Random variable:

A random variable may be defined as a functional relationship that assigns real numbers to each possible outcome of a random experiment.

→ A real variable x whose value is determined by the outcome of a random experiment is called a random variable.

→ The random variable concept is very important tool in solving the practical probability problems.

→ The random variable is represented by the X and Y etc. The values of random variable are represented by lower case letters x, y.

→ Suppose 'S' is a sample space of a random experiment X then any function $X: S \rightarrow R$ is called as random variable.

Ex: 1. Consider a sample space S consisting of the alphabets (vowels).

$$S = \{a, e, i, o, u\}$$

The random variable X can be consider to a function that maps all the elements of the sample space into real values.

$$X(S) = \{X(a), X(e), X(i), X(o), X(u)\}$$

$$= \{1, 5, 9, 15, 21\} \subseteq R$$

$$X(S) \subseteq R$$

$X: S \rightarrow R$ is a random variable.

2. Consider a random experiment consists of tossing a coin twice.

$$S = \{HH, HT, TH, TT\}$$

Define $x: S \to R$ by $x(s) =$ No. of heads occurs in s.

$x(HH) = 2$, $x(HT) = 1$, $x(TH) = 1$, $x(TT) = 0$.

$x(s) = \{2, 1, 1, 0\} = \{0, 1, 2\} \subseteq R$

$x(s) \subseteq R$

$x: S \to R$ is a random variable.

Types of random variable:

Random variables are classified into two types.

1. Discrete random variable
2. Continuous random variable

1. Discrete random variable:

A random variable x which can take only a finite no. of values in an interval of the domain is called discrete random variable.

In other words, a real valued function defined on discrete sample space is called a discrete random variable.

Ex: 1. No. of students in a college
2. Tossing a coin
3. Throwing a dice.

Probability distribution function:

Let X be a random variable then the function $F(x)$ is defined by $F(x) = P(X \leq x)$

$$F(x) = P[s : x(s) \leq x] \text{ where } -\infty < x < \infty$$

It is called distribution function of X.

Ex: In tossing 3 coins, the random variable X denotes the no. of heads ~~that~~ then the probability or distribution of x is given by

$S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Properties of distribution function:

1. If F_X is the distribution function of random variable X and if $a<b$ then

1. $P(a < X \leq b) = F(b) - F(a)$
2. $P(a \leq X \leq b) = P(X=a) + [F(b) - F(a)]$
3. $P(a < X < b) = F(b) - F(a) - P(X=b)$
4. $P(a \leq X < b) = P(X=a) + F(b) - F(a) - P(X=b)$

$\rightarrow \quad F(-\infty) = \underset{x \to -\infty}{Lt} F(x) = 0$

$F(\infty) = \underset{x \to \infty}{Lt} F(x) = 1$

Probability mass function (or) Probability distribution of discrete random variable:

Let X be a discrete random variable with possible outcomes $x_1, x_2, \ldots x_n$ with corresponding probabilities $p_1, p_2, \ldots, p_n$ respectively where

$P(x_i) = P(X = x_i) \quad i = 1, 2, 3, \ldots, n$

The probability function $P(x)$ is said to be probability mass function (or) probability distribution function if it satisfies following two conditions.

1. $P(x_i) \geq 0 \quad \forall x_i$
2. $\sum_{i=1}^{n} P(x_i) = 1$

It is called the discrete probability distribution of discrete random variable X.

The probability distribution of random variable x is given by means of following table:

X :	x_1	x_2	x_3 ...	x_i ...	x_n
$P(X=x)$:	$P(x_1)$	P_2	P_3 ...	P_i ...	P_n

$$P(X < x_i) = P(x_1) + P(x_2) + \cdots + P(x_{i-1})$$
$$P(X \le x_i) = P(x_1) + P(x_2) + \cdots + P(x_{i-1}) + P(x_i)$$
$$P(X > x_i) = 1 - P(X \le x_i)$$

Mathematical expectation:

1. Expectation (or) Mean of a discrete variable is denoted by $E(x)$ (or) μ.

$$\boxed{\mu = E(X) = \sum_{i=1}^{n} P_i x_i}$$

2. Variance (or) $V(X)$ (or) σ^2 (or) σ_x^2:

The variance of the random variable x is denoted by $V(x)$ (or) σ^2 (or) σ_x^2 and it is defined as

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$\boxed{\sigma^2 = E(X^2) - \mu^2}$$

3. Standard deviation (σ):

standard deviation $\boxed{\sigma = \sqrt{V(X)}}$

4. Covariance:

The covariance of X and Y is denoted by $\text{cov}(XY) = E(XY) - E(X)E(Y)$

Note:

1. $E(X+Y) = E(X) + E(Y)$
2. $E(XY) = E(X) \cdot E(Y)$
3. $E(kX) = k \cdot E(X)$.
4. $E(ax \pm b) = aE(X) \pm b$
5. $E(X+k) = E(X) + k$

A random variable x has the following probability function

$X=x$	1	2	3	4	5	6
$P(X)$	k	3k	5k	7k	9k	11k

then determine k.

We know that,

$$\sum_{i=1}^{6} P(x_i) = 1$$

$$k+3k+5k+7k+9k+11k = 1$$

$$36k = 1$$

$$k = 1/36$$

$$k = 0.0277$$

$$k = 0.028$$

The probability distribution of a finite random variable x is given by the following table.

X	-2	-1	0	1	2	3
$P(X)$	0.1	k	0.2	2k	0.3	k

Find the value of k, Mean and variance.

We know that,

$$\sum_{i=-2}^{3} P(x_i) = 1$$

$$\Rightarrow P(-2) + P(-1) + P(0) + P(1) + P(2) + P(3) = 1$$

$$\Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + k = 1$$

$$4k = 1 - 0.6$$

$4K = 0.4$

$\boxed{K = 0.1}$

X	-2	-1	0	1	2	3
P(X)	0.1	0.1	0.2	0.2	0.3	0.1

$$\text{Mean} = E(x) = \sum_{i=-2}^{3} P_i x_i$$

$$= (-2)(0.1) + (-1)(0.1) + 0 + 1(0.2) + 2(0.3) + 3(0.1)$$

$$= -0.2 - 0.1 + 0.2 + 0.6 + 0.3$$

$$\boxed{\mu = 0.8}$$

variance $V(X) = \sum_{i=} E(x^2) - [E(x)]^2$

$$V(X) = \sum_{i=-2}^{3} P_i x_i^2 - \mu^2$$

$$V(X) = 0.1(4) + 0.1(1) + 0.2(0) + 0.2(1) + 0.3(4) + 0.1(9) - \mu^2$$

$$= 0.4 + 0.1 + 0.2 + 1.2 + 0.9 - (0.8)^2$$

$$= 2.8 - 0.64$$

$$= 2.16$$

$$\boxed{V(X) = 2.16}$$

A random variable x has the following probability function

x :	0	1	2	3	4	5	6	7
$P(X)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) Determine k. ii) Evaluate $P(X<6), P(X\geq 6)$, $P(0<X<5)$, $P(0\leq X\leq 4)$

iii) Determine the distribution function of X

~~We know that~~

iv) Mean v) Variance vi) $P(X\leq k) > \frac{1}{2}$. Find the minimum value of k.

We know that,

$$\sum_{i=0}^{7} P(x_i) = 1$$

$\Rightarrow P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7)=1$

$\Rightarrow 0+k+2k+2k+3k+k^2+7k^2+k = 01$
$+2k^2$

$\Rightarrow 10k^2 + 10$

$\Rightarrow 10k^2+9k-1=0$

$10k^2+10k-k-1=0$

$10k(k+1)-(k+1)=0$

$(10k-1)(k+1)=0$

$k=1/10, \; k\neq -1$

$$\boxed{k=\frac{1}{10}}$$

$k\neq -1$ since, $P(x_i)$ must be greater than 0.

x :	0	1	2	3	4	5	6	7
$P(X)$:	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{17}{100}$

$P(X<6) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$= 0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\frac{1}{50}$

$= \frac{81}{100}$

$= 0.81$

$P(x \geq 6) = 1 - P(x < 6)$

$= 1 - 0.81$

$= 0.19$

$P(0 < x < 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= 0.1 + 0.2 + 0.2 + 0.3$

$= 0.8$

$P(0 \leq x \leq 4) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= 0 + 0.1 + 0.2 + 0.2 + 0.3$

$= 0.8$

Mean $E(x) = \sum_{i=0}^{7} P_i x_i$

$= 0(0) + 1(0.1) + 2(0.2) + 3(0.2) + 4(0.3) + 5(0.01) + 6(0.02) + 7(0.17)$

$= 0.1 + 0.4 + 0.6 + 1.2 + 0.05 + 0.12 + 1.19$

$= 3.66$

variance $V(x) = E(x^2) - [E(x)]^2$

$\Rightarrow V(x) = \sum_{i=0}^{7} P_i x_i^2 - \mu^2$

$= 0(0) + 1(0.1) + 0.2(4) + (0.2)(9) + 0.3(16) + 0.01(25) + (0.02)(36) + (0.17)(49) - \mu^2$

$= \cancel{12.48}\ 16.8 - 13.3956$

$= 3.4044$

The probability distribution.

x	$F(x) = P(X \le x)$
0	$P(X \le 0) = P(0) = 0$
1	$P(X \le 1) = P(0) + P(1) = 0.1$
2	$P(X \le 2) = 0.3$
3	$P(X \le 3) = 0.5$
4	$P(X \le 4) = 0.8$
5	$P(X \le 5) = 0.81$
6	$P(X \le 6) = 0.83$
7	$P(X \le 7) = 1.$

vi) $P(X \le k) > \frac{1}{2}$

Here $k = 4$.

$P(X \le 0) = P(0) = 0$

$P(X \le 1) = P(0) + P(1) = 0.1$

$P(X \le 2) = P(0) + P(1) + P(2) = 0.3$

$P(X \le 3) = P(0) + P(1) + P(2) + P(3) = 0.5$

$P(X \le 4) = P(0) + P(1) + P(2) + P(3) + P(4) = 0.8 > \frac{1}{2}$

$\therefore$ Minimum value of $k = 4$.

Let x denote the no. of heads in a single toss of four coins. determine

(1) $P(x \leq 2)$, (2) $P(1 < x \leq 3)$, (3) $P(x < 4)$

Four coins are tossed.

sample space = { HHHH, TTTT
HHHT TTTH
HHTH TTHT
HTHH THTT
THHH HTTT
HTHT THTH
HTTH THHT
HHTT TTHH }

Let x be the no. of heads.

X	0	1	2	3	4
$P(X=x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

1. $P(X<2) = P(X=0) + P(X=1)$

$= \frac{1}{16} + \frac{4}{16}$

$= \frac{5}{16}$

2. $P(1 < X \leq 3) = P(X=2) + P(X=3)$

$= \frac{6}{16} + \frac{4}{16}$

$= \frac{10}{16}$

$= \frac{5}{8}$

3. $P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16}$

$= \frac{15}{16}.$

Two dice are thrown. Let x assigns to each point (a,b) in S. The maximum of it's numbers i.e, $X(a,b) = \max(a,b)$. Find the probability distribution. X is a random variable with $X(S) = \{1,2,3,4,5,6\}$. Also find the mean and variance of the distribution.

Two dice are thrown.

$$\text{sample space} = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$$
$$(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$$
$$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$$
$$(4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$$
$$(5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6)$$
$$(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)\}$$

Let x be maximum of it's numbers.

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Mean $E(x) = \mu = \Sigma p_i x_i$

$$= 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{7}{36}\right) + 5\left(\frac{9}{36}\right) + 6\left(\frac{11}{36}\right)$$

$$= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36}$$

$$= \frac{161}{36} = 4.472$$

variance $V(x) = E(x^2) - [E(x)]^2$

$$= \Sigma p_i x_i^2 - \mu^2$$

$$= 1\left(\frac{1}{36}\right) + 4\left(\frac{2}{36}\right) + 9\left(\frac{5}{36}\right) + 16\left(\frac{7}{36}\right) + 25\left(\frac{9}{36}\right) + 36\left(\frac{11}{36}\right) - \left(\frac{161}{36}\right)^2$$

$$= \frac{791}{36} - 20.0007$$

$$= 21.972 - 20.0007$$

$$= 1.965$$

From a lot of 10 items containing 3 defective, a sample of 4 items is drawn at random. Let random variable x denote the no. of defective items in the sample. Find the probability distribution of x when the sample is drawn without replacement.

A lot consists of 10 items containing 3 defective.

Non-defective $= 10 - 3 = 7$

4 items are drawn from 10 items i.e, $n(s) = {}^{10}C_4$

Let x be random variable denotes no. of defective.

when $x = 0$ then $P(x=0) = \frac{{}^7C_4}{{}^{10}C_4} = \frac{35}{210} = 1/6$

when $x = 1$ then $P(x=1) = \frac{{}^3C_1 \, {}^7C_3}{{}^{10}C_4} = 1/2$

when $x = 2$ then $P(x=2) = \frac{{}^3C_2 \, {}^7C_2}{{}^{10}C_4} = 3/10$

when $x = 3$ then $P(x=3) = \frac{{}^3C_3 \, {}^7C_1}{{}^{10}C_4} = 1/30$

$x = X$	0	1	2	3
$P(x=X)$	$1/6$	$1/2$	$3/10$	$1/30$

A sample of 4 items is selected at random from a box containing 12 items of which 5 are defective. Find the expected no. of E of defective items.

Let x be the no. of defective.

When $x=0$ A box contains 12 items in which 5 are defective i.e, Non-defective $= 12 - 5 = 7$

$x = 0 \Rightarrow P(x=0) = \frac{{}^7C_4}{{}^{12}C_4} = \frac{7}{99}$

$x = 1 \Rightarrow P(x=1) = \frac{{}^5C_1 \, {}^7C_3}{{}^{12}C_4} = \frac{35}{99}$

$x = 2 \Rightarrow P(x=2) = \frac{{}^5C_2 \, {}^7C_2}{{}^{12}C_4} = \frac{14}{33}$

$x = 3 \Rightarrow P(x=3) = \frac{{}^5C_3 \, {}^7C_1}{{}^{12}C_4} = 14/99$

$x = 4 \Rightarrow P(x=4) = \frac{{}^5C_4 \, {}^7C_0}{{}^{12}C_4} = 1/99$

$X = x$	0	1	2	3	4
$P(X=x)$	$\frac{7}{99}$	$\frac{35}{99}$	$\frac{14}{33}$	$\frac{14}{99}$	$\frac{1}{99}$

$E(X) = \Sigma x P(x) = 0 + \frac{35}{99} + \frac{28}{33} + \frac{42}{99} + \frac{4}{99} = \frac{165}{99} = 1.666$

A variate x has the following probability distribution.

x :	-3	6	9
$p(x)$:	$1/6$	$1/2$	$1/3$

Find 1. $E(x)$ 2. $E(x^2)$ 3. $E[(2x+1)^2]$.

$$E(x) = \mu = \sum p_i x_i$$
$$= -3\left(\frac{1}{6}\right)+6\left(\frac{1}{2}\right)+9\left(\frac{1}{3}\right)$$
$$= -1/2 + 3 + 3$$
$$= 6 - 1/2$$
$$= 11/2 = 5.5$$

$$E(x^2) = \sum p_i x_i^2 = 9\left(\frac{1}{6}\right)+36\left(\frac{1}{2}\right)+81\left(\frac{1}{3}\right)$$
$$= \frac{3}{2} + 18 + 27$$
$$= \frac{3}{2} + 45$$
$$= 93/2$$
$$= 46.5$$

$$E[(2x+1)^2] = E[4x^2+1+4x]$$
$$= 4E(x^2) + 1 + 4E(x)$$
$$= 4 \times \frac{93}{2} + 1 + 4\left(\frac{11}{2}\right)$$
$$= 186 + 1 + 22$$
$$= 209$$

A fair coin is tossed until a head or ~~to~~ five tails occurs. Find the expected no. E of tosses of the coin.

If a head occurs first time there will be only one toss on the other hand if first one is tail second occurs. If head occurs then there will be only two tosses. suppose, second one is also tail, third occurs. If head occurs then there will be three tosses and so on.

$P(1) = P(H) = 1/2$
$P(2) = P(TH) = 1/4$
$P(3) = P(TTH) = 1/8$
$P(4) = P(TTTH) = 1/16$
$P(5) = P(TTTTH) + P(TTTTT) = 1/32 + 1/32 = 2/32 = 1/16$

x	1	2	3	4	5
$P(x)$	1/2	1/4	1/8	1/16	1/16

$$E(x) = \sum p_i x_i$$
$$= 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + 5\left(\frac{1}{16}\right)$$
$$= 1/2 + 1/2 + 3/8 + 4/16 + 5/16$$
$$= \frac{8+8+6+4+5}{16}$$
$$= \frac{31}{16}.$$

A discrete random variable x has the following distribution function

$$F(x) = \begin{cases} 0, & \text{for } x < 1 \\ 1/3, & \text{for } 1 \le x < 4 \\ 1/2, & \text{for } 4 \le x < 6 \\ 5/6, & \text{for } 6 \le x \le 10 \\ 1, & \text{for } x \ge 10. \end{cases}$$

Find 1. $P(2 < x \le 6)$ 2. $P(x = 5)$ 3. $P(x = 4)$ 4. $P(x \le 6)$ 5. $P(x = 6)$

1. $P(2 < x \le 6) = F(6) - F(2)$
$$= P(x \le 6) - P(x \le 2)$$
$$= \frac{5}{6} - 1/3$$
$$= 3/6 = 1/2.$$

2. $P(x = 5) = P(x \le 5) - P(x < 5)$
$$= 1/2 - 1/2 = 0$$

3. $P(X=4) = P(X\le 4) - P(X<4)$

$= \frac{1}{2} - \frac{1}{3}$

$= \frac{1}{6}$

4. $P(X\le 6) = F(6) = 5/6$

5. $P(X=6) = P(X\le 6) - P(X<6)$

$= \frac{5}{6} - \frac{1}{2}$

$= \frac{2}{6}$

$= \frac{1}{3}$

Find the distribution function which corresponds to probability distribution defined by $f(x) = x/15$ for $x = 1,2,3,4,5$.

Given that,

$f(x) = x/15$ for $x = 1,2,3,4,5$.

x :	1	2	3	4	5
$P(x)$:	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

x	$F(x) = P(X\le x)$
1	$P(X\le 1) = P(X=1) = \frac{1}{15}$
2	$P(X\le 2) = P(X=1) + P(X=2) = \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5}$
3	$P(X\le 3) = \frac{6}{15} = \frac{2}{5}$
4	$P(X\le 4) = \frac{10}{15} = \frac{2}{3}$
5	$P(X\le 5) = \frac{15}{15} = 1$

Continuous Random variable:

A random variable X which can take values continuously i.e, which takes all possible values in given interval is called a continuous random variable.

For example: Height, Age, weight

Probability density function:

For continuous variable, probability distribution function is called probability density function. Because it is defined for every point on the range not only for certain values.

Properties of the probability density function $f(x)$:

1. $f(x) \geq 0, \forall x \in R$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a<x<b) = \int_a^b f(x)\,dx$ = Area under the curve $f(x)$ between the ordinates $x=a$ and $x=b$

Cummulative distribution function of a continuous random variable:

The cummulative distribution function of X is denoted by $F(x)$ and it is ~~denoted~~ defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx.$$

Mean:

$$\text{Mean } \mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx.$$

$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x) f(x)\,dx$$

Variance:

Variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

(or)

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

suppose the variance is defined from a to b

then $\sigma^2 = \int_a^b (x-\mu)^2 f(x)\,dx$

If a random variable has the probability density function $f(x)$ as

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

Find the probabilities that it will take on a values (i) Between 1 and 3

(ii) Greater than 0.5

(i) $P(1 < x < 3) = \int_1^3 f(x)\,dx$

$$= \int_1^3 2e^{-2x}\,dx$$

$$= 2\int_1^3 e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_1^3$$

$$= -1\left[e^{-6} - e^{-2}\right]$$

$$= e^{-2} - e^{-6}$$

$$= 0.1328$$

(ii) $P(x > 0.5) = \int_{0.5}^{\infty} f(x)\,dx.$

$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-1}\right]$$

$$= e^{-1} - 0$$

$$= 0.3678$$

$$= 0.368.$$

A continuous random variable has the Probability density function

$$f(x) = \begin{cases} kxe^{-\lambda x} & \text{for } x \geq 0, \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine (i) k value (ii) Mean value (iii) variance

we know that,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1$$

$$0 + \int_{0}^{\infty} ke^{-\lambda x} x\, dx = 1$$

$$k\int_{0}^{\infty} xe^{-\lambda x} dx = 1$$

$u = x$ $\quad \int e^{-\lambda x} dx = \int dv$

$du = dx$ $\quad v = \frac{e^{-\lambda x}}{-\lambda}$

$$k\left[\left[\frac{-xe^{-\lambda x}}{\lambda}\right]_0^{\infty} + \frac{1}{\lambda}\int_0^{\infty} e^{-\lambda x} dx\right] = 1$$

$$k\left[0 + 0 + \frac{1}{\lambda}\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty}\right] = 1$$

$$\frac{-k}{\lambda^2}\left[0 - 1\right] = 1$$

$$\frac{k}{\lambda^2} = 1$$

$$k = \lambda^2$$

$$\therefore \boxed{k = \lambda^2}$$

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{for } x \geq 0, \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean } \mu = \int_{-\infty}^{\infty} x f(x)\, dx.$$

$$\mu = \int_{-\infty}^{0} x f(x)\, dx + \int_{0}^{\infty} x f(x)\, dx$$

$$\mu = 0 + \int_{0}^{\infty} x \cdot \lambda^2 x e^{-\lambda x}\, dx$$

$$\mu = \lambda^2 \int_{0}^{\infty} x^2 e^{-\lambda x}\, dx$$

$u = x^2$ $\quad$ $\int e^{-\lambda x} dx = \int dv$

$du = 2x\,dx$ $\quad$ $v = \frac{e^{-\lambda x}}{-\lambda}$

$$\mu = \lambda^2 \left[\left(x^2 \frac{e^{-\lambda x}}{-\lambda} \right)_0^{\infty} + \frac{2}{\lambda} \int_0^{\infty} x e^{-\lambda x} dx \right]$$

$u = x$ $\quad$ $\int e^{-\lambda x} dx = \int dv$

$du = dx$ $\quad$ $v = \frac{e^{-\lambda x}}{-\lambda}$

$$\mu = \lambda^2 \left[0 + \frac{2}{\lambda} \cdot \left[\left(-\frac{x e^{-\lambda x}}{\lambda} \right)_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx \right] \right]$$

$$\mu = \lambda^2 \left[0 + \frac{2}{\lambda} \left(0 - \left[\frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty} \right) \right]$$

$$\mu = \lambda^2 \left(-\frac{2}{\lambda^3} \right) (0 - 1)$$

$$\mu = \frac{2}{\lambda}.$$

$$\therefore \boxed{\text{Mean} = \frac{2}{\lambda}}$$

variance $v(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$$v(x) = \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$= 0 + \int_{0}^{\infty} x^2 \cdot \lambda^2 x e^{-\lambda x}\,dx - \mu^2$$

$$= \lambda^2 \int_{0}^{\infty} x^3 e^{-\lambda x}\,dx - \mu^2$$

$$= \lambda^2 \left[x^3 \frac{e^{-\lambda x}}{-\lambda} - 3x^2 \frac{e^{-\lambda x}}{\lambda^2} + 6x \frac{e^{-\lambda x}}{-\lambda^3} - 6 \frac{e^{-\lambda x}}{\lambda^4} \right]_0^{\infty} - \mu^2$$

$$= \lambda^2 \left[(0-0) - (0-0) + (0-0) - \frac{6}{\lambda^4}(0-1) \right] - \mu^2$$

$$= \frac{6\lambda^2}{\lambda^4} - \mu^2$$

$$= \frac{6\lambda^2}{\lambda^4} - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$= \frac{2}{\lambda^2}.$$

$$\therefore \boxed{\text{variance} = \frac{2}{\lambda^2}}$$

If the probability density of a random variable is given by $f(x) = \begin{cases} k(1-x^2) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Find the value of k and the probabilities that it will take on a value (i) between 0.1 and 0.2. (ii) Greater than 0.5.

We know that,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1.$$

$$\Rightarrow \int_{-\infty}^{0}(0)\,dx + \int_{0}^{1} k(1-x^2)\,dx + 0 = 1$$

$$0 + k\int_{0}^{1}(1-x^2)\,dx = 1$$

$$k\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$k\left[1 - 1/3 - 0\right] = 1$$

$$k\left(\frac{2}{3}\right) = 1$$

$$\boxed{k = \frac{3}{2}}$$

(i) $$P(0.1 < x < 0.2) = \int_{0.1}^{0.2} f(x)\,dx$$

$$= \int_{0.1}^{0.2} k(1-x^2)\,dx$$

$$f(x) = \begin{cases} \frac{3}{2}(1-x^2) & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0.1<x<0.2) = \int_{0.1}^{0.2} \frac{3}{2}(1-x^2)\,dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.1}^{0.2}$$

$$= \frac{3}{2}\left[0.2 - \frac{(0.2)^3}{3} - 0.1 + \frac{(0.1)^3}{3}\right]$$

$$= \frac{3}{2}\left[0.1 - \frac{0.007}{3}\right]$$

$$= 0.1465.$$

(ii) $$P(x>0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{1} \frac{3}{2}(1-x^2)\,dx + 0.$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.5}^{1}$$

$$= \frac{3}{2}\left[1 - \frac{1}{3} - 0.5 + \frac{1}{24}\right]$$

$$= \frac{3}{2}\left[0.5 - 0.291\right]$$

$$= \frac{3}{2}(0.491)$$

$$= \frac{3}{2}(0.209)$$

$$= 0.3135$$

If x is a continuous random variable and $y = ax + b$. prove that $E(y) = aE(x) + b$ and $V(y) = a^2 V(x)$ where V stands for variance and a, b are constants.

Given that,

$$y = ax + b$$

$$E(y) = E(ax + b)$$

$$= \int_{-\infty}^{\infty} (ax + b) f(x)\, dx$$

$$= a\int_{-\infty}^{\infty} x f(x)\, dx + b\int_{-\infty}^{\infty} f(x)\, dx.$$

$$E(y) = aE(x) + b(1)$$

$$\boxed{E(y) = aE(x) + b}$$

we have, $y = ax + b \rightarrow (1)$

$$E(y) = aE(x) + b \rightarrow (2)$$

$$(1) - (2) \Rightarrow y - E(y) = a[x - E(x)] + b - b$$

$$y - E(y) = a[x - E(x)]$$

squaring on both sides.

$$[y - E(y)]^2 = a^2[x - E(x)]^2$$

Taking Expectation on both sides.

$$E([Y-\{E(Y)]^2) = a^2 \, E[(x - E(x))^2]$$

$$\boxed{V(Y) = a^2 V(x)} \quad (\because V(x) = E[(x-\mu)^2]).$$

Hence, it is proved.

The cummulative distribution function for a continuous random variable x is

$$F(x) = \begin{cases} 1-e^{-2x} & ; x \geq 0 \\ 0 & ; x < 0. \end{cases}$$

Find the (i) density function $f(x)$ (ii) Mean (iii) variance of the density function.

The density function $f(x)$ is given by

$$f(x) = \frac{d}{dx} F(x)$$

$$f(x) = \frac{d}{dx}(1-e^{-2x}) = 0-(-2)e^{-2x} = 2e^{-2x}.$$

i) $$f(x) = \begin{cases} 2e^{-2x} & ; x \geq 0 \\ 0 & ; x < 0. \end{cases}$$

ii) Mean $$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{0} x f(x)\, dx + \int_{0}^{\infty} x f(x)\, dx$$

$$= 0 + \int_{0}^{\infty} x \cdot 2e^{-2x}\, dx.$$

$$\cancel{= 2\left[\frac{e^{-2x}}{-2}\right]_0^{\infty}} \qquad = 2\left[\frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4}\right]_0^{\infty}$$

$$\cancel{= -[e^{-\infty} - e^{0}]} \qquad = 2\left[(0-0) - \frac{(0-1)}{4}\right]$$

$$\cancel{= -(0-1)} \qquad = \frac{2}{4} = 1/2$$

$$= 1.$$

iii) Variance $V(x) = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

$$= \int_{-\infty}^{0} x^2 f(x)dx + \int_{0}^{\infty} x^2 f(x)dx - \mu^2$$

$$= 0 + \int_{0}^{\infty} x^2 \cdot 2e^{-2x} dx - \mu^2$$

$$= 2\int_{0}^{\infty} x^2 e^{-2x} dx - \mu^2$$

$$= 2\left[\frac{x^2 e^{-2x}}{-2} - \frac{2xe^{-2x}}{4} + \frac{2e^{-2x}}{-8}\right]_0^{\infty} - \mu^2$$

$$= 2\left[(0-0) - (0-0) - \frac{2}{8}(0-1)\right] - \mu^2$$

$$= 1/2 - 1/4$$

$$= 1/4$$

For the continuous random variable x whose probability density function is given by

$$f(x) = \begin{cases} cx(2-x) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find c, mean, variance.

We know that,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$$

$$0 + \int_{0}^{2} cx(2-x)dx + 0 = 1$$

$$c\int_{0}^{2} (2x - x^2) = 1$$

$$c\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$c\left(4 - \frac{8}{3}\right) = 1$$

$c = 3/4$.

$$f(x) = \begin{cases} \frac{3}{4}x(2-x) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Mean $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$\Rightarrow \int_{-\infty}^{\infty} \cancel{x \cdot \tfrac{3}{4}x(2-} = \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{2} x f(x)\,dx + \int_{2}^{\infty} x f(x)\,dx$$

$$= 0 + \frac{3}{4}\int_{0}^{2} x^2(2-x)\,dx + 0$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right]$$

$$= \frac{48}{4}\left(\frac{1}{12}\right)$$

$\mu = 1$.

variance $= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$$= \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{2} x^2 f(x)\,dx + \int_{2}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$= 0 + \frac{3}{4}\int_{0}^{2} x^3(2-x)\,dx + 0 - \mu^2$$

$$= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 + 0 - \mu^2$$

$$= \frac{3}{4}\left(\frac{32}{4} - \frac{32}{5}\right) - 1$$

$$= 24\left(\frac{1}{20}\right) = \frac{6}{5} - 1 = 1/5 = 0.2$$

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