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Equivalent Statements for Invertibility

Invertible Matrices (Theorem 2.4.7). Let **A** be a square matrix. The following statements are equivalent.

1. **A** is invertible.
2. **A** has a left inverse.
3. **A** has a right inverse.
4. The RREF of **A** is the identity matrix.
5. **A** can be expressed as a product of elementary matrices.
6. The homogeneous system $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.
7. For any $\boldsymbol{b}$, $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution.
8. The determinant of **A** is nonzero, $\det A \neq 0$
9. The columns/rows of **A** spans $\mathbb{R}^n$
10. The columns/rows of **A** are linearly independent.
11. **A** is of full rank, $rank(\mathbf{A}) = \text{n}$
12. $nullity(\mathbf{A}) = 0$
13. **0** is not an eigenvalue of **A**
14. The linear transformation T_A defined by **A** is injective, or $Ker(T_A) = \{\mathbf{0}\}$
15. The linear transformation T_A defined by **A** is surjective, or $R(T_A) = \mathbb{R}^n$

Notation 2.2.15.

Given **A** = $(a_{ij})_{m \times p}$ and **B** = $(b_{ij})_{p \times n}$, we can write

$\mathbf{A} = \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_m} \end{bmatrix}$ where $\boldsymbol{a_i} = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{ip}]$ and

$\mathbf{B} = [\boldsymbol{b_1} \quad \boldsymbol{b_2} \quad \cdots \quad \boldsymbol{b_n}]$ where $\boldsymbol{b_j} = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$ then

$\mathbf{AB} = \begin{bmatrix} \boldsymbol{a_1 b_1} & \boldsymbol{a_1 b_2} & \cdots & \boldsymbol{a_1 b_n} \\ \boldsymbol{a_2 b_1} & \boldsymbol{a_2 b_2} & \cdots & \boldsymbol{a_2 b_n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a_m b_1} & \boldsymbol{a_m b_2} & \cdots & \boldsymbol{a_m b_n} \end{bmatrix}$ where

$\boldsymbol{a_i b_j} = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{ip}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$

$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$

We can also write

$\mathbf{AB} = \mathbf{A}\,[\boldsymbol{b_1} \quad \boldsymbol{b_2} \quad \cdots \quad \boldsymbol{b_n}] = [\boldsymbol{Ab_1} \quad \boldsymbol{Ab_2} \quad \cdots \quad \boldsymbol{Ab_n}]$

or

$\boldsymbol{AB} = \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_m} \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} \boldsymbol{a_1 B} \\ \boldsymbol{a_2 B} \\ \vdots \\ \boldsymbol{a_m B} \end{bmatrix}$

Verifying Invertibility. Let **A** be a square matrix. If a REF of **A** has at least one zero row, **A** is singular.

Effects of ERO On Determinant

1) $\mathbf{A} \xrightarrow{kR_i} \mathbf{B_1}$: $\det(\mathbf{B_1}) = \text{k}\det(\mathbf{A})$

2) $\mathbf{A} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{B_1}$: $\det(\mathbf{B_1}) = -\det(\mathbf{A})$

3) $\mathbf{A} \xrightarrow{R_j + kR_i} \mathbf{B_1}$: $\det(\mathbf{B_1}) = \det(\mathbf{A})$

Furthermore, if **E** is an elementary matrix of the same size as **A**, then $\det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A})$

Adjoints. Let **A** be a square matrix of order n. Then

$$\mathbf{adj}(\mathbf{A}) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

where A_{ij} is the *(i, j)*-cofactor of **A**.

Inverse with Adjoints. If **A** is an invertible matrix, then $\mathbf{A}^{-1} = \frac{1}{\det(A)}\mathbf{adj}(\mathbf{A})$

Adjoint Identity. For *any* square matrix,

$$A\big(adj(A)\big) = \det(A)\,\boldsymbol{I}$$

Cramer's Rule. Suppose **Ax = b** is a linear system where **A** = $(a_{ij})_{n \times n}$, **x** = $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and **b** = $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. Let $\mathbf{A}_i$ be the n × n matrix obtained from **A** by replacing the i^{th} column of **A** and **B**.

If **A** is invertible, then the system has only one solution

x = $\frac{1}{\det(A)} \begin{bmatrix} \det(\boldsymbol{A_1}) \\ \det(\boldsymbol{A_2}) \\ \vdots \\ \det(\boldsymbol{A_n}) \end{bmatrix}$. In general, $\text{x}_i = \frac{\det(\mathbf{A_i})}{\det(\mathbf{A})}$

When $\boldsymbol{span(S)} = \mathbb{R}^n$. Suppose $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$. Let $\boldsymbol{A} = (\boldsymbol{u_1}\ \boldsymbol{u_2} \cdots \boldsymbol{u_k})$.

1. If a REF of $\boldsymbol{A}$ has no zero rows, then the linear system is always consistent. Hence $span(S) = \mathbb{R}^n$
2. If a REF of $\boldsymbol{A}$ has zero rows, then $span(S) \subset \mathbb{R}^n$

From the result above, we conclude that if $|S| < n$, S cannot span $\mathbb{R}^n$

Properties of Linear Span. Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\} \subseteq \mathbb{R}^n$.

(i) (Contains the origin)

$$\mathbf{0} \in span(S)$$

(ii) (Closed under Linear Combination)

$$\forall \boldsymbol{u}, \boldsymbol{v} \in span(S), \alpha, \beta \in \mathbb{R}, \alpha\boldsymbol{u} + \beta\boldsymbol{v} \in span(S)$$

When $\boldsymbol{span(S_1)} \subseteq \boldsymbol{span(S_2)}$. Let $S_1 = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ and $S_2 = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}\}$ be subsets of $\mathbb{R}^n$. Then $span(S_1) \subseteq span(S_2)$ if and only if each $\boldsymbol{u_i}$ is a linear combination of $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}$

$$span\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\} \subseteq span\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}\}$$
$$\leftrightarrow$$
$$\left(\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m} \,\middle|\, \boldsymbol{u_1} \,\middle|\, \boldsymbol{u_2} \,\middle|\, \cdots \,\middle|\, \boldsymbol{u_k}\right) \text{ is consistent}$$

Linear Dependence.

A set $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ is linearly dependent if there exists $c_1, c_2, \cdots, c_k \in \mathbb{R}$, not all zero such that

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0}$$

A set $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ is linearly independent if whenever $c_1, c_2, \cdots, c_k \in \mathbb{R}$ is such that

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0}$$

necessarily $c_1 = \cdots = c_k = 0$

You can use this to prove linear independence as well.

Testing for Linear Independence. Let $A = (\boldsymbol{u_1}\ \boldsymbol{u_2} \cdots \boldsymbol{u_k})$.

$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\} \subseteq \mathbb{R}^n$ is linearly dependent

$\leftrightarrow Ax = 0$ has only the trivial solution

$\leftrightarrow$ all columns of REF of A are pivot

Equivalent ways to check for basis. To prove that S is a basis for V

1. By definition
   - (i) $V = span(S)$
   - (ii) S is linearly independent
2. B1
   - (i) $|S| = \dim(V)$
   - (ii) $S \subseteq V$
   - (iii) S is linearly independent.
3. B2
   - (i) $|S| = \dim(V)$
   - (ii) $V \subseteq span(S)$

Transition Matrix. Suppose $V \subseteq \mathbb{R}^n$ is a subspace with dimension k. $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$ are basis for V. Then the transition matrix from S to T, denoted as $\boldsymbol{P}$, is

$$\boldsymbol{P} = [[\boldsymbol{u_1}]_T, \cdots, [\boldsymbol{u_k}]_T]$$

such that

$$[\boldsymbol{w}]_T = \boldsymbol{P}[\boldsymbol{w}]_S$$
$$[\boldsymbol{w}]_S = \boldsymbol{P^{-1}}[\boldsymbol{w}]_T$$

The transition matrix from S to T can be found by

$$(T \mid S) \xrightarrow{G.J.E} \left(\begin{array}{c|c} I_k & P \\ 0\cdots0 & 0\cdots0 \end{array}\right)$$

Orthogonal Basis to Relative Coordinates.

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ is a $\begin{cases} (i) \text{ orthogonal} \\ (ii) \text{ orthonormal} \end{cases}$ basis for $V \subseteq \mathbb{R}^n$ subspace.

(i) $\boldsymbol{v} = \frac{\boldsymbol{v}\cdot\boldsymbol{u_1}}{\boldsymbol{u_1}\cdot\boldsymbol{u_1}}\boldsymbol{u_1} + \frac{\boldsymbol{v}\cdot\boldsymbol{u_2}}{\boldsymbol{u_2}\cdot\boldsymbol{u_2}}\boldsymbol{u_2} + \cdots + \frac{\boldsymbol{v}\cdot\boldsymbol{u_k}}{\boldsymbol{u_k}\cdot\boldsymbol{u_k}}\boldsymbol{u_k}$

(ii) $\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{u_1})\boldsymbol{u_1} + (\boldsymbol{v}\cdot\boldsymbol{u_2})\boldsymbol{u_2} + \cdots + (\boldsymbol{v}\cdot\boldsymbol{u_k})\boldsymbol{u_k}$

Orthogonal Projection. Let $V \subseteq \mathbb{R}^n$. Every $\boldsymbol{w} \in \mathbb{R}^n$ can be decomposed uniquely as

$$\boldsymbol{w} = \boldsymbol{w_p} + \boldsymbol{w_n}$$

where $\boldsymbol{w_p} \in V$ and $\boldsymbol{w_n} \perp V$. The unique vector $\boldsymbol{w_p} \in V$ is called the orthogonal projection of $\boldsymbol{w}$ onto V.

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ is a $\begin{cases} (i) \text{ orthogonal} \\ (ii) \text{ orthonormal} \end{cases}$ basis for $V \subseteq \mathbb{R}^n$ subspace.

(iii) $\boldsymbol{w_p} = \frac{\boldsymbol{w}\cdot\boldsymbol{u_1}}{\boldsymbol{u_1}\cdot\boldsymbol{u_1}}\boldsymbol{u_1} + \frac{\boldsymbol{w}\cdot\boldsymbol{u_2}}{\boldsymbol{u_2}\cdot\boldsymbol{u_2}}\boldsymbol{u_2} + \cdots + \frac{\boldsymbol{w}\cdot\boldsymbol{u_k}}{\boldsymbol{u_k}\cdot\boldsymbol{u_k}}\boldsymbol{u_k}$

(iv) $\boldsymbol{w_p} = (\boldsymbol{w}\cdot\boldsymbol{u_1})\boldsymbol{u_1} + (\boldsymbol{w}\cdot\boldsymbol{u_2})\boldsymbol{u_2} + \cdots + (\boldsymbol{w}\cdot\boldsymbol{u_k})\boldsymbol{u_k}$

Gram-Schmidt Process. $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ be linearly independent.

$$\boldsymbol{v_1} = \boldsymbol{u_1}$$

$$\boldsymbol{v_2} = \boldsymbol{u_2} - \frac{\boldsymbol{v_1}\cdot\boldsymbol{u_2}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\,\boldsymbol{v_1}$$

$$\boldsymbol{v_3} = \boldsymbol{u_3} - \frac{\boldsymbol{v_1}\cdot\boldsymbol{u_3}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\,\boldsymbol{v_1} - \frac{\boldsymbol{v_2}\cdot\boldsymbol{u_3}}{\boldsymbol{v_2}\cdot\boldsymbol{v_2}}\,\boldsymbol{v_2}$$

$$\boldsymbol{v_k} = \boldsymbol{u_k} - \frac{\boldsymbol{v_1}\cdot\boldsymbol{u_k}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\,\boldsymbol{v_1} - \frac{\boldsymbol{v_2}\cdot\boldsymbol{u_k}}{\boldsymbol{v_2}\cdot\boldsymbol{v_2}}\,\boldsymbol{v_2} - \cdots - \frac{\boldsymbol{v_{k-1}}\cdot\boldsymbol{u_2}}{\boldsymbol{v_{k-1}}\cdot\boldsymbol{v_{k-1}}}\,\boldsymbol{v_{k-1}}$$

Then $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$ is an orthogonal set and hence

$$\left\{\frac{\boldsymbol{v_1}}{\|\boldsymbol{v_1}\|}, \frac{\boldsymbol{v_2}}{\|\boldsymbol{v_2}\|}, \cdots, \frac{\boldsymbol{v_k}}{\|\boldsymbol{v_k}\|}\right\}$$

is an orthonormal basis for $span(S)$

Obtaining the Least Squares Solution.

$\boldsymbol{u}$ is a least square solution to $\boldsymbol{Ax} = \boldsymbol{b}$ $\Leftrightarrow$ $\boldsymbol{Au}$ is the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$, $Col(\boldsymbol{A})$

$\Leftrightarrow$ $\boldsymbol{u}$ is a solution to $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$

Finding Projection using Least Squares. Let . $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\} \subseteq \mathbb{R}^n$, $V = span(S)$. For any $\boldsymbol{w} \in \mathbb{R}^n$, the projection of **w** onto V is **Au,** where $\boldsymbol{A} = (\boldsymbol{u_1} \quad \cdots \quad \boldsymbol{u_k})$, and $\mathbf{u} \in \mathbb{R}^k$ is a solution to $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{w}$

Finding Least Squares using Shortcut. Projection of $\boldsymbol{w}$ onto $\boldsymbol{V}$ is the formula below (although it is not proven)

$$\boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{w}$$

Transition Matrix between Two Orthogonal Basis.

Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$, $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$ orthonormal basis for subspace $W \subseteq \mathbb{R}^n$

1. Transition matrix $P: S \to T$ is an orthogonal matrix
2. The transition matrix $T \to P^T$

where

$$P = [\boldsymbol{v_1}\ \boldsymbol{v_2}\ \boldsymbol{v_3}]^T[\boldsymbol{u_1}\ \boldsymbol{u_2}\ \boldsymbol{u_3}]$$

Equivalent Statements for Diagonalizability.

(i) $\boldsymbol{A}$ is diagonalizable.

(ii) There exists a *basis* $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_n}\} \subseteq \mathbb{R}^n$ of eigenvectors of $\boldsymbol{A}$

(iii) The sum of dimension of the eigenspaces of $\boldsymbol{A}$ is equal to its order,

$$\sum_{\lambda \text{ eigenvectors of } \boldsymbol{A}} \dim(E_\lambda) = n$$

(iv) The characteristic polynomial of $\boldsymbol{A}$ splits

$$\det(x\boldsymbol{I} - \boldsymbol{A}) = (x - \lambda_1)^{r_1}(x - \lambda_2)^{r_2} \cdots (x - \lambda_k)^{r_k}$$

where r_i is the multiplicity of eigenvalue λ_i, for $i = 1, \cdots, k$ and the eigenvalues are distinct, $\lambda_i \neq \lambda_j$ for $i \neq j$, and the dimension of each eigenspace is equal to its multiplicity

$$\dim(E_{\lambda_i}) = r_i$$

Standard Matrix. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then the standard matrix $\boldsymbol{A}$ can be denoted as

$$\boldsymbol{A} = [T(\boldsymbol{e_1}) \quad T(\boldsymbol{e_1}) \quad \cdots \quad T(\boldsymbol{e_1})]$$

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists a $m \times n$ matrix $\boldsymbol{A}$ s.t. $T(\boldsymbol{u}) = \boldsymbol{Au}$ for all $\boldsymbol{u} \in \mathbb{R}^{\boldsymbol{n}}$

Retrieving the Standard Matrix. Suppose $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_n}\} \subseteq \mathbb{R}^n$ is a basis and $T(u_1), T(u_2), \cdots, T(u_n)$ is given. Define the representation of T with respect to S as

$$[T]_S = [T(\boldsymbol{u_1}) \quad T(\boldsymbol{u_1}) \quad \cdots \quad T(\boldsymbol{u_1})]$$

Then for any $\boldsymbol{v} \in \mathbb{R}^n$,

$$\begin{aligned} T(\boldsymbol{v}) &= T(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_n}) \\ &= c_1T(\boldsymbol{u_1}) + c_2T(\boldsymbol{u_2}) + \cdots + c_kT(\boldsymbol{u_n}) \\ &= [T]_S[\boldsymbol{v}]_S \end{aligned}$$

So, the standard matrix of T is the representation of T with respect to E, the standard matrix, $\boldsymbol{A} = [T]_E$

$P = (\boldsymbol{u_1} \quad \boldsymbol{u_2} \quad \cdots \quad \boldsymbol{u_n})$ is the transition matrix from S to E such that $P^{-1}\boldsymbol{v} = [\boldsymbol{v}]_S$.

$$\boldsymbol{Av} = T(\boldsymbol{v}) = [T]_S[\boldsymbol{v}]_S = [T]_S P^{-1}\boldsymbol{v}$$

Therefore, $A = [T]_S P^{-1}$

Range and Rank. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The range of T, which is denoted by $R(T)$, is the set of images of T.

$$R(T) = \{T(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m = \{\boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^n\} = Col(\boldsymbol{A})$$

Hence,

$$rank(T) = \dim\big(R(T)\big) = \dim\big(Col(\boldsymbol{A})\big) = rank(\boldsymbol{A})$$

Kernel and Nullity. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The kernel of T, which is denoted by $Ker(T)$, is the set of vectors in $\mathbb{R}^n$, whose image is the zero vector in $\mathbb{R}^m$

$$\begin{aligned} Ker(T) &= \{\boldsymbol{u} \in \mathbb{R}^n \mid T(\boldsymbol{u}) = \mathbf{0}\} \subseteq \mathbb{R}^m \\ &= \{\boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{Au} = \mathbf{0}\} \\ &= Null(\boldsymbol{A}) \end{aligned}$$

Hence,

$$nullity(T) = \dim\big(Ker(T)\big) = \dim\big(Null(A)\big) = nullity(A)$$

Injective. A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is injective if whenever $T(\boldsymbol{u}) = T(\boldsymbol{v})$, necessarily $\boldsymbol{u} = \boldsymbol{v}$.

$$T \text{ is injective} \Leftrightarrow \text{Ker(T)} = \{\mathbf{0}\} \Leftrightarrow nullity(T) = 0$$

Surjective. A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is surjective if for any $\boldsymbol{w} \in \mathbb{R}^{\boldsymbol{m}}$, there is a $\boldsymbol{u} \in \mathbb{R}^n$ such that $T(\boldsymbol{u}) = \boldsymbol{w}$

$$T \text{ is surjective} \Leftrightarrow R(T) = \mathbb{R}^m \Leftrightarrow rank(\text{T}) = \text{m}$$

Linear Systems and Their Solutions

Linear Systems. A linear system with m equations and n variables can be written in the following form.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Augmented Matrix. An augmented matrix of a linear system with m equations and n variables can be written in the following form.

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right)$$

Solution set. The set of all solutions to a linear system.

$$\left\{ \begin{pmatrix} 1-2s \\ s \\ s \end{pmatrix} \middle| s \in \mathbb{R} \right\}$$

General Solution. An expression that gives us all the solutions to the equation.

E.g., $\begin{cases} x_1 = 1 - 2s \\ x_2 = s \\ x_3 = s \end{cases}$

Inconsistent. The situation when a system of linear equation has *no solutions.*

Consistent. The situation when a system of linear equations has *solutions.*

Remark 1.1.10. Every linear solution has either
(i) no solution
(ii) exactly one solution
(iii) infinitely many solution

Geometric Interpretation of the Solution.

No Solution:	Empty
One Solution:	A point
Infinitely Many Solutions:	Depends on the number of arbitrary parameters. Could be a line (1), a plane (2), 3D space (3)...

Elementary Row Operations (ERO).
(i) Multiply a row by a nonzero constant.
(ii) Interchange two rows.
(iii) Add a multiple of one row to another row.

EROs with Variables. There are some additional precautions when doing EROS with a matrix containing variables/unknowns.

1. $\frac{1}{\alpha}R_i, R_i + \frac{1}{\alpha}R_j$ are <u>not</u> allowed.
2. $\alpha R_i, (1+\alpha)R_i$ are <u>not allowed.</u>

Row Equivalent Matrices. Two augmented matrices are row equivalent if one can be obtained from the other by a series of elementary row operations. If two augment matrices are row equivalent, then they have the same set of solutions. The REF/RREF form of any matrix is row equivalent to the original matrix.

Row-Echelon Form (REF). A matrix is said to be in REF if they have properties 1 and 2.
1. there are any zero rows, they a grouped together at the bottom of the matrix.
2. in any two successive rows that do not consist entirely of zeros, leading entry in the lower row occurs further right than the leading entry in the higher row.

Reduced Row-Echelon Forms (RREF). A matrix is said to be in RREF if it is in REF and has properties 3 and 4.
3. The leading entry of every nonzero row is 1.
4. In each pivot column, except the pivot point, all other entries are zero.

Pivot Columns and Non-Zero Rows. A pivot column is a column with a pivot entry. A non-zero row is a row that contains a leading entry.

$$\#pivot\ columns = \#leading\ entries = \#non\text{-}zero\ rows$$

Gaussian Elimination. Algorithm that reduces an augmented matrix into REF using ERO.

Gauss-Jordan Elimination. Algorithm that reduces an augmented matrix into RREF using ERO.

Remark 1.4.5. Every matrix has a unique RREF but have many REFs

Remark 1.4.8.1. A linear system is inconsistent if the last column of a REF of the augmented matrix is a pivot column.

Remark 1.4.8.2. A linear system has only one solution if every column except the last column is a pivot column.

Remark 1.4.8.3. A consistent linear system has infinitely many solutions if apart from the last column, a REF of the augmented matrix has at least one more non-pivot column.

Notation of EROs.

1. cR_i : multiply the i^{th} row by constant c
2. $R_i \leftrightarrow R_j$: interchange the i^{th} and the j^{th} row
3. $R_i + cR_j$: add c times of j^{th} row to the i^{th} row

Homogeneous Linear Systems. A system of linear equation is said to be homogeneous if all constant terms are zero.

Trivial Solution. $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution to the homogeneous system, hence it is called the trivial solution.

Non-trivial Solution. Any solution other than the trivial solution is called the non-trivial solution.

Solutions of Homogeneous System.
1. A *homogeneous* system of linear equations has either only the *trivial solution* or *infinitely many solutions* in addition to the trivial solution.
2. A *homogeneous* system of linear equations with more unknowns than equations has *infinitely many solutions*

Matrices

Matrix. A matrix is a rectangular array of numbers.

Entries. Are the numbers in the array.

Size of the Matrix. Size of the matrix is given by m × n where m is the number of rows and n is the number of columns.

(i, j)-Entry. The (i, j)-entry of a matrix is the number which is in the i^{th} row and the j^{th} column of the matrix.

Notation of Matrices. A m × n matrix can be written as $\mathbf{A} = (a_{ij})_{m \times n}$.

Square Matrices. A matrix is a square matrix if it has the same number of rows and columns. A n × n matrix is called a square matrix of order n.

$$\#rows = \#columns$$

Diagonal Entry. The a_{ii} entry is called the diagonal entry.

Diagonal Matrices. A *square matrix* is called a diagonal matrix if all its *non-diagonal entries are zero.*

$$a_{ij} = 0, \qquad \forall i \neq j$$

Scalar Matrices. A *diagonal matrix* is called a scalar matrix if all *diagonal entries are the same.*

$$a_{ii} = c, \qquad \forall i$$

Identity Matrices. A *diagonal matrix* is called an identity matrix if all is *diagonal entries are 1.*

Zero Matrices. A matrix with all entries equals zero is called a zero matrix.

$$a_{ij} = 0, \qquad \forall i, j$$

Symmetric Matrices. A square matrix is symmetric if $a_{ij} = a_{ji}$ for all i, j. A matrix **A** is symmetric if and only if $\mathbf{A} = \mathbf{A}^T$

Triangular Matrices.
1. A square matrix (a_{ij}) is called *upper triangular* if $a_{ij} = 0$ for all i > j
2. A square matrix (a_{ij}) is called *lower triangular* if $a_{ij} = 0$ for all i < j

Both upper and lower triangular matrices are called triangular matrices

Equal Matrices. 2 matrices are said to be equal if
1. they have the same size.
2. their corresponding entries are equal.

Matrix Addition. Given $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathbf{B} = (b_{ij})_{m \times n}$,

$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})_{m \times n}$.

Matrix Subtraction. Given $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathbf{B} = (b_{ij})_{m \times n}$,

$\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})_{m \times n}$.

Scalar Multiplication. Given $\mathbf{A} = (a_{ij})_{m \times n}$ and a constant c,

$c\mathbf{A} = (ca_{ij})_{m \times n}$

Theorem 2.2.6. (Basic Properties)
1. $\mathbf{A + B = B + A}$
2. $\mathbf{A + (B + C) = (A + B) + C}$
3. $c(\mathbf{A + B}) = c\mathbf{A} + c\mathbf{B}$
4. $(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$
5. $(cd)\mathbf{A} = c(d\mathbf{A}) = d(c\mathbf{A})$
6. $\mathbf{A + 0 = 0 + A = A}$
7. $\mathbf{A - A = 0}$
8. $0\mathbf{A} = \mathbf{0}$

Matrix Multiplication. Given $\mathbf{A} = (a_{ij})_{m \times p}$ and $\mathbf{B} = (b_{ij})_{p \times n}$, the product of **AB** is defined to be a m × n matrix whose *(i, j)-entry* is

$a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{i1p}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$

for i = 1, 2, …, m and j = 1, 2, …, n

The number of columns in **A** must be equal to the number of rows in **B.**

Multiplication Is Not Commutative. In general, $\mathbf{AB} \neq \mathbf{BA}$.

Remark 2.2.10.4. When $\mathbf{AB = 0}$, it is not necessary that $\mathbf{A = 0}$ or $\mathbf{B = 0}$.

Theorem 2.2.11. (Basic Properties)
1. $\mathbf{A(BC) = (AB)C}$
2. $\mathbf{A(B_1 + B_2) = AB_1 + AB_2}$
   $\mathbf{(C_1 + C_2)A = C_1A + C_2A}$
3. $c(\mathbf{AB}) = (c\mathbf{A})\mathbf{B} = \mathbf{A}(c\mathbf{B})$
4. $\mathbf{A0 = 0}$
   $\mathbf{0A = 0}$
   $\mathbf{IA = AI = A}$

Powers of Square Matrices. Let **A** be a square matrix and n a nonnegative integer. We define $\mathbf{A}^n$ as follows:

$$\mathbf{A}^n = \begin{cases} \boldsymbol{I} & if\ n = 0 \\ \boldsymbol{AA \ldots A} & if\ n \geq 1 \\ (\boldsymbol{A}^{-1})^{-n} & if\ n < 0 \end{cases}$$

Note:
1. $\mathbf{A}^m\mathbf{A}^n = \mathbf{A}^{m+n}$
2. In general, $(\mathbf{AB})^2 \neq \mathbf{A}^2\mathbf{B}^2$

Notation 2.2.15.

Given $\mathbf{A} = (a_{ij})_{m \times p}$ and $\mathbf{B} = (b_{ij})_{p \times n}$, we can write

$\mathbf{A} = \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_m} \end{bmatrix}$ where $\boldsymbol{a_i} = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{ip}]$ and

$\mathbf{B} = [\boldsymbol{b_1} \quad \boldsymbol{b_2} \quad \cdots \quad \boldsymbol{b_n}]$ where $\boldsymbol{b_j} = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$ then

$\mathbf{AB} = \begin{bmatrix} \boldsymbol{a_1b_1} & \boldsymbol{a_1b_2} & \cdots & \boldsymbol{a_1b_n} \\ \boldsymbol{a_2b_1} & \boldsymbol{a_2b_2} & \cdots & \boldsymbol{a_2b_n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a_mb_1} & \boldsymbol{a_mb_2} & \cdots & \boldsymbol{a_mb_n} \end{bmatrix}$ where

$\boldsymbol{a_ib_j} = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{ip}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$

$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$

We can also write

$\mathbf{AB} = \mathbf{A}\,[\boldsymbol{b_1} \quad \boldsymbol{b_2} \quad \cdots \quad \boldsymbol{b_n}] = [\boldsymbol{Ab_1} \quad \boldsymbol{Ab_2} \quad \cdots \quad \boldsymbol{Ab_n}]$

or

$\boldsymbol{AB} = \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_m} \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} \boldsymbol{a_1B} \\ \boldsymbol{a_2B} \\ \vdots \\ \boldsymbol{a_mB} \end{bmatrix}$

Representation of Linear Systems. We can represent the system of linear equations as $\mathbf{Ax = b}$, where **A** is the coefficient matrix, **x** is the variable matrix and **b** is the constant matrix

Solution to Linear Systems. A n × 1 matrix **u** is said to be a solution to the linear system $\mathbf{Ax = b}$ if $\mathbf{Au = b}$

Transposes. Given $\mathbf{A} = (a_{ij})_{m \times n}$, then $\mathbf{A}^T = (a_{ji})_{n \times m}$

Theorem 2.2.22 (Basic Properties)

Let **A** be a m × n matrix.
1. $(\mathbf{A}^T)^T = \mathbf{A}$
2. If **B** is an m × n matrix, then $(\mathbf{A + B})^T = \mathbf{A}^T + \mathbf{B}^T$
3. If c is a scalar, then $(c\mathbf{A})^T = c\mathbf{A}^T$
4. If B is a n × p matrix, then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Inverses. Let **A** be a *square matrix* of order n. **A** is said to be invertible if there exists a square matrix **B** of order n such that $\mathbf{AB = I}$ and $\mathbf{BA = I}$. **B** is called the inverse of A.

Singular Matrix. A square matrix is called *singular* if it has no inverses.

Matrix Cancellation Laws. Let A be an invertible m × m matrix.
(a) If $\mathbf{B_1}$ and $\mathbf{B_2}$ are m × n matrices such that $\mathbf{AB_1 = AB_2}$, then $\mathbf{B_1 = B_2}$
(b) If $\mathbf{C_1}$ and $\mathbf{C_2}$ are n × m matrices such that $\mathbf{C_1A = C_2A}$, then $\mathbf{C_1 = C_2}$

If **A** is not invertible, the cancellation laws may not hold.

Uniqueness of Inverses. If **B** and **C** are inverses of a square matrix **A**, then $\mathbf{B = C}$. The inverse of **A** can be denoted as $\mathbf{A}^{-1}$.

Inverse of a 2 × 2 matrix. Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then **A** is invertible and $\mathbf{A}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

Theorem 2.3.9 (Basic Properties)

Let **A**, **B** be two invertible matrices and c a nonzero scalar

1. $c\mathbf{A}$ is invertible and $(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$
2. $\mathbf{A}^T$ is invertible and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$
3. $\mathbf{A}^{-1}$ is invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
4. $\mathbf{AB}$ is invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
5. $\mathbf{A}^n$ is invertible and $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$

By part 4, $(\mathbf{A_1A_2} \dots \mathbf{A_k})^{-1} = \mathbf{A_k}^{-1} \dots \mathbf{A_2}^{-1}\mathbf{A_1}^{-1}$

Elementary Matrices. A square matrix is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation. There are 3 types of elementary matrices and they are all invertible. Their inverses are also elementary matrices of the same type.

$$I_n \xrightarrow{\text{elementary row operation}} E$$

where E is the corresponding elementary matrix

Finding Inverses. Let A be an invertible matrix of order n. Then

$$(\mathbf{A} \mid \mathbf{I}) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} (\mathbf{I} \mid \mathbf{A^{-1}})$$

Verifying Invertibility. Let **A** be a square matrix. If a REF of **A** has at least one zero row, **A** is singular.

Theorem 2.4.12. Suppose **A** and **B** are square matrices of the same size. If **AB = I**, then

(i) **A** is invertible
(ii) **B** is invertible
(iii) $\mathbf{A}^{-1} = \mathbf{B}$
(iv) $\mathbf{B}^{-1} = \mathbf{A}$
(v) **BA = I**

Theorem 2.4.14. Suppose **A** and **B** are square matrices of the same size. If **A** is singular, then both **AB** and **BA** are singular.

Determinants. Let $\mathbf{A} = (a_{ij})_{n \times n}$. Let $\mathbf{M}_{ij}$ be the $(n-1) \times (n-1)$ matrix obtained from **A** by *deleting* the i^{th} row and the j^{th} column. Then the determinant of **A** is defined to be

$$\det(\mathbf{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

Where $\boldsymbol{A}_{ij} = (-1)^{i+j}\det(\mathbf{M_{ij}})$ which is called the *(i, j)*-cofactor of ***A***.

Cofactor Expansions. Let $\boldsymbol{A} = (a_{ij})_{n \times n}$.
$\det(\boldsymbol{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$
$= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$
Hence, you can expand along any column and row.

Determinant of Triangular Matrices. If **A** is a n × n triangular matrix, then $\det(\mathbf{A}) = a_{11}a_{22}a_{33}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$

Determinant of Transposes. If **A** is a square matrix, then $\det(\mathbf{A}^T) = \det(\mathbf{A})$

Determinant of Matrices with Identical Rows or Rolumns.

1. The determinant of a square matrix with two identical rows is zero.
2. The determinant of a square matrix with two identical columns is zero.

Determinant of Matrices with Zero Rows/Columns. The determinant of a square matrix with a zero row is 0.

Effects of ERO On Determinant

4) $\mathbf{A} \xrightarrow{kR_i} \mathbf{B_1}$: $\det(\mathbf{B_1}) = k\det(\mathbf{A})$
5) $\mathbf{A} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{B_1}$: $\det(\mathbf{B_1}) = -\det(\mathbf{A})$
6) $\mathbf{A} \xrightarrow{R_j + kR_i} \mathbf{B_1}$: $\det(\mathbf{B_1}) = \det(\mathbf{A})$

Furthermore, if **E** is an elementary matrix of the same size as **A**, then $\det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A})$

Invertible Matrices and Determinants. A square matrix **A** is invertible if and only if $\det(\mathbf{A}) \neq 0$

Scalar Multiplication and Determinants. If **A** is a square matrix of order n and c a scalar, then $\det(c\mathbf{A}) = c^n\det(\mathbf{A})$

Matrix Multiplication and Determinants. If **A** and **B** are square matrices of the same size, then $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

Invertible Matrices and Determinants. If **A** is an invertible matrix, then $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$

Adjoints. Let **A** be a square matrix of order n. Then

$$\mathbf{adj}(\mathbf{A}) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

where A_{ij} is the *(i, j)*-cofactor of **A**.

Inverse with Adjoints. If **A** is an invertible matrix, then $\mathbf{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\mathbf{adj}(\mathbf{A})$

Adjoint Identity. For *any* square matrix,

$$\boldsymbol{A}(adj(\boldsymbol{A})) = \det(\boldsymbol{A})\,\boldsymbol{I}$$

Cramer's Rule. Suppose **Ax = b** is a linear system where $\mathbf{A} = (a_{ij})_{n \times n}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. Let $\mathbf{A}_i$ be the n × n matrix obtained from **A** by replacing the i^{th} column of **A** and **B**.

If **A** is invertible, then the system has only one solution

$$\mathbf{x} = \frac{1}{\det(\boldsymbol{A})}\begin{bmatrix} \det(\boldsymbol{A_1}) \\ \det(\boldsymbol{A_2}) \\ \vdots \\ \det(\boldsymbol{A_n}) \end{bmatrix}. \text{ In general, } x_i = \frac{\det(\mathbf{A_i})}{\det(\mathbf{A})}$$

Vector Spaces

Geometric Vectors.

- A (nonzero) vector can be represented geometrically by an arrow.
- The zero vector, denoted by **0**, is represented by a point.

n-vectors. Let $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)$ be two n-vectors.

1. $\boldsymbol{u} = \boldsymbol{v}$ if and only if $u_i = v_i$ for all $i = 1, 2, \cdots, n$
2. The addition $\boldsymbol{u} + \boldsymbol{v}$ of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by
$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$
3. For any real number c, the scalar multiple $c\boldsymbol{u}$ of $\boldsymbol{u}$ is defined by
$$c\boldsymbol{u} = (cu_1, cu_2, \cdots, cu_n)$$
4. The n-vector $(0, 0, \cdots, 0)$ is called the zero vector and is denoted by **0**.
5. The negative of $\boldsymbol{u}$ is defined by $(-1)\boldsymbol{u}$ and is denoted by $-\boldsymbol{u}$.
6. The subtraction $\boldsymbol{u} - \boldsymbol{v}$ of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by $\boldsymbol{u} + (-\boldsymbol{v})$

Basic Properties of Vectors. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be n-vectors and c, d real numbers

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
2. $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
3. $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$
4. $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$
5. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$
6. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$
7. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$
8. if $a\boldsymbol{u} = \boldsymbol{0}$, then $a = 0$ or $\boldsymbol{u} = \boldsymbol{0}$

Euclidean *n*-space. The set of all n-vectors of real number is called the Euclidean n-space, denoted by $\mathbb{R}^n$

Implicit vs Explicit Solutions to Linear Systems.
Implicit: $\{\boldsymbol{v} \in \mathbb{R}^n \mid \boldsymbol{v} \text{ fulfils some condition}\}$
Explicit: $\{\boldsymbol{u} + s_1\boldsymbol{v_1} + s_2\boldsymbol{v_2} + \cdots + s_k\boldsymbol{v_k} \mid s_1, s_2, \cdots, s_k \in \mathbb{R}\}$

Linear Combinations. Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$ be vectors in $\mathbb{R}^n$. For any real numbers $c_1, c_2, \cdots, c_k$, the vector

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

is called a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$.

Linear Spans. Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$

$$\{c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} \mid c_1, c_2, \cdots, c_k \in \mathbb{R}\}$$

is called a linear span of S and is denoted by $span(S)$. The span can be thought of as the set of all possible linear combinations.

Vector in a Span.

$$\boldsymbol{w} \in span\{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\} \leftrightarrow (\boldsymbol{u_1}\ \boldsymbol{u_2} \cdots \boldsymbol{u_k} \mid \boldsymbol{w}) \text{ is consistent}$$

When $span(S) = \mathbb{R}^n$. Suppose $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$. Let $\boldsymbol{A} = (\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k})$.

1. If a REF of $\boldsymbol{A}$ has no zero rows, then the linear system is always consistent. Hence $span(S) = \mathbb{R}^n$
2. If a REF of $\boldsymbol{A}$ has zero rows, then $span(S) \subset \mathbb{R}^n$

From the result above, we conclude that if $|S| < n$, S cannot span $\mathbb{R}^n$

Properties of Linear Span. Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\} \subseteq \mathbb{R}^n$.

(iii) (Contains the origin)
$$\boldsymbol{0} \in span(S)$$
(iv) (Closed under Linear Combination)
$$\forall \boldsymbol{u}, \boldsymbol{v} \in span(S), \alpha, \beta \in \mathbb{R}, \alpha\boldsymbol{u} + \beta\boldsymbol{v} \in span(S)$$

When $span(S_1) \subseteq span(S_2)$. Let $S_1 = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ and $S_2 = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}\}$ be subsets of $\mathbb{R}^n$. Then $span(S_1) \subseteq span(S_2)$ if and only if each $\boldsymbol{u_i}$ is a linear combination of $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}$

$$span\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\} \subseteq span\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}\}$$
$$\leftrightarrow$$
$$(\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m} \mid \boldsymbol{u_1} \mid \boldsymbol{u_2} \mid \cdots \mid \boldsymbol{u_k}) \text{ is consistent}$$

Subspaces Definition. Let V be a subset of $\mathbb{R}^n$. V is a subspace if it satisfies the following properties

(i) (Contains the origin)
$$\boldsymbol{0} \in span(S)$$
(ii) (Closed under Linear Combination)
$$\forall \boldsymbol{u}, \boldsymbol{v} \in V, \alpha, \beta \in \mathbb{R}, \alpha\boldsymbol{u} + \beta\boldsymbol{v} \in V$$

Subspace Alternate Definition. Let V be a subset of $\mathbb{R}^n$.

$$V \text{ is a subspace} \leftrightarrow V = span\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$$

Solution Spaces. The solution set of a homogenous system of linear equations in n variables is a subspace of $\mathbb{R}^n$

$$V = \{v \mid \boldsymbol{A}v = \boldsymbol{b}\} \subseteq \mathbb{R}^n \text{ is a subspace } \leftrightarrow \boldsymbol{b} = \boldsymbol{0}$$

Redundant Vectors. If a set of vectors is linearly dependent, then there exits at least on redundant vector in the set. If a set of vectors is linearly independent, then there is no redundant vector in the set.

Linear Dependence.

A set $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ is linearly dependent if there exists $c_1, c_2, \cdots, c_k \in \mathbb{R}$, not all zero such that

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0}$$

A set $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ is linearly independent if whenever $c_1, c_2, \cdots, c_k \in \mathbb{R}$ is such that

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0}$$

necessarily $c_1 = \cdots = c_k = 0$
You can use this to prove linear independence as well.

Testing for Linear Independence. Let $A = (\boldsymbol{u_1}\ \boldsymbol{u_2} \cdots \boldsymbol{u_k})$.

$$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\} \subseteq \mathbb{R}^n \text{ is linearly dependent}$$
$$\leftrightarrow Ax = 0 \text{ has only the trivial solution}$$
$$\leftrightarrow \text{all columns of REF of } A \text{ are pivot}$$

Linear Dependence if $k > n$. Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\} \subseteq \mathbb{R}^n$. If $k > n$, then S is linearly dependent.

Linear Dependence Special Cases.

(i) $\{\boldsymbol{v}\} \subseteq \mathbb{R}^n$ is linearly independence $\leftrightarrow v \neq 0$
(ii) $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}, \boldsymbol{0}\}$ is linearly dependent.
(iii) $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is linearly dependent $\leftrightarrow \boldsymbol{v_1} = \alpha\boldsymbol{v_2}$ or $\boldsymbol{v_2} = \beta\boldsymbol{v_1}$
(iv) The empty set $\emptyset$ is linearly independent

Basis. Let V be a vector space and $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ a subset of V. Then S is called a basis for V if

1. S is linearly independent and
2. S spans V

Coordinate System. Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ be a basis for a vector space V. then every vector $\boldsymbol{v} \in V$ can be expressed in the form

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

in exactly one way, where $c_1, c_2, \cdots, c_k \in \mathbb{R}$

Basis for $\mathbb{R}^n$.

$$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\} \text{ is a basis for } \mathbb{R}^n$$
$$\leftrightarrow k = n \text{ and } A = (\boldsymbol{u_1}\ \boldsymbol{u_2} \cdots \boldsymbol{u_k}) \text{ is invertible.}$$

Solution Space and Basis. $V = \{\boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{Au} = \boldsymbol{0}\}$ as solution space and $s_1\boldsymbol{u_1} + \cdots + s_k\boldsymbol{u_k}, s_1, \cdots, s_k \in \mathbb{R}$ is a general solution such that s_i are parameters corresponding to the non-pivot columns in RREF of $\boldsymbol{A}$, then $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ is a basis for V

Dimension and Subspaces. Let $U, V \subseteq \mathbb{R}^n$ subspaces. Suppose $U \subseteq V$. Then $\dim(U) \leq \dim(V)$ with equality if and only if $U = V$

Relative Coordinates. $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ basis for subspace $V \subseteq \mathbb{R}^n$. For $v \in V$, $v = c_1\boldsymbol{u_1} + \cdots + c_k\boldsymbol{u_k}$, then

$$[v]_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \in \mathbb{R}^k$$

Obtaining Relative Coordinates. $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ basis for subspace $V \subseteq \mathbb{R}^n$. For $v \in V$,

$$(\boldsymbol{u_1}\ \boldsymbol{u_2} \cdots \boldsymbol{u_k} \mid \boldsymbol{v}) \xrightarrow{G.J.E} \left(\begin{array}{ccc|c} 1 & \cdots & 0 & c_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & c_k \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{array}\right) \Rightarrow [v]_S = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$$

Properties of Relative Coordinates.

1. $\boldsymbol{u} = \boldsymbol{v} \Leftrightarrow [\boldsymbol{u}]_S = [\boldsymbol{v}]_S$
2. $(c_1\boldsymbol{u_1} + \cdots + c_k\boldsymbol{u_k})_S = c_1[\boldsymbol{u_1}]_S + \cdots + c_k[\boldsymbol{u_k}]_S$

it follows that

$$T = \{\boldsymbol{v_1}, \cdots, \boldsymbol{v_m}\} \subseteq V \begin{cases} \text{linearly independent} \\ \text{spans } V \end{cases}$$
$$\Leftrightarrow$$
$$T = \{[\boldsymbol{v_1}]_S, \cdots, [\boldsymbol{v_m}]_S\} \subseteq \mathbb{R}^k \begin{cases} \text{linearly independent} \\ \text{spans } \mathbb{R}^k \end{cases}$$

Dimension. $V \subseteq \mathbb{R}^n$ subspace, $\dim(V) = |S|$ for any basis S.

Size of Basis. Let $V \subseteq \mathbb{R}^n$ be a k-dimensional subspace and $T \subseteq V$.

1. If $|T| > k \Rightarrow T$ is linearly dependent.
2. If $|T| < k \Rightarrow T$ cannot span V

Equivalent ways to check for basis. To prove that S is a basis for V

1. By definition
   (iii) $V = span(S)$
   (iv) S is linearly independent
2. B1
   (iv) $|S| = \dim(V)$
   (v) $S \subseteq V$
   (vi) S is linearly independent.
3. B2
   (iii) $|S| = \dim(V)$
   (iv) $V \subseteq span(S)$

Transition Matrix. Suppose $V \subseteq \mathbb{R}^n$ is a subspace with dimension k. $S = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\}$ and $T = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\}$ are basis for V. Then the transition matrix from S to T, denoted as $\mathbf{P}$, is

$$\mathbf{P} = [[\mathbf{u_1}]_T, \cdots, [\mathbf{u_k}]_T]$$

such that

$$[\mathbf{w}]_T = \mathbf{P}[\mathbf{w}]_S$$
$$[\mathbf{w}]_S = \mathbf{P^{-1}}[\mathbf{w}]_T$$

The transition matrix from S to T can be found by

$$(T \mid S) \xrightarrow{G.J.E} \left(\begin{array}{c|c} I_k & P \\ 0\cdots0 & 0\cdots0 \end{array}\right)$$

Vector Spaces Associated with Matrices

Row Space, Column Space and Null Space. Let

$$\mathbf{A} = \begin{array}{c} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{matrix} \\ \begin{matrix} c_1 & \quad c_2 & \quad \cdots & \quad c_n \end{matrix} \end{array}$$

Column space: $Col(\mathbf{A}) = span\{c_1, c_2, \cdots, c_n\} \subseteq \mathbb{R}^m$

Column space: $Col(\mathbf{A}) = Col(\mathbf{A}) = \{\mathbf{Au} \mid \mathbf{u} \in \mathbb{R}^n\}$

Row space: $Row(\mathbf{A}) = span\{r_1, r_2, \cdots, r_m\} \subseteq \mathbb{R}^n$

Null space: $Null(\mathbf{A}) = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{Au} = \mathbf{0}\} \subseteq \mathbb{R}^n$

Row Operation and Vector Spaces. Suppose $\mathbf{A} = (\mathbf{a_1} \quad \mathbf{a_2} \quad \cdots \quad \mathbf{a_n})$ and $\mathbf{B} = (\mathbf{b_1} \quad \mathbf{b_2} \quad \cdots \quad \mathbf{b_n})$ are row equivalent matrices.

1. Row operations preserved linear relations of the columns.
   $\forall c_1, c_2, \cdots, c_n \in \mathbb{R}, c_1\mathbf{a_1} + \cdots + c_n\mathbf{a_n} = \mathbf{0} \Leftrightarrow c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n} = \mathbf{0}$

2. Row operations preserves row space.
   $$Row(A) = Row(B)$$

Basis of Vector Spaces. Suppose $\mathbf{R}$ is a REF of $\mathbf{A}$.
(i) The columns of $\mathbf{A}$ corresponding to the pivot columns of $\mathbf{R}$ form a basis for $Col(\mathbf{A})$
(ii) The nonzero rows of $\mathbf{R}$ form a basis for $Row(\mathbf{A})$

Caution:
(i) Row operations do not preserve column space.
(ii) Row operations do not preserve linear relations of the rows.

Rank and Nullity of Matrices.

$$rank(\mathbf{A}) = \dim\big(Col(\mathbf{A})\big) = \dim\big(Row(\mathbf{A})\big)$$
$$nullity(\mathbf{A}) = \dim\big(null(\mathbf{A})\big)$$

Vector Spaces Summary

Subspace	Subspace of	Basis	Dimension
$Col(\mathbf{A})$	$\mathbb{R}^m$	Columns of $\mathbf{A}$ corresponding to pivot column in *REF*	$rank(\mathbf{A})$
$Row(\mathbf{A})$	$\mathbb{R}^n$	Nonzero rows in *REF*	$rank(\mathbf{A})$
$Null(\mathbf{A})$	$\mathbb{R}^n$	Vectors in a general solution	$nullity(\mathbf{A})$

Rank Nullity Theorem

$$rank(\mathbf{A}) + nullity(\mathbf{A}) = \#cols$$

Full Rank. $\mathbf{A}$ is full rank if $rank(a) = \min\{\#Cols, \#Rows\}$

Full Rank and Invertibility.

A square matrix $\mathbf{A}$ is of full rank
$\Leftrightarrow \det(\mathbf{A}) \neq 0$
$\Leftrightarrow A$ *is invertible*

Bounds on Rank.

$$rank(\mathbf{AB}) \leq \min\{rank(\mathbf{A}), rank(\mathbf{B})\}$$

Applications of Vector Spaces.
1. Finding a basis from a spanning set
2. Finding a "nicer" basis
3. Extending a linearly independent subset to a basis for $\mathbb{R}^n$

Orthogonality

Inner Product. Let $\mathbf{u} = (u_i), \mathbf{v} = (v_i) \in \mathbb{R}^n$.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_1v_1 + u_2v_2 + \cdots + u_nv_n \\ &= \sum_{i=1}^{n} u_iv_i \\ &= (u_1 \quad \cdots \quad u_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ &= \begin{cases} \mathbf{u}^T\mathbf{v} & \text{if } \mathbf{u}, \mathbf{v} \text{ are column vectors} \\ \mathbf{uv}^T & \text{if } \mathbf{u}, \mathbf{v} \text{ are row vectors} \end{cases} \end{aligned}$$

Norm. $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = (\sum_{i=1}^{n} u_i^2)^{\frac{1}{2}}$
Distance. $d(u, v) = \|\mathbf{u} - \mathbf{v}\| = (\sum_{i=1}^{n}(\mathbf{u_i} - \mathbf{v_i})^2)^{\frac{1}{2}}$
Angle. $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$

Properties of Inner Product. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$ and a, b, c scalars

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
3. $\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = a(\mathbf{u} \cdot \mathbf{v}) + b(\mathbf{u} \cdot \mathbf{w})$
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ with equality iff $\mathbf{u} = 0$
5. $\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$
6. (Cauchy-Schwarz Inequality) $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

Orthogonal Vectors. $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal if

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \mathbf{u} = 0 \text{ or } \mathbf{v} = 0 \\ \mathbf{u} \text{ and } \mathbf{v} \text{ are perpendicular} \end{cases}$$

Orthogonal Sets. A set $\{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\}$ is orthogonal if

$$\mathbf{u_i} \cdot \mathbf{u_j} = 0 \quad \forall i \neq j$$

Orthonormal Sets. A set $\{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\}$ is orthogonal if

$$\mathbf{u_i} \cdot \mathbf{u_j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Orthogonal to Orthonormal. Every orthogonal set of nonzero vectors can be normalised to an orthonormal set.

$$S = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\} \xrightarrow{normalised} T = \left\{\frac{\mathbf{u_1}}{\|\mathbf{u_1}\|}, \frac{\mathbf{u_2}}{\|\mathbf{u_2}\|}, \cdots, \frac{\mathbf{u_k}}{\|\mathbf{u_k}\|}\right\}$$

Orthogonal means Independent. An orthogonal set of nonzero vectors is linearly independent.

Orthogonal and Orthonormal Bases. To show that $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\}$ is an orthogonal/orthonormal basis of $V \subseteq \mathbb{R}^n$, we need to check
(i) S is orthogonal/orthonormal and
(ii) $V = span(S)$ or
(iii) $|S| = \dim(V)$ and $S \subseteq V$

Orthogonal Basis to Relative Coordinates.

$S = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\}$ is a $\begin{cases} (i) \text{ orthogonal} \\ (ii) \text{ orthonormal} \end{cases}$ basis for $V \subseteq \mathbb{R}^n$ subspace.

(v) $\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}}\mathbf{u_1} + \frac{\mathbf{v} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}}\mathbf{u_2} + \cdots + \frac{\mathbf{v} \cdot \mathbf{u_k}}{\mathbf{u_k} \cdot \mathbf{u_k}}\mathbf{u_k}$
(vi) $\mathbf{v} = (\mathbf{v} \cdot \mathbf{u_1})\mathbf{u_1} + (\mathbf{v} \cdot \mathbf{u_2})\mathbf{u_2} + \cdots + (\mathbf{v} \cdot \mathbf{u_k})\mathbf{u_k}$

Vectors Orthogonal to Subspaces. A vector $\mathbf{u} \in \mathbb{R}^n$ is orthogonal to a subspace $V = span\{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\} \subseteq \mathbb{R}^n$, denoted by $\mathbf{u} \perp V$, if $\forall \mathbf{v} \in V, \mathbf{u} \cdot \mathbf{v} = 0$

$$\begin{aligned} \mathbf{u} \perp V \quad &\Leftrightarrow \mathbf{u} \cdot \mathbf{u_i} \quad \forall i = 1, \cdots, k \\ &\Leftrightarrow \mathbf{A}^T\mathbf{u} = 0 \quad \mathbf{A} = (\mathbf{u_1} \quad \cdots \quad \mathbf{u_k}) \\ &\Leftrightarrow \mathbf{u} \in Null(\mathbf{A}^T) \end{aligned}$$

Orthogonal Projection. Let $V \subseteq \mathbb{R}^n$. Every $\mathbf{w} \in \mathbb{R}^n$ can be decomposed uniquely as

$$\mathbf{w} = \mathbf{w_p} + \mathbf{w_n}$$

where $\mathbf{w_p} \in V$ and $\mathbf{w_n} \perp V$. The unique vector $\mathbf{w_p} \in V$ is called the orthogonal projection of $\mathbf{w}$ onto V.

$S = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\}$ is a $\begin{cases} (i) \text{ orthogonal} \\ (ii) \text{ orthonormal} \end{cases}$ basis for $V \subseteq \mathbb{R}^n$ subspace.

(vii) $\mathbf{w_p} = \frac{\mathbf{w} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}}\mathbf{u_1} + \frac{\mathbf{w} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}}\mathbf{u_2} + \cdots + \frac{\mathbf{w} \cdot \mathbf{u_k}}{\mathbf{u_k} \cdot \mathbf{u_k}}\mathbf{u_k}$
(viii) $\mathbf{w_p} = (\mathbf{w} \cdot \mathbf{u_1})\mathbf{u_1} + (\mathbf{w} \cdot \mathbf{u_2})\mathbf{u_2} + \cdots + (\mathbf{w} \cdot \mathbf{u_k})\mathbf{u_k}$

Gram-Schmidt Process. $S = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\}$ be linearly independent.

$$\mathbf{v_1} = \mathbf{u_1}$$
$$\mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{v_1} \cdot \mathbf{u_2}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1}$$
$$\mathbf{v_3} = \mathbf{u_3} - \frac{\mathbf{v_1} \cdot \mathbf{u_3}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{v_2} \cdot \mathbf{u_3}}{\mathbf{v_2} \cdot \mathbf{v_2}}\mathbf{v_2}$$
$$\mathbf{v_k} = \mathbf{u_k} - \frac{\mathbf{v_1} \cdot \mathbf{u_k}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{v_2} \cdot \mathbf{u_k}}{\mathbf{v_2} \cdot \mathbf{v_2}}\mathbf{v_2} - \cdots - \frac{\mathbf{v_{k-1}} \cdot \mathbf{u_2}}{\mathbf{v_{k-1}} \cdot \mathbf{v_{k-1}}}\mathbf{v_{k-1}}$$

Then $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\}$ is an orthogonal set and hence

$$\left\{\frac{\mathbf{v_1}}{\|\mathbf{v_1}\|}, \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|}, \cdots, \frac{\mathbf{v_k}}{\|\mathbf{v_k}\|}\right\}$$

is an orthonormal basis for $span(S)$

Least Squares Solutions. A vector $\mathbf{u} \in \mathbb{R}^n$ is a least square solution to $\mathbf{Ax} = \mathbf{b}$ if for every $\mathbf{v} \in \mathbb{R}^n$

$$\|\mathbf{Au} - \mathbf{b}\| \leq \|\mathbf{Av} - \mathbf{b}\|$$

Obtaining the Least Squares Solution.

$\mathbf{u}$ is a least square solution to $\mathbf{Ax} = \mathbf{b}$ $\Leftrightarrow$ $\mathbf{Au}$ is the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$, $Col(\mathbf{A})$
$\Leftrightarrow$ $\mathbf{u}$ is a solution to $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$

Finding Projection using Least Squares. Let . $S = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\} \subseteq \mathbb{R}^n, V = span(S)$. For any $\mathbf{w} \in \mathbb{R}^n$, the projection of $\mathbf{w}$ onto V is $\mathbf{Au}$, where $\mathbf{A} = (\mathbf{u_1} \quad \cdots \quad \mathbf{u_k})$, and $\mathbf{u} \in \mathbb{R}^k$ is a solution to $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{w}$

Finding Least Squares using Shortcut. Projection of $\mathbf{w}$ onto $\mathbf{V}$ is the formula below (although it is not proven)

$$\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{w}$$

$\mathbf{A}^T\mathbf{A}$ of Orthogonal/Orthonormal Matrices. Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\}$ and $\mathbf{A} = (\mathbf{u_1} \quad \cdots \quad \mathbf{u_k})$
1. If S is an orthogonal set $\Leftrightarrow \mathbf{A}^T\mathbf{A}$ is a diagonal matrix
2. If S is an orthonormal set $\Leftrightarrow \mathbf{A}^T\mathbf{A} = \mathbf{I_k}$

Orthogonal Matrix. A square matrix of order n is an orthogonal matrix if $\mathbf{A}^T = \mathbf{A^{-1}}$, or $\mathbf{A}^T\mathbf{A} = \mathbf{I_k} = \mathbf{AA}^T$

Product of 2 Orthogonal Matrices. The product of 2 orthogonal matrices is an orthogonal matrix.

Equivalent Statements of Orthogonal Matrix
1. $\mathbf{A}$ is an orthogonal matrix
2. The columns of $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$
3. The rows of $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$

Transition Matrix between Two Orthogonal Basis.
Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}\}, T = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\}$ orthonormal basis for subspace $W \subseteq \mathbb{R}^n$
3. Transition matrix $P: S \to T$ is an orthogonal matrix
4. The transition matrix $T \to P^T$

where

$$P = [\mathbf{v_1} \; \mathbf{v_2} \; \mathbf{v_3}]^T[\mathbf{u_1} \; \mathbf{u_2} \; \mathbf{u_3}]$$

Diagonalization

Eigenvalues and Eigenvectors. Let $\mathbf{A}$ be a square matrix of order n. A nonzero column vector $\mathbf{u} \in \mathbb{R}^n$ is called an eigenvector of $\mathbf{A}$ if

$$\mathbf{Au} = \boldsymbol{\lambda}\mathbf{u} \quad \text{for some scalar } \lambda$$

The scalar λ is call an eigenvalue of $\mathbf{A}$ and $\mathbf{u}$ is said to be an eigenvector of $\mathbf{A}$ associated with the eigenvalue λ.

Eigenspace. Let λ is an eigenvalue of $\mathbf{A}$. The solution space to the homogeneous system $(\lambda\mathbf{I} - \mathbf{A})x = 0$ is called the eigenspace associated to λ and is denoted as

$$E_\lambda = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{Av} = \lambda\mathbf{v}\} = Null(\lambda\mathbf{I} - \mathbf{A})$$

Characteristic Polynomial. The characteristic polynomial of $\mathbf{A}$ is

$$char(A) = \det(x\mathbf{I} - \mathbf{A})$$

Finding Eigenvalues. λ is an eigenvalue of $\mathbf{A} \Leftrightarrow$ the homogeneous system $(\lambda\mathbf{I} - \mathbf{A})x = 0$ has nontrivial solution. The nontrivial solution solutions are the eigenvectors associated to λ.
λ is an eigenvalue of $\mathbf{A} \Leftrightarrow \lambda$ is a root of the characteristic polynomial of $\mathbf{A}$, $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$

Eigenvalues of Triangular Matrices. If $\mathbf{A}$ is an triangular matrix, then its diagonal entries are its eigenvalues.

Multiplicity. Let λ be an eigenvalue of $\mathbf{A}$. The multiplicity of λ is the largest integer r_λ such that

$$\det(x\mathbf{I} - \mathbf{A}) = (x - \lambda)^{r_\lambda}\, p(x)$$

for some polynomial $p(x)$

Suppose $\mathbf{A}$ is an order n square matrix such that $\det(x\mathbf{I} - \mathbf{A})$ can be factorized completely into linear factors. Then we can write

$$\det(x\mathbf{I} - \mathbf{A}) = (x - \lambda_1)^{r_1}(x - \lambda_2)^{r_2} \cdots (x - \lambda_k)^{r_k}$$

where $r_1 + r_2 + \cdots + r_k = n$ and $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalue of $\mathbf{A}$. Then, the multiplicity of λ_i is r_i for $i = 1, 2, \cdots, k$

Bounds for Dimension of Eigenspace.

$$1 \leq \dim(E_\lambda) \leq r_\lambda$$

Algorithm to Finding Eigenvalue, Eigenvector, Eigenspace.
1. Compute the characteristic polynomial of $\mathbf{A}$
   $$\det(\lambda\mathbf{I} - \mathbf{A})$$
2. Find all roots λ of the characteristic polynomial.
3. For each λ, solve the homogeneous system $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$
4. The vectors in a general solution form a basis for the eigenspace E_λ

Diagonalization. An order n square matrix $\mathbf{A}$ is diagonalisable if there exist an invertible matrix $\mathbf{P}$ such that

$$\mathbf{P^{-1}AP} = \mathbf{D}$$

for some diagonal matrix $\mathbf{D}$. Equivalently, if $\mathbf{A} = \mathbf{PDP}^{-1}$.

$\mathbf{P} = (\mathbf{u_1} \quad \cdots \quad \mathbf{u_n}), \mathbf{D} = \begin{pmatrix} \lambda_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \lambda_n \end{pmatrix}$ where $\mathbf{u_1}$ is an eigenvector associated with λ_i

Equivalent Statements for Diagonalizability.
(v) $\mathbf{A}$ is diagonalizable.
(vi) There exists a *basis* $\{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n}\} \subseteq \mathbb{R}^n$ of eigenvectors of $\mathbf{A}$
(vii) The sum of dimension of the eigenspaces of $\mathbf{A}$ is equal to its order,

$$\sum_{\lambda \text{ eigenvectors of } \mathbf{A}} \dim(E_\lambda) = n$$

(viii) The characteristic polynomial of $\mathbf{A}$ splits
$$\det(x\mathbf{I} - \mathbf{A}) = (x - \lambda_1)^{r_1}(x - \lambda_2)^{r_2} \cdots (x - \lambda_k)^{r_k}$$
where r_i is the multiplicity of eigenvalue λ_i, for $i = 1, \cdots, k$ and the eigenvalues are distinct, $\lambda_i \neq \lambda_j$ for $i \neq j$, and the dimension of each eigenspace is equal to its multiplicity

$$\dim(E_{\lambda_i}) = r_i$$

Algorithm to Diagonalization.

1. Compute the characteristic polynomial of $\boldsymbol{A}$
$$\det(\lambda \boldsymbol{I} - \boldsymbol{A})$$
2. Find all root λ of the characteristic polynomial.
3. For each λ, find a basis S_λ for the eigenspace
$$E_\lambda = Null(\lambda \boldsymbol{I} - \boldsymbol{A})$$
4. Let $S = \bigcup_\lambda S_\lambda \Rightarrow S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_n}\}$ basis for $\mathbb{R}^n$
5. Let $P = (\boldsymbol{u_1}\ \boldsymbol{u_2} \cdots \boldsymbol{u_k})$ and $D = \begin{pmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n \end{pmatrix}$ where $\boldsymbol{u_1}$ is an eigenvector associated with μ_i.

Then $\boldsymbol{A} = \boldsymbol{PDP^{-1}}$

Sufficient Condition of Diagonalization.

(i) $\boldsymbol{A}$ is a diagonal matrix.
(ii) $\boldsymbol{A}$ is a symmetric matrix.
(iii) $\boldsymbol{A}$ has n distinct eigenvalues.

$\boldsymbol{A}$ is not diagonalization if either

(i) $\det(\lambda \boldsymbol{I} - \boldsymbol{A})$ does not split into linear factors.
(ii) there exists eigenvalue λ s.t. $\dim(E_\lambda) < r_\lambda$

Orthogonally Diagonalizable

An order n square matrix $\boldsymbol{A}$ is orthogonally diagonalisable if there exist an orthogonal matrix $\boldsymbol{P}$ such that
$$\boldsymbol{P^T A P} = \boldsymbol{D}$$
for some diagonal matrix $\boldsymbol{D}$. Equivalently, if $\boldsymbol{A} = \boldsymbol{PDP^T}$.

Orthogonally diagonalizable ⇔ symmetric

Orthogonality of Eigenspaces of Symmetric $\boldsymbol{A}$. $\lambda_1 \neq \lambda_2$ are distinct eigenvalues of orthogonally diagonalizable matrix $\boldsymbol{A}$, then $E_{\lambda_1} \perp E_{\lambda_2}$, that is for any vector $\boldsymbol{v_1} \in E_{\lambda_1}, \boldsymbol{v_2} \in E_{\lambda_2}$, $\boldsymbol{v_1} \cdot \boldsymbol{v_2} = \boldsymbol{0}$.

Therefore we can apply Gram-Schmidt process to the basis within each eigenspace

Algorithm to Orthogonally Diagonalization. Suppose $\boldsymbol{A}$ is symmetric. Follow step 1-3 of algorithm to diagonalization.

4. Apply Gram-Schmidt process to the basis S_λ of the eigenspace E_λ to obtain an orthonormal basis T_λ
5. Let $T = \bigcup_\lambda T_\lambda \Rightarrow T = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_n}\}$ is an orthonormal basis for $\mathbb{R}^n$
6. Follow step 5 of algorithm to diagonalization.

Application to Diagonalization.

Suppose $\boldsymbol{A} = \boldsymbol{PDP^{-1}}$, then $\boldsymbol{A}^k = \boldsymbol{PD^kP^{-1}}$

Linear Transformation

Linear Transformation. A linear transformation is a mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ of the form
$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

if $n = m$, then T is also called a linear operator

Linear Transformation Alternate Definition. Let V and W be vector spaces. A mapping $T: V \to W$ is called a linear transformation if and only if
$$T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v}) \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in V \text{ and } c, d \in \mathbb{R}$$

Linear Transformation Basic Properties.

1. $T(\boldsymbol{0}) = \boldsymbol{0}$
2. $T(\alpha \boldsymbol{u}) = \alpha\, T(\boldsymbol{u})$
3. $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$
4. $T(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) = c_1T(\boldsymbol{u_1}) + c_2T(\boldsymbol{u_2}) + \cdots + c_kT(\boldsymbol{u_k})$

Standard Matrix. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then the standard matrix $\boldsymbol{A}$ can be denoted as
$$\boldsymbol{A} = [T(\boldsymbol{e_1}) \quad T(\boldsymbol{e_1}) \quad \cdots \quad T(\boldsymbol{e_1})]$$

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists a $m \times n$ matrix $\boldsymbol{A}$ s.t. $T(\boldsymbol{u}) = \boldsymbol{Au}$ for all $\boldsymbol{u} \in \mathbb{R}^{\boldsymbol{n}}$

Retrieving the Standard Matrix. Suppose $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_n}\} \subseteq \mathbb{R}^n$ is a basis and $T(u_1), T(u_2), \cdots, T(u_n)$ is given. Define the representation of T with respect to S as
$$[T]_S = [T(\boldsymbol{u_1}) \quad T(\boldsymbol{u_1}) \quad \cdots \quad T(\boldsymbol{u_1})]$$
Then for any $\boldsymbol{v} \in \mathbb{R}^n$,
$$\begin{aligned} T(\boldsymbol{v}) &= T(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_n}) \\ &= c_1T(\boldsymbol{u_1}) + c_2T(\boldsymbol{u_2}) + \cdots + c_kT(\boldsymbol{u_n}) \\ &= [T]_S[\boldsymbol{v}]_S \end{aligned}$$
So, the standard matrix of T is the representation of T with respect to E, the standard matrix, $\boldsymbol{A} = [T]_E$

$P = (\boldsymbol{u_1} \quad \boldsymbol{u_2} \quad \cdots \quad \boldsymbol{u_n})$ is the transition matrix from S to E such that $P^{-1}\boldsymbol{v} = [\boldsymbol{v}]_S$.
$$\boldsymbol{Av} = T(\boldsymbol{v}) = [T]_S[\boldsymbol{v}]_S = [T]_S P^{-1}\boldsymbol{v}$$
Therefore, $A = [T]_S P^{-1}$

Composition of Mappings. Let $S: \mathbb{R}^n \to \mathbb{R}^m$ and $T: \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations. The composition of T with S, denoted by

$T \circ S$, is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$ defined by
$$(T \circ S)(\boldsymbol{u}) = T(S(\boldsymbol{u})) \quad \text{for } \boldsymbol{u} \in \mathbb{R}^{\boldsymbol{n}}$$

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are the standard matrix for S and T respectively, then $\boldsymbol{BA}$ is the standard matrix for $T \circ S$

Range and Rank. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The range of T, which is denoted by $R(T)$, is the set of images of T.
$$R(T) = \{T(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathbb{R}^{\boldsymbol{n}}\} \subseteq \mathbb{R}^m = \{\boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^{\boldsymbol{n}}\} = Col(\boldsymbol{A})$$
Hence,
$$rank(T) = \dim(R(T)) = \dim(Col(\boldsymbol{A})) = rank(\boldsymbol{A})$$

Kernel and Nullity. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The kernel of T, which is denoted by $Ker(T)$, is the set of vectors in $\mathbb{R}^n$, whose image is the zero vector in $\mathbb{R}^m$
$$\begin{aligned} Ker(T) &= \{\boldsymbol{u} \in \mathbb{R}^{\boldsymbol{n}} \mid T(\boldsymbol{u}) = \boldsymbol{0}\} \subseteq \mathbb{R}^m \\ &= \{\boldsymbol{u} \in \mathbb{R}^{\boldsymbol{n}} \mid \boldsymbol{Au} = \boldsymbol{0}\} \\ &= Null(\boldsymbol{A}) \end{aligned}$$
Hence,
$$nullity(T) = \dim(Ker(T)) = \dim(Null(A)) = nullity(A)$$

Dimension Theorem for Linear Transformation Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
$$rank(T) + nullity(T) = n$$

Injective. A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is injective if whenever $T(\boldsymbol{u}) = T(\boldsymbol{v})$, necessarily $\boldsymbol{u} = \boldsymbol{v}$.
$$T \text{ is injective} \Leftrightarrow \text{Ker(T)} = \{\boldsymbol{0}\} \Leftrightarrow nullity(T) = 0$$

Surjective. A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is surjective if for any $\boldsymbol{w} \in \mathbb{R}^{\boldsymbol{m}}$, there is a $\boldsymbol{u} \in \mathbb{R}^{\boldsymbol{n}}$ such that $T(\boldsymbol{u}) = \boldsymbol{w}$
$$T \text{ is surjective} \Leftrightarrow R(T) = \mathbb{R}^m \Leftrightarrow rank(\text{T}) = \text{m}$$

Poll Everywhere

1. True. If a linear system has more variable than equations, then we must introduce parameters in the general solution.
2. False. If a linear system has more equations than variables, then the system has at most one solution.
3. True. If the trivial solution is the solution of the linear system, it must be a homogeneous system.
4. True. If the homogeneous system has a unique solution, it must be the trivial solution.
5. False. If the homogeneous system has the trivial solution, it must be the unique solution.
6. False. For any square matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size $(\boldsymbol{A} + \boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{B}^2 + 2\boldsymbol{AB}$ $(\boldsymbol{AB} \neq \boldsymbol{BA})$
7. True. For any diagonal matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size $(\boldsymbol{A} + \boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{B}^2 + 2\boldsymbol{AB}$ $(\boldsymbol{AB} = \boldsymbol{BA}$ for diagonal matrix)
8. True. Inverse of square matrices is unique.
9. True. For a square matrix $\boldsymbol{A}$, if there is a square matrix B such that $\boldsymbol{AB} = \boldsymbol{I}$, then necessarily $\boldsymbol{BA} = \boldsymbol{I}$
10. False. Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are $m \times n$ matrices such that there is an invertible matrix $\boldsymbol{P}$ of order n such that $\boldsymbol{AP} = \boldsymbol{B}$. Then $\boldsymbol{A}$ and $\boldsymbol{B}$ are equivalent.
11. True. Suppose $\boldsymbol{A}$ is an invertible matrix of order n. Then for any $\boldsymbol{b} \in \mathbb{R}^n$,
    (i) $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent.
    (ii) the solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is unique.
12. True. Suppose $\boldsymbol{A}$ is an invertible matrix of order n. Then $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
13. True. Every square matrix is row equivalent to a triangular matrix.
14. True. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two square matrices of the same size, then $\det(\boldsymbol{AB}) = \det(\boldsymbol{BA})$
15. False. If $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{B}$ is an $n \times m$ matrix, then $\det(\boldsymbol{AB}) = \det(\boldsymbol{BA})$
16. True. For any square matrix $\boldsymbol{A}$, $\boldsymbol{A}adj(\boldsymbol{A}) = \det(\boldsymbol{A})\,\boldsymbol{I}$
17. True. Suppose $\boldsymbol{A}$ is a singular matrix. Cramer's rule will not give any solution
18. True. If $\boldsymbol{A}$ is an $n \times k$ matrix with $k < n$, then any REF of $\boldsymbol{A}$ must have a zero row.
19. True. $\{\boldsymbol{0}\} \subseteq \mathbb{R}^n$ is a subspace.
20. True. If $\boldsymbol{A}$ is an $n \times k$ matrix with $k > n$, then any REF of $\boldsymbol{A}$ must have a nonpivot column.
21. False. $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly independent if none of them is a multiple of the other.
22. False. $span\{\boldsymbol{u}, \boldsymbol{v}\} \subset \mathbb{R}^n$ is always a plane for any $n > 0$
23. False. $\mathbb{R}^2 \subseteq \mathbb{R}^3$
24. True. The dimension of the zero space is 0.
25. True. $Col(B) \subseteq Null(A)$
26. True. $\lambda = 0$ can be an eigen value of square matrix $\boldsymbol{A}$
27. True. If λ is an eigenvalue of $\boldsymbol{A}$,
    (i) λ is an eigenvalue of $\boldsymbol{A}^T$
    (ii) λ^n is an eigenvalue of $\boldsymbol{A}^n$
    (iii) λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$ if $\boldsymbol{A}$ is invertible
28. Suppose $\boldsymbol{A}$ diagonalizable,
    False. (i) If the diagonal matrix $\boldsymbol{D}$ is fixed, then the invertible matrix $\boldsymbol{P}$ is unique
    True. (ii) If the diagonal matrix $\boldsymbol{P}$ is fixed, then the invertible matrix $\boldsymbol{D}$ is unique
29. True. If $\boldsymbol{A}$ is not a scalar matrix has only 1 eigenvalue, then $\boldsymbol{A}$ is not diagonalizable.
30. True. If $\boldsymbol{A}$ is an invertible and diagonalizable matrix, then $\boldsymbol{A}^{-1}$ is diagonalizable
31. True. If $\boldsymbol{A}$ is diagonalizable, then $\boldsymbol{A}^T$ is diagonalizable
32. False. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable, then $\boldsymbol{A} + \boldsymbol{B}$ is diagonalizable.
33. False. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable, then $\boldsymbol{AB}$ is diagonalizable.
34. True. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonally diagonalizable, then $\boldsymbol{A} + \boldsymbol{B}$ is diagonalizable.
35. False. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonally diagonalizable, then $\boldsymbol{AB}$ is diagonalizable.
36. False. $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if there exist a matrix $\boldsymbol{A}$ such that $\boldsymbol{A} = [T(\boldsymbol{e_1}) \quad T(\boldsymbol{e_1}) \quad \cdots \quad T(\boldsymbol{e_1})]$
37. False. If $T(\alpha\boldsymbol{v}) = \alpha T(\boldsymbol{v})$ for any $\alpha \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^n$, then T is a linear transformation.
38. False. If $\boldsymbol{A}$ is a square matrix such that $\boldsymbol{A^2} = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$
39. True. If $\boldsymbol{A}$ is a matrix such that $\boldsymbol{AA^T} = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$
40. True. $span(S_1 \cup S_2) = span(S_1) + span(S_2)$
41. True. Nullspace of $\boldsymbol{A}$ = nullspace of $\boldsymbol{A^TA}$
42. True. rank of $\boldsymbol{A}$ = rank of $\boldsymbol{A^TA}$
43. True. $rank(M + N) \leq rank(M) + rank(N)$

Equivalent Statements for Invertibility

Invertible Matrices (Theorem 2.4.7). Let **A** be a square matrix. The following statements are equivalent.

1. **A** is invertible.
2. **A** has a left inverse.
3. **A** has a right inverse.
4. The RREF of **A** is the identity matrix.
5. **A** can be expressed as a product of elementary matrices.
6. The homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
7. For any $\boldsymbol{b}$, $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution.
8. The determinant of **A** is nonzero, $\det A \neq 0$
9. The columns/rows of **A** spans $\mathbb{R}^n$
10. The columns/rows of **A** are linearly independent.
11. **A** is of full rank, $rank(\mathbf{A}) = \text{n}$
12. $nullity(\mathbf{A}) = 0$
13. **0** is not an eigenvalue of **A**
14. The linear transformation T_A defined by **A** is injective, or $Ker(T_A) = \{\boldsymbol{0}\}$
15. The linear transformation T_A defined by **A** is surjective, or $R(T_A) = \mathbb{R}^n$