6 Differential Equations

Introduction

There is an area of mathematics that deals with equations that contain derivatives of a variable with respect to another variable, or variables. These equations are called differential equations. The following are examples.

(1) $$\frac{1}{k}\frac{dx}{dt} + x = X$$

(2) $$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = Q$$

(3) $$1 + \left(\frac{dx}{dt}\right)^2 = 3\frac{d^3x}{dt^3}$$

(4) $$1 + \left(\frac{d^2x}{dt^2}\right)^2 = 3x + \frac{dx}{dt}$$

(5) $$x^2\frac{dx}{dt} + tx = 10$$

(6) $$\frac{\partial^2 x}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

The solution of a differential equation requires that an equation be obtained that has the variables in their natural form, that is, free of all derivatives. In the case of the first example above, the solution is:

$$x = X\left(1 - e^{-kt}\right)$$

This solution can be verified. If it is actually the original relation between x and t, then taking the derivative with respect to t of both sides gives

$$\frac{dx}{dt} = \frac{d}{dt}(X - Xe^{-kt}) = Xke^{-kt}.$$

Then the left side of the differential equation equals

$$\frac{1}{k}\frac{dx}{dt} + x = \frac{1}{k}Xke^{-kt} + X(1 - e^{-kt}) = Xe^{-kt} + X - Xe^{-kt} = X$$

which is the right side of the equation.

Philosophy

In many cases, a set procedure cannot be established for the solution of a particular differential equation. In fact, many differential equations, principally those that lack a certain degree of symmetry or orderliness, are incapable of solution. Solving differential equations is often as much an art as it is a science. Mathematical intuition and experience are valuable assets.

Solvable differential equations tend to fall into patterns, so that part of the skill required to solve a differential equation lies in being able to spot the pattern and in knowing the right procedure for dealing with it.

As an example, the motion of a mass suspended from a spring and caused to bounce up and down can be described by the differential equation in Example (2). Observing the motion of the mass reveals, first of all, that the solution must contain a cyclic factor such as sin ωt, which describes the up and down movement, and another factor e^{-at}, which describes the gradual dying out of the oscillations over a period of time. In dealing with this differential equation, it is helpful to know at the outset that the solution must look something like $y = Ce^{-at} \sin \omega t$.

Definitions

An *ordinary* differential equation is one with just one independent variable. The independent variable is nearly always the variable that appears in the denominator of the derivative term. Only total derivatives are present; there are no partial derivatives.

Examples 1 through 5 are ordinary differential equations, while Example 6 is a partial differential equation with two independent variables.

The *order* of a differential equation is the order (the number of times the derivative has been taken) of the highest order derivative in the equation.

Examples 1 and 5 are consequently differential equations of the first order. Examples 2, 4, and 6 are of the second order. Example 3 is of the third order.

The *degree* of the differential equation is the degree (power) to which the highest order derivative in the equation has been raised. Note that the degree of the equation is not necessarily established by the highest power term that appears in the equation.

Examples 1, 2, 5, and 6 are differential equations of the first degree. Example 3 is also a first degree equation because, although one of the derivative terms is raised to the second power, this derivative is not the highest order derivative in the equation. The highest order derivative,

$$\frac{d^3t}{dt^3}$$

is raised to the first power only. Example 4, however, is a second degree equation.

A differential equation is *linear* if the coefficients multiplying the various derivative terms are constants, or at worst, functions of the independent variable. In addition, the power of each of the derivative terms can be no higher than one. This means that a linear differential equation must be of the first degree.

Examples 1, 2, and 6 are all linear differential equations. Examples 3 and 4 are nonlinear because of the derivative terms which are squared. Example 5 is nonlinear because of the factor x^2 (a function of the dependent variable), which multiplies the first term.

A linear differential equation has a general form, which is

$$f_n(t)\frac{d^n x}{dt^n} + f_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + f_{n-2}(t)\frac{d^{n-2}x}{dt^{n-2}} + \ldots$$

$$\ldots + f_2(t)\frac{d^2x}{dt^2} + f_1(t)\frac{dx}{dt} + f_0(t)x = f(t).$$

Application

Control systems engineering pertains to the study of the dynamics of the components of control systems, including both the process being controlled and the items of control hardware, for the purpose of obtaining satisfactory overall dynamics when these components are combined into a control system. A control system that has good dynamic behavior will recover quickly from upsets and will generally result in close automatic control.

Dynamics is a descriptive term, which characterizes the reaction of control system components, or complete systems, to impulses that vary with time. The dynamic behavior of many control system components and systems can be described by differential equations in which the independent variable is time. A knowledge of how to resolve these differential equations is consequently valuable in control systems engineering analysis.

Fortunately, the less complicated types of differential equations are frequently the ones that are involved with control system analysis. The most commonly encountered equation is likely the linear differential equation with constant coefficients. In its most general form, this equation would appear as

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}x}{dt^{n-2}} + ... + a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = f(t).$$

In this equation the a's are constants.

Differential Equations of the First Order and First Degree

The general form of this type of differential equation is

$$P\frac{dx}{dt} + Q = 0$$

where P and Q are both functions of t and x.

Consequently, this type of equation, in its general form, is not necessarily linear.

It is not possible to solve the general form of this equation and arrive at a formula that will give an automatic answer for any particular problem. When the problem is specifically known, however, the solution (assuming this is possible) will usually be obtainable because the equation belongs in one of four categories.

Category 1: Exact Differentials

This means that the expression

$$P\frac{dx}{dt} + Q$$

is actually the derivative of some other expression R, where R is a function of t, x, or both. In other words,

$$P\frac{dx}{dt} + Q = \frac{dR}{dt} = 0.$$

The solution is R = C, where C is a constant.

This method of arriving at the solution to the differential equation is not particularly useful except for full time mathematicians who have the experience to spot the exact differential. Furthermore, it turns out that equations that are exact differentials can also be solved by other means, whether or not it is recognized that they actually are exact differentials.

Category 2: Variables Separable

It may be possible, through a rearrangement of the terms, to get all of the x (dependent variable) terms on the left side of the equals sign and all of the t (independent variable) terms on the right side. The variables are then separated, and the problem is reduced to integrating the expressions on either side of the equation.

Accordingly, a logical starting point in the solution of a first degree, first order, differential equation, is to determine if it is possible to separate the variables.

The first example in the set of examples on page 73 describes the output behavior of a control system component whose reaction, designated by x, goes from x = 0 to x = X as a result of an impulse which has been applied to it. The output change from 0 to X is not instantaneous, however. The rate of change of x at any time t is proportional to the difference between the value of x at that instant and its ultimate value X. Since as time goes on the difference (X – x) is diminishing, the rate of change of x will fall off accordingly. Components with this type of dynamic behavior are quite common in control systems and are generally referred to as *time constants*.

This differential equation can be rearranged as $\frac{dx}{dt} = k(X - x)$.

In this equation the variables can be separated.

$$\frac{dx}{X - x} = k\,dt, \text{ so that } \int\frac{1}{X - x}dx = -\int\frac{1}{X - x}d(X - x) = k\int dt.$$

Integrating both sides yields,

$$-\log_e(X - x) = kt + \log_e C$$

The constant of integration is required here. Since C is a constant, then $\log_e$ C will also be a constant. Inasmuch as there is already a logarithm in the equation, the log form of the constant will make it easier to manipulate the equation into its final form.

$$\log_e C + \log_e(X - x) = -kt = \log_e\{C \times (X - x)\}$$

$$C(X - x) = e^{-kt} \text{ and } x = X - \frac{1}{C}e^{-kt}.$$

If x = 0 when t = 0, then $0 = X - \frac{1}{C}$ and $\frac{1}{C} = X$. Therefore,

$x = X - Xe^{-kt} = X(1 - e^{-kt})$.

Category 3: Homogeneous Equations

When a differential equation has the form

$$\frac{dx}{dt} = f\left(\frac{x}{t}\right),$$

it is termed a *homogeneous* differential equation, for reasons that are presumably obvious to qualified mathematicians, but not to the average math student.

The test for a homogeneous equation is to substitute the product vt for x in the right side of the equation. Since vt = x, then the new variable v = x/t. If the right side expression is actually a function of x/t, then after substituting vt = x in this expression, the t's will cancel out, leaving an expression containing v only.

For example, given the differential equation $2t^2\frac{dx}{dt} - x^2 = t^2$, which in rearranged form is

$$\frac{dx}{dt} = \frac{t^2 + x^2}{2t^2}.$$

In this equation, the variables cannot be separated. However, replacing x with vt in the expression on the right side gives

$$\frac{dx}{dt} = \frac{t^2 + v^2t^2}{2t^2} = \frac{1 + v^2}{2}.$$

Since the t terms cancel out completely, leaving only the v terms, the equation is homogeneous.

The substitution vt = x, which is used as the test for homogeneity, is also worth trying as a solution for the differential equation. If x is made equal to vt, then,

$$\frac{dx}{dt} = \frac{d}{dt}(vt) = v + t\frac{dv}{dt}.$$

Substituting vt for x in both sides of the equation accordingly gives

$$v + t\frac{dv}{dt} = \frac{1 + v^2}{2},$$

in which the variables can be separated.

$$t\frac{dv}{dt} = \frac{1 + v^2}{2} - v = \frac{1 + v^2 - 2v}{2} = \frac{(1 - v)^2}{2}, \text{ so that}$$

$$\int \frac{dt}{t} = 2\int \frac{1}{(1 - v)^2}dv = -2\int \frac{1}{(1 - v)^2}d(1 - v) = -2\int (1 - v)^{-2}d(1 - v).$$

Performing the integration, with C as the constant of integration,

$$\log_e t = -2\left(\frac{-1}{1 - v}\right) + \log_e C, \text{ so } \log_e \frac{t}{C} = \frac{2}{1 - v}$$

$$1 - v = \frac{2}{\log_e \frac{t}{C}} \text{ and } v = 1 - \frac{2}{\log_e \frac{t}{C}} = \frac{x}{t}$$

$$\text{Therefore, } x = t\left(1 - \frac{2}{\log_e \frac{t}{C}}\right).$$

Category 4: Linear Differential Equations

A differential equation of the first order and first degree, which is linear in addition, would have the general form

$$\frac{dx}{dt} + Px = Q$$

where P and Q are constants or functions of t but *not* of x. In this case, it is possible to obtain a general solution by the following method.

It was shown previously that a differential equation to be solved will sometimes turn out to be an exact differential. To obtain a solution for the general equation

$$\frac{dx}{dt} + Px = Q,$$

the approach is to find a factor R, which is a function of t (only), such that when each of the terms of the equation is multiplied by R, the left side of the equation becomes an exact differential. Accordingly,

$$R\frac{dx}{dt} + RPx = RQ, \text{ and } R\frac{dx}{dt} + RPx$$

is to be an exact differential. Notice the similarity between

$$\left\{R\frac{dx}{dt} + RPx\right\} \text{ and } \left\{R\frac{dx}{dt} + x\frac{dR}{dt}\right\}, \text{ which happens to be } \frac{d}{dt}(Rx).$$

This suggests that the exact differential required is $\frac{d}{dt}(Rx)$, provided that $\frac{dR}{dt}$ is equal to RP.

$$\text{If } \frac{dR}{dt} = RP, \text{ then } \int\frac{dR}{R} = \int P\,dt, \text{ and } \log_e R = \int P\,dt.$$

$$\text{Therefore, } R = e^{\int P\,dt}.$$

The complete solution is $\frac{d}{dt}(Rx) = RQ$,

$$\int d(Rx) = \int RQdt$$

$$Rx = \int RQdt$$

$$\text{and } x = \frac{1}{R}\int RQdt, \text{ where } R = e^{\int Pdt}.$$

Example 1: Time Constant

The differential equation that has already been examined is that in which the dependent variable y, as a result of a step change impulse, is changing from its starting point y = 0 so as to eventually attain a new value Y. However, the rate of change of y is proportional to (Y – y), so that it is constantly diminishing as y approaches Y. The differential equation is

$$\frac{dy}{dt} = k(Y - y)$$

where k is the proportional constant. Rearranging this,

$$\frac{dy}{dt} + ky = kY.$$

Thus, in this relation, P = k, and Q = kY.

$$\int P\,dt = \int k\,dt = kt, \text{ so } R = e^{kt}.$$

Therefore, applying the formula,

$$y = \frac{1}{e^{kt}}\int e^{kt}kYdt = \frac{kY}{e^{kt}}\int e^{kt}dt = \frac{kY}{e^{kt}}\left(\frac{1}{k}e^{kt} + C\right) = Y\left(1 + \frac{kC}{e^{kt}}\right)$$

where C is the constant of integration.

Since it is known that y = 0 when t = 0, the value of C can be found by substituting these values in the equation for y.

$$0 = Y\left(1 + \frac{kC}{1}\right)$$

from which $C = -\frac{1}{k}$.

Therefore, the final solution is

$$y = Y\left[1 + k\left(\frac{-1}{k}\right)\left(\frac{1}{e^{kt}}\right)\right] \text{ or } y = Y(1 - e^{-kt}).$$

Linear Differential Equations with Constant Coefficients

A first order equation of this type would have the form

$$\frac{dx}{dt} + a_0 x = f(t) \quad (a_0 \text{ is a constant}).$$

The situation in which $f(t) = 0$ will be considered first.

The equation $\frac{dx}{dt} + a_0 x = 0$ can be solved by separating the variables.

$$\frac{dx}{dt} = -a_0 x, \text{ thus } \frac{dx}{x} = -a_0 dt, \text{ and } \int \frac{dx}{x} = -a_0 \int dt$$

$$\log_e x = -a_0 t + \log_e C \quad (\text{where C is the constant of integration})$$

$$\log_e \frac{x}{C} = -a_0 t$$

$$\frac{x}{C} = e^{-a_0 t} \text{ and } x = Ce^{-a_0 t}$$

Second Order Linear Differential Equation with Constant Coefficients

A second order linear differential equation with constant coefficients would have the form

$$a_2 \frac{d_2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t).$$

This differential equation actually describes systems in the real world that can oscillate. In closed loop control systems, part of the output of the process is fed back to the input of the system as a measurement signal. This sets up the conditions that are conducive to oscillation.

In control systems studies, however, it is generally considered that oscillations in the system are triggered by a single transient input at time zero. Everything that happens from then on depends on the nature of the system itself, not on any further external influences. The net result is that in the differential equation, f(t) can be considered to be zero.

Furthermore, if f(t) = 0, then all of the terms on the left side can be divided by the constant a_2, which reduces the number of constants from 3 to 2. The two new constants will be A (replacing a_1/a_2) and B, (replacing a_0/a_2).

A solution will consequently be sought for

$$\frac{d^2x}{dt^2} + A\frac{dx}{dt} + Bx = 0.$$

At this point the procedure becomes less orderly and more abstract, in that the will of the wisp of mathematical intuition gets involved. In fact, it frequently turns out that the solution to a particular differential equation is found because someone with the right mathematical background is able to guess at the answer, and then verify that he or she was right. In this case, it was already shown that the solution for the first order equation with f(t) = 0 was

$$x = Ce^{-a_0 t}.$$

This suggests that $x = Ce^{mt}$, with the value of m to be determined, may be a solution for the second order equation as well.

If this trial solution is valid, then

$$x = Ce^{mt}, \ \frac{dx}{dt} = mCe^{mt}, \text{ and } \frac{d^2x}{dt^2} = m^2Ce^{mt}.$$

Inserting these values in the original equation yields

$$m^2Ce^{mt} + AmCe^{mt} + BCe^{mt} = 0, \text{ which reduces to}$$

$$m^2 + Am + B = 0.$$

This means that the trial solution could be a solution, provided that it is possible to find a value for m that satisfies the algebraic equation $m^2 + Am + B = 0$. As it turns out, there are actually two values of m (to be designated m_1 and m_2), which will satisfy this requirement. These are

$$m_1 = \frac{-A + \sqrt{A^2 - 4B}}{2}, \text{ and } m_2 = \frac{-A - \sqrt{A^2 - 4B}}{2}.$$

A complete solution for

$$\frac{d^2x}{dt^2} + A\frac{dx}{dt} + Bx = 0$$

will then be $x = C_1e^{m_1t} + C_2e^{m_2t}$.

Realistically, this should be verified.

$$x = C_1e^{m_1t} + C_2e^{m_2t}, \frac{dx}{dt} = m_1C_1e^{m_1t} + m_2C_2e^{m_2t}, \text{and}$$

$$\frac{d^2x}{dt^2} = m_1^2C_1e^{m_1t} + m_2^2C_2e^{m_2t}$$

Inserting these expressions in the original equation results in

$$\left\{m_1^{\;2}\,C_1\,e^{m_1t} + m_2^{\;2}\,C_2\,e^{m_2t}\right\} + A\left\{m_1\,C_1\,e^{m_1t} + m_2\,C_2\,e^{m_2t}\right\}$$

$$+ B\left\{C_1\,e^{m_1t} + C_2\,e^{m_2t}\right\},$$

which is equal to

$$C_1\,e^{m_1t}\left(m_1^{\;2} + A_1\,m_1 + B\right) + C_2\,e^{m_2t}\left(m_2^{\;2} + A\,m_2 + B\right)$$

$$= \left(C_1\,e^{m_1t} \times 0\right) + \left(C_2\,e^{m_2t} \times 0\right)$$

= zero = right side of the equation.

Inasmuch as the solution of a second order differential equation will involve two integrations, the general solution should contain two arbitrary constants.

The solution $x = C_1e^{m_1t} + C_2e^{m_2t}$ satisfies this requirement. It can therefore be considered to be the general solution.

The equation $m^2 + A\,m + B = 0$, from which the values of m_1 and m_2 are determined, is called the *auxiliary equation*. The dynamic behavior that is described by the differential equation depends on what the values of m_1 and m_2 turn out to be. As has been shown, these values are

$$m_1 = \frac{-A + \sqrt{A^2 - 4B}}{2} \text{ and } m_2 = \frac{-A - \sqrt{A^2 - 4B}}{2}.$$

If A^2 is greater than 4B, then m_1 and m_2 will be real numbers, although either, or both, could be negative. In any event, the solution would be the sum of two expressions in t, which are changing exponentially with time. The absence of a sine or cosine function is an indication that the system does not oscillate.

There is also the possibility that $A^2 = 4B$, in which case m_1 and m_2 are equal. The solution then is

$$x = (C_1 + C_2)e^{mt}$$

where m is the common value of m_1 and m_2. Here again the system behaves in exponential fashion and does not oscillate. Consequently, neither of these solutions is of great interest to students of automatic control systems.

The Oscillatory Case

The final possibility is the situation in which A^2 is less than 4B. In this case, the square root of a negative number is involved, and both roots of the auxiliary equation are complex numbers, namely,

$$m_1 = \frac{-A + j\sqrt{4B - A^2}}{2} \text{ and } m_2 = \frac{-A - j\sqrt{4B - A^2}}{2}$$

with $j = \sqrt{-1}$.

In addition to being complex, the roots m_1 and m_2 occur in conjugate pairs; that is, they are of the form $m_1 = \alpha + j\omega$ and $m_2 = \alpha - j\omega$, where

$$\alpha = \frac{-A}{2}, \text{ and } \omega = \frac{\sqrt{4B - A^2}}{2}.$$

It will be easier to work with $m_1 = \alpha + j\omega$ and $m_2 = \alpha - j\omega$ until the final answer is reached, and then replace α and ω with the original factors A and B.

Proceeding with the solution:

$$x = C_1 e^{m_1 t} + C_2 e^{m_2 t} = C_1 e^{(\alpha + j\omega)t} + C_2 e^{(\alpha - j\omega)t}$$

$$= C_1 e^{\alpha t} e^{j\omega t} + C_2 e^{\alpha t} e^{-j\omega t} = e^{\alpha t}\left(C_1 e^{j\omega t} + C_2 e^{-j\omega t}\right).$$

In general, $e^{j\omega t} = \cos \omega t + j \sin \omega t$, and $e^{-j\omega t} = \cos \omega t - j \sin \omega t$. Using these relationships,

$$x = e^{\alpha t}\left[C_1(\cos\omega t + j\sin\omega t) + C_2(\cos\omega t - j\sin\omega t)\right]$$

$$= e^{\alpha t}\left[(C_1 + C_2)\cos\omega t + j(C_1 - C_2)\sin\omega t\right].$$

The Constant of Integration

The solution for a differential equation sooner or later requires the integration of some expression. For the integration to produce a general result, the result has to include a constant of integration. This is because mathematical expressions that differ only by a constant will all have the same derivative. The appropriate value for the constant (or constants, as in this case) is usually determined in the final step by applying initial conditions.

The fact that the integral must include the constant does not mean, however, that the constant has to be an ordinary number. Any form of the constant is valid, provided that the form itself is basically a constant. This simply means that if C is a constant, then so are $(-C)$, C^2, the square root of C, e^C, $\log_e C$, $\sin C$, and so on. It is also possible for C to be a complex number.

One edge that accomplished mathematicians have is the ability to visualize the format that the constant should take, so that the final solution of the differential equation will be in its most useful form.

In the problem at hand, it would be desirable to have the expression inside of the box brackets in the form $(\sin \phi \cos \omega t + \cos \phi \sin \omega t)$, since this is equal to $\sin(\omega t + \phi)$. This would be possible if the constants C_1 and C_2 were replaced in the solution by two new parameters X and ϕ, such that $C_1 + C_2 = X \sin \phi$, and $j(C_1 - C_2) = X \cos \phi$.

What now needs to be shown is that given the expressions $(C_1 + C_2)$ and $j(C_1 - C_2)$ above, C_1 and C_2 each has its own value, separate from the other. This can be verified if $\sin \phi$ and $\cos \phi$ are converted to their exponential form.

$$X \sin\phi = X\left(\frac{e^{j\phi} - e^{-j\phi}}{2j}\right) = C_1 + C_2$$

$$X \cos\phi = X\left(\frac{e^{j\phi} + e^{-j\phi}}{2}\right) = j(C_1 - C_2)$$

$$\text{From (1)} \quad Xe^{j\phi} - Xe^{-j\phi} = 2jC_1 + 2jC_2$$

$$\text{From (2)} \quad Xe^{j\phi} + Xe^{-j\phi} = 2jC_1 - 2jC_2$$

Adding these expressions: $2Xe^{j\phi} = 4jC_1$, and $C_1 = \dfrac{Xe^{j\phi}}{2j}$.

Subtracting, $-2Xe^{-j\varphi} = 4jC_2$, and $C_2 = -\dfrac{Xe^{-j\varphi}}{2j}$.

Once it has been verified that C_1 and C_2 have their own values, even though the values may be complex numbers, then the solution for the original differential equation becomes

$$x = e^{\alpha t}\left[(C_1 + C_2)\cos\omega t + j(C_1 - C_2)\sin\omega t\right]$$

$$= Xe^{\alpha t}(\sin\phi \cos\omega t + \cos\phi \sin\omega t)$$

$$= Xe^{\alpha t}\sin(\omega t + \phi).$$

Commentary on the Result

The expression $x = X e^{\alpha t}(\sin \omega t)$ is the oscillatory solution for the differential equation

$$\frac{d^2x}{dt^2} + A\frac{dx}{dt} + Bx = 0.$$

In the real world, x may represent a temperature, pressure, or voltage displacement of an object, or some other variable whose value must be tracked. The sine function in the result says that the value of x will oscillate up and down. The fact that the right side of the original differential equation is zero says that after the original disturbance that starts the oscillations takes place, the system is allowed to oscillate on its own. It is not driven or further disturbed by any external force or influence.

The values of the parameters α and ω are established by the constants A and B in the original equation. Their values are

$$\alpha = -\frac{1}{2}A, \text{ and } \omega = \frac{1}{2}\sqrt{4B - A^2}.$$

ω is the frequency at which the system will oscillate. If time is measured in seconds, the units of ω will be radians per second.

α is the modifier that determines whether the oscillations get bigger or smaller. If α is >1, the oscillations increase. If α is <1, but not zero, the oscillations decrease and eventually die out. This is the situation that is sought in control systems.

If $\alpha = 0$, then the $e^{\alpha t}$ term becomes 1 and the oscillations go on forever. Since α depends only on the constant A, this also implies that A is zero, and the original differential equation has the form

$$\frac{d^2x}{dt^2} + Bx = 0.$$

The X and ϕ terms will be determined by the initial conditions, that is, the conditions that exist at $t = 0$. X is the amplitude of the first oscillation, to be modified subsequently by the value of α.

Finally, ϕ is the phase displacement of the oscillations on the time scale. The value of ϕ establishes the point where the oscillations will be, at maximum, minimum, zero, or whatever, when t is zero.

Example 2: The Spring/Mass System

Figure 6-1 portrays a spring that is fastened at its top end, and with a mass attached to its lower end. Under this arrangement the mass is free to oscillate up and down. However, attached to the mass is a dashpot, which imposes some resistance to the movement of the mass, with the result that the oscillations will eventually die out. The following data are known.

- The mass of the moving mass is m kg.
- The constant of the spring is k Newtons per metre.
- The resistance coefficient of the dashpot is r Newtons per metre per second.
- The elevation of the mass at any instant is y metres with respect to the $y = 0$ base line.

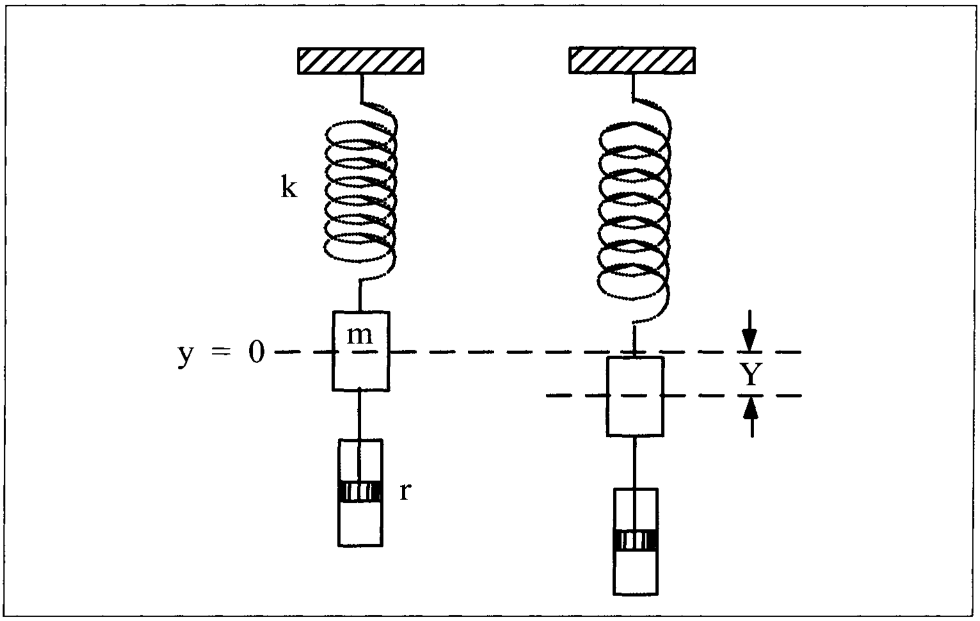

Figure 6-1. The Spring/Mass System.

The requirement is to find an expression of the form y = f(t), which will pinpoint the position of the mass at any time t.

At the outset, it will be specified that elevations above the base line and upward forces are positive, while elevations below the base line and downward forces are negative.

When the mass is at rest, y = 0, there are only two forces affecting it. These are the gravitational force (mg) downward and the upward spring force, which is equal to the spring constant (k) multiplied by the initial stretching of the spring (y_0). Since the system is in equilibrium at this time, these two forces will be equal and opposite. Thus mg = ky_0.

Now consider the moment when the mass is below the base line at distance y but moving upward.

- The gravitational force is downward and equals (–mg).
- The initial (steady state) force exerted by the spring will be upward and equal to (+ k y_0).
- The spring force due to the additional stretch y will be upward and equal to (–ky). At first, it would appear that the negative sign is an error. However, at the selected point in the motion of the mass, y has a negative value, so that the product of y and k would indicate a negative (downward) force, which would be incorrect. The negative

sign in front of the product ky is required to compensate for the negative value of y.

- The force applied by the dashpot is always opposite in direction to the motion. If it were not, there would be no braking action. Since the mass is moving upward, the resistance force will be downward, and equal to

$$-r\frac{dy}{dt}.$$

All of these forces in combination produce the acceleration of the mass. Therefore,

$$m\frac{d^2y}{dt^2} = -mg + ky_0 - ky - r\frac{dy}{dt}.$$

Since mg and ky0 are equal, the differential equation becomes

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + ky = 0.$$

This differential equation describes the motion of a mass suspended from a spring, with damping present. In DC electrical circuitry, there is an equivalent differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0,$$

where L, R, and C are the inductance, resistance, and capacitance of the circuit, respectively, and q is the electrical charge.

If the differential equation is written

$$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{k}{m}y = 0,$$

then referring back to the general solution already worked out,

$$A = \frac{r}{m}, \text{ and } B = \frac{k}{m}.$$

$$\text{Then, } \alpha = -\frac{A}{2} = -\frac{1}{2}\frac{r}{m}, \text{ and}$$

$$\omega = \frac{1}{2}\sqrt{4B - A^2} = \frac{1}{2}\sqrt{4\frac{k}{m} - \frac{r^2}{m^2}} = \sqrt{\frac{4km - r^2}{4m^2}} = \sqrt{\frac{k}{m} - \left(\frac{r}{2m}\right)^2}.$$

The end result is

$$y = Ye^{-\frac{r}{2m}t}\sin\left[\left(\sqrt{\frac{k}{m} - \left(\frac{r}{2m}\right)^2}\right)t + \phi\right].$$

What the solution reveals is that the initiating disturbance would cause a displacement Y of the mass. The following displacement y is then modified over a period of time by the cyclic sine function and the damping exponential function. The ϕ term identifies where the mass is in its cycle when $t = 0$. For the oscillations to begin, the mass must be given an initial displacement downward. If $y = -Y$ when $t = 0$, substituting these values in the expression for y results in

$$-Y = Y \times 1 \times \sin(0 + \phi).$$

Thus, $\sin\phi = -1$ and $\phi = \frac{3\pi}{2}$ (the low point in the oscillation).

Figure 6 – 2 on the following page is a plot of the mass position y with time for

$$\frac{r}{m} = 0.4, \frac{k}{m} = 12.6, Y = 1.5, \text{ and } \phi = \frac{3\pi}{2}.$$

Units

It is advisable to verify that the units of the factors in the expression have turned out to be correct, bearing in mind that the exponent and argument of the exponential and the sine function are required to be dimensionless. In both cases, the exponent and argument are factors multiplied by t, which has units of seconds. Therefore, these factors should have units of frequency, or the inverse of seconds (per second).

The units of mass are kg. (Note: Within the units the character m stands for metres).

The units of k are force per unit displacement or $kg\frac{m}{s^2}\frac{1}{m} = \frac{kg}{s^2}$.

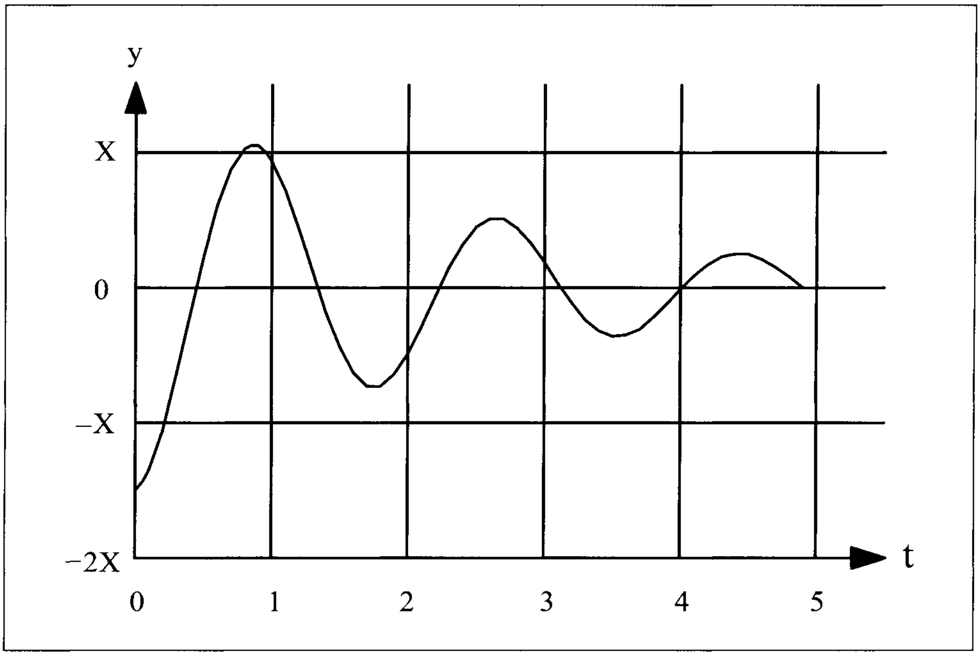

Figure 6-2. Time Displacement of a Mass on a Spring with Damping.

The units of r are force per unit velocity or $kg\frac{m}{s^2}\frac{1}{\frac{m}{s}} = \frac{kg}{s}$.

The exponent of e is $-\frac{r}{2m}$. The units are $\frac{kg}{s}\frac{1}{kg} = \frac{1}{s}$, which is correct.

The units of $\frac{r}{m}$ are also correct for the argument of the sine function.

While $\left(\frac{r}{m}\right)^2$ is involved, it is under the square root sign.

The other expression under the square root sign is $\frac{k}{m}$, which will have units of $\frac{kg}{s^2}\frac{1}{kg}$ or $\frac{1}{s^2}$, which is correct considering the square root function.

A final observation is that if the oscillations of the spring were not damped, then r would be zero. The differential equation would be

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

and the solution would be

$$y = Y\sin\left(\sqrt{\frac{k}{m}} + \phi\right)t.$$

Partial Differential Equations

The differential equations analyzed so far have been of the type that have only one dependent variable and one independent variable (usually time). Equations of this degree of complexity are generally adequate for describing the oscillatory behavior which occurs in control systems.

In the real world, however, there are systems in which a single dependent variable may be influenced by more than one independent variable. These systems have to be described by partial differential equations. They may very well apply in operating plants since, for example, analyzing how heat is transferred often requires the use of partial differential equations.

A prominent phenomenon in physics is wave motion. Wave motion in an outward direction occurs when a stone is dropped into a pond. Sound waves travel outward when a bell is truck. Electromagnetic waves travel outward from a radio antenna.

The magnitude of the wave, which is the dependent variable, depends on the point of observation relative to the source of the wave, the direction having three components, and time. The wave equation, familiar to physicists, has the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

In this equation, u is the dependent variable, that is, the local magnitude of the wave in whatever form it exists; x, y, and z are the positions outward along each of the axes of a 3 dimensional system; c is a constant with units of velocity; and t is time.

A partial derivative, $\frac{\partial u}{\partial t}$ for example, means that when the derivative of u with respect to t is determined, all of the other independent variables involved (in this case x, y, and z) are considered to be constants.

We will likely agree that the solution of the ordinary differential equation, which approximates the behavior of a control system reasonably well, can be sufficiently tedious mathematically without having to resort to the resolution of partial differential equations.