MA1521 Cheat Sheet

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1 MF26 Magic

Trigo

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$
$$\sin P + \sin Q = 2 \sin \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$$
$$\sin P - \sin Q = 2 \sin \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$$
$$\cos P + \cos Q = 2 \sin \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$$
$$\cos P - \cos Q = -2 \sin \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$$

Derivatives

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$
$$\frac{d}{dx} \csc x = -\csc x \cot x$$
$$\frac{d}{dx} \sec x = \sec x \tan x$$
$$\frac{d}{dx} \tan x = \sec^2 x$$
$$\frac{d}{dx} \cot x = -\csc^2 x$$

Integrals

Take note of the absolute sign, and always remember to $+c$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + c$$
$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(\frac{x}{a}) + c$$
$$\int \frac{1}{x} dx = \ln |x| + c$$

2 Basics

Extreme Values

Points where f can have an extreme value:

- Interior point where $f'(x) = 0$
- Interior points where $f'(x)$ doesn't exist
- End points of the domain of f

L'Hospital's Rule

The $\frac{0}{0}$ form: (1) f and g are differentiable in a neighborhood of x_0, (2) $f(x_0) = g(x_0) = 0$, (3) $g'(x) \neq 0$ except possibly at x_0

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

E.g. $\lim_{x \to 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1}|_{x=0} = 2$

The $\frac{\infty}{\infty}$ form: when $x \to a$, $f(x), g(x) \to \infty$, and both differentiable,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Else, change to these two forms. (e.g $\lim_{x \to 0^+} x \cot x = \lim_{x \to 0^+} \frac{x}{\tan x} = \lim_{x \to 0^+} \frac{1}{\sec^2 x} = 1$)

Fundamental Theorem of Calculus

$$\frac{d}{d\Box} \int_c^{\Box} f(t) dt = f(\Box)$$

3 Series

Geometric Series

Sum: $S_n = a\frac{1-r^n}{1-r}$, $r \neq 1$

Ratio test: $\lim_{n \to \infty} = \left|\frac{a_{n+1}}{a_n}\right| = \rho$;

1. $\rho < 1$: converge;
2. $\rho > 1$: diverge;
3. $\rho = 1$, no conclusion;

For convergent series: $S_n \to \frac{a}{1-r}$

Power Series

$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$, where a is the center of the power series

Convergence: $n \to \infty, S_n \to k$

1. $\sum c_n(x-a)^n$ converges at $x = a$ and diverges elsewhere
2. $h \in \mathbb{Z}$ that the series only converges in $(a-h, a+h)$
3. converges for every x

Finding Radius of Convergence

Apply ratio test and find

$$M = \lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| M < 1$$

and tranform it to the form of $|x - a| < b$; a is the center, b is the RoC

Or, if the series converges for all x, the RoC is ∞; if it only converges at a, the RoC is 0;

Some magic:

$$\frac{1}{1-\Box} = \sum_{n=0}^{\infty} \Box^n, |\Box| < 1 \tag{1}$$

Taylor Series

of f at a:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \ldots$$
$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Finding a specific high order derivative

1. given $\int f dx$
2. evaluate f in polynomial form and integrate the polynomial form
3. Compare the coefficient with the item that contains $f^{(100)}(0)$ in the Taylor expansion

4 Vectors

Angle between two vectors: $\cos\theta = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\|\vec{v_1}\| \|\vec{v_2}\|}$

Perpendicular vectors: $\vec{v_1} \cdot \vec{v_2} = 0$

5 Partial Diffrentiation

$$f_{xy}(a,b) = f_{yx}(a,b)$$
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Directional Derivative

$D_{\vec{u}} f(a,b) = f_x(a,b) \cdot u_1 + f_y(a,b) \cdot u_2$, where unit vector $u = u_1\vec{i} + u_2\vec{j}$

Gradient Vector: $\nabla f = f_x \vec{i} + f_y \vec{j}$

Thus,

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u} = \|\nabla f(a,b)\| \cos\theta$$

f increases most rapidly in $\nabla f(a,b)$, decreases most rapidly in $-\nabla f(a,b)$

Max value of $D_u f(a,b) = \|\nabla f(a,b)\|$, when $\vec{u}$ and ∇f in the same direction, since $\cos\theta = 0$

Increament in f (approx.): $\Delta f \approx [D_{\vec{u}} f(\vec{p})](\Delta t)$, where p is the origin, u is the unit direction.

Finding Duf

1. Find the direction $\vec{p}$
2. Find the unit vector $\vec{u} = \vec{p}/|\vec{p}|$
3. Find ∇f, then find $D_u f = \nabla f \cdot \vec{u}$

Critical Points

A point of f that satisfies either is a critical point:

1. $f_x(a,b) = 0$ and $f_y(a,b) = 0$
2. $f_x(a,b)$ or $f_y(a,b)$ doesn't exist

Perform Second Derivative Test: let $f_x(a,b) = 0$ and $f_y(a,b) = 0$

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

- $D > 0, f_{xx} > 0$, f has a local minimum at (a,b)
- $D > 0, f_{xx} < 0$, f has a local maximum at (a,b)
- $D < 0$, f has a saddle point at (a,b)
- $D = 0$, no conclusion

6 Ordinary Differential Equation

Separable Equations

$$M(x) - N(y)y' = 0 \implies \int M(x)dx = \int N(y)dy + c$$

Reduction to Separable Form

Let $v = y/x \implies y = xv \to y' = v + xv'$, transform equations of $y' = g(\frac{y}{x})$ to $v + xv' = g(v)$ such that

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

Similarly, $y' = f(ax + by + c)$ can be solved by $u = ax + by + c$

Linear First Order ODE

To solve $y' + Py = Q$: find integration factor

$$R = e^{\int Pdx}$$

Then, answer

$$y = \frac{1}{R}\int RQdx$$

Reduction to Linear Form

A Bernoulli equation: $y' + P(x)y = Q(x)y^n$, where $n \in \mathbb{R}$;
(When $n = -1$, try Reduction to Separable Form) To solve it, let $v = y^{1-n}$;
Find and express dv/dx in dy/dx; find dy/dx and sub that in original equation; transform into

$$v' + (1-n)Pv = Q(1-n)$$

and solve the linear ODE.

Homogeneous Linear Second Order DE

For $y'' + ay' + by = 0$, the characteristic equation is $\lambda^2 + a\lambda + b = 0$
Find $\Delta = a^2 - 4b$:

1. $\Delta > 0$, $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
2. $\Delta = 0$, $y = (c_1 + c_2 x)e^{-\frac{ax}{2}}$
3. $\Delta < 0$, it has two complex roots;$\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$;
   $y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$

where,

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$$
$$\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$$

7 Modeling

Population Growth

Malthus's Model: not an accurate representation

$$\frac{dN}{dt} = kN, k = B - D$$
$$N(t) = N_0 e^{kt}$$

Logistic Model

Assume $D = sN$, where s is a constant:

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$

The curve approaches carrying capacity $N = B/S$; point of inflection is at $N = B/2s$

$$N = \frac{N_\infty}{1 + (\frac{N_\infty}{N_0} - 1)e^{-Bt}}, N_\infty = \frac{B}{s}$$

Harvesting

Basic harvesting model: $\frac{dN}{dt} = BN - sN^2 - E$, where E is fish catched/ year.

Desirable result: $E < \frac{B^2}{4s}$, approaches the second root $\beta_2 = \frac{B + \sqrt{B^2 - 4Es}}{2s}$, when $dN/dt = 0$

Strategies

- When given dx/dt, find x that $dx/dt = 0$, draw out the axis, determine the sign of dx/dt within each region, and find the flow (+ to the right, - to the left)
- To find E, draw the graph without E and find the line of symmetry; use the product of the roots to find E;

8 PDE

For a PDE in the form of,

$$u_x = f(x)g(y)u_y$$

Substitute $u(x,y) = X(x)Y(y)$ in the PDE, usually

$$u_x = X'Y, u_y = XY', u_{xy} = X'Y'$$

and arrange the DE into a from in which X', Y' both has power of 1; Let both sides be k, or let them be $k, 1/k$; and solve X, Y in k, c;