Linear Systems and Their Solutions

Solution set. The set of all solutions to the equation.

General Solution. An expression that gives us all the solutions to the equation.

E.g., $\begin{cases} x_1 = 5 + 4s - 7t \\ x_2 = s \\ x_3 = t \end{cases}$

Inconsistent. No solution to the system of linear equation

Consistent. At least one solution to the system of linear equation

Remark 1.1.10. Every linear solution has either
(i) no solution
(ii) exactly one solution
(iii) infinitely many solution

Elementary Row Operations (ERO).
(i) Multiply a row by a nonzero constant.
(ii) Interchange two rows.
(iii) Add a multiple of one row to another row.

Row Equivalent Matrices. Two augmented matrices are row equivalent if one can be obtained from the other by a series of elementary row operations. If two augment matrices are row equivalent, then they have the same set of solutions.

Row-Echelon Form (REF). A matrix is said to be in REF if they have properties 1 and 2.

1. there are any zero rows, they a grouped together at the bottom of the matrix.
2. in any two successive rows that do not consist entirely of zeros, leading entry in the lower row occurs further right than the leading entry in the higher row.

Reduced Row-Echelon Forms (RREF). A matrix is said to be in RREF if it is in REF and has properties 3 and 4.

3. The leading entry of every nonzero row is 1.
4. In each pivot column, except the pivot point, all other entries are zero

Gaussian Elimination. Algorithm that reduces an augmented matrix into REF using ERO.

Gauss-Jordan Elimination. Algorithm that reduces an augmented matrix into RREF using ERO.

Remark 1.4.5. Every matrix has a unique RREF but have many REFs

Remark 1.4.8.1. A linear system is inconsistent if the last column of a REF of the augmented matrix is a pivot column

Remark 1.4.8.2. A linear system has only one solution if every column except the last column is a pivot column.

Remark 1.4.8.3. A consistent linear system has infinitely many solutions if apart from the last column, a REF of the augmented matrix has at least one more non-pivot column.

Notation of EROs.

1. cR_i : multiply the i^{th} row by constant c
2. $R_i \leftrightarrow R_j$: interchange the i^{th} and the j^{th} row
3. $R_i + cR_j$: add c times of j^{th} row to the i^{th} row

Homogeneous Linear Systems. A system of linear equation is said to be homogeneous if all constant terms are zero.

Trivial Solution. $x_1=0, x_2=0, \ldots, x_n=0$ is always a solution to the homogeneous system and it is called the trivial solution.

Non-trivial Solution. Any solution other than the trivial solution is called the non-trivial solution.

Solutions of Homogeneous System.
1. A *homogeneous* system of linear equations has either only the *trivial solution* or *infinitely many solutions* in addition to the trivial solution.
2. A *homogeneous* system of linear equations with more unknowns than equations has *infinitely many solutions*

Matrices

Matrix. A matrix is a rectangular array of numbers.

Entries. Are the numbers in the array.

Size of the Matrix. Size of the matrix is given by $m \times n$ where m is the number of rows and n is the number of columns.

(i,j)-Entry. The (i,j)-entry of a matrix is the number which is in the i^{th} row and the j^{th} column of the matrix.

Notation of Matrices. A $m \times n$ matrix can be written as

$\mathbf{A} = (a_{ij})_{m \times n}$.

Square Matrices. A matrix is a square matrix if it has the same number of rows and columns. A $n \times n$ matrix is called a square matrix of order n.

Diagonal Entry. The a_{ii} entry is called the diagonal entry.

Diagonal Matrices. A *square matrix* is called a diagonal matrix if all its *non-diagonal entries are zero.*

Scalar Matrices. A *diagonal matrix* is called a scalar matrix if all *diagonal entries are the same*.

Identity Matrices. A *diagonal matrix* is called an identity matrix if all is *diagonal entries are 1*.

Zero Matrices. A matrix with all entries equals zero is called a zero matrix.

Symmetric Matrices. A square matrix is symmetric if $a_{ij} = a_{ji}$ for all i, j. A matrix **A** is symmetric if and only if $\mathbf{A} = \mathbf{A}^T$

Triangular Matrices.
1. A square matrix (a_{ij}) is called *upper triangular* if $a_{ij} = 0$ for all $i > j$
2. A square matrix (a_{ij}) is called *lower triangular* if $a_{ij} = 0$ for all $i < j$

Both upper and lower triangular matrices are called triangular matrices

Equal Matrices. 2 matrices are said to be equal if
1. they have the same size.
2. their corresponding entries are equal.

Matrix Addition. Given $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathbf{B} = (b_{ij})_{m \times n}$,

$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})_{m \times n}$.

Matrix Subtraction. Given $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathbf{B} = (b_{ij})_{m \times n}$,

$\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})_{m \times n}$.

Scalar Multiplication. Given $\mathbf{A} = (a_{ij})_{m \times n}$ and a constant c,

$c\mathbf{A} = (ca_{ij})_{m \times n}$

Theorem 2.2.6. (Basic Properties)

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
3. $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$
4. $(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$
5. $(cd)\mathbf{A} = c(d\mathbf{A}) = d(c\mathbf{A})$
6. $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$
7. $\mathbf{A} - \mathbf{A} = \mathbf{0}$
8. $0\mathbf{A} = \mathbf{0}$

Matrix Multiplication. Given $\mathbf{A} = (a_{ij})_{m \times p}$ and $\mathbf{B} = (b_{ij})_{p \times n}$, the product of **AB** is definite to be a m × n matrix whose *(i, j)-entry* is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{i1p}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

for i = 1, 2, …, m and j = 1, 2, …, n

The number of columns in **A** must be equal to the number of rows in **B.**

Multiplication Is Not Commutative. In general, $\mathbf{AB} \neq \mathbf{BA}$.

Remark 2.2.10.4. When $\mathbf{AB} = \mathbf{0}$, it is not necessary that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

Theorem 2.2.11. (Basic Properties)

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
2. $\mathbf{A}(\mathbf{B}_1 + \mathbf{B}_2) = \mathbf{AB}_1 + \mathbf{AB}_2$
   $(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{A} = \mathbf{C}_1\mathbf{A} + \mathbf{C}_2\mathbf{A}$
3. $c(\mathbf{AB}) = (c\mathbf{A})\mathbf{B} = \mathbf{A}(c\mathbf{B})$
4. $\mathbf{A0} = \mathbf{0}$
   $\mathbf{0A} = \mathbf{0}$
   $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$

Powers of Square Matrices. Let **A** be a square matrix and n a nonnegative integer. We define $\mathbf{A}^n$ as follows:

$$\mathbf{A}^n = \begin{cases} \boldsymbol{I} & if\ n = 0 \\ \boldsymbol{AA \ldots A} & if\ n \geq 1 \\ (\boldsymbol{A}^{-1})^{-n} & if\ n < 0 \end{cases}$$

Note:

1. $\mathbf{A}^m\mathbf{A}^n = \mathbf{A}^{m+n}$
2. In general, $(\mathbf{AB})^2 \neq \mathbf{A}^2\mathbf{B}^2$

Notation 2.2.15.

Given $\mathbf{A} = (a_{ij})_{m \times p}$ and $\mathbf{B} = (b_{ij})_{p \times n}$, we can write

$\mathbf{A} = \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_m} \end{bmatrix}$ where $\boldsymbol{a_i} = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{ip}]$ and

$\mathbf{B} = [\boldsymbol{b_1} \quad \boldsymbol{b_2} \quad \cdots \quad \boldsymbol{b_n}]$ where $\boldsymbol{b_j} = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$ then

$\mathbf{AB} = \begin{bmatrix} \boldsymbol{a_1b_1} & \boldsymbol{a_1b_2} & \cdots & \boldsymbol{a_1b_n} \\ \boldsymbol{a_2b_1} & \boldsymbol{a_2b_2} & \cdots & \boldsymbol{a_2b_n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a_mb_1} & \boldsymbol{a_mb_2} & \cdots & \boldsymbol{a_mb_n} \end{bmatrix}$ where

$$\boldsymbol{a_ib_j} = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{ip}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$$

We can also write

$\mathbf{AB} = \mathbf{A}\,[\boldsymbol{b_1} \quad \boldsymbol{b_2} \quad \cdots \quad \boldsymbol{b_n}] = [\boldsymbol{Ab_1} \quad \boldsymbol{Ab_2} \quad \cdots \quad \boldsymbol{Ab_n}]$

or

$$\boldsymbol{AB} = \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_m} \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} \boldsymbol{a_1B} \\ \boldsymbol{a_2B} \\ \vdots \\ \boldsymbol{a_mB} \end{bmatrix}$$

Representation of Linear Systems. We can represent the system of linear equations as $\mathbf{Ax} = \mathbf{b}$, where **A** is the coefficient matrix, **x** is the variable matrix and **b** is the constant matrix

Solution to Linear Systems. A n × 1 matrix **u** is said to be a solution to the linear system $\mathbf{Ax} = \mathbf{b}$ if $\mathbf{Au} = \mathbf{b}$

Transposes. Given $\mathbf{A} = (a_{ij})_{m \times n}$, then $\mathbf{A}^T = (a_{ji})_{n \times m}$

Theorem 2.2.22 (Basic Properties)

Let **A** be a m × n matrix.

1. $(\mathbf{A}^T)^T = \mathbf{A}$
2. If **B** is an m × n matrix, then $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
3. If c is a scalar, then $(c\mathbf{A})^T = c\mathbf{A}^T$
4. If B is a n × p matrix, then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Inverses. Let A be a *square matrix* of order n. **A** is said to be invertible if there exists a square matrix **B** of order n such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$. **B** is called the inverse of A.

Singular Matrix. A square matrix is called *singular* if it has no inverses.

Matrix Cancellation Laws. Let A be an invertible m × m matrix.

(a) If $\mathbf{B}_1$ and $\mathbf{B}_2$ are m × n matrices such that $\mathbf{AB}_1 = \mathbf{AB}_2$, then $\mathbf{B}_1 = \mathbf{B}_2$

(b) If $\mathbf{C}_1$ and $\mathbf{C}_2$ are n × m matrices such that $\mathbf{C}_1\mathbf{A} = \mathbf{C}_2\mathbf{A}$, then $\mathbf{C}_1 = \mathbf{C}_2$

If **A** is not invertible, the cancellation laws may not hold.

Uniqueness of Inverses. If **B** and **C** are inverses of a square matrix **A**, then $\mathbf{B} = \mathbf{C}$. The inverse of **A** can be denoted as $\mathbf{A}^{-1}$.

Inverse of a 2 × 2 matrix. Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then **A** is invertible and $\mathbf{A}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

Theorem 2.3.9 (Basic Properties)

Let **A**, **B** be two invertible matrices and c a nonzero scalar

1. $c\mathbf{A}$ is invertible and $(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$
2. $\mathbf{A}^T$ is invertible and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$
3. $\mathbf{A}^{-1}$ is invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
4. $\mathbf{AB}$ is invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
5. $\mathbf{A}^n$ is invertible and $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$

By part 4, $(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1} \ldots \mathbf{A}_2^{-1}\,\mathbf{A}_1^{-1}$

Elementary Matrices. A square matrix is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation. There are 3 types of elementary matrices and they are all invertible. Their inverses are also elementary matrices of the same type.

Invertible Matrices (Theorem 2.4.7). Let **A** be a square matrix. The following statements are equivalent.

1. **A** is invertible.
2. The linear system **Ax = 0** has only the trivial solution.
3. The RREF(**A**) = **I**
4. **A** can be expressed as a product of elementary matrices.
5. det(**A**) $\neq 0$
6.

Finding Inverses. Let A be an invertible matrix of order n. Then

$$(\mathbf{A} \mid \mathbf{I}) \xrightarrow[Elimination]{Gauss-Jordan} (\mathbf{I} \mid \mathbf{A}^{-1})$$

Verifying Invertibility. Let **A** be a square matrix. If a REF of **A** `has at least one zero row, **A** is singular.

Theorem 2.4.12. Suppose **A** and **B** are square matrices of the same size. If **AB = I**, then

(i) **A** is invertible
(ii) **B** is invertible
(iii) $\mathbf{A}^{-1} = \mathbf{B}$
(iv) $\mathbf{B}^{-1} = \mathbf{A}$
(v) **BA** = **I**

Theorem 2.4.14. Suppose **A** and **B** are square matrices of the same size. If **A** is singular, then both **AB** and **BA** are singular.

Determinants. Let $\mathbf{A} = (a_{ij})_{n \times n}$. Let $\mathbf{M}_{ij}$ be the $(n-1) \times (n-1)$ matrix obtained from **A** by *deleting* the i^{th} row and the j^{th} column. Then the determinant of **A** is defined to be

$$\det(\mathbf{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

Where $A_{ij} = (-1)^{i+j}\det(\mathbf{M}_{ij})$ which is called the *(i, j)*-cofactor of **A.**

Cofactor Expansions. Let $\mathbf{A} = (a_{ij})_{n \times n}$.

$\det(\mathbf{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$
$\quad = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$

Hence, you can expand along any column and row.

Determinant of Triangular Matrices. If **A** is a $n \times n$ triangular matrix, then $\det(\mathbf{A}) = a_{11}a_{22}\cdots a_{nn}$.

Determinant of Transposes. If **A** is a square matrix, then $\det(\mathbf{A}^T) = \det(\mathbf{A})$

Determinant of matrices with identical rows or columns.

1. The determinant of a square matrix with two identical rows is zero.
2. The determinant of a square matrix with two identical columns is zero.

Effects of ERO On Determinant

1) $\mathbf{A} \xrightarrow{kR_i} \mathbf{B_1}$: $\det(\mathbf{B_1}) = k\det(\mathbf{A})$
2) $\mathbf{A} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{B_1}$: $\det(\mathbf{B_1}) = -\det(\mathbf{A})$
3) $\mathbf{A} \xrightarrow{R_j + kR_i} \mathbf{B_1}$: $\det(\mathbf{B_1}) = \det(\mathbf{A})$

Furthermore, if **E** is an elementary matrix of the same size as **A**, then $\det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A})$

Invertible Matrices and Determinants. A square matrix **A** is invertible if and only if $\det(\mathbf{A}) \neq 0$

Scalar Multiplication and Determinants. If **A** is a square matrix of order n and c a scalar, then $\det(c\mathbf{A}) = c^n\det(\mathbf{A})$

Matrix Multiplication and Determinants. If **A** and **B** are square matrices of the same size, then $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

Invertible Matrices and Determinants. If **A** is an invertible matrix, then $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$

Adjoints. Let **A** be a square matrix of order n. Then

$$\mathbf{adj}(\mathbf{A}) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

where A_{ij} is the *(i, j)*-cofactor of **A**.

Inverse with Adjoints. If **A** is an invertible matrix, then $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{adj}(\mathbf{A})$

Cramer's Rule. Suppose **Ax = b** is a linear system where $\mathbf{A} = (a_{ij})_{n \times n}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. Let $\mathbf{A}_i$ be the $n \times n$ matrix obtained from **A** by replacing the i^{th} column of **A** and **B**.

If **A** is invertible, then the system has only one solution

$\mathbf{x} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} \det(\mathbf{A_1}) \\ \det(\mathbf{A_2}) \\ \vdots \\ \det(\mathbf{A_n}) \end{bmatrix}$. In general, $x_i = \frac{\det(\mathbf{A_i})}{\det(\mathbf{A})}$