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Tarea #3

Lista 3 de ejercicios

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1) Muestre que: $\int_{|z|=1} \frac{\log z}{z}\, dz = 0,$

aunque $(\log z)/z$ no sea analítica en $|z| \leq 1$. Qué resultado se obtiene si se integra:

$\int_\gamma \frac{\log z}{z}\, dz$ sobre $\gamma: z(t) = e^{it},\ 0 \leq t \leq 2\pi$? Explíquelo.

$$\int_{|z|=1} \frac{\log z}{z}\, dz \Rightarrow |z| = \sqrt{x^2+y^2} \rightarrow \sqrt{x^2+y^2} = 1 \rightarrow x^2+y^2 = 1$$

$$\oint \frac{\log z}{z}\, dz \rightarrow \oint \frac{1}{\log z(z)} \rightarrow \frac{A}{\log z} + \frac{B}{z} \rightarrow 1 = A(z) + B(\log z)$$

si $z=1$, $A=1$ si $z = e^{iA}$, $B=0$

$$\rightarrow J = \oint \frac{\log z}{z}$$

$$\rightarrow \int \frac{\frac{\ln z}{\ln z}}{z}\, dz \rightarrow \int A\, dA \rightarrow \text{para } z = e^{iA}$$

$$\rightarrow \int_{-\pi}^{\pi} \lambda\, d\lambda \rightarrow \int du A \Rightarrow \frac{\lambda^2}{2} \rightarrow \frac{\lambda^2}{2}\Big|_{-\pi}^{\pi}$$

$$\rightarrow \frac{\pi^2}{2} - \frac{(-\pi)^2}{2} = 0 \ \therefore \int_{-\pi}^{\pi} \lambda\, d\lambda = 0 \rightarrow \int_{|z|=1} \frac{\log z}{z}\, dz = 0$$

para $0 \leq \arg z \leq 2\pi$

3).- Muestre que $\int_{\partial G} y\, dz = -A$

$$= \int_{\partial G} y\,(dx + i\,dy)$$

$$= \int_{\partial G} \cancel{(0)dy} + y\,dx + i\int_{\partial G} y\,dy\ \cancel{+(0)dx}$$

$$\downarrow$$

$$\iint_G (-1)\,dx\,dy + i\iint_G (0-0)\,dx\,dy \rightarrow dx\,dy = dA$$

$$= -1\iint_G dA = -A \ \therefore \text{ se cumple que } \int_{\partial G} y\,dz = -A$$

4) Pruebe que: $\int_{\partial G} \bar{z}\,dz = 2iA$

$= \int_{\partial G} (x-iy)\,dx + i\,dy$

$= \int_{\partial G} x\,dx + y\,dy + i(-y\,dx + x\,dy) \rightarrow \int_{\partial G} y\,dy + x\,dx + i\int_{\partial G} x\,dy - y\,dx$

$= \iint_G (0+0)\,dx\,dy + i\iint_G (1+1)\,dx\,dy \longrightarrow dx\,dy = dA$

$\therefore \rightarrow 2i\iint_G dA = 2iA \longrightarrow \therefore$ se cumple que:

$\int_{\partial G} \bar{z}\,dz = 2iA$

9) Suponga que $0<b<1$ y aplique el teorema de Cauchy a la funcion $f(z) = (1+z^2)^{-1}$ a lo largo de la frontera del rectángulo, para mostrar que:

$$\int_{-\infty}^{\infty} \frac{(1-b^2+x^2)\,dx}{(1-b^2+x^2)^2+4x^2b^2} = \pi$$

$$0 = \int_{-a}^{a} \frac{dx}{1+x^2} + \int_0^b \frac{i\,dy}{1+(a+iy)^2} + \int_a^{-a} \frac{dx}{1+(x+ib)^2} + \int_b^0 \frac{i\,dy}{1+(-a+iy)^2}$$

$$= \int_{-a}^{a} \frac{dx}{1+x^2} - \int_{-a}^{a} \frac{dx}{1+(x^2+2bxi-b^2)} + i\int_0^b \frac{dy}{1+(a^2+2ayi-y^2)} - i\int_0^b \frac{dy}{1+(a^2-2ay\ldots}$$

$\downarrow$

$$\int_{-a}^{a} \frac{dx}{1-x^2} - \int_{-a}^{a} \frac{(1-b^2+x^2)-2bxi}{[1-b^2+x^2]^2+4x^2b^2}\,dx + i\int_0^b \frac{(1+a^2-y^2)-2ayi}{[1+a^2-y]^2+4y^2a^2}$$

$$-i\int_0^b \frac{(1-a^2-y^2)+2ayi}{[1+a^2-y^2]^2+4y^2a^2}\,dy$$

$$= \int_{-a}^{a} \frac{dx}{1+x^2} - \int_{-a}^{a} \frac{(1-b^2+x^2)\,dx}{(1-b^2+x^2)^2+4x^2b^2} + 2bi\int_{-a}^{a} \frac{x\,dx}{(1-b^2+x^2)^2+4x^2b^2}$$

$$-4ai\int_0^b \frac{x\,dx}{(1+a^2-y^2)^2+4y^2+x^2}$$

10) Pruebe que:

$$\int_{-\infty}^{\infty} e^{kx^2} \cos ax\,dx = \sqrt{\frac{\pi}{k}}\, e^{-\frac{a^2}{4k}}, \quad k > 0, \ a \text{ real},$$

si utiliza el mismo procedimiento con la función $f(z) = e^{kz^2}$.

$$0 = \int_{-a}^{a} e^{-kx^2} dx + \int_0^b e^{-k(a+iy)^2} i\,dy + \int_a^{-a} e^{-k(x+ib)^2} dx + \int_b^0 e^{-k(-a+iy)^2} i\,dy$$

$$\Rightarrow \int_{-a}^{a} e^{-kx^2} dx - e^{b^2}\int_{-a}^{a} e^{-kx^2}(\cos 2bx - i\,\text{sen}\,2bx)\,dx - ie^{-a^2}\int_0^b e^{ky^2}(e^{2aiy} - e^{-2ay i} \ldots$$

$$= \int_{-a}^{a} e^{-kx^2} + e^{b^2}\int_{-a}^{a} e^{-kx^2} \cos 2bx\,dx + 2e^{-a^2}\int_0^b e^{ky^2}\,\text{sen}\,2ay\,dy$$

$\downarrow$

$$\int_{-a}^{a} e^{-kx^2} dx = \int_0^{2\pi}\int_0^{\infty} e^{-kr^2} r\,dr\,d\theta = \frac{\pi}{k} \Rightarrow \int_{-a}^{a} e^{-kx^2} \cos ax\,dx$$

$$= e^{-b^2}\int_{-\infty}^{\infty} e^{-kx^2} dx$$

$$= \sqrt{\frac{\pi}{k}}\, e^{-\frac{a^2}{4k}}$$

donde $k = 1$, $a = 2b$ para el ejercicio original

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Sea $z(t) = 2e^{it} + 1$, $0 \le t \le 2\pi$. Evalúe las integrales.

4) $\int_\gamma \frac{e^z}{z} dz$

De la form. integral de Cauchy: $f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0} dz$

$f(z) = e^z$, analítica en todo plano complejo

$\downarrow$

$z_0 = 0 \rightarrow$ contenido en γ

$f(z_0) = e^0 = 1$

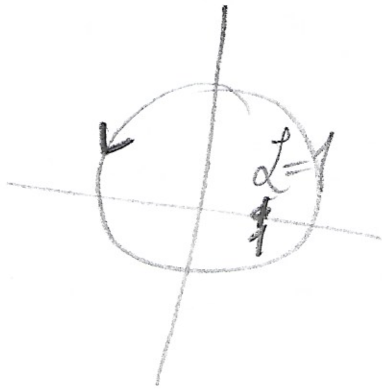
$$1 = \frac{1}{2\pi i}\int_\gamma \frac{e^z}{z} dz \longrightarrow \int_\gamma \frac{e^z}{z} dz = 2\pi i$$

5) $\int_\gamma \frac{\cos z}{z-1}\,dz$

$$\int_\gamma \frac{\cos z}{z-1}\,dz = \int \frac{f(z)}{z-\xi}\,dz = 2\pi i\,[f(\xi)]_{\xi \to 1}$$

$$= 2\pi i(\cos(1))$$

$$= 2\pi i \cos 1$$



9) Sea $z(t)=2e^{it}+1,\ 0\le t\le 2\pi$. Evalúe las integrales

$\int_\gamma \frac{\cos z}{(z-1)^2}\,dz$

$$f'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^2}\,dz$$

$\downarrow$

$f(z)=\cos z$, $z_0 = 1$; , $f'(z) = -\operatorname{sen} z$

$$\therefore \int_\gamma \frac{\cos z}{(z-1)^2}\,dz = -2\pi i \operatorname{sen} 1$$

ii) $\int_\gamma \frac{\operatorname{sen} z}{(z-1)^3}\,dz \quad \rightarrow f^{n}(z_0)\ \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

$f(z)=\operatorname{sen} z$

$z_0=1$

$n+1=3;\ n=2$

$f^{n}(z_0)=\frac{d^2}{dz}\operatorname{sen}(1)=-\operatorname{sen} 1$

$\rightarrow -\operatorname{sen} 1=\frac{2}{2\pi i}\int_\gamma \frac{\operatorname{sen}^z}{(z-1)^2}\,dz \rightarrow -\pi i \operatorname{sen} 1=$

$\downarrow$

$\int_\gamma \frac{\operatorname{sen}^z}{(z-1)^2}\,dz$

Pág 107 Obtenga las series de Mclaurin dadas

3) $\operatorname{sen} z=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}\ ,\ |z|<\infty$

$f(z)=\operatorname{sen} z\ ,\ f'(z)=\cos z\ ,\ f''(z)=-\operatorname{sen} z\ ,\ f'''=-\cos z\ ,\ f^{(4)}(z)=\operatorname{sen} z=f(z$

$f(0)=0\ ,\ f'(0)=1\ ,\ f''(0)=0\ ,\ f'''(0)=-1\ ,\ f^{(4)}(0)=0$

S. Mclaurin

$f(0)+f'(0)\,z+\frac{f''(0)}{2!}z^2+\frac{f'''(0)}{3!}z^3+\frac{f^{(4)}(0)}{4!}z^4+\ldots$

$=0+z+\cancel{\frac{0}{2!}z^2}-\frac{1}{3!}z^3+\cancel{\frac{0}{4!}z^4}+\frac{1}{5!}z^5+\cancel{\frac{0}{6!}z^6}+\ldots$

$\downarrow$

$\operatorname{sen} z=z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\frac{1}{7!}z^7+\frac{1}{9}z^9-\frac{1}{11!}z^{11}+\ldots$

$\therefore \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)!}z^{2n-1}$

4) $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \ |z|<\infty$

S. Mclaurin

$\rightarrow f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \frac{f^{(4)}(0)}{4!}z^4 + \dots$

$= 1 + \cancel{(0)z} - \frac{1}{2!}z^2 + \cancel{\frac{0}{3!}z^3} + \frac{1}{4!}z^4 + \cancel{\frac{0}{5!}z^5} - \frac{1}{6!}z^6 + \cancel{\frac{0}{7!}z^7} + \dots$

$\downarrow$

$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \frac{1}{8!}z^8 - \frac{1}{10!}z^{10} + \frac{1}{12!}z^{12} + \dots$

$\therefore \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} z^{2n}$

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Encuentre la serie de Laurent de la función $(z^2+z)^{-1}$ en las regiones del ejercicio

2) $0<|z-1|<1$

$f(z) = \frac{1}{z^2+z} = \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$

$\downarrow$

$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \rightarrow \text{si } |z|<1 \longrightarrow \frac{1}{1-(z-1)} = \sum_{n=0}^{\infty} (z-1)^n, \ |z-1|<1$

Fact. $\rightarrow f(z) = \left[\frac{1}{1+(z-1)}\right] - \left[\frac{1}{2+(z-1)}\right]$

$= \left\{1-(z-1)+(z-1)^2+\dots\right\} - \frac{1}{2}\left\{1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{2^2} + \dots\right\}$

$= \frac{1}{2} - \frac{3}{4}(z-1) + \frac{7}{8}(z-1)^2 - \frac{15}{16}(z-1)^3 + \dots$

$\therefore = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n-1}{2^n} (z-1)^{n-1}$

8) Encuentra la serie de Laurent de la función $z/(z^2+z-2)$ en las regiones dadas

$|z|<1$

$$f(z)=\frac{z}{z^2+z-2}=\frac{z}{(z-1)(z+2)}=\frac{A}{z-1}+\frac{B}{z+2}=\frac{Az+2A+Bz-B}{(z-1)(z+2)}=\frac{(A+B)z+2A-B}{(z-1)(z+2)}$$

$$\begin{cases}A+B=1\\2A-B=0\end{cases}\rightarrow \begin{matrix}A=1/3\\B=2/3\end{matrix}\longrightarrow \frac{1}{3}\,\frac{1}{z-1}+\frac{2}{3}\,\frac{1}{z+2}\longrightarrow \frac{1}{1-z}=\sum_{n=0}^{\infty} z^n$$

$$-\frac{1}{3}\left(\frac{1}{1-z}\right)+\frac{2}{3}\left[\frac{1}{1-(-z-1)}\right]=-\frac{1}{3}\left(\frac{1}{1-z}\right)+\frac{2}{3}\left[\frac{1}{1-(-z-1)}\right]$$

$$\therefore\ \frac{1}{3}\sum_{n=0}^{\infty} z^n+\frac{2}{3}\sum_{n=0}^{\infty}(-z-1)^n=\sum_{n=0}^{\infty}\left[\frac{2(z-1)^n}{3}-\frac{z^n}{3}\right]$$

11) $1<|z|<2$

$$\rightarrow \frac{z}{z^2+z-2}\rightarrow z^2+z-2=0$$

$$(z+2)(z-1)$$

$$z_1=-2$$
$$z_2=1$$

$\longrightarrow \frac{z}{z^2+z-2}$ es analítica en todo lugar excepto en $z=(-2,1)$

$$\frac{z}{(z+2)(z-1)}=\frac{2}{3(z+2)}+\frac{1}{3(z-1)}$$

$$=\frac{2}{3}\,\frac{1}{(z+2)}+\frac{1}{3}\,\frac{1}{(z-1)}\longrightarrow \frac{2}{3}\,\frac{1}{(1+\frac{2}{z})}+\frac{1}{3}\,\frac{1}{(1-\frac{1}{z})}$$

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n\ ,\quad \frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

$$=\frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n+\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$\therefore\ =\frac{1}{3}\sum_{n=0}^{\infty}\left[(-1)^n\left(\frac{2}{z}\right)^n+\left(\frac{1}{z}\right)^n\right]+\frac{1}{3}$$

Encuentre las series de Laurent de la función en la región $0<|z-1|<1$

1a) $\frac{1}{z}$ sen $\frac{1}{z-1}$

$$\text{Sen}\,\frac{1}{z-1} = \sum_{n=0}^{\infty} \frac{-1^n}{(2n+1)!}\left(\frac{1}{z-1}\right)^{2n+1}, \quad \text{con: } \frac{1}{z-1}<\infty$$

$$\downarrow$$

$$|z-1|<1<\infty \longrightarrow \frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{n=0}^{\infty} -1^n (z-1)^n, \text{ con } |z-1|<$$

$$\therefore f(z) = \left[\sum_{n=0}^{\infty} -1^n (z-1)^n\right]\left[\sum_{n=0}^{\infty} \frac{-1^n}{(2n+1)!}\left(\frac{1}{(z-1)^{2n+1}}\right)\right]$$