2 Differential Calculus

Mathematical relationships are constructed around variable quantities (called variables for short). The relationship shows the way that the value of one of the variables changes when the values of the other variables change.

This implies that in each relationship there is one variable whose value is dependent on the values of the other variables. It is consequently called the *dependent variable*, while the other variables are called the *independent variables*.

The way in which some mathematical relationships are structured often leaves doubt as to which of the variables is the dependent variable. The question becomes more difficult to answer as the number of variables in the relationship increases.

In the control systems engineering field, however, many relationships contain only two variables. Furthermore, one of the variables will be time (designated t). Since a unique characteristic of time is that it pursues its uniform and relentless course into eternity, unaffected by anything else, it is obvious that time cannot be dependent on any other physical variable. Consequently, in all control systems relationships that involve time, it must be the independent variable.

The relationships that are most common in control systems engineering generally show how some dependent variable, which could be distance, temperature, pressure, and so on, varies with time. If the dependent variable is represented by x, then it can be stated that "x is some function of time." In the shorthand of mathematics, this is written $x = f(t)$.

Notice that what this shorthand relationship is telling us is not only that the value of x is dependent on the value of t, but perhaps more important, that the value of x depends *only* on the value of t, and not on the value of any other variable, such as n, y, θ, etc., whatever these characters may represent in the real world. If a number of functions were involved, these might be distinguished by labeling them $f_1(t)$, $f_2(t)$, and so on.

It will often be useful to plot the values of the dependent variable over a range of values of time. When making the plot, it is customary to plot the values of the independent variable (t) along the horizontal axis and the corresponding values of the dependent variable (x, θ, or whatever) along the vertical axis. Figure 2-1 is an example. If the resulting plot is a curve that has no breaks or gaps, then the function is said to be *continuous*. The relationships that arise in control systems engineering can generally be counted on to be *single valued*, meaning that for each value of t there is only one value of the dependent variable.

Concept of Approaching a Limit

When we are first introduced to mathematics, we get the impression that it is an exact science. The rules of mathematics have no exceptions. Everything is based on the concept that something must be equal to something else.

Later, our confidence is shaken when we learn that mathematicians are also concerned with not only what a particular variable is equal to, but also what value that variable may be approaching. This may occur when some other variable, on which the first variable is dependent, approaches its limiting value, which often proves to be zero. Sometimes this apparently nebulous procedure is justified to get around certain complications, one of which might be division by zero. For example, suppose that

$$x = f(t) = \frac{t^2 - t - 6}{t - 3}.$$

What will be value of x when t approaches 3? The function is undefined for $t = 3$ since division by zero does not compute. However, it turns out that if the denominator $(t - 3)$ is divided into the numerator, the result is $(t + 2)$. This being the case, the value of x *approaches* 5, not infinity, as t *approaches* 3.

A small change (increment) in a variable such as t is usually called *delta t* and is written Δt. While Δt is a small change, it is nevertheless measurable, in whatever units are appropriate. As Δt is made smaller and smaller, a point is reached where we have to ask, "How much smaller can it get without being zero?" In the mathematical sense, it is at this point where Δt becomes the *differential of t*, which is identified by dt. In mathematics shorthand, the limit of the ratio $\Delta x/\Delta t$ as Δt approaches zero is written

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}.$$

This limit, if it exists, becomes dx/dt, and it is termed *the derivative of x with respect to t*. The function x = f(t) is then said to be differentiable. The ability to differentiate the function f(t) will require not only that the function be continuous, but also that when it is plotted, there are no corners in the graph.

When dealing with increments and with differentials in the determination of limiting values of variables and functions, there is a basic rule with which one need to be familiar. This rule says that in the determination of the limiting value of a derivative, as the value of the increment of the independent variable approaches zero, products and powers of incrementals become insignificant and can be discounted. Specifically, if terms such as Δx^2, Δt^2, or $(\Delta x \Delta t)$ emerge, then they can be ignored. This rule must be on board as other derivative functions are developed.

Figure 2-1 is a graph that shows the variation of a dependent variable x with values of the independent variable t. For any specified value of t within the range of the graph, there will be a corresponding value of x, as shown by point P in the graph. Suppose that the value of t increases a small amount. Once again in mathematical shorthand, this small change in t is generally designated Δt. The increase in t will cause a change in x, which will be designated Δx. Depending on the nature of the function f (t), Δx may be positive or negative. Point Q is the new point whose coordinates are $t + \Delta t$, and $x + \Delta x$.

From the graph it can be seen that the ratio $\Delta x/\Delta t$ is the slope of a line that passes through the points P and Q. Furthermore, if Δt, and consequently Δx, were made smaller and smaller, then point Q will approach point P, and the slope of the line through points P and Q will approach the slope of the graph x = f (t) at point P.

The slope of the graph x = f (t) at point P can actually be obtained by determining the value of the ratio $\Delta x/\Delta t$ as Δt approaches zero. When Δt is made smaller and smaller, point Q will approach point P, and the slope of the line through points P and Q will approach the slope of the tangent line to the graph x = f(t) at point P. Ultimately, the slope of the graph x = f(t) at point P will be equal to the rate of change of the dependent variable x with t at point P.

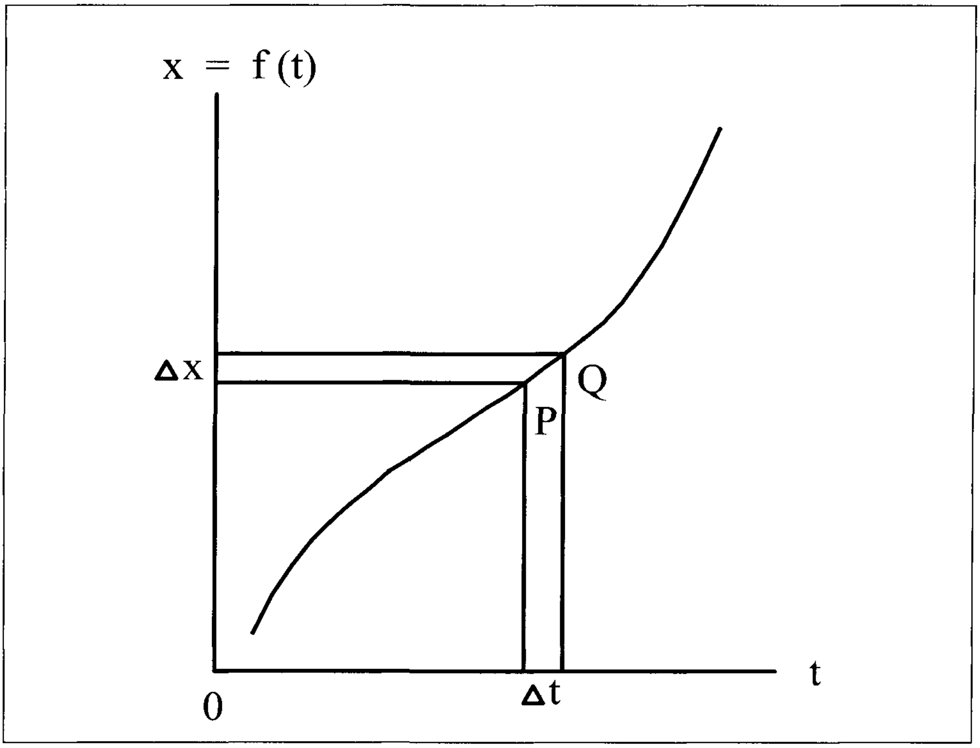

Figure 2-1. A Differentiable Function.

Example 1: $f(t) = t^2$

Suppose that $x = f(t) = t^2$.

Starting from any point t, x, as t changes by an amount Δt, x will change by an amount Δx, which produces a new point $t + \Delta t$, $x + \Delta x$. However, this new point is also on the graph $x = t^2$.

Therefore, $(x + \Delta x) = (t + \Delta t)^2$.

Expanding this, $x + \Delta x = t^2 + 2t\,\Delta t + (\Delta t)^2$.

Since $x = t^2$, these two items can be subtracted from the left and right sides, respectively, of the equation.

Therefore, $\Delta x = 2t\,\Delta t + (\Delta t)^2$, and dividing through by Δt,

$$\frac{\Delta x}{\Delta t} = 2t + \Delta t.$$

In the example above, as Δt approaches zero, the equation $\frac{\Delta x}{\Delta t} = 2t + \Delta t$

consequently becomes $\frac{dx}{dt} = 2t$.

An important characteristic of the derivative expression is that it can identify the slope of the graph x = f (t) at any point established by a selected value of t. Therefore, the slope of the curve $x = t^2$ will be equal to 2 t anywhere along the curve.

Procedure for Determining a Derivative

The general procedure for determining a derivative expression is to substitute (x + Δx) and (t + Δt) in the relation x = f (t), evaluate the ratio Δx/Δt, and then determine the limiting value of Δx/Δt as Δt approaches zero. The limiting value thus determined will be *dx/dt*, the derivative of x with respect to t.

Derivative of a Sum or Difference

Suppose that x is the sum of two differentiable functions designated u and v, in which $u = f_1$ (t), and $v = f_2$ (t). Since x, u, and v are all functions of t, when t becomes (t + Δt), x becomes x + Δx, u becomes u + Δu, and v becomes v + Δ, so that (x + Δx) = (u + Δu) + (v + Δv). Since x = u + v, they will drop out of this equation leaving Δx = Δu + Δv.

$$\text{Thus, } \Delta x = \Delta u + \Delta v. \text{ Dividing by } \Delta t, \frac{\Delta x}{\Delta t} = \frac{\Delta u}{\Delta t} + \frac{\Delta v}{\Delta t}.$$

$$\text{As } \Delta t \text{ approaches zero, the } \lim_{\Delta t \to 0} \frac{dx}{dt} = \frac{du}{dt} + \frac{dv}{dt}.$$

What this reveals is really a basic rule, namely, that the derivative of the sum of two functions is equal to the sum of the individual derivatives. This rule can be extended, without difficulty, to show that the derivative with respect to t (or any other independent variable) of the sum or difference of any number of functions of t, is equal to the sum or difference of the individual derivatives.

Derivative of a Product

Suppose that x = u × v, where x, u, and v are all differentiable functions of the independent variable t. With a small change in t, t becomes t + Δt. This results in incremental changes in x, u, and v, so that x + Δx = (u + Δu) (v + Δv) = uv + u Δv + v Δu + Δu Δv.

Since x = uv, Δx = u Δv + Δu (v + Δv).

$$\text{Dividing by } \Delta t, \frac{\Delta x}{\Delta t} = u\frac{\Delta v}{\Delta t} + \frac{\Delta u}{\Delta t}(v + \Delta v).$$

The function $v = f_1(t)$ is differentiable, so that as Δt approaches zero, Δv approaches zero. Therefore,

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta uv}{\Delta t} = \frac{dx}{dt} = u\frac{dv}{dt} + v\frac{du}{dt}.$$

Derivative of a Quotient

As before, x, u, and v are all differentiable functions of the independent variable t, and it is given that $x = u/v$. If t should change incrementally to $t+\Delta t$, x will become $x + \Delta x$, which will be equal to

$$\frac{u + \Delta u}{v + \Delta v}.$$

Therefore,

$$\Delta x = \frac{u + \Delta u}{v + \Delta v} - x = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{uv + v\Delta u - uv - u\Delta v}{v(v + \Delta v)} = \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}.$$

Dividing both sides by Δt:

$$\frac{\Delta x}{\Delta t} = \frac{v\frac{\Delta u}{\Delta t} - u\frac{\Delta v}{\Delta t}}{v(v + \Delta v)}.$$

As Δt approaches zero, Δv approaches zero. Consequently,

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}.$$

Dimensions and Units

A requirement of mathematics is that the argument of a trigonometric function, or the exponent of an exponential function, be dimensionless, that is, it should not have units. Specifically, in functions such as $\sin \theta$, $\cos \theta$, and e^{θ}, the variable θ must not have units associated with it.

This is why it is not appropriate to use the variable t as the argument or exponent in trigonometric or exponential functions because in control systems studies, t usually stands for time, and time has definite units of seconds, minutes, or hours. If time does appear in the argument or exponent of functions of these types, then it has to be compensated for by coupling it with a second variable such as omega (ω). For the combination ωt to be dimensionless, the units of ω must be the inverse of time. If the units of ω are radians per second or cycles per second, this will meet the requirement because angles in radians and cycles or revolutions have no units, and the "per second" dimension in the ω compensates for the "seconds" in the t.

Derivative of a Sine Function

Given the function $f(\theta) = x = \sin\theta$, as θ becomes $(\theta + \Delta\theta)$, $\sin(\theta + \Delta\theta)$ causes x to become $(x + \Delta x)$. Accordingly,

$$x + \Delta x = \sin(\theta + \Delta\theta) = \sin\theta \cos\Delta\theta + \cos\theta \sin\Delta\theta$$

$$\Delta x = \sin\theta \cos\Delta\theta + \cos\theta \sin\Delta\theta - x = \sin\theta \cos\Delta\theta + \cos\theta \sin\Delta\theta - \sin\theta.$$

It is a fact of mathematics that when the magnitude of an angle (θ) approaches zero, then the value of $\sin\theta$ approaches the value of θ. The numbers in the table following bear this out.

Table 2-1. Magnitude of an Angle (θ)

θ degrees	θ radians	$\sin\theta$	$\cos\theta$
10	0.1745	0.1736	0.9848
8	0.1396	0.1392	0.9903
6	0.1047	0.1045	0.9945
4	0.0698	0.0698	0.9976
2	0.0349	0.0349	0.9994

Also, as θ approaches zero, $\cos\theta$ approaches 1. Therefore, as $\Delta\theta$ approaches zero, Δx approaches $\sin\theta + \cos\theta(\Delta\theta) - \sin\theta = \cos\theta(\Delta\theta)$, and $\Delta x/\Delta\theta$ approaches $\cos\theta$.

Thus the derivative $\frac{d}{d\theta}\sin\theta = \cos\theta$.

Binomial Theorem

As a prerequisite to obtaining the expression for the derivative of the independent variable raised to a power other than one, some familiarity with the binomial theorem and the expansion for the power of a sum is helpful. Some of the terms in this expansion contain factorial quantities. A factorial applies only to positive integers, and in mathematical shorthand, the factorial operator consists of an exclamation mark (!) following the integer.

The factorial of a number (positive integer) is equal to the number multiplied by all of the numbers less than itself, in sequence, down to the number one. Thus, factorial 5 would be $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

In general, $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$, assuming that in this statement, $(n-2)$ is a number larger than 3.

The expansion for the expression $(a+b)^n$ using the binomial theorem is:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots$$

The expansion ends with the term that has $(a^{(n-n)} = a^0 = 1) \times b^n$.

As a test of the theorem, set $n = 4$. Then,

$$\begin{aligned}(a+b)^4 &= a^4 + 4a^3b + \frac{4 \times 3}{2 \times 1}a^2b^2 + \frac{4 \times 3 \times 2}{3 \times 2 \times 1}ab^3 + \frac{4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}b^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

which is the same result as would be obtained by multiplying $(a+b)$ by $(a+b)$ three times algebraically.

Derivative of a Power

Given that $x = t^n$, if t changes to $t + \Delta t$, then x becomes $x + \Delta x$, and the new relation is $(x + \Delta x) = (t + \Delta t)^n$. Using the binomial theorem to expand $(t + \Delta t)^n$,

$$x + \Delta x = (t + \Delta t)^n$$

$$= t^n + nt^{n-1}\Delta t + \frac{n(n-1)}{2!}t^{n-2}\Delta t^2 + \frac{n(n-1)(n-2)}{3!}t^{n-3}\Delta t^3 + \ldots + \Delta t^n.$$

Since $x = t^n$, x can be removed from the left side, and t^n from the right.

Dividing by Δt,

$$\frac{\Delta x}{\Delta t} = nt^{n-1} + \frac{n(n-1)}{2!}t^{n-2}\Delta t + \frac{n(n-1)(n-2)}{3!}t^{n-3}\Delta t^2 + \ldots + \Delta t^{n-1}.$$

As Δt approaches zero, all of the terms on the right side except the first term approach zero because they are multiplied by some positive power of Δt. Therefore the limit condition is

$$\frac{d}{dt}t^n = \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} = \frac{dx}{dt} = nt^{n-1}.$$

Thus, if $x = t^3$, then $\frac{dx}{dt} = 3t^2$.

This relationship is also valid if n is a fraction. Suppose that $x = \sqrt{t} = t^{\frac{1}{2}}$. Then,

$$\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{t}}.$$

In fact, n can be any real number.

Derivative of an Exponential

Given that $x = e^p$, what will be the value of $\frac{dx}{dp}$?

The easiest way to determine the derivative is to express e^p in its power series form. (This is developed in Chapter 4.)

$$e^p = 1 + p + \frac{p^2}{2!} + \frac{p^3}{3!} + \frac{p^4}{4!} + \ldots$$

Taking the derivative of the series, term by term,

$$\frac{dx}{dp} = 0 + 1 + \frac{2p}{2 \times 1} + \frac{3p^2}{3 \times 2 \times 1} + \frac{4p^3}{4 \times 3 \times 2 \times 1} + \ldots = 1 + p + \frac{p^2}{2!} + \frac{p^3}{3!} + \ldots$$

which is the original function. Consequently,

$$\frac{d}{dp}e^p = e^p.$$

The power series for $\sin\theta$ and for $\cos\theta$ are both developed in Chapter 4. By taking the derivative of the series for $\sin\theta$, term by term, it becomes the series for $\cos\theta$, which verifies that the derivative of $\sin\theta$ is $\cos\theta$. If the derivative of the series for $\cos\theta$ is taken, term by term, it becomes the series for $\sin\theta$, multiplied by (-1). Hence the derivative of $\cos\theta$ is $-\sin\theta$.

A table of derivatives of selected functions of t is on page 29.

A Function Within a Function

In working out derivative expressions, there is a trap into which an unwary student can fall, and it is associated with taking the derivative of a function that has a second function within it.

If $x = \sin\theta$, then $\frac{dx}{d\theta} = \cos\theta$. But if $x = \sin\theta^2$, $\frac{dx}{d\theta}$ is not equal to $\cos\theta^2$.

In the second case, θ^2 is not θ but a function of θ within the sine function.

Similarly, if $x = e^p$, then $\frac{dx}{dp} = e^p$; but if $x = e^{ap}$, $\frac{dx}{dp}$ is not equal to e^{ap}.

Even though $a \times p$ is the simplest possible function of p, it is nevertheless a function.

The first step in dealing with the problem of the derivative of a function that has another function imbedded in it is to recognize that there is a complication, avoiding the trap described. Then, the procedure is to introduce an intermediate variable (u or whatever) and set it equal to the function inside of the base function. The required derivative can then be worked out by using the relationship

$$\frac{dx}{dp} = \frac{dx}{du} \times \frac{du}{dp}.$$

Given that $x = \sin\theta^2$, what is the derivative?

Designate the intermediate function $u = \theta^2$. Then,

$$\frac{du}{d\theta} = 2\theta.$$

Also, $x = \sin u$, and $\frac{dx}{du} = \cos u$.

$$\frac{dx}{d\theta} = \frac{dx}{du} \times \frac{du}{d\theta} = \cos u \times 2\theta = 2\theta\cos\theta^2.$$

Similarly, if $x = e^{at}$, designate $u = at$.

$$\frac{du}{dt} = a. \text{ Also } x = e^u, \text{ and } \frac{dx}{du} = e^u.$$

$$\frac{dx}{dt} = \frac{dx}{du} \times \frac{du}{dt} = e^u \times a = ae^{at}.$$

Example 2: Detecting a Maximum or Minimum

One use of the derivative function is that it can often detect the presence of a maximum or minimum point within the original function. This is due to the fact that when the function passes through a maximum or minimum, the slope of the tangent to the curve is horizontal, and the value of the derivative is zero. Consider the function

$$y = \frac{6x - x^2 - 1}{4}.$$

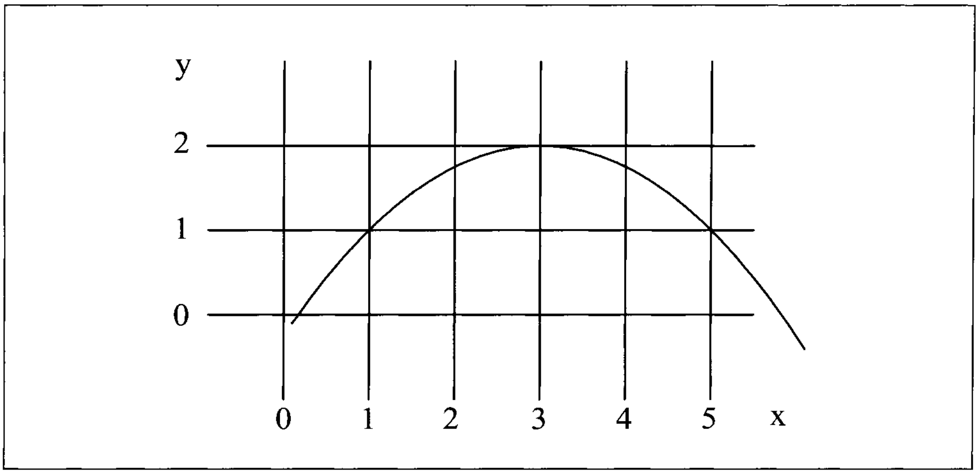

Figure 2-2. Graph of the function $y = \frac{1}{4}(6x - x^2 - 1)$.

This is a parabolic function in the variable x. As can be seen from Figure 2-2, it peaks at x = 3. The peak value is y = 2. The peak point and its value can be calculated by taking the derivative with respect to x and equating it to zero.

$$\frac{dy}{dx} = \frac{1}{4}(6 - 2x)$$

from which (6 – 2x) = 0, and x = 3.

Substituting x = 3 into the original function,

$$y = \frac{18 - 9 - 1}{4} = 2.$$

Example 3: Watering the Lawn

In another example, suppose that you wish to water your lawn, but your garden hose will stretch only to the midpoint of the lawn. You know from experience that to get the water to reach farther, you need to tip the nozzle upward. Intuition tells you that the water will reach the farthest point possible in the horizontal direction if you tilt the nozzle upward at an angle of 45°. At an angle less than 45°, the water jet falls short. If the angle is greater than 45°, the jet travels upward rather than outward and again falls short. The optimum angle appears to be 45°, but can this be proven?

In Figure 2-3, the nozzle is tilted upward at an angle θ to the horizontal. The water jets from the nozzle with a velocity w. The horizontal component of the velocity is $w \cos \theta$. Thus the distance S the jet will travel outward will be the product of its velocity and its time of travel, that is $S = w\cos\theta \times t$.

The value of t can be determined from the upward component of the velocity, which is $w \sin\theta$. The basic relation is $v = u + at$, where u is the initial velocity of the object, v is its final velocity, and a is its acceleration over the timed interval. Rearranging this,

$$t = \frac{v - u}{a}.$$

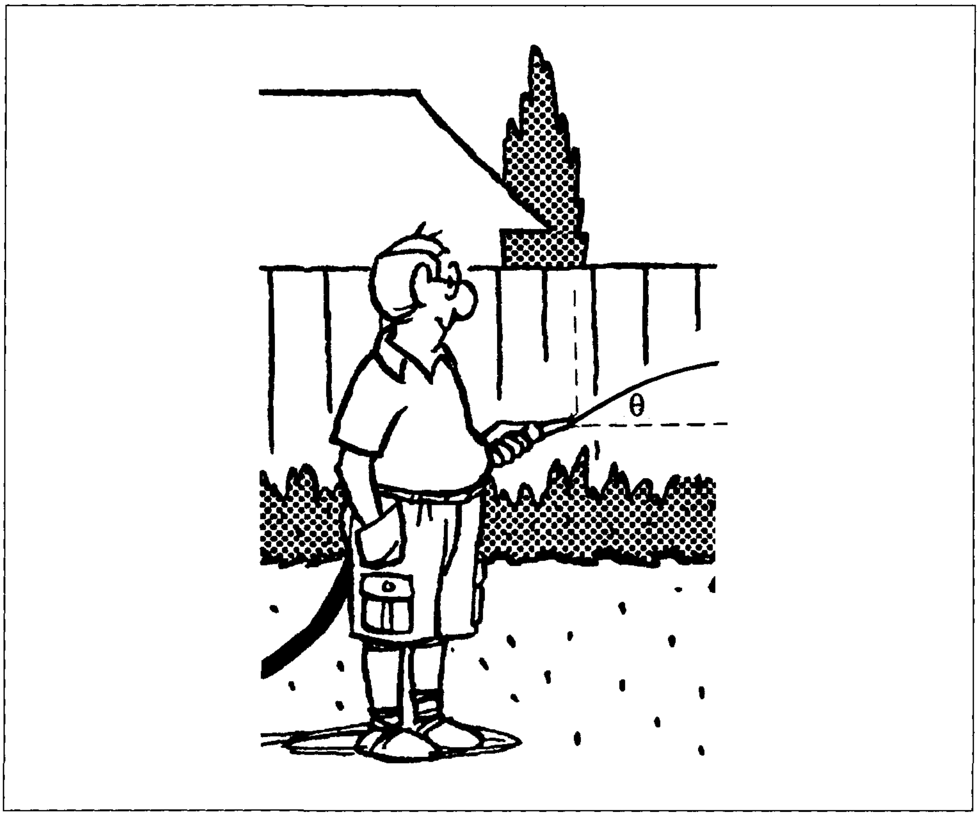

Figure 2-3. The optimum angle (© Washington Post Writers Group).

For the jet of water, the initial velocity is $w \sin\theta$, assuming that velocities in the upward direction are positive. The acceleration due to gravity ($-g$) will cause the velocity of the jet to decline to zero and then fall back to earth. Its velocity on returning will be the same as when it started out, but in the opposite direction, that is, $-w \sin\theta$. Accordingly,

$$t = \frac{v-u}{a} \text{ becomes } \frac{(-w\sin\theta)-w\sin\theta}{-g} = \frac{2w\sin\theta}{g}.$$

The required relationship between the horizontal distance the jet will travel, (S), and the angle θ of the nozzle, will be

$$S = w\cos\theta \times \frac{2w\sin\theta}{g} = \frac{w^2}{g}(2\sin\theta\cos\theta) = \frac{w^2}{g}\sin 2\theta.$$

The derivative of S with respect to θ will be $\frac{dS}{d\theta} = \frac{w^2}{g} \times \cos 2\theta \times 2.$

Since w, g, and the factor 2 are all constants, the derivative will be zero when $\cos 2\theta$ is zero, which will be when 2θ is 90°. Therefore the distance S will be maximum when θ is 45°, which fortunately verifies the intuitive conclusion.

Table 2-2. Some Common Derivatives

Function F(t)	Derivative
a (constant)	0 (zero)
a t	a
t^n	$n\,t^{(n-1)}$
$\sin\omega t$	$\omega\cos\omega t$
$\cos\omega t$	$-\omega\sin\omega t$
$\tan\omega t$	$\frac{\omega}{\cos^2\omega t}$
$e^{\omega t}$	$\omega e^{\omega t}$
$e^{-\omega t}$	$-\omega e^{-\omega t}$
$\log_e t = \lvert\ln\rvert t$	$\frac{1}{t}$
For u and v both functions of t:	
$u+v$	$\frac{du}{dt}+\frac{dv}{dt}$
$u\,v$	$u\frac{dv}{dt}+v\frac{du}{dt}$
$\frac{u}{v}$	$\frac{v\frac{du}{dt}-u\frac{dv}{dt}}{v^2}$