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Trigonometry and Cyclic Functions

Trigonometry is a branch of mathematics concerned with functions that describe angles. Although knowledge of trigonometry is valuable in surveying and navigation, in control systems engineering its virtue lies in the fact that trigonometric functions can be used to describe the status of objects that exhibit repeatable behavior. This includes the motion of the planets, pendulums, a mass suspended on a spring, and perhaps most relevant here, the oscillation of process variables under control.

Units of Measurement

The most common unit of measurement for angles is the *degree*, which is 1/360 of a whole circle.

A lesser used unit is the *radian*. Although the radian is not ordinarily used in angular measurement, it should be understood because when differential equations, which occur in control systems engineering, are solved, the angles emerge in radians.

On the circumference of a circle, if an arc equal in length to the radius of the circle is marked off, then the arc will subtend, at the center of the circle, an angle of 1 radian. The angle θ (or POB) in Figure 1-1, illustrates this.

In line with this definition of a radian, the relationship between radians and degrees can be worked out. The full circumference of the circle (length 2π r) subtends an angle of 360° at the center of the circle. An arc of length r will subtend an angle of

$$\frac{r}{2\pi r} \times 360° = \frac{180}{\pi} \text{ degrees.}$$

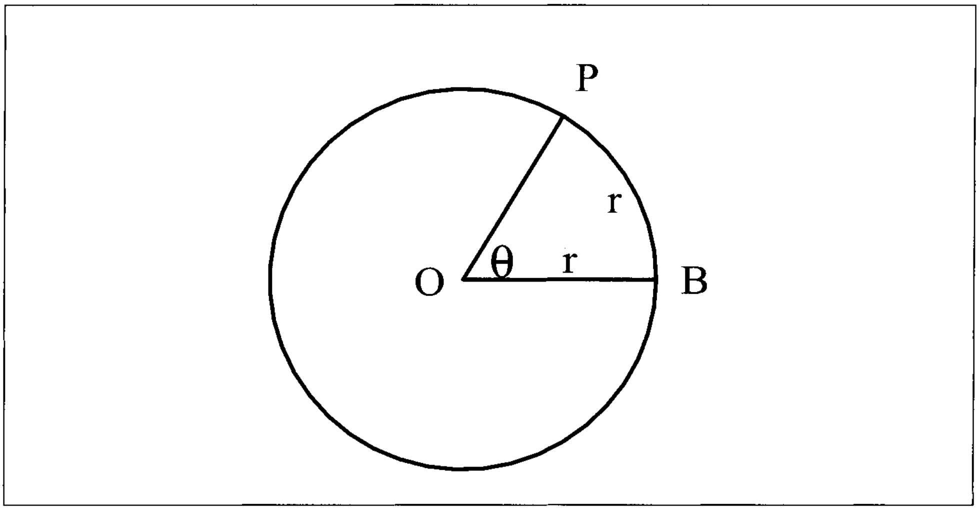

Figure 1-1. A radian defined.

Therefore 1 radian = $\frac{180}{\pi}$ deg, or π radians = $180°$.

The actual value of a radian is 57°17′45″, although this value is hardly ever required in control systems analysis.

If the base line OB in Figure 1-1 remains fixed and the radius OP is allowed to rotate counterclockwise around the center O, then the angle θ (or POB) increases. If the starting point for OP is coincident with OB, and OP rotates one complete rotation (or cycle) until it is again coincident with OB, then the angle θ will be 360°. From this it is evident that 1 cycle = 360° = 2π radians.

Functions of Angles

Let θ be any acute angle for which OB is the base, and P be any point on the inclined side of the angle, as in Figure 1-2. A perpendicular from P down to the base OB meets OB at point A.

First, the ratio of any one of the three sides of the triangle POA, to either of the other sides, is a characteristic of the angle θ. In other words, if any of the ratios PA/OP, OA/OP, or PA/OA is known, then the angle θ can be determined from the appropriate tables.

Note that the values of these three ratios do not depend on the position of P. As P moves out along the inclined side of the angle, OP increases, but PA and OA also increase in the same proportion. The values of the ratios depend on the size of the angle θ, but not on the location of P.

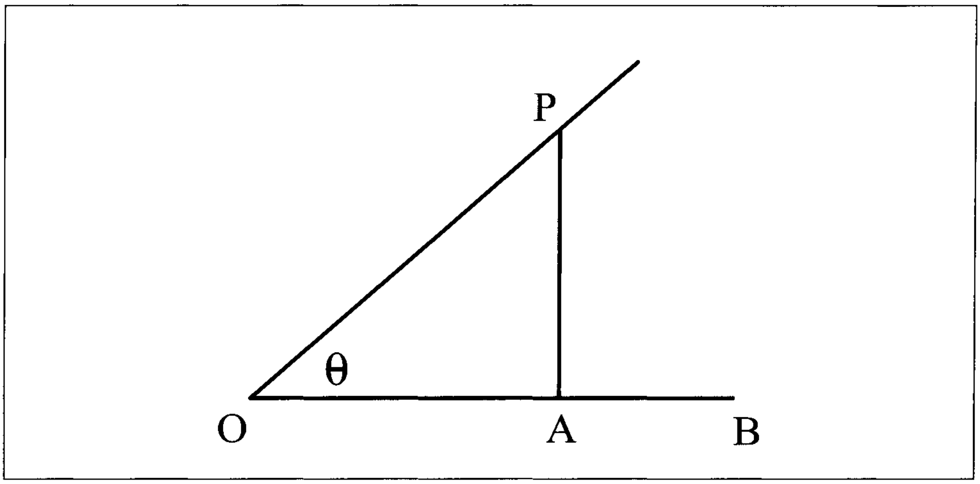

Figure 1-2. Functions of angles.

Definitions

In the triangle POA, the ratio of the length of the side opposite the angle θ to the length of the hypotenuse, or PA/OP, is called the *sine* of the angle θ. The abbreviation *sin* is generally used, that is, PA/OP = sin θ.

The ratio of the side adjacent to the angle θ to the hypotenuse, or OA/OP, is called the *cosine* of the angle θ. This is usually abbreviated *cos*, that is, OA/OP = cos θ.

The ratio of the opposite side to the adjacent side, or PA/OA, is called the *tangent* of the angle θ. This is abbreviated *tan*, so that PA/OA = tan θ.

There are, in addition, some other less common functions of the angle θ. These are defined as follows.

$$\frac{OP}{PA} = \frac{1}{\sin\theta} = \text{cosecant } \theta \text{ (abbreviated cosec } \theta\text{)}$$

$$\frac{OP}{OA} = \frac{1}{\cos\theta} = \text{secant } \theta \text{ (abbreviated sec } \theta\text{)}$$

$$\frac{OA}{PA} = \frac{1}{\tan\theta} = \text{cotangent } \theta \text{ (abbreviated cot } \theta\text{)}$$

Quadrants

The complete circle is divided into four equal parts (called *quadrants*) by horizontal and vertical axes that intersect at the center of the circle. These quadrants are numbered 1 to 4, starting with the upper right quadrant and proceeding counterclockwise, as diagrammed in Figure 1-3.

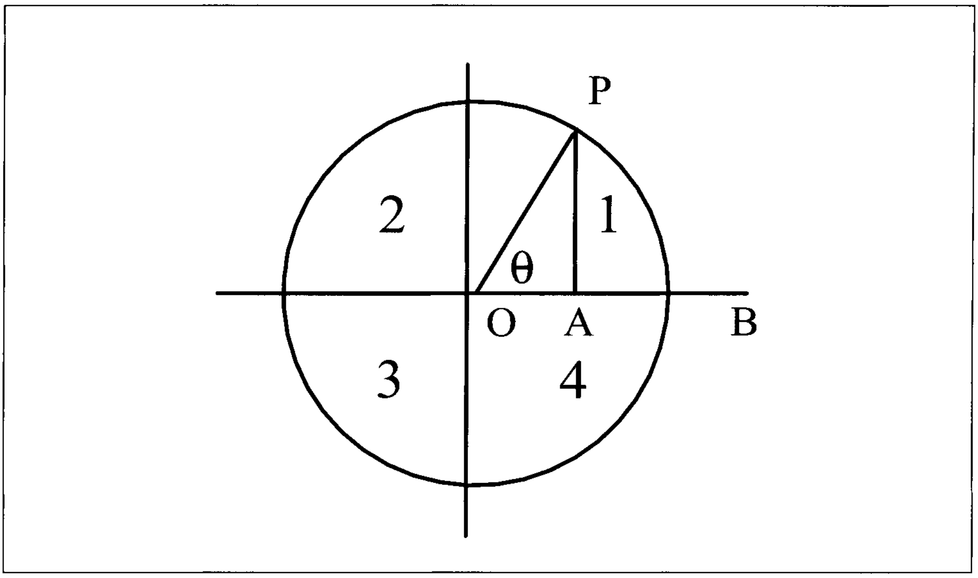

Figure 1-3. The quadrants.

When θ is an acute angle, the radius OP lies in the first quadrant. As the radius rotates counterclockwise and θ increases beyond 90°, OP lies in the second quadrant. For θ between 180° and 270°, OP is in the third quadrant. For θ between 270° and 360°, OP is in the fourth quadrant.

By definition, the measurement of the radius OP is always positive. The measurement OA is defined to be positive when the point A is on the right side of the vertical axis, and negative when A is on the left side of the vertical axis. The measurement PA is defined to be positive when P is above the horizontal axis, and negative when P is below the horizontal axis.

This means that $\sin\theta$, $\cos\theta$, and $\tan\theta$ can have positive or negative values depending on the quadrant in which OP lies, which in turn depends on the magnitude of the angle θ. The following values consequently prevail.

When $\theta = 0°$, PA = 0, and OA = OP. Therefore $\sin\theta = 0$, $\cos\theta = 1$, and $\tan\theta = 0$.

When $\theta = 90°$, PA = OP, and OA = 0. Therefore $\sin\theta = 1$, $\cos\theta = 0$, and $\tan\theta$ becomes infinite.

When $\theta = 180°$, PA = 0, and OA = OP in magnitude, but OA is negative and consequently $OA/OP = -1$. Therefore $\sin\theta = 0$, $\cos\theta = -1$, and $\tan\theta = 0$.

When $\theta = 270°$, PA = OP in magnitude, but PA is negative, so PA/OP = –1. Therefore, $\sin \theta = -1$, $\cos \theta = 0$, and $\tan \theta$ becomes infinite in the negative direction. The following table summarizes these points.

Table 1-1. Sine, Cosine and Tangent Functions

Quadrant	θ	sin θ	cos θ	tan θ
1	0° to 90°	0 to 1	1 to 0	0 to ∞
2	90° to 180°	1 to 0	0 to –1	–∞ to 0
3	180° to 270°	0 to –1	–1 to 0	0 to ∞
4	270° to 360° (= 0°)	–1 to 0	0 to 1	–∞ to 0

The values in the table above show that the sine, cosine, and tangent functions repeat themselves with time and are therefore cyclic. *Periodic* is another term that is sometimes used. The sine and cosine functions both cycle between +1 and –1, while the tangent function cycles between +∞ and –∞.

Therefore, sine and cosine functions are useful in describing the behavior of objects and systems that are cyclic. As an example, an object might be known to cycle between the limits of 0 and 10. Its behavior y could be described using the sine function, as

$$y = 5 \sin \theta + 5 = 5 (\sin \theta + 1).$$

In this relationship, when $\sin \theta = 1$, $y = 10$, and when $\sin \theta = -1$, $y = 0$.

Frequency of Cycling

If the motion of an object is linear, the distance traveled is equal to the average velocity of the object multiplied by the elapsed time. In symbol form,

$$s = v\,t$$

where s is in metres, v is in metres per second, and t is in seconds.

The equivalent relation for rotational motion, as in the case of the radius rotating around the center of its circle, is that the angle swept through is equal to its angular velocity multiplied by the elapsed time, or in symbol form,

$$\theta = \omega t$$

where θ is in radians, ω (the Greek letter often used for angular velocity) is in radians per second, and t is in seconds.

Thus, the function sin ωt rather than sin θ may be used to describe the behavior of objects and systems that cycle on a time basis. These include the motion of the planets, pendulums, masses suspended on springs, and the variation with time of temperature, p, and other plant variables that are being controlled.

It is a rule of mathematics that the argument of a sine or cosine function does not have units, or in mathematical terms, it is dimensionless. Consequently if t, which has units of seconds, is in the argument, then the second factor ω must have units that are the inverse of time to balance off the time units. In the basic form, these units will be radians per second. However, radians per second are not particularly appropriate for many cyclic applications. Cycles per second (Hertz, Hz), which are the units for frequency, would be more practical.

If the radius is rotating continuously around its circle, then one complete revolution from zero back to the starting point constitutes one cycle. Its rotational speed is ω radians per second, while its frequency of rotation will be f cycles per second, or Hz.

However, it takes 2π radians to fill out one complete circle, or one cycle. Therefore,

$$f = \frac{\omega}{2\pi} \text{ or } \omega = 2\pi f.$$

Therefore, it is possible to write sin θ = sin ωt = sin 2π f t. In this way, the frequency of the cyclic behavior is identified within the sine function. Functions involving cyclic behavior with time will invariably have a sin ωt or sin ($2\pi f$ t) term or a cosine term with similar arguments.

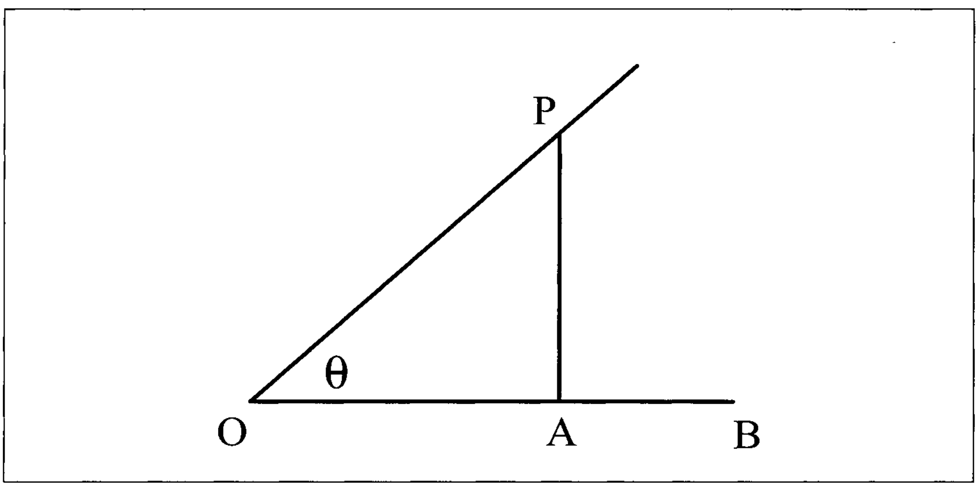

Figure 1-4. Trigonometric function relationships.

Interrelationships

If any one of the trigonometric functions (the sine, cosine, or tangent) is known, then the angle is known. It follows that if any one of the functions is given, the others can be found from it. Consequently, relationships must exist between the various trigonometric functions.

In Figure 1-4,

1. $$\tan\theta = \frac{PA}{OA} = \frac{\frac{PA}{OP}}{\frac{OA}{OP}} = \frac{\sin\theta}{\cos\theta}$$

2. Since OAP is a right angle triangle, it follows that,

$$(OP)^2 = (PA)^2 + (OA)^2.$$

Therefore,

$$\frac{(PA)^2}{(OP)^2} + \frac{(OA)^2}{(OP)^2} = 1 \text{ and } \left(\frac{PA}{OP}\right)^2 + \left(\frac{OA}{OP}\right)^2 = 1$$

Thus, $\sin^2\theta + \cos^2\theta = 1$.

One relationship that is *not* valid is that the sine of the sum of two angles is equal to the sum of the sines of the individual angles. In other words, if the angles are called X and Y, then sin (X + Y) is *not* equal to sin X + sin Y. The same is true for the cosine and tangent functions.

Sine of the Sum of Two Angles

In Figure 1-5, the angle POA is the sum of an angle X and another angle Y. PC is a perpendicular from point P to the common side OC of the angles X and Y. CD is a perpendicular from point C to the side PA at D. CE is the perpendicular from point C to the extension of OA at point E.

From Figure 1-5, $\sin(X + Y) = \sin(\text{angle } POA) = \frac{PA}{OP}$.

Developing this further, in triangles OFA and PFC,

Angle OFA = angle PFC,

Angle OAF = angle FCP = 90°.

Therefore, angle FPC = angle FOA = angle X.

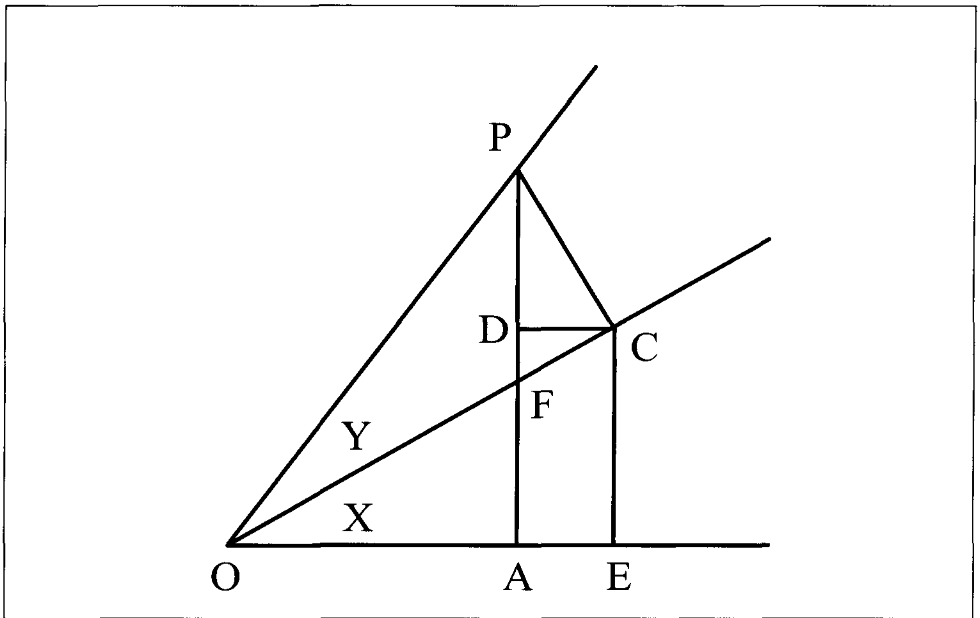

Figure 1-5. Sum of two angles.

$$\frac{PA}{OP} = \frac{PD + DA}{OP} = \frac{CE + PD}{OP} = \frac{CE}{OP} + \frac{PD}{OP} = \left[\frac{CE}{OP} \times \frac{OC}{OC}\right] + \left[\frac{PD}{OP} \times \frac{PC}{PC}\right]$$

$$= \left[\frac{CE}{OC} \times \frac{OC}{OP}\right] + \left[\frac{PD}{PC} \times \frac{PC}{OP}\right]$$

Consequently, $\frac{PD}{PC} = \cos X$. In addition,

$$\frac{PC}{OP} = \sin Y, \frac{CE}{OC} = \sin X, \text{ and } \frac{OC}{OP} = \cos Y.$$

Therefore,

$$\sin(X + Y) = \frac{PA}{OP} = \left[\frac{CE}{OC} \times \frac{OC}{OP}\right] + \left[\frac{PD}{PC} \times \frac{PC}{OP}\right] = \sin X \cos Y + \cos X \sin Y.$$

Cosine of a Sum

In Figure 1-5, cos (X + Y) = cos (angle POA) = $\frac{OA}{OP}$.

$$\frac{OA}{OP} = \frac{OE - AE}{OP} = \frac{OE}{OP} - \frac{DC}{OP} = \left[\frac{OE}{OP} \times \frac{OC}{OC}\right] - \left[\frac{DC}{OP} \times \frac{PC}{PC}\right]$$

$$= \left[\frac{OE}{OC} \times \frac{OC}{OP}\right] - \left[\frac{DC}{PC} \times \frac{PC}{OP}\right] = \cos X \cos Y - \sin X \sin Y$$

Sine of a Difference Between Two Angles

The sine of the difference between two angles can be determined in a similar manner.

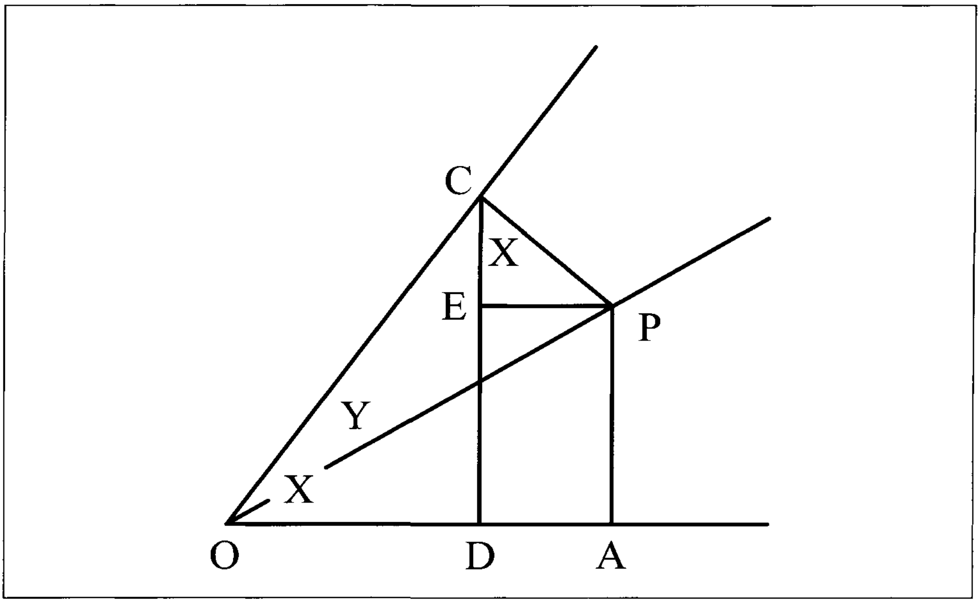

Figure 1-6. Difference between two angles.

In Figure 1-6, the angle POA is the difference between an angle X (COA) and another angle Y (COP). PC is the perpendicular from P to the common side OC of angles X and Y. CD is the perpendicular from C to OA. PE is the perpendicular from P to CD.

Examining this further,

$$\frac{PA}{PO} = \frac{ED}{PO} = \frac{CD - CE}{PO} = \frac{CD}{PO} - \frac{CE}{PO}$$

$$\left[\frac{CD}{PO} \times \frac{OC}{OC}\right] - \left[\frac{CE}{PO} \times \frac{CP}{CP}\right] = \left[\frac{CD}{OC} \times \frac{OC}{PO}\right] - \left[\frac{CE}{CP} \times \frac{CP}{PO}\right]$$

which is equal to $\sin X \cos Y - \left[\frac{CE}{CP} \times \sin Y\right]$.

Triangle ODC is a right angle triangle, therefore Angle COD + angle OCD = 90^{O} = Angle OCD + angle ECP, from which Angle ECP = angle COD = X.

$\frac{CE}{CP}$ = cos (angle ECP) = cos X, and $\sin(X - Y) = \sin X \cos Y - \cos X \sin Y$.

Cosine of a Difference

In Figure 1-6, $\cos(X - Y) = \cos(\text{angle POA}) = \dfrac{OA}{OP}$.

$$\frac{OA}{OP} = \frac{OD + DA}{OP} = \frac{OD + EP}{OP} = \frac{OD}{OP} + \frac{EP}{OP} = \left[\frac{OD}{OP} \times \frac{OC}{OC}\right] + \left[\frac{EP}{OP} \times \frac{CP}{CP}\right]$$

$$= \left[\frac{OD}{OC} \times \frac{OC}{OP}\right] + \left[\frac{EP}{CP} \times \frac{CP}{OP}\right] = \cos X \cos Y + \sin X \sin Y$$

Potentially Useful Relationships

The following three relationships often figure in the resolution of problems and, as such, are worth remembering.

1. $\sin^2 X + \cos^2 X = 1$ (developed previously).
2. $\sin 2X = \sin (X + X) = \sin X \cos X + \cos X \sin X = 2 \sin X \cos X$.
   A useful variation of this is

   $$\sin X = 2 \sin \frac{X}{2} \cos \frac{X}{2}.$$

3. $\cos 2X = \cos (X + X) = \cos X \cos X - \sin X \sin X = \cos^2 X - \sin^2 X$.

Example 1: Tan (X + Y)

Evaluate $\tan (X + Y)$.

$$\tan (X + Y) = \frac{\sin (X + Y)}{\cos (X + Y)} = \frac{\sin X \cos Y + \cos X \sin Y}{\cos X \cos Y - \sin X \sin Y}$$

$$= \frac{\dfrac{\sin X \cos Y}{\cos X \cos Y} + \dfrac{\cos X \sin Y}{\cos X \cos Y}}{\dfrac{\cos X \cos Y}{\cos X \cos Y} - \dfrac{\sin X \sin Y}{\cos X \cos Y}} = \frac{\tan X + \tan Y}{1 - \tan X \tan Y}$$

Example 2: sin 2X – sin 2Y

Show that $(\sin 2X - \sin 2Y) = 2 \cos (X + Y) \sin (X - Y)$.

Evaluating the right side of the equation:

$$RS = 2[(\cos X \cos Y - \sin X \sin Y)(\sin X \cos Y - \cos X \sin Y)]$$

$$= 2\begin{bmatrix}(\cos X \cos Y)(\sin X \cos Y) - (\cos X \cos Y)(\cos X \sin Y) \\ - (\sin X \sin Y)(\sin X \cos Y) + (\sin X \sin Y)(\cos X \sin Y)\end{bmatrix}$$

$$= 2\begin{bmatrix}(\sin X\cos X)\cos^2 Y - (\sin Y\cos Y)\cos^2 X - (\sin Y\cos Y)\sin^2 X \\ + (\sin X\cos X)\sin^2 Y\end{bmatrix}$$

$$= 2[(\sin X\cos X)(\cos^2 Y + \sin^2 Y) - (\sin Y\cos Y)(\cos^2 X + \sin^2 X)]$$

$$= 2\sin X\cos X - 2\sin Y\cos Y = \sin 2X - \sin 2Y.$$

Example 3: Radius of the Earth

Many people believe that the fact that the earth is round was first promoted by Christopher Columbus, and that on the basis of this knowledge he sailed off westward to discover the new world. In reality, the credit should go to a Greek scholar, Erathosthenes, who lived in the third century B.C.

As the story goes, about 800 km from Alexandria, Egypt, a shaft had been dug that was exactly plumb. This shaft was used by astronomers as a check on the calendar, since there was just one day out of the entire year in which one could stand at the bottom of the shaft and see the sun. Erathosthenes knew about the shaft.

On the specific day that the sun's rays lit the bottom of the shaft, Erathosthenes was in Alexandria and he noticed that a statue was casting a shadow. This was somewhat surprising because if the surface of the earth were flat, there should be no shadow at all. The only plausible explanation was that the earth's surface, or at least that portion of it between Alexandria and the shaft, must be curved.

Erathosthenes measured the height of the statue and the length of the shadow. Since these two measurements formed the tangent of the angle between the statue and the rays of the sun that fell upon it, he could then calculate the angle, which was 7°.

From Figure 1-7 (out of proportion), it can be seen that the angle formed by the statue and the sun's rays is the same as the angle subtended by the two radii emanating from the center of the arc to the shaft and the statue. This would be dependent on the correctness of the assumption that the sun is so much larger than the earth that its rays are all parallel when they arrive at the earth, which is reasonable enough.

It was then only a matter of applying the relation: a (arc) = r $\times$ θ (radians). The arc is 800 km. θ is 7° or 0.122 radians. From this, r = 6550 km.

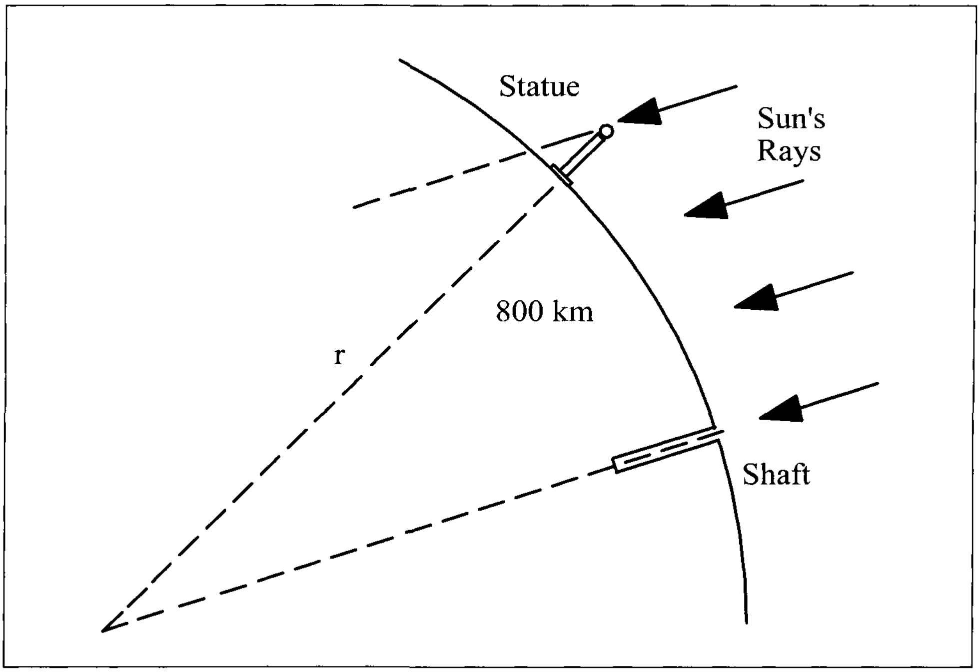

Figure 1-7. Erathosthenes' brilliant deduction.

Using more sophisticated modern techniques, the mean radius of the earth has been computed to be 6371 km, so the error in Erathosthenes calculation was less than 3%. As an afterthought to the story, Columbus likely knew about this gem of knowledge, which was brought to light 17 centuries previously by Erathosthenes. In fact, in the time of Columbus, it is probable that many educated people knew that the earth had to be round, not flat.

Example 4: The Third Side

In the triangle in Figure 1-8, the lengths of two of the sides, a and b, are known, as is the angle θ between them. What is required is to develop an expression involving a, b, and θ, from which the length of the third side, c, can be calculated.

In the triangle, draw a perpendicular from the apex down to the base. This divides the base a into two segments, z_1 and z_2, so that $z_1 + z_2 = a$. The length of the perpendicular is z_3.

$$\frac{z_1}{b} = \cos\theta\text{, so that } z_1 = b\cos\theta\text{. Similarly, } z_3 = b\sin\theta$$

$$z_2 = a - z_1 = a - b\cos\theta$$

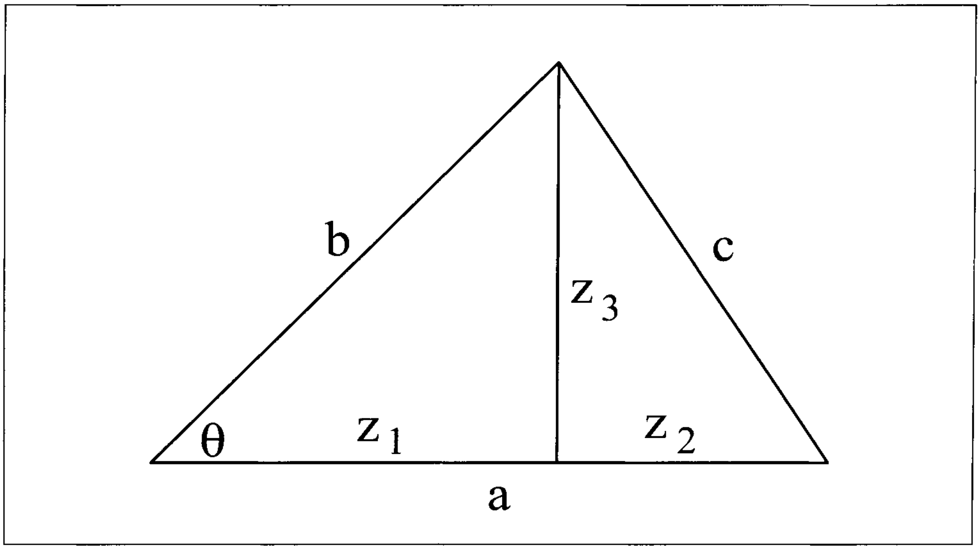

Figure 1-8. Third side problem.

$$c = \sqrt{z_2^2 + z_3^2} = \sqrt{(a - b\cos\theta)^2 + (b\sin\theta)^2}$$

$$= \sqrt{a^2 - 2ab\cos\theta + b^2\cos^2\theta + b^2\sin^2\theta}$$

$$= \sqrt{a^2 - 2ab\cos\theta + b^2(\sin^2\theta + \cos^2\theta)}$$

$$= \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

Example 5: Gain and Phase Lag

This example is potentially useful in designing a computer program that will determine the gain and phase lag in a closed loop control system. In Figure 1-9, OA is a vector whose magnitude is G units and whose orientation with respect to the horizontal axis is the angle a. AB is a vector that is 1 unit in length and has no phase angle. It always lies parallel to the horizontal axis and points in the positive direction.

OB is a vector that is the vector sum of the vector G and the unit vector. The objective is to develop two expressions that a computer can use to calculate the magnitude G_1 and phase angle a_1 of the vector OB, if given the magnitude of G and its phase angle a.

When writing a computer program of this type, it should be kept in mind that computer programs that do mathematics deal with angles in units of

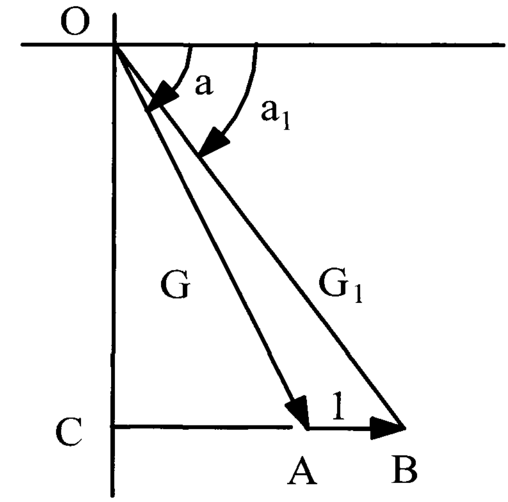

Figure 1-9. Determining closed loop gain and phase lag.

radians, not degrees. Thus, if angle a is specified in degrees, it must be multiplied by the factor 0.0174 5 (π/180).

In the diagram, the angle OAC is equal to angle a.

$$\frac{OC}{G} = \sin a$$

Therefore, $OC = G \sin a$. Similarly, $CA = G \cos a$.

$$CB = CA + 1$$

$$G_1 \text{ magnitude} = \sqrt{(OC)^2 + (CB)^2} = \sqrt{(G \sin a)^2 + (G \cos a + 1)^2}$$

$$a_1 = \text{Angle whose tangent is } \frac{OC}{CB} = \tan^{-1}\left(\frac{G \sin a}{G \cos a + 1}\right)$$

Once again, when the computer calculates the value of the angle a_1, the units of a_1 will be radians not degrees.