Tarea 8
Espacio generado

7.- ¿Es $\underline{v}$ una combinación lineal de los vectores dados?

a) $\underline{v} = (2,1,5)$, $v_1 = (1,2,1)$, $v_2 = (1,0,2)$, $v_3 = (1,1,0)$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 5 \end{array}\right) \begin{array}{c} \\ R_2-2R_1 \\ R_3-R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & -3 \\ 0 & 1 & -1 & 3 \end{array}\right)(-1/2) \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1/2 & 3/2 \\ 0 & 1 & -1 & 3 \end{array}\right)$$

$$\begin{array}{c} \\ \\ R_3-R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1/2 & 3/2 \\ 0 & 0 & -3/2 & 3/2 \end{array}\right)(-2/3) \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1/2 & 3/2 \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{c} R_1-R_3 \\ R_2-\frac{1}{2}R_3 \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\begin{array}{c} R_1-R_2 \\ \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$\therefore$ $\underline{v}$ si es una c. l. de $(1,2,1)$, $(1,0,2)$ y $(1,1,0)$

b) $\underline{v} = \begin{pmatrix} 5 & 1 \\ -1 & 9 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 \\ 3 & 2 & 1 & 9 \end{array}\right) \begin{array}{c} \\ R_2+R_1 \\ \\ R_4-3R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & -5 & -6 \end{array}\right)(1/2) \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & -5 & -6 \end{array}\right)$$

$$\begin{array}{c} \\ \\ \\ R_4+R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -3 & -3 \end{array}\right)(-1) \left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right) \begin{array}{c} R_1-2R_3 \\ R_2-2R_3 \\ \\ R_4+3R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} R_1-R_2 \\ \\ \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ $\underline{v}$ si es una c. l. de $\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ y $\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

8- Calcula el ó los valores de k tales que:

a) $\begin{pmatrix} k \\ 2 \\ -2k \end{pmatrix}$ sea una combinación lineal de $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ y $\begin{pmatrix} 1 \\ 0 \\ k \end{pmatrix}$

$$\left(\begin{array}{cc|c} 0 & 1 & k \\ 2 & 0 & 2 \\ 1 & k & -2k \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_2} \left(\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 1 & k \\ 1 & k & -2k \end{array}\right)(1/2) \quad \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & k \\ 1 & k & -2k \end{array}\right)$$

$$R_3 + R_1 \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & k \\ 0 & k & -2k-1 \end{array}\right) \qquad R_3 - kR_2 \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & k \\ 0 & 0 & -k^2-2k-1 \end{array}\right)$$

$(-1)\ -k^2-2k-1=0\ (-1)$

$= k^2+2k+1=0 \rightarrow \dfrac{-2 \pm \sqrt{2^2-4(1)(1)}}{2(1)} = \dfrac{-2 \pm \sqrt{4-4}}{2} = \dfrac{-2 \pm 0}{2}$

$$k = -\frac{2}{2} = -1$$

∴ Para que sea una c.l. $k=-1$

b) $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ sea una combinación lineal de las columnas de $\begin{pmatrix} 1 & -k & 1 \\ 0 & k & 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -k & 1 & 0 \\ 0 & k & 0 & -1 \end{array}\right)(1/k) \qquad \left(\begin{array}{ccc|c} 1 & -k & 1 & 0 \\ 0 & 1 & 0 & -1/k \end{array}\right)$$

$$R_1 + kR_2 \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1/k \end{array}\right)$$

$x_1 = -1 - x_3$
$x_2 = -1/k$
$x_3 = x_3$

$k \neq 0$

∴ Para que sea una c.l. $k \neq 0$

9- En P_2 sean $v_1 = 2t^2 + t + 2$, $v_2 = t^2 - 2t$, $v_3 = 5t^2 - 5t + 2$, $v_4 = t^2 - 3t - 2$

Si $\underline{u} = t^2 + t + 2$ ¿Esta $\underline{u}$ en gen$\{v_1, v_2, v_3, v_4\}$

$$\left(\begin{array}{cccc|c} 2 & 1 & 5 & -1 & 1 \\ 1 & -2 & -5 & -3 & 1 \\ 2 & 0 & 2 & -2 & 2 \end{array}\right)(1/2) \quad \left(\begin{array}{cccc|c} 1 & 1/2 & 5/2 & -1/2 & 1/2 \\ 1 & -2 & -5 & -3 & 1 \\ 2 & 0 & 2 & -2 & 2 \end{array}\right)$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 1/2 & 5/2 & -1/2 & 1/2 \\ 0 & -5/2 & -15/2 & -5/2 & 1/2 \\ 0 & -1 & -3 & -1 & 1 \end{array}\right)(-2/5) \quad \left(\begin{array}{cccc|c} 1 & 1/2 & 5/2 & -1/2 & 1/2 \\ 0 & 1 & 3 & 1 & -1/5 \\ 0 & -1 & -3 & -1 & 1 \end{array}\right)$$

$$R_3 + R_1 \left(\begin{array}{cccc|c} 1 & 1/2 & 5/2 & -1/2 & 1/2 \\ 0 & 1 & 3 & 1 & -1/5 \\ 0 & 0 & 0 & 0 & \underline{4} \end{array}\right) \quad \therefore \underline{u} \text{ no está en gen}\{v_1, v_2, v_3, v_4\}$$

$\underline{u}$

10- Prueba que si $S_1 = \{v_1, v_2, \dots v_n, v_{n+1}\}$ genera un espacio vectorial V y v_{n+1} es una c.l de $\{v_1, v_2, \dots, v_n\}$ entonces $S_2 = \{v_1, v_2, \dots, v_n\}$ también genera a V.

Sean $\alpha, \beta \in \mathbb{R}$ y $\underline{u}, \underline{v} \in V$

$V = \text{gen}\{S_1\} = \{U \in V \mid \underline{U} = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n + \alpha_{n+1} v_{n+1}\}$

Sabemos que: $v_{n+1} = (\beta_1 v_1 + \beta_2 v_2 + \dots \beta_n v_n \dots)$

$\underline{U} = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n + \alpha_{n+1} v_{n+1}$

$= \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n + \alpha_{n+1}(\beta_1 v_1) + \beta_2 v_2 + \dots \beta_n v_n)$

$= (\alpha_1 + \alpha_{n+1}\beta_1) v_1 + (\alpha_2 + \alpha_{n+1}\beta_2) v_2 + \dots + (\alpha_n + \alpha_{n+1}\beta_n) v_n$

Si $\gamma_i = \alpha_i + \alpha_{n+1}\beta_i$

$\underline{U} = \gamma_1 v_1 + \gamma_2 v_2 + \dots \gamma_n v_n \Rightarrow \underline{U} \in \text{gen}\{v_1, v_2, \dots v_n\} = \text{gen}\{S_2\} = V$

11.- Demuestra que para cualesquiera vectores $\underline{u}, \underline{v}$ de un espacio vectorial V $gen\{\underline{u},\underline{v}\} = gen\{\underline{u}+\underline{v}, \underline{u}-\underline{v}\}$

Sabemos que $gen\{\underline{u},\underline{v}\} = \{X \in V \mid \underline{x} = \alpha_1 u_1 + \alpha_2 v\}$

y también sabemos que:

$$gen\{\underline{u}+v, \underline{u}-\underline{v}\} = \{y \in V \mid y = \beta_1(\underline{u}+\underline{v}) + \beta_2(\underline{u}+v\}$$

$$y = \beta_1(u+v) + \beta_2(u-v)$$

$$\underline{x} = \gamma_1 u + \gamma_2 u \in gen\{u,v\}$$

12.- Sea U el subespacio de $\mathbb{R}^3$ generado por los vectores $(1,2,3)$ y $(-1,2,5)$. Sea V el subespacio de $\mathbb{R}^3$ generado por los vectores $(1,6,11)$ y $(2,0,-2)$. Demuestra que $U=V$

$$\left(\begin{array}{cc|c} 1 & -1 & x \\ 2 & 2 & y \\ 3 & 5 & z \end{array}\right) \quad \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array} \left(\begin{array}{cc|c} 1 & -1 & x \\ 0 & 4 & -2x+y \\ 0 & 8 & -3x+z \end{array}\right)(1/4) \quad \left(\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & \frac{-2x+y}{4} \\ 0 & 8 & -3x+z \end{array}\right)$$

$$R_3-8R_2 \left(\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & \frac{-2x+y}{4} \\ 0 & 0 & x-2y+z \end{array}\right) \quad R_1+R_2 \left(\begin{array}{cc|c} 1 & 0 & \frac{2x+y}{4} \\ 0 & 1 & \frac{-2x+y}{4} \\ 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} x_1 = \frac{2x+y}{4} \\ x_2 = \frac{-2x+y}{4} \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 2 & x \\ 6 & 0 & y \\ 11 & -2 & z \end{array}\right) \quad \begin{array}{l} R_2-6R_1 \\ R_3-11R_1 \end{array} \left(\begin{array}{cc|c} 1 & 2 & x \\ 0 & -12 & -6x+y \\ 0 & -24 & -11x+z \end{array}\right)(-1/12) \quad \left(\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & \frac{6x-y}{12} \\ 0 & -24 & -11x+z \end{array}\right)$$

$$R_3-24R_2 \left(\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & \frac{6x-y}{12} \\ 0 & 0 & x-2y+z \end{array}\right) \quad R_1-2R_2 \left(\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & \frac{6x-y}{12} \\ 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} x_1 = \frac{y}{6} \\ x_2 = \frac{6x-y}{12} \end{array}$$

$$x-2y+z=0 \qquad x-2y+z=0$$

$$x-2y+z = x-2y+z \longrightarrow U=V$$

Independencia lineal

13- Determina si los vectores son linealmente independientes o linealmente dependientes

a) $v_1 = (-1, 1, 0, 0)$ $v_2 = (-2, 0, 1, 1)$

$$\left(\begin{array}{cc|c} -1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right)(-1) \quad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \quad \begin{array}{c} \\ R_2-R_1 \\ \\ \\ \end{array}\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right)(-1/2) \quad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

$$\begin{array}{c} R_1-R_2 \\ \\ R_3-R_2 \\ R_4-R_2 \end{array}\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

v_1 y v_2 son linealmente independientes

b) $v_1 = (1, 2, -1)$ $v_2 = (1, -2, 1)$ $v_3 = (-3, 2, 1)$ $v_4 = (2, 0, 0)$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 2 & -2 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \end{array}\right) \quad \begin{array}{c} \\ R_2-2R_1 \\ R_3+R_1 \end{array}\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{array}\right)(-1/4) \quad \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{array}\right)$$

$$\begin{array}{c} \\ \\ R_3-2R_2 \end{array}\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right)(1/2) \quad \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} R_1+3R_3 \\ R_2+2R_3 \\ \\ \end{array}\left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \quad \begin{array}{c} R_1-R_2 \\ \\ \\ \end{array}\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

$\alpha_1 = -2\alpha_4$
$\alpha_2 = -\alpha_4$ $\therefore$ v_1, v_2, v_3 y v_4 son linealmente dependientes
$\alpha_3 = 0$
$\alpha_4 = \alpha_4$

c) $P_1 = t^2+t+2 \quad P_2 = 2t^2+t \quad P_3 = 3t^2+2t+2$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right) \begin{array}{c} \\ R_2-R_1 \\ R_3-2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right) (-1) \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right)$$

$$\begin{array}{c} \\ \\ +4R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} R_1-2R_2 \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} \alpha_1 = -\alpha_3 \\ \alpha_2 = -\alpha_3 \\ \alpha_3 = \alpha_3 \end{array}$$

$\therefore$ P_1, P_2 y P_3 son linealmente dependientes

14- Encuentra el ó los valores de t para los cuales son L.i los conjuntos siguientes:

a) $(3, t), (6, t-1)$ $\left(\begin{array}{cc|c} 3 & 6 & 0 \\ t & t-1 & 0 \end{array}\right) (1/3) \left(\begin{array}{cc|c} 1 & 2 & 0 \\ t & t-1 & 0 \end{array}\right)$

$$\begin{array}{c} \\ -tR_1 \end{array} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -t-1 & 0 \end{array}\right) (-1) \; -t-1 = 0 \; (-1) \; = t+1 = 0 \quad \begin{array}{c} R_1-2R_2 \\ \\ \end{array} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

$\therefore$ para que sean l.i. $\rightarrow t \neq -1$

b) $(2, -t), (2t+6, 4t)$ $\left(\begin{array}{cc|c} 2 & 2t+6 & 0 \\ -t & 4t & 0 \end{array}\right) (1/2) \left(\begin{array}{cc|c} 1 & t+3 & 0 \\ -t & 4t & 0 \end{array}\right)$

$$\begin{array}{c} \\ +tR_1 \end{array} \left(\begin{array}{cc|c} 1 & t+3 & 0 \\ 0 & t^2+7t & 0 \end{array}\right) \frac{1}{t^2+7t} \left(\begin{array}{cc|c} 1 & t+3 & 0 \\ 0 & 1 & 0 \end{array}\right) \quad R_1-(t+3) \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

$t^2+7t = 1$

$t(t+7) = 1$

$t = \frac{1}{t+7}$ $\quad \therefore$ Para que sean l.i

$t \neq -7 \quad t \neq 0$

15.- Sea (v_1, v_2, v_3) l.i. en V. Sea α un escalar distinto de cero, demuestra que los siguientes conjuntos son Li

a) (v_1, v_1+v_2, v_3) Sean $\alpha, \beta \in \mathbb{R}$

$\beta_1 v_1 + \beta_2(v_1+v_2)\beta_3 v_3 = \underline{0}$

$\beta_1 v_1 + \beta_2 v_1 + \beta_2 v_2 + \beta_3 v_3 = \underline{0}$

$(\beta_1+\beta_2)v_1 + \beta_2 v_2 + \beta_3 v_3 = \underline{0}$

Para que v_1, v_2 y v_3 sean li se necesita que

$\beta_1+\beta_2 = 0$

$\beta_2 = 0$

$\beta_3 = 0$

$\beta_1+\beta_2=0 \Rightarrow \beta_2=0$

$\beta_1=0$

$\therefore v_1, v_1+v_2, v_3$ son li

b) $(v_1, \alpha v_2, v_3)$

$\beta_1 v_1 + \beta_2(\alpha v_2) + \beta_3 v_3 = \underline{0}$

$\beta_1 v_1 + \alpha(\beta_2 v_2) + \beta_3 v_3 = \underline{0}$

Se necesita que

$\beta_1 = \underline{0}$

$\beta_2 = \underline{0}$

$\beta_3 = \underline{0}$

$\alpha = \underline{0}$

$\therefore v_1, \alpha v_2$ y v_3 son li

c) $(v_1, v_1+\alpha v_2, v_3)$

$\beta_1 v_1 + \beta_2(v_1+\alpha v_2) + \beta_3 v_3 = \underline{0}$

$\beta_1 v_1 + \beta_2 v_1 + \beta_2\alpha v_2 + \beta_3 v_3 = \underline{0}$

$(\beta_1+\beta_2)v_1 + \beta_2\alpha v_2 + \beta_3 v_3 = \underline{0}$

Se necesita que

$\beta_1+\beta_2 = \underline{0}$

$\beta_2 = \underline{0}$

$\alpha = \underline{0}$

$\beta_3 = \underline{0}$

$\beta_1 = -\beta_2$

$\beta_1 = -0$

$\beta_1 = 0$

$\therefore v_1, v_1+\alpha v_2, v_3$ son li