1

Laplace Transform of Periodic function :

A function $f(t)$ is said to be Periodic function with Period T

$T > 0$,

If $f(t) = f(t+T) = f(t+2T) = \cdots = f(t+nT) = \cdots$

where T is the smallest Positive value.

For ex :-

Tan x & Cot x are Per π.

Sin x, Cosec x, sec x and Cos x 2π.

(i) Sin t, Cos t are Periodic functions with Period "2π"

(ii) Cosec, sec, Tan t, Cot t are Periodic functions with Period "π"

Theorem :- If $f(t)$ is a Periodic function with Period T then

$$\boxed{L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}}$$

(1) Find the Laplace Transform of square wave function of Period $2a$ defined as

$$f(t) = K \quad \text{when } 0 < t < a$$
$$= -K \quad \text{when } a < t < 2a$$

$T = 2a$

Sol :- Since $f(t)$ is a periodic with period $2a$ - Periodic function with Period $T = 2a$

$$\therefore L\{f(t)\} = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt.$$

$$\boxed{L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt}$$

$\boxed{T = 2a}$

$f(t+\underset{T}{\underline{\underline{2a}}}) = f(t)$

$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a K e^{-st}\,dt + \int_a^{2a} (-K) e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[K\left(\frac{e^{-st}}{-s}\right)_0^a - K\left(\frac{e^{-st}}{-s}\right)_a^{2a} \right]$$

$$= \frac{K}{s(1-e^{-2as})} \left[-e^{-as} + 1 + e^{-2as} - e^{-as} \right]$$

$$= \frac{K}{s(1-e^{-2as})} \left[1^2 - 2e^{-as} + (e^{-as})^2 \right]$$

$a^2 - 2ab + b^2$

$$= \frac{K}{s(1-e^{-2as})} (1-e^{-as})^2$$

$1^2 - (e^{-as})^2$

$= (1-e^{-as})(1+e^{-as})$

$$= \frac{K}{s} \frac{1-e^{-as}}{1+e^{-as}}$$

$$= \frac{K}{s} \frac{e^{-\frac{as}{2}}}{e^{-\frac{as}{2}}} \frac{(e^{\frac{as}{2}} - e^{-\frac{as}{2}})}{(e^{\frac{as}{2}} + e^{-\frac{as}{2}})}$$

$\operatorname{Tanh} ax = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

$$= \frac{K}{s} \operatorname{Tanh} \frac{as}{2}$$

$\operatorname{Tanh} \frac{as}{2} = \frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{e^{\frac{as}{2}} + e^{-\frac{as}{2}}}$

$$\frac{K}{s(1-e^{-2as})} (1-e^{-as})(1+e^{-as})$$

(2)

(ii) Find the L.T of the Rectified Semi-wave function defined by

$$f(t) = \begin{cases} \sin\omega t & ; \ 0 < t < \frac{\pi}{\omega} \\ 0 & ; \ \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases} \quad \text{with Period } \underbrace{\frac{2\pi}{\omega}}_{T}.$$

Sol :- Since $f(t)$ is a Periodic function with Period $\frac{2\pi}{\omega}$.

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt \qquad \boxed{T = \frac{2\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \left[\int_0^{\frac{\pi}{\omega}} e^{-st} \sin\omega t\, dt + \underbrace{\int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st} (0)\, dt}_{0} \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \left[\int_0^{\pi/\omega} e^{-st} \sin\omega t\, dt \right]$$

$$\boxed{\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b\cos bx)}$$

On Comparing we have

$a = -s$

$x = t$

$b = \omega$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \int_0^{\pi/\omega} e^{-st} \sin\omega t \, dt .$$

$$\boxed{\int e^{ax}\sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}$$

$a = -s$

$x = t \qquad b = \omega .$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t) \right]_0^{\pi/\omega} \qquad (t)$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \left[\frac{e^{-\frac{\pi}{\omega}s}}{s^2+\omega^2}(0+\omega) - \frac{1}{s^2+\omega^2}(0-\omega) \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \left[\frac{\omega e^{-\frac{\pi}{\omega}s}}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2} \right]$$

$$= \frac{\omega\left(1+e^{-\frac{\pi}{\omega}s}\right)}{\left(1-e^{-\frac{2\pi}{\omega}s}\right)\left(s^2+\omega^2\right)}$$

$$= \frac{\omega}{(s^2+\omega^2)\left(1+e^{-\frac{\pi s}{\omega}}\right)}$$

3

3) Find $L\{f(t)\}$ where

$$f(t) = \begin{cases} \sin t \;; & 0 < t < \pi \\ 0 \;; & \pi < t < 2\pi \end{cases}$$

with Period 2π

Sol :- given Period $T = 2\pi$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-s(2\pi)}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt \right]$$

$$\boxed{\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}$$

$e^{0} = 1$

On Comparing we have. $a = -s$
$t = x$
$b = 1$

w.r.to t

$\cos\pi = -1$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{s^2+1} \left[-s\sin t - \cos t\right] \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-\pi s}}{s^2+1} (\cancel{}) (0-(-1)) - \frac{1}{s^2+1}(-1) \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-\pi s}}{s^2+1}(1) + \frac{1}{s^2+1} \right]$$

$\cos(-\theta) = \cos\theta$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-\pi s}+1}{s^2+1}\right] \qquad 1^2-\left(e^{-\pi s}\right)^2$$

$$a^2-b^2=(a-b)(a+b)$$

$$1^2-\left(e^{-\pi s}\right)^2=\left(1-e^{-\pi s}\right)\left(1+e^{-\pi s}\right)$$

$$= \frac{1}{\cancel{\left(1+e^{-\pi s}\right)}\left(1-e^{-\pi s}\right)}\;\frac{\cancel{\left(1+e^{-\pi s}\right)}}{s^2+1}$$

$$= \frac{1}{\left(s^2+1\right)\left(1-e^{-\pi s}\right)}$$

(4)

Initial Value Theorem :-

If $L\{f(t)\} = \bar{f}(s)$ Then

$$\boxed{\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,\bar{f}(s)}$$

Proof :- Let $L\{f(t)\} = \bar{f}(s)$ $\quad L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

we know that

$$\boxed{L\{f'(t)\} = s\,\bar{f}(s) - f(0)}$$

$$= \int_0^\infty e^{-st} f'(t)\,dt = s\bar{f}(s) - f(0)$$

$e^{-\infty} = 0$

now taking $\lim_{s\to\infty}$ on b.s

$$= \int_0^\infty \lim_{s\to\infty} e^{-st} f'(t)\,dt = \lim_{s\to\infty} s\,\bar{f}(s) - f(0)$$

$$(0) = \lim_{s\to\infty} s\bar{f}(s) - f(0)$$

$$f(0) = \lim_{s\to\infty} s\,\bar{f}(s)$$

($\because$ Continuous Limit = function value at 0)

$$\boxed{\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,\bar{f}(s)}$$

Hence Proved.

$\therefore$ Initial value theorem is Proved

Final value Theorem

If $L\{f(t)\} = \bar{f}(s)$ Then

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,\bar{f}(s)}$$

Proof: Let $L\{f(t)\} = \bar{f}(s)$

we know that

$$\boxed{L\{f'(t)\} = s\bar{f}(s) - f(0)}$$

$$= \int_0^\infty e^{-st} f'(t)\,dt = s\bar{f}(s) - f(0)$$

$\lim_{s\to 0} e^{-st} = e^0 = 1$

now taking $\lim_{s\to 0}$ on b.s

$$= \int_0^\infty \lim_{s\to 0} e^{-st} f'(t)\,dt = \lim_{s\to 0} s\bar{f}(s) - f(0)$$

$$= \int_0^\infty f'(t)\,dt = \lim_{s\to 0} s\bar{f}(s) - f(0)$$

$$= [f(t)]_0^\infty = \lim_{s\to 0} s\bar{f}(s) - f(0)$$

$$= f(\infty) - \cancel{f(0)} = \lim_{s\to 0} s\bar{f}(s) - \cancel{f(0)}$$

$$= f(\infty) = \lim_{s\to 0} s\bar{f}(s)$$

$$= \boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s)}$$

$\therefore$ Final value theorem is Proved

⑤

① verify Initial value theorem for

$(2t+3)^2$

Sol: Initial value Theorem.

If $L\{f(t)\} = \bar{f}(s)$ Then.

$$\boxed{\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ s\,\bar{f}(s)}$$

Let

$f(t) = (2t+3)^2$

$$\underset{t\to 0}{Lt}\ (2t+3)^2 = 9$$

$= 2(0)+(3)^2$

$= 9$.

$f(t) = (2t+3)^2$

$(a+b)^2 = a^2+b^2+2ab$

$(2t+3)^2 = 4t^2+9+12t$

$$\underset{s\to\infty}{Lt}\ s\,\bar{f}(s) = \underset{s\to\infty}{Lt}\ s\,L\{f(t)\}$$

$$= \underset{s\to\infty}{Lt}\ s\,L\{(2t+3)^2\}$$

$$= \underset{s\to\infty}{Lt}\ s\,L\{4t^2+9+12t\}$$

$$= \underset{s\to\infty}{Lt}\ s\left[4\,\frac{2}{s^3} + \frac{9}{s} + 12\,\frac{1}{s^2}\right]$$

$$= \underset{s\to\infty}{Lt}\ s\left[\frac{8}{s^3} + \frac{9}{s} + \frac{12}{s^2}\right]$$

$$= \underset{s\to\infty}{Lt}\left[\frac{8}{s^2} + 9 + \frac{12}{s}\right]$$

$$= \frac{8}{\infty^2} + 9 + \frac{12}{\infty}.$$

$\frac{1}{\infty} = 0$

$\frac{\text{any num}}{\infty} = 0$

$$= \underline{9}.$$

$\therefore$ Initial value theorem is verified.

(2) Verify Final value theorem for.

$t^3 e^{-2t}$

L-Hospital rule: numerator & Denom are seperately differentiated. Until we get defined value as we substitute given limit.

Sol :- If $L\{f(t)\} = \bar{f}(s)$ Then

$$\boxed{\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt}\, s\,\bar{f}(s)}$$

$$\underset{t\to\infty}{Lt}\, t^3 e^{-2t} = \underset{t\to\infty}{Lt} \frac{t^3}{e^{2t}} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \underset{t\to\infty}{Lt} \frac{3t^2}{2e^{2t}} \quad (\because \text{L-Hospital rule})$$

$$= \underset{t\to\infty}{Lt} \frac{6t}{4e^{2t}} \quad (\because \text{L-Hospital rule})$$

$$= \underset{t\to\infty}{Lt} \frac{6}{8e^{2t}} = \underline{0}. \quad \left(\frac{6}{\infty} = 0\right)$$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$

$$\underset{s\to 0}{Lt}\, s\,\bar{f}(s) = \underset{s\to 0}{Lt}\, s\,L\{f(t)\}$$

$$= \underset{s\to 0}{Lt}\, s\,L\{t^3 e^{-2t}\}$$

$$= \underset{s\to 0}{Lt}\, s\left\{\frac{6}{(s+2)^4}\right\} = \underline{0}.$$

$$L\{t^3\} = \left\{\frac{6}{s^4}\right\}$$

$$L\{t^3 e^{-2t}\} = \left\{\frac{6}{s^4}\right\}_{s \text{ replace } s+2}$$

$$= \frac{6}{(s+2)^4}.$$

$$\therefore \underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt}\, s\,\bar{f}(s)$$

$\therefore$ Final value theorem is Proved.

⑥

1) ~~Verify Initial Value theorem for~~

~~① $(2t+3)^2$.~~

~~Sol: we know that Initial value theorem is~~

① $f(t) = K$ when $0<t<a$

$\quad\quad -K$ when $a<t<2a$. $T=2a$.

Sol:
$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a K e^{-st}\,dt + \int_a^{2a} (-K) e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[K\int_0^a e^{-st}\,dt - K\int_a^{2a} e^{-st}\,dt\right]$$

$$= \frac{K}{1-e^{-2as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$

$$= \frac{K}{s(1-e^{-2as})}\left[-e^{-as} + 1 + e^{-2as} - e^{-as}\right]$$

$$= \frac{K}{s(1-e^{-2as})}\left[1^2\right.$$

$$f(t) = \begin{cases} K & ; \ 0 < t < a \\ -k & ; \ a < t < \underline{\underline{2a}} \end{cases} \qquad T = 2a.$$

$\underline{\underline{\text{Sol}}}$:- $$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} K\,dt + \int_a^{2a} (-k)\, e^{-st}\,dt \right]$$

$$\Rightarrow \frac{K}{1-e^{-2as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^a + \left(\frac{e^{-st}}{s} \right)_a^{2a} \right]$$

$e^0 = 1$

$$= \frac{K}{s(1-e^{-2as})} \left[\underbrace{-e^{-as}} + \underline{\underline{1}} + e^{-2as} - \underbrace{e^{-as}} \right]$$

$$= \frac{K}{s(1-e^{-2as})} \left[1 - 2e^{-as} + e^{-2as} \right]$$

$$= \frac{K}{s(1-e^{-2as})} \left[\underset{a^2}{1} \underset{-}{-} \underset{2ab}{2e^{-as}} \underset{+}{+} \underset{b^2}{(e^{-as})^2} \right]$$

$$= \frac{K}{s(1-e^{-2as})}\left[\left(1-\frac{e^{-as}}{(a-b)^2}\right)^2\right]$$

$$= \frac{K}{s\,\cancel{(1-e^{-as})}(1+e^{-as})}\cdot(1-e^{-as})^{\cancel{2}}$$

$$= \frac{K}{s}\cdot\frac{(1-e^{-as})}{(1+e^{-as})}$$

f as half $f\frac{as}{2}$

$e^{m}\cdot e^{-n} = e^{m-n}$

$e^{as/2}\cdot e^{-as} = e^{\frac{as}{2}-as}$

$= e^{\frac{as-2as}{2}}$

$= e^{-\frac{as}{2}}$.

$$= \frac{K}{S}\cdot\frac{e^{as/2}}{e^{as/2}}\cdot\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$= \frac{K}{S}\left(\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right)$$

$$= \frac{K}{S}\cdot\operatorname{Tanh}\frac{as}{2}.$$

$\therefore \operatorname{Tanh}ax = \frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}$

$\operatorname{Sinh}ax = \frac{e^{ax}-e^{-ax}}{2}$

$\operatorname{Tan} = \frac{\sin}{\cos}$

$\operatorname{Cosh}ax = \frac{e^{ax}+e^{-ax}}{2}$

(1) b
(2) d
(3) a
(4) b
(5) d
(6) a
(7) b
(8) b
(9) a
(10) c

11) a
12) a
13) b
14) a
15) a
16) b
17) c
18) d.
19) a
20) b

21) a
22) a
23) d
24) b
25) a
26) $\frac{2!}{(s+2)^3}$
27) $\frac{1}{(s-2)^2+1}$
28) $1+3t+7\frac{t^2}{2!}$
29) $\frac{t}{2a}\sin at$
30) $\frac{1}{4}\cosh 2t$.

$\frac{1}{3}\cdot f\left(\frac{s}{3}\right)$

$= \frac{1}{3}\,\frac{e^{-\frac{3}{s}}}{\frac{s}{3}}$

$$L^{-1}\left(\frac{1}{s^{3/2}}\right)$$

$$= \frac{t^{\frac{3}{2}-1}}{\Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{t^{1/2}}{\Gamma\left(\frac{1}{2}+1\right)}$$

$$= \frac{\sqrt{t}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{\sqrt{t}}{\frac{1}{2}\sqrt{\pi}} = 2\sqrt{\frac{t}{\pi}}$$

(26')

*** (3) Find $L^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right)$

Sol:- $L^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right) = L^{-1}\left(\frac{1}{s^2+1}\right) + L^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right)$

$= \sin t + L^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right)$

$\therefore L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = f(t)$ (say)

By Second Shifting Theorem

$$\therefore L^{-1}\left(e^{-as}\bar{f}(s)\right) = \begin{cases} f(t-a) & ; t>a \\ 0 & ; t<a \end{cases}$$

$$\therefore L^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) = \begin{cases} f(t-\pi) & ; t>\pi \\ 0 & ; t<\pi \end{cases}$$

$$= \begin{cases} \sin(t-\pi) & ; t>\pi \\ 0 & ; t<\pi \end{cases}$$

$$= \sin(t-\pi)\cdot u(t-\pi)$$

$$\therefore L^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right) = \sin t + \sin(t-\pi)\cdot u(t-\pi)$$

Inverse Laplace Transform of Derivatives :-

Theorem :- If $L^{-1}(\bar{f}(s)) = f(t)$ Then

$$L^{-1}\left(\bar{f}^{(n)}(s)\right) = (-1)^n t^n f(t)$$

where $\bar{f}^{(n)}(s) = \frac{d^n}{ds^n}(\bar{f}(s))$

Proof :- we know that

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$$

$$L\{t^n f(t)\} = (-1)^n f^{(n)}(s)$$

$$t^n f(t) = L^{-1}\left((-1)^n f^{(n)}(s)\right)$$

ie :- $t^n f(t) = (-1)^n L^{-1}\left(f^{(n)}(s)\right)$

$$(-1)^n t^n f(t) = (-1)^{2n} L^{-1}\left(\bar{f}^{(n)}(s)\right)$$

$$\therefore L^{-1}\{\bar{f}^{(n)}(s)\} = (-1)^n t^n f(t) \quad //.$$

Problems

1) Find Inverse Laplace Transform of $\log\left(\frac{s+1}{s-1}\right)$ (or)

Find $L^{-1}\left(\log\left(\frac{s+1}{s-1}\right)\right)$

Sol :- Let $\bar{f}(s) = \log\left(\frac{s+1}{s-1}\right)$

$$L\{f(t)\} = \log\left(\frac{s+1}{s-1}\right)$$

$$L\{t f(t)\} = -\frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

(27)

$$= (-1)\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \frac{1}{s-1} - \frac{1}{s+1}$$

$$\Rightarrow t f(t) = L^{-1}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

$$= t f(t) = e^{t} - e^{-t}$$

$$f(t) = \frac{e^{t} - e^{-t}}{t}$$

$$\therefore L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\} = \frac{2}{t}\sinh t$$

(2) Find $L^{-1}\left\{\log\left(\frac{s-a}{s^2+a^2}\right)\right\}$

Sol: Let $L^{-1}\left\{\log\left(\frac{s-a}{s^2+a^2}\right)\right\} = f(t)$

$$\Rightarrow L\{f(t)\} = \log\left(\frac{s-a}{s^2+a^2}\right)$$

$$\Rightarrow L\{f(t)\} = \log(s-a) - \log(s^2+a^2)$$

$$\Rightarrow L\{t\, f(t)\} = -\frac{d}{ds}\left(\bar{f}(s)\right)$$

$$= -\frac{d}{ds}\left[\log(s-a) - \log(s^2+a^2)\right]$$

$$= -\left[\frac{1}{s-a} - \frac{2s}{s^2+a^2}\right]$$

$$= \frac{2s}{s^2+a^2} - \frac{1}{s-a}$$

$$\therefore L\{t f(t)\} = \frac{2s}{s^2+a^2} - \frac{1}{s-a}$$

P.T.O.

$$t\,f(t) = L^{-1}\left(\frac{2s}{s^2+a^2}\right) - L^{-1}\left(\frac{1}{s-a}\right)$$

$$= 2\cos at - e^{at}$$

$$f(t) = \frac{2\cos at - e^{at}}{t}$$

$$\therefore L^{-1}\left\{\log\left(\frac{s-a}{s^2+a^2}\right)\right\} = \frac{2\cos at - e^{at}}{t}$$

(3) Find $L^{-1}\left\{\log\left(1+\frac{\omega^2}{s^2}\right)\right\}$

Sol: $L^{-1}\left\{\log\left(1+\frac{\omega^2}{s^2}\right)\right\} = f(t)$

$$L\{f(t)\} = \log\left(\frac{s^2+\omega^2}{s^2}\right)$$

$$= \log(s^2+\omega^2) - \log s^2$$

$$L\{t f(t)\} = -\frac{d}{ds}\left[\log(s^2+\omega^2) - \log s^2\right]$$

$$= -\left[\frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}\right]$$

$$= \frac{2}{s} - \frac{2s}{s^2+\omega^2}$$

$$t\{f(t)\} = L^{-1}\left(\frac{2}{s}\right) - L^{-1}\left(\frac{2s}{s^2+\omega^2}\right)$$

$$= 2 - 2\cos\omega t$$

$$f(t) = \frac{2}{t}(1-\cos\omega t)$$

$$\therefore L^{-1}\left\{\log\left(1+\frac{\omega^2}{s^2}\right)\right\} = \frac{2}{t}(1-\cos\omega t)$$

(28)

(4) Find $L^{-1}(\cot^{-1}s)$

Sol: Let $\bar{f}(s) = \cot^{-1}s$

$$\bar{f}'(s) = \frac{d}{ds}\cot^{-1}s = -\frac{1}{1+s^2}$$

$$L^{-1}(\bar{f}'(s)) = -t f(t) \Rightarrow f(t) = -\frac{1}{t} L^{-1}(\bar{f}'(s))$$

ie: $L^{-1}(\bar{f}(s)) = -\frac{1}{t} L^{-1}\left(\frac{-1}{1+s^2}\right)$

$$L^{-1}(\cot^{-1}s) = +\frac{1}{t} L^{-1}\left(\frac{1}{1+s^2}\right)$$

$$L^{-1}(\cot^{-1}s) = \frac{\sin t}{t}$$

(5) Find $L^{-1}(\tan^{-1}(s+1))$

Sol: Let $\bar{f}(s) = \tan^{-1}(s+1)$

$$\bar{f}'(s) = \frac{1}{1+(s+1)^2}$$

we have $L^{-1}(\bar{f}'(s)) = -t f(t)$

$$f(t) = -\frac{1}{t} L^{-1}(\bar{f}'(s))$$

$$= -\frac{1}{t} L^{-1}\left(\frac{1}{1+(s+1)^2}\right)$$

$$= -\frac{1}{t} e^{-t} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$\therefore f(t) = -\frac{1}{t} e^{-t}\sin t$$

Hence $L^{-1}(\tan^{-1}(s+1)) = -\frac{1}{t} e^{-t}\sin t$

Method of Partial Fractions

If $\bar{f}(s)$ is given in the form $\frac{g(s)}{h(s)}$

where g and h are Polynomials in s

Then $f(t)$ can be Obtained by resolving $\bar{f}(s)$ into Partial fractions and apply Inverse Laplace transform.

(1) Evaluate $L^{-1}\left(\frac{s^2}{(s+1)(s+2)(s+3)}\right)$

Sol:- $\frac{s^2}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$

$$\Rightarrow s^2 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) \longrightarrow (1)$$

Put $\boxed{s=-2}$; $4 = B(-1)(1)$; $\boxed{B=-4}$

Put $\boxed{s=-1}$; $1 = A(2)$; $\boxed{A=\frac{1}{2}}$

Put $\boxed{s=-3}$; $9 = C(-2)(-1)$; $\boxed{C=\frac{9}{2}}$

$$\therefore \frac{s^2}{(s+1)(s+2)(s+3)} = \frac{1}{2(s+1)} - \frac{4}{s+2} + \frac{9}{2(s+3)}$$

$$\therefore L^{-1}\left(\frac{s^2}{(s+1)(s+2)(s+3)}\right) = \frac{1}{2}e^{-t} - 4e^{-2t} + \frac{9}{2}e^{-3t}$$

29

(2) Find $L^{-1}\left(\frac{1}{(s+1)(s^2+1)}\right)$

Sol :- $\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$

$$1 = A(s^2+1) + (s+1)(Bs+C)$$

Put $\boxed{s=-1}$

$1 = 2A$

$\boxed{A = \frac{1}{2}}$

Equating Co-eff of "s^2" on b.s

$0 = A+B$

$B = -A$

$\boxed{B = -\frac{1}{2}}$

Equating Constants on b.s

$1 = A + C$

$\Rightarrow C = 1 - A = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore \boxed{C = \frac{1}{2}}$

$$\therefore \frac{1}{(s+1)(s^2+1)} = \frac{\frac{1}{2}}{s+1} + \frac{\left(-\frac{1}{2}\right)s + \left(\frac{1}{2}\right)}{s^2+1}$$

Applying L^{-1} on b.s

$$\therefore L^{-1}\left(\frac{1}{(s+1)(s^2+1)}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{2}L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$$= \frac{1}{2}\left(e^{-t} - \cos t - \sin t\right)$$

(3) Find $L^{-1}\left(\frac{1}{(s+1)^2(s^2+4)}\right)$

Sol : $\frac{1}{(s+1)^2(s^2+4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+4}$

$$1 = A(s+1)(s^2+4) + B(s^2+4) + (Cs+D)(s+1)^2 \longrightarrow (1)$$

Put $\boxed{s=-1}$

$1 = B(5)$

$\boxed{B = \frac{1}{5}}$

Equating Co-eff of

" s^2 " $0 = A+B+2C+D$

" s " $0 = 4A + C + 2D$

"Const" $1 = 4A + 4B + D$

" s^3 " $0 = A + C$

On Solving $A = \frac{2}{25}$ $B = \frac{1}{5}$ $C = \frac{-2}{25}$

$D = \frac{-3}{25}$

$$\therefore L^{-1}\left(\frac{1}{(s+1)^2(s^2+4)}\right) = L^{-1}\left(\frac{2}{25(s+1)}\right) + \frac{1}{5}L^{-1}\left(\frac{1}{(s+1)^2}\right) + L^{-1}\left(\frac{-\frac{2}{25}s}{s^2+4}\right) + \left(\frac{-3}{25}\right)L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{2}{25}e^{-t} + \frac{1}{5}e^{-t}t - \frac{2}{25}\cos 2t - \frac{3}{25}\frac{\sin 2t}{2}$$

(30)

Multiplication by "S"

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $f(0)=0$ Then

$$L^{-1}(s\bar{f}(s)) = f'(t)$$

Proof :- we Know that

$$L\{f'(t)\} = s\,\bar{f}(s) - f(0)$$
$$= s\bar{f}(s) - 0$$
$$= s\bar{f}(s)$$

$$\therefore L\{f'(t)\} = s\bar{f}(s)$$

$$\Rightarrow L^{-1}\{s\,\bar{f}(s)\} = f'(t)$$

In General $L^{-1}\{s^n\bar{f}(s)\} = f^{(n)}(t)$

where $f^{(n)}(0)=0$ for $n=1,2,3 \ldots n-1$.

(1) Find $L^{-1}\left\{\dfrac{s}{(s+2)^2+4}\right\}$

Sol :- $L^{-1}\left\{\dfrac{s}{(s+2)^2+4}\right\} = \dfrac{d}{dt} L^{-1}\left\{\dfrac{1}{(s+2)^2+4}\right\}$

$$= \frac{d}{dt}\left\{e^{-2t} L^{-1}\left\{\frac{1}{s^2+4}\right\}\right\}$$

$$= \frac{d}{dt}\left(e^{-2t}\,\frac{\sin 2t}{2}\right) = \frac{1}{2}\left(e^{-2t}\,2\cos 2t - 2e^{-2t}\sin 2t\right)$$

$$= e^{-2t}(\cos 2t - \sin 2t)$$

(2) Find $L^{-1}\left\{\dfrac{s}{(s+2)^2}\right\}$

Sol :- $L^{-1}\left\{\dfrac{s}{(s+2)^2}\right\} = \dfrac{d}{dt}\left\{L^{-1}\left(\dfrac{1}{(s+2)^2}\right)\right\} = \dfrac{d}{dt}\left(e^{-2t}\cdot t\right)$

$$= e^{-2t}(1) + t(-2e^{-2t})$$
$$= e^{-2t}(1-2t)$$

(3) Find $L^{-1}\left\{\frac{s^2}{(s-2)^2}\right\}$

<u>Sol</u> :- $L^{-1}\left\{\frac{s^2}{(s-2)^2}\right\} = \frac{d}{dt}\left\{L^{-1}\left(\frac{s}{(s-2)^2}\right)\right\}$

$$= \frac{d}{dt}\left\{\frac{d}{dt}\left\{L^{-1}\left(\frac{1}{(s-2)^2}\right)\right\}\right\}$$

$$= \frac{d}{dt}\left\{\frac{d}{dt}\left(e^{2t}t\right)\right\}$$

$$= \frac{d}{dt}\left(e^{2t} + 2te^{2t}\right)$$

$$= 2e^{2t} + 4te^{2t} + 2e^{2t}$$

$$= 4\left(e^{2t} + te^{2t}\right)$$

Division By 'S' :-

If $L^{-1}(\bar{f}(s)) = f(t)$ Then $L^{-1}\left(\frac{\bar{f}(s)}{s}\right) = \int_0^t f(t)\,dt$

<u>Proof</u> :- we Know that

$$L\left(\int_0^t f(t)\,dt\right) = \frac{\bar{f}(s)}{s}$$

$$\Longrightarrow L^{-1}\left(\frac{\bar{f}(s)}{s}\right) = \int_0^t f(t)\,dt$$

(1) Find $L^{-1}\left(\frac{1}{s(s+3)}\right)$

<u>Sol</u> :- $L^{-1}\left(\frac{1}{s(s+3)}\right) = \int_0^t L^{-1}\left(\frac{1}{s+3}\right)dt = \int_0^t e^{-3t}\,dt = \left[\frac{e^{-3t}}{-3}\right]_0^t$

$$= \frac{e^{-3t}}{-3} + \frac{1}{3}$$

$$= \frac{1 - e^{-3t}}{3}$$

(31)

(2) Find $L^{-1}\left(\frac{1}{s(s^2+a^2)}\right)$

Sol:- $L^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \int_0^t L^{-1}\left(\frac{1}{s^2+a^2}\right) dt$

$$= \int_0^t \frac{\sin at}{a}\, dt = \frac{1}{a}\left(-\frac{\cos at}{a}\right)_0^t$$

$$= \frac{1-\cos at}{a^2}$$

(3) Find $L^{-1}\left(\frac{1}{s(s^2-2s+5)}\right)$

Sol:- $L^{-1}\left(\frac{1}{s(s^2-2s+5)}\right) = L^{-1}\left(\frac{\frac{1}{s^2-2s+5}}{s}\right)$

$$= \int_0^t L^{-1}\left(\frac{1}{(s-1)^2+4}\right) dt = \int_0^t e^t\, \frac{\sin 2t}{2}\, dt$$

$$= \frac{1}{2}\left[\frac{e^t}{1^2+2^2}\left(\sin 2t - 2\cos 2t\right)\right]_0^t$$

$$= \frac{1}{2}\left[\frac{e^t}{5}\left(\sin 2t - 2\cos 2t\right)\right]_0^t$$

$$= \frac{1}{2}\left[\frac{e^t}{5}\left(\sin 2t - 2\cos 2t\right) + 2\right]$$

(4) Find $L^{-1}\left(\frac{1}{s(s^2-1)(s^2+1)}\right)$

Sol:- $L^{-1}\left(\frac{1}{(s^2-1)(s^2+1)}\right) = L^{-1}\left(\frac{1}{2}\left(\frac{1}{s^2-1} - \frac{1}{s^2+1}\right)\right)$

$$= \frac{1}{2}\left(\sinh t - \sin t\right)$$

$$= f(t) \text{ (say)}$$

$$L^{-1}\left(\frac{\frac{1}{(s^2-1)(s^2+1)}}{s}\right) = \int_0^t f(t)\,dt$$

$$= \frac{1}{2}\int_0^t (\sinh t - \sin t)\,dt = \frac{1}{2}\left[\cosh t + \cos t\right]_0^t$$

$$= \frac{1}{2}\left(\cosh t + \cos t - 2\right)$$

Applications of Laplace (Imp Problems)

1) Solve the DE Using L.T

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{-t}, \quad x(0)=0, \quad x'(0)=1$$

Sol :- Given DE $x'' + 3x' + 2x = e^{-t}$

Taking L.T on b.s

$$L(x'') + 3L(x') + 2L(x) = L(e^{-t})$$

$$= \left[s^2 L(x) - s x(0) - x'(0)\right] + 3\left[sL(x) - x(0)\right] + 2L(x) = \frac{1}{s+1}$$

Since $x(0)=0$, $x'(0)=1$

$$= \left[s^2 L(x) - 1\right] + 3\left[sL(x)\right] + 2L[x] = \frac{1}{s+1}$$

$$(s^2+3s+2)L(x) = \frac{1}{s+1} + \frac{1}{1} = \frac{1+s+1}{s+1} = \frac{s+2}{s+1}$$

$$L(x) = \frac{s+2}{(s^2+3s+2)(s+1)} = \frac{\cancel{s+2}}{(s+1)^2(\cancel{s+2})}$$

$$= \frac{1}{(s+1)^2}$$

$$\therefore x = L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$x = te^{-t}$$

(32)

2) Solve $\frac{dy}{dt} + 3y + 2\int_0^t y(t)\,dt = t$ by L.T

Sol :- Given $\frac{dy}{dt} + 3y + 2\int_0^t y(t)\,dt = t$

Taking L.T on b.s

$$L(y'(t)) + 3L(y(t)) + 2L\left(\int_0^t y(t)\,dt\right) = L(t)$$

$$\Rightarrow [s\bar{y}(s) - y(0)] + 3\bar{y}(s) + 2\frac{\bar{y}(s)}{s} = \frac{1}{s^2}$$

Let $y(0) = a$

$$\Rightarrow \left(s + 3 + \frac{2}{s}\right)\bar{y}(s) = \frac{1}{s^2} + a$$

$$\Rightarrow \frac{s^2+3s+2}{\cancel{s}}\,\bar{y}(s) = \frac{as^2+1}{s^{\cancel{2}}}$$

$$\Rightarrow (s+1)(s+2)\,\bar{y}(s) = \frac{as^2+1}{s}$$

$$\Rightarrow \bar{y}(s) = \frac{as^{\cancel{2}}}{\cancel{s}(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)}$$

$$L(y(t)) = \frac{as}{(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)}$$

$$y(t) = aL^{-1}\left(\frac{s}{(s+1)(s+2)}\right) + L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right)$$

$$\therefore y(t) = \frac{1}{2} - e^{-t}(1+a) + e^{-2t}\left(\frac{1}{2} + 2a\right)$$

(3) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 2e^{-x}$

given $y=0$, $\frac{dy}{dx} = -1$ at $x=0$

Sol :- Given eqn Can be written as

$$y'' + 4y' + 13y = 2e^{-x}$$

Taking LT on b.s

$$L(y'') + 4L(y') + 13L(y) = 2L(e^{-x})$$

$$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 13\bar{y}(s) = 2\left(\frac{1}{s+1}\right)$$

given that $y(0)=0$, $y'(0)=-1$

$$[s^2\bar{y}(s)+1) + 4(s\bar{y}(s)) + 13\bar{y}(s) = \frac{2}{s+1} - 1$$

$$(s^2+4s+13)\bar{y}(s) = \frac{2}{s+1} - 1 = \frac{1-s}{s+1}$$

$$\bar{y}(s) = \frac{1-s}{(s+1)(s^2+4s+13)}$$

$$L(y(x)) = \frac{1-s}{(s+1)(s^2+4s+13)}$$

$$y(x) = L^{-1}\left(\frac{1-s}{(s+1)(s^2+4s+13)}\right)$$

$$\therefore y(x) = \frac{1}{5}e^{-x} - \frac{1}{5}e^{-2x}\cos 3x - \frac{2}{5}e^{-2x}\sin 3x$$

Laplace Transform of Periodic Function :- (33)

Periodic Function :- A Function $f(t)$ is said to be Periodic function with Period $T > 0$

If $f(t) = f(t+T) = f(t+2T) = \cdots = f(t+nT) = \cdots$

where T is the Smallest Positive Value

For ex :-

(i) $\sin t$, $\cos t$ are Periodic functions with Period "2π"

(ii) $|\sin t|$, $\tan t$, $\cot t$ are Periodic functions with Period π

Theorem :- If $f(t)$ is a Periodic function with Period T then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

(1) Find the L.T of Square-wave function of Period $2a$ defined as

$$f(t) = K \quad \text{when } 0 < t < a \qquad T = 2a$$

$$= -K \quad \text{when } a < t < 2a \qquad \text{(or) } f(t+2a) = f(t)$$

Sol :- Since $f(t)$ is a Periodic function with Period $T = 2a$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a K e^{-st}\,dt + \int_a^{2a} (-K)\, e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[K \left(\frac{e^{-st}}{-s} \right)_0^a - K \left(\frac{e^{-st}}{-s} \right)_a^{2a} \right]$$

$$= \frac{K}{s(1-e^{-2as})}\left[-e^{-as}+1+e^{-2as}-e^{-as}\right]$$

$$= \frac{K}{s(1-e^{-2as})}\left[1^2-2e^{-as}+(e^{-as})^2\right]$$

$$= \frac{K}{s(1-e^{-2as})}\left(1-e^{-as}\right)^2$$

$$= \frac{K}{s(1-e^{-2as})}\left[1^2-2e^{-as}+(e^{-as})^2\right]$$

$a^2-2ab+b^2$

$(a-b)^2$

$(1-e^{-as})^2$

$$= \frac{K}{s}\,\frac{1-e^{-as}}{1+e^{-as}}$$

$1^2-(e^{-as})^2$

$=(1-e^{-as})(1+e^{-as})$

$$= \frac{K}{s}\,\frac{e^{-\frac{as}{2}}\left(e^{\frac{as}{2}}-e^{-\frac{as}{2}}\right)}{e^{-\frac{as}{2}}\left(e^{\frac{as}{2}}+e^{-\frac{as}{2}}\right)}$$

$$= \frac{K}{s}\,\mathrm{Tanh}\,\frac{as}{2} \;//$$

(2) Find the L.T of the rectified Semi-wave function defined by

$$f(t) = \begin{cases} \sin\omega t & ; \; 0<t<\frac{\pi}{\omega} \\ 0 & ; \; \frac{\pi}{\omega}<t<\frac{2\pi}{\omega} \end{cases} \quad \text{with Period } \frac{2\pi}{\omega}.$$

Sol :- Since $f(t)$ is a Periodic function with Period $\frac{2\pi}{\omega}$ (T).

$$\therefore\; L\{f(t)\} = \frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-st}}{s^2+\omega^2}\left(-s\sin\omega t-\omega\cos\omega t\right)\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-\frac{\pi}{\omega}s}}{s^2+\omega^2}(0+\omega)-\frac{1}{s^2+\omega^2}(0-\omega)\right]$$

(34)

$$= \frac{\omega\left(1+e^{-\frac{\pi}{\omega}s}\right)}{\left(1-e^{-\frac{2\pi}{\omega}s}\right)\left(s^2+\omega^2\right)}$$

$$= \frac{\omega}{\left(s^2+\omega^2\right)\left(1+e^{-\frac{\pi s}{\omega}}\right)}$$

Evaluate the Following Integrals Using Laplace Transforms

1) Show that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$ (Starting Prob)

Sol :- we have $\int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$

Let $f(t) = \frac{\sin t}{t}$

$$L\{f(t)\} = L\left(\frac{\sin t}{t}\right) = \cot^{-1}s = \bar{f}(s)$$

$$\therefore \int_0^\infty e^{-0t} f(t)\,dt = \bar{f}(0) = \cot^{-1}(0) = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

2) Show that $\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = \frac{\pi}{4}$

Sol :- we have $L\left(\frac{\sin t}{t}\right) = \cot^{-1}s = \bar{f}(s)$

$$\therefore \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$$

$$\Rightarrow \int_0^\infty e^{-t} f(t)\,dt = \bar{f}(1) \Rightarrow \int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = \cot^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = \frac{\pi}{4}$$

(3) Evaluate $\int_0^{\infty} \frac{e^{-t}-e^{-3t}}{t}\,dt$

<u>Sol</u> :- $L(e^{-t}-e^{-3t}) = \frac{1}{s+1} - \frac{1}{s+3}$

$$L\left(\frac{e^{-t}-e^{-3t}}{t}\right) = \int_s^{\infty}\left(\frac{1}{s+1}-\frac{1}{s+3}\right)ds$$

$$= \left[\log(s+1) - \log(s+3)\right]_s^{\infty}$$

$$= \left[\log\left(\frac{s+1}{s+3}\right)\right]_s^{\infty}$$

$$= \log\left(\frac{s+3}{s+1}\right) = \bar{f}(s) \text{ (say)}.$$

$$\therefore \int_0^{\infty} \frac{e^{-t}-e^{-3t}}{t}\,dt = \bar{f}(0) = \log\left(\frac{0+3}{0+1}\right)$$

$$= \underline{\underline{\log 3}}.$$

(4) S.T $\int_0^{\infty} \frac{\cos 5t - \cos 3t}{t}\,dt = \log\frac{3}{5}$

<u>Sol</u> :- Let $f(t) = \frac{\cos 5t - \cos 3t}{t}$

$$L\{f(t)\} = L\left(\frac{\cos 5t - \cos 3t}{t}\right)$$

$$= \int_s^{\infty}\left[L(\cos 5t) - L(\cos 3t)\right]ds$$

$$= \int_s^{\infty}\left(\frac{s}{s^2+25} - \frac{s}{s^2+9}\right)ds$$

(35)

$$= \frac{1}{2}\left[\log(s^2+25) - \log(s^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+25}\right)$$

$$= \bar{f}(s) \text{ (say)}$$

$$\therefore \int_0^\infty e^{-0t}\,\frac{\cos 5t - \cos 3t}{t}\,dt = \bar{f}(0)$$

ie :- $$\int_0^\infty \frac{\cos 5t - \cos 3t}{t}\,dt = \frac{1}{2}\log\left(\frac{0^2+9}{0^2+25}\right)$$

$$= \frac{1}{2}\log\left(\frac{9}{25}\right)$$

$$= \frac{1}{\cancel{2}}(\cancel{2})\log\left(\frac{3}{5}\right)$$

$$= \log\left(\frac{3}{5}\right)$$

$$\therefore \int_0^\infty \frac{\cos 5t - \cos 3t}{t}\,dt = \log\left(\frac{3}{5}\right)$$

(5) S.T $\int_0^\infty t e^{-2t}\sin t\,dt = \frac{4}{25}$

Sol :- Let $f(t) = t\sin t$

$$L\{f(t)\} = L\{t\sin t\} = -\frac{d}{ds}L\{\sin t\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -\left[\frac{-1}{(s^2+1)^2}\,2s\right]$$

$$= \frac{2s}{(s^2+1)^2} = \bar{f}(s) \text{ (say)}$$

$$\therefore \int_0^\infty t\, e^{-2t} \sin t \, dt = \bar{f}(2)$$

$$= \frac{2(2)}{(2^2+1)^2} = \frac{4}{25}$$

(36)

Formulas

1) $\cos(-\theta) = \cos\theta$.

2) $\sin(-\theta) = -\sin\theta$.

3) ~~sin(A+~~

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$-2\sin A \sin B = \cos(A+B) - \cos(A-B)$

(or)

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\Gamma_{n+1} = n\Gamma(n)$
$= n!$

① $L\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$ Γ (gamma) ✓

Proof: Consider $L\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$.

Put $st = x \Rightarrow t = \frac{x}{s}$ when $t = 0 \Rightarrow x = 0$.
diff b.s $t = \infty \Rightarrow x = \infty$.
$s\,dt = dx$
$dt = \frac{dx}{s}$.

Substitute these in above we get $L\{t^n\}$

Note

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$.

$\Gamma(n+1) = \int_0^\infty e^{-x} x^{(n+1)-1} dx$

$n = n+1$

$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$.

$\left(\frac{1}{s}\right)^n \cdot \frac{1}{s} = \frac{1}{s^{n+1}}$ Com

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{(n+1)-1} dx$.

$\Gamma_{n+1} = n\Gamma(n) = n!$

$= \frac{\Gamma(n+1)}{s^{n+1}}$ // $= \frac{n!}{s^{n+1}}$

(1) Find the Laplace transform of
$t^3 + 5\cos t$

Sol :- Given function $t^3 + 5\cos t$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Let $f(t) = t^3 + 5\cos t$

$L\{f(t)\} = L\{t^3 + 5\cos t\}$

$= L\{t^3\} + 5\{\cos t\}$

$L\{t^n\} = \dfrac{n!}{s^{n+1}}$

$L\{t^3\} = \dfrac{3!}{s^{3+1}} = \dfrac{6}{s^4}$

$= \dfrac{6}{s^4} + 5\dfrac{s}{s^2+1}$

$L\{\cos at\} = \dfrac{s}{s^2+a^2}$

$= \dfrac{6}{s^4} + \dfrac{5s}{s^2+1}$

$L\{\cos t\} = \dfrac{s}{s^2+1}$

Inverse Laplace Transform $\underline{\underline{Q}} \longrightarrow \underline{\underline{Ans}}$
$$s \longrightarrow t$$

Definition :- If $L\{f(t)\} = \bar{f}(s)$ (or)

If $\bar{f}(s)$ is the Laplace transform of $f(t)$ Then $f(t)$ is Called the Inverse Laplace transform of $\bar{f}(s)$ and is denoted by $L^{-1}\{\bar{f}(s)\}$

ie :- $L^{-1}\{\bar{f}(s)\} = f(t)$

If $L\{f(t)\} = \bar{F}(s)$ then $f(t)$ is called the Inverse Laplace transform of $\bar{F}(s)$ and is denoted by $L^{-1}\{\bar{F}(s)\} = f(t)$

L^{-1} is Called Inverse Laplace Operator.

ex :- (1) $L\{t\} = \frac{1}{s^2} \Rightarrow L^{-1}\left\{\frac{1}{s^2}\right\} = t$ $\quad \Rightarrow L\{t\} = \frac{1}{s} \Rightarrow L^{-1}\left\{\frac{1}{s}\right\} = t$.

(2) $L\{\sin t\} = \frac{1}{s^2+1} \Rightarrow L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

Inverse Laplace transform of some standard functions :-

1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$

2) $L^{-1}\left\{\frac{1}{s^2}\right\} = t$

3) $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

4) $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$ $(n > -1)$

5) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

6) $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

7) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$ (or)

8) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

9) $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

10) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$

11) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

12) $L^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at}$

13) $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = e^{at}\frac{\sin bt}{b}$.

14) $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

15) $L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{e^{at}}{b}\sinh bt$

16) $L^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\} = e^{at}\cosh bt$

17) $L^{-1}\left\{\frac{2as}{(s^2+a^2)^2}\right\} = t\sin at$.

18) $L^{-1}\left\{\frac{s^2-a^2}{(s^2+a^2)^2}\right\} = t\cos at$

1) Linearity Property :-

$$L^{-1}\{c_1\bar{f}_1(s)+c_2\bar{f}_2(s)\} = c_1L^{-1}\{\bar{f}_1(s)\} + c_2L^{-1}\{\bar{f}_2(s)\}.$$

① Find $L^{-1}\left\{\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-4}\right\}$

Sol :- $L^{-1}\left\{\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-4}\right\}$

$= L^{-1}\left\{\frac{1}{s-3}\right\} + L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{s}{s^2-4}\right\}$

$= e^{3t}+1+\cosh 2t$

② Find $L^{-1}\left\{\frac{1}{s^2}+\frac{1}{s^2+4}+\frac{1}{s+4}+\frac{s}{s^2-9}\right\}$

Sol :- $L^{-1}\left\{\frac{1}{s^2}+\frac{1}{s^2+4}+\frac{1}{s+4}+\frac{s}{s^2-9}\right\}$

$\Rightarrow L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{1}{s^2+4}\right\} + L^{-1}\left\{\frac{1}{s+4}\right\} + L^{-1}\left\{\frac{s}{s^2-9}\right\}$

$= t + \frac{\sin 2t}{2} + e^{-4t} + \cosh 3t$

2) First Shifting theorem in Inverse Laplace Transform :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{\bar{f}(s-a)\} = e^{at}f(t)$

① Find $L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\}$

Sol :- $L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\} = L^{-1}\{\bar{f}(s+2)\} = e^{-2t}f(t)$

$= e^{-2t}L^{-1}\{\bar{f}(s)\}$

$= e^{-2t}L^{-1}\left\{\frac{1}{s^2+16}\right\}$

$= e^{-2t}\frac{1}{4}\sin 4t$

$s+a$
$s+2$ $= e^{-2t}\cdot L^{-1}\left\{\frac{1}{s^2+16}\right\}$

$= e^{-2t}\frac{1}{4}\sin 4t$

(2) Find $L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$

<u>Sol</u>: $L^{-1}\left\{\frac{1}{(s+1)^3}\right\} = e^{-t} L^{-1}\left\{\frac{1}{s^3}\right\} = e^{-t}\cdot\frac{t^2}{2!}$ //.

<u>Second</u> <u>Shifting</u> <u>theorem</u> <u>of</u> <u>Inverse</u> <u>Laplace</u> <u>Transform</u>.

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{e^{-as}\bar{f}(s)\} = G(t)$

where $G(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$ $\qquad L^{-1}\{e^{\text{power}}\cdot\bar{f}(s)\}$

second shift

(1) Find $L^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right)$

<u>Sol</u>: $L^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right) = L^{-1}\left(\frac{1}{s^2+1}\right) + L^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right)$

$= \sin t + L^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right)$

$\therefore L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = f(t)$ (say)

by Second Shifting theorem we have.

$L\left(e^{-as}\bar{f}(s)\right) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a. \end{cases}$

$\therefore L^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) = \begin{cases} f(t-\pi) & ; t > \pi \\ 0 & ; t < \pi \end{cases}$

$= \begin{cases} \sin(t-\pi) & ; t > \pi \\ 0 & ; t < \pi \end{cases}$

$= \sin(t-\pi)\cdot u(t-\pi)$

$\therefore L^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right) = \sin t + \sin(t-\pi)\cdot u(t-\pi)$ //.

(2) Find $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}$

Sol :- $L^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t}.t = f(t)$ (say)

$$\therefore L\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = \begin{cases} e^{4(t-3)}.(t-3) & ; t>3 \\ 0 & ; t<3 \end{cases}$$

(or)

$$L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = (t-3)e^{4(t-3)}u(t-3)$$

(3) Find $L^{-1}\left\{\frac{e^{-3s}}{s^2-2s+5}\right\}$

Sol :- $L^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = e^t L^{-1}\left\{\frac{1}{s^2+4}\right\}$

$$= \frac{e^t}{2}\sin 2t.$$

$$\therefore L^{-1}\left\{\frac{e^{-3s}}{s^2-2s+5}\right\} = \begin{cases} f(t-3) & ; t>3 \\ 0 & ; t<3 \end{cases}$$

$$= \begin{cases} \frac{1}{2}e^{t-3}\sin 2(t-3) & ; t>3 \\ 0 & ; t<3 \end{cases}$$

$$= \frac{1}{2}e^{t-3}\sin 2(t-3)H(t-3)$$

Inverse Laplace Transform :-

1) **First Shifting Theorem** :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t)$$

2) **Second Shifting Theorem** :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L^{-1}\{e^{-as}\bar{f}(s)\} = G(t)$$

where $G(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a. \end{cases}$

(3) **Change of scale Property**

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L^{-1}\{\bar{f}(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

4) **Inverse Laplace Transform of Derivatives** :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then.

$$L^{-1}\left\{\frac{d^n}{ds^n}\bar{f}(s)\right\} = (-1)^n t^n f(t)$$

5) **Inverse Laplace Transform of Integrals** :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\} = \frac{f(t)}{t}$$

6) Multiplication by S :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ Then

$$L^{-1}\left(s\bar{f}(s)\right) = f'(t) \text{ Provided } f(0) = 0.$$

7) Division by S :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ Then.

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt \cdot \text{ Provided the Integral exists.}$$

Convolution Theorem

If $L\{f_1(t)\} = \bar{f_1}(s)$

$L\{f_2(t)\} = \bar{f_2}(s)$ Then

$$L\left\{\int_0^t f_1(u)\, f_2(t-u)\, du\right\} = \bar{f_1}(s) \cdot \bar{f_2}(s)$$

Hence $L^{-1}\{\bar{f_1}(s)\,\bar{f_2}(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du.$

1) Using Convolution Theorem

Find $L^{-1}\left(\frac{1}{s(s^2+1)}\right)$

Sol :- Let $\bar{f_1}(s) = \frac{1}{s} \Rightarrow L^{-1}\{\bar{f_1}(s)\} = L^{-1}\left\{\frac{1}{s}\right\} = 1 = f_1(t)$

$\bar{f_2}(s) = \frac{1}{s^2+1} \Rightarrow L^{-1}\{\bar{f_2}(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f_2(t)$

$$\therefore L^{-1}\{\bar{f_1}(s)\,\bar{f_2}(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du.$$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t 1\cdot \sin(t-u)\, du = \left[\cancel{-}\frac{\cos(t-u)}{\cancel{-1}}\right]_0^t$$

$$[\cos(t-u)]_0^t \qquad = \cos(t-t) - \cos(t-0)$$

$$= \cos 0 - \cos t$$

$$= 1 - \cos t$$

$$\therefore L^{-1}\left(\frac{1}{s(s^2+1)}\right) = 1 - \cos t$$

(d) Using Convolution Theorem.

Find $L^{-1}\left(\dfrac{1}{s(s^2-1)(s^2+1)}\right)$

Sol :- Let

$\bar{f}_1(s) = \dfrac{1}{(s^2-1)(s^2+1)}$ $= L^{-1}\{\bar{f}_1(s)\} = L^{-1}\left\{\dfrac{1}{(s^2-1)(s^2+1)}\right\}$

$\bar{f}_2(s) = \dfrac{1}{s}$ $= \dfrac{1}{2}L^{-1}\left\{\dfrac{1}{s^2-1} - \dfrac{1}{s^2+1}\right\}$

$L^{-1}\{\bar{f}_2(s)\} = L^{-1}\left\{\dfrac{1}{s}\right\}$ $= \dfrac{1}{2}(\sinh t - \sin t)$

$= 1$ $= f_1(t)$ (say)

$= f_2(t)$ (say)

$$\therefore L^{-1}\left\{\frac{1}{s(s^2-1)(s^2+1)}\right\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t \frac{1}{2}(\sinh u - \sin u)\cdot 1 \; du$$

$$= \frac{1}{2}\left[\cosh u + \cos u\right]_0^t$$

$$= \frac{1}{2}\left[\cosh t + \cos t - 2\right]$$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2-1)(s^2+1)}\right\} = \frac{1}{2}\left[\cosh t + \cos t - 2\right]$$

3) Using Convolution Theorem

Find $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Sol: $\bar{f}_1(s) = \dfrac{s}{(s^2+a^2)} \Rightarrow L^{-1}\{\bar{f}_1(s)\} = \cos at = f_1(t)$ (say)

$\bar{f}_2(s) = \dfrac{s}{(s^2+b^2)} \Rightarrow L^{-1}\{\bar{f}_2(s)\} = \cos bt = f_2(t)$ (say)

$$\therefore L^{-1}\{\bar{f}_1(s)\bar{f}_2(s)\} = \int_0^t f_1(u) f_2(t-u)\,du.$$

$$= \int_0^t \cos au \cos b(t-u)\,du.$$

$$= \frac{1}{2}\int_0^t 2\cos au \cos b(t-u)\,du.$$

$$\boxed{2\cos A\cos B = \cos(A+B) + \cos(A-B)}$$

$$= \frac{1}{2}\int_0^t \left[\cos(a-b)u + bt\right] + \left[\cos(a+b)u - bt\right] du$$

$$= \frac{1}{2}\left[\frac{\sin(a-b)u + bt}{a-b} + \frac{\sin(a+b)u - bt}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{a-b}(\sin at - \sin bt) + \frac{1}{a+b}(\sin at - \sin bt)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b} + \frac{1}{a+b}\right) + \sin bt\left(\frac{1}{a+b} - \frac{1}{a-b}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2a}{a^2-b^2}\sin at - \frac{2b}{a^2-b^2}\sin bt\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Method of Partial fractions

If $\bar{f}(s)$ is given in the form $\frac{g(s)}{h(s)}$

where g and h are Polynomials in s

Then $f(t)$ can be Obtained by resolving $\bar{f}(s)$ into Partial fractions and Apply Inverse Laplace Transform

1) Evaluate $L^{-1}\left(\frac{s^2}{(s+1)(s+2)(s+3)}\right)$

Sol:- $\frac{s^2}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$

(un repeated)

$\Rightarrow s^2 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) \rightarrow (1)$

Put $\boxed{s=-1}$

$1 = A(-1+2)(-1+3)$

$1 = A(2)$

$\boxed{A = \frac{1}{2}}$

Put $\boxed{s=-2}$

$4 = B(-2+1)(-2+3)$

$4 = B(-1)(1)$

$\boxed{B=-4}$

Put $\boxed{s=-3}$

$9 = C(-3+1)(-3+2)$

$9 = C(-2)(-1)$

$9 = 2C$

$\boxed{C = \frac{9}{2}}$

$\therefore \frac{s^2}{(s+1)(s+2)(s+3)} = \frac{1}{2(s+1)} - \frac{4}{s+2} + \frac{9}{2(s+3)}$

$\therefore L^{-1}\left(\frac{s^2}{(s+1)(s+2)(s+3)}\right) = \frac{1}{2}e^{-t} - 4e^{-2t} + \frac{9}{2}e^{-3t}$

(2) Find $L^{-1}\left(\frac{1}{(s+1)(s^2+1)}\right)$

Sol :- $\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$

$$1 = A(s^2+1) + (s+1)(Bs+C) \longrightarrow (1)$$

Put $\boxed{s=-1}$ in (1)

$1 = A(1+1)$

$1 = 2A$

$\boxed{A = \frac{1}{2}}$

Equating Co-eff of s^2 on b.s

$0 = A+B$

$B = -A$

$\boxed{B = -\frac{1}{2}}$

Equating Constants on b.s.

$1 = A+C$

$C = 1 - A$

$= 1 - \frac{1}{2}$

$\boxed{C = \frac{1}{2}}$

$$\therefore \frac{1}{(s+1)(s^2+1)} = \frac{\frac{1}{2}}{s+1} + \frac{\left(-\frac{1}{2}\right)s + \frac{1}{2}}{s^2+1}$$

Applying L^{-1} on b.s.

$$\therefore L^{-1}\left(\frac{1}{(s+1)(s^2+1)}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{2}L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t.$$

$$= \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

(3) Find $L^{-1}\left(\frac{1}{(s+1)^2(s^2+4)}\right)$

Sol:- $\frac{1}{(s+1)^2(s^2+4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+4}$.

repeted.

$$1 = A(s+1)(s^2+4) + B(s^2+4) + (Cs+D)(s+1)^2 \longrightarrow (1)$$

Put $\boxed{s=-1}$

$1 = B(5)$

$\boxed{B = \frac{1}{5}}$

Equating Coeff of

$s^2 = 0 = A + B + 2C + D$

$s = 0 = 4A + C + 2D$

Const $1 = 4A + 4B + D$

$s^3 \quad 0 = A + C$

On solving $A = \frac{2}{25}$ $B = \frac{1}{5}$ $C = -\frac{2}{25}$ $D = \frac{-3}{25}$.

$$\therefore L^{-1}\left(\frac{1}{(s+1)^2(s^2+4)}\right) = L^{-1}\left(\frac{2}{25(s+1)}\right) + \frac{1}{5}L^{-1}\left(\frac{1}{(s+1)^2}\right) +$$

$$L^{-1}\left(\frac{\frac{-2}{25}s}{s^2+4}\right) + \left(\frac{-3}{25}\right)L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{2}{25}e^{-t} + \frac{1}{5}e^{-t}t - \frac{2}{25}\cos 2t - \frac{3}{25}\frac{\sin 2t}{2}$$

Periodic function of Laplace

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

① $f(t) = \begin{cases} \sin t & ; \ 0 < t < \pi \\ 0 & ; \ \pi < t < 2\pi \end{cases}$ $\quad T = 2\pi$

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt \Rightarrow \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st}\sin t\,dt + \int_{\pi}^{2\pi} e^{-st}(0)\,dt\right]$$

$$\Rightarrow \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st}\sin t\,dt \qquad \left(\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right)$$

On Comp $a = -s$, $x = t$, $b = 1$ $\qquad \sin\pi = 0$, $\cos\pi = -1$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-st}}{s^2+1}(-s\sin t - 1\cos t)\right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\left(\frac{e^{-\pi s}}{s^2+1}\left((0)-(-1)\right)\right) - \left(\frac{1}{s^2+1}\left((0)-(1)\right)\right)\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{1+e^{-\pi s}}{s^2+1}\right]$$

$$= \frac{1}{\cancel{(1+e^{-\pi s})}(1-e^{-\pi s})} \cdot \frac{\cancel{1+e^{-\pi s}}}{s^2+1} = \frac{1}{(s^2+1)(1-e^{-\pi s})}$$

(2) $f(t) = \begin{cases} K & ; 0 < t < a \\ -K & ; a < t < 2a \end{cases}$ $\quad T = 2a.$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt \Rightarrow \frac{1}{1-e^{-2as}}\left[\int_0^a K(e^{-st}) f(t)\,dt + \int_a^{2a} (-K)(e^{-st}) f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[K\left(\frac{e^{-st}}{-s}\right)_0^a \cancel{-} K\left(\frac{e^{-st}}{\cancel{-}s}\right)_a^{2a}\right]$$

$$= \frac{K}{s(1-e^{-2as})}\left[\underline{\underline{-e^{-as}}} + \underline{\underline{1}} + e^{-2as} - \underline{\underline{e^{-as}}}\right]$$

$$= \frac{K}{s(1-e^{-2as})}\left[1 - 2e^{-as} + e^{-2as}\right]$$

$\cancel{a^2 - 2ab + b^2}$.

$$= \frac{K}{s(1-e^{-2as})}\left[1 - 2e^{-as} + (e^{-as})^2\right]$$

$a^2 - 2ab + b^2$

$$= \frac{K}{s(1-e^{-2as})}\left[(1 - e^{-as})^2\right]$$

$$= \frac{K}{s\,\cancel{(1-e^{-as})}(1+e^{-as})} (1-e^{-as})^{\cancel{2}} = \frac{K}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right)$$

$$= \frac{K}{s} \cdot \frac{e^{as/2}}{e^{as/2}} \cdot \frac{(1-e^{-as})}{(1+e^{-as})}$$

$$= \frac{K}{s}\left(\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}\right) = \underline{\underline{\frac{K}{s}\,\tanh{as/2}}}$$

Convolution theorem (or) Convolution Product theorem

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and

$L^{-1}\{\bar{g}(s)\} = g(t)$ then

$$\boxed{L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\cdot g(t-u)\,du}$$

t place u
t place t−u
w.r.to u Integration

1) Using Convolution Theorem

Find $L^{-1}\left(\frac{1}{s(s^2+1)}\right)$

Sol :- Let $\bar{f}(s) = \frac{1}{s} \Rightarrow L^{-1}\{\bar{f}(s)\} = L^{-1}\{\frac{1}{s}\} = 1 = f(t)$

$\bar{g}(s) = \frac{1}{s^2+1} \Rightarrow L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = g(t)$

we know that by Convolution theorem we have

$$\boxed{L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)g(t-u)\,du}$$

$f(t) = 1$
$f(u) = 1$
$g(t) = \sin t$
$g(t-u) = \sin(t-u)$
$\sin t - (-\cos t)_0^t$
$-\cos t + (1)$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t f(u)\,g(t-u)\,du.$$

$$= \int_0^t (1)\sin(t-u)\,du = \int_0^t \sin(t-u)\,du$$

$$= \left[\frac{\cancel{-}\cos(t-u)}{\cancel{-}1}\right]_0^t$$

$$= \{\cos(t-u)\}_0^t$$

$$= [\cos\{t-t\} - \cos(t-0)]$$

$$= \cos 0 - \cos t$$

$$= 1 - \cos t$$

$\int \sin ax\,dx = -\frac{\cos ax}{a}$

$\int \sin(t-u)\,du = \frac{-\cos(t-u)}{-1}$

(8) Using Convolution theorem $L^{-1}\{\bar{f}(s)\} = f(t)$

$L^{-1}\{\bar{g}(s)\} = g(t)$

Find $L^{-1}\left\{\frac{1}{s(s^2-a^2)}\right\}$

Sol :- let $\bar{f}(s) = \frac{1}{s^2-a^2} \Rightarrow L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a} = f(t)$

Let $\bar{g}(s) = \frac{1}{s} \Rightarrow L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s}\right\} = 1 = g(t)$

$f(t) = \frac{\sinh at}{a}$ $\quad g(t) = 1$

$f(u) = \frac{\sinh au}{a}$ $\quad g(t-u) = 1$

By Convolution theorem

$$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du.$$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2-a^2)}\right\} = \int_0^t \frac{\sinh au}{a}(1)\, du = \int_0^t \frac{\sinh au}{a}\, du.$$

$$= \frac{1}{a}\int_0^t \sinh au\, du. = \frac{1}{a}\left[\frac{\cosh au}{a}\right]_0^t$$

$$= \frac{1}{a^2}\left[\cosh at - 1\right]$$

$$\int \sinh ax\, dx = \frac{\cosh ax}{a}$$

3) Using Convolution theorem

Find $L^{-1}\left(\frac{1}{s(s^2-1)(s^2+1)}\right)$

$\bar{f}(s)$ – big
$\bar{g}(s)$ – small.

Sol:- Let $\bar{f}(s) = \frac{1}{(s^2-1)(s^2+1)} \Rightarrow L^{-1}\{\bar{f}(s)\}$

$\Rightarrow L^{-1}\left\{\frac{1}{(s^2-1)(s^2+1)}\right\}$ — Partial fraction

$\frac{As+B}{s^2-1} + \frac{Cs+D}{s^2+1}$

$A=0 \quad B=1/2$

$C=0 \quad D=-1/2$

$\Rightarrow \frac{1}{2}L^{-1}\left\{\frac{1}{s^2-1} - \frac{1}{s^2+1}\right\}$ (trial and error method)

$= \frac{1}{2}(\sinh t - \sin t) = f(t)$

Let $\bar{g}(s) = \frac{1}{s} \Rightarrow L^{-1}\{\bar{g}(s)\} = L^{-1}\{\frac{1}{s}\} = 1 = g(t)$

$f(t) = \frac{1}{2}(\sinh t - \sin t) \Rightarrow f(u) = \frac{1}{2}(\sinh u - \sin u)$

$g(t) = 1 \Rightarrow g(t-u) = 1$

By Convolution theorem

$$L^{-1}\left(\frac{1}{s(s^2-1)(s^2+1)}\right) = \int_0^t f(u)\,g(t-u)\,du = \int_0^t \frac{1}{2}(\sinh u - \sin u)(1)\,du.$$

$$= \int_0^t \frac{1}{2}(\sinh u - \sin u)\,du.$$

$$= \frac{1}{2}\left[\cosh u + \cos u\right]_0^t \Rightarrow \frac{1}{2}\left[(\cosh t + \cos t) - (1+1)\right]$$

$$= \frac{1}{2}\left[\cosh t + \cos t - 2\right]$$

$$\therefore L^{-1}\left(\frac{1}{s(s^2-1)(s^2+1)}\right) = \frac{1}{2}\left[\cosh t + \cos t - 2\right]$$

4) Using Convolution.

a) Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$

5) Using Convolution

Find $L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} = \frac{1}{8}\left[\frac{\sin 2t}{2} - t\cos 2t\right]$

→ Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Sol :- $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at} = f(t)$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t f(t)\,dt = \int_0^t e^{-at}\,dt$$

$$= \left(\frac{e^{-at}}{-a}\right)_0^t$$

$$= \left[\frac{e^{-at}}{-a} \cancel{+} \frac{1}{\cancel{+}a}\right]_0^t$$

$$= \frac{1}{a}\left(1 - e^{-at}\right) //.$$

→ Find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

$\int \sin at = -\frac{\cos at}{a}$

Sol :- $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a} = f(t)$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t f(t)\,dt = \int_0^t \frac{\sin at}{a}\,dt$$

$$= \frac{1}{a}\int_0^t \sin at\,dt = \frac{1}{a}\left[-\frac{\cos at}{a}\right]_0^t$$

$$= \frac{1}{a^2}\left[1 - \cos at\right] //.$$

Find $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}$

Sol: $L^{-1}\left(\frac{1}{s^2}\right) = t$

$L^{-1}\left(\frac{1}{(s+3)^2}\right) = te^{-3t}$

$$L^{-1}\left[\frac{s}{(s+3)^2}\right] = \frac{d}{dt}\left(e^{-3t}\cdot t\right)$$

$$= t\cdot -3e^{-3t} + (1)\,e^{-3t}$$

$$= e^{-3t}(1-3t)$$

① Using Convolution theorem find

$$L^{-1}\left(\frac{1}{s(s^2+1)}\right)$$

Sol :- Let $\bar{f}(s) = \frac{1}{s^2+1} \Rightarrow L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$

$\bar{g}(s) = \frac{1}{s} \Rightarrow L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s}\right\} = 1 = g(t)$

we know that by Convolution theorem

$$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du.$$

$f(t) = \sin t$

$f(u) = \sin u.$

$g(t) = 1$

$g(t-u) = 1$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\,(1)\,du.$$

$\int \sin t = -\cos t$

$$= (-\cos u)_0^t = -(\cos u)_0^t$$

$$= \cancel{(-\cos t)-(1)} = -\left[\cos t - \cos(0)\right]$$

$$= -(\cos t - 1)$$

$$= \left(-\cos t - (-\cos 0)\right)$$

$$= 1 - \cos t$$

$$= -\left[\cos t - \cos(0)\right]$$

$$= 1 - \cos t$$

(5) Using Convolution find.

$A = au$
$B = bt - bu$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$$

$(A+B) = au + bt - bu$
$= (a-b)u + bt$

$(A-B) = au - bt + bu$
$= (a+b)u - bt$

$\sin[(a-b)u + bt]$

$$= L^{-1}\left\{\frac{s}{(s^2+a^2)} \cdot \frac{s}{(s^2+b^2)}\right\}$$

Let $\bar{f}(s) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t)$

Let $\bar{g}(s) = \frac{s}{s^2+b^2} \Rightarrow L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt = g(t)$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos au \cdot \cos b(t-u)\,du.$$

$$= \frac{1}{2}\int_0^t 2\cos \underbrace{au}_{A} \cos \underbrace{b(t-u)}_{B}\,du. \quad \text{(2 mult 2 divide)}$$

$A = au$
$B = b(t-u) = bt - bu$

$$\boxed{2\cos A \cos B = \cos(A+B) + \cos(A-B)}$$

$$= \frac{1}{2}\int_0^t \left[\cos\left((a-b)u + bt\right) + \cos\left((a+b)u - bt\right)\right] du \qquad \int \cos ax\,dx = \frac{\sin ax}{a}$$

w.r.to $u \to t$ is Const ab Const

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u + bt]}{a-b} + \frac{\sin[(a+b)u - bt]}{a+b}\right]_0^t$$

$\sin(-\theta) = -\sin\theta$.

limits $= \frac{1}{2}\left[\left(\frac{\sin(a-b)t + bt}{a-b} - \frac{\sin bt}{a-b}\right) + \frac{\sin(a+b)t - bt}{a+b} - \frac{-\sin bt}{a+b}\right]$

$$= \frac{1}{2}\left[\frac{1}{a-b}\left(\sin at - \sin bt\right) + \frac{1}{a+b}\left(\sin at - \sin bt\right)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b} + \frac{1}{a+b}\right) + \sin bt\left(\frac{1}{a+b} - \frac{1}{a-b}\right)\right]$$

$\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a}$

$\int \sin(ax+b)\,dx = \frac{-\cos(ax+b)}{a}$

$$= \frac{1}{2}\left[\frac{2a}{a^2-b^2}\sin at - \frac{2b}{a^2-b^2}\sin bt\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

$$\therefore L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right) = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

$$\longrightarrow L^{-1}\left(\frac{s^2}{(s^2+4)(s^2+9)}\right) = \frac{2\sin 2t - 3\sin 3t}{-5}$$

(6) Using Convolution Find $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

Let $\bar{f}(s) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\{\bar{f}(s)\} = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at = f(t)$

$\bar{g}(s) = \frac{1}{s^2+a^2} \Rightarrow L^{-1}\{\bar{g}(s)\} = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a} = g(t)$

$$\therefore L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \int_0^t \cos au \, \frac{\sin a(t-u)}{a}\, du$$

$$= \frac{1}{2a}\left[\int_0^t [\sin(at-au+au) + \sin(at-au-au)]\, du\right]$$

$$= \frac{1}{2a}\left[\int_0^t [\sin at + \sin a(t-2u)]\, du\right]$$

$$= \frac{1}{2a}\left[u\sin at - \frac{\cos a(t-2u)}{a(-2)}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \cancel{\frac{\cos at}{2a}} - \cancel{\frac{\cos at}{2a}}\right] = \frac{t}{2a}\sin at$$

$$\therefore L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{t}{2a}\sin at, \quad L^{-1}\left(\frac{s}{(s^2+1^2)^2}\right) = \frac{t}{2}\sin t$$

Applications to Ordinary Differential Equations

Working Rule :-

(1) Write down the given equation and apply the Laplace Transpose

(2) Transpose all the terms without $L(y(t))$ on the right hand side

(3) Divide both sides by the Co-efficient of $L(y(t))$

(4) Take the Inverse Laplace Transform on both sides to get the solution for the given D.E with Constant Co-efficients.

$L(x) = L(x)$
$L(x') = sL(x) - x(0)$
$L(x'') = s^2 L(x) - sx(0) - x'(0)$

Formulas :-

(1) $L\{y(t)\} = \bar{y}(s)$

(2) $L\{y'(t)\} = s\bar{y}(s) - y(0)$

(3) $L\{y''(t)\} = s^2\bar{y}(s) - sy(0) - y'(0)$

(4) $L\{y'''(t)\} = s^3\bar{y}(s) - s^2 y(0) - sy'(0) - y''(0)$

(5) $L\{y^{(n)}(t)\} = s^n\bar{y}(s) - s^{n-1}y'(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$

Problems

(1) $L\{x(t)\} = \bar{x}(s)$

2) $L(x') = sL\{x\} - x(0)$

3) $L(x'') = s^2 L\{x\} - sx(0) - x'(0)$

Bound Condit (at 1, 2, 3 - -)

(vii) Solve the DE Using L.T

Initial value Conditions (at 0)

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{-t}, \quad x(0)=0, \quad x'(0)=1$$

Sol :- given D.E $x'' + 3x' + 2x = e^{-t}$

Taking L.T on b.s

$$L(x'') + 3L(x') + 2L(x) = L(e^{-t})$$

$$= \left(s^2L(x) - s\underset{0}{x(0)} - \underset{1}{x'(0)}\right) + 3\left(sL(x) - \underset{0}{x(0)}\right) + 2L(x) = \frac{1}{s+1}$$

Since $x(0)=0$, $x'(0)=1$

$$= \left(s^2L(x) - 1\right) + 3\left(sL(x)\right) + 2L(x) = \frac{1}{s+1}$$

$$= \left(s^2 + 3s + 2\right)L(x) = \frac{1}{s+1} + \frac{1}{1}$$

$$= \frac{1 + s + 1}{s+1} = \frac{s+2}{s+1}$$

$$\div \quad = \frac{s+2}{(s^2+3s+2)(s+1)} = \frac{\cancel{s+2}}{(s+1)^2(\cancel{s+2})}$$

$$\frac{s+2}{(s+2)(s+1)(s+1)}$$

$$= \frac{\cancel{s+2}}{(s+1)^2\cancel{(s+2)}}$$

$$= \frac{1}{(s+1)^2}$$

$$L(x) = \frac{1}{(s+1)^2}$$

$$x = L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$x = te^{-t}$$

$s^2 + 3s + 2$

$s^2 + 2s + s + 2$

$s(s+2) + 1(s+2)$

$(s+2)(s+1)$

$2s^2$

$2s \quad s$

(2) Solve the ~~DE~~ DE Using L.T (not in syllabus)

$$y''' + 2y'' + y' - 2y = 0$$

given that $y(0) = y'(0) = 0$, $y''(0) = 6$

Sol:- given DE

$$y''' + 2y'' + y' - 2y = 0 \longrightarrow (1)$$

Taking L.T on b.s

$$L(y''') + 2L(y'') + L(y') - 2L(y) = 0.$$

$$= \left[\left(s^3\bar{y}(s) - s^2\underbrace{y(0)}_{0} - s\underbrace{y'(0)}_{0} - \underbrace{y''(0)}_{6}\right] + 2\left[s^2\bar{y}(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right]$$

$$- \left[s\bar{y}(s) - \underbrace{y(0)}_{0}\right] - 2\bar{y}(s) = 0.$$

But given $y(0) = y'(0) = 0$, $y''(0) = 6$.

$$= (s^3 + 2s^2 - s - 2)\,\bar{y}(s) - 6 = 0.$$

$$\bar{y}(s) = \frac{6}{s^3 + 2s^2 - s - 2}$$

$$\bar{y}(s) = \frac{6}{s^2(s+2) - (s+2)}$$

$$L\{y(t)\} = \frac{6}{(s+2)(s^2-1)}$$

$$\therefore y(t) = L^{-1}\left(\frac{6}{(s+2)(s-1)(s+1)}\right)$$

$$= 6L^{-1}\left(\frac{1/3}{s+2} + \frac{1/6}{s-1} - \frac{1/2}{s+1}\right)$$

$$= L^{-1}\left(\frac{2}{s+2} + \frac{1}{s-1} - \frac{3}{s+1}\right) = 2e^{-2t} + e^{t} - 3e^{-t}$$

$\therefore y(t) = 2e^{-2t} + e^{t} - 3e^{-t}$

Is the required solution of DE of Order 3

$\frac{6}{(-3)(-1)}$ $\frac{6}{3}$

$(-1+2)(-1-1) = \frac{6}{-2} = 3$ $\frac{6}{6} = 1$

$(1)(-2) = \frac{6}{-2}$ 6/

(with 6.)

$\frac{2}{s+2}$ $s=-2$ $+ \frac{1}{s-1}$ $s=1$ $+ \frac{-3}{s+1}$ $s=-1$

(3) Using L.T Solve the DE

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=e^{-t}\sin t$$

given $y(0)=0$, $y'(0)=1$

$L\{e^{-t}\sin t\}$

$L\{\sin t\}=\left\{\frac{1}{s^2+1}\right\}_{s \text{ chng } s+1}$

$=\frac{1}{(s+1)^2+1}$.

Sol :- given eqn Can be written as

$$y''+2y'+5y=e^{-t}\sin t$$

Take the L.T on b.s of given DE and Using given Conditions we have.

$$L(y'')+2L(y')+5L(y)=L(e^{-t}\sin t)$$

$$[s^2L(y)-sy(0)-y'(0)]+2[sL(y)-y(0)]+5L(y)$$

$$=\frac{1}{(s+1)^2+1}\quad (\because \text{ 1st shifting})$$

Since $y(0)=0$, $y'(0)=1$

$$s^2L(y)-1+2sL(y)+5L(y)=\frac{1}{s^2+2s+2}\quad \text{(or)}$$

$$(s^2+2s+5)L(y)=\frac{1}{s^2+2s+2}+1$$

$$\therefore\ L(y)=\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$y=L^{-1}\left(\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right)$$

$$=L^{-1}\left(\frac{1/3}{s^2+2s+2}+\frac{2/3}{s^2+2s+5}\right)$$

(on resolving into Partial fractions)

$$= \frac{1}{3} L^{-1}\left(\frac{1}{(s+1)^2+1}\right) + \frac{2}{3} L^{-1}\left(\frac{1}{(s+1)^2+4}\right)$$

$$= \frac{1}{3} e^{-t} L^{-1}\left(\frac{1}{s^2+1}\right) + \frac{2}{3} e^{-t} L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$\therefore\ y(t) = \frac{e^{-t}}{3}\left[\sin t + \sin 2t\right]$$

is the required solution of DE.

(4) Solve the DE

$$(D^2+5D-6)y = x^2 e^{-x}$$

$y(0)=a$, $y'(0)=b$ Using L.T

<u>Sol</u>: given eqn can be written as

$$y'' + 5y' - 6y = x^2 e^{-x}$$

Taking L.T b.s we get

$$L(y'') + 5L(y') - 6L(y) = L(x^2 e^{-x})$$

$$\left[s^2\bar{y}(s) - sy(0) - y'(0)\right] + 5\left[s\bar{y}(s) - y(0)\right] - 6\left[\bar{y}(s)\right] = \left[\frac{2}{s^3}\right]_{s \to s+1}$$

$$\Rightarrow \left[s^2 L(y) - s(a) - b\right] + 5\left[s\, L(y) - a\right] - 6L(y) = \frac{2}{(s+1)^3}$$

$$\Rightarrow (s^2+5s-6)L(y) - as - 5a - b = \frac{2}{(s+1)^3}$$

$$\Rightarrow (s^2+5s-6)L(y) = \frac{2}{(s+1)^3} + as + 5a + b$$

$$\Rightarrow \quad L(y) = \frac{2}{(s+1)^3(s+6)(s-1)} + \frac{as}{(s+6)(s-1)} + \frac{5a+b}{(s+6)(s-1)}$$

$$y = L^{-1}\left(\frac{2}{(s+6)(s-1)(s+1)^3}\right) + a\,L^{-1}\left(\frac{s}{(s+6)(s-1)}\right)$$

$$+ (5a+b)\,L^{-1}\left(\frac{1}{(s+6)(s-1)}\right)$$

$$= L^{-1}\left(\frac{2}{(s+6)(s-1)(s+1)^3}\right) + \frac{a}{7}L^{-1}\left(\frac{s}{s-1} - \frac{s}{s+6}\right) +$$

$$(5a+b)\frac{1}{7}\left(e^{t} - e^{-6t}\right)$$

(5) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 2e^{-x}$

given $y(0)=0$ $\frac{dy}{dx} = -1$ at $x=0$. $(y'(0) = -1)$

Sol: given eqn can be written as

$$y'' + 4y' + 13y = 2e^{-x}$$

taking LT on b.s

$$L(y'') + 4L(y') + 13L(y) = 2L(e^{-x})$$

$$= \left[s^2\bar{y}(s) - sy(0) - y'(0)\right] + 4\left[s\bar{y}(s) - y(0)\right] + 13\bar{y}(s) = 2\left(\frac{1}{s+1}\right)$$

given that $y(0)=0$, $y'(0) = -1$

$$= \left[s^2\bar{y}(s) - 1\right) + 4\left(s\bar{y}(s)\right) + 13\bar{y}(s) = \frac{2}{s+1}$$

$$= \left(s^2+4s+13\right)\bar{y}(s) = \frac{2}{s+1} - 1 = \frac{1-s}{s+1}$$

$$\bar{y}(s) = \frac{1-s}{(s+1)(s^2+4s+13)}$$

$$L\{y(x)\} = \frac{1-s}{(s+1)(s^2+4s+13)}$$

$$y(x) = L^{-1}\left(\frac{1-s}{(s+1)(s^2+4s+13)}\right)$$

$$y(x) = \frac{1}{5}e^{-x} - \frac{1}{5}e^{-2x}\cos 3x - \frac{2}{5}e^{-2x}\sin 3x$$

(1) Using Convolution theorem

Find $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$L^{-1}\left(\frac{s}{(s^2+a^2)(s^2+b^2)}\right)$

$= L^{-1}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)$

$= \frac{1}{b^2-a^2} L^{-1}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)$

$(a)\cos at - \cos bt$

$= \frac{1}{b^2-a^2}$

Sol :- $L^{-1}\left\{\frac{s}{(s^2+a^2)} \cdot \frac{s}{(s^2+b^2)}\right\}$

Let $\bar{f}(s) = \frac{s}{(s^2+a^2)} \Rightarrow L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t)$

Let $\bar{g}(s) = \frac{s}{(s^2+b^2)} \Rightarrow L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt = g(t)$

By Convolution theorem

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos au \cdot \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t 2\underbrace{\cos au}_{A} \cdot \underbrace{\cos b(t-u)}_{B}\,du$$

$A = au$

$B = bt - bu$

$A+B = au + bt - bu$

$= (a-b)u + bt$

$= au + bt - bu$

$A-B = au - bt + bu$

$= (a+b)u - bt$

$= au - bt + bu$

$$\boxed{2\cos A\cos B = \cos(A+B) + \cos(A-B)}$$

$$= \frac{1}{2}\int_0^t \left[\cos(au + bt - bu) + \cos(au - bt + bu)\,du\right]$$

$$= \frac{1}{2}\int_0^t \cos((a-b)u + bt) + \cos((a+b)u - bt)\, du$$

$$= \frac{1}{2}\left[\frac{\sin((a-b)u+bt)}{a-b} + \frac{\sin((a+b)u-bt)}{a+b}\right]_0^t \qquad u = t$$

$$= \frac{1}{2}\left[\left(\frac{\sin((a-b)t+bt)}{a-b} + \frac{\sin((a+b)t-bt)}{a+b}\right) - \left(\frac{\sin bt}{a-b} + \frac{\sin(-bt)}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin(at-\cancel{bt}+\cancel{bt})}{a-b} + \frac{\sin(at+\cancel{bt}-\cancel{bt})}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin(at)}{a-b} + \frac{\sin(at)}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b} + \frac{1}{a+b}\right) - \sin bt\left(\frac{1}{a-b} - \frac{1}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{a+\cancel{b}+a-\cancel{b}}{(a-b)(a+b)}\right) - \sin bt\left(\frac{\cancel{a}+b-\cancel{a}+b}{(a-b)(a+b)}\right)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{2a}{a^2-b^2}\right) - \sin bt\left(\frac{2b}{a^2-b^2}\right)\right]$$

$$= \frac{a\sin at}{a^2-b^2} - \frac{b\sin bt}{a^2-b^2}$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

$$\therefore L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos au \cdot \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t 2\cos \underbrace{au}_{A} \cos \underbrace{b(t-u)}_{B}\,du$$

$A = au$

$B = bt - bu$.

$A+B = au + bt - bu$

$= (a-b)u + bt$

$A-B = au - bt + bu$

$= (a+b)u - bt$

$$\boxed{2\cos A \cos B = \cos(A+B) + \cos(A-B)}$$

$$= \frac{1}{2}\int_0^t \left[\cos(\underbrace{(a-b)u}_{a} + \underbrace{bt}_{b}) + \cos(\underbrace{(a+b)u}_{a} - \underbrace{bt}_{b})\right] du$$

$$\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a}$$

$$= \frac{1}{2}\left[\frac{\sin((a-b)u+bt)}{(a-b)} + \frac{\sin((a+b)u-bt)}{a+b}\right]_0^t \quad (u \to t)$$

$-\sin bt$

$\sin(-bt)$

$$= \frac{1}{2}\left[\left(\frac{\sin((a-b)t+bt}{a-b} + \frac{\sin(a+b)t-bt}{a+b}\right) - \left(\frac{\sin((0)+bt)}{a-b} + \frac{\sin((0)-bt}{a+b}\right)\right]$$

$$= \frac{1}{2}\left(\left(\frac{\sin(at-\cancel{bt}+\cancel{bt})}{a-b} + \frac{\sin(at+\cancel{bt}-\cancel{bt})}{a+b}\right) - \left(\frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right)\right)$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(a-b)}(\sin at - \sin bt) + \frac{1}{a+b}(\sin at + \sin bt)\right]$$

$$L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right) = ?$$

$$\cancel{f(s)} = \cancel{s} = \int_0^t \cos au \cos b(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos(au+bt-bu) + \cos(au-bt+bu)]\,du$$

$$= \frac{1}{2}\int_0^t \cos((a-b)u+bt) + \cos((a+b)u-bt)\,du$$

$$= \frac{1}{2}\left[\frac{\sin((a-b)u+bt)}{a-b} + \frac{\sin((a+b)u-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin((a-b)t+bt)}{a-b} + \frac{\sin((a+b)t-bt)}{a+b}\right) - \left(\frac{\sin bt}{a-b} + \frac{\sin(-bt)}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$\cancel{= \frac{1}{2}\left[\frac{1}{a-b}(\sin at - \sin bt) + \frac{1}{a+b}(\sin at + \sin bt)\right]}$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b} + \frac{1}{a+b}\right) - \sin bt\left(\frac{1}{a-b} - \frac{1}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\sin at\cdot\left(\frac{a+\cancel{b}+a-\cancel{b}}{(a-b)(a+b)}\right) - \sin bt\left(\frac{\cancel{a}+b-\cancel{a}+b}{(a-b)(a+b)}\right)\right]$$

$$= \frac{1}{2}\left[\sin at\cdot\left(\frac{2a}{a^2-b^2}\right) - \sin bt\cdot\left(\frac{2b}{a^2-b^2}\right)\right]$$

$$= \frac{a\sin at}{a^2-b^2} - \frac{b\sin bt}{a^2-b^2}$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

If $L\{f(t)\} = \bar{f}(s)$ then $f(t)$ is called the Inverse Laplace transform of $\bar{f}(s)$ and is denoted by $L^{-1}\{\bar{f}(s)\} = f(t)$