UNIT - IV

Numerical Solutions of Ordinary Differential Equations:

Consider an ordinary differential equation of 1st order & 1st degree of the form $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$.

In this chapter mainly concentrate on numerical solutions of ordinary differential equations & discuss the following methods.

(i) Taylor's Series Method.

(ii) Picard's Method of successive approximation

(iii) Euler's Method.

(iv) Modified Euler's Method

(v) RK Method.

Type-1 : ~~Par~~ Euler's Method

Consider an ordinary differential equation $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$.

then;

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$y_3 = y_2 + h\, f(x_2, y_2)$$

Problems:

1. Solve $\frac{dy}{dx} = xy^2$ using Euler's method for $x = 1.2, 1.4$ given $y(1)=1$

Sol:

Given; $\frac{dy}{dx} = xy^2$

WKT; $\frac{dy}{dx} = f(x,y)$

x	1	1.2	1.4
y'	1		

$\therefore f(x,y) = xy^2$

$y(1) = 1 \Rightarrow y(x_0) = y_0$

Equations:

st order &

ndition.

solutions of

methods.

$\frac{dy}{dx} = f(x,y)$

iven $y(1)=1$.

2 1.4

	x_0	x_1	x_2
x	1	1.2	1.4
y	1 (y_0)	? (y_1)	? (y_2)

$h = 0.2$

1. $y_1 = y_0 + h\, f(x_0, y_0)$

$= 1 + 0.2\,(1\cdot(1)^2)$

$= 1 + 0.2$

$y_1 = 1.2$

2. $y_2 = y_1 + h\, f(x_1, y_1)$

$= 1.2 + 0.2\, f(1.2, 1.2)$

$= 1.2 + [0.2[((1.2)(1.2)^2)]]$

$= 1.2 + 0.2\,(1.728)$

$= 1.2 + 0.3456$

$y_2 = 1.5456$

2. Using Euler's method, solve numerically the equation $\frac{dy}{dx} = 3x^2+1$, $y(1)=2$ for y at $x=1.25$ taking step size 0.25.

Sol:

Given; $\frac{dy}{dx} = 3x^2+1$

WKT; $\frac{dy}{dx} = f(x,y)$

$\therefore f(x,y) = 3x^2+1$

$y(1) = 2 \Rightarrow y(x_0) = y_0$

	x_0	x_1
x	1	1.25
y	2 (y_0)	? (y_1)

1. $y_1 = y_0 + h\, f(x_0, y_0)$

$= 2 + 0.25\, f(1, 2)$

$= 2 + 0.25\,(3+1)$

$= 3$

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if so solve them. $4x+2y+z+3w=0$, ... $2x+y+w=0$

3. Solve by Euler's method, $y' = x+y$, $y(0) = 1$ and find $y(0.3)$ taking step size $h = 0.1$

Sol: Given: $y' = x+y$.

$$\frac{dy}{dx} = \frac{dy}{dx} = x+y$$

WKT; $\frac{dy}{dx} = f(x,y)$

$\therefore f(x,y) = x+y$

$y(0) = 1 \Rightarrow y(x_0) = y_0$

	x_0	x_1	x_2	x_3
x	0	0.1	0.2	0.3
y	1 (y_0)	y_1 ? 1.1	? y_2 1.22	? y_3 1.362

1. $y_1 = y_0 + h\, f(x_0, y_0)$

$= 1 + 0.1\,(0+1)$

$= 1 + 0.1$

$y_1 = 1.1$

2. $y_2 = y_1 + h\, f(x_1, y_1)$

$= 1.1 + 0.1\,(0.1 + 1.1)$

$= 1.1 + 0.1\,(1.2)$

$y_2 = 1.22$

3. $y_3 = y_2 + h\, f(x_2, y_2)$

$= 1.22 + 0.1\,(0.2 + \overset{1.22}{\cancel{1.22}})$

$= \cancel{1.12 + 0.132}\ 1.22 + 0.142$

$y_3 = \cancel{1.252}\ 1.362$

$\therefore y(0.3) = 1.362$

4. Illustrate Euler's method to calculate $y(0.1)$ for $y' = x+y$ with condition $y(0)=1$ & $h=0.05$

Sol:

Given: $y' = x+y$.

$$\frac{dy}{dx} = x+y$$

WKT; $\frac{dy}{dx} = f(x,y)$

$$\therefore f(x,y) = x+y.$$

$y(0)=1 \Rightarrow y(x_0)=y_0$

	x_0	x_1	x_2
x	0	0.05	0.1
y	1 (y_0)	? (y_1) 1.05	? (y_2)

1. $y_1 = y_0 + h\, f(x_0, y_0)$

$= 1 + 0.05\ (0+1)$

$= 1+0.05$

$y_1 = 1.05$

2. $y_2 = y_1 + h\, f(x_1, y_1)$

$= 1.05 + 0.05\ (0.05+1.05)$

$= 1.05 + (0.055)$

$y_2 = 1.105$

5. Given $\frac{dy}{dx} = -y$ & $y(0)=1$ find y at $x = 0.01, 0.02, 0.04$ by using Euler's method.

Sol:

Given; $\frac{dy}{dx} = -y$.

WKT; $\frac{dy}{dx} = f(x,y)$

$$\therefore f(x,y) = -y.$$

$y(0)=1 \Rightarrow y(x_0)=y_0$.

	x_0	x_1	x_2	x_3	x_4
x	0	0.01	0.02	0.03	0.04
y	y_0 1	0.99 y_1	y_2 0.9801	y_3 0.9702	y_4 0.9604

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 1 + 0.01\,(-1)$$
$$= 1 - 0.01$$
$$y_1 = 0.99$$

$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$= 0.99 + 0.01\,(-0.99)$$
$$= 0.99 - 0.0099$$
$$y_2 = 0.9801$$

$$y_3 = y_2 + h\, f(x_2, y_2)$$
$$= 0.9801 + 0.01\,(-0.9801)$$
$$= 0.9801 - 0.009801$$
$$y_3 = 0.9702$$

$$y_4 = y_3 + h\, f(x_3, y_3)$$
$$= 0.9702 + 0.01\,(-0.9702)$$
$$= 0.9702 - 0.009702$$
$$y_4 = 0.96049$$

$$\therefore\ y(0.04) = 0.96049.$$

Type - II: Modified Euler's Method

Consider the 1st order ordinary differential equation is $\frac{dy}{dx} = f(x,y)$ with initial condition. $y(x_0) = y_0$.

1. $y_1^{(0)} = y_0 + h\, f(x_0, y_0)$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$$

2. $y_2^{(0)} = y_1 + h\, f(x_1, y_1)$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)})\right]$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]_{\prime\prime}$$

Problems:

1. Using Modified Euler's method find $y(0.1)$ and $y(0.2)$ given that $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$.

Sol:

Given; $\frac{dy}{dx} = x^2 - y$.

2

WKT; $\frac{dy}{dx} = f(x,y)$

$\therefore f(x,y) = x^2 - y$.

$y(0) = 1 \Rightarrow y(x_0) = y_0$

	x_0	x_1	x_2
x	0	0.1	0.2
y	1		
	y_0	y_1	y_2

1. (i) $y_1^{(0)} = y_0 + h\left[f(x_0, y_0)\right]$

$$= 1 + 0.1\,(0^2 - 1)$$

$$= 1 - 0.1$$

$$\boxed{y_1^{(0)} = 0.9}$$

(ii) $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$

$$= 1 + \frac{0.1}{2}\left[(0-1) + ((0.1)^2 - 0.9)\right]$$

$$\boxed{y_1^{(1)} = 0.9055}$$

(iii) $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$

$= 1 + \frac{0.1}{2}\left[(0-1) + ((0.1)^2 - 0.9055)\right]$

$= 1 + \frac{0.1}{2}\left[-1.8955\right]$

$\boxed{y_1^{(2)} = 0.9052}$ $\therefore$ $\boxed{y(0.1) = 0.9052}$

(2)

(i) $y_2^{(0)} = y_1 + h\left[f(x_1, y_1)\right]$

$= 0.9052 + 0.1\left[(0.1)^2 - 0.9052\right]$

$y_2^{(0)} = 0.81568$

(ii) $y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)})\right]$

$= 0.9052 + \frac{0.1}{2}\left[\left[(0.1)^2 - 0.9052\right] + ((0.2)^2 - 0.81568)\right]$

$= 0.9052 + (-0.83544)$

$y_2^{(1)} = 0.82165$

(iii) $y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$

$= 0.9052 + \frac{0.1}{2}\left[\left[(0.1)^2 - 0.9052\right] + ((0.2)^2 - 0.82165)\right]$

$= 0.9052 - 0.08384$

$y_2^{(2)} = 0.82135$

$\therefore$ $\boxed{y(0.2) = 0.82135}$

2. Solve the differential equation $y' = e^x + y$, $y(0) = 0$ by M.E.M & hence evaluate $y(0.2)$ & $y(0.4)$

Sol: Given; $y' = e^x + y$

$\frac{dy}{dx} = e^x + y.$

WKT; $\frac{dy}{dx} = f(x,y)$

$f(x,y) = e^x + y$

$y(0)=0 \Rightarrow y(x_0) = y_0$

	x_0	x_1	x_2
x	0	0.2	0.4
y	0 (y_0)	y_1	y_2

1. (i) $y_1^{(0)} = y_0 + h\, f(x_0, y_0)$

$= 0 + (0.2)\,[e^0 + 0]$

$y_1^{(0)} = 0.2$

(ii) $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$

$= 0 + \frac{0.2}{2}\left[e^0 + (e^{0.2} + 0.2)\right]$

$y_1^{(1)} = 0.24214$

(iii) $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$

$= 0 + \frac{0.2}{2}\left[e^0 + (e^{0.2} + 0.24214)\right]$

$y_1^{(2)} = 0.24635$

(iv) $y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$

$= 0 + \frac{0.2}{2}\left[e^0 + (e^{0.2} + 0.24635)\right]$

$y_1^{(3)} = 0.24677$

$\therefore \boxed{y(0.2) = 0.24677}$

2. (i) $y_2^{(0)} = y_1 + h\, f(x_1, y_1)$

$= 0.24677 + (0.2)\,[e^{0.2} + 0.24677]$

$y_2^{(0)} = 0.54040$

(ii) $y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f[x_2, y_2^{(0)}]\right]$

$= 0.24677 + \frac{0.2}{2}\left[(e^{0.2} + 0.24677) + (e^{0.4} + 0.54040)\right]$

$y_2^{(1)} = 0.59680$

(iii) $y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$

$= 0.24677 + \frac{0.2}{2}\left[(e^{0.2} + 0.24677) + (e^{0.4} + 0.59680)\right]$

$y_2^{(2)} = 0.60244$

(iv) $y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(2)})\right]$

$= 0.24677 + \frac{0.2}{2}\left[(e^{0.2} + 0.24677) + (e^{0.4} + 0.60244)\right]$

$y_2^{(3)} = 0.60301$

(v) $y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(3)})\right]$

$= 0.24677 + \frac{0.2}{2}\left[(e^{0.2} + 0.24677) + (e^{0.4} + 0.60244)\right]$

$y_2^{(4)} = 0.60307$

$\therefore y(0.4) = 0.60307.$

3. Given $\frac{dy}{dx} = -xy^2$ and $y(0)=2$ find $y(0.2)$ taking $h=0.1$

Sol:

Given; $\frac{dy}{dx} = -xy^2$

WKT; $\frac{dy}{dx} = f(x,y)$

$f(x,y) = -xy^2$

$y(0)=2 \Rightarrow y(x_0)=y_0$

	x_0	x_1	x_2
x	0	0.1	0.2
y	2 (y_0)	y_1	y_2

1. (i) $y_1^{(0)} = y_0 + h\, f(x,y)$

$= 2 + 0.1\left[-0(2)^2\right]$

$y_1^{(0)} = 2$

(ii) $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$

$= 2 + \frac{0.1}{2}\left[-0 + \left[-(0.1)(2)^2\right]\right]$

$$= 2 + \frac{0.1}{2}\left[-0 + (-0.1)(4)\right]$$

$$y_1^{(1)} = 1.98$$

(iii) $$y_1^{(2)} = y_0 + \frac{h}{2}\, f(x_0,y_0) + \left[(-0.1)(1.98)^2\right]$$

$$= 2 + \frac{0.1}{2}\left[-0 + (-0.1)(1.98)^2\right]$$

$$y_1^{(2)} = 1.9803$$

(iv) $$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 2 + \frac{0.1}{2}\left[-0 + (-0.1)(1.9803)^2\right]$$

$$y_1^{(3)} = 1.9803$$

2. (i) $$y_2^{(0)} = y_1 + h\, f(x_1,y_1)$$

$$= 1.9803 + 0.1\left[-(0.1)(1.9803)^2\right]$$

$$y_2^{(0)} = 1.9410$$

(ii) $$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 1.9803 + \frac{0.1}{2}\left[-(0.1)(1.9803)^2 + -(0.2)(1.9410)^2\right]$$

$$y_2^{(1)} = 1.9230$$

(iii) $$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.9803 + \frac{0.1}{2}\left[-(0.1)(1.9803)^2 + (-0.2)(1.9230)^2\right]$$

$$y_2^{(2)} = 1.9237.$$

$$\therefore\ y(0.2) = 1.9237$$

4. Given $y' = \log_{10}(x+y)$ and $y(0)=2$ find $y(0.2)$, $y(0.4)$, $h=0.2$

Sol:

$$y' = \log(x+y)$$

$$\frac{dy}{dx} = \log(x+y)$$

WKT; $$\frac{dy}{dx} = f(x,y)$$

$$f(x,y) = \log_{10}(x+y)$$

$$y(0)=2 \Rightarrow y(x_0) = y_0$$

	x_0	x_1	x_2
x	0	0.2	0.4
y	2		
	y_0	y_1	y_2

1\. (i) $y_1^{(0)} = y_0 + h\, f(x_0, y_0)$

$$= 2 + 0.2\, f(0,2)$$

$$= 2 + 0.2\,[\log(0+2)]$$

$$y_1^{(0)} = 2.0602$$

(ii) $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$

$$[f(0,2) + f(0.2,\ 2.0602)]$$

$$= 2 + \frac{0.2}{2}\left[\log(0+2) + \log(0.2 + 2.0602)\right]$$

$$y_1^{(1)} = \cancel{2+}\ 2.0655$$

(iii) $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f[x_1, y_1^{(1)}]\right]$

$$= 2 + \frac{0.2}{2}\left[f(0,2) + f(0.2,\ 2.0655)\right]$$

$$= 2 + \frac{0.2}{2}\left[\log(0+2) + \log(0.2 + 2.0655)\right]$$

$$y_1^{(2)} = 2.0656$$

$$\boxed{y(0.2) = 2.0656}$$

2\. $y_2^{(0)} = y_1 + h\ f(x_1, y_1)$

$$= 2.0656 + 0.2\,[f(0.2, 2.0656)]$$

$$= 2.0656 + 0.2\,[\log(0.2 + 2.0656)]$$

$$y_2^{(0)} = 2.1366$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f\{x_2, y_2^{(0)}\}\right]$$

$$= 2.0656 + \frac{0.2}{2}\left[f(0.2, 2.0656) + f(0.4, 2.1366)\right]$$

$$= 2.0656 + \frac{0.2}{2}\left[\log(0.2+2.0656) + \log(0.4+2.1366)\right]$$

$$y_2^{(1)} = 2.1415$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 2.0656 + \frac{0.2}{2}\left[f(0.2, 2.0656) + f(0.4, 2.1415)\right]$$

$$= 2.0656 + \frac{0.2}{2}\left[\log(0.2+2.0656) + \log(0.4+2.1415)\right]$$

$$y_2^{2} = 2.1416$$

$$\boxed{y(0.4) = 2.1416}$$

Type-III - RK Method:

RK Method of 1st Order:

1. $y_1 = y_0 + k_1$

   $k_1 = h\, f(x_0, y_0)$

2. $y_2 = y_1 + k_1$

   $k_1 = h\, f(x_1, y_1)$

RK Method of 2nd Order:

1. $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

   $k_1 = h\, f(x_0, y_0)$

   $k_2 = h\, f(x_0 + h, y_0 + k_1)$

2. $y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

   $k_1 = h\, f(x_1, y_1)$

   $k_2 = h\, f(x_1 + h, y_1 + k_1)$

RK Method of 3rd Order:

1. $y_1 = y_0 + \frac{1}{6}\left[k_1 + 4k_2 + k_3\right]$

$$k_1 = h\ f(x_0, y_0)$$

$$k_2 = h\ f\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right]$$

$$k_3 = h\ f\left[x_0 + h,\ y_0 + 2k_2 - k_1\right]$$

RK Method of 4th Method:

1. $y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$$k_1 = h\ f(x_0, y_0)$$

$$k_2 = h\ f\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right]$$

$$k_3 = h\ f\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right]$$

$$k_4 = h\ f\left[x_0 + h,\ y_0 + k_3\right]$$

1. Using RK 1st order method find $y(2.5)$ from $y' = \frac{x+y}{x}$, $y(2) = 2$ using $h = 0.25$

Sol:

Given; $\frac{dy}{dx} = \frac{x+y}{x}$

WKT; $\frac{dy}{dx} = f(x,y)$

$\therefore\ f(x,y) = \frac{x+y}{x}$

$y(2) = 2 \Rightarrow y(x_0) = y_0$

	x_0	x_1	x_2
x	2	2.25	2.5
y	2 (y_0)	y_1	y_2

RK 1st Order:

1. $y_1 = y_0 + k_1$

$$k_1 = h\ f(x_0, y_0)$$

$$k_1 = 0.25\ f(2, 2)$$

$$k_1 = 0.25\left(\frac{2+2}{2}\right)$$

$$k_1 = 0.5$$

$$y_1 = y_0 + k_1$$

$$y_1 = 2 + 0.5$$

$$\boxed{y_1 = 2.5}$$

2. $y_2 = y_1 + k_1$

$$k_1 = h\ f(x_1, y_1)$$

$$k_1 = 0.25\ f\ (2.25, 2.5)$$

$$k_1 = 0.25\left[\frac{2.25 + 2.5}{2.25}\right]$$

$$k_1 = 0.5277$$

$$y_2 = y_1 + k_1$$

$$y_2 = 2.5 + 0.5277$$

$$\boxed{y_2 = 3.0277}$$

2. Solve $\frac{dy}{dx} = x + y$, $y(0) = 1$ find $y(0.1)$, $y(0.2)$ by using RK-1st order

Sol:

Given; $\frac{dy}{dx} = x + y$

WKT; $\frac{dy}{dx} = f(x, y)$

$$f(x,y) = x + y$$

$y(0) = 1 \Rightarrow y(x_0) = y_0$

	x_0	x_1	x_2
x	0	0.1	0.2
y	1 (y_0)	y_1	y_2

RK 1st order:

1. $y_1 = y_0 + k_1$

$$k_1 = h\ f(x_0, y_0)$$

$$k_1 = 0.1\ f\ (0, 1)$$

$$k_1 = 0.1\ (0 + 1)$$

$$k_1 = 0.1$$

$y_1 = y_0 + k_1$

$y_1 = 1 + 0.1$

$\boxed{y_1 = 1.1}$

2. $y_2 = y_1 + k_1$

$k_1 = h\ f(x_1, y_1)$

$k_1 = 0.1\ f(0.1, 1.1)$

$k_1 = 0.1\ (0.1 + 1.1)$

$k_1 = 0.12$

$y_2 = 1.1 + 0.12$

$\boxed{y_2 = 1.22}$

RK Method of second order:

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$$

$$k_1 = h\ f(x_0, y_0)$$

$$k_2 = h\ f(x_0 + h, y_0 + k_1).$$

1. Given that $y' + y = 0$ & $y(0) = 1$ find y value at $x = 0.1$ & 0.2 by using RK- second order.

Sol: Given; $y' = -y$

$f(x, y) = -y$

$y(0) = 1 \Rightarrow y(x_0) = y_0$

i.e. $x_0 = 0,\ y_0 = 1$

	x_0	x_1	x_2
x	0	0.1	0.2
y	1 (y_0)	y_1 0.905	y_2

$h = 0.1$

RK- 2nd order:

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$$

$k_1 = h\ f(x_0, y_0)$

$k_1 = 0.1\ f(0, 1)$

$k_1 = 0.1\ (-1) \Rightarrow k_1 = -0.1$

$K_2 = h\ f(x_0+h,\ y_0+k_1)$

$K_2 = 0.1\ f(0+0.1,\ 1-0.1)$

$K_2 = 0.1\ f(0.1, +0.9)$

$K_2 = 0.1(-0.9) \Rightarrow \quad K_2 = -0.09.$

$y_1 = y_0 + \frac{1}{2}\ [K_1+K_2]$

$y_1 = 1 + \frac{1}{2}\ [-0.1-0.09]$

$y_1 = 0.905$

2. $y_2 = y_1 + \frac{1}{2}\ [K_1+K_2]$

$K_1 = h\ f(x_1, y_1)$

$K_1 = 0.1\ f(0.1, 0.905)$

$K_1 = 0.1\ (-0.905) \Rightarrow \quad K_1 = -0.0905.$

$K_2 = h\ f(x_1+h,\ y_1+k_1)$

$K_2 = 0.1\ f(0.1+0.1,\ 0.905-0.0905)$

$K_2 = 0.1\ (0.2,\ 0.8145)$

$K_2 = 0.1\ (-0.8145) \Rightarrow \quad K_2 = -0.08145.$

$y_2 = y_1 + \frac{1}{2}\ [K_1+K_2]$

$y_2 = 0.905 + \frac{1}{2}\ (-0.0905 - 0.08145)$

$y_2 = 0.819025$

2. Using RK- second order find $y(0.1)$ & $y(0.2)$ where $\frac{dy}{dx} = x+y$ & $y(0) = 1$

Sol:

Given; $\frac{dy}{dx} = x+y.$

$f(x,y) = x+y.$

$y(0) = 1 \Rightarrow \quad y(x_0) = y_0$

i.e. $x_0 = 0,\ y_0 = 1$

$h = 0.1$

	x_0	x_1	x_2
x	0	0.1	0.2
y	1 (y_0)	y_1	y_2

RK - 2^{nd} order:

1. $y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

$k_1 = h\ f(x_0, y_0)$

$k_1 = 0.1\ f(0,1)$

$k_1 = 0.1\ (1) \Rightarrow k_1 = 0.1$

$k_2 = h\ f[x_0 + h,\ y_0 + k_1]$

$k_2 = 0.1\ f[0 + 0.1,\ 1 + 0.1]$

$k_2 = 0.1\ f[0.1, 1.1] \Rightarrow k_2 = 0.1\ (0.1 + 1.1) \Rightarrow k_2 = 0.12$

$y_1 = 1 + \frac{1}{2}[0.1 + 0.12]$

$\boxed{y_1 = 1.11}$

2. $y_2 = y_1 + \frac{1}{2}[k_1 + k_2]$

$k_1 = h\ f(x_1, y_1)$

$k_1 = 0.1\ f(0.1, 1.11)$

$k_1 = 0.1\ (0.1 + 1.11) \Rightarrow k_1 = 0.121$

$k_2 = h\ f[x_1 + h,\ y_1 + k_1]$

$k_2 = 0.1\ f[0.1 + 0.1,\ 1.11 + 0.121]$

$k_2 = 0.1\ f[0.2,\ 1.231]$

$k_2 = 0.1\ [0.2 + 1.231] \Rightarrow k_2 = 0.1431$

$y_2 = y_1 + \frac{1}{2}[k_1 + k_2]$

$y_2 = 1.1 + \frac{1}{2}[0.121 + 0.1431]$

$\boxed{y_2 = 1.23205}$

RK-Method of 3rd Order:

$$y_1 = y_0 + \frac{1}{6}\left[K_1 + 4K_2 + K_3\right]$$

$$K_1 = h\ f(x_0, y_0)$$

$$K_2 = h\ f\left[x_0 + \frac{h}{2},\ y_0 + K_1/2\right]$$

$$K_3 = h\ f\left[x_0 + h,\ y_0 + 2K_2 - K_1\right]$$

1. Given $y' = x - y$ and $y(1) = 0.4$ find $y(1.2)$ by using RK-3rd order.

Sol: Given; $y' = x - y$

$$\frac{dy}{dx} = x - y$$

$$f(x, y) = x - y.$$

$y(x_0) = y_0 \Rightarrow y(1) = 0.4.$

$x_0 = 1,\ y_0 = 0.4$

	x_0	x_1
x	1	1.2
y	0.4 (y_0)	y_1

$h = 0.2$

RK 3rd order:

$$y_1 = y_0 + \frac{1}{6}\left[K_1 + 4K_2 + K_3\right]$$

$$K_1 = h\ f(x_0, y_0)$$

$$K_1 = 0.2\ f(1, 0.4)$$

$$K_1 = 0.2\ (1 - 0.4) \Rightarrow K_1 = 0.12$$

$$K_2 = h\ f\left[x_0 + h/2,\ y_0 + K_1/2\right]$$

$$K_2 = 0.2\ f\left[1 + \frac{0.2}{2},\ 0.4 + \frac{0.12}{2}\right]$$

$$K_2 = 0.2\ f(1.1, 0.46)$$

$$K_2 = 0.2\ (1.1 - 0.46) \Rightarrow K_2 = 0.128$$

$$K_3 = h\ f\left[x_0 + h,\ y_0 + 2K_2 - K_1\right]$$

$$K_3 = 0.2\ f\left[1 + 0.2,\ 0.4 + 2(0.128) - 0.12\right]$$

$$K_3 = 0.2\ f(1.2, 0.536)$$

$$K_3 = 0.2\ (1.2 - 0.536) \Rightarrow K_3 = 0.1328$$

$$y_1 = y_0 + \frac{1}{6}\left[K_1 + 4K_2 + K_3\right]$$

$$y_1 = 0.4 + \frac{1}{6}\left[0.12 + 4(0.128) + 0.1328\right]$$

$$y_1 = 0.4 + \frac{1}{6}(0.7648)$$

$$\boxed{y_1 = 0.52746.}$$

Order:

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_2 + k_4\right]$$

$$k_1 = h\ f(x_0, y_0)$$

$$k_2 = h\ f\left[x_0 + h/2,\ y_0 + k_1/2\right]$$

$$k_3 = h\ f\left[x_0 + h/2,\ y_0 + k_2/2\right]$$

$$k_4 = h\ f\left[x_0 + h,\ y_0 + k_3\right].$$

1. Using RK of 4th order to find value of y when $x = 1.2$, in steps of 0.1, given that $\frac{dy}{dx} = x^2 + y^2$, $y(1) = 1.5$.

Sol:

Given; $\frac{dy}{dx} = x^2 + y^2$

$$f(x, y) = x^2 + y^2$$

1. $y(1) = 1.5 \Rightarrow y(x_0) = y_0$.

$x_0 = 1$, $y_0 = 1.5$ & $h = 0.1$

	x_0	x_1	x_2
x	1	1.1	1.2
y	1.5 (y_0)	y_1 1.895518	y_2

RK-4th order:

(i) $$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_2 + k_4\right]$$

$$k_1 = h\ f(x_0, y_0)$$

$$k_1 = 0.1\left[(1)^2 + (1.5)^2\right]$$

$$k_1 = 0.325$$

$$k_2 = h\ f\left[x_0 + h/2,\ y_0 + k_1/2\right]$$

$$k_2 = 0.1\ f\left[1 + \frac{0.1}{2},\ 1.5 + \frac{0.325}{2}\right]$$

$$k_2 = 0.1\ f\left[1.05,\ 1.6625\right]$$

$$k_2 = 0.1\left[(1.05)^2 + (1.6625)^2\right]$$

$$k_2 = \cancel{0.301390}\ \ 0.38664$$

$k_3 = h \; f\left[x_0+h/2,\ y_0+k_2/2\right]$

$k_3 = 0.1 \; f\left[1.05,\ 1.5+\frac{\cancel{0.301390}\ 0.38664}{2}\right]$

$k_3 = 0.1 \; f(1.05,\ 1.69332)$

$k_3 = 0.1\left[(1.05)^2+(1.69332)^2\right]$

$k_3 = 0.39698$

$k_4 = h \; f\left[x_0+h,\ y_0+k_3\right]$

in steps of 0.1,

$k_4 = 0.1 \; f\left[1+0.1,\ 1.5+0.39698\right]$

$k_4 = 0.1 \; f(1.1,\ 1.89698)$

$k_4 = 0.1\left[(1.1)^2+(1.89698)^2\right]$

$k_4 = 0.48085$

Now; $y_1 = 1.5+\frac{1}{6}\left[0.325+2(0.38664)+2(0.39698)+0.48085\right]$

$y_1 = 1.5+\frac{1}{6}(2.37309)$

$y_1 = 1.895515$

(ii) $y_2 = y_1+\frac{1}{6}\left[k_1+2k_2+2k_3+k_4\right]$

$k_1 = h \; f(x_1,y_1)$

$k_1 = 0.1 \; f(1.1,1.89551)$

$k_1 = 0.1\left[(1.1)^2+(1.89551)^2\right] \Rightarrow k_1 = 0.48029$

$k_2 = h \; f\left[x_1+\frac{h}{2},\ y_1+\frac{k_1}{2}\right]$

$k_2 = 0.1 \; f\left[1.1+\frac{0.1}{2},\ 1.89551+\frac{0.48029}{2}\right]$

$k_2 = 0.1 \; f\left[1.15,\ 2.13565\right]$

$k_2 = 0.1\left[(1.15)^2+(2.13565)^2\right] \Rightarrow k_2 = 0.58835$

$k_3 = h \; f\left[x_1+\frac{h}{2},\ y_1+\frac{k_2}{2}\right]$

$k_3 = 0.1 \; f\left[1.15,\ 1.89551+\frac{0.58835}{2}\right]$

$k_3 = 0.1 \; f(1.15,\ 2.18968)$

$k_3 = 0.1\left[(1.15)^2+(2.18968)^2\right] \Rightarrow k_3 = 0.61171$

$K_4 = h\ f\left[x_1+h,\ y_1+K_3\right]$

$K_4 = 0.1\ f\ (1.1+0.1,\ 1.89551 + 0.61171)$

$K_4 = 0.1\ \ (1.2,\ 2.50722)$

$K_4 = 0.1\ \left[(1.2)^2 + (2.50722)^2\right] \Rightarrow K_4 = 0.77261$

$y_2 = 1.89551 + \frac{1}{6}\left[0.48029 + 2\,(0.58835) + 2\,(0.61171) + 0.77261\right]$

$y_2 = 1.89551 + \frac{1}{6}\left[3.653308\right]$

$y_2 = 2.50439$

2. Using RK-method find $y(0.1)$, $y(0.2)$ for $\frac{dy}{dx} = x+y^2$; $y(0)=1$

Sol:

Given; $\frac{dy}{dx} = x+y^2$

$f(x,y) = x+y^2$

$y(0)=1 \Rightarrow y(x_0)=y_0$

$x_0=0,\ y_0=1,\ \ h=0.1$

	x_0	x_1	x_2
x	0	0.1	0.2
y	1		
	y_0	y_1	y_2
		1.11648	

RK - 4th order:

(i) $y_1 = y_0 + \frac{1}{6}\left[K_1+2K_2+3K_3+K_4\right]$

$K_1 = h\ f\ (x_0,y_0)$

$K_1 = 0.1\ f\ (0,1)$

$K_1 = 0.1\ (0+1^2) \Rightarrow K_1 = 0.1$

$K_2 = h\ f\left[x_0 + h/2,\ y_0 + \frac{K_1}{2}\right]$

$K_2 = 0.1\ f\left[0 + \frac{0.1}{2},\ 1.0 + \frac{0.1}{2}\right]$

$K_2 = 0.1\ f\ (0.05,\ 1.05)$

$K_2 = 0.1\ \left[0.05 + (1.05)^2\right] \Rightarrow K_2 = \cancel{0.24525}\ 0.11525$

$) + 0.77861]$

$$k_3 = h\, f\left[x_0 + h/2,\ y_0 + k_2/2\right]$$

$$k_3 = 0.1\, f\left[0 + \frac{0.1}{2},\ 1 + \frac{0.11525}{2}\right]$$

$$k_3 = 0.1\, f\,(0.05,\ 1.05762)$$

$$k_3 = 0.1\left[(0.05) + (1.05762)^2\right] \Rightarrow k_3 = 0.11685$$

$$k_4 = h\, f\left[x_0 + h,\ y_0 + k_3\right]$$

$$k_4 = 0.1\left[0 + 0.1,\ 1 + 0.11685\right]$$

$$k_4 = 0.1\left[0.1,\ 1.11685\right]$$

$$k_4 = 0.13473.$$

$y(0)=1$

$$y_1 = 1 + \frac{1}{6}\left[0.1 + 2(0.11525) + 2(0.11685) + 0.13473\right]$$

$$y_1 = 1 + \frac{1}{6}\,(0.69893)$$

$$y_1 = 1.11648.$$

(ii) $$y_2 = y_1 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$k_1 = h\, f\,(x_1, y_1)$$

$$k_1 = 0.1\, f\left[0.1,\ 1.11648\right]$$

$$k_1 = 0.1\left[(0.1) + (1.11648)^2\right] \Rightarrow k_1 = 0.13465$$

$$k_2 = h\, f\left[x_1 + h/2,\ y_1 + \frac{k_1}{2}\right]$$

$$k_2 = 0.1\, f\left[0.1 + \frac{0.1}{2},\ 1.11648 + \frac{0.13465}{2}\right]$$

$$k_2 = 0.1\, f\left[0.15,\ 1.18380\right]$$

$$k_2 = 0.1\left[0.15 + (1.18380)^2\right] \Rightarrow k_2 = 0.15513$$

$$k_3 = h\, f\left[x_1 + h/2,\ y_1 + \frac{k_2}{2}\right]$$

$$k_3 = 0.1\, f\left[0.15,\ 1.11648 + \frac{0.15513}{2}\right]$$

$$k_3 = 0.1\left[(0.15) + (1.19404)^2\right]$$

$$k_3 = 0.15757$$

$K_4 = h\ f[x_1+h,\ y_1+K_3]$

$K_4 = 0.1\ f[0.1+0.1,\ 1.11648+0.15757]$

$K_4 = 0.1\ [0.2+(1.27405)^2]$

$K_4 = 0.18232$

$y_2 = 1.11648 + \frac{1}{6}\ [0.13465 + 2(0.15513) + 2(0.15757) + 0.18232]$

$y_2 = 1.11648 + \frac{1}{6}\ \cancel{(0.62967)}\ \cancel{(0.62967)(4)}\ (0.86237)$

~~$y_2 = 1.27354$~~ $y_2 =)$

$y_2 = 1.11648 + \frac{1}{6}(0.94237)$

$y_2 = 1.27354$

3. Solve $y' = x^2y+y$, $y(0)=1$ by 4th RK to find value of $y(0.1)$ & $y(0.2)$

Sol:

Given; $\frac{dy}{dx} = x^2y+y$

$f(x,y) = x^2y+y$

$y(0)=1 \Rightarrow y(x_0)=y_0$

$x_0=0;\ y_0=1;\ h=0.1$

	x_0	x_1	x_2
x	0	0.1	0.2
y	1 (y_0)	y_1 1.10553	y_2

RK - 4th order:

(i) $y_1 = y_0 + \frac{1}{6}\ [K_1+2K_2+2K_3+K_4]$

$K_1 = h\ f(x_0, y_0)$

$K_1 = 0.1\ f(0,1)$

$K_1 = 0.1\ [0^2(1)+1] \Rightarrow K_1 = 0.1$

$K_2 = h\ f[x_0+h/2,\ y_0+\frac{K_1}{2}]$

$K_2 = 0.1\ f[0+\frac{0.1}{2},\ 1+\frac{0.1}{2}]$

$K_2 = 0.1\ f(0.05,\ 1.05)$

$k_2 = 0.1 \; [(0.05)^2 (1.05) + 1.05]$

$k_2 = \cancel{1.052} \; 0.10526.$

$k_3 = h \; f\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right]$

$k_3 = 0.1 \; f\left[0.05, \; 1 + \frac{0.10526}{2}\right]$

$k_3 = 0.1 \; f(0.05, \; 1.05263)$

$k_3 = 0.1 \; [(0.05)^2 (1.05263) + (1.05263)]$

$k_3 = 0.10552$

$k_4 = h \; f[x_0 + h, y_0 + k_3]$

$k_4 = 0.1 \; f[0 + 0.1, \; 1 + 0.10552]$

$k_4 = 0.1 \; f(0.1, \; 1.10552)$

$k_4 = 0.1 \; [(0.1)^2 (1.10552) + 1.10552]$

$k_4 = 0.11165.$

$y_1 = y_0 + \frac{1}{6} \; [k_1 + 2k_2 + 2k_3 + k_4]$

$y_1 = 1 + \frac{1}{6} \; [0.1 + 2(0.10526) + 2(0.10552) + 0.11165]$

$y_1 = 1.10553$

(ii) $y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$k_1 = h \; f(x_1, y_1)$

$k_1 = 0.1 \; f(0.1, \; 1.10553)$

$k_1 = 0.1 \; [(0.1)^2 (1.10553) + 1.10553]$

$k_1 = 0.11165$

$k_2 = h \; f(x_0 + h/2, \; y_1 + \frac{k_1}{2})$

$k_2 = 0.1 \; f\left[0.1 + \frac{0.1}{2}, \; 1.10553 + \frac{0.11165}{2}\right]$

$k_2 = 0.1 \; f(0.15, \; 1.161356)$

$k_2 = 0.1 \; [(0.15)^2 (1.161356) + 1.161356]$

$$K_3 = 0.1 \quad f\left(0.15,\ 1.10553 + \frac{0.11874}{2}\right)$$

$$K_3 = 0.1 \quad f(0.15,\ 1.1649)$$

$$K_3 = 0.1 \quad [(0.15)^2(1.1649) + 1.1649]$$

$$K_3 = 0.11911$$

$$K_4 = 0.1 \quad f[x_1 + h,\ y_1 + k_3]$$

$$K_4 = 0.1 \quad f[0.1 + 0.1,\ 1.10553 + 0.11911]$$

$$K_4 = 0.1 \quad f(0.2,\ 1.22464)$$

$$K_4 = 0.1 \quad [(0.2)^2(1.22464) + 1.22464]$$

$$K_4 = 0.12736.$$

$$y_2 = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$y_2 = 1.10553 + \frac{1}{6}[0.11165 + 2(0.11874) + 2(0.11911) + 0.12736]$$

$$y_2 = 1.22464$$

4. Solve $\frac{dy}{dx} = xy$ and $y(0) = 1$. find $y(0.2)$ taking $h = 0.1$ by using 4^{th} order.

Sol:

Given $\frac{dy}{dx} = xy$

$$y(x_0) = y_0 \Rightarrow y(0) = 1$$

$x_0 = 0,\ y_0 = 1,\ h = 0.1$

	x_0	x_1	x_2
x	0	0.1	0.2
y	1 (y_0)	y_1 1.00501	y_2

RK – 4^{th} order:

(i) $K_1 = h\,f(x_0, y_0)$

$$K_1 = 0.1\ f(0,1)$$

$$K_1 = 0.1\ [0(1)]$$

$$K_1 = 0$$

$k_2 = h\ f\left[x_0 + h/2,\ y_0 + \frac{k_1}{2}\right]$

$k_2 = 0.1\ f\left[0 + \frac{0.1}{2},\ 1 + \frac{0}{2}\right]$

$k_2 = 0.1\ f\ [0.05, 1]$

$k_2 = 0.1\ (0.05) \Rightarrow k_2 = 0.005$

$k_3 = h\ f\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right]$

$k_3 = 0.1\ f\left[0.05,\ 1 + \frac{0.005}{2}\right]$

$k_3 = 0.1\left[(0.05)(1.0025)\right]$

$k_3 = 0.00501$

$k_4 = h\ f\ [x_0 + h,\ y_0 + k_3]$

$k_4 = 0.1\ f\ [0 + 0.1,\ 1 + 0.00501]$

$k_4 = 0.1\left[(0.1)(1.00501)\right]$

$k_4 = 0.01005$

$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$y_1 = 1 + \frac{1}{6}\left[0 + 2(0.005) + 2(0.00501) + 0.01005\right]$

$y_1 = 1.00501$

(ii) $k_1 = h\ f\ (x_1, y_1)$

$k_1 = 0.1\ (0.1, 1.00501)$

$k_1 = 0.1\left[(0.1)(1.00501)\right]$

$k_1 = 0.01005$

$k_2 = h\ f\left[x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right]$

$k_2 = 0.1\ f\left[0.1 + \frac{0.1}{2},\ 1.00501 + \frac{0.01005}{2}\right]$

$k_2 = 0.1\ f\ [0.15,\ 1.01003]$

$k_2 = 0.1\left[(0.15)(1.01003)\right]$

$k_2 = 0.01515$

$$k_3 = 0.1 \quad f\left[(0.15),\ 1.00501 + \frac{0.01515}{2}\right]$$

$$k_3 = 0.1\ [(0.15)\ (1.01258)]$$

$$k_3 = 0.01518.$$

$$k_4 = h\ f[x_1+h,\ y_1+k_3]$$

$$k_4 = 0.1\ f\ [0.1+0.1,\ 1.00501+0.01518]$$

$$k_4 = 0.1\ [(0.2)\ (1.02019)]$$

$$k_4 = 0.02040$$

$$y_2 = 1.00501 + \frac{1}{6}\ [0.01005 + 2(0.01515) + 2(0.01518) + 0.02040]$$

$$y_2 = 1.02019.$$

Taylor series Method:

Consider the 1st order differential equ^n $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$. Here we have to find the values of y by using taylor series Method.

1. $y(x_1) = y_1 = y(x_0) + y'(x_0)\frac{(x-x_0)}{\lfloor 1} + y''(x_0)\frac{(x-x_0)^2}{2!} + y'''(x_0)\frac{(x-x_0)^3}{\lfloor 3!} + \dots$

2. $y(x_2) = y_2 = y(x_1) + y'(x_1)\frac{(x-x_1)}{\lfloor 1} + y''(x_1)\frac{(x-x_1)^2}{\lfloor 2} + y'''(x_1)\frac{(x-x_1)^3}{\lfloor 3!} + \dots$

Problems:

1. Using Taylor series method, solve the problem for $x = 0.1$ given $y' = x^2 - y$, $y(0) = 1$.

Sol:

Given; $y' = x^2 - y \longrightarrow (1)$

$y(0) = 1 \Rightarrow y(x_0) = y_0$

$x_0 = 0,\ y_0 = 1$

	x_0	x_1
x	0	0.1
y	1	0.90516
	y_0	y_1

using Taylor series:

$$y(x_1) = y_1 = y(x_0) + y'(x_0)\frac{(x-x_0)}{\lfloor 1} + y''(x_0)\frac{(x-x_0)^2}{\lfloor 2} + y'''(x_0)\frac{(x-x_0)^3}{\lfloor 3!} + \cdots$$

1. $y(x_0) = y_0 = 1$; $x = 0.1$

2. $y' = x^2 - y \Rightarrow y'(x_0) = y_0' = x_0^2 - y_0 = 0^2 - 1 = -1$

3. $y'' = 2x - y' \Rightarrow y''(x_0) = y_0'' = 2x_0 - y_0' = 2(0) - (-1) = 1$

4. $y''' = 2 - y'' \Rightarrow y'''(x_0) = y_0''' = 2 - y_0'' = 2 - 2(1) = 1$

$$y_1 = 1 + (-1)\frac{(0.1-0)}{1} + 1\,\frac{(0.1-0)^2}{2} + 1\cdot\frac{(0.1-0)^3}{6}$$

$y_1 = 0.90516$

2. Solve: $y' = x^2 - y$, $y(0) = 1$ by using Taylor series, hence evaluate $y(0.2)$ & $y(0.4)$

sol: Given; $y' = x^2 - y \longrightarrow (1)$

$y(0) = 1 \Rightarrow y(x_0) = y_0$

$x_0 = 0,\ y_0 = 1$

	x_0	x_1	x_2
x	0	0.2	0.4
y	1 (y_0)	y_1	y_2

Using Taylor series:

(i) $$y(x_1) = y_1 = y(x_0) + y'(x_0)\frac{(x-x_0)}{\lfloor 1} + y''(x_0)\frac{(x-x_0)^2}{\lfloor 2} + y'''(x_0)\frac{(x-x_0)^3}{\lfloor 3} + \cdots$$

1. $y(x_0) = y_0 = 1$; $x_1 = 0.2$

2. $y' = x^2 - y \Rightarrow y'(x_0) = y_0' = x_0^2 - y_0 = 0^2 - 1 = -1$

3. $y'' = 2x - y' \Rightarrow y''(x_0) = y_0'' = 2x_0 - y_0' = 2(0) - (-1) = 1$

4. $y''' = 2 - y'' \Rightarrow y'''(x_0) = y_0''' = 2 - y_0'' = 2 - 1 = 1$

$$y_1 = 1 + (-1)\frac{(0.2-0)}{1} + 1\,\frac{(0.2-0)^2}{2} + 1\,\frac{(0.2-0)^3}{6} + \cdots$$

$y_1 = 0.82133$

(ii) $y_2 = y(x_1) + y'(x_1)\dfrac{(x-x_1)}{\lfloor 1} + y''(x_1)\dfrac{(x-x_1)^2}{\lfloor 2} + y'''(x_1)\dfrac{(x-x_1)^3}{\lfloor 3} + \ldots$

1. $y(x_1) = y_1 = 0.82133$

2. $y' = x^2 - y \Rightarrow y'(x_1) = y_1' = x_1^2 - y_1 = (0.2)^2 - 0.82133 = -0.78133$

3. $y'' = 2x - y' \Rightarrow y''(x_1) = y_1'' = 2x_1 - y_1' = 2(0.2) - (-0.78133)$

$= 1.18133$

4. $y''' = 2 - y'' \Rightarrow y'''(x_1) = y_1''' = 2 - y_1'' = 2 - 1.18133 = 0.81867.$

$$y_2 = 0.82133 - 0.78133\frac{(0.4-0.2)}{1} + 1.18133\frac{(0.4-0.2)^2}{2} + 0.81867\frac{(0.4-0.2)^3}{6}$$

$y_2 = 0.68978.$

3. Using Taylor, find approx value of y at $x = 0.2$ for $y' - 2y = 3e^x$, $y(0) = 0$.

Sol:

Given; $y' - 2y = 3e^x$

$y' = 3e^x + 2y.$

$y(0) = 0 \Rightarrow y(x_0) = y_0$

$x_0 = 0,\ y_0 = 0,\ x = 0.2$

	x_0	x_1
x	0	0.2
y	0 (y_0)	y_1

Using Taylor series:

(i) $y_1 = y(x_0) + y'(x_0)\dfrac{(x-x_0)}{\lfloor 1} + y''(x_0)\dfrac{(x-x_0)^2}{\lfloor 2} + y'''(x_0)\dfrac{(x-x_0)^3}{\lfloor 3} + \ldots$

1. $y(x_0) = y_0 = 0.$

2. $y' = 3e^x + 2y \Rightarrow y'(x_0) = y_0' = 3e^{x_0} + 2y_0 = 3e^0 + 2(0) = 3$

3. $y'' = 3e^x + 2y' \Rightarrow y''(x_0) = y_0'' = 3e^{x_0} + 2y_0' = 3e^0 + 2(3) = 9$

4. $y''' = 3e^x + 2y'' \Rightarrow y'''(x_0) = y_0''' = 3e^{x_0} + 2y_0'' = 3e^0 + 2(9) = 21$

$+ y'''(x_1) \frac{(x-x_1)^3}{\lfloor 3} + \ldots$

$)^2 - 0.82133 = -0.78133$

$0.2) - (-0.78133)$

18133

$1.18133 = 0.81867.$

$)^2 +$

$867 \frac{(0.4-0.2)^3}{6}$

2 for $y' - 2y = 3e^x,$

$y'''(x_0) \frac{(x-x_0)^3}{\lfloor 3} + \ldots$

$3e^0 + 2(0) = 3$

$= 3e^0 + 2(3) = 9$

$= 3e^0 + 2(9) = 21$

$$y_1 = 0 + 3 \frac{(0.2-0)}{1} + 9 \frac{(0.2-0)^2}{2} + 21 \frac{(0.2-0)^3}{8}$$

$$y_1 = 0.808$$

4. Solve $y' = 3x + y^2$, $y(0) = 1$ using Taylor method, find approx value of $y(0.1)$ & $y(0.2)$

Sol: Given, $y' = 3x + y^2$

$y(0) = 1 \Rightarrow y(x_0) = y_0.$

$x_0 = 0,\ y_0 = 1$

	x_0	x_1	x_2
x	0	0.1	0.2
y	1	1.127	
	y_0	y_1	y_2.

Using Taylor series:

(i) $$y_1 = y(x_0) + y'(x_0) \frac{(x-x_0)}{\lfloor 1} + y''(x_0) \frac{(x-x_0)^2}{\lfloor 2} + y'''(x_0) \frac{(x-x_0)^3}{\lfloor 3} + \ldots$$

1. $y(x_0) = y_0 = 1$

2. $y' = 3x + y^2 \Rightarrow y'(x_0) = y_0' = 3x_0 + y_0^2 = 3(0) + 1 = 1$

3. $y'' = 3\cancel{x} + 2y \cdot y' \Rightarrow y''(x_0) = y_0'' = 3\cancel{x_0} + 2 \cdot y_0 \cdot y_0' = 3\cancel{0} + 2(1)\cdot(1) = 2+3 = 5$

4. $y''' = 0 + 2y'y' + 2yy'' \Rightarrow y_0''' = 2y_0' y_0' + 2y_0 y_0'' = 2(1)(1) + 2(1)(5) = 12$

$$y_1 = 1 + 1 \frac{(0.1-0)}{1} + 5 \frac{(0.1-0)^2}{2} + 12 \frac{(0.1-0)^3}{6}$$

$$y_1 = 1.127$$

(ii) $$y_2 = y(x_1) + y'(x_1) \frac{(x-x_1)}{\lfloor 1} + y''(x_1) \frac{(x-x_1)^2}{\lfloor 2} + y'''(x_1) \frac{(x-x_1)^3}{\lfloor 3} + \ldots$$

1. $y(x_1) = y_1 = 1.127$

2. $y' = 3x + y^2 \Rightarrow y'(x_1) = y_1' = 3x_1 + y_1^2 = 3(0.1) + (1.127)^2 = 1.57012$

3. $y'' = 3 + 2yy' \Rightarrow y''(x_1) = y_1'' = 3 + 2y_1 y_1' = 3 + 2(1.127)(1.57012) = 6.53905$

4. $y''' = 2y'y' + 2yy'' \Rightarrow y_1''' = 2y_1' y_1' + 2y_1 y_1'' = 2(1.57012)(1.57012) + 2(1.127)(6.5390$

$$y_1''' = 19.66957$$

$$y_2 = 1.127 + 1.57012 \frac{(0.2-0.1)}{1} + 6.539\ldots \frac{\ldots}{2} + 19.66\ldots \frac{(0.2-0.1)^3}{8}$$

$$y_2 = 1.319980$$

5. Employ the Taylor's series to obtain the approx value of $y(1.1)$ & $y(1.2)$ from equn $y' = xy^{1/3}$, $y(1) = 1$

Sol:

Given; $y' = xy^{1/3}$

$y(1) = 1 \Rightarrow y(x_0) = y_0$

$x_0 = 1$; $y_0 = 1$

	x_0	x_1	x_2
x	1	1.1	1.2
y	1 (y_0)	y_1	y_2

Using Taylor series :-

(i) $$y_1 = y(x_0) + y'(x_0)\frac{(x-x_0)}{\lfloor 1} + y''(x_0)\frac{(x-x_0)^2}{\lfloor 2} + \cdots$$

1. $y(x_0) = y_0 = 1$

2. $y' = xy^{1/3} \Rightarrow y'(x_0) = y_0' = x_0 y_0^{1/3} = 1(1)^{1/3} = 1$

3. $y'' = y^{1/3} + x\frac{1}{3}y^{1/3-1}y'$

$= y^{1/3} + \frac{1}{3}xy^{-2/3}y' \Rightarrow y_0'' = y_0^{1/3} + \frac{1}{3}x_0 y_0^{-2/3}y_0'$

$y_0'' = (1)^{1/3} + \frac{1}{3}(1)(1)^{-2/3}(1) = 1 + 1/3.$

$= 1.3333$

$$y_1 = 1 + 1\frac{(1.1-1)}{1} + 1.33333\frac{(1.1-1)^2}{2}$$

$$y_1 = 1.10666$$

(ii) $$y_2 = y(x_1) + y'(x_1)\frac{(x-x_1)}{\lfloor 1} + y''(x_1)\frac{(x-x_1)^2}{\lfloor 2} + \cdots$$

1. $y(x_1) = y_1 = 1.10666$

2. $y' = xy^{1/3} \Rightarrow y'(x_1) = y_1' = x_1 y_1^{1/3} = (1.1)(1.10666)^{1/3}$

$= 1.13779$

(iii) $y'' = y^{1/3} + x\frac{1}{3}y^{-2/3}y'$

$= y^{1/3} + \frac{1}{3}xy^{-2/3}y' \Rightarrow y_1'' = y_1^{1/3} + \frac{1}{3}x_1 y_1^{-2/3}y_1'$

$y_1'' = (1.10666)^{1/3} + \frac{1}{3}(1.1)(1.10666)^{-2/3}(1.13779)$

$y_1'' = 1.42429$

$$y_2 = 1.10666 + 1.13779\frac{(1.2-1.1)}{1} + 1.42429\frac{(1.2-1.1)^2}{2}$$

$y_2 = 1.22756.$

6. Solve $\frac{dy}{dx} = x+y$, $y(0)=1$ by Taylor method & Hence Compute $y(0.2)$ & $y(0.4)$. Compare the result with exact solution

Sol:

Given; $y' = x+y$

$y(0)=1 \Rightarrow y(x_0)=y_0$

$x_0 = 0;\ y_0 = 1$

	x_0	x_1	x_2
x	0	0.2	0.4
y	1 (y_0)	y_1	y_2

Using Taylor series:

(i) $$y_1 = y(x_0) + y'(x_0)\frac{(x-x_0)}{\lfloor 1} + y''(x_0)\frac{(x-x_0)^2}{\lfloor 2} + y'''(x_0)\frac{(x-x_0)^3}{\lfloor 3} + \cdots$$

1. $y(x_0) = y_0 = 1$

2. $y' = x+y \Rightarrow y'(x_0) = y_0' = x_0+y_0 = 0+1 = 1$

3. $y'' = 1+y' \Rightarrow y''(x_0) = y_0'' = 1+y_0' = 1+1 = 2$

4. $y''' = y'' \Rightarrow y'''(x_0) = y_0''' = y_0'' = 2.$

$$y_1 = 1 + 1\frac{(0.2-0)}{1} + 2\times\frac{(0.2-0)^2}{2} + 2\cdot\frac{(0.2-0)^3}{3} + \cdots$$

$y_1 = 1.24266$

(ii) $y_2 = y(x_1) + y'(x_1)\frac{(x-x_1)}{\underline{1}} + y''(x_1)\frac{(x-x_1)^2}{\underline{2}} + y'''(x_1)\frac{(x-x_1)^3}{\underline{3}} + \dots$

1. $y(x_1) = y_1 = 1.24266$

2. $y' = x+y \Rightarrow y'(x_1) = y_1' \Rightarrow x_1 + y_1 = 0.2 + 1.24266$
$= 1.44266$

3. $y'' = 1+y' \Rightarrow y''(x_1) = y_1'' \Rightarrow 1+y_1' = 1 + \cancel{1.24266}\ 1.44266$
$= 2.44266$

4. $y''' = y'' \Rightarrow y'''(x_1) = y_1''' \Rightarrow y_1'' = 2.44266.$

$$y_2 = 1.24266 + 1.44266\frac{(0.4-0.2)}{1} + 2.44266\frac{(0.4-0.2)^2}{2} + 2.44266\frac{(0.4-0.2)^3}{6}$$

$y_2 = 1.58330$

Exact Solution:

$y' = x+y$

$y' - y = x$

$y' + P(x)y = Q(x)$

$P = -1, \quad Q = x$

$I.F = e^{\int P dx} = e^{\int -1 dx} = e^{-\int dx} = e^{-x}$

$y(I.F) = \int I.F\ Q(x)\,dx + c$

$y(e^{-x}) = \int \underbrace{e^{-x}}_{v} \underbrace{x}_{u}\,dx + c$

$ye^{-x} = x\frac{e^{-x}}{-1} - \int \frac{e^{-x}}{-1}\,dx + c$

$ye^{-x} = -x\,e^{-x} + \frac{e^{-x}}{-1} + c$

$y\cancel{e^{-x}} = \cancel{e^{-x}}\left[-x - 1 + c/e^{-x}\right]$

$y = -x - 1 + \frac{c}{e^{-x}} \rightarrow (1)$

$\frac{dy}{dx} + P(x)y = Q(x)$

$y' + P(x)y = Q(x)$

$I.F = e^{\int P dx}$

Soln

$y(IF) = \int I.F\ Q(x)\,dx + c$

$u = x$

$du = dx$

$\int dv = \int e^{-x}dx$

$v = e^{-x}/-1$

$y(0)=1$

$x=0,\ y=1$

sub in eq-①

$1 = -0-1+ce^{0}$

$1=-1+ce^{0}$

$c=2$

$\therefore \boxed{y = -x-1+2e^{x}}$

	Practical	Exact value.
$x=0.2$	$y_1 = 1.24266$	$y_1 = 1.24280$
$x=0.4$	$y_2 = 1.58330$	$y_2 = 1.58364$

16. Using Taylor series of $\frac{dy}{dx} = x^2+y^2$, $y(0)=0$, find $y(0.1)$

Sol:

Given; $y' = x^2+y^2$

$y(0)=0 \Rightarrow y(x_0)=y_0.$

$x_0=0,\ y_0=0$

	x_0	x_1
x	0	0.1
y	0 y_0	y_1

Using Taylor series:

(i) $y_1 = y(x_0) + y'(x_0)\frac{(x-x_0)}{\lfloor 1} + y''(x_0)\frac{(x-x_0)^2}{\lfloor 2} + y'''(x_0)\frac{(x-x_0)^3}{\lfloor 3} + \cdots$

1. $y(x_0) = y_0 = 0$

2. $y' = x^2+y^2 \Rightarrow y'(x_0)=y_0' \Rightarrow x_0^2+y_0^2 = 0+0=0.$

3. $y'' = 2x + 2\cdot y\cdot y' \Rightarrow y''(x_0)=y_0'' \Rightarrow 2x_0 + 2y_0y_0'$

$= 2(0)+2(0)(0) = 0$

4. $y''' = 2 + 2y'y' + 2yy'' \Rightarrow y'''(x_0) = y_0''' \Rightarrow 2+2(0)(0)+2(0)(0)$

$\Rightarrow 2.$

$y_1 = 0 + 0 + 0 + 2\,\frac{(0.1-0)^3}{6}$

$y_1 = 0.00033$

Picard's method of successive Approximation:

Consider 1^{st} order differential equation $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$.

Let; $y = f(x)$ be the solution of the given equation.

Using picard's method $\boxed{y = y_0 + \int_{x_0}^{x} f(x,y)\, dx}$

Note:

1. $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$

2. $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$.

Problems:

1. If $\frac{dy}{dx} = x+y$, $y=1$ at $x=1$, using Picard's method find the first approximation.

Sol:

Given; $y' = x+y$

$$y' = f(x,y) = x+y.$$

$$x_0 = 1,\ y_0 = 1$$

Using Picard's method:

1.
$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$
$$= 1 + \int_{1}^{x} f(x,1)\, dx$$
$$= 1 + \int_{1}^{x} (x+1)\, dx$$
$$= 1 + \left[\frac{x^2}{2} + x\right]_1^x$$
$$= 1 + \left[\frac{x^2}{2} + x - \frac{1}{2} - 1\right]$$
$$= \cancel{1+} \frac{x^2}{2} + x - \frac{1}{2} \cancel{-1}$$

$$y_1 = \frac{x^2}{2} + x - \frac{1}{2}$$

2. Solve the initial value problem $\frac{dy}{dx} = x+y$, $y=1$ at $x=0$ & obtain $y(0.1)$ and $y(0.2)$ using picard's & check exact solution.

Sol: Given; $y' = f(x,y) = x+y$

$x_0 = 0, \quad y_0 = 1$

		x_0	x_1	x_2
x	:	0	0.1	0.2
y	:	1		
		y_0	y_1	y_2

① At $x = 0.1$

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_0^x f(x, 1)\,dx$$

$$= 1 + \int_0^x (x+1)\,dx$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x$$

$$y_1 = 1 + \frac{x^2}{2} + x$$

$$y(0.1) = 1 + \frac{(0.1)^2}{2} + 0.1$$

$$\underline{y(0.1) = 1.105}$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$y_2 = 1 + \int_0^x f\left[x, 1 + \frac{x^2}{2} + x\right]dx$$

$$= 1 + \int_0^x \left[x + 1 + \frac{x^2}{2} + x\right]dx$$

$$= 1 + \int_0^x \left[1 + 2x + \frac{x^2}{2}\right]dx$$

$$= 1 + \left[x + \cancel{2}\,\frac{x^2}{\cancel{2}} + \frac{x^3}{6}\right]$$

$$y_2 = 1 + x + x^2 + \frac{x^3}{6}$$

$$\underline{y(0.1) = 1.11016}$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= 1 + \int_0^x f\left[x, 1 + x + x^2 + \frac{x^3}{6}\right]dx$$

$$= 1 + \int_0^x \left[x + 1 + x + x^2 + \frac{x^3}{6}\right]dx$$

$$= 1 + \int_0^x \left[1 + 2x + x^2 + \frac{x^3}{6}\right]dx$$

$$= 1 + x + \frac{\cancel{2}x^2}{\cancel{2}} + \frac{x^3}{3} + \frac{x^4}{24}$$

$$y_3 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{24}$$

$$\underline{y(0.1) = 1.11013}$$

$$\boxed{\therefore\ y(0.1) = 1.11013}$$

② At $x=0.2$

1. $y(0.2) = 1 + \frac{(0.2)^2}{2} + 0.2$

$y_1 = y(0.2) = 1.22$

2. $y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{6}$

$y_2 = y(0.2) = 1.24133$

3. $y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24}$

$y_3 = y(0.2) = 1.24273.$

Exact Solution

$x+y$

3. Solve the $\frac{dy}{dx} = y - x^2$, given $y(0)=1$ by picard's up to 3rd approximation & find $y(0.1)$ & $y(0.2)$ & check exact solution.

sol: given; $y' = f(x,y) = y - x^2$

$y(x_0) = y_0 \Rightarrow y(0)=1$

4

x:	x_0 0	x_1 0.1	x_2 0.2
y:	1 y_0	y_1	y_2

① At $x=0.1$

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_0^x f(x,1)\,dx$$

$$= 1 + \int_0^x \left[1 - x^2\right] dx$$

$$= 1 + \left[x - \frac{x^3}{3}\right]_0^x$$

$$y_1 = 1 + x - \frac{x^3}{3}$$

$$y(0.1) = 1 + 0.1 - \frac{(0.1)^3}{3}$$

$$y(0.1) = 1.09966$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 1 + \int_0^x f\left[x, 1 + x - \frac{x^3}{3}\right] dx$$

$$= 1 + \int_0^x \left[1 + x - \frac{x^3}{3} - x^2\right] dx$$

$$= 1 + \int_0^x \left[x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3}\right]_0^x$$

$$y_2 = 1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3}$$

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} - \frac{(0.1)^4}{12} - \frac{(0.1)^3}{3}$$

$$y(0.1) = 1.10465$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= 1 + \int_0^x f\left(x,\ 1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3}\right) dx$$

$$= 1 + \int_0^x \left[1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3} - x^2\right] dx$$

$$= 1 + \int_0^x \left[1 + x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12}\right] dx$$

$$= 1 + \left[1 + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{60}\right]_0^x$$

$$y_3 = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{60}$$

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} - \frac{(0.1)^3}{6} - \frac{(0.1)^4}{12} - \frac{(0.1)^5}{60}$$

$$y(0.1) = 1.1048$$

② At $x = 0.2$

i) $y_1 = y(0.2) = 1 + 0.2 - \frac{(0.2)^3}{3}$

$$y_1 = y(0.2) = 1.19733$$

ii) $y_2 = y(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2} - \frac{(0.2)^4}{12} - \frac{(0.2)^3}{3}$

$$y_2 = y(0.2) = 1.2172.$$

iii) $y_3 = y(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{6} - \frac{(0.2)^4}{12} - \frac{(0.2)^5}{60}$

$$y_3 = y(0.2) = 1.21852$$

Exact Solution:

Given; $y' = y - x^2$.

$y' - y = -x^2$.

Compare to $y' + Py = Qx$.

$P = -1$; $Q = -x^2$

$$I.F = e^{\int P dx} = e^{\int -1 dx} = e^{-x}$$

$$y(I.F) = \int IF\ Q(x) + c$$

$$y e^{-x} = \int e^{-x} (-x^2) + c.$$

$$y e^{-x} = -\int \underbrace{e^{-x}}_{dv} \underbrace{x^2}_{u} dx + c$$

$$ye^{-x} = -\left[x^2 \frac{e^{-x}}{-1} - \int \frac{e^{-x}}{-1} 2x\,dx\right] + c \qquad u = x^2,\ du = 2x\,dx \qquad \int dv = \int e^{-x}dx,\ v = e^{-x}/-1$$

$$ye^{-x} = x^2 e^{-x} - 2\int \underbrace{e^{-x}}_{dv} \underbrace{x}_{u}\, dx + c$$

$$ye^{-x} = x^2 e^{-x} - 2\left[\frac{xe^{x}}{-1} - \int \frac{e^{-x}}{-1} dx\right]dx + c \qquad u = x,\ du = dx \qquad \int dv = \int e^{-x}dx,\ v = \frac{e^{-x}}{-1}$$

$$ye^{-x} = x^2 e^{-x} + 2xe^{x} - 2\frac{e^{-x}}{-1} + c.$$

$$ye^{-x} = e^{-x}\left[x^2 + 2x + 2 + \frac{c}{e^{-x}}\right]$$

$$y = x^2 + 2x + 2 + \frac{c}{e^{-x}}$$

$$y = x^2 + 2x + 2 + ce^{x}$$

$y(0) = 1 \Rightarrow x = 0, y = 1$

$1 = 0 + 2(0) + 2 + ce^{0}$

$1 = 2 + c$

$\boxed{c = -1}$

$$y = x^2 + 2x + 2 - e^{x}$$

x	Practical	Theoritical.
0.1	$y_1 = 1.1048$	$y_1 = 1.1048$
0.2	$y_2 = 1.2185$	$y_2 = 1.2185$

v.Imp.

4. Find value of y at $x = 0.1$ by Picard's method, given

$$\frac{dy}{dx} = \frac{y-x}{y+x}, \quad y(0) = 1$$

Sol:

Given; $y' = f(x, y) = \dfrac{y-x}{y+x}$

$y(0) = 1 \Rightarrow y(x_0) = y_0$

$x_0 = 0,\ y_0 = 1$

	x_0	x_1
x :	0	0.1
y :	1	
	y_0	y_1

At $x=0.1$:-

1. $$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx.$$

$$= 1 + \int_0^x f(x, 1)\, dx$$

$$= 1 + \int_0^x \frac{1-x}{1+x}\, dx$$

$$= 1 + \int_0^x \frac{1+1-x-1}{1+x}\, dx$$

$$= 1 + \int_0^x \frac{2-1-x}{1+x}\, dx$$

$$= 1 + \int_0^x \frac{2-(1+x)}{1+x}\, dx$$

$$= 1 + \int_0^x \left[\frac{2}{1+x} - 1\right] dx$$

$$= 1 + 2\left[\log(1+x)\right]_0^x - (x)_0^x$$

$$y_1 = 1 + 2\log(1+x) - x$$

$$y_1(0.1) = 1 + 2\log(1+0.1) - 0.1$$

$$y_1 = 1.09062.$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx.$$

$$= 1 + \int_0^x \left[f(x,\ 1+2\log(1+x) - x)\right] dx$$

$$= 1 + \int_0^x \left[\frac{1+2\log(1+x) - x - x}{1+2\log(1+x) \cancel{-x} + \cancel{x}}\right] dx$$

$$= 1 + \int_0^x \left[\frac{1+2\log(1+x)}{1+2\log(1+x)} - \frac{2x}{1+2\log(1+x)}\right] dx$$

$$y_2 = 1 + \int_0^x \left[1 - \frac{2x}{1+2\log(1+x)}\right] dx$$

which is difficult to integrate. Hence, take first approximation only.

$$\therefore\ y(0.1) = 1.09062$$

5. $y' = x^2+y^2$ and $y(0)=0$ find y value at $x=0.4$ using Picard's method.

Sol:

Given; $y' = f(x,y) = x^2+y^2$

$y(0)=0 \to y(x_0)=y_0$

$x_0=0, \quad y_0=0$

	x_0	x_1
$x:$	0	0.4
$y:$	0	$?$
	y_0	y_1

At $x=0.4$:-

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 0 + \int_0^x f(x,0)\,dx$$

$$= \int_0^x x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^x$$

$$y_1 = \frac{x^3}{3}$$

$$y(0.4) = 0.02133$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 0 + \int_0^x f\left[x, \frac{x^3}{3}\right]dx$$

$$= \int_0^x \left[x^2 + \left(\frac{x^3}{3}\right)^2\right]dx$$

$$= \int_0^x \left[x^2 + \frac{x^6}{9}\right]dx$$

$$= x^2 + \frac{x^7}{63}$$

$$y_2 = x^2 + \frac{x^7}{63}$$

~~y(0.4) = 0.02135~~

$$y(0.4) = 0.02135$$

6. solve

y at

Sol:

6. solve $\frac{dy}{dx} = xy+1$ and $y(0)=1$ using picard's method find y at $x=0.1$

Sol: Given; $y' = f(x,y) = xy+1$

$$y(0)=1 \Rightarrow y(x_0)=y_0$$

$$x_0=0,\ y_0=1$$

	x_0	x_1
x :	0	0.1
y :	1	?
	y_0	y_1

At $x=0.1$:-

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_0^x f(x,1)\,dx$$

$$= 1 + \int_0^x (x+1)\,dx$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x$$

$$y_1 = 1 + x + \frac{x^2}{2}$$

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2}$$

$$y(0.1) = 1.105$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 1 + \int_0^x f\left[x,\ 1+x+\frac{x^2}{2}\right]dx$$

$$= 1 + \int_0^x \left[x\left(1+x+\frac{x^2}{2}\right)+1\right]dx$$

$$= 1 + \int_0^x \left[x + x^2 + \frac{x^3}{2} + 1\right]dx$$

$$= 1 + \left[x^2 + \frac{x^3}{3} + \frac{x^4}{8} + x\right]_0^x$$

$$y_2 = 1 + x^2 + \frac{x^3}{3} + \frac{x^4}{8} + x$$

$$y(0.1) = 1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + 0.1$$

$$y(0.1) = 1.1053$$