3 Integral Calculus

Integral calculus has as its basis the mathematical operation of integration, which is generally considered to be the reverse of the operation of taking the derivative of a function. What this means is that in some problems, the derivative dx/dt is known, and the requirement is to determine the original function f (t) for which dx/dt is the derivative. This can often, though not always, be done through integration.

Integration is always performed on the differential of a variable. The quantities dt and dx are the differentials of the variables t and x, respectively. If the relation x = f (t) is given, then the function of t, which is obtained by evaluating the derivative dx/dt, is customarily designated f′(t). That is,

$$x = f(t), \frac{dx}{dt} = f'(t), \frac{d^2x}{dt^2} = f''(t), \text{ and so on.}$$

The relationship between the differentials of x and t, (dx and dt), is consequently

$$dx = f'(t)dt.$$

Differentiation is the process of obtaining the differential of a function. *Integration*, the inverse operation, involves obtaining the original function from the differential. The integration operation is flagged by the $\int$ integration sign.

In mathematical symbology, $dx = f'(t)\,dt$ identifies the differentiation operation, while $\int f'(t)\,dt = f(t) = x$ identifies the operation of integration.

Problem Areas

Integration differs from differentiation in one notable respect—while it is always possible to differentiate any function involving the independent variable, it is not possible to integrate all such functions. Certain functions cannot be integrated. This is because although every function has a derivative, not every function is the derivative of some other function.

Unlike differentiation, integration is not a straightforward mechanical procedure. In fact, the basis of performing integration in most cases is having available a table of integration formulas, which has been prepared over time by inverting various formulas for differentiation. For example, the fact that

$$\int \cos x \, dx = \sin x$$

is only known because it is known that

$$\frac{d}{dx}(\sin x) = \cos x .$$

If the function f′(t) dt is to be integrated, the problem is usually one of manipulating the function so that it is compatible with some formula in the table of integrals.

Another complication is that any function that can be integrated will have more than one integral; in fact, it will have many integrals. It can been proven, fortunately, that if two separate functions are both integrals of another function, then these two functions can differ only by a constant. Since it is obligatory to state the solution to a problem of integration in its most general form, it is customary to add on an arbitrary constant to the function obtained by integrating. That is,

$$\int f'(t) dt = f(t) + C .$$

Note that

$$\frac{d}{dt}[f(t) + C] = \frac{d}{dt}f(t) + \frac{d}{dt}C = f'(t)$$

since the derivative of a constant is zero.

In a specific problem, the constant C may have a particular value, which can be determined by applying initial conditions.

Practical Uses of Integration

Integration has practical value in at least three areas.

- If the rate of change of the dependent variable is known, integration over a specified range of the independent variable will yield the cumulative value of the dependent variable over the chosen range.
- An offshoot of this is the ability to calculate the area under a specified section of a curve. This area, divided by the length of the base under the curve, will give the true average value of the dependent variable represented by the curve over the given range.
- The solution of differential equations that describe the dynamic behavior of certain control system components requires the capability to do integration.

A limited list of integration formulas is contained on page 50.

Example 1: Powers and Constants

The formulas in the table on page 50 show that integration is no problem if the expression to be integrated involves only constants and powers of the variable.

Integrate the expression $a + bx + cx^2$, where a, b, and c are constants.

$$\int(a + bx + cx^2)dx = \int a dx + \int bx dx + \int cx^2 dx$$

$$= a\int dx + b\int x dx + c\int x^2 dx = ax + b\frac{x^2}{2} + c\frac{x^3}{3} + C$$

Example 2: $\sin^3 x$

Evaluate $\int \sin^3 x dx$.

This example can clarify two important points regarding the integration process. First, the integration rule which applies to *powers* of the variable involved, *does not apply to powers of functions of that variable.*

Specifically,

$$\int x^3 dx = \frac{x^4}{4} + C\text{, but } \int \sin^3 x dx \text{ is not } \frac{\sin^4 x}{4} + C.$$

This is a trap into which many students have fallen.

The second point concerns a maneuver that sometimes is needed to effect the integration. It involves forcing part of the function to be integrated past the differential sign, so that instead of integrating with respect to the variable involved, the integration is carried out with respect to a function of that variable. This sounds complicated, but the example will help to simplify the process.

Note that $\int \sin^3 x\,dx = \int \sin^2 x \times \sin x\,dx$. By forcing the function sin x past the differential sign, sin x dx becomes d (–cos x). The expression inside the brackets is in fact $\int \sin x\,dx$. To confirm that this procedure is mathematically correct,

$$\frac{d}{dx}(-\cos x) = \sin x \text{, so that } d(-\cos x) = \sin x dx .$$

Since $\sin^2 x + \cos^2 x = 1$, then $\sin^2 x = 1 - \cos^2 x$. Therefore,

$$\int \sin^2 x \sin x dx \text{ becomes } \int (1 - \cos^2 x) d(-\cos x) .$$

The integration can now be done on a term by term basis. For convenience, let cos x = u. On substituting, the expression becomes

$$-\int (1 - u^2) du = -\int 1 du + \int u^2 du = \frac{u^3}{3} - u + C .$$

Therefore,

$$\int \sin^3 x dx = \frac{\cos^3 x}{3} - \cos x + C .$$

To verify that the integration has been done correctly, the result should be differentiated.

$$\frac{d}{dx}\left(\frac{\cos^3 x}{3} - \cos x + C\right) = \frac{(3\cos^2 x)(-\sin x)}{3} + \sin x$$

$$= (1 - \sin^2 x)(-\sin x) + \sin x = -\sin x + \sin^3 x + \sin x = \sin^3 x$$

Example 3: cos 2x

This example illustrates still another trap to be avoided. The table of integrals shows that $\int \cos x \, dx = \sin x + C$. From this one could conclude that $\int \cos 2x \, dx = \sin 2x + C$, but this is not correct.

Substitute u=2x. Then

$$x = \frac{u}{2} \text{ and } dx = \frac{1}{2}du$$

$$\int \cos 2x dx = \int \cos u \frac{1}{2} du = \frac{1}{2}\int \cos u du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C.$$

The fact that any differential will be a product of a function f′ (x) multiplied by dx, and that these two expressions must be compatible when integration is performed, must be respected. Failure to do this is probably the biggest single cause of mistakes when working out integrals. For the differential term d ( ), what is inside the brackets has to be correct. Cos 2x dx cannot be integrated as it stands. Cos 2x d(2x) is required to compute.

Example 4: Substitution of Variables

Success in integrating a function often depends on an artful substitution of variables, which in turn depends on experience.

Integrate $\int \frac{1}{a-x} dx$ (a is a constant.

Substitute $(a - x) = v$. Then $x = a - v$, $\frac{dx}{dv} = -1$, and $dx = -dv$.

The integral becomes

$$\int \frac{1}{v}(-dv) = -\int \frac{dv}{v} = -\ln|v| + C = C - \ln|a-x|.$$

Example 5: Fractions

If the expression to be integrated is a fraction, consideration should be given to trying to put the entire numerator under the differential sign. The method of solution may then become apparent from the new appearance of the expression. Bear in mind that a constant can be added or subtracted to a function under the differential sign d( ) without altering the mathematical correctness. Specifically, $dx = d(x + a) = d(x - a)$.

Integrate $\int \frac{(1+2x)}{2+x+x^2}dx$.

$$\int (1+2x)dx = x + 2\frac{x^2}{2} + C = x + x^2 + C.$$

The choice of a value for C is arbitrary; hence C can be 2. Therefore,

$$(1+2x)dx = d(2+x+x^2), \text{ and}$$

$$\int \frac{(1+2x)}{2+x+x^2}dx = \int \frac{1}{2+x+x^2}d(2+x+x^2).$$

This is one of the standard forms $\int \frac{1}{u}du$, with $u = 2+x+x^2$.

Therefore, the solution is

$$\int \frac{(1+2x)}{2+x+x^2}dx = \ln|2+x+x^2| + A,$$ *A* being an arbitrary constant.

Example 6: Using Partial Fractions

A fraction in which the numerator and the denominator both consist only of powers of the independent variable, is called a *rational function*. The expression that was integrated in Example 4 is in this category. If the numerator of the rational function is of a lower degree than the denominator, another approach to integrating the expression may be to break the expression into partial fractions.

For example, it is required to integrate $\frac{10x+6}{x^2-2x-3}$.

In this case the denominator will factor into $(x-3)$ and $(x+1)$.

$$\frac{10x+6}{x^2-2x-3} = \frac{P}{x-3} + \frac{Q}{x+1}$$

provided that values for P and Q can be determined. Then,

$$\frac{P}{x-3} + \frac{Q}{x+1} = \frac{Px+P+Qx-3Q}{x^2-2x-3} = \frac{(P+Q)x+(P-3Q)}{x^2-2x-3}.$$

Comparing this numerator with the original numerator 10 x + 6, it is clear that (P + Q) = 10, and (P – 3Q) = 6. From these two equations, P = 9, and Q = 1.

Consequently,

$$\int \frac{10x+6}{x^2-2x-3}dx = \int \frac{9}{x-3}dx + \int \frac{1}{x+1}dx$$

which can be easily integrated.

Example 7: Numerator Higher Order than Denominator

If the numerator of the rational function is of a higher order than the denominator, then the approach is to divide the denominator into the numerator and integrate the results.

Integrate $\frac{x^3+3x}{x^2-2x-3}$. First, divide the denominator into the numerator.

$$\begin{array}{r|l}
 & \quad x+2 \\
\hline
x^2-2x-3 & x^3 \qquad\quad +3x \\
 & \underline{x^3-2x^2-3x} \\
 & \qquad 2x^2+6x \\
 & \qquad \underline{2x^2-4x-6} \\
 & \qquad\qquad 10x+6
\end{array}$$

$\int \frac{x^3+3x}{x^2-2x-3}dx$ becomes $\int x dx + \int 2 dx + \int \frac{10x+6}{x^2-2x-3}dx$.

Example 8: Changing to an Angular Mode

If an expression contains factors such as x, a (a constant), and $\sqrt{a^2-x^2}$, it might be useful to switch to a new variable θ, where θ is an acute angle in a right angle triangle for which

x, a, and $\sqrt{a^2-x^2}$ are the sides.

For example, $\int \frac{x}{\sqrt{a^2-x^2}}dx$ is to be determined.

The ratio $\frac{x}{\sqrt{a^2 - x^2}}$ could be represented as the tangent of the angle θ in Figure 3-1. Consequently, let $\frac{x}{\sqrt{a^2 - x^2}} = \tan\theta$.

Then $\frac{x}{a} = \sin\theta$, $x = a\sin\theta$, $\frac{dx}{d\theta} = a\cos\theta$, and $dx = a\cos\theta d\theta$.

Therefore, $\int \frac{x}{\sqrt{a^2 - x^2}} dx = \int \tan\theta \times a\cos\theta d\theta = a\int \sin\theta d\theta$.

$$= a(-\cos\theta) + C = a\left(-\frac{\sqrt{a^2 - x^2}}{a}\right) + C = C - \sqrt{a^2 - x^2}.$$

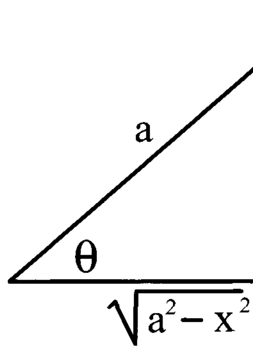

Figure 3-1. Switching to functions of an angle θ.

Example 9: $\sin^2 x$

Evaluate $\int \sin^2 x dx$.

A word of caution here that is worth repeating: The rule that governs integrating powers of a variable does *not* apply to integrating powers of functions of the variable. Specifically, while

$\int x^2 dx$ is equal to $\frac{x^3}{3}$, $\int \sin^2 x dx$ is not equal to $\frac{\sin^3 x}{3}$.

To integrate the function $\sin^2 x$, the function has to be converted into a more workable form, which turns out to be

$$\sin^2 x = \frac{1 - \cos 2x}{2}.$$

In Chapter 1 on trigonometric functions, the expression for the cosine of a sum was developed.

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

Consequently, $\cos 2x = \cos(x + x) = \cos x \cos x - \sin x \sin x$

$$= \cos^2 x - \sin^2 x.$$

Since $\sin^2 x + \cos^2 x = 1$, $\cos 2x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$.

Rearranging this, $\sin^2 x = \frac{1 - \cos 2x}{2}$.

Thus, $\int \sin^2 x dx = \int \left(\frac{1 - \cos 2x}{2}\right) dx = \int \frac{1}{2} dx - \int \frac{1}{2} \cos 2x dx$.

$$\int \frac{1}{2} dx = \frac{1}{2} \int dx = \frac{x}{2}$$

For $\int \frac{1}{2} \cos 2x dx$, substitute $u = 2x$. Then, $\frac{du}{dx} = 2$, and $dx = \frac{du}{2}$.

$$\int \frac{1}{2} \cos 2x dx = \frac{1}{2} \int \cos u \frac{du}{2} = \frac{1}{4} \int \cos u du = \frac{1}{4} \sin u = \frac{1}{4} \sin 2x$$

Finally, $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4} \sin 2x + C$.

This result should be verified by taking the derivative with respect to x.

$$\frac{d}{dx}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) = \frac{1}{2} - \frac{\cos 2x}{4} \times 2 = \frac{1}{2}(1 - \cos 2x) = \sin^2 x$$

Example 10: Square Root of $(a^2 - x^2)$

Sometimes the solution for one integral will provide a means of solving another. The integral

$$\int \sqrt{a^2 - x^2} dx$$

looks as though it should be fairly simple to evaluate but proves to be otherwise. The approach is to convert to the trigonometric mode.

Figure 3-2 is a right angled triangle with its hypotenuse equal to a, and the side opposite the angle θ equal to x.

This makes the remaining side $\sqrt{a^2 - x^2}$.

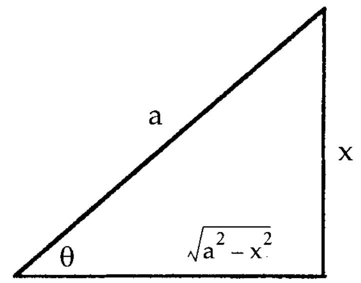

Figure 3-2. The trigonometric mode.

In this triangle,

$$\frac{\sqrt{a^2 - x^2}}{a} = \cos\theta \text{, and } \sqrt{a^2 - x^2} = a\cos\theta .$$

$$\frac{x}{a} = \sin\theta,\ x = a\sin\theta, \text{ and } dx = a\cos\theta d\theta$$

$$\begin{aligned}
\therefore \int\sqrt{a^2 - x^2}dx &= \int a\cos\theta \,.\, a\cos\theta d\theta = a^2\int\cos^2\theta d\theta \\
&= a^2\left[\int(1 - \sin^2\theta)d\theta\right] = a^2\left[\int 1\,d\theta - \int\sin^2 d\theta\right] \\
&= a^2\left[\theta - \left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\right] \quad \text{(from Example 9)} \\
&= a^2\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)
\end{aligned}$$

However, the solution should be in terms of x and a, not θ and a.

For the first term in the brackets, $\theta = \sin^{-1}\frac{x}{a}$

$$\frac{1}{4}\sin 2\theta = \frac{1}{4}(2\sin\theta\cos\theta) = \frac{1}{2} \times \frac{x}{a} \times \frac{\sqrt{a^2 - x^2}}{a} \quad \text{(from Figure 3-2)}$$

$$\therefore \int\sqrt{a^2 - x^2}\,dx = a^2\left(\frac{\sin^{-1}\frac{x}{a}}{2} + \frac{1}{2a^2}x\sqrt{a^2 - x^2}\right)$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C = \frac{1}{2}\left(a^2\sin^{-1}\frac{x}{a} + x\sqrt{a^2 - x^2}\right) + C.$$

Integration Over a Specified Range

So far, the integration process has resulted in integrals that are general in nature, in the sense that after integration has been performed, it is necessary to add a constant to the integral. This recognizes the fact that if two functions of the same independent variable differ only by a constant, then they will have the same derivative.

In some problems, however, the requirement is to integrate the expression over a specified range of the independent variable. In these cases, the constant does not apply, but it is necessary to identify what the upper and lower limits of the integration are to be. These limits are specified mathematically by placing them at the top and bottom of the integral sign. After the integral of the function has been determined, the final result will be the value of the integral at the upper limit of the independent variable, minus the value of the integral at the lower limit of the independent variable. The example below should help to illustrate this.

Example 11: Tank Filling Case

Figure 3-3 shows a 200 litre tank, which is being filled with a liquid by gravity from a large reservoir. The reservoir is large enough that drawing off 200 litre will not appreciably change its level. The driving force that moves the liquid is proportional to the difference in the levels of the tank and the reservoir. When the valve is opened, the initial flow rate is 2 l/s into the tank, but as the tank fills, this difference decreases, the driving force diminishes, and the rate of flow falls off.

Figure 3-4 shows how the flow rate decreases over time, starting at 2 l/s at $t = 0$. The relationship shown by the graph is typical of situations in which the driving force that creates the change in the dependent variable is in proportion to the difference between the instantaneous value of the dependent variable and its ultimate value over time. As the dependent variable approaches its ultimate value, in this case zero flow, the driving force falls off, as does the rate of change of the dependent variable.

If x is the liquid flow rate, the equation of the graph will be $x = 2.0e^{-.005t}$.

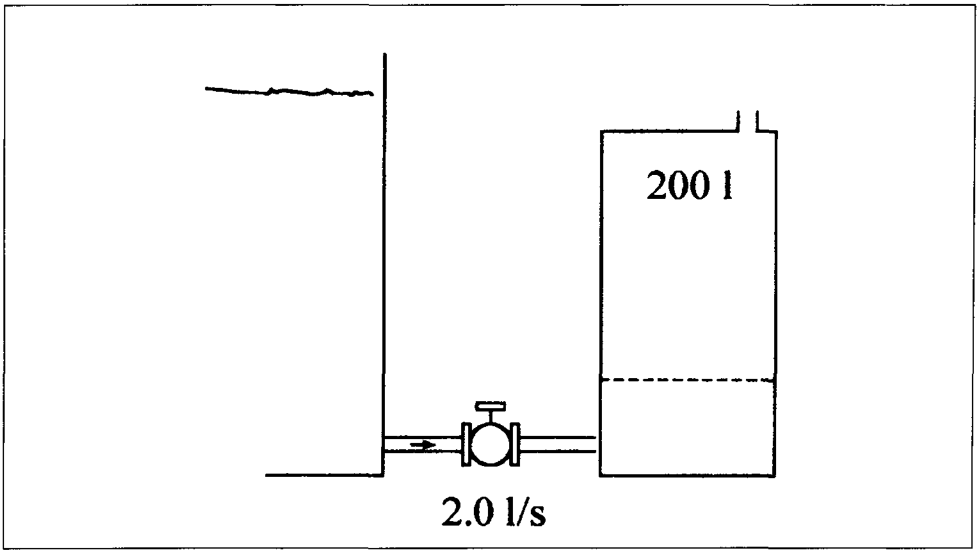

Figure 3-3. Tank Filling.

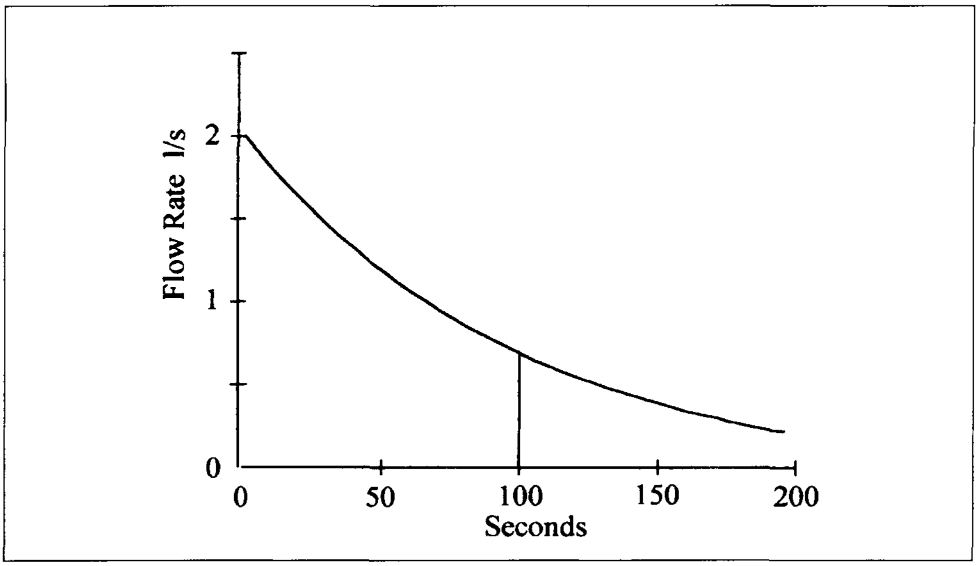

Figure 3-4. Flow into the tank decreases with time.

In this relation, x is in litres per second and t is in seconds. The factor 0.005 is set by the resistance to flow of the valve and piping and the cross sectional area of the tank.

If the flow rate into the tank were maintained at the initial rate of 2 *l*/s, then the tank would fill up in 100 s. A logical question might be how much liquid will actually be in the tank after 100 s, considering that the flow rate decreases as time goes on? To determine this, it is necessary to integrate the function x = f (t) over the range t = 0 to t = 100. The cumulative flow value is actually represented graphically by the area under the curve from t = 0 to t = 100. It is this area value that the integration will hopefully reveal.

$$\int_0^{100} f(t)dt = \int_0^{100} 2e^{-.005t}dt = 2\left[\frac{1}{-.005}e^{-.005t}\right]_0^{100} = -400[e^{-.005t}]^{100}$$

$$-400(0.607 - 1) = -400 \times -.393 = 157 \text{ litres}.$$

The average flow rate over this time period would be 157 *l* in 100 s or 1.57 *l*/s.

The solution to this example is fraught with mathematical shorthand. To avoid possible misunderstanding,

$[e^{-.005t}]_0^{100}$ means the value of $e^{-.005t}$ when $t = 100$, minus the value of

$e^{-.005t}$ when $t = 0$.

The next set of examples illustrates the principle of arriving at the desired solution by first setting up a differential increment of the entity to be determined, then integrating that increment over the relevant range to obtain the complete entity.

Example 12: Area of a Circle

The circumference of a circle is equal to 2π × the radius of the circle. Starting with this basic fact, the area of a circle can be determined by integration without much difficulty.

Figure 3-5 shows a circle with a radius r. Inside this circle is an elemental ring of width dx at a distance x from the center of the circle. The area inside the elemental ring will be the product of its length and width, that is, 2π x times dx. If this expression can be integrated from x = 0 to x = r, this will determine the area of the circle itself.

$$\text{Area} = \int_0^r 2\pi x dx = 2\pi\int_0^r x dx = 2\pi\left[\frac{x^2}{2}\right]_0^r = 2\pi\left[\frac{r^2}{2} - 0\right] = \pi r^2$$

This result, of course, is a revelation to no one.

Example 13: Surface Area of a Sphere

A similar procedure can be used to determine the surface area of a sphere.

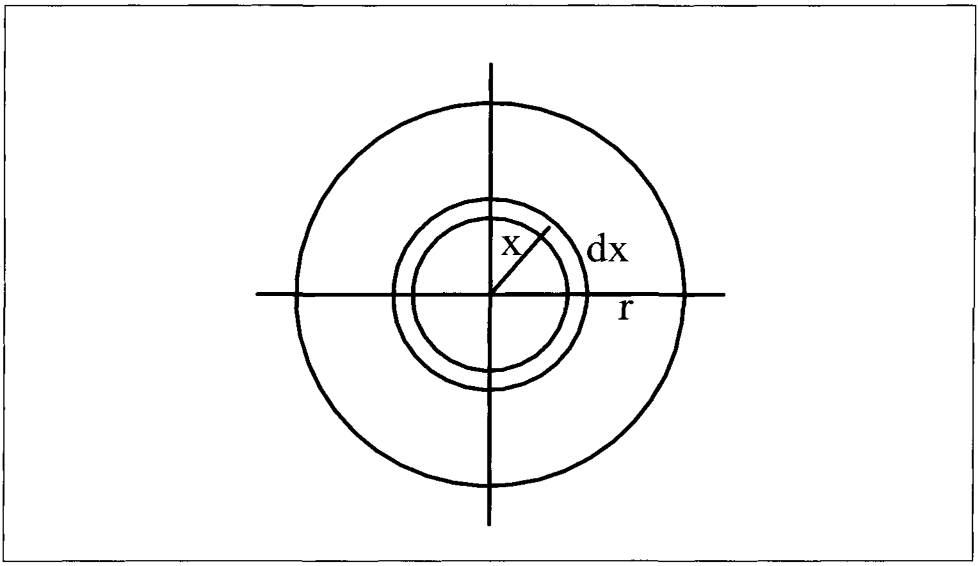

Figure 3-5. Area of a circle.

Figure 3-6 shows a sphere with a radius r. Distance on the surface of the sphere is represented by the variable a. In this case, a = arc PB. The elemental surface is a ring around the outside of the sphere. Its width will be da. Its length will be 2π times the distance AP.

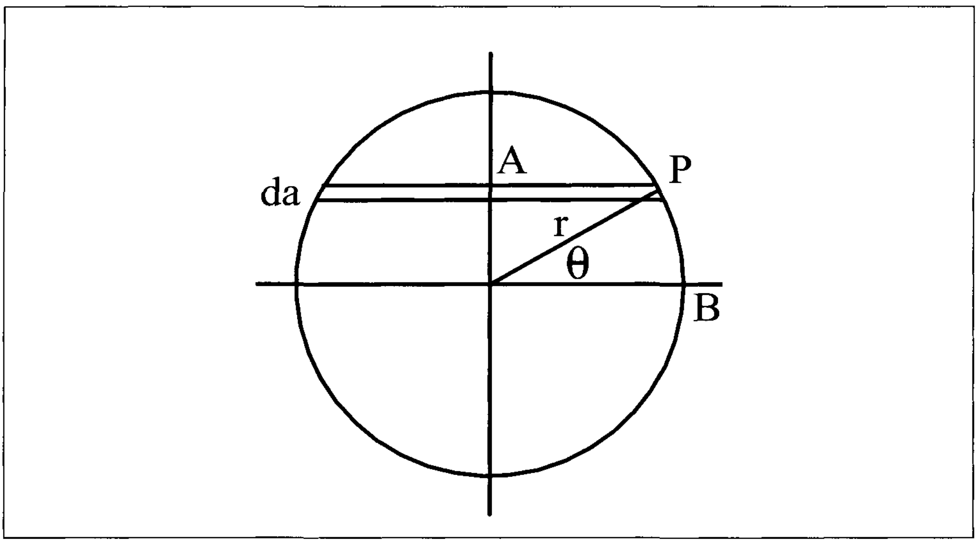

Figure 3-6. Area of a sphere.

If the radius vector makes an angle θ with the horizontal axis, then for an incremental change $d\theta$ in θ,

$$da = r \times d\theta.$$

$$\text{Also, } \frac{AP}{r} = \cos\theta, \text{ and } AP = r\cos\theta.$$

Consequently, the expression to be integrated is $2\pi \times r \cos \theta \times r\, d\theta$. The expression should be integrated between the limits $\theta = -90$ degrees to $\theta = +90$ degrees. In radians, this means $-\pi/2$ to $+\pi/2$.

$$\text{Area} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi r^2 \cos\theta d\theta = 2\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta = 2\pi r^2 \Big[\sin\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 2\pi r^2[1-(-1)] = 2\pi r^2 \times 2 = 4\pi r^2$$

Example 14: Volume of a Sphere

The integration process can also be used to determine the volume of a sphere.

Figure 3-7 shows a sphere with a radius r. The incremental element is a disc cut through the sphere at distance y from the origin. The radius of the disc is the distance AB, and its thickness is dy.

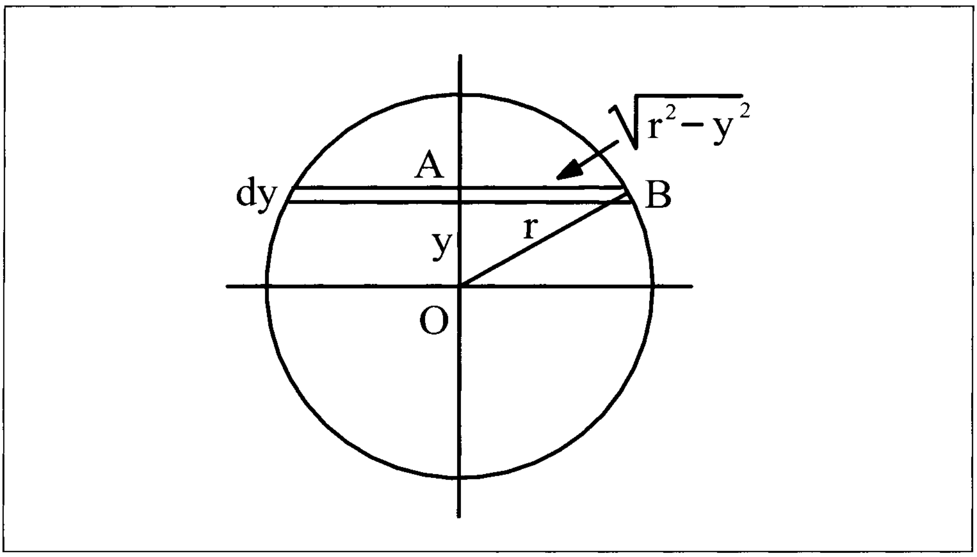

Figure 3-7. Volume of a sphere.

Since the triangle OAB is right angled,

$$(AB)^2 + y^2 = r^2, \text{ and } (AB)^2 = r^2 - y^2.$$

The volume of the elemental disc will be $\pi\,(AB)^2 \times dy = \pi\,(r^2 - y^2)\,dy$.

To arrive at the volume of the sphere, this expression must be integrated from the bottom of the sphere to its top, which means from – r to + r.

$$\int_{-r}^{r} \pi(r^2 - y^2)dy = \pi r^2 \int_{-r}^{r} dy - \pi \int_{-r}^{r} y^2 dy = \pi r^2 [y]_{-r}^{r} - \pi \left[\frac{y^3}{3}\right]_{-r}^{r}$$

$$= \pi r^2[r - (-r)] - \frac{\pi}{3}[r^3 - (-r^3)] = \pi r^2(2r) - \frac{\pi}{3}(2r^3) = 2\pi r^3 - \frac{2}{3}\pi r^3$$

$$= \frac{4}{3}\pi r^3.$$

Example 15: The Escape Velocity

Determine the vertical velocity that a mass m would have to be given at the surface of the earth to break free from the gravitational pull of the earth and remain in orbit. This velocity is sometimes called the escape velocity.

It is a fact of physics that work and energy are interchangeable. Accordingly, the approach to this problem is to calculate the work that would have to be done in moving the mass from the earth's surface out into space where the gravitational pull of the earth is no longer effective. Then the kinetic energy that the mass must have as it starts out must equal the required amount of work. If the kinetic energy is known, then the velocity can be calculated.

The relationship governing the gravitational force F between 2 masses is

$$F = k\,\frac{m_1 m_2}{d^2}$$

where

m_1 and m_2 are the masses of the mutually attracted objects, in kilograms. If m_1 is the mass of the earth, then $m_1 = 5.983 \times 10^{24}$ kg.

d is the distance between the centers of gravity of the objects, in metres.

k is the gravitational constant (6.670×10^{-11}).

With k at this value, the m's in kilograms, and d in metres, F will be in Newtons.

Let the vertical elevation of the object be y, measured from the center of the earth. This puts the starting point of the object at the radius r of the earth, or $y = 6.371 \times 10^6$ m.

For a small change in elevation dy, the work done will be the force times the distance.

$$dw = k\frac{m_1 m}{y^2}dy$$

To find the total work required, this function should be integrated from y = earth's radius ($r = 6.371 \times 10^6$ metres) to y = infinity.

$$\int dw = \int_r^{\infty} k\frac{m_1 m}{y^2}dy = km_1 m\int_r^{\infty} \frac{1}{y^2}dy = km_1 m\int_r^{\infty} y^{-2}dy$$

$$= km_1 m\left[-\frac{1}{y}\right]_{6.731 \times 10^6}^{\infty} = km_1 m\left(-0-\left(-\frac{1}{6.731 \times 10^6}\right)\right)$$

$$= (6.670 \times 10^{-11}) \times (5.983 \times 10^{24}) \times m \times \left\{\frac{1}{6.371 \times 10^6}\right\}$$

$$= 6.264 \times 10^7 \times m$$

Therefore, the kinetic energy that must be given to the object will be

$\frac{1}{2}mv^2 = 6.264 \times 10^7 \times m$, from which $v^2 = 1.2527 \times 10^8$, and

$v = 1.119 \times 10^4 \text{m/s}$, or 6.95 miles/s.

Example 16: Area of a Segment of a Circle

The problem in this example is to develop a formula for the area of a segment of a circle, as shown in Figure 3–8.

In the circle shown, the segment of concern is that bounded by the chord AD, which is a distance d from the center of the circle, and the arc ACD. The formula that is derived should be in terms of the radius r of the circle and the distance d.

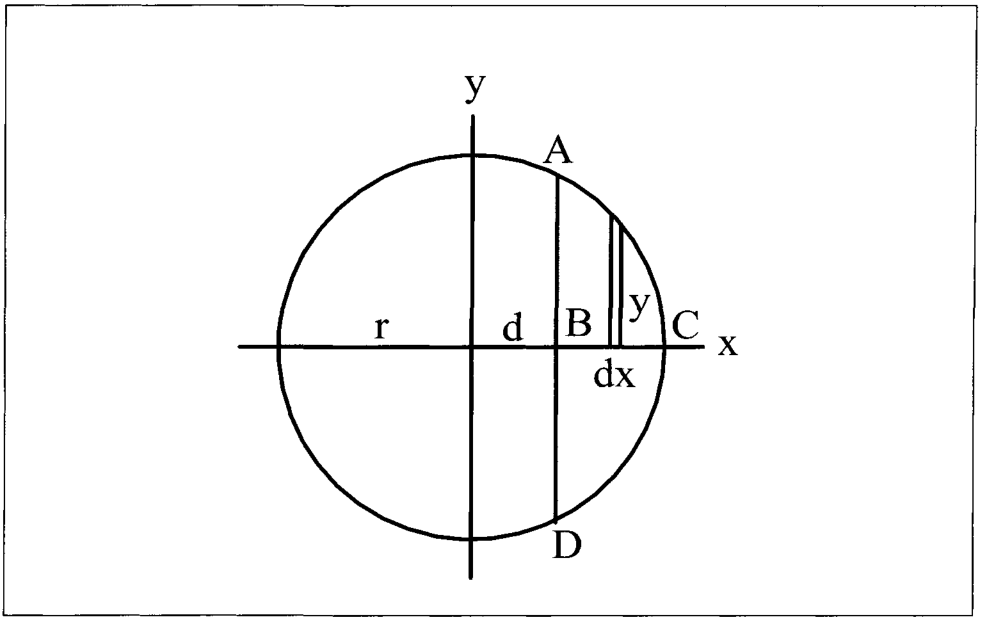

Figure 3-8. Area of a segment of a circle.

From the figure, it is apparent that the area of the segment will be twice the area of the half segment described by the corners A, B, and C. Within that half segment is an elemental strip that is dx in width and y in height, so that its area is y dx. The task, therefore, will be to integrate the expression 2 y dx from x = d to x = r.

The equation for the circle is $x^2 + y^2 = r^2$. Therefore, $y = \sqrt{r^2 - x^2}$.

The integral to be computed, then, is $2\int_d^r \sqrt{r^2 - x^2}\,dx$.

Example 10 showed that

$$\int\sqrt{r^2 - x^2}\,dx = \frac{1}{2}\left(r^2\sin^{-1}\frac{x}{r} + x\sqrt{r^2 - x^2}\right)$$

$$\therefore 2\int_d^r \sqrt{r^2 - x^2}\,dx = 2\left[\frac{1}{2}\left(r^2\sin^{-1}\frac{x}{r} + x\sqrt{r^2 - x^2}\right)\right]_d^r$$

$$= 2\left[\frac{1}{2}\left(r^2\sin^{-1}\frac{r}{r} + r\sqrt{r^2 - r^2}\right) - \frac{1}{2}\left(r^2\sin^{-1}\frac{d}{r} + d\sqrt{r^2 - d^2}\right)\right]$$

$$\sin^{-1}\frac{r}{r} = \text{(angle whose sine is 1)} = 90 \text{ deg, or } \frac{\pi}{2} \text{ in radians.}$$

$$\therefore \text{Area of the segment} = \frac{\pi}{2}r^2 - \left(r^2 \sin^{-1}\frac{d}{r} + d\sqrt{r^2 - d^2}\right)$$

This equation looks weird for an area expression, but it can be tested.

If d = 0, the chord that forms the segment will lie on the vertical (y) axis, and the segment will occupy half of the circle. With d = 0,

$\sin^{-1}\frac{d}{r} = \sin^{-1}0 = 0$, as does $d\sqrt{r^2 - d^2}$, and the area of the segment $= \frac{\pi r^2}{2}$.

If d = r, the area should be zero. For d = r,

$$\sin^{-1}\frac{r}{r} = \sin^{-1}1 = \frac{\pi}{2} \text{ radians, and } r\sqrt{r^2 - r^2} = 0$$

The area of the segment becomes $\frac{\pi r^2}{2} - r^2\frac{\pi}{2} = 0$, which computes.

Table of Basic Integrals

In this table, a, n, and C are constants, and x, u, and v are functions of t.

1. The integral of a sum is the sum of the integrals.

$$\int(dx + du + dv) = \int dx + \int du + \int dv$$

2. A constant multiplying the function to be integrated can be moved outside of the integral sign.

$$\int a\,dx = a\int dx$$

3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, provided that n is not –1.

4. $\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$ (Note 1)

5. $\int \cos x\,dx = \sin x + C$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \tan x\,dx = -\ln|\cos x| + C$ (Note 1)

8. $\int e^{ax} dx = \frac{1}{a}e^{ax} + C$

9. $\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

10. $\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

11. $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left(a^2 \sin^{-1}\frac{x}{a} + x\sqrt{a^2 - x^2}\right) + C$

Note 1: In mathematical notation, the natural logarithm of a number x, that is, its logarithm to the base e, is written ln | x |.