is difficult to work with
- e.g. median of two sets cannot be defined in terms of the individual medians

Note - Sample Variance vs. Variance

Given sample $\{x_1, x_2, .., x_N\}$

$$\mu \approx \bar{x} = \frac{1}{N}\sum_i x_i \tag{5}$$

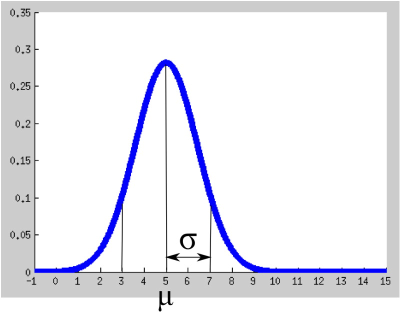
$$\sigma^2 \approx s^2 = \frac{1}{N-1}\sum_i (x_i - \bar{x})^2 \tag{6}$$

- These are only estimates of the 'true' mean and variance
- $N - 1$ gives unbiased estimate of the variance
- As $N \to \infty$
  - $\bar{x} \to \mu$
  - $s^2 \to \sigma^2$

Normal Distribution (Reminder)

For a normal distribution $\mathcal{N}(\mu, \sigma^2)$ in one dimension, the probability density function (pdf) can be calculated as

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{7}$$



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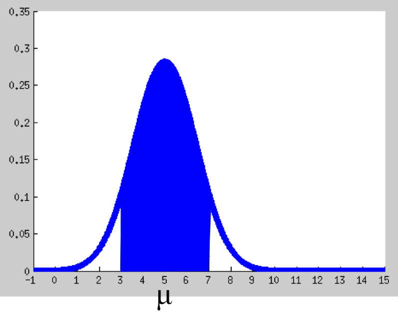
Normal Distribution (Reminder)

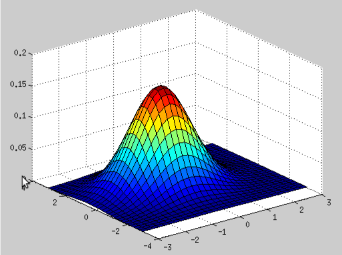
- 68% of the sample *should* lies within one standard deviation of the mean
- 95% of that area lies within two standard deviations of the mean
- 99.9% of that area lies within three standard deviations of the mean

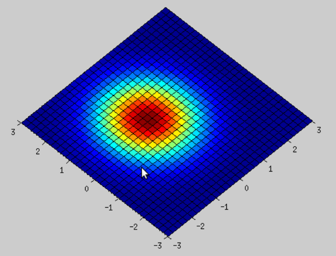


Normal Distribution - Multi-dimensional

For multi-dimensional normal distribution $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$ in M dimensions, the probability density function (pdf) can be calculated as

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^M|\Sigma|}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})} \tag{8}$$





WARNING: Σ is the capital letter of σ, not the summation sign!

Problem class tomorrow!

- Problem Class Tomorrow (Thur 1-2): Data Acquisition
- Prepare your answers in advance [problem sheet on github SPS page]

Further Reading

- Fundamentals of Multimedia
  Li and Drew (2004)
  - Section 6.1 Digitization of Sound
- Applied Multivariate Statistical Analysis
  Hardle and Simar (2003)
  - Section 1.2
  - Section 1.4
  - Section 3.1
  - Section 3.2
- Linear Algebra and its applications
  Lay (2012)
  - Section 6.5
  - Section 6.6
- Advances in Data Mining Knowledge Discovery and applications
  Karahoca (Ed.) (2012)
  - Chapter 3. Similarity Measures and Dimensionality Reduction Techniques for Time Series Data Mining