4
Infinite Series

It sometimes happens that a function f(t) of the variable t appears as the sum of a number of terms, each of which in itself is a function of t. One such example is

$$f(t) = \sin \omega t + 2 \sin (\omega t)^2 + 3 \sin (\omega t)^3 + \text{etc.}$$

This type of function can become useful under the following circumstances.

- There is no limit to the number of terms.
- The format of the terms follows a discernible pattern.

In other words, if n designates the number of an individual term in the function (n = 1, 2, 3, 4, etc.), then a formula for the general, or n^{th}, term can be written. In the example above, the n^{th} term is $n \sin (\omega t)^n$.

If these two conditions are met, the function f(t) is called an *infinite series.*

An infinite series will often prove productive if it is the type for which the sum of the terms never exceeds a certain finite limit, no matter how many terms are added on. A series of this type is said to be *convergent.* An infinite series whose sum eventually becomes infinite as more and more terms are added is called *divergent.*

For example, the infinite series

$$f(t) = 1 + t + t^2 + t^3 + t^4 + \ldots + t^n + \ldots$$

may be convergent or divergent, depending on the value of t. If $t = 1$, then the sum goes to infinity as the number of terms becomes larger. If $t < 1$, however, the sum will be limited to a definite finite number, no matter how many terms are included. If $t = 1/2$, for instance, the sum will never be greater than 2.

An allied characteristic of a series that is convergent is that as the number of terms (n) gets larger, the value of the n^{th} term approaches zero. This is illustrated for the series above in Figure 4-1, with $t = ½$ and $t = 2$. When the n^{th} term is becoming larger as n increases, it is an indication that the sum of the series is going to infinity.

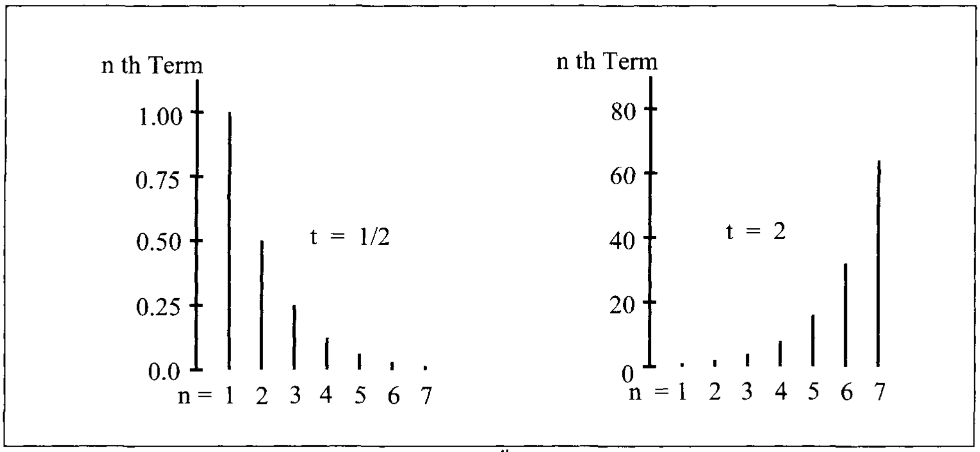

Figure 4-1. Effects of the value of t on the n^{th} term.

In the graphs in Figure 4-1, it should be noted that a curve joining the tips of the vertical bars representing the values of the terms has no significance. This is because n is an integer. No intermediate values lie between the whole numbers.

Power Series

A power series is a special form of infinite series, in which the terms are ascending powers of the variable, multiplied by a constant coefficient. The general form of a power series is

$$f(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + \ldots\ldots + a_nt^n + \text{etc.}$$

The n^{th} Term

If the first few terms of an infinite series are given, it may be possible to develop the formula for the n^{th} term. There is at least one reason for doing this, namely, the test for the convergence of the series (described later) requires the expression for the n^{th} term.

Usually the expression (formula) can be deduced from inspecting the terms progressively. Arranging the various factors in a table will be helpful. In a power series, it is necessary to establish three things: (1) the power, (2) the coefficient, and (3) the sign, in order to define the n^{th} term completely. An example will help to illustrate the method.

Evaluate the n^{th} term of the series

$$f(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7} + \frac{t^9}{9} - \text{etc.}$$

The exclamation mark (!) here is the factorial symbol.

The factors tabulate as follows:

Table 4-1. n^{th} Term Factors

Term	Power of t	Coefficient	Sign
1	1	1	+
2	3	1/3!	–
3	5	1/5!	+
4	7	1/7!	–
5	9	1/9!	+
n	2n – 1	$\frac{1}{(2n-1)!}$	$(-1)^{(n+1)}$

The n^{th} term of this series is consequently

$$(-1)^{(n+1)} \frac{1}{(2n-1)!} t^{(2n-1)}.$$

Test for Convergence

There are three established tests for convergence of an infinite series. Only one of these, the *ratio test,* will be mentioned here. It is based on determining the ratio of the $(n + 1)^{th}$ term to the n^{th} term. These two terms are, of course, consecutive terms. The ratio test is prescribed in the following way.

An infinite series $f(t) = u_1 + u_2 + u_3 + \ldots$ is convergent if the ratio

$$\frac{u_{(n+1)}}{u_n}$$

is numerically <1 as n approaches infinity, and divergent if the ratio is > or = 1 as n approaches infinity.

Example 1: Convergence Test 1

Test the series $f(t) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \text{etc.}$ for convergence.

The n^{th} term in this series is

$$\frac{1}{2^{(n-1)}}.$$

Substituting (n + 1) for n in the expression for the n^{th} term, gives

$$\frac{1}{2^n}$$

which is the $(n + 1)^{th}$ term. The ratio of the $(n + 1)^{th}$ term to the n^{th} term is

$$\frac{\frac{1}{2^n}}{\frac{1}{2^{(n-1)}}}$$

which simplifies to $\frac{1}{2}$. Since the result turns out to be <1, the series converges.

Example 2: Convergence Test 2

Taking the general case of Example 1, for the series

$$f(t) = 1 + t + t^2 + t^3 + t^4 + \text{etc.},$$

the n^{th} term is $t^{(n-1)}$, and the $(n + 1)^{th}$ term is t^n. The required ratio is

$$\frac{t^n}{t^{(n-1)}} = t.$$

This is <1 for values of $t < 1$. Therefore, if $t < 1$, the series converges. If $t > 1$, it diverges.

Example 3: Convergence Test 3

Test the series $f(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} -$ etc. for convergence.

As was previously deduced, the n^{th} term of this series is

$$(-1)^{(n+1)} \frac{t^{(2n-1)}}{(2n-1)!}.$$

The $(n + 1)^{th}$ term, which is obtained by replacing n with (n + 1), is consequently,

$$(-1)^{(n+2)} \frac{t^{(2n+1)}}{(2n+1)!}.$$

When determining the ratio, the magnitude only is relevant. The sign has no influence on whether or not the series converges.

The ratio of consecutive terms is $\frac{t^{(2n+1)}}{(2n+1)!} \div \frac{t^{(2n-1)}}{(2n-1)!}$

$$\frac{(2n-1)!}{(2n+1)!} \times \frac{t^{(2n+1)}}{t^{(2n-1)}} = \frac{t^2}{(2n+1) \times 2n}.$$

As n approaches infinity, this ratio approaches zero, a quantity obviously <1, provided that t is not infinite. Therefore the series

$f(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$ is convergent for all finite values of t.

When determining the $(n + 1)^{th}$ term, it is important to realize that the correct procedure is to substitute "(n + 1)" for "n" in the expression for the n^{th} term. This is not the same thing as adding 1 to the expression that involves the factor n. The fact that this second procedure sometimes yields the same answer is deceiving.

Example 4: The $(n+1)^{th}$ Term

It was previously determined that the n^{th} term of the series

$f(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$ is $(-1)^{(n+1)} \frac{1}{(2n-1)!} t^{(2n-1)}$.

To obtain the $(n+1)^{th}$ term, n is replaced by $(n+1)$ at each place where n appears. That is, the $(n+1)^{th}$ term will be

$$(-1)^{[(n+1)+1]}\frac{1}{[2(n+1)-1]!}t^{[2(n+1)-1]} = (-1)^{(n+2)}\frac{1}{(2n+1)!}t^{(2n+1)}.$$

Notice that this is not the same as adding 1 in each expression in which n appears. The $(n+1)^{th}$ term *is not*

$$(-1)^{(n+1+1)}\frac{1}{(2n-1+1)!}t^{(2n-1+1)} = (-1)^{(n+2)}\frac{1}{2n!}t^{2n}.$$

Maclaurin's Series

Some mathematical functions can be represented by an infinite series, provided that the function is to be evaluated for values of the variable quantity in the series which will cause the series to converge. In the language of mathematicians, this is referred to as being "within the region of convergence" of the series.

Suppose that $f(t) = a_O + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$, and that the values of t are restricted to those that cause the series to converge. The problem is to find the values of the various constant coefficients $a_0, a_1, a_2, a_3, \ldots$. Since the function $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \ldots$ holds for any value of t that produces convergence, the coefficients can be determined by taking successive derivatives with respect to t for both sides of the equation, and then substituting $t = 0$ in the result. ($t = 0$ is a value within the region of convergence.)

This process can be illustrated as follows.

Suppose that f(t) is $\sin t = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots$ Taking the derivative with respect to t for both side results in:

$$\frac{d}{dt} = \cos t = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \ldots$$

$$\frac{d^2}{dt^2} = -\sin t = 1\times 2a_2 + 2\times 3a_3 t + 3\times 4a_4 t^2 + 4\times 5a_5 t^3 \ \ldots$$

$$\frac{d^3}{dt^3} = -\cos t = 1\times 2\times 3a_3 + 2\times 3\times 4a_4 t + 3\times 4\times 5a_5 t^2 + \ldots$$

$$\frac{d^4}{dt^4} = \sin t = 1\times2\times3\times4 a_4 + 2\times3\times4\times5 a_5 t + \ldots$$

$$\frac{d^5}{dt^5} = \cos t = 1\times2\times3\times4\times5 a_5 \ldots$$

In each of these equations in t, substitute t = 0.

For f(t), sin (0) = 0 = a_O, and all of the terms following a_O will be zero because they contain t to some power. Thus $a_O = 0$.

$$\text{For } \frac{d}{dt};\ \cos(0) = 1 = a_1.\ \therefore a_1 = 1.$$

$$\text{For } \frac{d^2}{dt^2};\ -\sin(0) = 0 = 2!\times a_2.\ \therefore a_2 = 0.$$

$$\text{For } \frac{d^3}{dt^3};\ -\cos(0) = -1 = 3!\times a_3.\ \therefore a_3 = -\frac{1}{3!}.$$

$$\text{For } \frac{d^4}{dt^4};\ \sin(0) = 0 = 4!\times a_4.\ \therefore a_4 = 0\cdot$$

$$\text{For } \frac{d^5}{dt^5};\ \cos(0) = 1 = 5!\times a_5.\ \therefore a_5 = +\frac{1}{5!}\cdot$$

From the pattern of the values of the coefficients, the conclusion is that

$$\sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \ldots$$

The general case can now be developed. Designate f(t) = F.

$$F = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \ldots$$

$$\frac{dF}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \ldots$$

$$\frac{d^2F}{dt^2} = 1\times2\times a_2 + 2\times3\times a_3 t + 3\times4\times a_4 t^2 + 4\times5\times a_5 t^3 + \ldots$$

$$\frac{d^3F}{dt^3} = 1 \times 2 \times 3 \times a_3 + 2 \times 3 \times 4 \times a_4 t + 3 \times 4 \times 5 \times a_5 t^2 + \ldots$$

$$\frac{d^4F}{dt^4} = 1 \times 2 \times 3 \times 4 \times a_4 + 2 \times 3 \times 4 \times 5 \times a_5 t + \ldots$$

Substituting t = 0 in each of these relations, results in

$$F(0) = a_O \text{ and } a_O = F(0)$$

$$\frac{dF}{dt}(0) = a_1 \text{ and } a_1 = \frac{dF}{dt}(0)$$

$$\frac{d^2F}{dt^2}(0) = 1 \times 2 \times a_2 \text{ and } a_2 = \frac{1}{2!}\frac{d^2F}{dt^2}(0)$$

$$\frac{d^3F}{dt^3}(0) = 1 \times 2 \times 3 \times a_3 \text{ and } a_3 = \frac{1}{3!}\frac{d^3F}{dt^3}(0)$$

$$\frac{d^4F}{dt^4}(0) = 1 \times 2 \times 3 \times 4 \times a_4 \text{ and } a_4 = \frac{1}{4!}\frac{d^4F}{dt^4}(0).$$

The general expression for f(t) (= F) is consequently

$$F = F(0) + \frac{dF}{dt}(0)t + \frac{1}{2!}\frac{d^2F}{dt^2}(0)t^2 + \frac{1}{3!}\frac{d^3F}{dt^3}(0)t^3 + \frac{1}{4!}\frac{d^4F}{dt^4}(0)t^4 + \frac{1}{5!}\frac{d^5F}{dt^5}(0)t^5 + \ldots$$

This expansion is called Maclaurin's series.

Example 5: $e^{\omega t}$

Expand $f(t) = e^{\omega t}$ and determine the region of convergence.

For convenience, designate f(t) = F.

$$F = e^{\omega t} \text{ and } F(0) = 1$$

$$\frac{dF}{dt} = \omega e^{\omega t} \text{ and } \frac{dF}{dt}(0) = \omega$$

$$\frac{d^2F}{dt^2} = \omega^2 e^{\omega t} \text{ and } \frac{d^2F}{dt^2}(0) = \omega^2$$

$$\frac{d^3F}{dt^3} = \omega^3 e^{\omega t} \text{ and } \frac{d^3F}{dt^3}(0) = \omega^3$$

$$\frac{d^4F}{dF^4} = \omega^4 e^{\omega t} \text{ and } \frac{d^4F}{dF^4}(0) = \omega^4$$

Accordingly,

$$e^{\omega t} = 1 + \omega t + \frac{1}{2!}\omega^2 t^2 + \frac{1}{3!}\omega^3 t^3 + \frac{1}{4!}\omega^4 t^4 + \ldots$$

The region of convergence can be determined from the table of successive terms, and the form of the n$^{\text{th}}$ term.

Term	Power	Coefficient	Sign
1	0	1	+
2	1	ω	+
3	2	$\frac{1}{2!}\omega^2$	+
4	3	$\frac{1}{3!}\omega^3$	+
5	4	$\frac{1}{4!}\omega^4$	+
n	(n – 1)	$\frac{1}{(n-1)!}\omega^{(n-1))}$	+

The n$^{\text{th}}$ term is consequently

$$+\frac{\omega^{(n-1)}}{(n-1)!}t^{(n-1)}$$

The (n + 1)$^{\text{th}}$ term will be

$$+\frac{\omega^{[(n+1)-1]}}{[(n+1)-1]!}t^{(n+1)-1} = +\frac{1}{n!}\omega^n t^n.$$

The ratio of consecutive terms is

$$\frac{\omega^n t^n}{n!} \div \frac{\omega^{(n-1)}t^{(n-1)}}{(n-1)!} = \frac{(n-1)!}{n!} \times \frac{\omega^n t^n}{\omega^{(n-1)}t^{(n-1)}} = \frac{1}{n}\omega t.$$

As n approaches infinity, the limit of

$$\frac{\omega t}{n}$$

approaches zero for all finite values of ω and t. Therefore, the expansion for $e^{\omega t}$ is valid for all finite values of ω and t.

Practical Disadvantage of Maclaurin's Series

The value of the function f(t) can be evaluated by expanding it in a Maclaurin's series provided that the value of t is fairly small so that the t^2, t^3, t^4, etc. terms diminish rapidly. Thus, the value for f(t) would be obtainable to the desired degree of accuracy without having to calculate very many terms. On the other hand, if t were close to the maximum value it could assume without causing the series to diverge, then the values of a great many terms would have to be calculated to obtain the value of f(t) to the desired accuracy.

Taylor's Series

To get around the problem with the Maclaurin's series, one approach could be to develop a different series, not in powers of t, but as a series of powers of (t – a), where a is a constant. That is,

$$f(t) = F = a_O + a_1 (t - a) + a_2 (t - a)^2 + a_3 (t - a)^3 + a_4 (t - a)^4 + \dots$$

Successively taking the derivative with respect to t, gives

$$\frac{dF}{dt} = a_1 + 2a_2(t-a) + 3a_3(t-a)^2 + 4a_4(t-a)^3 + \dots$$

$$\frac{d^2F}{dt^2} = 1\times2\times a_2 + 2\times3\times a_3(t-a) + 3\times4\times a_4(t-a)^2 + \dots$$

$$\frac{d^3F}{dt^3} = 1\times2\times3\times a_3 + 2\times3\times4\times a_4(t-a) + \dots$$

$$\frac{d^4F}{dt^4} = 1\times2\times3\times4\times a_4 + 2\times3\times4\times5\times a_5(t-a) + \dots$$

Setting t = a in all of these expressions, so that (t – a) = 0 gives,

$$F(a) = a_0 \text{ and } a_0 = F(a)$$

$$\frac{dF}{dt}(a) = a_1 \text{ and } a_1 = \frac{dF}{dt}(a)$$

$$\frac{d^2F}{dt^2}(a) = 1\times2\times a_2 \text{ and } a_2 = \frac{1}{2!}\frac{d^2F}{dt^2}(a)$$

$$\frac{d^3F}{dt^3}(a) = 1\times2\times3\times a_3 \text{ and } a_3 = \frac{1}{3!}\frac{d^3F}{dt^3}(a)$$

$$\frac{d^4F}{dt^4}(a) = 1\times2\times3\times4\times a_4 \text{ and } a_4 = \frac{1}{4!}\frac{d^4F}{dt^4}(a).$$

Therefore,

$$f(t) = F = F(a) + \frac{dF}{dt}(a)(t-a) + \frac{1}{2!}\frac{d^2F}{dt^2}(a)(t-a)^2 + \frac{1}{3!}\frac{d^3F}{dt^3}(a)(t-a)^3 + \ldots.$$

In this expression, F(a) is the value of f(t) when t = a. $\frac{dF}{dt}(a)$ is the value of the first derivative of f(t) when t = a. The higher derivatives follow in the same manner.

This expansion is Taylor's series. To apply it, it is necessary to know the value of f(t) and all of its relevant derivatives at t = a. Furthermore, if t = b is the value at which f(t) is to be evaluated, then the value of a should be selected as close as possible to the value of b so that the $(b-a)^2$, $(b-a)^3$, etc. terms will diminish quickly, and fewer terms will be required to calculate the value of f(t) to the desired accuracy.

The Taylor series is an expansion of the function f(t) using the value t = a as the origin or starting point, rather than zero. Comparing the Taylor series with the Maclaurin series shows that the Maclaurin series is the particular case of the Taylor series in which a = zero.

Example 6: Value of a Sine Function Using Taylor's Series

Determine the value of sin 32° to three decimal places. This could best be done by developing a Taylor series using a = 30° as the origin, since this value is close to the desired value, and the values of sin 30° and cos 30° are known to be

$$\frac{1}{2} \text{ and } \frac{\sqrt{3}}{2}, \text{ respectively.}$$

For the calculations to be correct, the values of the angles involved must be expressed in their fundamental units of radians, rather than degrees. Thus a = 30° = 0.524 rad. For f(t) = F = sin t,

$$F(0.524) = \sin 0.524 = 0.500$$

$$\frac{dF}{dt}(0.524) = \cos 0.524 = 0.866$$

$$\frac{d^2F}{dt^2}(0.524) = -\sin 0.524 = -0.500$$

$$\frac{d^3F}{dt^3}(0.524) = -\cos 0.524 = -0.866\cdot$$

Therefore,

$$\sin t = 0.500 + 0.866(t - 0.524) - \frac{0.500}{2!}(t - 0.524)^2 - \frac{0.866}{3!}(t - 0.524)^3 + \ldots$$

Calculating for t = 32° = 0.559 rad,

The first term is + 0.500.

The second term is + 0.866 x (0.559 – 0.524) = 0.030.

The third term is $-\frac{0.500}{2}(0.559 - 0.524)^2 = -0.00031\cdot$

There is no merit to calculating additional terms since they are not large enough to have any bearing on the value of sin 32° to three decimal places.

Accordingly, sin 32° = 0.500 + 0.030 = 0.530.