OBSERVERS

$$x_{k+1} = Ax_k + Bu_k + v_k^x$$

$$y_k = Cx_k + v_k^y$$

GOAL:

Compute an estimate $\hat{x}_t$ of the state x_t on the basis of the measurements

ASSUMPTIONS:

- Process Disturbance v_k^x is Gaussian noise with $E[v_k^x] = 0$ and covariance $E[(v_k^x - 0)(v_k^x - 0)^T] = Q$
- Sensor Noise v_k^y is Gaussian noise with $E\left[v_k^y\right] = 0$ and covariance $E\left[(v_k^y - 0)(v_k^x - 0)^T\right] = R$
- x_0 is Gaussian with mean $E[x_0] = \bar{x}_0$ and covariance $E\left[(v_k^y - \bar{x}_0)(v_k^x - \bar{x}_0)^T\right] = P_0$

KALMAN FILTER

$$\hat{x}_{k|h} = E\left[x_k \mid y_0,\ldots,y_h,u_0,\ldots,u_h\right]$$

$$P_{k|h} = E\left[\left(x_k - \hat{x}_{k|h}\right)\left(x_k - \hat{x}_{k|h}\right)^T \mid y_0,\ldots,y_h,u_0,\ldots,u_h\right]$$

1. Prediction

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$

$$P_{k+1|k} = AP_{k|k}A^T + Q$$

Q always increases

covariance of the next state with info only on the last inputs and outputs and thus last estimated state = prediction

2. Correction

$$K_{k+1|k+1} = P_{k+1|k}C^T\left(CP_{k+1|k}C^T + R\right)^{-1}$$

changes at each time instant

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1|k+1}\left(y_{k+1} - C\hat{x}_{k+1|k}\right)$$

Cx^ is the predicted output and K is either very big (meaning prediction unreliable but sensors are) or very small (meaning prediction reliable and not sensors) -> if very big then very big K to come back to reality, otherwise it's ok because our sensing ain't good anyway so let's not move too much from where we predicted we would be

$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1|k+1}CP_{k+1|k}$$

same thing for Pk+1|k+1

With **initial conditions:**

$$\hat{x}_{0|-1} = E\left[x_0\right]$$

$$P_{0|-1} = \hat{P}_0$$

Comparison with Classical Observers

1. Prediction

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$

$$P_{k+1|k} = AP_{k|k}A^T + Q$$

2. Correction

$$K_{k+1|k+1} = P_{k+1|k}C^T\left(CP_{k+1|k}C^T + R\right)^{-1}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1|k+1}\left(y_{k+1} - C\hat{x}_{k+1|k}\right)$$

$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1|k+1}CP_{k+1|k}$$

1. Prediction

$$\hat{x}_{k+1|k} = A\,\hat{x}_{k|k} + Bu_k$$

2. Correction

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + L(y_{k+1} - C\,\hat{x}_{k+1|k})$$

Remark 1: Similar but different !
Remark 2: The Kalman filter converges to a classical observer !

Kalman Filter with intermittent observations

$$x_{k+1} = Ax_k + Bu_k + v_k^x$$
$$y_k = \Gamma_k(Cx_k + v_k^y)$$

$$\Gamma_k = \left[e_i^T\right]_{if\ the\ i-th\ sensor\ is\ available}$$

1. Prediction

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$

$$P_{k+1|k} = AP_{k|k}A^T + Q$$

2. Correction

$$K_{k+1} = P_{k+1|k}C^T\Gamma_{k+1}^T\left(\Gamma_{k+1}\left(CP_{k+1|k}C + R\right)\Gamma_{k+1}^T\right)^{-1}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1|k+1}\left(y_{k+1} - \Gamma_{k+1}C\,\hat{x}_{k+1|k}\right)$$
$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1|k+1}\Gamma_{k+1}CP_{k+1|k}$$

Extended Kalman Filter

$$x_{k+1} = f\left(x_{k+1}, u_k\right) + v_k^x$$
$$y = h\left(x_k\right) + v_k^x$$

1. Prediction

$$\hat{x}_{k+1|k} = f\left(\hat{x}_{k|k}, u_k\right)$$
$$P_{k+1|k} = A_k P_{k+1|k} A_k^T + Q$$

$$A_k = \frac{df}{dx}\bigg|_{\substack{x=\hat{x}_{k+1|k} \\ u=u_k}}$$

2. Correction

$$K_{k+1} = P_{k+1|k} C_k^T \left(C_k P_{k+1|k} C_k + R\right)^{-1}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1|k+1}\left(y_{k+1} - h\left(\hat{x}_{k+1|k}\right)\right)$$
$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1|k+1} C_k P_{k+1|k}$$

$$C_k = \frac{dh}{dx}\bigg|_{x=\hat{x}_{k+1|k}}$$