4CM1

2do examen departamental

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Matemáticas Avanzadas para la ingeniería

1.- Pruebe que $\cot^{-1} z = \frac{i}{2} \log\left(\frac{i+z}{1-z}\right)$

$z = \cot w \rightarrow \cot w = \frac{\cos w}{\operatorname{sen} w}$, $\cot^{-1} \cot w = w$

$$\cot w = \frac{\frac{e^{iv}+e^{-iv}}{2}}{\frac{e^{iv}-e^{iv}}{2i}} \rightarrow \cot w = \frac{2i(e^{iv}+e^{-iv})}{2(e^{iv}-e^{iv})} = z = \frac{\cancel{2}i(e^{iv}+e^{-iv})}{\cancel{2}(e^{iv}-e^{iv})}$$

$$= \frac{i(e^{iv}+e^{-iv})}{e^{iv}-e^{iv}} \rightarrow i\,\frac{v+\frac{1}{v}}{v-\frac{1}{v}} = i\,\frac{\frac{v^2+1}{\cancel{v}}}{\frac{v^2-1}{\cancel{v}}} \rightarrow i\,\frac{v^2+1}{v^2-1}$$

Ahora:

$$z = i\,\frac{v^2+1}{v^2-1} \rightarrow zv^2 - z = iv^2 + i \rightarrow zv^2 - iv^2 = z + i$$

$$v^2(z-i) = z+i \rightarrow v^2 = \frac{z+i}{z-i} \rightarrow = e^{i2w} = \frac{z+i}{z-i}$$

$$\therefore \ln e^{i2w} = \ln \frac{z+i}{z-i} \rightarrow i2w = \ln \frac{z+i}{z-i}$$

$$w = \frac{1}{2i}\,\frac{i}{i}\,\ln\frac{z+i}{z-i} \quad \left(\frac{i}{2}\ln\frac{z-i}{z+i}\right)^{-1}$$

$$= \frac{i}{2}\ln\left(\frac{z+i}{z-i}\right)$$

2.- Use las ecuaciones de Cauchy-Riemann para probar que $f(z)=|z|$ no es derivable en C

Ec Cauchy-Riemann

① $U_x = V_y$

② $U_y = -V_x$; $z = x + iy$, $|z| = \sqrt{x^2+y^2}$

$f(z) = u(z) + iv(z)$; $z \in \mathbb{R}$

$\downarrow$

$f(z) = |z|$

$\downarrow$

$U(z) = |z| \longrightarrow U(z) = \sqrt{x^2+y^2}$

$V(z) = 0$

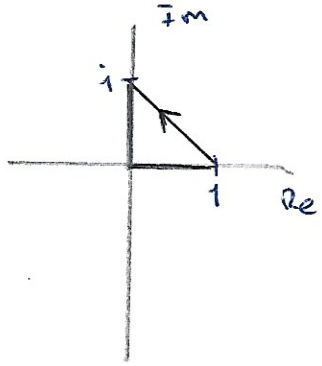
$\therefore U_x = V_y \longrightarrow \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \rightarrow (x^2+y^2)^{1/2} dx = \frac{d}{dU} U^{1/2}(2x)$

$= \frac{U^{-1/2}}{2}(2x) = \frac{\cancel{2}x}{\cancel{2}U} = \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial V}{\partial y} = 0\, dy^{\to 0} \longrightarrow \frac{x}{\sqrt{x^2+y^2}} \neq 0$

$\therefore$ Se prueba que $f(z) = |z|$, no es derivable en C

3. Parametrice un arco suave por partes representado por un triángulo con vertices en 0, 1 e i



$z(t) = (1-t) + it \quad , t\ [0,1]$

$dz = -1 + i\,dt$

$= \int_0^1 e^{(1-t)+it}(-1+i)\,dt$

$= -\int_0^1 e^{(1-t)+it}\,dt + i\int_0^1 e^{(1-t)+it}\,dt$

$= -(-\tfrac{1}{2} - \tfrac{1}{2}i)\, e^{(1-t)+it}\Big|_0^1 + i(-\tfrac{1}{2} - \tfrac{1}{2}i)\, e^{(1-t)+it}\Big|_0^1$

$= (\tfrac{1}{2} + \tfrac{1}{2}i)\, e^{(1-t)+it} - (\tfrac{1}{2} + \tfrac{1}{2})\, e^{(1-t)+it}\Big|_0^1$

$= (\tfrac{1}{2})\, e^{(1-t)+it} \cancel{+ \tfrac{1}{2}i(e^{(1-t)+it}) - \tfrac{1}{2}i(e^{(1-t)+it})}$

$+ \tfrac{1}{2}\, e^{(1-t)+it}\Big|_0^1$

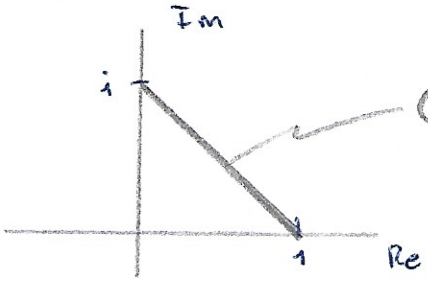
$= \tfrac{1}{2}(e^{(1-t)+it}) + \tfrac{1}{2}(e^{(1-t)+it}\Big|_0^1$

$= e^{(1-t)+it}\Big|_0^1 \rightarrow e^{0+i} - e^{1+0}$

$\therefore = e^{i} - e$

4. Evalue $\int_C e^z dz$, donde C es la recta que une a 1 con i

$y = mx + b$



$z(t) = (1-t) + it \longrightarrow t[0,1]$

$dz = -1 + i\, dt$

$= \int_0^1 e^{(1-t)+it} dt + i\int_0^1 e^{(1-t)+it}(-1+i)dt$

$= -\int_0^1 e^{(1-t)+it} dt + i\int_0^1 e^{(1-t)+it} dt$

$= -(-\frac{1}{2} - \frac{1}{2} i) e^{(1-t)+it}\Big|_0^1 + i(-\frac{1}{2}-\frac{1}{2} i)e^{(1-t)+it}\Big|_0^1$

$+(\frac{1}{2}+\frac{1}{2} i)e^{(1-t)+it} - \frac{1}{2} + \frac{1}{2}(e^{(1-t)+it})\Big|_0^1$

$= \cancel{\frac{1}{2} e^{(1-t)+it}} + \frac{1}{2} i (e^{(1-t)+it} \cancel{- \frac{1}{2} i e^{(1-t)+it}} + \frac{1}{2} e^{(1-t)+it}\Big|_0$

$\downarrow$

$\frac{1}{2} e^{(1-t)+it} + \frac{1}{2} e^{(1-t)+it}\Big|_0^1 = e^{(1-t)+it}\Big|_0^1$

$\therefore e^{0+i} - e^{1+0} \longrightarrow \underline{e^i - e}$

5. Evalue $\int_1^i (z-1)^3 dz$ $\qquad (z-1) = w$

$$\int_1^i w^3 dw$$

$$\downarrow$$

$$\frac{w^4}{4}\Big|_0^1 \longrightarrow \frac{(z-1)^4}{4}\Big|_1^i$$

$$= \frac{(i-1)^4}{4} - \cancel{\frac{(1-1)^4}{4}} = \frac{(i-1)^{2+2}}{4} = \frac{(i-1)^2 (i-1)^2}{4}$$

$$= \frac{(i^2-2i+1)(i^2-2i+1)}{4} = \frac{(\cancel{-1}-2i\cancel{+1})(\cancel{-1}-2i\cancel{+1})}{4}$$

$$= \frac{(-2i)(-2i)}{4} \longrightarrow \frac{4i^2}{4} = \frac{4(-1)}{4} = \frac{-4}{4} = \underline{-1}$$