Lectures 2 & 3: Time Series, ARIMA Models

This lesson is based on material by [Robert Nau, Duke University]

Time Series Data

A time series consists of repeated observations of a single variable, y, at various times, t.

$$\mathbf{y} = \{y_1, y_2, y_3, \ldots, y_t\}$$

We seek to predict y_{t+1} using the information from previous observations **y**.

Time Series Data

In order to estimate y_{t+1}, we need to find the effect of previous observations of y on the upcoming period. We might write this model as

$$y_{t+1} = \alpha + \sum_{s=1}^{t} \beta_s \cdot y_s + \epsilon$$

Time Series Data

If we choose to base our model solely on the previous period, then the model would be written

$$y_{t+1} = \alpha + \beta_t \cdot y_t + \epsilon$$

Critically, OLS estimates of this model are invalid.

Autocorrelation

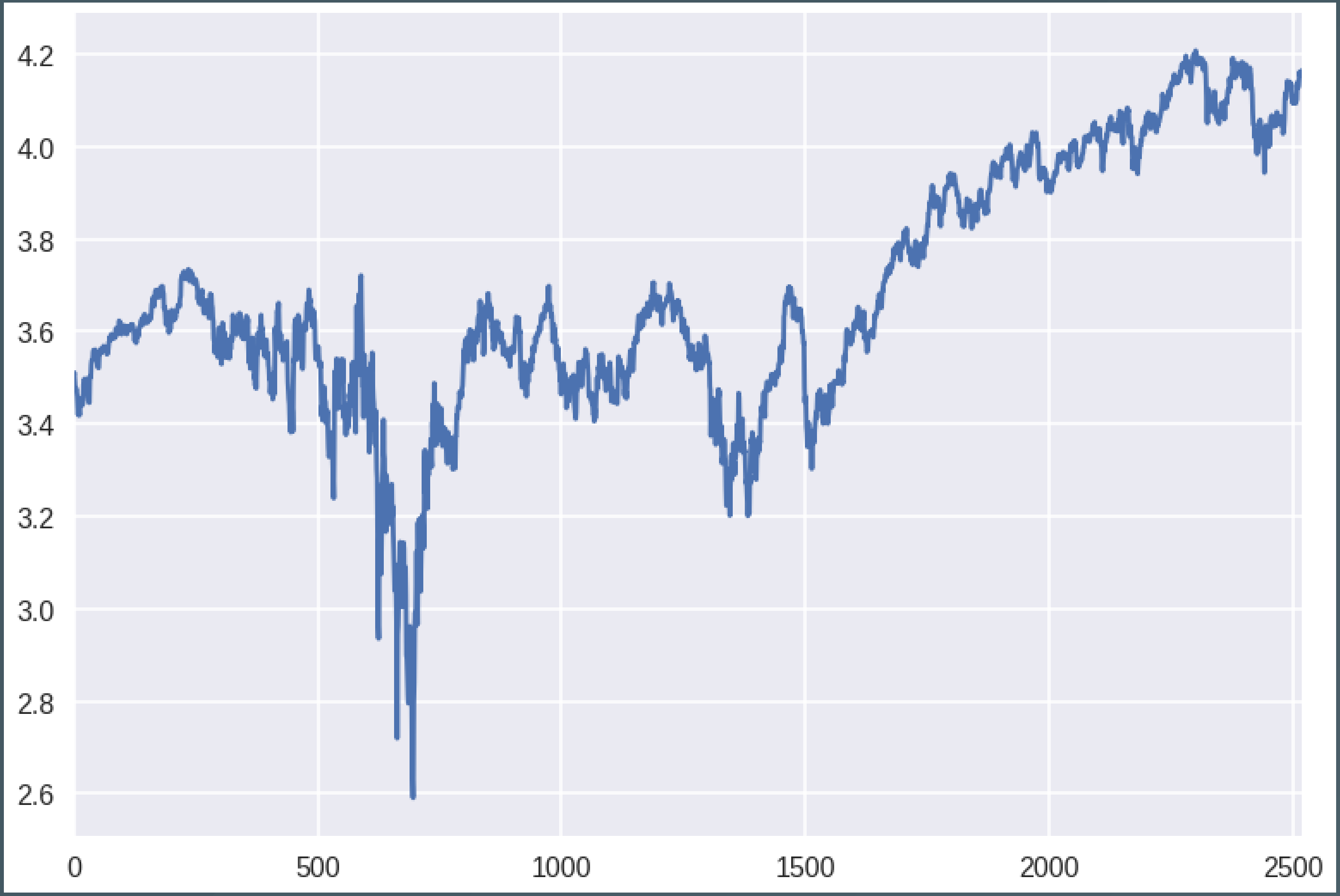
One of the primary assumptions of the OLS model is that

$$Cov(\epsilon_t, \epsilon_s) = 0,\ \forall\ t \neq s$$

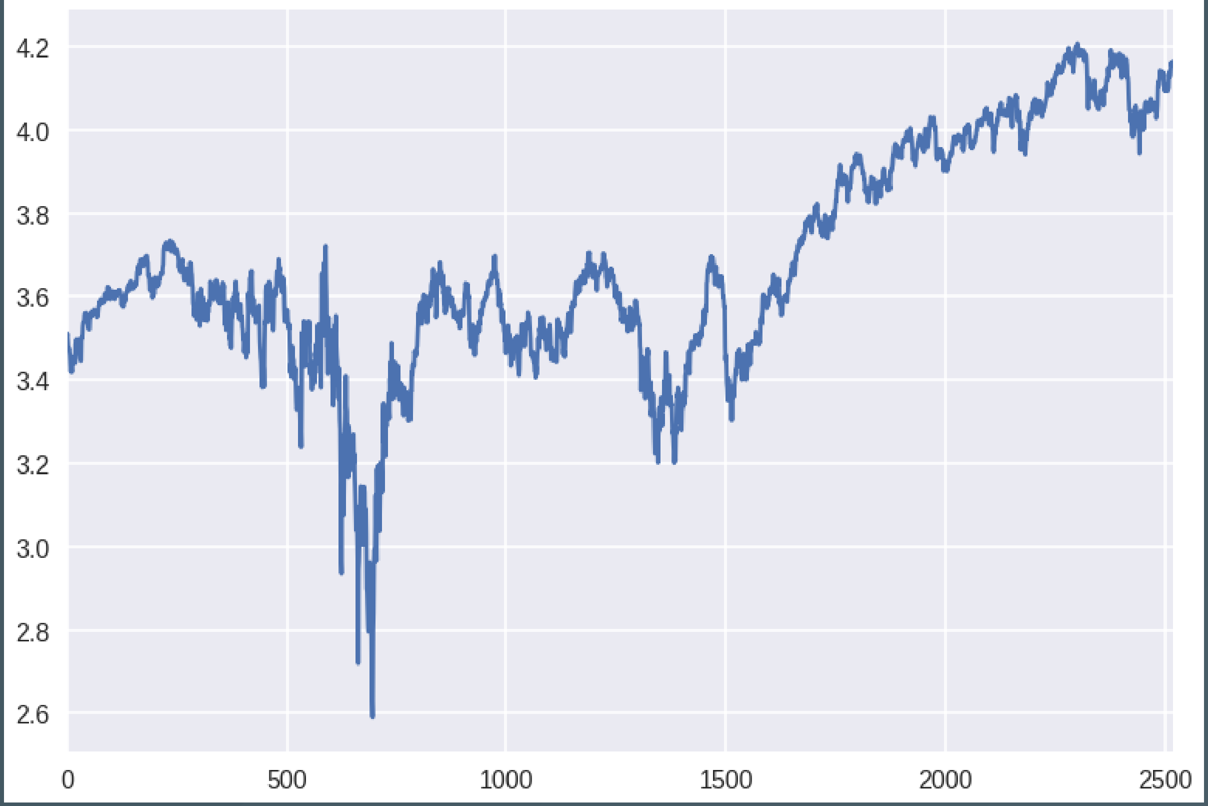
This assumption is clearly **not** valid in the case of time series data.

Let's look at some data to find out why.

Autocorrelation



Autocorrelation



We need to find a model that can eliminate the autocorrelation almost always seen in time series data.

Autoregressive Models

AR models are based on the premise that deviation from the underlying trend in the data persists in all future observations.

$$y_t = \alpha + \sum_{i=1}^{p} \rho_i \cdot y_{t-i} + \epsilon_t$$

Where ρ is the correlation term between periods and ϵ is an error (shock) term

AR Models

- We need to consider lagged observations of y in order to predict future outcomes
- The number of lags that we include is the **order** of our AR model
  - The model is an AR(p) Model, where p is the order of the model

AR Models

- The AR coefficients tell us how quickly a model returns to its mean
    - If the coefficients on AR variables add up to close to 1, then the model reverts to its mean **slowly**
    - If the coefficients sum to near zero, then the model reverts to its mean **quickly**

Integrated Models

Integration occurs when a process is non-stationary. A non-stationary process is one that contains a linear time trend. One example might be a long-term series of stock prices:

Integrated Models

We need to ensure that our data is stationary. To do so, we need to remove the time-trend from the data.

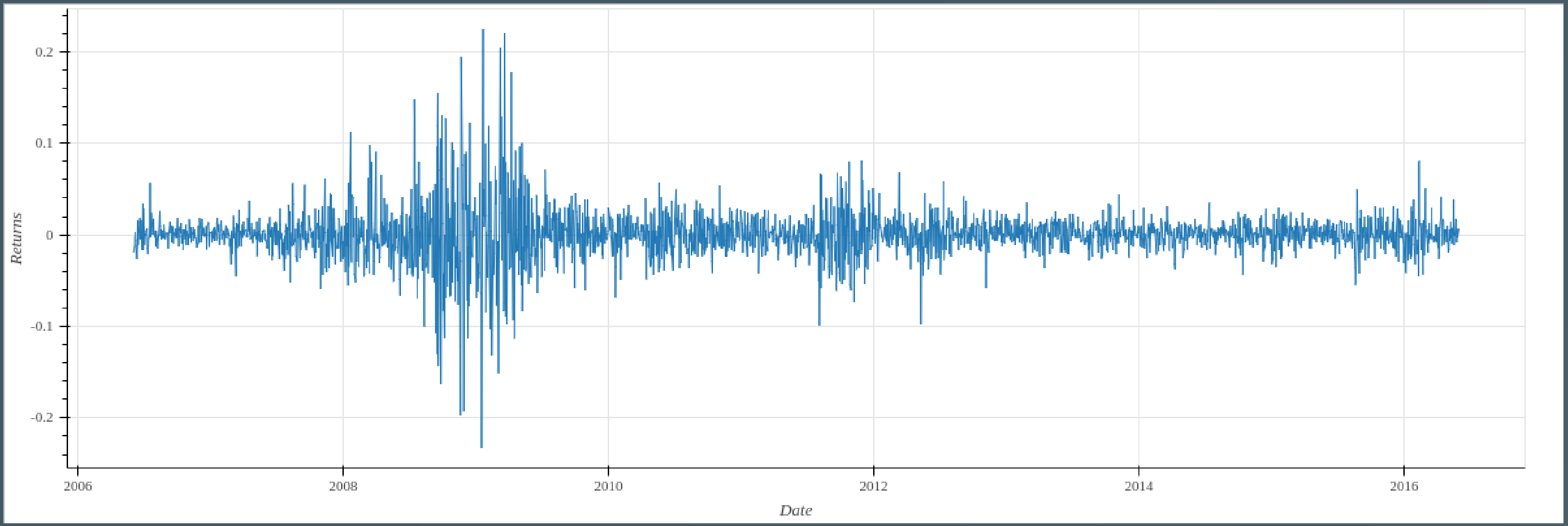
- This is typically done through differencing

$$y_i^s = y_i - y_{i-1}$$

where y_t^s is the stationary time series based on the original series y_t

Integrated Models

Here, the time trend has been differenced out of the data from the previous plot



Integrated Models

The Integration term d represents the number of differencing operations performed on the data:

- I(1): $y^s_t = y_t - y_{t-1}$
- I(2): $y^s_t = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$

Where an I(2) model is analogous to a standard difference-in-differences model applied to time-series data.

Moving Average Models

While an AR(·) model accounts for previous values of the dependent variable, MA(·) models account for previous values of the **error** terms:

$$AR(p) = \alpha + \sum_{i=1}^{p} \rho_i \cdot y_{t-i} + \epsilon_t$$

$$MA(q) = \alpha + \sum_{i=1}^{q} \theta_i \cdot \epsilon_{t-i} + \epsilon_t$$

Moving Average Models

An MA model suggests that the current value of a time-series depends linearly on previous error terms.

- Current value depends on how far away from the underlying trend previous periods fell
- The larger θ becomes, the more persistent those error terms are

Moving Average Models

- AR models' effects last infinitely far into the future
  - Each observation is dependent on the observation before
- In an MA model, the effect of previous periods only persists q periods into the past
  - Each error is uncorrelated with previous errors

Putting it Together

In order to account for all the problems that we might encounter in time series data, we can make use of ARIMA models.

Auto**R**egressive **I**ntegrated **M**oving **A**verage models allow us to

- Include lags of the dependent variable
- Take differences to eliminate trends
- Include lagged error terms

Putting it Together

Even better, we can use ARIMA**X** models to include exogenous regressors in our estimations!

Now we just need to understand how to decide on the correct specifications for our model.

The ARIMA(X) Model

ARIMA models are often referred to as ARIMA(p, d, q) models, where p, d, and q are the parameters denoting the order of the autoregressive terms, integration terms, and moving average terms, respectively.

- It is often a matter of guessing and checking to find the correct specification for a model
- We can use the ACF and PACF graphs to visually determine the order of our model

The Autocorrelation Function (ACF)

The ACF illustrates the correlation between a dependent variable and its lags.

- Choose how many lags to explore (based on nature of data)
- **Reminder**: correlations will vary between -1 and 1, with 1 being perfect correlation, and -1 being perfect inverse correlation
- Correlation can be cyclical!

The Autocorrelation Function (ACF)

The Partial Autocorrelation Function

The PACF illustrates the correlation between a dependent variable and its lags, **after controlling for lower-order lags**.

- Choose how many lags to explore (based on nature of data)

The Partial Autocorrelation Function (PACF)

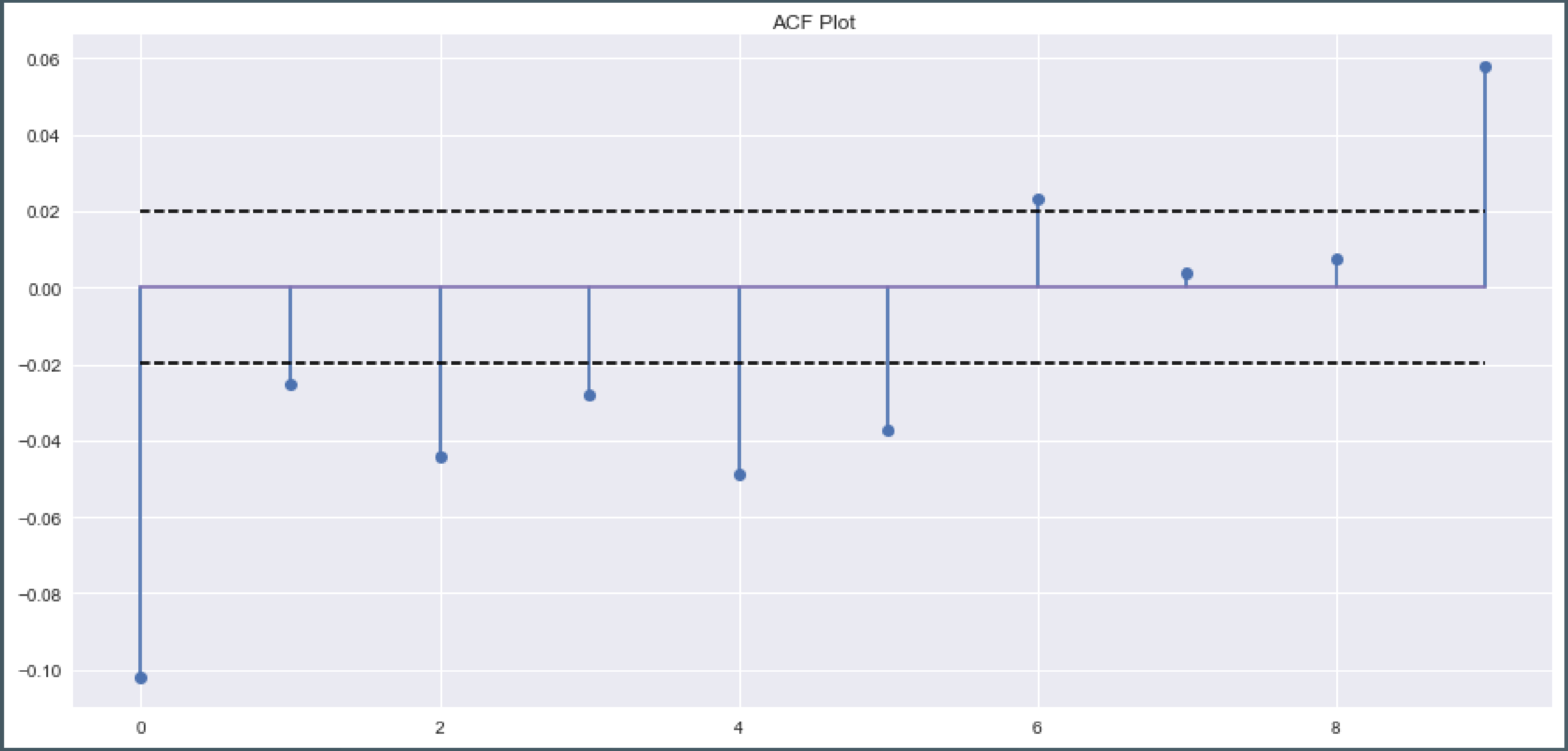
Building the Model

1. Make the series **stationary**
    - When the ACF falls "quickly" to zero at higher lags, the series is stationary
    - Can also use a **unit root test** to check for stationarity

Building the Model

Building the Model

Stationary:



Building the Model

1. Make the series **stationary**
2. Use ACF and PACF plots to decide if you should include **AR** or **MA** terms in your model
    - Typically, we do not use both in the same model

Building the Model

Signatures of **AR** and **MA** models:

AR Model: ACF dies out gradually, and the PACF cuts off sharply after a few lags

MA Model: ACF cuts off sharply, and PACF dies off more gradually (remember that **MA** models are based on previous *errors*)

Building the Model

1. Make the series **stationary**
2. Use ACF and PACF plots to decide if you should include **AR** or **MA** terms in your model
3. Fit the model, and check residual ACF and PACF for lingering significance
4. If there are significant terms in residual ACF or PACF, add **AR** or **MA** terms, and try again

ARIMA(X) in Python

```
# Import pandas, numpy, and libraries for ARIMA models,
#     for tools such as ACF and PACF functions, plotting,
#     and for using datetime formatting
import pandas as pd
import numpy as np
from statsmodels.tsa.arima_model import ARIMA
import statsmodels.tsa.stattools as st
import matplotlib.pyplot as plt
from datetime import datetime

# Import the pandas datareader function
from pandas_datareader.data import DataReader

# Collect data
a = DataReader('AAPL', 'yahoo', datetime(1990,6,1),
                datetime(2016,6,1))
```

ARIMA(X) in Python

```
# Generate DataFrames from raw data
a_ts = pd.DataFrame(np.log(a['Adj Close'].values))
a_ts.columns = ["Index"]
a_ts['date'] = a.index.values
```

Here, we generate the time-series data that we will work with as we explore our ARIMA(X) models

- Take the logged price data, and put it into a separate DataFrame for analysis

ARIMA(X) in Python

```
# Plot the data
p = figure(plot_width = 1200, plot_height=400,
        y_axis_label="Log Value",
        x_axis_label="Date",
        x_axis_type="datetime")
p.line(a_ts['date'], a_ts['Index'])
show(p)
```

ARIMA(X) in Python

```
# Generate plot from ACF
acf, aint=st.acf(a_ts['Index'], nlags=10, alpha=.05)
# Create figure, add ACF values
p = figure(plot_width = 800, plot_height = 600)
p.vbar(x = list(range(1,11)), width = 0.5, top = acf[1:],
        bottom = 0)
# Confidence Intervals
p.line(list(range(1,11)), [1/np.sqrt(len(a_ts))]*10,
        color = 'black', line_dash = "dashed")
p.line(list(range(1,11)), [-1/np.sqrt(len(a_ts))]*10,
        color = 'black', line_dash = "dashed")
show(p)
```

ACF Plot

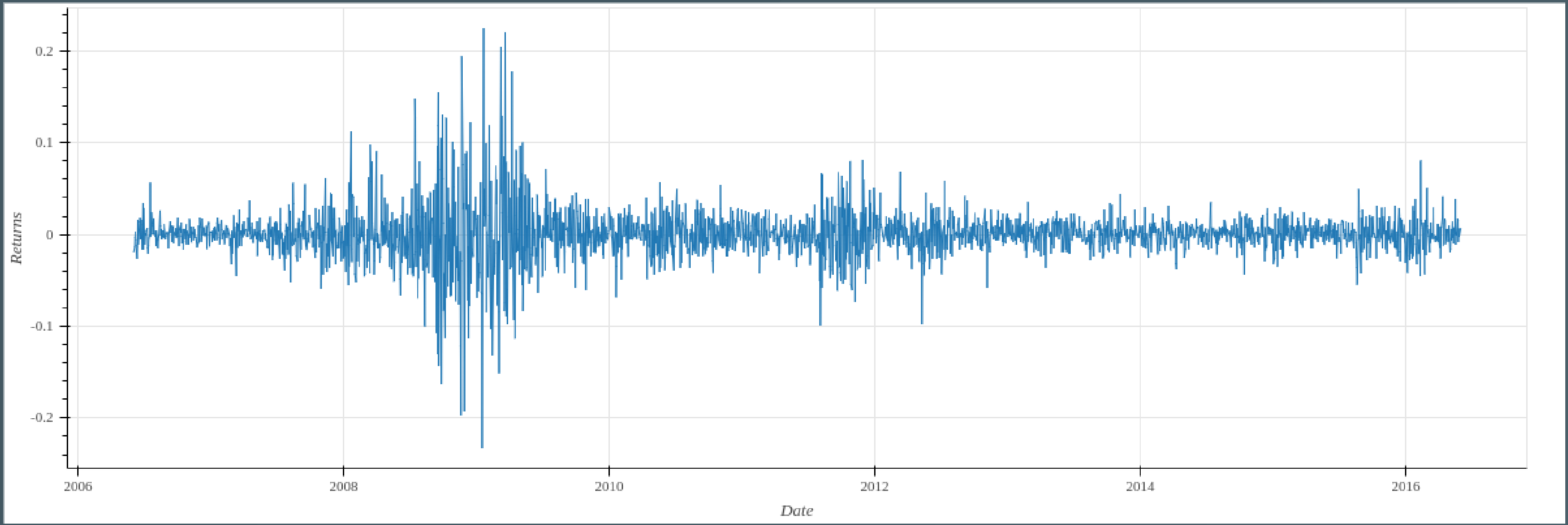
ARIMA(X) in Python

```
# Generate plot from PACF
pacf, paint=st.pacf(a_ts['Index'], nlags=10, alpha=.05)
# Create figure, add ACF values
p = figure(plot_width = 800, plot_height = 600)
p.vbar(x = list(range(1,11)), width = 0.5, top = pacf[1:],
        bottom = 0)
# Confidence Intervals
p.line(list(range(1,11)), [1/np.sqrt(len(a_ts))]*10,
        color = 'black', line_dash = "dashed")
p.line(list(range(1,11)), [-1/np.sqrt(len(a_ts))]*10,
        color = 'black', line_dash = "dashed")
show(p)
```

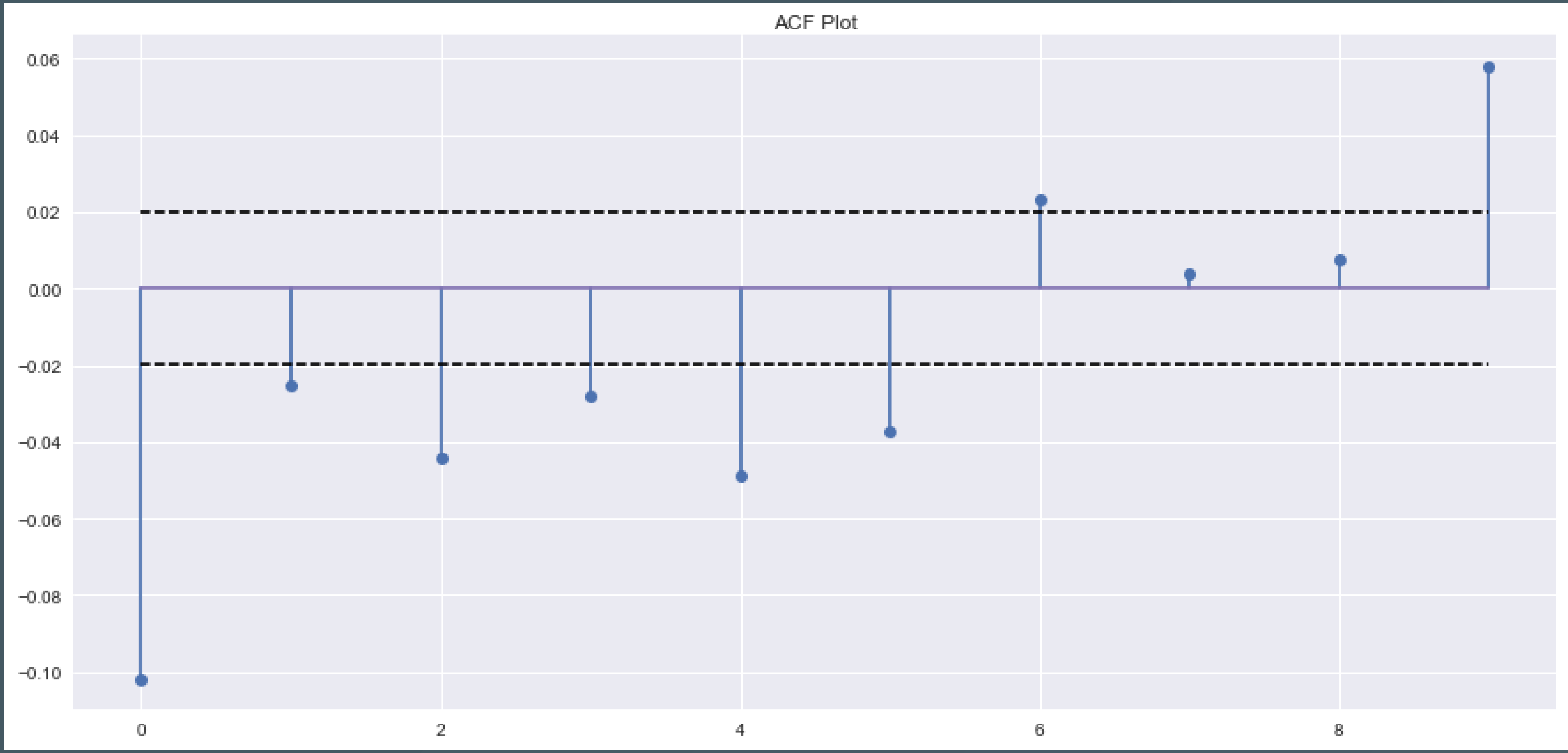
PACF Plot

ARIMA(X) in Python

```
# Plot first differences
plt.figure(figsize=(15, 5))
plt.ylabel("Returns")
plt.plot(np.diff(a_ts["Index"])[1:]) # plot differenced
plt.show()                           # log price series
```



Differenced ACF Plot



This looks a lot more like white noise than the undifferenced ACF plot!

Finding the Right Fit

- Time series models are unique in Econometrics: we need to **visually** diagnose the proper specifications for our model
  - This takes practice
  - This takes repetition and iteration for any given model

Fitting the ARIMA(X) model

```
from statsmodels.tsa.arima_model import ARIMA

model = ARIMA(a_ts, (1,1,1)) # Use a_ts data to fit an
                             # ARIMA(1,1,1) model
reg = model.fit() # Fit the model using standard params
res = reg.resid # store the residuals as res
```

Once we fit the ARIMA model using our selected specification, we can then explore the residual ACF and PACF of the model.

Fitting the ARIMA(X) model

Residual ACF (the text is dark)

Fitting the ARIMA(X) model

Residual PACF (the text is dark) - nearly identical to the ACF plot (what does that mean?)

Looking Ahead

Now that we have a fitted model, we can start to make predictions

```
fcst = reg.forecast(steps=10) # Generate forecast
future = pd.DatetimeIndex(start=datetime(2016,6,2),
                          freq='D', periods=10) # Index
predicted = pd.DataFrame(fcst[0], columns = ['Index'],
                         index = future) # Map forecast
upper = fcst[2][:,1] # Specify upper 95% CI
lower = fcst[2][:,0] # Specify lower 95% CI
```

We make our out-of-sample forecast, and store it as a DataFrame, with dates as index values

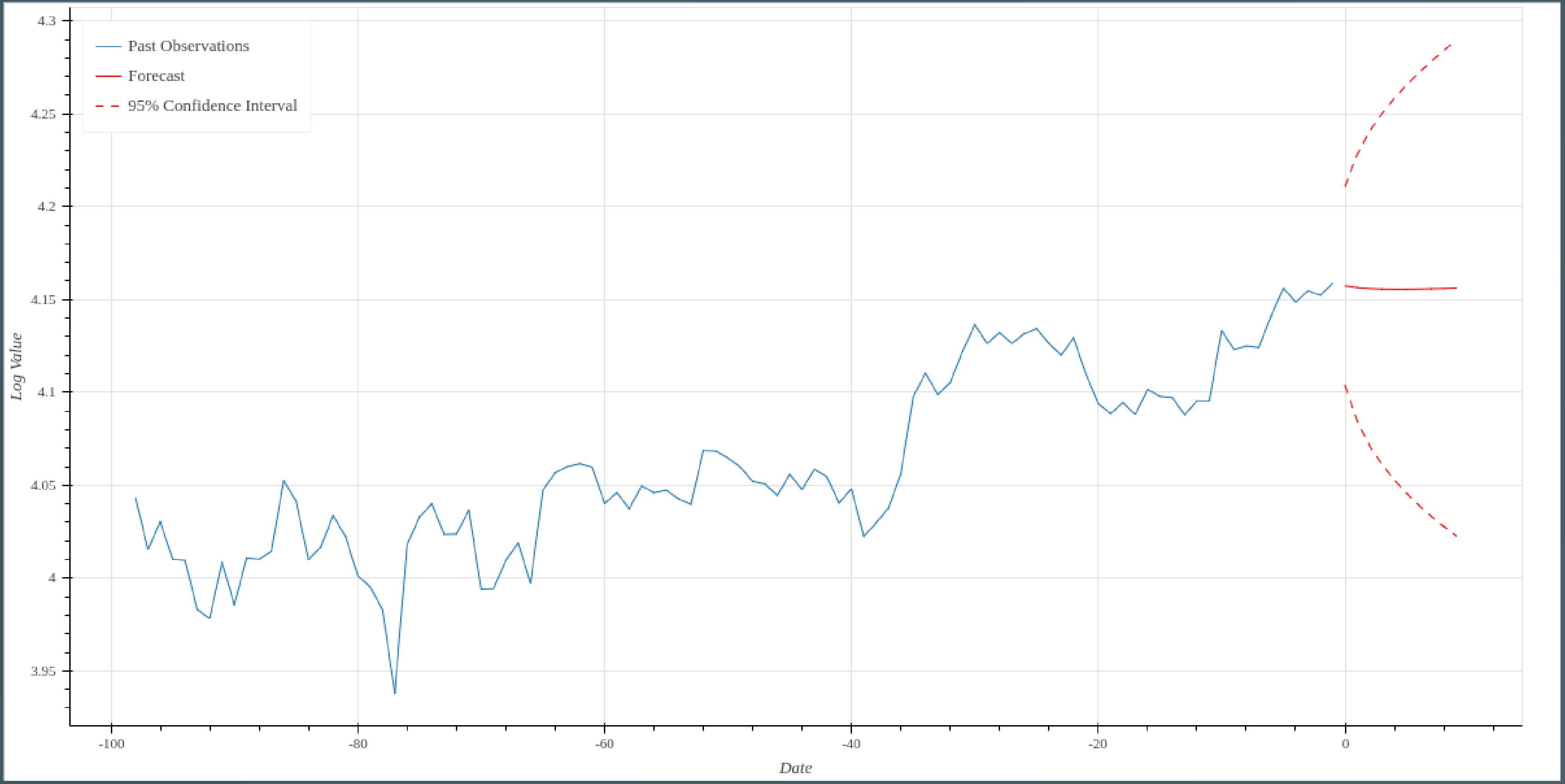
Looking Ahead

```
p = figure(plot_width = 1200, plot_height=400,
        y_axis_label="Log Value",
        x_axis_label="Date")
p.line(list(range(-98,0)), a_ts['Index'][-98:],
        legend="Past Observations")
rng = list(range(0,10))
p.line(rng, predicted['Index'], color = 'red',
        legend="Forecast")
p.line(rng, upper, color = 'red', line_dash = 'dashed',
        legend="95% Confidence Interval")
p.line(rng, lower, color = 'red', line_dash = 'dashed')
p.legend.location="top_left"
show(p)
```

We can then take a look at how our prediction follows the pattern from our time series

Looking Ahead

Plotting the forecast,



Finding the Right Fit

In order to more carefully choose the proper specification, we will actually select a preliminary model, and then investigate the **residual** ACF and PACF plots

- We can start with an ARIMA(0,1,0) model, or perhaps an ARIMA(1,0,0) model
- Base our starting point on the shape of our time series, or intuition about the data

For lab today:

Working with your group, use the Omaha historic weather data (using all but the final 10 days) to:

- Plot the data
- Make the data stationary
- Fit an ARIMA model
- Validate the model by plotting residuals
- Forecast temperature 10 days into the future, and send me your forecast, so we can compare to the real temperature