5

Complex Quantities

Background

A quadratic equation involving the variable "x" can be written in its general form and can then be solved using an algebraic procedure. The general form of the quadratic equation in x, is

$$a\,x^2 + b\,x + c = 0.$$

Because of the quadratic (second power) nature of the equation, two values of x will satisfy it. These values of x are called, in mathematical parlance, the "roots of the equation." They can be designated m_1 and m_2, and their values are

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The solution for any quadratic equation can consequently be found by applying this formula for m_1 and m_2. For example, given that

$$x^2 + x - 6 = 0 \text{ (i.e., } a = 1, b = 1, c = -6\text{)},$$

$$\text{then } m_1 = \frac{-1 + \sqrt{1 + 24}}{2} = \frac{-1 + 5}{2} = 2,$$

$$\text{and } m_2 = \frac{-1 - \sqrt{1 + 24}}{2} = \frac{-1 - 5}{2} = -3.$$

Therefore, $x = 2$ and $x = -3$ are the solutions for $x^2 + x - 6 = 0$.

The determination of the roots of a quadratic equation is straightforward, provided that the quantity under the square root sign ($b^2 - 4ac$) does not turn out to be negative, as it would if the equation to be solved were

$$x^2 - 6x + 13 = 0.$$

In this case,

$$m_1 = \frac{+6 + \sqrt{36 - 52}}{2} = \frac{6 + \sqrt{-16}}{2} \text{ and } m_2 = \frac{6 - \sqrt{-16}}{2}.$$

The term $\sqrt{-16}$ can be simplified one step further by taking the factor 4 outside of the square root sign, leaving only the factor (– 1). That is, $\sqrt{-16} = 4\sqrt{-1}$. The roots of the equation then become

$$m_1 = \frac{6 + 4\sqrt{-1}}{2} = 3 + 2\sqrt{-1}, \text{ and } m_2 = \frac{6 - 4\sqrt{-1}}{2} = 3 - 2\sqrt{-1}$$

This introduces the concept of a "number" whose value is $\sqrt{-1}$. This number will be identified by the letter j, that is, $j = \sqrt{-1}$. Since $\sqrt{-1}$ cannot be evaluated, or from another viewpoint, it is not possible to draw a line that is $\sqrt{-1}$ units long, the quantity j must be imaginary.

It is important to understand, however, that as far as mathematics is concerned, stating that a number is *imaginary* is entirely different from stating that there is *no such number.* The quantity j is imaginary because it cannot be observed in nature, but it definitely exists because $x = (3 + 2j)$ and $x = (3 - 2j)$ are values that satisfy the equation $(x^2 - 6x + 13) = 0$. This can be proven by substituting the value $x = (3 + 2j)$ into the original equation, bearing in mind that $j^2 = -1$.

$$x^2 = 9 + 12j + 4j^2 = 9 + 12j - 4 = 5 + 12j$$

$$-6x = -18 - 12j$$

$$+13 = 13$$

$$x^2 - 6x + 13 = 5 + 12j - 18 - 12j + 13 = 0$$

The numbers $(3 + 2j)$ and $(3 - 2j)$, which in this case are the roots of $(x^2 - 6x + 13) = 0$, are called *complex numbers*. The characteristic of a complex number is that it contains an imaginary part, that is, a part that contains the number j. A complex number usually contains a real part as well, as it does in the case of the complex number $(3 + 2j)$. In this instance, the real part is 3 and the imaginary part is 2 j.

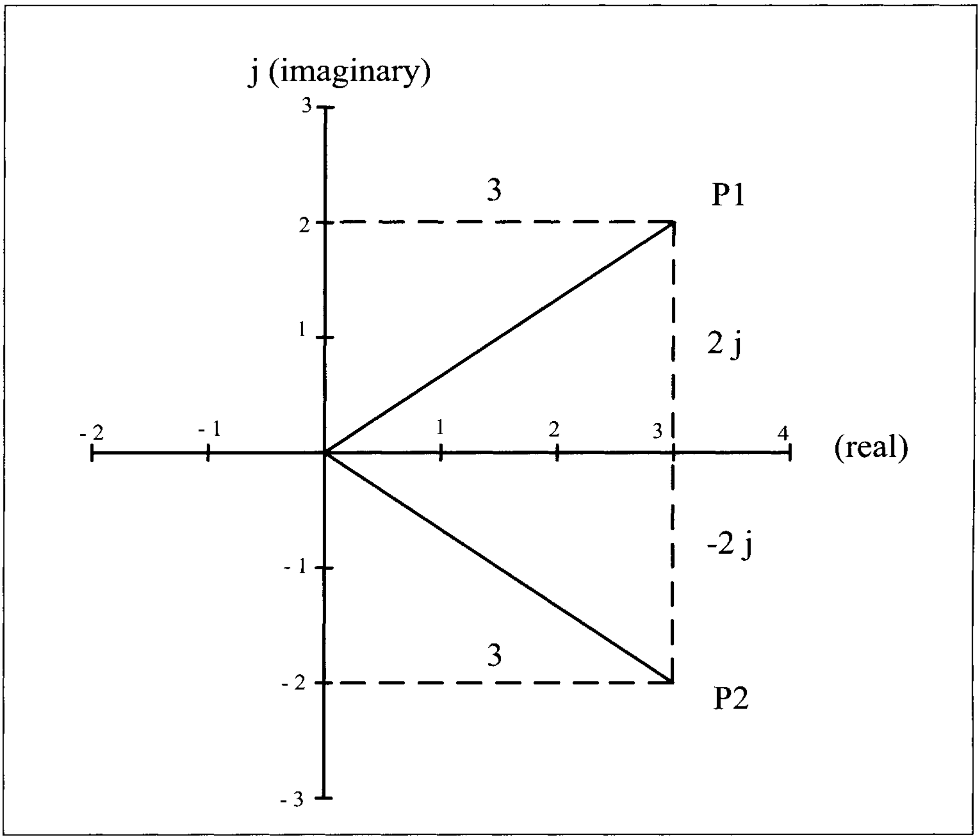

Figure 5-1. Graphical Representation of Complex Numbers.

The real part of a complex number may be zero, leaving only the imaginary part. If the imaginary part were zero, however, only the real part would remain, and the number would be a real number rather than a complex number.

Graphical Representation

A complex number can be represented graphically by plotting it on a complex plane, in which the axis for the real part is horizontal, while the axis for the imaginary part is vertical. In Figure 5-1, the point P1 represents the complex number (3 + 2 j), while P2 represents (3 – 2 j).

When the roots of a quadratic equation with real coefficients are complex numbers, they will always occur in pairs, called *conjugate pairs*. This means that the roots are of the form (a + jb) and (a – jb). In any conjugate pair, the real parts of both numbers are the same, while the imaginary parts differ in sign only.

The Complex Variable

Variables, as well as numbers, can be complex, which means they can have an imaginary part. If z is a complex variable, then it *may* have a real part, but it *will* have an imaginary part. The real and imaginary parts can be designated by x and y, respectively, so that $z = (x + j\,y)$.

Figure 5-2 is the graphical representation of a complex variable in both rectilinear and polar coordinates. In rectilinear coordinates, the x component is shown measured along the horizontal (real) axis, while the y component is measured along the vertical (imaginary or j) axis. The sum of x horizontally and jy vertically, which is z, creates a vector, which starts at the origin O and ends at P.

Since z has not only the quality of length but also the quality of direction, depending on the values of the real and imaginary parts x and y, the vector z is designated $\overline{OP}$. The bar over the letters is the shorthand symbol that indicates that OP is a vector.

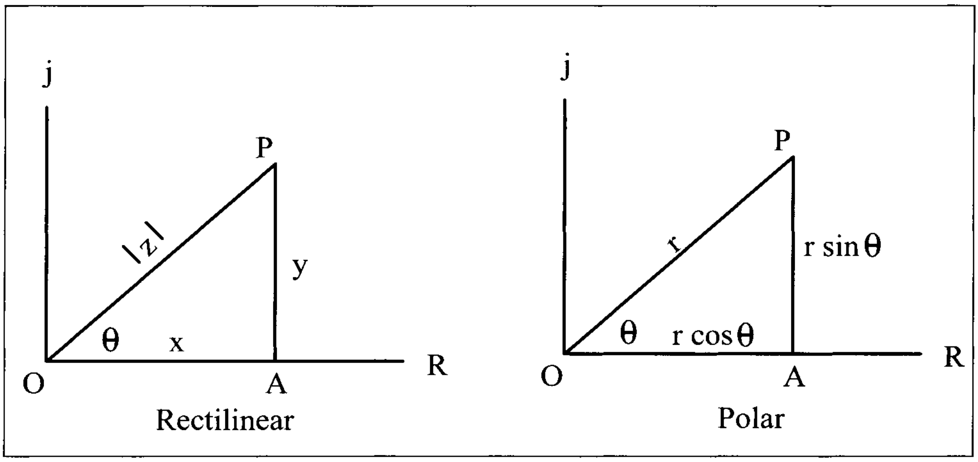

Figure 5-2. z = x + j y in Rectilinear and Polar Coordinates.

The length, or magnitude, of the vector OP is equal to the distance OP and is customarily identified by $|z|$. In mathematics terminology, $|z|$ is called the *modulus* of the complex number z. From the triangle OAP, $|z|^2 = x^2 + y^2$. Therefore,

$$|z| = \sqrt{x^2 + y^2}.$$

The vector OP is vector the sum of $x + j\,y$ in a graphical representation using rectilinear coordinates. Certain problems may be dealt with more conveniently by representing a complex number in a system of polar coordinates. In the polar system, the modulus of z is equal to r. The direction of the vector OP is shown by the angle θ, which is the angle formed by OP and the horizontal or real axis. This angle is defined by the relation

$$\tan\theta = \frac{y}{x}, \text{ or } \theta = \tan^{-1}\frac{y}{x} \left(\text{the angle whose tangent is } \frac{y}{x}\right).$$

The angle θ is called the *argument* of the vector z, or *arg z* for short.

These two diagrams show that $x = r\cos\theta$, and $y = r\sin\theta$.

Trigonometric and Exponential Functions

If the MacLaurin series expansion is used to develop an infinite series expression for $e^{j\theta}$ (bearing in mind that $j^2 = -1$), the result is

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - j\frac{\theta^7}{7!} + \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right).$$

The expression in first set of brackets is the MacLaurin series for $\cos\theta$, while the expression in the second set of brackets is the series for $\sin\theta$. Consequently,

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

By using the same technique, it can be shown that $e^{-j\theta} = \cos\theta - j\sin\theta$.

If these expressions are added, $e^{j\theta} + e^{-j\theta} = 2\cos\theta$, and

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}.$$

If the second expression is subtracted from the first,

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta, \text{ and } \sin\theta = e\frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

These relationships frequently come in handy in the solution of differential equations.

Sum of Two Complex Quantities

If $z_1 = x_1 + j\,y_1$, and $z_2 = x_2 + j\,y_2$, then

$$z_1 + z_2 = x_1 + j\,y_1 + x_2 + j\,y_2 = (x_1 + x_2) + j\,(y_1 + y_2).$$

This indicates that to find the sum of two complex quantities, the real and imaginary parts should be added separately. Figure 5-3 illustrates this.

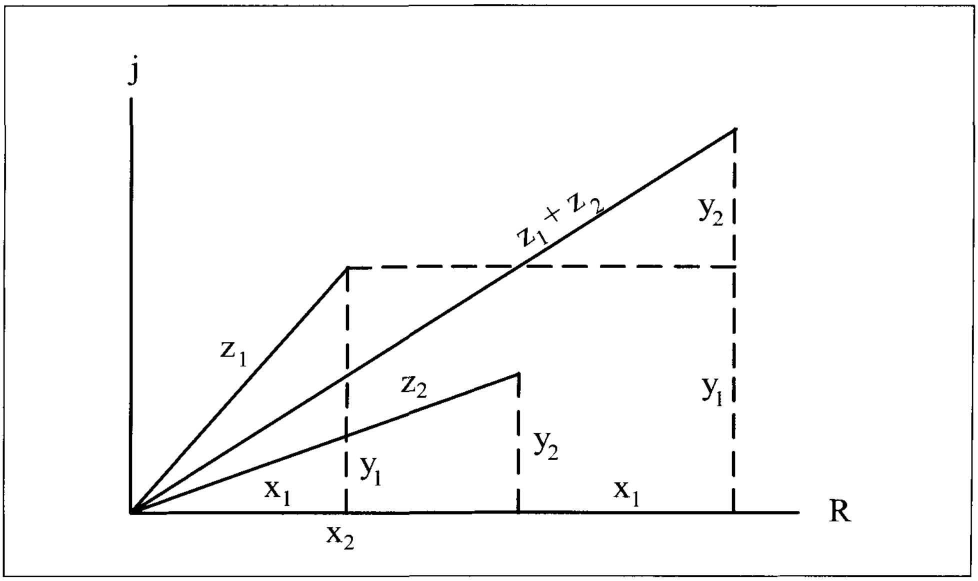

Figure 5-3. Graphical Representation of the Sum of Two Complex Quantities.

Product of Two Complex Quantities

Figure 5-4 is a graphical illustration of the product of two complex quantities.

The product of two complex quantities $z_1 = x_1 + j\,y_1$ and $z_2 = x_2 + j\,y_2$ is most easily determined by converting to the polar form.

If $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then

$$\begin{aligned} z_1 \times z_2 &= r_1 e^{j\theta_1} \times r_2 e^{j\theta_2} \\ &= r_1 r_2 e^{j(\theta_1 + \theta_2)} \\ &= |z_1||z_2| e^{j(\theta_1 + \theta_2)} \end{aligned}$$

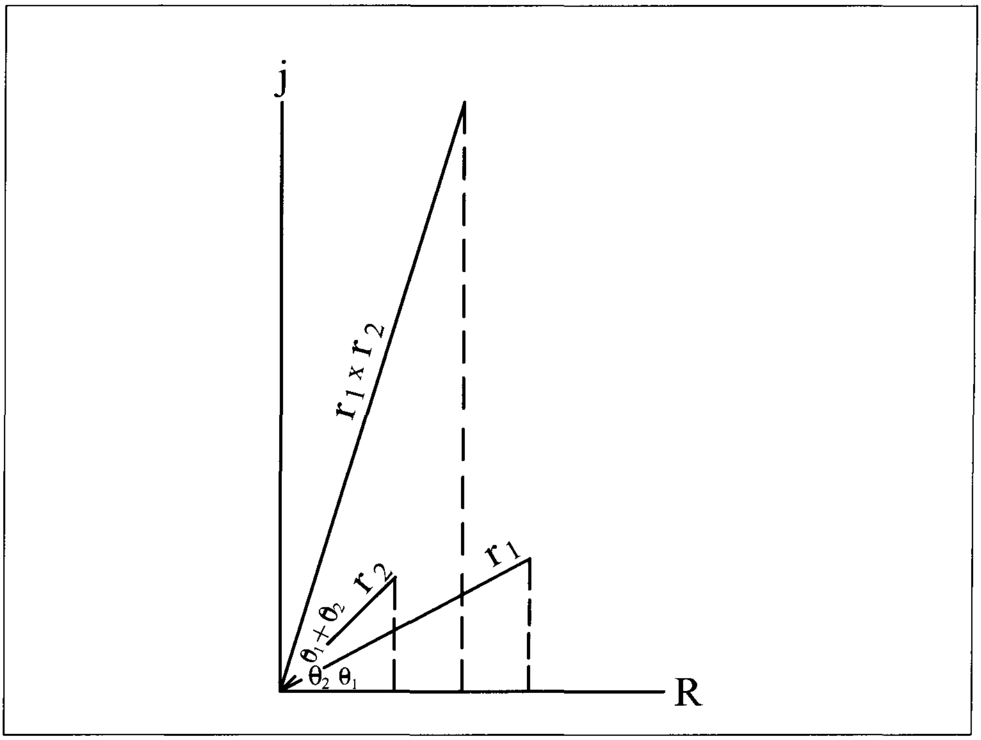

Figure 5-4. Multiplication (z_1 x z_2).

This verifies that the product of two complex quantities is obtained by multiplying the magnitudes and adding the arguments.

Separating the Real and Imaginary Parts

It is often necessary to rearrange the terms of a complex quantity to determine the magnitude and the argument. This is done by collecting the terms that are *not* multiplied by j into one group, and the terms that *are* multiplied by j into another. For example, suppose z = (a + jb) (c + jd). Multiplying this yields

$$z = ac + jad + jbc - bd.$$

The real part is (ac – bd), while the imaginary part is (ad + bc).

The modulus $|z| = \sqrt{(ac-bd)^2 + (ad+bc)^2}$.

The argument $\theta = \tan^{-1}\left(\frac{ad+bc}{ac-bd}\right)$.

Separation of the complex quantity into its real and imaginary parts is straightforward, except in cases where the complex quantity is a fraction. When this happens, the algebraic relation $(a - b)(a + b) = (a^2 - b^2)$ is used.

Notice that if b were an imaginary number, (= jc), the imaginary part would disappear when the product is taken. That is,

$$(a + jc)(a - jc) = (a^2 - j^2 c^2) = (a^2 + c^2).$$

In this way the denominator of the fraction can be cleared of imaginary numbers, and the fraction can be separated in real and imaginary parts using the procedure

$$\frac{x + jy}{a + jb} = \frac{x + jy}{a + jb} \times \frac{a - jb}{a - jb} = \frac{xa - jxb + jya + yb}{a^2 + b^2}$$

$$= \frac{xa + yb + j(ya - xb)}{a^2 + b^2} = \frac{xa + yb}{a^2 + b^2} + j\left(\frac{ya - xb}{a^2 + b^2}\right).$$

Example 1: Magnitude and Argument of a Complex Expression

Determine the modulus and argument of the complex quantity

$$z = \frac{A}{j\omega t + 1}.$$

To clear the denominator of the imaginary number, multiply the numerator and denominator by $(j\omega t - 1)$.

$$z = \frac{A}{j\omega t + 1} \times \frac{j\omega t - 1}{j\omega t - 1} = \frac{Aj\omega t - A}{-\omega^2 t^2 - 1} = \frac{A - Aj\omega t}{\omega^2 t^2 + 1}$$

$$= \frac{A}{\omega^2 t^2 + 1} - j\frac{A\omega t}{\omega^2 t^2 + 1}.$$

$$\text{The modulus } |z| = \sqrt{\frac{A^2}{(\omega^2 t^2 + 1)^2} + \frac{A^2 \omega^2 t^2}{(\omega^2 t^2 + 1)^2}}$$

$$= \frac{A}{\omega^2 t^2 + 1}\sqrt{1 + \omega^2 t^2} = \frac{A}{\sqrt{1 + \omega^2 t^2}}.$$

The argument $\theta = \tan^{-1}\left(-\dfrac{A\omega t}{\omega^2 t^2 + 1} \div \dfrac{A}{\omega^2 t^2 + 1}\right)$

$$= \tan^{-1}\left(-\frac{A\omega t}{A}\right) = \tan^{-1}(-\omega t).$$