7 Laplace Transforms

History

The origin of Laplace transforms dates back to the era of a British civil engineer named Oliver Heaviside, who lived from 1850 to 1925. An accomplished mathematician, Heaviside was experimenting with what came to be called mathematical operators. In operational calculus, for example, the letter p might replace the operation of taking a derivative, so that if $x = f(t)$, then px was equivalent to

$$\frac{dx}{dt}.$$

Oliver Heaviside's contemporaries ridiculed his work, not so much because of the operators themselves, but because he actually moved operators around in algebraic expressions as if they were ordinary terms. This did not deter him, however, because as far as he was concerned, the method worked.

Eventually, results prevailed, and from Oliver Heaviside's beginnings, mathematicians developed a set of operational transforms that came to be known as the *Laplace transforms.* Just as the use of logarithms can reduce multiplication and division to addition and subtraction, Laplace transforms, where they can be applied, can reduce the problem of solving a differential equation to one of solving an algebraic equation.

In the analysis of control systems, process variables vary with time. The observed behavior may be described by a differential equation, which has time as the independent variable. In such cases, what is required is an expression $x = f(t)$, which describes the behavior of the dependent variable on a time basis, and which is clear of any derivatives. Laplace transforms is a mathematical technique through which this may be achieved.

In Laplace transforms, time is replaced as the independent variable by a new variable, which has been designated s. This has been done by multiplying the function f(t) by e^{-st}, and then integrating the product between the lower and upper limits of zero and infinity. The resulting integral appears as

$$\int_0^\infty e^{-st} f(t)dt.$$

The values that s can acquire must necessarily be limited to those that will cause the integral to assume some finite value. The types of differential equations with which this text deals are essentially linear, which means that s will have a real value and will be greater than zero. Most functions f(t) that occur in control systems engineering are Laplace transformable, since with s real and positive, e^{-st} decreases rapidly as t approaches infinity, which more than compensates for an increasing value of f(t).

Inasmuch as the domain of the integral is from zero to infinity, all activity starts at $t = 0$ and proceeds in the positive direction. Negative values of t have no meaning.

There are two further restrictions on the application of Laplace transforms.

1. The function f(t) must be single valued, that is, any value of t greater than zero produces only one value of the dependent variable.
2. More must be known about the relation $x = f(t)$ than just its differential equation. Specifically, the value of f(t) at $t = 0$ must be known if the differential equation is of the first order. If the differential equation is of the second order, then the value of the first derivative of f(t), that is, the rate of change of the dependent variable must be known at $t = 0$ as well.

The Laplace transform of a function f(t), denoted L {f(t)} in mathematical shorthand, is defined as

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \text{ and is generally denoted F(s).}$$

If these conditions are satisfied, then the function f(t) is *Laplace transformable.*

Example 1: Step Change

It is worthwhile to work out the Laplace transform for one particular function f(t), since this function becomes important in the study of transfer functions. Fortunately, it is the easiest one to evaluate.

In the process of testing control system components to determine their static and dynamic properties, the input that is used most often is a step input, that is, a sudden jump at t equals zero from a zero signal to a constant signal of finite value. Note that this conforms to the requirement that the initial value of the function be zero. If the input signal suddenly assumes a finite value C at $t = 0$, then $f(t) = C$ is the function to be transformed. The Laplace transform for the step change will then be,

$$F(s) = \int_0^\infty Ce^{-st}\,dt = C\int_0^\infty e^{-st}\,dt = C\left[\frac{1}{-s}e^{-st}\right]_0^\infty$$
$$= C\left[0 - \left(\frac{1}{-s}\right)\right] = \frac{C}{s}.$$

Transforms of Derivatives

The following shorthand symbols are generally used when working with Laplace transforms.

The finite value that $x = f(t)$ assumes at $t = 0$ is designated x_0. In other words, $x_0 = f(0)$.

The value of $\frac{dx}{dt}$ at $t = 0$ is designated $\left(\frac{dx}{dt}\right)_0$.

Without going through the mathematical chores involved, we have the following relationships.

1. The Laplace transform of the function f(t) is designated F(s).

2. The Laplace transform of the first derivative $\frac{dx}{dt}$ will be $sF(s) - x_0$.

3. The Laplace transform of the second derivative $\frac{d^2x}{dt^2}$ will be

$$s^2F(s) - sx_0 - \left(\frac{dx}{dt}\right)_0.$$

These facts will be required for the solution of differential equations using Laplace transforms.

Example 2: Time Constant

Figure 7–1 describes the type of dynamic response that is typical of many elements that appear in control systems. The primary characteristic of this component is that at any point on its response curve, the rate of change of the dependent variable with time is proportional to the distance remaining for it to attain its ultimate value. In this case, the dependent variable is y, and in response to a step change input, y begins to change from zero to eventually reach a new value Y.

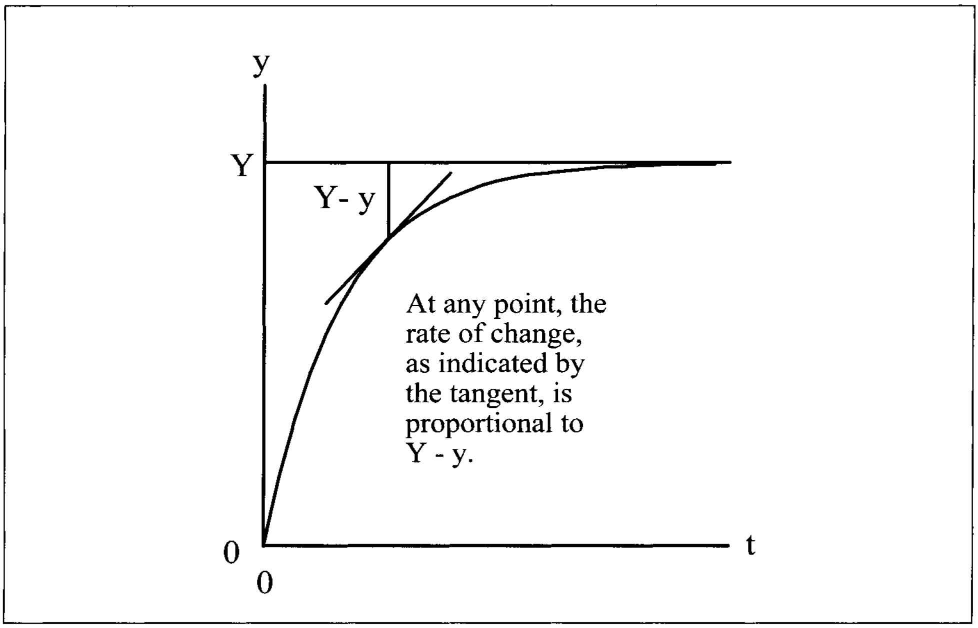

Figure 7-1. Behavior of a Time Constant Element.

At any point in the response of the variable y, the distance remaining is (Y – y). Describing the behavior as a differential equation,

$$\frac{dy}{dt}\alpha(Y-y),\text{ or }\frac{dy}{dt} = \frac{1}{T}(Y-y)$$

where T is a constant. In this expression, the units turn out to be more realistic if 1/T is chosen as the constant rather than T.

Rearranging, $T\dfrac{dy}{dt} + y = Y.$

Then, applying the Laplace transformation term by term,

$$T\{sF(s) - y_0\} + F(s) = \frac{Y}{s}.$$

Since $y_0 = 0$, the expression becomes

$$F(s)(Ts + 1) = \frac{Y}{s}, \text{ and } F(s) = Y\frac{1}{s(Ts + 1)}.$$

A table of Laplace transforms reveals that the transform

$$F(s) = \frac{1}{s(Ts + 1)}$$

originates from

$$f(t) = 1 - e^{-\frac{1}{T}t}.$$

Accordingly, the solution to the differential equation,

$$\frac{dy}{dt} = \frac{1}{T}(Y - y) \text{ is } y = Y\left(1 - e^{-\frac{1}{T}t}\right).$$

Since the exponent of e must be dimensionless, and t is in time units, T must also be in units of time. T is, in fact, the time constant of the component.

Example 3: Pendulum

Figure 7–2 shows a pendulum of length L, which oscillates about its fixed center O. At any time t, the angle that the shaft of the pendulum makes with the vertical is θ. Once the free end of the pendulum has been raised so that the pendulum starts from a position θ_0, the pendulum will begin to swing under the influence of gravity.

If we can assume that the mass of the pendulum is m, and that the mass is mainly concentrated at the center of gravity of the bob at the free end of the pendulum, then the force of gravity will be mg downward from its center of gravity.

The path BP is perpendicular to the shaft of the pendulum and is the instantaneous path along which the bob is moving. The angle APB is actually equal to the angle θ. Therefore the component of mg in the direction BP is equal to

mg × the cosine of the angle that BP makes with the vertical

$$= mg \times \cos(90 - \theta) = mg(\cos 90 \cos \theta + \sin 90 \sin \theta)$$

$$= mg(0 + \sin \theta) = mg \sin \theta.$$

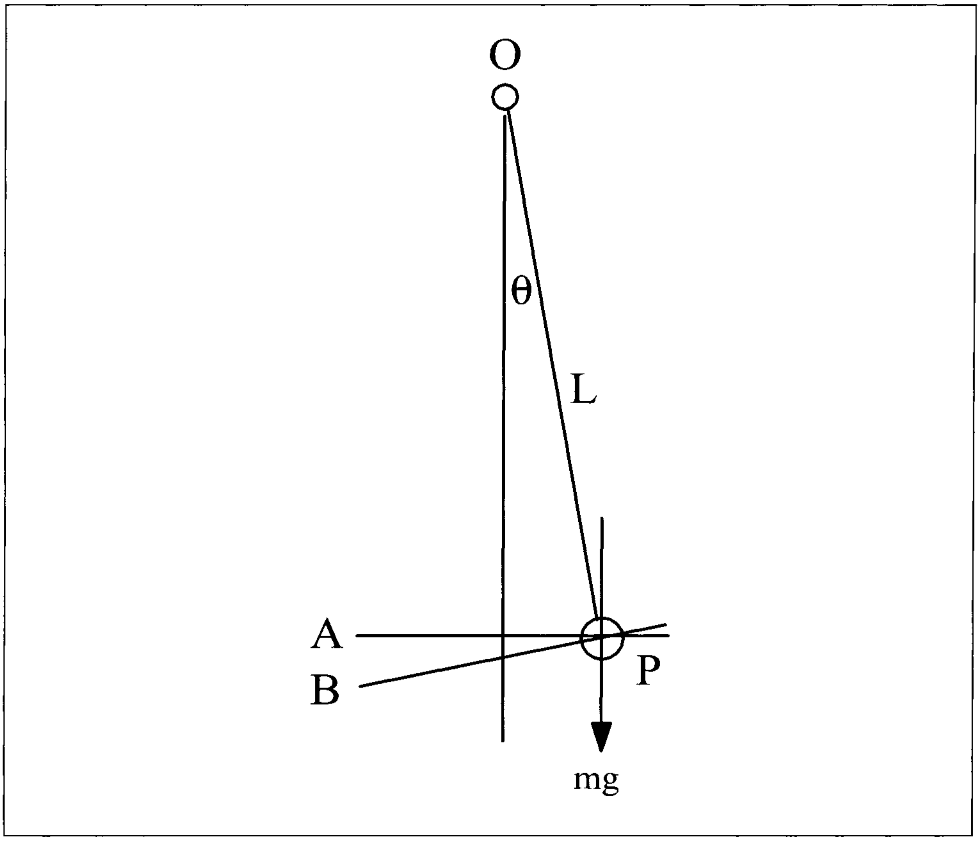

Figure 7-2. A Simple Pendulum.

If the pendulum is made to swing so that the angle θ is kept small, then the value of sin θ becomes virtually the same as the value of θ. Consequently, the component of the gravitational force mg in the direction of motion of the bob will have a magnitude $mg\theta$. However, if we agree that values of the angle θ, and of forces, are positive to the right side of the vertical and negative to the left, then the tangential force exerted on the bob will be $(-mg\theta)$.

The force $-mg\theta$ will be equal to the mass m of the bob multiplied by its acceleration along the path BP. If distances along the arc are denoted by the variable z, then

$$-mg\theta = m\frac{d^2z}{dt^2}.$$

However, we are not so much interested in the position of the weight in its track, with time, as we are in the angle that the shaft makes with the vertical.

The basic relation (arc) z = (radius) L × θ can be differentiated twice, which gives

$$\frac{d^2z}{dt^2} = L\frac{d^2\theta}{dt^2}.$$

$$\therefore\ -mg\theta = mL\frac{d^2\theta}{dt^2},\ \text{or}\ \frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta,\ \text{and}\ \frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0.$$

What remains now is the solution of this differential equation using Laplace transforms.

Inasmuch as there is a second order derivative involved, it is necessary to know the starting (t = 0) values of θ and of $\frac{d\theta}{dt}$.

Since the motion of the pendulum is cyclical, the starting point can be chosen at any point in the cycle. From the viewpoint of knowing the required starting point values, the best point from which to start is shown in Figure 7-2. With the pendulum in this position, the initial value of θ will be whatever angle is given to the pendulum to start it off. This will be the eventual amplitude of the pendulum and can be designated θ_0.

The angular velocity of the pendulum, $\frac{d\theta}{dt}$, will be zero at this point.

Applying the Laplace transformations term by term,

$$\{s^2 F(s) - s\theta_0 - 0\} + \frac{g}{L}F(s) = 0,$$

$$\text{and}\ F(s) = \frac{s\theta_0}{s^2 + \frac{g}{L}} = \theta_0\left(\frac{s}{s^2 + \frac{g}{L}}\right).$$

The table of Laplace transforms shows that the transform $\frac{s}{s^2 + \omega^2}$ originates from the function $f(t) = \cos\omega t$. In this case, $\omega^2 = \frac{g}{L}$

and the function sought is consequently

$$\theta = \theta_0 \cos\sqrt{\frac{g}{L}}t.$$

This result is not really as relevant as the period of oscillation of the pendulum. The ω term in the cosine expression has to have units of seconds^{-1} so that the argument ωt is dimensionless. The ω term is actually the angular velocity of the shaft, with units in radians per second.

Since $\omega = 2\pi f$, the frequency f will be equal to $\frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$.

The period of oscillation $P = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}}$.

Example 4: Second Order Linear Differential Equations with Constant Coefficients

Example 2 in Chapter 6 (Differential Equations) dealt with the oscillations of a weight suspended on a spring. The resulting differential equation was

$$\frac{d^2y}{dt^2} + A\frac{dy}{dt} + By = 0$$

where A and B are constants.

In the spring/mass problem worked as Example 2 in Chapter 6, A stood for r/m, while B stood for k/m. The k term was the constant of the spring, which induces the oscillations. The r term was the damping effect, which tends to reduce or eliminate the oscillations, while the m term was the mass of the weight.

However, it is desirable that the differential equation represents more real world systems than simply that of a weight dangling on a spring. Consequently, the B term in the equation will now be regarded as the "driving force" behind the oscillations, and the A term as the "damping effect." What these two parameters stand for should be kept in mind. The relative magnitudes of the parameters A and B will have a significant impact on the behavior of the system.

The differential equation is linear with constant coefficients, and furthermore, the right side is zero. Accordingly, the resolution of this equation by standard techniques is relatively straightforward, as was shown in Chapter 6. What should now be attempted is the resolution of this differential equation by using Laplace transforms.

Applying the Laplace transforms term by term,

$$\left\{s^2F(s) - sf(0) - \frac{df}{dt}(0)\right\} + \{A(sF(s) - f(0))\} + BF(s) = 0.$$

To continue, it is necessary to declare the initial conditions. Since there is no way of knowing what the velocity of the weight will be when the weight is moving, we are obliged to designate t = 0 at some point where the velocity of the weight is zero. This means at one end or the other of its travel. If the weight is oscillating up and down across the y = 0 base line, then we can specify that it starts out at t = 0 with zero velocity from a position Y below the base line.

With these initial conditions, the transform equation becomes

$$\{s^2F(s) - sY - 0\} + \{AsF(s) - AY\} + BF(s) = 0.$$

Collecting F(s) and Y terms: $F(s)[s^2 + As + B] - Y[s + A] = 0$

which leads to $F(s) = Y\left(\dfrac{s + A}{s^2 + As + B}\right)$.

The procedure now calls for consulting a table of Laplace transforms to obtain the function f(t), which has the expression F(s) determined above as its transform. Unfortunately, the transform

$$\frac{s + A}{s^2 + As + B}$$

does not appear anywhere in the table. Is the conclusion, therefore, that this problem cannot be resolved using the Laplace transform technique?

In fact, Laplace transforms can produce the answer, but here again, the difference between success and failure is the required mathematical experience. The denominator of the transform can be rearranged to complete the square of the first two terms.

$$s^2 + As + B = \left(s^2 + As + \frac{A^2}{4}\right) - \frac{A^2}{4} + B = \left(s + \frac{A}{2}\right)^2 + \left(B - \frac{A^2}{4}\right)$$

There are now three possibilities to be considered. The first is the case in which

$$\frac{A^2}{4} = B.$$

This would imply that the driving force and the damping effect balance each other off. With

$$B - \frac{A^2}{4} = 0$$

the transform becomes

$$\frac{s+A}{\left(s+\frac{A}{2}\right)^2}$$

which can be rearranged as

$$\frac{s+\frac{A}{2}+\frac{A}{2}}{\left(s+\frac{A}{2}\right)^2} = \frac{s+\frac{A}{2}}{\left(s+\frac{A}{2}\right)^2} + \frac{\frac{A}{2}}{\left(s+\frac{A}{2}\right)^2}$$

$$= \frac{1}{s+\frac{A}{2}} + \frac{A}{2}\frac{1}{\left(s+\frac{A}{2}\right)^2}.$$

An important fact of mathematics, one which is not disclosed in many texts on Laplace transforms, now emerges: *If a Laplace transform, which cannot be inverted as it stands, can be expressed as the sum of two parts, each of which is capable of being inverted, then the solution will be the sum of the inverted parts.*

From the table of Laplace transforms, the inverse transforms (in mathematics shorthand, L^{-1}) for this problem are

$$L^{-1}\left(\frac{1}{s+\frac{A}{2}}\right) = e^{-\frac{A}{2}t}, \text{ and } L^{-1}\frac{1}{\left(s+\frac{A}{2}\right)^2} = te^{-\frac{A}{2}t}$$

$$\therefore \text{ the solution } f(t) = Y\left(e^{-\frac{A}{2}t} + te^{-\frac{A}{2}t}\right) = Ye^{-\frac{A}{2}t}(1+t).$$

Since the solution does not contain either a sine or cosine function, the system is not going to oscillate. Furthermore, this system had as its basis that

$$\frac{A^2}{4} = B.$$

Note that if $\frac{A^2}{4}$ were greater than B, then the damping effect that A represents would be even more dominant.

Consequently, for any value of $\frac{A^2}{4}$ that is equal to or greater than B, the system will not oscillate.

The Oscillatory Case

The remaining possibility, therefore is the one in which B is greater than $\frac{A^2}{4}$, and the driving force is dominant over the damping effect.

The Laplace transform $F(s) = Y\left(\dfrac{s + A}{s^2 + As + B}\right)$

$$= Y\left[\frac{s + A}{\left(s + \frac{A}{2}\right)^2 + \left(B - \frac{A^2}{4}\right)}\right].$$

To simplify the various expressions, let $A/2 = a$, and let

$$\omega^2 = \left(B - \frac{A^2}{4}\right).$$

Since B is greater than $A^2/4$, ω^2 will be a positive quantity. Later it will be seen that using ω^2 instead of ω will make it easier to invert the transform.

Accordingly, $F(s) = Y\left(\dfrac{s + \frac{A}{2} + \frac{A}{2}}{\left(s + \frac{A}{2}\right)^2 + \left(B - \frac{A^2}{4}\right)}\right)$

$$= Y\left(\frac{s + a}{(s + a)^2 + \omega^2} + \frac{a}{(s + a)^2 + \omega^2}\right)$$

$$= Y\left(\frac{s + a}{(s + a)^2 + \omega^2} + \frac{a}{\omega}\frac{\omega}{(s + a)^2 + \omega^2}\right).$$

From the table of Laplace transforms:

$$L^{-1}\left(\frac{s+a}{(s+a)^2+\omega^2}\right) = e^{-at}\cos\omega t, \text{ and } L^{-1}\left(\frac{\omega}{(s+a)^2+\omega^2}\right) = e^{-at}\sin\omega t$$

$$\therefore \text{the solution is } y = f(t) = Y\left[e^{-at}\left(\frac{a}{\omega}\sin\omega t + \cos\omega t\right)\right]$$

$$= Y e^{-\frac{A}{2}t}\left(\frac{A}{2\omega}\sin\omega t + \cos\omega t\right),$$

where $\omega = \sqrt{B - \frac{A^2}{4}}$, or $\frac{1}{2}\sqrt{4B - A^2}$ as in Chapter 6.

Consistency of Results

An astute observer would notice that the solution obtained through the use of Laplace transforms is not the same as that obtained in Chapter 6 for the second order linear differential equation with constant coefficients, which was

$x = X e^{\alpha t} \sin(\omega t + \phi)$ (where α was substituted for $-A/2$).

In this expression, x is a general variable, not necessarily the displacement of a mass on a spring. X will be the original amplitude of the oscillations, whereas M, where it appears, is the value of x at t = 0.

When the differential equation was solved using the Laplace transforms, it was necessary to specify two initial conditions, and this fact must be recognized when comparing the two apparently different results. Specifically, this means that the initial conditions, which were specified when using the Laplace transforms, should be applied to the solution obtained in Chapter 6.

The initial conditions were:

1. When t = 0, x = M.

2. When t = 0, $\frac{dx}{dt} = 0$.

Applying the initial condition (1) to the solution $x = X e^{\alpha t} \sin(\omega t + \phi)$:

$$M = X \times 1 \times \sin(0 + \phi), \text{ from which } X = \frac{M}{\sin\phi}.$$

To apply initial conditions (2), it will be necessary to take the derivative of x with respect to t for $f(t) = X e^{\alpha t} \sin(\omega t + \phi)$. The rules for taking the derivative of a product, and of a function within a function, are needed (refer to Chapter 2).

$$\frac{d}{dt}\left[X e^{\alpha t} \sin(\omega t + \phi)\right] = X\left[\frac{d}{dt} e^{\alpha t} \times \sin(\omega t + \phi) + e^{\alpha t} \times \frac{d}{dt}\sin(\omega t + \phi)\right]$$

$$= X\left[\alpha e^{\alpha t} \sin(\omega t + \phi) + e^{\alpha t} \cos(\omega t + \phi) \times \omega\right]$$

$$= X e^{\alpha t}\left[\alpha \sin(\omega t + \phi) + \omega \cos(\omega t + \phi)\right] = \frac{dx}{dt}.$$

Since $\frac{dx}{dt} = 0$ when $t = 0$, $X \times 1 \times \left[\alpha \sin(0 + \phi) + \omega \cos(0 + \phi)\right] = 0$ and

$\alpha \sin\phi + \omega \cos\phi = 0$. Rearranging this:

$$\frac{\sin\phi}{\cos\phi} = -\frac{\omega}{\alpha} = \tan\phi.$$

Now, $X e^{\alpha t} \sin(\omega t + \phi) = X e^{\alpha t}(\sin\omega t \cos\phi + \cos\omega t \sin\phi)$

$$= \frac{M}{\sin\phi} e^{\alpha t}(\sin\omega t \cos\phi + \cos\omega t \sin\phi)$$

$$= M e^{\alpha t}\left(\sin\omega t \frac{\cos\phi}{\sin\phi} + \cos\omega t \frac{\sin\phi}{\sin\phi}\right)$$

$$= M e^{\alpha t}\left(\sin\omega t \frac{1}{\tan\phi} + \cos\omega t\right) = M e^{\alpha t}\left(\sin\omega t \times \left(-\frac{\alpha}{\omega}\right) + \cos\omega t\right).$$

$$= M e^{\alpha t}\left(\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\right).$$

Finally, since

$$\alpha = -\frac{A}{2}, \quad x = M e^{-\frac{A}{2}t}\left(\frac{A}{2\omega}\sin\omega t + \cos\omega t\right),$$

which is the solution obtained through the use of Laplace transforms.

Table of Laplace Transforms

A table containing some of the more common Laplace transforms is contained in Table 7-1.

Table 7-1. Short Table of Laplace Transforms

Function x = f(t) (**t > 0**)	**Laplace Transform F(s) = L {f(t)}**
C (cons tan t)	$\frac{C}{s}$
t	$\frac{1}{t^2}$
t^2	$\frac{2}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-\omega t}$	$\frac{1}{s+\omega}$
$\frac{1}{T}e^{-\frac{t}{T}}$	$\frac{1}{Ts+1}$
$1-e^{-\frac{t}{T}}$	$\frac{1}{s(Ts+1)}$
$f(t-L)$	$e^{-Ls}F(s)$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-\alpha t}\sin\omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos\omega t$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
dx/dt	$s\,F(s) - x_0$
d^2x/dt^2	$s^2\,F(s) - s\,x_0 - (dx/dt)_0$

x_0 and $(dx/dt)_0$ are the values of x and dx/dt when t = 0.