The Algebras of a Tangent Bundle Monad

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Talk Outline

- The tangent bundle functor is a monad
- Example: the category of commutative rings
- Example: the category of affine schemes (my project)

Recall: Tangent Categories

A *tangent category* is a category $\mathscr{C}$ and functor $T : \mathscr{C} \to \mathscr{C}$, equipped with five natural transformations.

$$
\begin{array}{llcccl}
(1) & \textit{projection} & p & : & T & \longrightarrow 1 \\
(2) & \textit{zero} & 0 & : & 1 & \longrightarrow T \\
(3) & \textit{sum} & + & : & T_2 & \longrightarrow T \\
(4) & \textit{lift} & \ell & : & T & \longrightarrow T^2 \\
(5) & \textit{flip} & c & : & T^2 & \longrightarrow T^2
\end{array}
$$

There are a bunch of properties these must satisfy. We're going to ignore most of them.

Constructing a monad

Our goal is to turn the *tangent functor* T into *monad*. We'll only need three of our natural transformations to get this done.

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I haven't defined this "T_2" functor, yet. Let's do that now.

One tangent category axiom says that for each $R : \mathscr{C}$, the *pullback* of p_R with itself must exist. We label this object T_2R.

$$\begin{tikzcd} T_2R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TR \arrow[d, "p_R"] \\ TR \arrow[r, "p_R"'] & R \end{tikzcd}$$

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Now, our *zero* and *sum* look a lot like a *unit* and *join*.

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Unfortunately T_2 isn't T^2. Can we define a $u : T^2 \longrightarrow T_2$? Well, by the naturality of p and the universal condition on T_2

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\begin{tikzcd}
T^2R \arrow[rr, "p_{TR}"] \arrow[dr, dashed, "u_R"'] \arrow[dd, "Tp_R"'] & & TR \arrow[dd, "p_R"] \\
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Constructing a monad

$$\eta \ : \ 1 \xrightarrow{0} T$$

$$\mu \ : \ T^2 \xrightarrow{u} T_2 \xrightarrow{+} T$$

CRing is a Tangent Category

We're going to equip CRing with a tangent structure and compute its *monad*. First, I'll specify the *tangent functor*.

$$\perp : R \longmapsto R[\varepsilon]$$

$$\perp(f)(a + b\varepsilon) = f(a) + f(b)\varepsilon$$

Along with a *projection* $p : \perp \to 1$ and a *zero* $0 : 1 \to \perp$.

$$p_R \ : \ a + b\varepsilon \longmapsto a$$

$$0_R \ : \ a \longmapsto a + 0\varepsilon$$

Before we can give the *sum*, we need the *pullback* of p_R with itself.

$$\begin{aligned} \perp_2 R \ &= \ \left\{(r,s) \in (\perp R)^2 : p_R(r) = p_R(s)\right\} \\ &= \ \left\{(a + b\varepsilon, a + c\varepsilon) \in R[\varepsilon] \times R[\varepsilon]\right\} \end{aligned}$$

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CRing is a Tangent Category

We define the *sum* $+_R : \perp_2 R \to R[\varepsilon]$ as

$$+_R \ : \ (a + b\varepsilon, a + c\varepsilon) \longmapsto a + (b + c)\varepsilon$$

Now that we have the *projection*, *zero*, and *sum*, we can compute the *monad*. Starting with $u : R[\varepsilon][\varepsilon'] \to \perp_2 R$

$$\begin{aligned} u(a + b\varepsilon + (c + d\varepsilon)\varepsilon') &= \langle p_{\perp R}, \perp p_R \rangle \left(a + b\varepsilon + (c + d\varepsilon)\varepsilon' \right) \\ &= (a + b\varepsilon, a + c\varepsilon) \end{aligned}$$

We get our *monad*:

$$\begin{aligned} \eta_R \ &: \ a \longmapsto a + 0\varepsilon \\ \mu_R \ &: \ a + b\varepsilon + (c + d\varepsilon)\varepsilon' \longmapsto a + (b + c)\varepsilon \end{aligned}$$

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The Algebras of the CRing Tangent Monad

A $\perp$-algebra is pair $R : \mathsf{CRing}$ and $\alpha : R[\varepsilon] \to R$, such that

$$\begin{tikzcd} \perp R \arrow[r, "\alpha"] & R \\ R \arrow[u, "\eta_R"] \arrow[ur, equal] & \end{tikzcd} \qquad \begin{tikzcd} \perp\perp R \arrow[r, "\perp(\alpha)"] \arrow[d, "\mu_R"'] & \perp R \arrow[d, "\alpha"] \\ \perp R \arrow[r, "\alpha"'] & R \end{tikzcd}$$

From the left diagram,

$$\begin{aligned} \alpha(r + s\varepsilon) &= \alpha(r) + \alpha(s)\alpha(\varepsilon) \\ &= \alpha(\eta_R(r)) + \alpha(\eta_R(s))\alpha(\varepsilon) \\ &= r + s \cdot \alpha(\varepsilon) \end{aligned}$$

$$\alpha(\varepsilon)^2 = \alpha(\varepsilon^2) = \alpha(0) = 0$$

So, α is entirely determined by a choice of nilpotent $\alpha(\varepsilon) \in R$.

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The Algebras of the CRing Tangent Monad

What about the maps between $\perp$-algebras? A morphism $f : R \to S$ is a map between $\perp$-algebras (R, α) and (S, β) if

$$\begin{tikzcd}
\perp R \arrow[r, "\alpha"] \arrow[d, "\perp(f)"'] & R \arrow[d, "f"] \\
\perp S \arrow[r, "\beta"'] & S
\end{tikzcd}$$

Expanding,

$$\begin{aligned}
f(\alpha(a + b\varepsilon)) &= \beta(f(a) + f(b)\varepsilon) \\
f(a) + f(b)f(\alpha(\varepsilon)) &= f(a) + f(b)\beta(\varepsilon) \\
\iff f(\alpha(\varepsilon)) &= \beta(\varepsilon)
\end{aligned}$$

It turns out the Eilenberg-Moore category of $\perp$ is isomorphic to NIL, the category of pointed rings with nilpotent points.

$\mathsf{CRing}^{\mathrm{op}}$ is a Tangent Category

We need to define, given a ring R, the *symmetric algebra* of the module of *Kähler differentials*,

$$\begin{aligned}\mathrm{Sym}_R(\Omega R) = R[\, d(r) : r \in R\,] \;/ \\ \langle\, d(r+s) = d(r) + d(s)\,\rangle \\ \langle\, d(rs) = rd(s) + sd(r)\,\rangle\end{aligned}$$

That is, we take the ring consisting of formal symbols $d(r)$ for each element $r \in R$ with relations imposed that make $d(-)$ a differential. This is the *tangent functor*

$$\top : R \longmapsto \mathrm{Sym}_R(\Omega R)$$

$$\top(f) = \begin{cases} r & \longmapsto f(r) \\ d(r) & \longmapsto d(f(r)) \end{cases}$$

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Along with a *projection* $p : 1 \to \top$ and *zero* $0 : \top \to 1$

$$p_R(r) = r + 0 + 0 + \cdots$$

$$0_R = \begin{cases} r & \longmapsto r \\ d(r) & \longmapsto 0 \end{cases}$$

What is the *pullback* of p_R with itself?

$$\top_2 R = \top R \otimes_R \top R$$

And a *sum* $+_R : \top R \to \top R \otimes_R \top R$

$$+_R = \begin{cases} r & \longmapsto \; r \otimes 1 \\ d(r) & \longmapsto \; d(r) \otimes 1 + 1 \otimes d(r) \end{cases}$$

$\mathsf{CRing}^{\mathrm{op}}$ is a Tangent Category

Let's construct u. What do elements of $\top^2 R$ look like?

$$\begin{matrix} r & d(r) \\ d'(r) & d'(d(r)) \end{matrix}$$

We compute $u : \top_2 \to \top^2$,

$$u_R = \langle p_{\top R}, \top p_R \rangle = \begin{cases} r \otimes 1 \longmapsto r \\ 1 \otimes r \longmapsto r \\ d(r) \otimes 1 \longmapsto d(r) \\ 1 \otimes d(r) \longmapsto d'(r) \end{cases}$$

and we get our *(co)monad*.

$$\begin{aligned} \epsilon_R &: r \longmapsto r,\ d(r) \longmapsto 0 \\ \delta_R &: r \longmapsto r,\ d(r) \longmapsto d(r) + d'(r) \end{aligned}$$

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The Algebras of the $\mathsf{CRing}^{\mathrm{op}}$ Tangent Monad

A $\top$-coalgebra is a pair $R : \mathsf{CRing}$ and $\gamma : R \to \mathrm{Sym}_R(\Omega R)$ such that

$$\begin{tikzcd} \top R \arrow[r, "\epsilon"] & R \\ R \arrow[u, "\gamma"] \arrow[ur, equal] & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \top R \arrow[d, "\top(\gamma)"] \\ \top R \arrow[r, "\delta_R"'] & \top\top R \end{tikzcd}$$

Write $\gamma(r) = \gamma_0(r) + \gamma_1(r) + \ldots$ where $\gamma_n(r) \in \mathrm{Sym}^n(\Omega R)$

$$r = \epsilon(\gamma(r)) = \gamma_0(r)$$

So, the left diagram says $\gamma(r) = r + \gamma_1(r) + \ldots$

The right diagram tells us that any "$d'(d(r))$s" must cancel.

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Example: take $\gamma(r) = r + \lambda d(r)$.

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\delta_R(\gamma(r)) &= \top(\gamma)(\gamma(r)) \\
\delta_R(r + \lambda d(r)) &= \top(\gamma)(r + \lambda d(r)) \\
r + \lambda d(r) + \lambda d'(r) &= \gamma(r) + \gamma(\lambda) d(\gamma(r)) \\
\lambda d'(r) &= \gamma(\lambda) d(\gamma(r)) \\
\lambda d'(r) &= (\lambda + \lambda d(\lambda))\big(d'(r) + d'(\lambda d(r))\big)
\end{aligned}
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Take $\lambda = 2e$. Notice $0 = d(0) = d(e^2) = 2ed(e)$

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\begin{aligned}
2ed'(r) &= (2e + 2ed(2e))\big(d'(r) + d'(2ed(r))\big) \\
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For nilpotent $e \in R$, $\gamma(r) = r + 2ed(r)$ is a $\top$-algebra. This constitutes a functor $\mathrm{Alg}(\perp) \to \mathrm{CoAlg}(\top)$

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The Algebras of the $\mathsf{CRing}^{\mathrm{op}}$ Tangent Monad

Facts about $\top$:

- There is a functor $\mathrm{Alg}(\perp) \to \mathrm{CoAlg}(\top)$
- $\top$ is representable in $\mathsf{CRing}^{\mathrm{op}}$, specifically $\top = (-)^{\perp\mathbb{Z}}$
- There is an adjunction $\top \dashv \perp$
- There is a mixed distributive law $\lambda : \perp\top \longrightarrow \top\perp$
- For every ring R the *projection* p_R is a $\top$-coalgebra.
- p_R is the only $\top$-coalgebra on $\mathbb{Z}[x]$
- There are many $\top$-coalgebras on $\mathbb{Z}[x, y]$

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- $\top$ is representable in $\mathsf{CRing}^{\mathrm{op}}$, specifically $\top = (-)^{\perp\mathbb{Z}}$
- There is an adjunction $\top \dashv \perp$
- There is a mixed distributive law $\lambda : \perp\top \longrightarrow \top\perp$
- For every ring R the *projection* p_R is a $\top$-coalgebra.
- p_R is the only $\top$-coalgebra on $\mathbb{Z}[x]$
- There are many $\top$-coalgebras on $\mathbb{Z}[x, y]$

The Algebras of the $\mathsf{CRing}^{\mathrm{op}}$ Tangent Monad

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