T6. Fracciones Parciales

CASO 1:

(1) $\int \frac{x-1}{x^3 - x^2 - 2x}\, dx = \int \frac{x-1}{x(x^2-x-2)} = \int \frac{x-1}{x(x+1)(x-2)}$

$$\left[\frac{x-1}{x(x+1)(x-2)} = \frac{A_1}{x} + \frac{A_2}{x+1} + \frac{A_3}{x+2} \right] (x)(x+1)(x-2)$$

$$x-1 = A_1(x+1)(x-2) + A_2(x)(x-2) + A_3(x)(x+1)$$

$$x-1 = A_1(x^2-x-2) + A_2(x^2-2x) + A_3(x^2+x)$$

$A_1 + A_2 + A_3 = 0$ $\qquad \therefore\ A_1 = \frac{1}{2}$

$-A_1 - 2A_2 + A_3 = 1$

$-2A_1 = -1$

$A_3 = -\frac{1}{2} - A_2$

$A_2 + A_3 = -\frac{1}{2}$

$-2A_2 + A_3 = \frac{3}{2} \rightarrow -2A_2 + \left(-\frac{1}{2} - A_2\right) = \frac{3}{2}$

$-3A_2 = 2 \qquad A_2 = -\frac{2}{3}$

$A_3 = -\frac{1}{2} - \left(-\frac{2}{3}\right) = \frac{1}{6}$

$$\int \frac{x-1}{x(x+1)(x-2)}\, dx = \frac{1}{2}\int \frac{1}{x}\, dx - \frac{2}{3}\int \frac{1}{x+1}\, dx + \frac{1}{6}\int \frac{1}{x-2}\, dx$$

$$= \boxed{\frac{1}{2}\ln(x) - \frac{2}{3}\ln(x+1) + \frac{1}{6}\ln(x-2) + C}$$

(2) $\int \frac{5x-2}{x^2-4}\, dx = \int \frac{5x-2}{(x+2)(x-2)} \Longrightarrow \left[\frac{5x-2}{(x+2)(x-2)} = \frac{A_1}{x+2} + \frac{A_2}{x-2} \right](x+2)(x-2)$

$$5x-2 = A_1(x-2) + A_2(x+2)$$

$A_1 + A_2 = 5$ $\quad | \quad A_1 = 5 - A_2 \Longrightarrow A_1 = 5 - 2$, $\boxed{A_1 = 3}$

$-2A_1 + 2A_2 = -2$ $\quad | \quad -2(5-A_2) + 2A_2 = -2$

$4A_2 = 8 \Longrightarrow \boxed{A_2 = 2}$

$$\int \frac{5x-2}{(x+2)(x-2)}\,dx = 3\int \frac{1}{x+2}\,dx + 2\int \frac{1}{x-2}\,dx$$
$$= \boxed{3\ln(x+2) + 2\ln(x-2) + C}$$

(3) $\int \frac{6x^2-2x-1}{4x^3-x} = \int \frac{6x^2-2x+1}{x(4x^2-1)} = \int \frac{6x^2-2x-1}{x(2x-1)(2x+1)}\,dx$

$$\left[\frac{6x^2-2x-1}{x(2x-1)(2x+1)} = \frac{A_1}{x} + \frac{A_2}{2x-1} + \frac{A_3}{2x+1}\right](x)(2x-1)(2x+1)$$

$6x^2-2x+1 = A_1(2x-1)(2x+1) + A_2(x)(2x+1) + A_3(x)(2x-1)$

$6x^2-2x-1 = A_1(4x^2-1) + A_2(2x^2+x) + A_3(2x^2-x)$

$4A_1 + 2A_2 + 2A_3 = 6$

$A_2 - A_3 = -2$

$-A_1 = -1$

$\boxed{A_1 = 1}$

$2A_2 + 2A_3 = 2$

$A_2 = -2 + A_3$

$\boxed{A_2 = -2 + 3/2 = -\frac{1}{2}}$

$2(-2+A_3) + 2A_3 = 2$

$4A_3 = 6$

$\boxed{A_3 = \frac{3}{2}}$

$$\int \frac{6x^2-2x-1}{x(2x-1)(2x+1)}\,dx = \int \frac{1}{x}\,dx - \frac{1}{4}\int \frac{1}{2x-1}\,dx + \frac{3}{4}\int \frac{1}{2x+1}\,dx$$

$$= \boxed{\ln(x) - \frac{1}{4}\ln(2x-1) + \frac{3}{4}\ln(2x+1) + C}$$

$u = 2x-1$

$du = 2dx$

$\frac{du}{2} = dx$

(II) CASO 2:

(1) $\int \frac{5x^2-11x+5}{x^3-4x^2+5x-2}\,dx = \int \frac{5x^2-11x+5}{(x-2)(x-1)^2}\,dx$

$$\left[\frac{5x^2-11x+5}{(x-2)(x-1)^2} = \frac{A_1}{x-2} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2}\right](x-2)(x-1)^2$$

$5x^2-11x+5 = A_1(x-1)^2 + A_2(x-2)(x-1) + A_3$

$5x^2-11x+5 = A_1(x^2-2x+1) + A_2(x^2-3x+2) + A_3$

$A_1 + A_2 = 5$

$-2A_1 - 3A_2 = -11$

$A_1 + 2A_2 + A_3 = 5$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ -2 & -3 & 0 & -11 \\ 1 & 2 & 1 & 5 \end{array}\right] \begin{array}{l} R_2+2R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_1+R_2 \\ R_3+R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] -R_2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\boxed{\therefore \quad A_1 = 4 \qquad A_2 = 1 \qquad A_3 = -1}$$

$$\int \frac{5x^2-11x+5}{(x-2)(x-1)^2} = 4\int \frac{1}{x-2}\,dx + \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^2}\,dx$$

$u = x-1$
$du = dx$

$$= 4\ln(x-2) + \ln(x-1) - \int u^{-2}\,du$$

$$\boxed{= 4\ln(x-2) + \ln(x-1) + \frac{1}{x-1} + C}$$

(2) $$\left[\int \frac{x^3-1}{x^2(x-2)^2} = \int \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x-2} + \frac{A_4}{(x-2)^2}\right](x^2)(x-2)^2$$

$$= x^3 - 1 = A_1(x)(x-2)^2 + A_2(x-2)^2 + A_3(x^2)(x-2) + A_4(x^2)$$

$$x^3 - 1 = A_1(x^3-4x^2+4x) + A_2(x^2-4x+4) + A_3(x^3-2x^2) + A_4(x^2)$$

$A_1 + A_3 = 1$
$-4A_1 + A_2 - 2A_3 + A_4 = 0$
$4A_1 - 4A_2 = 0$
$4A_2 = -1$

$\boxed{A_2 = -\frac{1}{4}}$

$4A_1 = 4(-\frac{1}{4})$

$\boxed{A_1 = -\frac{1}{4}}$

$A_1 + A_3 = 1$
$-\frac{1}{4} + A_3 = 1$
$\boxed{A_3 = 5/4}$

$-4(-\frac{1}{4}) + (-\frac{1}{4}) - 2(5/4) + A_4 = 0$
$1 - \frac{1}{4} - 10/4 + A_4 = 0$
$\boxed{A_4 = 7/4}$

$$\therefore \int \frac{x^3-1\,dx}{x^2(x-2)^2} = -\frac{1}{4}\int \frac{1}{x}\,dx - \frac{1}{4}\int \frac{1}{x^2}\,dx + \frac{5}{4}\int \frac{1}{x-2}\,dx + 7/4\int \frac{1}{(x-2)^2}\,dx$$

$$\boxed{\int \frac{x^3-1}{x^2(x-2)^2}\,dx = -\frac{1}{4}\ln(x) + \frac{1}{4x} + \frac{5}{4}\ln(x-2) + \frac{7}{4(x-2)} + C}$$

③ $\int \frac{6x+7}{x^2+4x+4}\,dx = \int \frac{6x+7}{(x+2)^2}$

$$\left[\frac{6x+7}{(x+2)^2} = \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2}\right](x+2)^2$$

$$6x+7 = A_1(x+2) + A_2 \Longrightarrow \boxed{A_1 = 6} \quad 2A_1 + A_2 = 7 \Rightarrow \boxed{A_2 = -5}$$

$$\therefore \int \frac{6x+7}{x^2+4x+4}\,dx = 6\int\frac{1}{x+2}\,dx - 5\int\frac{1}{(x+2)^2}\,dx$$

$$= \boxed{6\ln(x+2) + \frac{5}{x+2} + C}$$

CASO 3:

① $$\left[\int \frac{2x^3+9x}{(x^2+3)(x^2-2x+3)}\,dx = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2-2x+3)}\right](x^2+3)(x^2-2x+3)$$

$$2x^3+9x = Ax+B(x^2-2x+3) + Cx+D(x+3)$$
$$= Ax^3 - 2Ax^2 + 3Ax + Bx^2 - 2Bx + 3B$$
$$+ Cx^2 + 3Cx + Dx + 3D$$

x^3: $A = 2$

x^2: $-2A + B + C = 0$

x: $3A - 2B + 3C + D = 9$

$3B + 3D = 0$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & 1 & 0 & 0 \\ 3 & -2 & 3 & 1 & 9 \\ 0 & 3 & 0 & 3 & 0 \end{array}\right] \begin{array}{l} \\ R_2+2R_1 \\ R_3-3R_1 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & -2 & 3 & 1 & 3 \\ 0 & 3 & 0 & 3 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3+2R_2 \\ R_4-3R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 5 & 1 & 11 \\ 0 & 0 & -3 & 3 & -6 \end{array}\right] \begin{array}{l} \\ 5R_2-R_3 \\ \\ 5R_4+3R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 5 & 0 & -1 & 9 \\ 0 & 0 & 5 & 1 & 11 \\ 0 & 0 & 0 & 18 & 3 \end{array}\right] \begin{array}{l} \\ 18R_2+R_4 \\ 18R_3-R_4 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 90 & 0 & 0 & 165 \\ 0 & 0 & 90 & 0 & 195 \\ 0 & 0 & 0 & 18 & 3 \end{array}\right]$$

$$\begin{array}{l} R_1 \\ R_2/90 \\ R_3/90 \\ R_4/18 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & \frac{11}{6} \\ 0 & 0 & 1 & 0 & \frac{13}{6} \\ 0 & 0 & 0 & 1 & \frac{1}{6} \end{array}\right] \begin{array}{l} A \\ B \\ C \\ D \end{array}$$

$$\boxed{\begin{aligned} A &= 2 \\ B &= 11/6 \\ C &= 13/6 \\ D &= 1/6 \end{aligned}}$$

$$\int \frac{2x^3+9x}{(x^2+3)(x^2-2x+3)}\,dx = \int \frac{2x+\frac{11}{6}}{(x^2+3)}\,dx + \int \frac{\frac{13}{6}x+\frac{1}{6}}{(x^2-2x+3)}\,dx$$

$$= 2\int \frac{x}{x^2+3}\,dx + \frac{11}{6}\int \frac{1}{x^2+3}\,dx + \frac{13}{6}\int \frac{x}{x^2-2x+3}\,dx + \frac{1}{6}\int \frac{1}{x^2-2x+3}\,dx$$

$u = x^2+3$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \ln(x^2+3) + \frac{11}{6}\left(\int\right.$$

2 $$\left[\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx = \int \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+2)}\right](x^2+1)(x^2+2)$$

$$x^3+x^2+2x+1 = Ax+B(x^2+2) + Cx+D(x^2+1)$$
$$= Ax^3+2Ax+Bx^2+2B+Cx^3+Cx+Dx^2+D$$

$A+C=1$

$B+D=1$

$2A+C=2$

$2B+D=1$

$A = 1-C$

$2(1-C)+C=2 \Rightarrow -C=0$

$\boxed{A=1}$

$\boxed{C=0}$

$B=1-D$

$2(1-D)+D=1$

$-D=-1$

$\boxed{D=1}$

$\rightarrow \boxed{B=0}$

$$\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+2}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$\frac{1}{2}\int \frac{du}{u}$

$\theta = \tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$x = \sqrt{2}\tan\theta$

$dx = \sqrt{2}\sec^2\theta\,d\theta$

$$\int \frac{\sqrt{2}\sec^2\theta}{2\tan^2\theta+2}\,d\theta = \int \frac{\sqrt{2}\sec^2\theta}{2(\tan^2\theta+1)}\,d\theta$$

$$= \frac{\sqrt{2}}{2}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \frac{\sqrt{2}}{2}\theta$$

$$\therefore = \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$$

$$\boxed{= \frac{1}{2}\ln(x^2+1) + \frac{\sqrt{2}}{2}\arctan\left(\frac{x}{\sqrt{2}}\right) + C}$$

EJERCICIOS

① $\int \frac{3x^2 - x + 1}{x^2(x-1)} dx$

$$\left[\frac{3x^2 - x + 1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}\right] (x^2)(x-1)$$

$$3x^2 - x + 1 = Ax(x-1) + B(x-1) + Cx^2$$
$$= Ax^2 - Ax + Bx - B + Cx^2$$

$$\begin{aligned} A + C &= 3 \\ -A + B &= -1 \\ -B &= 1 \end{aligned} \quad \Bigg| \quad \begin{aligned} &\boxed{B = -1} \\ &-A = 0 \Rightarrow \boxed{A = 0} \\ &\boxed{C = 3} \end{aligned}$$

$$\int \frac{3x^2 - x + 1}{x^2(x-1)} dx = -\int \frac{1}{x^2} dx + 3\int \frac{1}{x-1} dx$$
$$= \boxed{\frac{1}{x} + 3\ln(x-1) + C}$$

② $\int \frac{1}{x - x^2} dx = \int \frac{1}{x(1-x)}$

$$\left[\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}\right] x(1-x) \Rightarrow 1 = A(1-x) + Bx$$
$$= A - Ax + Bx$$

$$\begin{aligned} -A + B &= 0 \\ &\boxed{A = 1} \end{aligned} \quad \Big| \quad \boxed{B = 1} \quad \Big| \quad \int \frac{1}{x - x^2} dx = \int \frac{1}{x} dx + \int \frac{1}{1-x} dx$$
$$= \boxed{\ln(x) - \ln(1-x) + C}$$

③ $\int \frac{1}{9x^4 + x^2} dx = \int \frac{1}{x^2(9x^2+1)} dx$

$$\left[\frac{1}{x^2(9x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{9x^2+1}\right] (x^2)(9x^2+1)$$

$$1 = Ax(9x^2+1) + B(9x^2+1) + (Cx + D)(x^2)$$
$$= 9Ax^3 + Ax + 9Bx^2 + B + Cx^3 + Dx^2$$

$9A + C = 0$

$9B + D = 0$

$\boxed{A = 0}$

$B = 1$

$\boxed{C = 0}$

$\boxed{B = 1}$

$\boxed{D = -9}$

$9 + D = 0$

$D = -9$

$x = \tan\theta \quad \theta = \arctan(x)$

$dx = \sec^2\theta \, d\theta$

$\frac{1}{9}\int \frac{\sec^2\theta}{\tan^2\theta + 1} d\theta$

$= \frac{1}{9}\theta = \frac{1}{9}\arctan(x)$

$$\int \frac{1}{x^2(9x^2+1)} dx = \int \frac{1}{x^2} dx - 9 \int \frac{1}{9x^2+1} dx$$

$$= \boxed{-\frac{1}{x} - \arctan(x) + C}$$

4) $\int \frac{1}{x^3+x^2+x} = \int \frac{1}{x(x^2+x+1)} dx$

$$\left[\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}\right] x(x^2+x+1)$$

$$1 = Ax^2 + Ax + A + Bx^2 + Cx$$

$A + B = 0$

$A + C = 0$

$\boxed{A = 1}$

$\boxed{B = -1}$

$\boxed{C = -1}$

$1 = x + 1$

$$= \int \frac{1}{x} dx - \int \frac{x+1}{x^2+x+1} dx$$

$u = x^2 + x + 1$

$du = (2x+1)dx$

$$\int \frac{\frac{1}{2}(2x+1) + \frac{1}{2}}{x^2+x+1} dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1} dx + \frac{1}{2}\int \frac{1}{x^2+x+1} dx$$

$$= \frac{1}{2}\int \frac{1}{u} du + \frac{1}{2}\int \frac{1}{x^2+x+1} dx = \frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\int \frac{1}{x^2+x+1} dx$$

$$= \ln(x) - \frac{1}{2}\ln(x^2+x+1) - \frac{1}{2}\int \frac{1}{x^2+x+1} dx$$

$$\int \frac{1}{x^2+x+1} dx =$$

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