KMY

Mathematics - II

Unit - V

Z - Transform

Syllabus :-

→ Z - Transform - Inverse Z - Transform - Properties - Damping rule - Shifting rule - Initial and Final value Theorems.

Convolution Theorem - Solution of difference equations by Z - Transforms.

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Unit-V

Z-Transform

→ The technique of Z-Transform is useful in solving the difference equations, digital signal Processing and digital filter.

→ The Z-Transform have Properties similar to that of Laplace Transform

The main difference is Laplace transform defined for functions of Continuous Variables whereas Z-Transform defined for functions of descrete variables.

→ The development of Communication branch is based on discrete analysis.

→ The Z-Transform plays an important role in the field of Communication engineering and Control engineering at the stage of Analysis.

→ The Z-Transform play the role in Descrete Analysis as Laplace Transform play the role in Continuous systems

Z-Transform :-

Consider a function $f(n)$ defined for $n = 0, 1, 2, 3 \ldots$

Then the Z-Transform of $f(n)$ is defined as

$$\boxed{Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n) Z^{-n}}$$

→ If the Right hand side series is Convergent then we write $\boxed{Z\left[f(n)\right] = F(Z)}$

→ The Inverse Z-Transform of $F(Z)$ is written as $\boxed{Z^{-1}[F(Z)] = f(n)}$

whenever $Z[f(n)] = F(Z)$

Expansions of some functions :-

1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$

2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$

3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

5) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

6) $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$

7) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

8) $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

9) $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$, $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

10) $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

Z-Transforms of some standard functions

(1) $Z(1) = \frac{Z}{Z-1}$

(2) $Z(a^n) = \frac{Z}{Z-a}$

(3) $Z(n) = \frac{Z}{(Z-1)^2}$

(4) $Z(n^2) = \frac{Z^2+Z}{(Z-1)^3}$

(5) $Z(na^n) = \frac{aZ}{(Z-a)^2}$

(6) $Z\left(\frac{1}{n!}\right) = e^{1/Z}$

(2)

(7) $Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)$

(8) $Z\left(\frac{1}{n+1}\right) = z\log\left(\frac{z}{z-1}\right)$

9) $Z(n^2 a^n) = \frac{a(z^2+az)}{(z-a)^3}$

(10) $z^{-n} = \frac{1}{z^n}$

11) $Z(\cos nt) = \frac{z(z-\cos t)}{z^2-2z\cos t+1}$

12) $Z(\sin nt) = \frac{z\sin t}{z^2-2z\cos t+1}$

$x^{-1} = \frac{1}{x}$

(1) $\boxed{Z(1) = \frac{z}{z-1}}$

Proof :- Let $f(n) = 1$ for $n = 0, 1, 2, 3 \ldots$

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z(1) = Z[f(n)] = \sum_{n=0}^{\infty} (1) z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots$$

$z^{-0} = 1$

$z^{-1} = \frac{1}{z}$

$z^{-2} = \frac{1}{z^2}$

$z^{-3} = \frac{1}{z^3} \ldots$

$z^{-n} = \frac{1}{z^n}$

$$= \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

$\therefore \boxed{Z(1) = \frac{z}{z-1}}$

(or)

$[1-x]^{-1} = \frac{1}{1-\frac{1}{x}}$

$$Z(1) = Z[f(n)] = \sum_{n=0}^{\infty} (1) z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \ldots$$

$$= 1 + z^{-1} + (z^{-1})^2 + (z^{-1})^3 + \ldots$$

$\boxed{1 + x + x^2 + x^3 + \ldots = (1-x)^{-1}}$

$$= [1 - z^{-1}]^{-1} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

$\therefore \boxed{Z(1) = \frac{z}{z-1}}$

2) $\boxed{Z(a^n) = \dfrac{Z}{Z-a}}$

$$\frac{1}{1-\frac{a}{z}} = \frac{1}{\frac{z-a}{z}} = \frac{z}{z-a}$$

Proof :- Let $f(n) = a^n$ for $n = 0, 1, 2, 3 \ldots$

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) Z^{-n}$$

$$Z[a^n] = \sum_{n=0}^{\infty} a^n \cdot Z^{-n}$$

$$\boxed{\left(\frac{a}{Z}\right)^0 = 1}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{Z}\right)^n$$

$$= 1 + \frac{a}{Z} + \left(\frac{a}{Z}\right)^2 + \left(\frac{a}{Z}\right)^3 + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

$$= \left(1 - \frac{a}{Z}\right)^{-1} = \frac{1}{1-\frac{a}{Z}} = \frac{Z}{Z-a}$$

$$\therefore \boxed{Z(a^n) = \frac{Z}{Z-a}}$$

Special Case -1

If we take $\boxed{a = -1}$ in the above we get.

$$Z[a^n] = \frac{Z}{Z-a}.$$

$$\Rightarrow Z[(-1)^n] = \frac{Z}{Z-(-1)} = \frac{Z}{Z+1}$$

$$\therefore \boxed{Z[(-1)^n] = \frac{Z}{Z+1}}$$

(3)

3) $\boxed{Z(n) = \frac{z}{(z-1)^2}}$

$z^{-0} = 1$

Sol: Let $f(n) = n$ for $n = 0, 1, 2, 3, \dots$

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z[n] = \sum_{n=0}^{\infty} n \cdot z^{-n}$$

$$= z^{-1} + 2 \cdot z^{-2} + 3 \cdot z^{-3} + 4 \cdot z^{-4} + \dots$$

$$= \frac{1}{z} + 2 \cdot \frac{1}{z^2} + 3 \cdot \frac{1}{z^3} + 4 \cdot \frac{1}{z^4} + \dots$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \dots\right]$$

$$= \frac{1}{z}\left[\left(1 - \frac{1}{z}\right)^{-2}\right]$$

$z^{-1} = \frac{1}{z}$

$z^{-2} = \frac{1}{z^2}$

$z^{-3} = \frac{1}{z^3}$

$\vdots$

$z^{-n} = \frac{1}{z^n}$.

$$\boxed{(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots}$$

$$= \frac{1}{z} \cdot \frac{1}{\left(1 - \frac{1}{z}\right)^2} = \frac{1}{\cancel{z}} \cdot \frac{z^{\cancel{2}}}{(z-1)^2} = \frac{z}{(z-1)^2}$$

$$\therefore \boxed{Z(n) = \frac{z}{(z-1)^2}}$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

(4) $\boxed{Z(n^p) = -z\frac{d}{dz}[Z(n^{p-1})]}$

Sol :- we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

- let $f(n) = n^p$ (where p is a +ve integer)

$$Z[f(n)] = Z[n^p] = \sum_{n=0}^{\infty} n^p . z^{-n}.$$

$n^p = n^{p-1}. n^1$

$z^{-n} = z^1 . z^{-(n+1)}$

$$= z\sum_{n=0}^{\infty} n^{p-1}.n . z^{-(n+1)} \longrightarrow (1).$$

we have

$$Z[f(n)] = Z[n^{p-1}] = \sum_{n=0}^{\infty} n^{p-1} . z^{-n}$$

$\frac{d}{dx}x^n = nx^{n-1}$

$\frac{d}{dz}z^{-n} = -n . z^{-(n+1)}$

D.w.r.to 'z' we get.

$$\frac{d}{dz}[Z(n^{p-1})] = \sum_{n=0}^{\infty} n^{p-1}. (-n) z^{-(n+1)}$$

$$= -\sum_{n=0}^{\infty} n . n^{p-1}. z^{-(n+1)} \longrightarrow (2)$$

$$\Rightarrow \sum_{n=0}^{\infty} n.n^{p-1}. z^{-(n+1)} = -\frac{d}{dz} Z(n^{p-1}) \longrightarrow (2).$$

Substitute (2) in (1) we get.

$$\boxed{Z[n^p] = -z\frac{d}{dz} Z(n^{p-1})}$$

This is also Called as Recurrence formula.

(4)

we have:

$$\boxed{Z[n^p] = -z\frac{d}{dz}\left[Z(n^{p-1})\right]} \longrightarrow (I)$$

$n^0 = 1$

$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

(5)

on putting $\boxed{p=1}$ in (I) we get.

$$Z[n] = -z\frac{d}{dz}[Z(1)] = -z\frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$= -z\left[\frac{(\cancel{z}-1)-\cancel{z}}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

$$\therefore \boxed{Z[n] = \frac{z}{(z-1)^2}}$$

on putting $\boxed{p=2}$ in (I) we get.

$$Z[n^2] = -z\frac{d}{dz}[Z(n)] = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$\Rightarrow \boxed{Z[n^2] = \frac{z^2+z}{(z-1)^3}}$$

Ily on Putting $\boxed{p=3}$ in (I) we get.

$$\boxed{Z[n^3] = \frac{z(z^2+4z+1)}{(z-1)^4}}$$

(6) $\boxed{Z[na^n] = \dfrac{az}{(z-a)^2}}$

Sol: we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z[na^n] = \sum_{n=0}^{\infty} na^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} n\left(\frac{a}{z}\right)^n$$

$$= \frac{a}{z} + 2\left(\frac{a}{z}\right)^2 + 3\left(\frac{a}{z}\right)^3 + 4\left(\frac{a}{z}\right)^4 + \cdots$$

$$= \frac{a}{z}\left[1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + 4\left(\frac{a}{z}\right)^3 + \cdots\right]$$

$$= \frac{a}{z}\left[1 - \frac{a}{z}\right]^{-2} \qquad (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \frac{a}{z}\left[\frac{1}{\left(1-\frac{a}{z}\right)^2}\right] = \frac{a}{\cancel{z}}\left[\frac{z^{\cancel{2}}}{(z-a)^2}\right]$$

$$Z[na^n] = \frac{az}{(z-a)^2}$$

(7) $\boxed{Z\left[\dfrac{1}{n!}\right] = e^{1/z}}$

$z^{-0} = 1$, $z^{-1} = \frac{1}{z}$

$0! = 1$, $\frac{1}{0!} = 1$

$\frac{1}{1!} = 1$

$\frac{1}{2!} = \frac{1}{2}$

Sol: we know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= 1 + z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \cdots$$

$$= 1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \frac{(1/z)^4}{4!} + \cdots$$

$$Z\left[\frac{1}{n!}\right] = e^{1/z} \qquad \left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

(8) $\boxed{Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)}$ $\quad \left(\frac{1}{0} = \text{un-def}\right)$ (5)

Sol: we know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} \quad \left(\text{Here we take } \sum_{n=1}^{\infty}\right)$$

$$= z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{3} z^{-3} + \frac{1}{4} z^{-4} + \cdots$$

$$= \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3}\frac{1}{z^3} + \frac{1}{4}\frac{1}{z^4} + \cdots \rightarrow = \frac{\left(\frac{1}{z}\right)}{1} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \frac{\left(\frac{1}{z}\right)^4}{4} + \cdots$$

$$= -\log\left(1 - z^{-1}\right) \quad \cancel{\log(1-x)} =$$

$$= -\log\left(1 - \frac{1}{z}\right)$$

$$= -\log\left(\frac{z-1}{z}\right) = \log\left(\frac{z}{z-1}\right)$$

$x = z^{-1}$

$\log(1-x)$

$-\log(1-x)$

$$\therefore \boxed{Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)}$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

(9) $\boxed{Z\left(\frac{1}{n+1}\right) = z\log\left(\frac{z}{z-1}\right)}$

Sol: we know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} \quad \left(z^{-0} = 1\right)$$

$$= 1 + \frac{1}{2} z^{-1} + \frac{1}{3} z^{-2} + \frac{1}{4} z^{-3} + \cdots$$

$$= 1 + \frac{1}{2}\frac{1}{z} + \frac{1}{3}\frac{1}{z^2} + \frac{1}{4}\frac{1}{z^3} + \cdots \longrightarrow$$

$$= z\left[\frac{1}{z} + \frac{1}{2} z^{-2} + \frac{1}{3} z^{-3} + \frac{1}{4} z^{-4} + \cdots\right]$$

$$1 + \frac{1}{2}\frac{1}{z} + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \frac{1}{4}\left(\frac{1}{z}\right)^3 + \cdots$$

$$= -z\log\left[1 - z^{-1}\right] = -z\log\left[1 - \frac{1}{z}\right]$$

$$1 + \frac{1/z}{2} + \frac{\left(\frac{1}{z}\right)^2}{3} + \frac{\left(\frac{1}{z}\right)^3}{4} + \cdots$$

$$= -z\log\left(\frac{z-1}{z}\right) = z\log\left(\frac{z}{z-1}\right)$$

$$\therefore \boxed{Z\left(\frac{1}{n+1}\right) = z\log\left(\frac{z}{z-1}\right)}$$

(10) Unit - step function (or) Z-Transform of Discrete Unit step function

The Unit step Sequence (or) function is defined by

$$f(n) = \begin{cases} 0 & ; n < 0 \\ 1 & ; n \geq 0 \end{cases}$$

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum z^{-n} = \frac{z}{z-1}$$

(or)

$z^{-1} = \frac{1}{z}$

$$* \; Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} (1) z^{-n} \qquad z^0 = 1 \quad z^{-2} = \frac{1}{z^2}$$

(This is in G.P its sum is $\frac{a}{1-r}$

$1+r+r^2+r^3+\cdots = \frac{1}{1-r}$

$r \longrightarrow \frac{1}{z}$.)

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^n} + \cdots$$

$$= \frac{1}{1-\frac{1}{z}} = \frac{1}{\frac{z-1}{z}} = \frac{z}{z-1}$$

Properties of Z-Transforms :-

(1) Linearity Property :- If a, b are any Constants and f(n) and g(n) be any discrete functions then

$$Z[af(n) + bg(n)] = aZ[f(n)] + bZ[g(n)]$$

Proof :- we know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$.

$$Z[g(n)] = \sum_{n=0}^{\infty} g(n) z^{-n}$$

by definition

$$Z[af(n)+bg(n)] = \sum_{n=0}^{\infty} [af(n) + bg(n)] z^{-n}$$

$$= \sum_{n=0}^{\infty} a f(n) z^{-n} + \sum_{n=0}^{\infty} b g(n) z^{-n}$$

$$= a \sum_{n=0}^{\infty} f(n) z^{-n} + b \sum_{n=0}^{\infty} g(n) z^{-n}$$

$$= aZ[f(n)] + bZ[g(n)]$$

$$\therefore Z[af(n)+bg(n)] = aZ[f(n)] + bZ[g(n)]$$ ⑥

$\therefore$ Linearity Property is Proved

*** 2) Change of Scale Property (or) Damping Rule :-

(i) If $Z[f(n)] = F(z)$ then

$$Z[a^{-n} f(n)] = F(az)$$

Proof :- we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z)$$

Consider $Z[a^{-n} f(n)] = \sum_{n=0}^{\infty} a^{-n} f(n) z^{-n}$

$$= \sum_{n=0}^{\infty} f(n)(az)^{-n}$$

$$= F(az) \quad (\because Z[f(n)] = F(z))$$

$$\therefore Z[a^{-n} f(n)] = F(az)$$

This is called as Damping rule.

(ii) If $Z[f(n)] = F(z)$ then

$$Z[a^{n} f(n)] = F\left(\frac{z}{a}\right)$$

Sol :- we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z)$$

$a^n = \frac{1}{a^{-n}}$

$z^{-n} = \frac{1}{z^n}$

Consider $Z[a^{n} f(n)] = \sum_{n=0}^{\infty} a^{n} f(n) z^{-n}$

$$= \sum_{n=0}^{\infty} f(n)\left(\frac{z}{a}\right)^{-n}$$

$$= F\left(\frac{z}{a}\right)$$

$$\therefore Z[a^{n} f(n)] = F\left(\frac{z}{a}\right)$$ Hence Proved

① Find. $Z[{}^nc_k]$ $(0 \le k \le n)$

Sol :- Let $f(k) = {}^nc_k$.

$$Z[{}^nc_k] = \sum_{k=0}^{n} f(k) z^{-k}$$

$$= \sum_{k=0}^{n} {}^nc_k z^{-k}$$

$$= {}^nc_0 + {}^nc_1 z^{-1} + {}^nc_2 z^{-2} + {}^nc_3 z^{-3} + \cdots + {}^nc_n z^{-n}.$$

$$= (1+z^{-1})^n \quad (\text{using Binomial theorem})$$

$$\therefore Z[{}^nc_k] = (1+z^{-1})^n$$

$$\boxed{(1+x)^n = {}^nc_0 + {}^nc_1 x + {}^nc_2 x^2 + {}^nc_3 x^3 + \cdots \Rightarrow \text{Binomial Theorem}}$$

(2) $$Z\left(\frac{a^n}{n!}\right) = \sum_{n=0}^{\infty} \left(\frac{a^n}{n!}\right) z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(az^{-1})^n}{n!}$$

$$= 1 + \frac{az^{-1}}{1!} + \frac{(az^{-1})^2}{2!} + \frac{(az^{-1})^3}{3!} + \frac{(az^{-1})^4}{4!} + \cdots$$

$$= e^{az^{-1}}$$

$$\therefore Z\left(\frac{a^n}{n!}\right) = e^{\frac{a}{z}} \quad \left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

(3) $Z(a^n)$ we know that

$$Z(1) = \frac{z}{z-1} \qquad \text{Let } f(n) = 1$$

Using Change of scale

$$Z[a^n f(n)] = F\left(\frac{z}{a}\right)$$

$$Z(a^n) = Z(a^n \cdot 1)$$

$$= \frac{z/a}{\left(\frac{z}{a}-1\right)} = \frac{z}{z-a}$$

$$\left[\frac{z}{z-1}\right]_{z \to z/a} = \frac{\frac{z}{a}}{\frac{z}{a}-1} = \frac{\frac{z}{\cancel{a}}}{\left(\frac{z-a}{\cancel{a}}\right)} = \frac{z}{z-a}$$

$$\boxed{Z(a^n) = \frac{z}{z-a}}$$

(7)

(4) Evaluate $Z\left[(\cos\theta + i\sin\theta)^n\right]$

Hence evaluate (a) $Z(\cos n\theta)$

(b) $Z(\sin n\theta)$

Sol :- we know that $\boxed{\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{array}}$

$Z\left[(\cos\theta + i\sin\theta)^n\right]$

$\Rightarrow Z\left[(e^{i\theta})^n\right] = \dfrac{z}{z - e^{i\theta}} \quad \left(Z(a^n) = \dfrac{z}{z-a}\right)$

Using De-Moivre's Theorem

$Z\left[(\cos\theta + i\sin\theta)^n\right] = \dfrac{z}{z - e^{i\theta}} \cdot \dfrac{z - e^{-i\theta}}{z - e^{-i\theta}}$ (general multip) $z \cdot z = z^2$

$\Rightarrow Z\left[\cos n\theta + i\sin n\theta\right] = \Rightarrow \dfrac{z(z - e^{-i\theta})}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1}$

$\Rightarrow \dfrac{z(z - e^{-i\theta})}{z^2 - z(e^{-i\theta}) - e^{i\theta}z + 1} \Rightarrow \dfrac{z(z - e^{-i\theta})}{z^2 - 2z\cos\theta + 1}$

$(e^{i\theta})(e^{-i\theta})$

$= e^0 = 1$

$\Rightarrow \dfrac{z(z - \cos\theta + i\sin\theta)}{z^2 - 2z\cos\theta + 1}$

$Z[\cos n\theta] + iZ[\sin n\theta] \Rightarrow \dfrac{z(z - \cos\theta) + i(z\sin\theta)}{z^2 - 2z\cos\theta + 1}$

Comparing Real and Imaginary Parts we get

$$\boxed{\begin{array}{l} Z[\cos n\theta] = \dfrac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1} \\ Z[\sin n\theta] = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1} \end{array}}$$

$e^{i\theta} + e^{-i\theta} =$

$\cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta}$

$= 2\cos\theta$

(5) $Z[(-2)^n] = \sum_{n=0}^{\infty} (-2)^n z^{-n}$

$$= \sum_{n=0}^{\infty} \left(\frac{-2}{z}\right)^n$$

$$= 1 + \left(-\frac{2}{z}\right) + \left(\frac{-2}{z}\right)^2 + \left(\frac{-2}{z}\right)^3 + \cdots$$

This is a Geometric Series

whose sum $= \frac{a}{1-r}$

Hence $Z[(-2)^n] = \frac{1}{1-\left(-\frac{2}{z}\right)} = \frac{z}{z+2}$

$$\therefore \boxed{Z[(-2)^n] = \frac{z}{z+2}}$$

(or)

$Z[(-2)^n]$ we know that

$$\boxed{Z[a^n] = \frac{z}{z-a}} \quad (a = -2)$$

$$\Rightarrow Z[(-2)^n] = \frac{z}{z-(-2)} = \frac{z}{z+2}$$

$$\therefore \boxed{Z[(-2)^n] = \frac{z}{z+2}}$$

(6) Find Z-Transform of Unit Impulse function

$$\delta(n) = \begin{cases} 1 & ; n = 0 \\ 0 & ; n \neq 0 \end{cases}$$

Sol:- $Z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n) z^{-n}$

$$= 1 + 0 + 0 + \cdots = 1$$

(7) Find Z-Transform of discrete Unit step function (8)

$$u(n) = \begin{cases} 0 & ; \ n < 0 \\ 1 & ; \ n \geq 0 \end{cases}$$

Sol: $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z[u(n)] = \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \frac{1}{1-\frac{1}{z}} = \frac{1}{\frac{z-1}{z}}$$

$$= \frac{z}{z-1}$$

$$\therefore \boxed{Z[u(n)] = \frac{z}{z-1}}$$

(8) Find the Z-Transform of $\frac{a^n}{n!} e^{-a}$

Sol: $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$Z\left[\frac{a^n}{n!} e^{-a}\right]$ Hear e^{-a} is Constant

$$Z\left[\frac{a^n}{n!} e^{-a}\right] = e^{-a} Z\left[\frac{a^n}{n!}\right] \longrightarrow (1)$$

Let $f(n) = \frac{1}{n!} \Rightarrow Z[f(n)] = Z\left[\frac{1}{n!}\right] = e^{1/z} = F(z)$

by Damping rule we have

$$Z[a^n f(n)] = F\left(\frac{z}{a}\right)$$

$$Z\left[a^n \frac{1}{n!}\right] = e^{1/z/a} = e^{a/z}$$

$$\therefore Z\left[\frac{a^n}{n!} e^{-a}\right] = e^{-a} \cdot e^{a/z} = e^{a/z - a}$$

$$= e^{a\left(\frac{1}{z} - 1\right)}$$

(9) Find the Z-Transform of

$$Z[n^2 a^n] = \frac{az(z+a)}{(z-a)^3}$$

Sol :- we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

Here $f(n) = n^2$

$$Z[f(n)] = Z[n^2] = \frac{z(z+1)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3} = F(z)$$

by Damping rule we have

$$Z[a^n f(n)] = F(z/a)$$

$$Z[a^n n^2] = \frac{(z/a)^2 + (z/a)}{\left(\frac{z}{a}-1\right)^3}$$

$$= \frac{\frac{z}{a}\left(\frac{z}{a}+1\right)}{\left(\frac{z-a}{a}\right)^3} = \frac{\frac{z}{a}\left(\frac{z+a}{a}\right)}{\left(\frac{z-a}{a}\right)^3}$$

$$= \frac{\frac{z(z+a)}{a^2}}{\frac{(z-a)^3}{a^3}} = \frac{az(z+a)}{(z-a)^3}$$

(9)

10) Find $Z\left[\cos\frac{n\pi}{2}\right]$

Sol: we know that $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

Put $\theta = \frac{\pi}{2}$ we get

$$Z\left[\cos\frac{n\pi}{2}\right] = \frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\underbrace{\cos\frac{\pi}{2}}_{0}+1} = \frac{z^2}{z^2+1}$$

$\cos\frac{\pi}{2} = 0$.

$\sin\frac{\pi}{2} = 1$.

~~Cos 180~~

$\cos\frac{\cancel{180}\ 90^\circ}{\cancel{2}\ 1} = 0$.

$\cos 90^\circ = 0$.

$\sin 90^\circ = 1$.

Illy. we know that

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Put $\theta = \frac{\pi}{2}$

$$Z\left(\sin\frac{n\pi}{2}\right) = \frac{z\sin\frac{\pi}{2}}{z^2-2z\underbrace{\cos\frac{\pi}{2}}_{0}+1} = \frac{z}{z^2+1}$$

11) Find $Z\left[(n-1)^2\right]$

Sol: $Z\left[n^2-2n+1\right]$

$$\Rightarrow Z[n^2] - 2Z[n] + Z[1]$$

$$\Rightarrow \frac{z^2+z}{(z-1)^3} - \frac{2z}{(z-1)^2} + \frac{z}{z-1}$$

$$\Rightarrow \frac{(z^2+z) - 2z(z-1) + z(z-1)^2}{(z-1)^3} \Rightarrow \frac{z^2+z-2z^2+2z+z(z^2-2z+1)}{(z-1)^3}$$

$$\Rightarrow \frac{z^2+z-2z^2+2z+z^3-2z^2+z}{(z-1)^3} = \frac{z^3-3z^2+4z}{(z-1)^3}$$

$$Z\left[(n-1)^2\right] = \frac{z^3-3z^2+4z}{(z-1)^3}$$

12) Find $Z(an^2+bn+c)$

$$Z(n) = \frac{z}{(z-1)^2}$$

$$Z(n^2) = \frac{z^2+z}{(z-1)^3}$$

Sol: $a Z[n^2] - b Z[n] + c Z[1]$

$$= \frac{a(z^2+z)}{(z-1)^3} - b\frac{z}{(z-1)^2} + c\frac{z}{z-1}$$

$$\Rightarrow \frac{1}{(z-1)^3}\left[a(z^2+z) - bz(z-1) + cz(z-1)^2\right]$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

13) Find $Z\left[2n - 5\sin\frac{n\pi}{4} + 3a^4\right]$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Sol: $2Z(n) - 5Z\left(\sin\frac{n\pi}{4}\right) + 3a^4 Z(1)$

$$= 2\left(\frac{z}{(z-1)^2}\right) - \frac{5z\sin\left(\frac{\pi}{4}\right)}{z^2 - 2z\cos\frac{n\pi}{4} + 1} + 3a^4\left(\frac{z}{z-1}\right)$$

$$= \frac{2z}{(z-1)^2} - \frac{5z/\sqrt{2}}{z^2 - 2z\left(\frac{1}{\sqrt{2}}\right)+1} + \frac{3a^4 z}{z-1}$$

$$= \frac{2z}{(z-1)^2} - \frac{5z/\sqrt{2}}{z^2 - \sqrt{2}z + 1} + \frac{3a^4 z}{z-1}$$

$$\frac{2}{\sqrt{2}} = \frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = \sqrt{2}$$

① we written

14) Find the Z Transform of $\frac{a^n}{n!}e^{-a}$ (or) P.T $Z\left[\frac{a^n}{n!}e^{-a}\right] = e^{a\left(\frac{1}{z}-1\right)}$

Sol: we know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$.

$$Z\left[\frac{a^n}{n!}e^{-a}\right] = \sum_{n=0}^{\infty}\frac{a^n}{n!}e^{-a}\cdot z^{-n}$$

$$= e^{-a}\sum_{n=0}^{\infty}\frac{a^n}{n!}z^{-n}$$

(10)

$$= e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$= e^{-a} \sum_{n=0}^{\infty} \frac{(a/z)^n}{n!}$$

$$= e^{-a}\left[1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \cdots\right]$$

$$= e^{-a} \cdot e^{a/z}$$

$$\Rightarrow e^{a\left(\frac{1}{z} - 1\right)}$$

***

15) Find the Z-Transform of the sequence $\{x(n)\}$ where $x(n)$ is $n \cdot 2^n$.

<u>Sol</u> :- we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z(n \cdot 2^n) = \sum_{n=0}^{\infty} n \cdot 2^n z^{-n}$$

$$= \sum_{n=0}^{\infty} n \cdot \left(\frac{2}{z}\right)^n$$

$$= \frac{2}{z} + 2\left(\frac{2}{z}\right)^2 + 3\left(\frac{2}{z}\right)^3 + \cdots$$

$$= \frac{2}{z}\left[1 + 2\left(\frac{2}{z}\right) + 3\left(\frac{2}{z}\right)^2 + \cdots\right]$$

$$= \frac{2}{z}\left[1 - \frac{2}{z}\right]^{-2}$$

$$= \frac{2}{z}\left[\frac{1}{\left(1 - \frac{2}{z}\right)^2}\right] = \frac{2}{z}\left[\frac{1}{\left(\frac{z-2}{z}\right)^2}\right]$$

$$= \frac{2}{\cancel{z}} \cdot \frac{z^{\cancel{2}}}{(z-2)^2} = \frac{2z}{(z-2)^2}$$

(or)

Alternative method :-

we have $Z(2^n) = \frac{z}{z-2} = F(z)$

$$Z(n \cdot 2^n) = (-z)\frac{d}{dz}[F(z)]$$

$$= (-z)\frac{d}{dz}\left(\frac{z}{z-2}\right)$$

$$= (-z)\left[\frac{-2}{(z-2)^2}\right] = \frac{2z}{(z-2)^2}$$

16) Find the Z-Transform of 2^{2k+3}.

Sol :- let $f(k) = 2^{2k+3}$

$$= 2^{2k} \cdot 2^3 = 8 \cdot 2^{2k}$$

$$Z[f(k)] = Z[8 \cdot 2^{2k}] = 8\,Z[2^{2k}]$$

$$= 8 \sum_{k=0}^{\infty} 2^{2k} \cdot z^{-k}$$

$$= 8\left[1 + 2^2 z^{-1} + 2^4 \cdot z^{-2} + 2^6 \cdot z^{-3} + \cdots\right]$$

$$= 8\left[1 + 2^2 z^{-1} + (2^2 \cdot z^{-1})^2 + (2^2 z^{-1})^3 + (2^2 \cdot z^{-1})^4 + \cdots\right]$$

$$= 8\left[(1 - 2^2 z^{-1})^{-1}\right]$$

$$= 8 \frac{1}{(1 - 2^2 z^{-1})} = 8 \cdot \frac{1}{1 - \frac{4}{z}}$$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$z^{-1} = \frac{1}{z}$

$z^{-2} = \frac{1}{z^2}$

$z^{-3} = \frac{1}{z^3}$

$$= 8 \cdot \frac{1}{\frac{z-4}{z}} = \frac{8z}{z-4}$$

(11)

**(17) Find $Z\left[\frac{1}{n(n+1)}\right]$

$Z\left[\frac{\cancel{n}+1-\cancel{n}}{n(n+1)}\right]$

Sol :- by Linearity Property

$$Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n} - \frac{1}{n+1}\right]$$ (Partial fractions Trial & error method.)

$$= Z\left[\frac{1}{n}\right] - Z\left[\frac{1}{n+1}\right]$$

$$= \log\left(\frac{z}{z-1}\right) - z\log\left(\frac{z}{z-1}\right)$$

$$\Rightarrow (1-z)\log\left(\frac{z}{z-1}\right)$$

18) Using $Z(n^2) = \frac{z^2+z}{(z-1)^3}$ P.T $Z[(n+1)^2] = \frac{z^3+z^2}{(z-1)^3}$

Sol :- Let $Z[f(n)] = Z[n^2] = \frac{z^2+z}{(z-1)^3} = F(z)$

where $f(n) = n^2$

$$Z[f(n+1)] = Z\{(n+1)^2\}$$

$f(n) = n^2$
$f(n+1) = (n+1)^2$

by shifting theorem

$$\boxed{Z[f(n+1)] = Z[F(z) - f(0)]} \longrightarrow (1)$$

Substituting

$F(z) = Z[f(n)] = \frac{z^2+z}{(z-1)^3}$ and $f(n) = n^2 \Rightarrow f(0) = 0$ in (1) we get

$$Z[f(n+1)] = Z[(n+1)^2]$$

$$= Z\left[\frac{z^2+z}{(z-1)^3} - 0\right] = \frac{z^3+z^2}{(z-1)^3}$$

* **Initial value Theorem**

If $Z[f(n)] = F(Z)$ then

$$\underset{Z\to\infty}{Lt} F(Z) = f(0)$$

$\frac{1}{\infty} = 0$ $\frac{1}{\infty^2} = 0$

Proof :- we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) Z^{-n} = F(Z)$$

$$= f(0) + f(1)Z^{-1} + f(2)Z^{-2} + f(3)Z^{-3} + \cdots$$

$$= f(0) + \underbrace{f(1)\frac{1}{Z} + f(2)\frac{1}{Z^2} + f(3)\frac{1}{Z^3} + \cdots}_{0}$$

now taking limit as $Z \to \infty$ we get

$$\boxed{\underset{Z\to\infty}{Lt} F(Z) = f(0)}$$

ie :- $f(0) = \underset{Z\to\infty}{Lt} F(Z)$

$\longrightarrow$ $f(0) = \underset{Z\to\infty}{Lt} F(Z)$

$f(1) = \underset{Z\to\infty}{Lt} Z[F(Z) - f(0)]$

$f(2) = \underset{Z\to\infty}{Lt} Z^2[F(Z) - f(0) - f(1)Z^{-1}]$

$f(3) = \underset{Z\to\infty}{Lt} Z^3[F(Z) - f(0) - f(1)Z^{-1} - f(2)Z^{-2}]$

and so on

*** **Final Value Theorem** :-

If $Z[f(n)] = F(Z)$ then

$$\underset{n\to\infty}{Lt} f(n) = \underset{z\to 1}{Lt} (z-1) F(Z)$$

(12)

***Final value Theorem :- (v.v.Imp)

If $Z[f(n)] = F(Z)$ then

$$\lim_{n \to \infty} f(n) = \lim_{Z \to 1} (Z-1) F(Z)$$

$Z \to 1$
Z^{-n}
$1^{-n} = 1$
$1^{\infty} = 1$
$1^{1000} = 1$

Sol:- we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) Z^{-n}$$

we have $Z[f(n+1) - f(n)] = \sum_{n=0}^{\infty} [f(n+1) - f(n)] Z^{-n}$

$$= Z[f(n+1)] - Z[f(n)] = \sum_{n=0}^{\infty} [f(n+1) - f(n)] Z^{-n}$$

by shifting

$$Z[F(Z) - f(0)] - F(Z) = \sum_{n=0}^{\infty} [f(n+1) - f(n)] Z^{-n}.$$

$$= F(Z)(Z-1) - Z f(0) = \sum_{n=0}^{\infty} [f(n+1) - f(n)] Z^{-n}$$

Z^{-n}
$(1)^{-n} = 1$

now taking the Limits on b.s as $Z \to 1$ we get

$$\lim_{Z \to 1} [F(Z)(Z-1)] - f(0) = \sum_{n=0}^{\infty} [f(n+1) - f(n)]$$

$$= \lim_{n \to \infty} \sum_{n=0} f(n+1) - f(n)$$

$$= \lim_{n \to \infty} \left[(\cancel{f(1)} - f(0)) + (\cancel{f(2)} - \cancel{f(1)}) + (\cancel{f(3)} - \cancel{f(2)}) + \cdots f(n+1) - \cancel{f(n)} \right]$$

$$= \lim_{n \to \infty} f(n+1) - f(0)$$

$$\lim_{Z \to 1} (Z-1) F(Z) - \cancel{f(0)} = \lim_{n \to \infty} f(n+1) - \cancel{f(0)}$$

$$\cancel{\Rightarrow \lim_{Z \to 1} (Z-1) F(Z) = f(0)}$$

$$\lim_{Z \to 1} (Z-1) F(Z) = \lim_{n \to \infty} f(n+1) = \lim_{n \to \infty} f(n)$$

In general

$$\therefore \boxed{\lim_{n \to \infty} f(n) = \lim_{Z \to 1} (Z-1) F(Z)}$$

Hence Proved.

Shifting Properties

(1) Shifting $f(n)$ to the Right :-

If $Z[f(n)] = F(z)$ then

$$Z[f(n-k)] = z^{-k}F(z)$$

$n-k=0$
$n=k$

Sol :- we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$0 \quad k \quad k+1 \quad k+2 \quad \infty$

Consider $Z[f(n-k)] = \sum_{n=0}^{\infty} f(n-k) z^{-n}$ (k, n diff Power)

$$= \sum_{n=k}^{\infty} f(n-k) z^{-k} \quad (\because \text{we are shifting } f(n) \text{ to right})$$

$$= f(0) z^{-k} + f(1) \cdot z^{-(k+1)} + f(2) z^{-(k+2)} + \cdots$$

$$= z^{-k}[f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots]$$

$$= z^{-k} \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= z^{-k} F(z)$$

$$\therefore Z[f(n-k)] = z^{-k} F(z)$$

Note :- $Z[f(n-k)] = z^{-k} F(z)$

Putting $\boxed{k=1}$ we have $Z[f(n-1)] = z^{-1} F(z)$

Putting $\boxed{k=2}$ we have $Z[f(n-2)] = z^{-2} F(z)$

Putting $\boxed{k=3}$ we have $Z[f(n-3)] = z^{-3} F(z)$

13

(2) Shifting $f(n)$ to the Left :-

If $Z[f(n)] = F(z)$ then

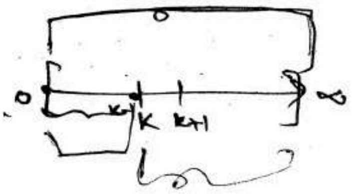
$$Z[f(n+k)] = z^{k}\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right].$$

Proof :- we have $Z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

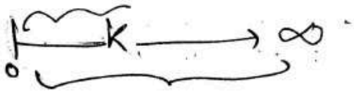
$$z^{-n} = z^{k} \cdot z^{-(n+k)}$$
$$= z^{k} \cdot z^{-n-k}$$

Consider $Z[f(n+k)] = \sum_{n=0}^{\infty} f(n+k)\, z^{-n}$

$$= z^{k} \sum_{n=0}^{\infty} f(n+k)\, z^{-(n+k)} \quad \left(\text{same sup by } n\right)$$



$$= z^{k} \sum_{n=k}^{\infty} f(n)\, z^{-n} \quad \left(\text{Replacing } (n+k) \text{ by } n\right)$$



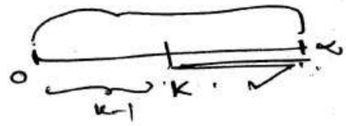
$$= z^{k}\left[\sum_{n=0}^{\infty} f(n) z^{-n} - \sum_{n=0}^{k-1} f(n) z^{-n}\right]$$

$$= z^{k}\left[Z[f(n)] - \sum_{n=0}^{k-1} f(n) z^{-n}\right]$$

$$\therefore Z[f(n+k)] = z^{k}\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right]$$

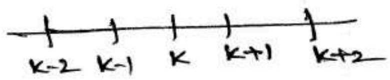
which is a Recurrence formula.

In Particular :



(a) If $\boxed{k=1}$ then

$$Z[f(n+1)] = z[F(z) - f(0)]$$



(b) If $\boxed{k=2}$ then

$$Z[f(n+2)] = z^{2}\left[F(z) - f(0) - f(1)z^{-1}\right]$$

(c) If $\boxed{k=3}$ then

$$Z[f(n+3)] = z^{3}\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2}\right]$$

. . . . and so on .

(31)

$$Z\left(\frac{1}{n+1}\right) = Z\log\left(\frac{Z}{Z-1}\right) \qquad \frac{1}{Z^0} = 1$$

let, $f(n) = \frac{1}{n+1} \Rightarrow Z[f(n)] = \sum_{n=0}^{\infty} f(n) Z^{-n}$

$$Z[f(n)] = Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1}\frac{1}{z^n}$$

$$= \cancel{\frac{1}{1} + \frac{1}{2}\frac{1}{z} + \frac{1}{3}\frac{1}{z^2} + \frac{1}{4}\frac{1}{z^3} + \cdots}$$

$$= \frac{1}{1}\frac{1}{1} + \frac{1}{2}\frac{1}{z} + \frac{1}{3}\frac{1}{z^2} + \frac{1}{4}\frac{1}{z^3} + \cdots$$

Expansion needs 'z' in denominator's,
For this, multiply & divide with 'z'

$$= z\left(\frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3}\frac{1}{z^3} + \frac{1}{4}\frac{1}{z^4} + \cdots\right)$$

$$= z\left[\frac{\frac{1}{z}}{1} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots\right]$$

$$= z\cancel{\left[\frac{\frac{1}{z}}{1} + \frac{\left(\frac{1}{z}\right)^2}{2} + \left(\frac{1}{z}\right)\right]}$$

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\log(1-x)$$

$$= z\left[-\log\left(1-\frac{1}{z}\right)\right]$$

$$= z\left[\log\left(1-\frac{1}{z}\right)^{-1}\right]$$

$$= z\log\left(\frac{z-1}{z}\right)^{-1}$$

$$= z\log\left(\frac{z}{z-1}\right)$$

(14)

$$Z\left(\frac{1}{n+1}\right) = z\log\left(\frac{z}{z-1}\right)$$

<u>Sol</u> :- Let $f(n) = \frac{1}{n+1}$

we know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{1}{z^n}$$

$$= \frac{1}{1}\cdot\frac{1}{1} + \frac{1}{2}\cdot\frac{1}{z} + \frac{1}{3}\cdot\frac{1}{z^2} + \cdots$$

Expansion needs z in denominators, for this multiply and divide with z.

$$= z\left(\frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3}\frac{1}{z^3} + \frac{1}{4}\frac{1}{z^4} + \cdots\right)$$

$$= z\left[\frac{\frac{1}{z}}{1} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots\right]$$

$$\boxed{x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\log(1-x)}$$

$$= z\left[-\log\left(1-\frac{1}{z}\right)\right]$$

$$= z\left[\log\left(1-\frac{1}{z}\right)^{-1}\right]$$

$$= z\log\left(\frac{z-1}{z}\right)^{-1}$$

$$= z\log\left(\frac{z}{z-1}\right)$$

(16) Find $Z\left[\frac{1}{n!}\right]$ and Using shifting theorem.

Evaluate.

(i) $Z\left(\frac{1}{(n+1)!}\right)$ (ii) $Z\left(\frac{1}{(n+2)!}\right)$

Sol :- we know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

Let $f(n) = \frac{1}{n!}$ for $n = 0, 1, 2, 3 \ldots$

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= 1 + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \ldots$$

$$= 1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \ldots$$

$$= e^{1/z} \quad \left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)$$

$$= F(z) \text{ (say)}$$

By Shifting Theorem

$$Z[f(n+1)] = z[F(z) - F(0)]$$

$$Z[f(n+2)] = z^2[F(z) - F(0) - F(1) z^{-1}]$$

(i) $Z\left[\frac{1}{(n+1)!}\right] = z\left[e^{1/z} - 1\right]$ $\left(\because f(0) = \frac{1}{0!} = 1\right)$

(ii) $Z\left[\frac{1}{(n+2)!}\right] = z^2\left[e^{1/z} - 1 - \frac{1}{1!} z^{-1}\right]$

$$= z^2\left[e^{1/z} - 1 - \frac{1}{1!}\left(\frac{1}{z}\right)\right]$$

$$= z^2\left[e^{1/z} - 1 - z^{-1}\right]$$

$\longrightarrow$ $f(n) = \frac{1}{n!}$

$f(n+1) = \frac{1}{(n+1)!}$

$f(n+2) = \frac{1}{(n+2)!}$

(15)

Multiplication by 'n' :-

If $Z[f(n)] = F(Z)$ then

$$Z[n f(n)] = -Z \frac{d}{dZ}[F(Z)]$$

Proof :- we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) Z^{-n} = F(Z)$$

$$Z[n f(n)] = \sum_{n=0}^{\infty} n f(n) Z^{-n}$$

$Z^{-n} = Z^{1} \cdot Z^{-n-1}$

$\frac{d}{dZ}(Z^{-n}) = (-n) Z^{-n-1}$

$$= -Z \sum_{n=0}^{\infty} f(n) (-n) Z^{-n-1}$$

$$= -Z \sum_{n=0}^{\infty} \frac{d}{dZ}[f(n) Z^{-n}]$$

$$= -Z \frac{d}{dZ}\left[\sum_{n=0}^{\infty} f(n) Z^{-n}\right]$$

$$= -Z \frac{d}{dZ}[Z f(n)]$$

$$= -Z \frac{d}{dZ} F(Z)$$

$$\therefore \boxed{Z[n f(n)] = -Z \frac{d}{dZ}[F(Z)]}$$

(1) If $F(Z) = \dfrac{5Z^2 + 3Z + 12}{(Z-1)^4}$ then find the values of $f(2)$ and $f(3)$.

Sol :- given $F(Z) = \dfrac{5Z^2 + 3Z + 12}{(Z-1)^4}$

$$= \frac{\cancel{Z^2}(5 + 3Z^{-1} + 12 Z^{-2})}{\cancel{Z^4}^{\,2}(1 - Z^{-1})^4}$$

$$= \frac{1}{Z^2} \frac{(5 + 3Z^{-1} + 12 Z^{-2})}{(1 - Z^{-1})^4} \longrightarrow (1)$$

By Initial value theorem we have

$$f(0) = \lim_{z\to\infty} F(z) = \underline{\underline{0}} \qquad \left(\tfrac{1}{\infty} = 0\right)$$

$$(z-1)^4 = z^4(1-z^{-1})^4$$

$$f(1) = \lim_{z\to\infty} z\left[F(z) - f(0)\right] = \underline{\underline{0}}.$$

$$f(2) = \lim_{z\to\infty} z^2\left[F(z) - f(0) - f(1)z^{-1}\right]$$

$$= 5 - 0 - 0 = \underline{\underline{5}}$$

$$f(3) = \lim_{z\to\infty} z^3\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2}\right]$$

$$= \lim_{z\to\infty} z^3\left[F(z) - (0) - (0) - 5z^{-2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{5z^2+3z+12}{(z-1)^4} - \frac{5}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{5z^4+3z^3+12z^2-5(z-1)^4}{z^2(z-1)^4}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{5z^4+3z^3+12z^2-5(z^4-4z^3+6z^2-4z+1)}{z^2(z-1)^4}\right]$$

$$(z-1)^4 = (z-1)^2(z-1)^2$$
$$\Rightarrow (z^2+1-2z)(z^2+1-2z)$$
$$\Rightarrow z^4+z^2-2z^3+z^2+1-2z-2z^3-2z+4z^2$$
$$= z^4+6z^2-4z^3-4z+1$$

$$= \lim_{z\to\infty} z^3\left[\frac{\cancel{5z^4}+3z^3+12z^2-\cancel{5z^4}+20z^3-30z^2+20z-5}{z^2(z-1)^4}\right]$$

(16)

$$= \lim_{z \to \infty} \frac{\cancel{z^3}\left[23z^3 - 18z^2 + 20z - 5\right]}{\cancel{z^{n}} \cdot \cancel{z^4}\,(1 - z^{-1})^4}$$

$$= \lim_{z \to \infty} \frac{\cancel{z^3}\left[23 - 18z^{-1} + 20z^{-2} - 5z^{-3}\right]}{\cancel{z^3}\left[1 - z^{-1}\right]^4}$$

$$= 23$$

1

Problems on Convolution theorem

1) Using Convolution Theorem evaluate

$$Z^{-1}\left[\left(\frac{z}{z-a}\right)^2\right]$$

Sol :- we know that $Z^{-1}\left(\frac{z}{z-a}\right) = a^n$

By Using Convolution Theorem

$$Z^{-1}\left[\left(\frac{z}{z-a}\right)^2\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n$$

$$= \sum_{m=0}^{n} f(m)\,g(n-m)$$

$$= \sum_{m=0}^{n} a^m \cdot a^{n-m}$$

$$a^m \cdot a^{n-m} = a^{\cancel{m}+n-\cancel{m}} = a^n$$

$$= \sum_{m=0}^{n} a^n$$

$$= a^n \sum_{m=0}^{n} 1$$

$$= a^n\left[1+1+1+\cdots+1\right]$$

$$= a^n(n+1)$$

(sum of n+1 terms n+1 times including zero term)

$$\therefore Z^{-1}\left[\left(\frac{z}{z-a}\right)^2\right] = a^n(n+1)$$

(2) Find $Z^{-1}\left[\left(\frac{z}{z-a}\right)^3\right]$

Sol :- $= Z^{-1}\left[\left(\frac{z}{z-a}\right)^2 \cdot \left(\frac{z}{z-a}\right)\right]$

$= a^n(n+1) * a^n$

$= \sum_{m=0}^{n} f(m)\, g(n-m)$

$= \sum_{m=0}^{n} a^m(m+1) \cdot a^{n-m}$

$= \sum_{m=0}^{n} \cancel{a^m}(m+1) \cdot \frac{a^n}{\cancel{a^m}}$

$= a^n \sum_{m=0}^{n} (m+1)$

$= a^n\left[1+2+3+ \cdots (n+1)\right]$

$$\boxed{\begin{gathered} 1+2+3+\cdots+n = \frac{n(n+1)}{2} \text{ by (mathem Induct)} \\ \text{sum of } n \text{ natural numbers.} \end{gathered}}$$

replace n by $n+1$

$= a^n \frac{(n+1)(n+2)}{2}$

$= \frac{1}{2} a^n (n+1)(n+2)$

(2)

(3) Find $Z^{-1}\left(\frac{z^2}{(z-a)(z-b)}\right)$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] ; Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$\therefore Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \sum_{m=0}^{n} f(m)\, g(n-m)$$

$$= \frac{b^{n+1}-a^{n+1}}{b-a}$$

$$= \sum_{m=0}^{n} a^m \cdot b^{(n-m)}$$

$$= \sum_{m=0}^{n} a^m \cdot \frac{b^n}{b^m}$$

$$= b^n \sum_{m=0}^{n} \frac{a^m}{b^m}$$

$$\left(\frac{a}{b}\right)^0 = 1$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n\left[1+\left(\frac{a}{b}\right)+\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)^3+\cdots+\left(\frac{a}{b}\right)^n\right]$$

$a + ar + ar^2 + \cdots + ar^n = \frac{a(1-r^{n+1})}{1-r}$

Finite G.P $(n+1)$ Including 1st term $(n+1)$ term sum

because ending point

$a = 1$ (1st term) $\quad r = \frac{a}{b}$ Common ratio.

a = 1st term
r = Common ration

$$= b^n\left[\frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}}\right] = b^n\left[\frac{1-\frac{a^{n+1}}{b^{n+1}}}{\frac{b-a}{b}}\right]$$

$$= b^n\left[\frac{\frac{b^{n+1}-a^{n+1}}{b^{n+1}}}{\frac{b-a}{b}}\right] = \cancel{b^n}\cdot\frac{b^{n+1}-a^{n+1}}{\cancel{b^{n+1}}}\cdot\frac{\cancel{b}}{b-a} = \frac{b^{n+1}-a^{n+1}}{b-a}$$

(4) Find $Z^{-1}\left[\dfrac{z^2}{(z-3)(z-4)}\right]$

we know that

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{b^{n+1} - a^{n+1}}{b-a}$$

Put $\boxed{a=3}$ and $\boxed{b=4}$ we have

$$Z^{-1}\left[\frac{z^2}{(z-3)(z-4)}\right] = \frac{4^{n+1} - 3^{n+1}}{4-3} = \underline{\underline{4^{n+1} - 3^{n+1}}}$$

(5) Find $Z^{-1}\left[\dfrac{z^2}{(z-4)(z-5)}\right]$

we know that

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{b^{n+1} - a^{n+1}}{b-a}$$

Put $\boxed{a=4}$ and $\boxed{b=5}$ we have

$$Z^{-1}\left[\frac{z^2}{(z-4)(z-5)}\right] = \frac{5^{n+1} - 4^{n+1}}{5-4} = \underline{\underline{5^{n+1} - 4^{n+1}}}$$

(3)

6) Evaluate $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$ Using Convolution theorem.

Sol :- $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right]$

$= Z^{-1}\left[\frac{z}{z-1}\right]\cdot Z^{-1}\left[\frac{z}{z-3}\right]$

$= 1^n * 3^n$

$= 1 * 3^n$

$= \sum_{m=0}^{n} f(m)\,g(n-m)$

$= \sum_{m=0}^{n} 1^m \cdot 3^{n-m}$

$= 3^n \sum_{m=0}^{n} 3^{-m}$

$= 3^n \sum_{m=0}^{n} \frac{1}{3^m}$

$= 3^n\left(\frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^n}\right)$

$= 3^n\left(1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\cdots+\left(\frac{1}{3}\right)^n\right]$

$$\boxed{a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \frac{a(1-r^n)}{1-r} \qquad \boxed{a=1 \quad r=\frac{1}{3}}}$$

$= 3^n\left[\frac{1\left(1-\left(\frac{1}{3}\right)^n\right]}{1-\frac{1}{3}} = \frac{3^n\left[1-\frac{1^n}{3^n}\right]}{\frac{3-1}{3}} = \frac{3^n\left[\frac{3^n-1^n}{3^n}\right]}{\frac{2}{3}}$

$= \cancel{3^n}\left[\frac{3^n-1^n}{\cancel{3^n}}\right]\cdot\frac{3}{2}$

4

Partial Fractions Method

*** 1) Find $Z^{-1}\left[\dfrac{z}{z^2+11z+24}\right]$ (non-repeated Linear Factors)

Sol: Let $F(z) = \dfrac{z}{z^2+11z+24} = \dfrac{z}{(z+3)(z+8)}$

Then $\dfrac{F(z)}{z} = \dfrac{1}{(z+3)(z+8)} = \dfrac{A}{z+3} + \dfrac{B}{z+8} \longrightarrow (1)$

$$= \frac{1}{\cancel{(z+3)(z+8)}} = \frac{A(z+8)+B(z+3)}{\cancel{(z+3)(z+8)}}$$

$$1 = A(z+8) + B(z+3) \longrightarrow (2)$$

Put $\boxed{z=-8}$ in (2) $\Rightarrow 1 = A(-8+8) + B(-8+3)$

$$1 = -5B \Rightarrow \boxed{B = -\frac{1}{5}}$$

Put $\boxed{z=-3}$ in (2) $\Longrightarrow 1 = A(-3+8) + B(-3+3)$

$$1 = 5A \Rightarrow \boxed{A = \frac{1}{5}}$$

now Substitute A and B values in (1) we get

$$\frac{F(z)}{z} = \frac{1}{5(z+3)} - \frac{1}{5(z+8)}$$

$$\therefore F(z) = \frac{z}{5(z+3)} - \frac{z}{5(z+8)}$$

$Z^{-1}[a^n] = \dfrac{z}{z-a} \Rightarrow$

$Z^{-1}\left[\dfrac{z}{z-a}\right] = a^n$

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{5(z+3)} - \frac{z}{5(z+8)}\right]$$

$$= \frac{1}{5}\left[Z^{-1}\left[\frac{z}{z+3}\right] - Z^{-1}\left[\frac{z}{z+8}\right]\right]$$

$$= \frac{1}{5}\left[(-3)^n - (-8)^n\right]$$

$$\therefore Z^{-1}\left[\frac{z}{z^2+11z+24}\right] = \frac{1}{5}\left[(-3)^n - (-8)^n\right]$$

(2) Find $Z^{-1}\left[\dfrac{z}{z^2+8z+15}\right]$

Sol :- Let $F(z) = \dfrac{z}{z^2+8z+15} = \dfrac{z}{(z+3)(z+5)}$

$$\therefore \frac{F(z)}{z} = \frac{1}{(z+3)(z+5)} = \frac{A}{z+3} + \frac{B}{z+5} \longrightarrow (1)$$

$$\Rightarrow \frac{1}{\cancel{(z+3)(z+5)}} = \frac{A(z+5)+B(z+3)}{\cancel{(z+3)(z+5)}}$$

$$\Rightarrow 1 = A(z+5) + B(z+3) \longrightarrow (2)$$

Put $\boxed{z=-5}$ in (2) we get

$$1 = A(-5+5) + B(-5+3)$$

$$1 = -2B \Rightarrow \boxed{B = -\frac{1}{2}}$$

Put $\boxed{z=-3}$ in (2) we get

$$1 = A(-3+5) + B(-3+3)$$

$$1 = A2 \Rightarrow \boxed{A = \frac{1}{2}}$$

now Substitute A and B values in (1) we get

$$\frac{F(z)}{z} = \frac{1}{2(z+3)} - \frac{1}{2(z+5)}$$

$Z[a^n] = \dfrac{z}{z-a} \Rightarrow$

$Z^{-1}\left[\dfrac{z}{z-a}\right] = a^n$

$$\therefore F(z) = \frac{z}{2(z+3)} - \frac{z}{2(z+5)}$$

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{2(z+3)} - \frac{z}{2(z+5)}\right]$$

$$= \left[\frac{1}{2}\left[Z^{-1}\left[\frac{z}{z+3}\right]\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z+5}\right]\right]$$

$$\Rightarrow \frac{1}{2}\left[(-3)^n - (-5)^n\right] //.$$

$$\therefore Z^{-1}\left[\frac{z}{z^2+8z+15}\right] = \frac{1}{2}\left[(-3)^n - (-5)^n\right] //.$$

(8)

(3) Find the Inverse Z-Transform of

$$\frac{z}{(z-1)(z-2)}$$ Using Partial fractions

Sol :- Let $F(z) = \dfrac{z}{(z-1)(z-2)}$

Here we can resolve $F(z)$ into Partial fractions directly as follows

$$F(z) = z\left[\frac{1}{(z-1)(z-2)}\right] = z\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$F(z) = \left[\frac{z}{z-2} - \frac{z}{z-1}\right]$$

Here $z^{-1}[F(z)] = z^{-1}\left[\frac{z}{z-2} - \frac{z}{z-1}\right]$

$$= z^{-1}\left[\frac{z}{z-2}\right] - z^{-1}\left[\frac{z}{z-1}\right]$$

$$= 2^n - 1^n \quad \left(\because z^{-1}\left[\frac{z}{z-a}\right] = a^n\right)$$

4) Find $z^{-1}\left[\dfrac{z}{(z+3)^2(z-2)}\right]$ (repeted Linear factor of form $(ax+b)$ 2-times)

Sol :- Let $F(z) = \dfrac{z}{(z+3)^2(z-2)}$ not Quadratic exp. ext'd

$$\frac{F(z)}{z} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2} = \frac{1}{(z+3)^2(z-2)} \rightarrow (1)$$

$$\frac{1}{\cancel{(z+3)^2(z-2)}} = \frac{A(z+3)^2 + B(z-2)(z+3) + C(z-2)}{\cancel{(z-2)(z+3)^2}}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2) \longrightarrow (2)$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2) \longrightarrow ②$$

Put $\boxed{z=2}$ in ② $\Rightarrow$ $1 = A(2+3)^2$

$1 = 25A \Rightarrow \boxed{A = \frac{1}{25}}$

Put $\boxed{z=-3}$ in ② $\Rightarrow$ $1 = C(-3-2)$

$1 = -5C \Rightarrow \boxed{C = -\frac{1}{5}}$

$z-2=0$
$z=2$
$z+3=0$
$z=-3$

Comparing Coeff of z^2 on b.s

$0 = A + B$

$A = -B \Rightarrow B = -A \Rightarrow -\frac{1}{25}$

$\boxed{B = -\frac{1}{25}}$

$$z^{-1}\left[\frac{z}{(z+3)^2(z-2)}\right] = z^{-1}\left[\frac{z}{25(z-2)} - \frac{z}{25(z+3)} - \frac{z}{5(z+3)^2}\right]$$

$$\Rightarrow \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5} z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$\Rightarrow \frac{1}{25}(2^n) - \frac{1}{25}(-3)^n - \frac{1}{5} n(-3)^n$$

$z^{-1}\left[\frac{z}{(z+3)^2}\right] = \frac{1}{-3} z^{-1}\left[\frac{(-3)z}{(z-(-3))^2}\right]$

$= -\frac{1}{3} n (-3)^n$

$z\left(n a^n\right) = \frac{az}{(z-a)^2}$

(5) Find $z^{-1}\left[\frac{3z^2+z}{(5z-1)(5z+2)}\right]$

Sol :- Let $F(z) = \frac{z(3z+1)}{(5z-1)(5z+2)}$ then.

$$\frac{F(z)}{z} = \frac{3z+1}{(5z-1)(5z+2)} = \frac{A}{5z-1} + \frac{B}{5z+2} \text{ (by Part Fot} \longrightarrow ①$$

$$\frac{3z+1}{\cancel{(5z-1)(5z+2)}} = \frac{A(5z+2) + B(5z-1)}{\cancel{(5z-1)(5z+2)}}$$

(6)

$$3z+1 = A(5z+2) + B(5z-1) \longrightarrow (2)$$

$$\Rightarrow \boxed{A = \frac{8}{15}} \quad \boxed{B = \frac{1}{15}}$$

Substitute A & B values in (1) we get.

$$\frac{F(z)}{z} = \frac{8}{75} \cdot \frac{1}{\left(z - \frac{1}{5}\right)} + \frac{1}{75} \cdot \frac{1}{z + \left(\frac{2}{5}\right)}$$

Hence $F(z) = \frac{8}{75}\left(\frac{z}{z - \frac{1}{5}}\right) + \frac{1}{75}\left(\frac{z}{z + \frac{2}{5}}\right)$

Hence $z^{-1}[F(z)] = \frac{8}{75} z^{-1}\left(\frac{z}{z - 0.2}\right) + \frac{1}{75} z^{-1}\left[\frac{z}{z - (-0.4)}\right]$

$$\therefore z^{-1}\left[\frac{3z^2+z}{(5z-1)(5z+2)}\right] = \underline{\frac{8}{75}(0.2)^n + \frac{1}{75}}(-0.4)^n$$

(1')

Inverse Z-Transform

we have $Z[f(n)] = F(Z)$ which can also be written as

$f(n) = Z^{-1}[F(Z)]$

Then $f(n)$ is called the Inverse Z-Transform of $F(Z)$

Thus finding the sequence $\{f(n)\}$ from $F(Z)$ is defined as Inverse Z-Transform

The symbol Z^{-1} is the Inverse Z-Transform

Convolution Theorem :- (V.V. Imp)

If $Z^{-1}[F(Z)] = f(n)$ and

$Z^{-1}[G(Z)] = g(n)$ then

$$Z^{-1}[F(Z)\cdot G(Z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m)g(n-m)$$

where $*$ denotes the Convolution operator.

$Z \cdot Z^{-1} = 1$

Proof :- we have $F(Z) = \sum_{n=0}^{\infty} f(n)Z^{-n}$ and

$G(Z) = \sum_{n=0}^{\infty} g(n)Z^{-n}$ then

$$F(Z)\cdot G(Z) = \left[f(0) + f(1)Z^{-1} + f(2)Z^{-2} + \cdots + f(n)Z^{-n} + \cdots\right] * \left[g(0) + g(1)Z^{-1} + g(2)Z^{-2} + \cdots + g(n)Z^{-n} + \cdots\right]$$

$$= \sum_{n=0}^{\infty}\left[f(0)g(n) + f(1)g(n-1) + f(2)g(n-2) + \cdots + f(n)g(0)\right]Z^{-n}$$

$$= Z\left[f(0)g(n) + f(1)g(n-1) + \cdots + f(n)g(0)\right]$$

$$Z^{-1}[F(Z)\cdot G(Z)] = f(0)g(n) + f(1)g(n-1) + \cdots + f(n)g(0)$$

$$= \sum_{m=0}^{n} f(m)g(n-m)$$

$$\therefore Z^{-1}[F(Z)\cdot G(Z)] = \sum_{m=0}^{n} f(m)\cdot g(n-m)$$

Problems

1) Evaluate $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$ Using Convolution theorem.

Sol :- we know that

$$Z(a^n) = \frac{z}{z-a} \quad (\text{or}) \quad Z^{-1}\left(\frac{z}{z-a}\right) = a^n$$

$$\therefore Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right]$$

By Convolution theorem we have

$$\boxed{Z^{-1}[F(z)\cdot G(z)] = \sum_{m=0}^{n} f(m)g(n-m)}$$

Here $F(z) = \frac{z}{z-1} \Rightarrow f(n) = Z^{-1}\left[\frac{z}{z-1}\right] = 1^n = \underline{1}$

$G(z) = \frac{z}{z-3} \Rightarrow g(n) = Z^{-1}\left[\frac{z}{z-3}\right] = \underline{3^n}$

$$\therefore Z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right] = \sum_{m=0}^{n} 1^m \cdot 3^{n-m}$$

$= \sum_{m=0}^{n} 3^{n-m}$ (Using Convolution theorem)

$= 3^n \sum_{m=0}^{n} 3^{-m}$

$= 3^n \sum_{m=0}^{n} \frac{1}{3^m}$

$= 3^n\left(\frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^n}\right)$ $= 3^n + 3^{n-1} + 3^{n-2} + \cdots + 3^0$ ($3^0 = 1$)

$= 3^n\left(1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\cdots+\left(\frac{1}{3}\right)^n\right)$ $= \frac{1-3^{n+1}}{1-3} = \frac{1}{2}\left(3^{n+1}-1\right)$

r = Common ratio

$a=1$ $r=\frac{1}{3}$

G.P $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \frac{a(1-r^n)}{1-r}$ $a=1$, $r=3$

S_n = sum of n terms $\frac{a(1-r^n)}{1-r}$ $r<1$

$1 + 1\cdot 3 + 1\cdot 3^2 + 1\cdot 3^3 + \cdots$

$a=1$ $ar = 1\cdot 3$ $= \frac{a(r^n-1)}{r-1}$ $r>1$

$\underline{r=3}$

$= \frac{1-(1-(3)^n)}{1-3}$

$= 3^n\left[\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}\right]$

$=$

(2')

(2) Evaluate (i) $Z^{-1}\left[\left(\frac{z}{z-a}\right)^2\right]$ Using Convolution theorem

(ii) $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Sol: (i) $Z^{-1}\left[\left(\frac{z}{z-a}\right)^2\right]$ $\qquad Z^{-1}[F(z)] = f(n)$

$= Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right]$

$a^m \cdot a^{n-m}$
$= a^{\cancel{m}+n-\cancel{m}}$
$= a^n$

$F(z) = \frac{z}{z-a} \Rightarrow f(n) = Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

$G(z) = \frac{z}{z-a} \Rightarrow g(n) = Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

$$\boxed{\text{By Convolution theorem} \quad Z^{-1}[F(z)\cdot G(z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m)\,g(n-m)}$$

$Z^{-1}[F(z)\cdot G(z)] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right] = \sum_{m=0}^{n} a^m \cdot a^{n-m}$

$a^n * a^n = \quad = \sum_{m=0}^{n} a^n$

$= a^n \sum_{m=0}^{n} 1$

n+1 times
including Zero term
1+1+1 = 3 times

$= a^n[1+1+\cdots+1]$ (n+1) times

Sum of n+1 terms

$= (n+1)a^n$

(ii) $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$

$F(z) = \frac{z}{z-a} \Rightarrow f(n) = Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

$G(z) = \frac{z}{z-b} \Rightarrow g(n) = Z^{-1}\left(\frac{z}{z-b}\right) = b^n$

By Convolution theorem we have.

$$Z^{-1}\left[F(z)\cdot G(z)\right] = \sum_{m=0}^{n} f(m)\,g(n-m)$$

$$= \sum_{m=0}^{n} a^m \cdot b^{n-m}$$

$$= \sum_{m=0}^{n} b^n \left(\frac{a}{b}\right)^m$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n\left[\left(\frac{a}{b}\right)^0 + \left(\frac{a}{b}\right)^1 + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n \left[\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}}\right] \quad \text{(This is in G.P)}$$

$$= b^n \left[\frac{1 - \frac{a^{n+1}}{b^{n+1}}}{\frac{b-a}{b}}\right]$$

$$= b^n \left[\frac{\frac{b^{n+1} - a^{n+1}}{b^{n+1}}}{\frac{b-a}{b}}\right]$$

$$= \cancel{b^n} \cdot \frac{b^{n+1} - a^{n+1}}{\cancel{b^{n+1}}} \cdot \frac{\cancel{b}}{b-a} = \frac{b^{n+1} - a^{n+1}}{b-a}$$

$$\therefore\ Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{b^{n+1} - a^{n+1}}{b-a}$$

→ Put $a = 3$ and $b = 4$ we get $Z^{-1}\left[\frac{z^2}{(z-3)(z-4)}\right] = \frac{4^{n+1} - 3^{n+1}}{4-3} = 4^{n+1} - 3^{n+1}$

(3)

(3) Using Convolution theorem

Find $Z^{-1}\left[\dfrac{z^2}{(z-4)(z-5)}\right]$

Sol :-

we know that

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{b^{n+1}-a^{n+1}}{b-a}$$

Put $a=4$
$b=5$

$$Z^{-1}\left[\frac{z^2}{(z-4)(z-5)}\right] = \frac{b^{n+1}-a^{n+1}}{b-a} = \frac{5^{n+1}-4^{n+1}}{5-4}$$

$$= 5^{n+1}-4^{n+1}$$

(4) Using Convolution theorem

S.T $Z^{-1}\left[\dfrac{1}{n!} * \dfrac{1}{n!}\right] = \dfrac{2^n}{n!}$

where $*$ is the Convolution operator.

Sol :- $f(n)=\dfrac{1}{n!}$ $g(n)=\dfrac{1}{n!}$

$$\frac{1}{(n-1)!} = \frac{n}{n(n-1)!} = \frac{1}{(n-1)!}$$

$$f(n) * g(n) = \sum_{m=0}^{n} f(m)\, g(n-m)$$

$$= \sum_{m=0}^{n} \frac{1}{m!}\cdot\frac{1}{(n-m)!}$$

$$= 1\cdot\frac{1}{n!} + \frac{1}{1!}\,\frac{1}{(n-1)!} + \frac{1}{2!}\,\frac{1}{(n-2)!} + \cdots + \frac{1}{n!}\,\frac{1}{0!}$$

$$= \frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{2!}\left(\frac{1}{n-2}\right) + \cdots + \frac{1}{n!}$$

$$= \frac{1}{n!} + \frac{1}{1!}\,\frac{n}{n!} + \frac{1}{2!}\,\frac{n(n-1)}{n!} + \cdots + \frac{1}{n!}$$

$$= \frac{1}{n!}\left[1 + \frac{n}{1!} + \frac{n(n-1)}{2!} + \cdots \text{ to } (n+1) \text{ terms}\right]$$

$$= \frac{1}{n!}(1+1)^n$$

$$= \frac{2^n}{n!}$$

(5) $Z^{-1}\left[\frac{z^2}{(z-4)(z-5)}\right]$

$$= Z^{-1}\left[\frac{z}{z-4} - \frac{z}{z-5}\right]$$

we know that $Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

$F(z) = \frac{z}{z-4} \Rightarrow f(n) = Z^{-1}\left(\frac{z}{z-4}\right) = 4^n$

$G(z) = \frac{z}{z-5} \Rightarrow g(n) = Z^{-1}\left(\frac{z}{z-5}\right) = 5^n$

By Convolution $Z^{-1}\left[F(z)\cdot G(z)\right] = \sum_{m=0}^{n} f(m)\,g(n-m) = f(n) * g(n)$

$$= \sum_{m=0}^{n} 4^m \cdot 5^{n-m}$$

$\left(\frac{4}{5}\right)^0 = 1$

$$= 5^n \sum_{m=0}^{n} 4^m \cdot 5^{-m}$$

$$= 5^n \sum_{m=0}^{n} \left(\frac{4}{5}\right)^m$$

$$= 5^n\left[1 + \frac{4}{5} + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{5}\right)^3 + \cdots + \left(\frac{4}{5}\right)^n\right]$$

which is in G.P

$$= 5^n \cdot \frac{1-\left(\frac{4}{5}\right)^{n+1}}{1-\frac{4}{5}} = 5^n \cdot \frac{1-\frac{4^{n+1}}{5^{n+1}}}{\frac{5-4}{5}}$$

$$= 5^n \cdot \frac{\frac{5^{n+1}-4^{n+1}}{5^{n+1}}}{\frac{1}{5}} = \cancel{5^n} \cdot \frac{5^{n+1}-4^{n+1}}{\cancel{5^{n+1}}} \cdot \cancel{5}$$

$$= 5^{n+1} - 4^{n+1}$$

(4)

Partial Fractions Method

*** 1) Find $Z^{-1}\left[\dfrac{z}{z^2+11z+24}\right]$ (non repeated Linear factors)

[JNTU 2002, 2004, 2005, 2007, 2009, 2010, 2011, 2012, 2013]

Sol: Let $F(z) = \dfrac{z}{z^2+11z+24} = \dfrac{z}{(z+3)(z+8)}$

Then $\dfrac{F(z)}{z} = \dfrac{1}{(z+3)(z+8)} = \dfrac{A}{z+3} + \dfrac{B}{z+8} \longrightarrow (1)$

$$= \dfrac{1}{\cancel{(z+3)(z+8)}} = \dfrac{A(z+8)+B(z+3)}{\cancel{(z+3)(z+8)}}$$

$\therefore\ 1 = A(z+8)+B(z+3) \longrightarrow (2)$

Put $\boxed{z=-8}$ in (2) $\Rightarrow 1 = A(-8+8)+B(-8+3)$

$1 = B(-8+3)$

$1 = -5B \Rightarrow \boxed{B = -\frac{1}{5}}$

Put $\boxed{z=-3}$ in (2) $\Rightarrow 1 = A(-3+8)+B(-3+3)$

$1 = A(-3+8)$

$1 = 5A \Rightarrow \boxed{A = \frac{1}{5}}$

now substitute A and B values in (1) we get.

$$\dfrac{F(z)}{z} = \dfrac{1}{5(z+3)} - \dfrac{1}{5(z+8)}$$

$$\therefore F(z) = \dfrac{z}{5(z+3)} - \dfrac{z}{5(z+8)}$$

$$\boxed{Z[a^n] = \frac{z}{z-a} \Rightarrow Z^{-1}\left[\frac{z}{z-a}\right] = a^n.}$$

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{5(z+3)} - \frac{z}{5(z+8)}\right]$$

$$= \frac{1}{5}\left[Z^{-1}\left[\frac{z}{z+3}\right] - Z^{-1}\left[\frac{z}{z+8}\right]\right]$$

$$= \frac{1}{5}\left[(-3)^n - (-8)^n\right]$$

$$\therefore Z^{-1}\left[\frac{z}{z^2+11z+24}\right] = \frac{1}{5}\left[(-3)^n - (-8)^n\right]$$

(8) Find $Z^{-1}\left[\frac{z}{z^2+8z+15}\right]$

Sol :- $\frac{F(z)}{z} = \frac{1}{z^2+8z+15} = \frac{1}{(z+3)(z+5)} = \frac{1}{2}\left[\frac{1}{z+3} - \frac{1}{z+5}\right]$

(by Partial fractions)

$$\therefore F(z) = \frac{1}{2}\left[\frac{z}{z+3} - \frac{z}{z+5}\right]$$

Hence $Z^{-1}[F(z)] = \frac{1}{2} Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{2} Z^{-1}\left[\frac{z}{z+5}\right]$

$$= \frac{1}{2}(-3)^n - \frac{1}{2}(-5)^n$$

$$\therefore Z^{-1}\left[\frac{z}{z^2+8z+15}\right] = \frac{1}{2}\left[(-3)^n - (-5)^n\right]$$

5

(3) Find the Inverse Z-Transform of

$$\frac{z}{(z-1)(z-2)}$$

Sol :- Let $F(z) = \dfrac{z}{(z-1)(z-2)}$

Here we can resolve $F(z)$ into Partial fractions directly as follows

$$F(z) = z\left[\frac{1}{(z-1)(z-2)}\right] = z\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$F(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

Hence $z^{-1}[F(z)] = z^{-1}\left[\dfrac{z}{z-2}\right] - z^{-1}\left[\dfrac{z}{z-1}\right]$

$$= \underline{2^n - 1^n} \quad \left(\because \underline{z^{-1}\left[\frac{z}{z-a}\right] = a^n}\right)$$

(4) Find $z^{-1}\left[\dfrac{3z^2+z}{(5z-1)(5z+2)}\right]$

Sol :- Let $F(z) = \dfrac{z(3z+1)}{(5z-1)(5z+2)}$ then

$$\frac{F(z)}{z} = \frac{3z+1}{(5z-1)(5z+2)} = \frac{A}{5z-1} + \frac{B}{5z+2} \quad \text{(by Partial fractions)} \rightarrow (1)$$

$$= \frac{3z+1}{\cancel{(5z-1)(5z+2)}} = \frac{A(5z+2)+B(5z-1)}{\cancel{(5z-1)(5z+2)}}$$

$\Rightarrow 3z+1 = A(5z+2) + B(5z-1)$

$\Rightarrow \boxed{A = \frac{8}{15}} \quad \boxed{B = \frac{1}{15}}$

Substitute A & B values in (1) we get

$$\therefore \frac{F(z)}{z} = \frac{8}{75} \cdot \frac{1}{(z-\frac{1}{5})} + \frac{1}{75} \cdot \frac{1}{z+(\frac{2}{5})}$$

Hence $F(z) = \frac{8}{75}\left(\frac{z}{z-\frac{1}{5}}\right) + \frac{1}{75}\left(\frac{z}{z+\frac{2}{5}}\right)$

Hence $z^{-1}[F(z)] = \frac{8}{75}\left(\frac{z}{z-0.2}\right) + \frac{1}{75}\left(\frac{z}{(z+0.4)}\right)$

ie: $z^{-1}\left[\frac{3z^2+z}{(5z-1)(5z+2)}\right] = \frac{8}{75} z^{-1}\left(\frac{z}{z-0.2}\right) + \frac{1}{75} z^{-1}\left[\frac{z}{z-(-0.4)}\right]$

$$= \frac{8}{75}(0.2)^n + \frac{1}{75}(-0.4)^n$$

||ly we can solve

(5) $z^{-1}\left[\frac{3z^2}{(5z-1)(5z+2)}\right]$

(6)

Finite :- **G.P.**

$$a + ar + ar^2 + \cdots + ar^{n-1} + ar^n = \frac{a(1-r^{n+1})}{1-r}.$$

Infinite :-

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{a-r}$$

$$a = 1$$

$$1 + r + r^2 + \cdots + r^n + \cdots = \frac{1}{1-r}.$$

6) Find $z^{-1}\left[\frac{z}{(z+3)^2(z-2)}\right]$ (repeted Linear factor of form $(ax+b)$ 2 times)

Sol :- $$\frac{F(z)}{z} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2} = \frac{1}{(z+3)^2(z-2)} \longrightarrow (1)$$

$$\frac{1}{\cancel{(z+3)^2(z-2)}} = \frac{A(z+3)^2 + B(z-2)(z+3) + C(z-2)}{\cancel{(z-2)(z+3)^2}}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2) \longrightarrow (2)$$

$z-2=0$
$z=2$

$z+3=0$
$z=-3$

Put $\boxed{z=2} \Rightarrow 1 = A(2+3)^2$

$1 = 25A \Rightarrow \boxed{A = \frac{1}{25}}$

Put $\boxed{z=-3} \Rightarrow 1 = C(-3-2)$

$1 = -5C \Rightarrow \boxed{C = -\frac{1}{5}}$

now Comparing the Co-eff of z^2 on both sides

$0 = A + B \Rightarrow \boxed{B = -\frac{1}{25}}$

$$z^{-1}\left[\frac{z}{(z+3)^2(z-2)}\right] = z^{-1}\left[\frac{z}{25(z-2)} - \frac{z}{25(z+3)} - \frac{z}{5(z+3)^2}\right]$$

$$= \frac{1}{25}(2^n) - \frac{1}{25}(-3)^n + \frac{1}{15}n(-3)^n$$

Z - Transforms (Last topic)

Difference Equations :- Solutions of Difference eqn

Just as the Differential equations are used for dealing with Continuous Processes in nature, the difference eqns are used for dealing of descrete Processes.

Definition :-

A difference eqn is a relation between the difference of an unknown function at one (or) more general values of the argument.

Thus $\Delta y_n + 2y_n = 0$ and
$\Delta^2 y_n + 5\Delta y_n + 6y_n = 0$ are difference equations.

Solution :- The solution of a difference eqn is an expression for y_n which satisfies the given difference equation.

General Solution :-

The general solution of a difference eqn is that in which the number of arbitrary Constants is equal to the Order of the difference eqn.

Linear difference equation :-

The Linear difference eqn is that in which y_{n+1}, y_{n+2}, y_{n+3} ---- etc occur to the 1st degree only and are not multiplied together.

The difference eqn is Called Homogeneous if $f(n) = 0$

Otherwise it is Called as Non-Homogeneous eqn

$(\text{ie}; f(n) \neq 0)$

(7)

Working Rule (or) Working Procedure :-

To Solve a given Linear difference eqn with Constant Co-efficient by Z-Transforms.

Step-1 :- Let $Z(y_n) = Z[y(n)] = Y(z)$

Step-2 :- Take Z-Transform on bothsides of the given difference eqn

Step-3 :- Use the formulae. $Z(y_n) = Y(z)$

$$Z[y_{n+1}] = Z[Y(z) - y_0]$$

$$Z[y_{n+2}] = z^2[Y(z) - y_0 - y_1 z^{-1}]$$

Step-4 :- Simplify and find $Y(z)$ by transposing the terms to the right and dividing by the Co-efficient of $Y(z)$

Step-5 :- Take the Inverse Z-Transform of $Y(z)$ and Find the Solution of y_n

This gives y_n as a function of n which is the desired Solution.

1) Solve $y_{n+1} - 2y_n = 0$ Using Z-Transform.

Sol :- Let $Z[y_n] = Y(z)$

$$Z[y_{n+1}] = Z[Y(z) - y_0]$$

Taking Z-Transform of the given eqn we get.

$$Z[y_{n+1}] - 2Z[y_n] = 0$$

$$Z[Y(z) - y_0] - 2Y(z) = 0 \qquad \left[\because Z(a^n) = \frac{z}{z-a}\right]$$

$$\Rightarrow Y(z)(z-2) - z(y_0) = 0$$

$$\Rightarrow Y(z) = \frac{z}{z-2} y_0 \qquad Z[y(n)] = Y(z)$$

$$\Rightarrow Z^{-1}[Y(z)] = Z^{-1}\left[\frac{z}{z-2}\right] y_0 \qquad \Rightarrow Z^{-1}[Y(z)] = y_n$$

$$y_n = 2^n y_0$$

(2) Solve the difference eqn Using Z-Transforms

$u_{n+2} - 3u_{n+1} + 2u_n = 0$

given that $u_0 = 0$, $u_1 = 1$

Sol: Let $Z(u_n) = U(z)$

$Z(u_{n+1}) = z(U(z) - u_0]$

$Z(u_{n+2}) = z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$

now taking Z-Transform of both sides of the given eqn we get.

$Z(u_{n+2}) - 3Z(u_{n+1}) + 2Z(u_n) = 0$

$\Rightarrow z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] - 3z[U(z) - 0] + 2U(z) = 0$

Using the given Conditions it reduces to.

$\Rightarrow z^2\left[U(z) - 0 - \frac{1}{z}\right] - 3z[U(z)] + 2U(z) = 0$

ie: $U(z)\left[z^2 - 3z + 2\right] = z$ (or)

$U(z) = \frac{z}{z^2 - 3z + 2} = \frac{z}{(z-1)(z-2)} = z\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$

(reducing into Partial fractions) $= \frac{z}{z-2} - \frac{z}{z-1}$

On taking Inverse Z-Tranf of both sides we get.

$z^{-1}\{U(z)\} = z^{-1}\left[\frac{z}{z-2} - \frac{z}{z-1}\right]$

$U(z) = z^{-1}\left\{\frac{z}{z-2}\right\} - z^{-1}\left\{\frac{z}{z-1}\right\}$

$u_n = 2^n - 1$.

(3) Solve the difference eqn Using Z-Transform (8)

$$y_{n+2} - 4y_{n+1} + 3y_n = 0$$

given that $y_0 = 2$ and $y_1 = 4$

Sol :- Let $Z(y_n) = y(z)$

$$Z(y_{n+1}) = z[y(z) - y_0]$$

$$Z(y_{n+2}) = z^2[y(z) - y_0 - y_1 z^{-1}]$$

Taking Z-Transform of b.s of the given eqn we get

$$Z(y_{n+2}) - 4Z(y_{n+1}) + 3Z(y_n) = 0.$$

$$\Rightarrow z^2[y(z) - y_0 - y_1 z^{-1}] - 4z[Y(z) - y_0] + 3Y(z) = 0$$

Using the given Condition it reduces to

$$\Rightarrow z^2\left[y(z) - 2 - \frac{4}{z}\right] - 4z[Y(z) - 2] + 3Y(z) = 0.$$

ie :- $y(z)[z^2 - 4z + 3] - 2z^2 - 4z + 8z = 0$

$$\Rightarrow y(z)[z^2 - 4z + 3] = z(2z - 4)$$

$$\therefore \frac{y(z)}{z} = \frac{2z-4}{z^2-4z+3} = \frac{2z-4}{(z-1)(z-3)} = \frac{1}{z-1} + \frac{1}{z-3} \quad \text{(reducing into Partial fract)}$$

$$y(z) = \frac{z}{z-1} + \frac{z}{z-3}$$

On taking Inverse Z-Transf on b.s we Obtain.

$$Z^{-1}[y(z)] = z^{-1}\left[\frac{z}{z-1}\right] + z^{-1}\left[\frac{z}{z-3}\right]$$

$$y_n = 1 + 3^n$$

4) Solve the difference eqn Using Z-Transform

$$u_{n+2} - u_n = 2^n$$

where $u_0 = 0$ and $u_1 = 1$

Sol :- Let $Z(u_n) = u(z)$

$$Z(u_{n+1}) = z\left[u(z) - u_0\right]$$

$$Z(u_{n+2}) = z^2\left[u(z) - u_0 - \frac{u_1}{z}\right]$$

Taking Z-Transform on b.s

$$Z[u_{n+2}] - Z[u_n] = Z(2^n)$$

$$\Rightarrow z^2\left[u(z) - u_0 - \frac{u_1}{z}\right] - u(z) = \frac{z}{z-2}$$

Using given Conditions it reduces to

$$\Rightarrow z^2\left[u(z) - 0 - \frac{1}{z}\right] - u(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2\left[u(z) - \frac{1}{z}\right] - u(z) = \frac{z}{z-2}$$

$$\Rightarrow (z^2 - 1)u(z) = z + \frac{z}{z-2} = \frac{z(z-1)}{z-2}$$

(or)

$$u(z) = \frac{z(z-1)}{(z-2)(z-1)(z+1)} = \frac{z}{(z-2)(z+1)}$$

$$= \frac{z}{3}\left(\frac{1}{z-2} - \frac{1}{z+1}\right) = \frac{1}{3}\left[\frac{z}{z-2} - \frac{z}{z+1}\right]$$

On inversion we Obtain.

$$u_n = \frac{1}{3}\left[z^{-1}\left[\frac{z}{z-2}\right] - z^{-1}\left[\frac{z}{z+1}\right]\right]$$

$$\Rightarrow \frac{1}{3}\left[2^n - (-1)^n\right] \qquad \left(\because Z(a^n) = \frac{z}{z-a}\right)$$

(9)

Using Convolution evaluate

$$Z^{-1}\left[\left[\frac{z}{z-a}\right]^2\right]$$

Sol: we know that

$$Z^{-1}\left(\frac{z}{z-a}\right) = a^n$$

By Using Convolution Theorem

$$Z^{-1}\left[\left(\frac{z}{z-a}\right)^2\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right]$$

$$= z^{-1}\left(\frac{z}{z-a}\right) * z^{-1}\left(\frac{z}{z-a}\right)$$

$$= a^n * a^n$$

$$= \sum_{m=0}^{n} f(m)\cdot g(n-m)$$

$$= \sum_{m=0}^{n} a^m \cdot a^{n-m}$$

$$= \sum_{m=0}^{n} a^n$$

$$= a^n \sum_{m=0}^{n} 1$$

$$= a^n[1+1+\cdots+1]$$

$$= a^n(n+1)$$ (sum of $n+1$ terms)

($n+1$ times including zero term)

$$Z^{-1}\left[\left(\frac{z}{z-a}\right)^3\right] = Z^{-1}\left[\left(\frac{z}{z-a}\right)^2 \cdot \left(\frac{z}{z-a}\right)\right]$$

$$= a^n(n+1) * a^n$$

$$= \sum_{m=0}^{n} a^m(m+1) \cdot a^{n-m}$$

$$= \sum_{m=0}^{n} \cancel{a^m}(m+1) \cdot \frac{a^n}{\cancel{a^m}}$$

$$= a^n \sum_{m=0}^{n} (m+1)$$

$$= a^n\left[1+2+3+\cdots(n+1)\right]$$

$$\boxed{1+2+3+\cdots+n = \frac{n(n+1)}{2} \quad (\text{by Mat Ind}) \quad \text{Sum of } n \text{ natural number}}$$

repl n by n+1

$$= a^n \frac{(n+1)(n+2)}{2}$$

$$= \frac{1}{2} a^n (n+1)(n+2)$$

(3) Find $Z^{-1}\left(\frac{z^2}{(z-a)(z-b)}\right)$

Sol: Let $F(z) = \dfrac{z^2}{(z-a)(z-b)}$

$$Z^{-1}(F(z)) = Z^{-1}\left(\frac{z}{z-a} \cdot \frac{z}{z-b}\right)$$

$$= Z^{-1}\left(\frac{z}{z-a}\right) \cdot Z^{-1}\left(\frac{z}{z-b}\right)$$

(9)

$$= a^n * b^n.$$

$$= \sum_{m=0}^{n} f(m)\, g(n-m)$$

$$= \sum_{m=0}^{n} a^m \cdot b^{(n-m)}$$

$$= \sum_{m=0}^{n} a^m \cdot \frac{b^n}{b^m}.$$

$$= b^n \sum_{m=0}^{n} \frac{a^m}{b^m}$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right]$$

$$\boxed{a + ar + ar^2 + \cdots + ar^n = \frac{a(1 - r^{n+1})}{1-r}}$$

(n+1) Includ 1st term. (Finite G.P). (n+1 terms sum

$$= \quad a = 1 \quad r = \frac{a}{b}$$

1st term Common ratio.

$$= b^n \left[\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}}\right] = b^n \left[\frac{1 - \frac{a^{n+1}}{b^{n+1}}}{\frac{b-a}{b}}\right]$$

$$= b^n \left[\frac{\frac{b^{n+1} - a^{n+1}}{b^{n+1}}}{\frac{b-a}{b}}\right] = \cancel{b^n} \cdot \frac{b^{n+1} - a^{n+1}}{\cancel{b^{n+1}}} \cdot \frac{\cancel{b}}{b-a} = \frac{b^{n+1} - a^{n+1}}{b-a}$$