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B-Tech (I st → 2-Sem)

Mathematics - II

Unit - III

Fourier Transforms

Syllabus :-

→ Fourier Integral theorem (only statement)

→ Fourier Sine and Cosine Integral

→ Fourier Transform :- Fourier Sine and Cosine Transforms.

→ Properties - Inverse Transform

→ Finite Fourier Transforms.

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(1)

Unit-III

Fourier Transforms

A transformation is a Mathematical device which Converts (or) changes one function into another function.

for ex :- differentiation and Integration are transformations.

Definition :-

The integral transform of a function $f(t)$ is defined by

$$\boxed{\mathcal{I}[f(t)] = \int_a^b f(t)\, K(p,t)\, dt = \bar{f}(p)}$$

where $K(p,t)$ is called the Kernal of the Integral transform and is a function of p and t

Examples of different Kernals are as follows :-

(1) If $K(p,t) = e^{-pt}$, $a=0$ and $b\to\infty$ then we get Laplace transform of $f(t)$

$$\boxed{L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt = \bar{f}(p)}$$

(2) If $K(p,t) = e^{ipt}$, $a\to-\infty$ and $b\to\infty$ then we get the Infinite Fourier transform of $f(t)$

$$\boxed{F\{f(t)\} = \int_{-\infty}^{\infty} e^{ipt} f(t)\, dt = \bar{f}(p)}$$

3) If $K(p,t) = t^{p-1}$, $a \to 0$ and $b \to \infty$ then we have Mellin transform of $f(t)$

$$\boxed{M[f(t)] = \int_0^\infty t^{p-1} f(t)\,dt = \bar{f}(p)}$$

Fourier Integral Theorem :-

The Fourier Integral theorem states that

$$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \cos p(t-x)\, f(t)\,dt\,dp}$$

Fourier Sine and Fourier Cosine Integral :-

The Fourier Sine Integral is

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \sin px \int_0^\infty f(t) \sin pt\, dt\, dp}$$

The Fourier Cosine Integral is

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \cos px \int_0^\infty f(t) \cos pt\, dt\, dp}$$

(2)

1) Using Fourier Integral Show that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin\lambda x \, d\lambda}{(\lambda^2+a^2)(\lambda^2+b^2)} \quad a, b > 0$$

Sol :- Since the integrand on RHS Contains Sine term, we Use Fourier Sine Integral Formula.

we know that the Fourier Sine Integral for f(x) is given by

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \sin px \int_0^\infty f(t)\sin pt \, dt \, dp}$$

Replacing p with λ we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t \, dt \, d\lambda} \longrightarrow (1)$$

Here $f(x) = e^{-ax} - e^{-bx}$ (or)

$f(t) = e^{-at} - e^{-bt} \longrightarrow (2)$

now Substituting (2) in (1) we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \left\{\int_0^\infty (e^{-at} - e^{-bt}) \sin\lambda t \, dt\right\} d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left\{\int_0^\infty e^{-at}\sin\lambda t \, dt - \int_0^\infty e^{-bt}\sin\lambda t \, dt\right\} d\lambda$$

$$\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)}$$

$x = t$
$b = \lambda$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\left[\frac{e^{-at}}{\lambda^2+a^2}\left(-a\sin\lambda t - \lambda\cos\lambda t\right)\right]_0^\infty - \left[\frac{e^{-bt}}{\lambda^2+b^2}\left(-b\sin\lambda t - \lambda\cos\lambda t\right)\right]_0^\infty\right]d\lambda$$

$e^{-\infty} = 0$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda}{\lambda^2+a^2} - \frac{\lambda}{\lambda^2+b^2}\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\cdot\lambda\left[\frac{1}{\lambda^2+a^2} - \frac{1}{\lambda^2+b^2}\right]d\lambda$$

$$\frac{\cancel{\lambda^2}+b^2-\cancel{\lambda^2}-a^2}{(\lambda^2+a^2)(\lambda^2+b^2)} = \frac{b^2-a^2}{(\lambda^2+a^2)(\lambda^2+b^2)}$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\cdot\frac{\lambda(b^2-a^2)}{(\lambda^2+a^2)(\lambda^2+b^2)}\,d\lambda$$

$$= \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\,d\lambda \quad (\text{or})$$

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\,d\lambda.$$

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2) Using Fourier Integral

Show that $\int_0^\infty \frac{1 - \cos\pi\lambda}{\lambda} \sin x\lambda \, d\lambda = \underbrace{\begin{cases} \frac{\pi}{2} & ; \text{ If } 0 < x < \pi \\ 0 & ; \text{ If } x > \pi \end{cases}}_{f(x)}$

Sol: Let $f(x) = \begin{cases} \frac{\pi}{2} & ; \text{ If } 0 < x < \pi \\ 0 & ; \text{ If } x > \pi \end{cases}$

then $f(t) = \left\{ \begin{array}{ll} \frac{\pi}{2} & ; \text{ If } 0 < t < \pi \\ 0 & ; \text{ If } t > \pi \end{array} \right\} \longrightarrow (1)$

we know that the Fourier Sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[\int_0^\infty f(t) \sin\lambda t \, dt \right] d\lambda \longrightarrow (2)$$

Substituting (1) in (2) we get

$\int \sin ax \, dx = \frac{-\cos ax}{a}$

$\int_0^\infty$ splited as. 0 to π, π to ∞.

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[\int_0^\pi \frac{\pi}{2} \sin\lambda t \, dt \right] d\lambda$$

$$= \cancel{\frac{2}{\pi}} \cdot \cancel{\frac{\pi}{2}} \int_0^\infty \sin\lambda x \left(-\frac{\cos\lambda t}{\lambda} \right)_0^\pi d\lambda$$

$$= \int_0^\infty \sin\lambda x \left(-\frac{\cos\lambda\pi}{\lambda} - -\frac{\cos 0}{\lambda} \right) d\lambda$$

$$= \int_0^\infty \sin\lambda x \left(\frac{1 - \cos\lambda\pi}{\lambda} \right) d\lambda \quad \text{ie, } \int_0^\infty \left(\frac{1 - \cos\lambda\pi}{\lambda} \right) \sin\lambda x \, d\lambda = f(x)$$

$$(\text{or}) \int_0^\infty \frac{1-\cos\lambda\pi}{\lambda} \sin x\lambda \, d\lambda = \begin{cases} \frac{\pi}{2} & ; \text{ if } 0<x<\pi \\ 0 & ; \text{ if } x>\pi \end{cases}$$

(3) Using Fourier Integral

Show that $e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\lambda x}{\lambda^2+a^2} d\lambda \quad (a>0, x\geq 0)$.

Sol:- Since the integrand contains Cosine terms we use Fourier Cosine Integral formula.

By Fourier Cosine Integral formula.

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \cos px \int_0^\infty f(t)\cos pt \, dt \, dp} \longrightarrow (1)$$

Here replacing p with λ we get.

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t \, dt \, d\lambda}$$

Let $f(x) = e^{-ax}$

$f(t) = e^{-at}$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty\int_0^\infty e^{-at}\cos\lambda t \cos\lambda x \, dt \, d\lambda.$$

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$$e^{-ax} = \frac{2}{\pi}\int_0^\infty\int_0^\infty e^{-at}\cos\lambda t\cos\lambda x\,dt\,d\lambda.$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\infty e^{-at}\cos\lambda t\,dt\right]d\lambda.$$

$$\boxed{\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)}$$

Comparing $x=t$, $b=\lambda$, $a=-a$.

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\frac{e^{-at}}{a^2+\lambda^2}\left(-a\cos\lambda t + \lambda\sin\lambda t\right)\right]_0^\infty$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\frac{e^{-\infty}}{a^2+\lambda^2}(0)(0) - \frac{1}{a^2+\lambda^2}(-a)\right]$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\cdot\frac{a}{a^2+\lambda^2}\,d\lambda$$

$$= \frac{2a}{\pi}\int_0^\infty \frac{\cos\lambda x}{\lambda^2+a^2}\,d\lambda$$

$$\therefore\ \boxed{e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\lambda x}{\lambda^2+a^2}\,d\lambda.}$$

Hence Proved

(4) Using Fourier Integral Show that

$$e^{-x} \cos x = \frac{2}{\pi} \int_0^\infty \frac{\lambda^2+2}{\lambda^4+4} \cos \lambda x \, d\lambda$$

Sol: Since the integrand Contains Cosine terms we use Fourier Cosine Integral formula.

The Fourier Cosine Integral formula is

$$\boxed{f(x) = \frac{2}{\pi} \int_0^\infty \cos px \int_0^\infty f(t) \cos pt \, dt \, dp}$$

now replacing p with λ we get

$$\boxed{f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t \, dt \, d\lambda}$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos \lambda t \cos \lambda x \, dt \, d\lambda$$

Let $f(x) = e^{-x} \cos x$ then

$f(t) = e^{-t} \cos t$

$$e^{-x} \cos x = \frac{2}{\pi} \int_0^\infty \int_0^\infty e^{-t} \cos t \cos \lambda t \cos \lambda x \, dt \, d\lambda$$

$$= \frac{\cancel{2}}{\cancel{2}\pi} \int_0^\infty \left[\int_0^\infty e^{-t} (2 \cos t \cos \lambda t) dt \right] \cos \lambda x \, d\lambda$$

$$\boxed{2 \cos A \cos B = \cos (A+B) + \cos (A-B)}$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_0^\infty \left[\int_0^\infty e^{-t}\left(2\cos t\cos\lambda t\right)dt\right]\cos\lambda x\,d\lambda$$

$$\boxed{2\cos A\cos B = \cos(A+B)+\cos(A-B)}$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_0^\infty e^{-t}\left(\cos(t+\lambda t)+\cos(t-\lambda t)\right)dt\right]\cos\lambda x\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_0^\infty e^{-t}\left\{\cos(1+\lambda)t+\cos(1-\lambda)t\right\}dt\right]\cos\lambda x\,d\lambda$$

$\cos(-\theta) = \cos\theta$ ($-$ Common)

$$= \frac{1}{\pi}\int_0^\infty \left[\int_0^\infty e^{-t}\left\{\cos(\lambda+1)t+\cos(\lambda-1)t\right\}dt\right]\cos\lambda x\,d\lambda$$

$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$$

$a=1$

$b=\lambda+1$

$$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{1}{(\lambda+1)^2+1}+\frac{1}{(\lambda-1)^2+1}\right]\cos\lambda x\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{1}{\lambda^2+2\lambda+2}+\frac{1}{\lambda^2-2\lambda+2}\right]\cos\lambda x\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{2(\lambda^2+2)}{[(\lambda^2+2)+2\lambda]\,[(\lambda^2+2)-2\lambda]}\right]\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(\lambda^2+2)\cos\lambda x}{(\lambda^2+2)^2-(2\lambda)^2}\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4}\cos\lambda x\,d\lambda \quad //$$

$$\therefore\ \boxed{e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4}\cos\lambda x\,d\lambda} \quad //$$

Fourier Integral in Complex form :-

The Fourier Integral in Complex form is

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ip(t-x)} f(t)\,dt\,dp}$$

Infinite Fourier Transform :-

The Fourier Transform of a function $f(x)$ is given by

$$\boxed{F\{f(x)\} \text{ (or) } F(p) = \int_{-\infty}^{\infty} f(x)\,e^{ipx}\,dx}$$

The Inverse Fourier transform of $F(p)$ is given by

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\,e^{-ipx}\,dp}$$

Fourier Sine transform

$$\boxed{f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_s(p)\,\sin px\,dp}$$

where

$$\boxed{F_s(p) \text{ (or) } F_s\{f(x)\} = \int_{0}^{\infty} f(x)\,\sin px\,dx}$$

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Fourier Cosine Transform :-

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty F_c(p)\cos px\,dp}$$

where $\boxed{F_c(p) = F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx}$

Properties of Fourier Transform :-

1) Linearity Property of Fourier Transform :-

If $F(p)$ and $G(p)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively then

$$F[af(x)+bg(x)] = aF(p)+bG(p)$$

where a and b are Constants.

Proof :- we know that

$$F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)dx \qquad G(p) = \int_{-\infty}^{\infty} e^{ipx} g(x)dx$$

Consider $F[af(x)+bg(x)] = \int_{-\infty}^{\infty} [af(x)+bg(x)]\,e^{ipx}\,dx$

$$= a\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx + b\int_{-\infty}^{\infty} e^{ipx} g(x)\,dx$$

$$= aF(p)+bG(p)$$

$$\therefore F[af(x)+bg(x)] = aF(p)+bG(p)$$

Hence Linearity Property of Fourier transform is Proved//

→ Fourier transform of $f(x)$ is denoted by $F(p)$ instead of writing $F\{f(x)\}$

Illy (i) $F_s\{af(x)+bg(x)\} = aF_s(p)+bG_s(p)$

(ii) $F_c\{af(x)+bg(x)\} = aF_c(p)+bG_c(p)$

(2) Change of Scale Property :-

If $F(p)$ is the Complex Fourier transform of $f(x)$

then $\boxed{F\{f(ax)\} = \frac{1}{a}F\left(\frac{p}{a}\right), \ a>0.}$

Proof : we have

$$F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$

Put $ax = t \Rightarrow x = \frac{t}{a}$

$a\,dx = dt$

$dx = \frac{dt}{a}$

$x \to -\infty \Rightarrow t \to -\infty$

$x \to \infty \Rightarrow t \to \infty$

Consider $F\{f(ax)\} = \int_{-\infty}^{\infty} e^{ipx} f(ax)\,dx$

$$= \int_{-\infty}^{\infty} e^{ip\left(\frac{t}{a}\right)} f(t)\,\frac{dt}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} e^{i\left(\frac{p}{a}\right)t} f(t)\,dt$$

$$= \frac{1}{a}F\left(\frac{p}{a}\right)$$

$$\therefore F\{f(ax)\} = \frac{1}{a}F\left(\frac{p}{a}\right)$$

(definite integral is a function of limits only the value of definite integral changes only when limits are to be changed)

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→ If $F_s(p)$ and $F_c(p)$ are the Fourier Sine and Cosine transforms of $f(x)$ respectively then

$$F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{p}{a}\right) \text{ (or) } F_s\left\{f\left(\frac{x}{a}\right)\right\} = a F_s(ap)$$

$$F_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{p}{a}\right) \text{ (or) } F_c\left\{f\left(\frac{x}{a}\right)\right\} = a F_c(ap)$$

→ Fourier transform of $f(x)$ is denoted by $F(p)$ instead of writing $F\{f(x)\}$

***(3) Shifting Property :-

If $F(p)$ is the Complex Fourier transform of $f(x)$ then the Complex Fourier transform of

$f(x-a)$ is $e^{ipa} F(p)$

Proof :- we have $F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$

Put $x-a=t$
$\Rightarrow x = a+t$
$dx = dt$
$x \to -\infty \Rightarrow t \to -\infty$
$x \to \infty \Rightarrow t \to \infty$

$$F[f(x-a)] = \int_{-\infty}^{\infty} e^{ipx} f(x-a)\, dx$$

$$= \int_{-\infty}^{\infty} e^{ip(a+t)} f(t)\, dt$$

$$= e^{ipa} \int_{-\infty}^{\infty} e^{ipt} f(t)\, dt$$

$$= e^{ipa} F(p)$$

$$F[f(x-a)] = e^{ipa} F(p)$$

→ $F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{p}{a}\right)$

→ $F_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{p}{a}\right)$

4) Modulation Theorem :-

If $F(p)$ is the Complex Fourier transform of $f(x)$ then the Complex Fourier transform of $f(x)\cos ax$ is

$$\frac{1}{2}\left[F(p+a)+F(p-a)\right]$$

ie :- $$\boxed{F\{f(x)\cos ax\} = \frac{1}{2}\left[F(p+a)+F(p-a)\right]}$$

Proof :- we have $F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$

$$\therefore F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{ipx} f(x)\cos ax\, dx$$

$$\sin ax = \frac{e^{iax}-e^{-iax}}{2i}$$

$$\boxed{\cos ax = \frac{e^{iax}+e^{-iax}}{2}}$$

$$\cosh ax = \frac{e^{ax}+e^{-ax}}{2}$$

$$= \int_{-\infty}^{\infty} e^{ipx} f(x) \left(\frac{e^{iax}+e^{-iax}}{2}\right) dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(p+a)x} f(x)\, dx + \int_{-\infty}^{\infty} e^{i(p-a)x} f(x)\, dx\right]$$

$$= \frac{1}{2}\left[F(p+a)+F(p-a)\right]$$

$$\therefore \boxed{F\{f(x)\cos ax\} = \frac{1}{2}\left[F(p+a)+F(p-a)\right]}$$

Hence Proved

Note :-

$F_s\left[f(x)\cos ax\right] = \int_{-\infty}^{\infty} e^{ipx} f(x)\cos ax \sin px\, dx$ (2) $F_c\{f(x)\sin ax\} = \int_{-\infty}^{\infty} e^{ipx} f(x)\sin ax \cos px\, dx$

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(5) If $F_s(p)$ and $F_c(p)$ are the Fourier Sine and Cosine transform of $f(x)$ respectively then

(i) $F_s\{f(x)\cos ax\} = \frac{1}{2}\left[F_s(p+a) + F_s(p-a)\right]$

(ii) $F_c\{f(x)\sin ax\} = \frac{1}{2}\left[F_s(p+a) - F_s(p-a)\right]$

Proof :- (i) we have $F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx$

$$F_s\{f(x)\cos ax\} = \int_0^\infty e^{ipx} f(x)\cos ax \sin px\,dx$$

$$= \frac{1}{2}\int_0^\infty e^{ipx} f(x)\, 2\cos ax \sin px\,dx$$

$2\sin px\cos ax\,dx$

$F_s(p) = \int_0^\infty f(x)\sin px\,dx$

$\boxed{2\sin A\cos B = \sin(A+B) + \sin(A-B)}$

$A = px$
$B = ax$

$$= \frac{1}{2}\int_0^\infty e^{ipx} f(x)\left[\sin(px+ax) + \sin(px-ax)\right]dx.$$

$$= \frac{1}{2}\int_0^\infty e^{ipx} f(x)\left[\sin(p+a)x + \sin(p-a)x\right]dx.$$

$$= \frac{1}{2}\left[\int_0^\infty e^{ipx} f(x)\sin(p+a)x\,dx + \int_0^\infty e^{ipx} f(x)\sin(p-a)x\,dx\right]$$

$$= \frac{1}{2}\left[F_s(p+a) + F_s(p-a)\right] //.$$

$$\therefore \boxed{F_s\{f(x)\cos ax\} = \frac{1}{2}\left[F_s(p+a) + F_s(p-a)\right]} //.$$

Hence Proved.

(ii) $F_c[f(x)\sin ax] = \int_{-\infty}^{\infty} e^{ipx} f(x) \sin ax \cos px \, dx.$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{ipx} f(x)\, 2\sin ax \cos px \, dx \qquad A = px, \; B = ax.$$

$$\boxed{2\cos A \sin B = \sin(A+B) - \sin(A-B)}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{ipx} f(x) \left[\sin(px+ax) - \sin(px-ax)\right] dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{ipx} f(x) \sin(p+a)x \, dx - \int_{-\infty}^{\infty} e^{ipx} f(x) \sin(p-a)x \, dx\right]$$

$$= \frac{1}{2}\left[F_s(p+a) - F_s(p-a)\right].$$

$$\therefore \; F_c[f(x)\sin ax] = \frac{1}{2}\left[F_s(p+a) - F_s(p-a)\right]$$

(6) Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} e^{iqx} & ; \; \alpha < x < \beta. \\ 0 & ; \; x < \alpha \text{ and } x > \beta. \end{cases}$$

(or)

$$f(x) = \begin{cases} e^{ikx} & ; \; a < x < b \\ 0 & ; \; x < a \text{ and } x > b. \end{cases}$$

Sol :- we have

$$F\{f(x)\} = F[p] = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx.$$

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$$= \int_{\alpha}^{\beta} e^{ipx} \cdot e^{iqx} \, dx = \int_{\alpha}^{\beta} e^{i(p+q)x} \, dx$$

$$= \left[\frac{e^{i(p+q)x}}{i(p+q)} \right]_{\alpha}^{\beta}$$

$$= \frac{e^{i(p+q)\beta} - e^{i(p+q)\alpha}}{i(p+q)}$$

(or)

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x) \, dx$$

$$= \int_{a}^{b} e^{ipx} \cdot e^{ikx} \, dx = \int_{a}^{b} e^{i(p+k)x} \, dx$$

$$= \left[\frac{e^{i(p+k)x}}{i(p+k)} \right]_{a}^{b}$$

$$= \frac{e^{i(p+k)b} - e^{i(p+k)a}}{i(p+k)}$$

(7) Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 0 & ; \ -\infty < x < \alpha \\ x & ; \ \alpha \le x \le \beta . \\ 0 & ; \ x > \beta \ (\infty) \end{cases}$$

Sol :- we have .

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx .$$

$$= \left[\underbrace{\int_{-\infty}^{\alpha} (0) e^{ipx} dx}_{\to 0} + \int_{\alpha}^{\beta} x\, e^{ipx} dx + \underbrace{\int_{\beta}^{\infty} (0) e^{ipx} dx}_{\to 0} \right]$$

$$= \int_{\alpha}^{\beta} x\, e^{ipx} dx \qquad \begin{array}{llll} D & x & 1 & 0 \\ I & e^{ipx} & \dfrac{e^{ipx}}{ip} & \dfrac{e^{ipx}}{i^2 p^2} \end{array} \qquad (i^2 = -1)$$

$$= \left[x\left(\frac{e^{ipx}}{ip}\right) - \left(\frac{e^{ipx}}{i^2 p^2}\right) \right]_{\alpha}^{\beta} = \left[x\left(\frac{e^{ipx}}{ip}\right) + \left(\frac{e^{ipx}}{p^2}\right) \right]_{\alpha}^{\beta}$$

$$= \left[\beta\left(\frac{e^{ip\beta}}{ip}\right) + \left(\frac{e^{ip\beta}}{p^2}\right) \right] - \left[\alpha\left(\frac{e^{ip\alpha}}{ip}\right) + \left(\frac{e^{ip\alpha}}{p^2}\right) \right]$$

$$= \frac{i}{p}\left(\alpha e^{ip\alpha} - \beta e^{ip\beta} \right) + \frac{1}{p^2}\left(e^{ip\beta} - e^{ip\alpha} \right)$$

$$\frac{1}{i} = -i$$

$$\frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

(8) Find the Fourier transform of $f(x) = x e^{-x}$, $0 \le x < \infty$ (10)

Sol: given $f(x) = x e^{-x}$, $0 \le x < \infty$

by def $F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x) e^{ipx}\, dx$

$$F\{f(x)\} = \int_0^{\infty} x e^{-x} e^{ipx}\, dx$$

$$= \int_0^{\infty} x e^{(ip-1)x}\, dx$$

$e^{-\infty} = 0$

$e^{\infty} = \infty$.

D: x, 1, 0

I: $e^{(ip-1)x}$, $\frac{e^{(ip-1)x}}{ip-1}$, $\frac{e^{(ip-1)x}}{(ip-1)^2}$

$(i^2 = -1)$

$$= \left[x\left(\frac{e^{(ip-1)x}}{ip-1}\right) - \left(\frac{e^{(ip-1)x}}{(ip-1)^2}\right)\right]_0^{\infty}$$

$$= \frac{1}{(ip-1)^2} = \frac{(ip+1)^2}{(ip-1)^2 (ip+1)^2}$$

$\frac{(ip+1)^2}{(ip+1)^2} = 1$

$$= \frac{(ip+1)^2}{[(ip-1)(ip+1)]^2} = \frac{(1+ip)^2}{(i^2p^2-1)^2} = \frac{(1+ip)^2}{(1+p^2)^2}$$

$(a-b)(a+b)$

$$\boxed{\int_0^{\infty} x e^{(ip-1)x}\, dx \Rightarrow \int_0^{\infty} x e^{-(1-ip)x}\, dx \Rightarrow e^{-\infty} = 0.}$$

*** (9) Find the Fourier Cosine transform of the function $f(x)$ defined by

$$f(x) = \begin{cases} \cos x & ; \; 0 < x < a \\ 0 & ; \; x \geq a \end{cases}$$

(JNTU 2005 (Set-3), 2006, 2007, 2009, 2011)

Sol :- we have Fourier Cosine transform of the function $f(x)$ defined as

$$F_c\{f(x)\} = \int_0^{\infty} f(x) \cos px \, dx$$

$$= \int_0^{a} \cos x \cos px \, dx$$

$$= \frac{1}{2}\int_0^{a} 2\cos x \cos px \, dx \qquad \begin{matrix} A = x \\ B = px \end{matrix}$$

$$\boxed{2\cos A \cos B = \cos(A+B) + \cos(A-B)}$$

$$= \frac{1}{2}\int_0^{a} \cos(x+px) + \cos(x-px)\, dx \qquad \int \cos ax = \frac{\sin ax}{a}$$

$$= \frac{1}{2}\int_0^{a} [\cos(1+p)x + \cos(1-p)x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(1+p)x}{1+p} + \frac{\sin(1-p)x}{1-p}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(1+p)a}{1+p} + \frac{\sin(1-p)a}{1-p}\right]$$

10) Find the Fourier Sine transform of

(i) $f(x) = \begin{cases} \sin x & ; \ 0 < x < a \\ 0 & ; \ x \geq a \end{cases}$ (ii) $f(x) = e^{-x}$

Sol :- (i) $f(x) = \begin{cases} \sin x & ; \ 0 < x < a \\ 0 & ; \ x \geq a \end{cases}$

we know that the Fourier Sine transform is

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin px \, dx$$

$$= \int_0^a \sin x \sin px \, dx + \int_a^\infty 0 \sin px \, dx$$

$$= \int_0^a \sin x \sin px \, dx.$$

$$= \frac{1}{2}\int_0^a 2 \sin x \sin px \, dx$$

$$\boxed{2\sin A \sin B = \cos(A-B) - \cos(A+B)}$$

$A = x$
$B = px$

$$= \frac{1}{2}\int_0^a \cos(x - px) - \cos(x + px) \, dx$$

$$\int \cos ax = \frac{\sin ax}{a}$$

$$= \frac{1}{2}\int_0^a \cos(1-p)x - \cos(1+p)x \, dx$$

$$= \frac{1}{2}\left[\frac{\sin(1-p)x}{1-p} - \frac{\sin(1+p)x}{1+p}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(1-p)a}{1-p} - \frac{\sin(1+p)a}{1+p}\right]$$

(2) $f(x) = e^{-x}$

we have Fourier Sine transform is

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx$$

$$F_s\{f(x)\} = \int_0^\infty e^{-x}\sin px\,dx$$

we know that $\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}$

On Comparing $a = -1$ $b = p$

$$= \left[\frac{e^{-x}}{p^2+1}(-\sin px - p\cos px)\right]_0^\infty$$

$$= \left[\frac{e^{-\infty}}{p^2+1}(-\sin\infty - p\cos\infty)\right] - \left\{\frac{e^0}{p^2+1}(-\sin 0 - p\cos 0)\right\}$$

$\rightarrow 0$

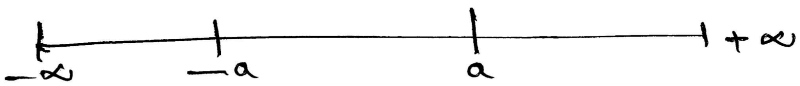
$$= -\frac{1}{p^2+1}(-p) = \frac{p}{p^2+1}$$

11) Find the Fourier transform of

$$f(x) = \begin{cases} \cos x & ; \text{ for } |x| \le a \quad (x \text{ lies b/w } -a \text{ to } a) \\ 0 & ; \text{ for } |x| > a > 0 \quad (x > a \text{ (or) } -x > a) \end{cases}$$

Sol: The Fourier transform of $f(x)$ is given by

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$



(12)

$$= \underbrace{\int_{-\infty}^{-a} f(x)\, e^{ipx}\, dx}_{\to 0} + \int_{-a}^{a} f(x)\, e^{ipx}\, dx + \underbrace{\int_{a}^{\infty} f(x)\, e^{ipx}\, dx}_{\to 0}$$

$$= \int_{-a}^{a} f(x)\, e^{ipx}\, dx = \int_{-a}^{a} \cos x\, e^{ipx}\, dx$$

$$= \int_{-a}^{a} e^{ipx} \cos x\, dx$$

$$\boxed{\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)}$$

On Comparing we have $a = ip$, $b = 1$

$$= \left[\frac{e^{ipx}}{i^2p^2+1}\left(ip\cos x + \sin x\right)\right]_{-a}^{a}$$

$i^2 = -1$

$$= \left[\frac{e^{ipa}}{1-p^2}\left(ip\cos a + \sin a\right)\right] - \left\{\frac{e^{-ipa}}{1-p^2}\left(ip\cos a - \sin a\right)\right]$$

$$= \frac{1}{1-p^2}\left[\left(e^{ipa} - e^{-ipa}\right) ip\cos a + \left(e^{ipa} + e^{-ipa}\right)\sin a\right]$$

$$e^{ipa} = \cos pa + i\sin pa$$
$$e^{-ipa} = \cos pa - i\sin pa$$
$$e^{ipa} - e^{-ipa} = \cancel{\cos pa} + i\sin pa - \cancel{\cos pa} + i\sin pa$$
$$= \underline{2i\sin pa}$$
$$e^{ipa} + e^{-ipa} = \cos pa + \cancel{i\sin pa} + \cos pa - \cancel{i\sin pa}$$
$$= \underline{2\cos pa}$$

$$= \frac{1}{1-p^2}\left[(2i\sin pa)(ip\cos a) + (2\cos pa)\cdot \sin a\right]$$

$$= \frac{2}{1-p^2}\left[\sin a\cos pa - p\cos a\sin pa\right]$$

12) Find the Fourier transform of

$$f(x) = \begin{cases} \sin x & ; \ 0 < x < \pi \\ 0 & ; \ \text{Otherwise} \end{cases}$$

Sol :- $\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$ $\boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}}$

Ans :- $\frac{1 + e^{ip\pi}}{1-p^2}$

13) Find the Fourier transform of $f(x)$ defined by

$$f(x) = e^{\frac{-x^2}{2}} \quad -\infty < x < \infty$$

(or)

Show that the Fourier transform of $e^{\frac{-x^2}{2}}$ is Reciprocal.

(or)

Show that Fourier transform of $e^{\frac{-x^2}{2}}$ is $\sqrt{2\pi}\cdot e^{\frac{-p^2}{2}}$

Sol :- given $f(x) = e^{\frac{-x^2}{2}} \quad -\infty < x < \infty$

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx$$

$$= \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}\, e^{ipx}\, dx.$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-ip)^2} \cdot e^{\frac{-p^2}{2}}\, dx$$

x వేరు Const

$$e^{\frac{-x^2}{2} + ipx}$$

$a = x$
$b = ip$

$$= e^{-\frac{1}{2}(x^2 - 2ipx)}$$

$a^2 - 2ab$

$$= e^{-\frac{1}{2}}\left(x^2 - 2x(ip) + (ip)^2 - (ip)^2\right)$$

$$= e^{-\frac{1}{2}}\left[(x-ip)^2 + p^2\right]$$

$$= e^{-\frac{p^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-ip)^2}\, dx$$

$$= e^{-p^2/2} \int_{-\infty}^{\infty} e^{-\left(\frac{1}{\sqrt{2}}(x-pi)\right)^2} dx$$

Put $\frac{1}{\sqrt{2}}(x-ip) = t$

$x - ip = \sqrt{2}\, t$

$dx = \sqrt{2}\, dt$

$x \to -\infty \Rightarrow t \to -\infty$

$x \to \infty \Rightarrow t \to \infty$.

$$= e^{-p^2/2} \int_{-\infty}^{\infty} e^{-t^2} \cdot \sqrt{2}\, dt$$

$$= \sqrt{2} \cdot e^{-p^2/2} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$

by gamma function

$\int_{0}^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$.

$$= \sqrt{2} \cdot e^{-p^2/2} \cdot \sqrt{\pi}$$

$$= \sqrt{2\pi} \cdot e^{-p^2/2}$$

$\therefore$ Fourier transform of $e^{-\frac{x^2}{2}}$ is $\sqrt{2\pi} \cdot e^{-p^2/2}$.

Thus Fourier transform of $e^{-x^2/2}$ is $e^{-p^2/2}$

Hence $f(x)$ is Self Reciprocal.

Hence Proved

14) Find Fourier Cosine and Sine transform of e^{-ax}, $a > 0$ and hence deduce the inversion formula (or)

Deduce the integrals (i) $\int_0^{\infty} \frac{\cos px}{a^2+p^2}\, dp$ (ii) $\int_0^{\infty} \frac{p \sin px}{a^2+p^2}\, dp$

(JNTU 2002, 2004, 2006, 2005, 2008, 2011)

Sol: given $f(x) = e^{-ax}$, $a > 0$

(1) Fourier Cosine transform :-

we know that Fourier Cosine transform is

$$F_c\{f(x)\} = \int_0^{\infty} f(x) \cos px\, dx \quad .$$

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx$$

$$= \int_0^\infty e^{-ax}\cos px\,dx$$

$$\boxed{\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)}$$

$$\boxed{a=-a,\quad b=p}$$

$$e^{-\infty}=0$$
$$e^0=1$$

$$= \left[\frac{e^{-ax}}{a^2+p^2}(-a\cos px + p\sin px)\right]_0^\infty$$

$$= \left[\frac{e^{-\infty}}{a^2+p^2}(-a\cos\infty + p\sin\infty)\right] - \left[\frac{e^0}{a^2+p^2}(-a\cos 0 + p\sin 0)\right]$$

$\to 0$

$$= \frac{a}{a^2+p^2}$$

$$\therefore \boxed{F_c\{f(x)\} = \frac{a}{a^2+p^2}}$$

Fourier Sine Transform :

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx$$

$$= \int_0^\infty e^{-ax}\sin px\,dx$$

$$\therefore \boxed{F_s\{f(x)\} = \frac{p}{a^2+p^2}}$$

$$\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}$$

on Comparing $a=-a$
$b=p$

$$= \left[\frac{e^{-\infty}}{a^2+p^2}(-a\sin\infty - p\cos\infty)\right] - \left[\frac{e^0}{a^2+p^2}(-a\sin 0 - p\cos 0)\right]$$

$\to 0$

$$= \frac{p}{a^2+p^2}$$

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$\therefore$ we have:

$$\boxed{F_c\{f(x)\} = \frac{a}{a^2+p^2}} \qquad \boxed{F_s\{f(x)\} = \frac{p}{a^2+p^2}}$$

① now by Inverse Fourier Cosine Transform:

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty F_c\{f(x)\}\cos px\, dp}$$

$$= \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+p^2}\cos px\, dp \qquad \left(\because f(x) = e^{-ax}\right)$$

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos px}{a^2+p^2}\, dp \quad (\text{or}) \quad \int_0^\infty \frac{\cos px}{a^2+p^2}\, dp = \frac{\pi}{2a} e^{-ax}$$

② now by Inverse Fourier Sine transform we get:

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty F_s\{f(x)\}\sin px\, dp}$$

$$= \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\, dp$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\, dp$$

$$\therefore \int_0^\infty \frac{p\sin px}{a^2+p^2}\, dp = \frac{\pi}{2} e^{-ax}$$

(15) Find Fourier Sine and Cosine transform of

$$2e^{-5x} + 5e^{-2x}$$

Sol :- given $f(x) = 2e^{-5x} + 5e^{-2x}$

(1) Fourier Sine Transform of $f(x)$ is :-

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx$$

$$= \int_0^\infty \left(2e^{-5x} + 5e^{-2x}\right)\sin px\,dx$$

$$= 2\int_0^\infty e^{-5x}\sin px\,dx + 5\int_0^\infty e^{-2x}\sin px\,dx$$

$$\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}$$

on Comparing $\boxed{a = -5}$ $\boxed{b = p}$ $\boxed{a = -2}$ $\boxed{b = p}$

$$= 2\left[\frac{e^{-5x}}{p^2+25}(-5\sin px - p\cos px)\right]_0^\infty + 5\left[\frac{e^{-2x}}{p^2+4}(-2\sin px - p\cos px)\right]_0^\infty$$

$$= 2\left[(0) - \frac{1}{p^2+25}(-p)\right] + 5\left[(0) - \frac{1}{p^2+4}(-p)\right]$$

$$= 2\left[\frac{p}{p^2+25}\right] + 5\left(\frac{p}{p^2+4}\right) = \frac{2p}{p^2+25} + \frac{5p}{p^2+4}$$

(17)

17) Find Fourier Sine and Cosine transform of

$$f(x) = \begin{cases} K & ; \text{ if } 0 < x < a \\ 0 & ; \text{ if } x > a \end{cases}$$

Sol : Fourier Sine Transform :-

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx = \left[\int_0^a K\sin px\,dx + \underbrace{\int_a^\infty (0)\sin px\,dx}_{\to 0}\right]$$

$$= \int_0^a K\sin px\,dx = K\int_0^a \sin px\,dx = K\left(\frac{-\cos px}{p}\right)_0^a$$

$$= K\left(\frac{-\cos pa}{p} - \frac{-\cos 0}{p}\right) = \frac{-K}{p}(\cos pa - 1)$$

$$= \frac{K}{p}(1 - \cos ap)$$

Fourier Cosine Transform :-

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx = \int_0^a K\cos px\,dx + \underbrace{\int_a^\infty 0\cos px\,dx}_{\to 0}$$

$$= K\int_0^a \cos px\,dx = K\left(\frac{\sin px}{p}\right)_0^a = K\left(\frac{\sin pa}{p} - \frac{\sin 0}{p}\right)$$

$$= \frac{K}{p}(\sin pa) = \frac{K\sin pa}{p} \quad (\text{or}) \quad \frac{K\sin ap}{p}$$

Formulas

(1) Fourier Sine Integral for $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin px \int_0^\infty f(t) \sin pt \, dt \, dp.$$

(2) Fourier Cosine Integral for $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos px \int_0^\infty f(t) \cos pt \, dt \, dp$$

(3) Fourier Transform of $f(x)$

$$F\{f(x)\} \text{ (or) } F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x) \, dx$$

(4) Fourier Cosine Transform of $f(x)$ is

$$F_c\{f(x)\} = F_c(p) = \int_0^\infty f(x) \cos px \, dx$$

(5) Fourier Sine Transform of $f(x)$ is

$$F_s\{f(x)\} = F_s(p) = \int_0^\infty f(x) \sin px \, dx$$

(6) Inverse Fourier Sine Transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s\{f(x)\} \sin px \, dp$$

(7) Inverse Fourier Cosine Transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c\{f(x)\} \cos px \, dp.$$

(8) $$\int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$$

(9) $$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$$

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18) Find $f(x)$ if its Fourier Sine transform is e^{-ap}

Sol :- The Inverse Fourier Sine transform of $F_s\{f(x)\}$ (or) $F_s(p)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(p)\sin px\,dx$$

$$= \frac{2}{\pi}\int_0^\infty e^{-ap}\sin px\,dx$$

$$\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}$$

on Comparing we have
$a = -a$
$x = p$
$b = p$

$$= \frac{2}{\pi}\left[\frac{e^{-pa}}{a^2+p^2}(-a\sin px - p\cos px)\right]_0^\infty$$

$$= \frac{2}{\pi}\cdot\frac{p}{a^2+p^2} = \frac{2p}{\pi(a^2+p^2)}$$

Finite Fourier Transforms.

Finite Fourier Sine and Cosine Transform.

Finite Fourier Sine Transform :-

$$F_s\{f(x)\} = F_s(n) = \int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Finite Fourier Cosine Transform :-

$$F_c\{f(x)\} = F_c(n) = \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

(1) Find the Finite Fourier Sine and Cosine transform of $f(x) = 1$ in $[0, \pi]$

Sol :- we usually take range in $[0, l]$ → $[0, \pi]$

(1) Finite Fourier Sine Transform :-

$$\boxed{F_s\{f(x)\} = F_s(n) = \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx}$$

$$= \int_0^\pi 1 \cdot \sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right) dx = \int_0^\pi \sin nx \, dx = \left(-\frac{\cos nx}{n}\right)_0^\pi$$

$$= \left(-\frac{\cos n\pi}{n} - -\frac{\cos 0}{n}\right) = -\frac{1}{n}(\cos n\pi - 1)$$

$$= \frac{1 - \cos n\pi}{n} = \frac{1-(-1)^n}{n}$$

(2) Finite Fourier Cosine Transform :-

$$\boxed{F_c\{f(x)\} = F_c(n) = \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx}$$

$$= \int_0^\pi 1 \cdot \cos\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right) dx = \int_0^\pi \cos nx \, dx = \left(\frac{\sin nx}{n}\right)_0^\pi$$

$$= \left(\frac{\sin n\pi}{n} - \frac{\sin 0}{n}\right) = \frac{1}{n}(0 - 0) = 0$$

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(2) Find the Finite Fourier Sine and Cosine transform of $f(x)$ defined by $f(x) = x$ where $0 < x < 4$

Sol :- given $f(x) = x$ $\quad 0 < x < 4$, $\quad 0 < x < l$ $\quad l = 4$

(1) Finite Fourier Sine transform :-

$$F_s\{f(x)\} = F_s(n) = \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_0^4 x \sin\left(\frac{n\pi x}{4}\right) dx$$

D	x	1	0
I	$\sin\frac{n\pi x}{4}$	$-\dfrac{\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}$	$-\dfrac{\sin\frac{n\pi x}{4}}{\frac{n^2\pi^2}{16}}$

(signs: $+$, $-$)

$$= \left[-x\left(\frac{\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right) + \left(\frac{\sin\frac{n\pi x}{4}}{\frac{n^2\pi^2}{16}}\right)\right]_0^4$$

$$= \left[-4\left(\frac{\cos\frac{n\pi \cancel{4}}{\cancel{4}}}{\frac{n\pi}{4}}\right) + \underbrace{\left(\frac{\sin\frac{n\pi \cancel{4}}{\cancel{4}}}{\frac{n^2\pi^2}{16}}\right)}_{\to 0}\right] - \left[\underbrace{-(0)\left(\frac{\frac{\cos 0}{4}}{\frac{n\pi}{4}}\right)}_{\to 0} + \underbrace{\left(\frac{\frac{\sin 0}{4}}{\frac{n^2\pi^2}{16}}\right)}_{\to 0}\right]$$

$$= \frac{-16}{n\pi}\cos n\pi = \frac{-16}{n\pi}(-1)^n //$$

(2) Finite Fourier Cosine transform :-

$$F_c\{f(x)\} = F_c(n) = \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_0^4 x \cos\left(\frac{n\pi x}{4}\right) dx \qquad D\ x \quad 1 \quad 0$$

$$\mathcal{E}\ \frac{\cos n\pi x}{4} \qquad \frac{\frac{\sin n\pi x}{4}}{\frac{n\pi}{4}} \qquad -\frac{\frac{\cos n\pi x}{4}}{\frac{n^2\pi^2}{16}}$$

$$= \left[x\left(\frac{\frac{\sin n\pi x}{4}}{\frac{n\pi}{4}}\right) + \left(\frac{\frac{\cos n\pi x}{4}}{\frac{n^2\pi^2}{16}}\right)\right]_0^4$$

$$= \left[(4)\underbrace{\left(\frac{\frac{\sin n\pi \cancel{4}}{\cancel{4}}}{\frac{n\pi}{4}}\right)}_{0} + \left(\frac{\frac{\cos n\pi \cancel{4}}{\cancel{4}}}{\frac{n^2\pi^2}{16}}\right)\right] - \left[\underbrace{(0)\left(\frac{\frac{\sin 0}{4}}{\frac{n\pi}{4}}\right)}_{0} + \underbrace{\left(\frac{\frac{\cos 0}{4}}{\frac{n^2\pi^2}{16}}\right)}_{0}\right]$$

$$= \frac{16}{n^2\pi^2}\left[\cos n\pi - \cos 0\right] = \frac{16}{n^2\pi^2}\left[(-1)^n - 1\right] \quad \text{where } n > 0 .$$

If $\boxed{n=0}$ $\Rightarrow \begin{cases} 0 & ; \text{ when } n \text{ is even.} \\ \frac{-32}{n^2\pi^2} & ; \text{ when } n \text{ is odd} \end{cases}$

$$F_c(n) = \int_0^4 x\,dx = \left(\frac{x^2}{2}\right)_0^4 = \frac{\cancel{16}\,8}{\cancel{2}} = 8$$

(3) Find the Finite Fourier Sine and Cosine transform of $f(x)$ defined by

$$f(x) = 2x \quad \text{where} \quad 0 < x < 2\pi, \quad 0 < x < l.$$

Sol: given $f(x) = 2x \qquad 0 < x < 2\pi$, $0 < x < l$.

(20)

(1) Finite Fourier Sine Transform :-

$$F_s\{f(x)\} = F_s(n) = \int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\sin\left(\frac{n\cancel{\pi}x}{2\cancel{\pi}}\right) = \sin\left(\frac{nx}{2}\right)$$

$$= \int_0^{2\pi} 2x\sin\left(\frac{nx}{2}\right)dx$$

$$= 2\int_0^{2\pi} x\sin\left(\frac{nx}{2}\right)dx$$

$$= 2\left[-x\left(\frac{\cos\frac{nx}{2}}{\frac{n}{2}}\right) + \left(\frac{\sin\frac{nx}{2}}{\frac{n^2}{4}}\right)\right]_0^{2\pi}$$

	x	1	0
I	$\sin\frac{nx}{2}$	$-\frac{\cos\frac{nx}{2}}{\frac{n}{2}}$	$-\frac{\sin\frac{nx}{2}}{\frac{n^2}{4}}$

(signs: $+$, $-$)

$$= 2\left[-2\pi\left(\frac{\cos\frac{\cancel{2}n\pi}{\cancel{2}}}{\frac{n}{2}}\right) + \underbrace{\left(\frac{\sin\frac{\cancel{2}n\pi}{\cancel{2}}}{\frac{n^2}{4}}\right)}_{\to 0}\right] - \left[-(0)\underbrace{\left(\frac{\cos\frac{0}{2}}{\frac{n}{2}}\right)}_{\to 0} + \underbrace{\left(\frac{\sin\frac{0}{2}}{\frac{n^2}{4}}\right)}_{\to 0}\right]$$

$$= -\frac{8\pi}{n}\cos n\pi = -\frac{8\pi}{n}(-1)^n \quad (\text{or}) \quad \frac{8\pi}{n}(-1)^{n+1}$$

(2) Finite Fourier Cosine Transform :-

$$F_c\{f(x)\} = F_c(n) = \int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^{2\pi} 2x\cos\left(\frac{nx}{2}\right)dx = 2\int_0^{2\pi} x\cos\left(\frac{nx}{2}\right)dx$$

$$= 2\int_0^{2\pi} x \cos\left(\frac{nx}{2}\right)dx$$

D x $\quad$ 1 $\quad$ 0

I $\cos\frac{nx}{2}$ $\quad \dfrac{\sin\frac{nx}{2}}{\frac{n}{2}}$ $\quad -\dfrac{\cos\frac{nx}{2}}{\frac{n^2}{4}}$

$$= 2\left[x\left(\frac{\sin\frac{nx}{2}}{\frac{n}{2}}\right) + \left(\frac{\cos\frac{nx}{2}}{\frac{n^2}{4}}\right)\right]_0^{2\pi}$$

$$= 2\left[2\pi\underbrace{\left(\frac{\sin\frac{\cancel{2}n\pi}{\cancel{2}}}{\frac{n}{2}}\right)}_{0} + \left(\frac{\cos\frac{\cancel{2}n\pi}{\cancel{2}}}{\frac{n^2}{4}}\right)\right] - \left[(0)\underbrace{\left(\frac{\sin\frac{0}{2}}{\frac{n}{2}}\right)}_{0} + \left(\frac{\cos\frac{0}{2}}{\frac{n^2}{4}}\right)\right]$$

$$= \frac{8}{n^2}\left[\cos n\pi - \cos 0\right] = \frac{8}{n^2}\left[(-1)^n - 1\right]$$

(4) Find the Finite Fourier Sine transform of

$f(x) = x^3$ in $(0,\pi)$
$(0,l)$

Sol :- The Finite Fourier Sine transform is

$$F_S\{f(x)\} = F_S(n) = \int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^{\pi} f(x)\sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)dx = \int_0^{\pi} x^3 \sin nx\, dx$$

D x^3 $\quad 3x^2$ $\quad 6x$ $\quad 6$ $\quad 0$

I $\sin nx$ $\quad -\dfrac{\cos nx}{n}$ $\quad -\dfrac{\sin nx}{n^2}$ $\quad \dfrac{\cos nx}{n^3}$ $\quad -\dfrac{\sin nx}{n^4}$

(21)

$$= \left[-x^3 \left(\frac{\cos nx}{n}\right) + 3x^2 \left(\frac{\sin nx}{n^2}\right) + 6x \left(\frac{\cos nx}{n^3}\right) + 6 \left(\frac{\sin nx}{n^4}\right) \right]_0^{\pi}$$

$$= \left[-\pi^3 \left(\frac{\cos n\pi}{n}\right) + 3\pi^2 \underbrace{\left(\frac{\sin n\pi}{n^2}\right)}_{\to 0} + 6\pi \left(\frac{\cos n\pi}{n^3}\right) + 6 \underbrace{\left(\frac{\sin n\pi}{n^4}\right)}_{\to 0} \right] -$$

$$\left[-0^3 \underbrace{\left(\frac{\cos 0}{n}\right)}_{\to 0} + 3(0)^2 \underbrace{\left(\frac{\sin 0}{n^2}\right)}_{\to 0} + 6(0) \underbrace{\left(\frac{\cos 0}{n^3}\right)}_{\to 0} + 6 \underbrace{\left(\frac{\sin 0}{n^4}\right)}_{\to 0} \right]$$

$$= \frac{-\pi^3}{n} \cos n\pi + \frac{6\pi}{n^3} \cos n\pi$$

$$= \left(\frac{6\pi}{n^3} + \frac{-\pi^3}{n} \right) \cos n\pi$$

(5) Find the Finite Fourier Sine Transform of $f(x) = \frac{x}{\pi}$, $0 < x < \pi$

Sol: Finite Fourier Sine Transform $0 < x < l$.

$$F_s\{f(x)\} = F_s(n) = \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_0^{\pi} \frac{x}{\pi} \sin\left(\frac{n\pi x}{\pi}\right) dx = \frac{1}{\pi} \int_0^{\pi} x \sin nx \, dx$$

D	x	1	0
I	$\sin nx$	$-\frac{\cos nx}{n}$	$-\frac{\sin nx}{n^2}$

$$= \frac{1}{\pi} \left[-x \left(\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right) \right]_0^{\pi}$$

$$\boxed{(-1)\frac{1}{n}(-1)^n, \quad \frac{1}{n}(-1)^{n+1}}$$

$$= \frac{1}{\pi} \left[-\pi \left(\frac{\cos n\pi}{n}\right) + \underbrace{\left(\frac{\sin n\pi}{n^2}\right)}_{\to 0} \right] - \left[-(0) \underbrace{\left(\frac{\cos 0}{n}\right) + \left(\frac{\sin 0}{n^2}\right)}_{\to 0} \right] = \frac{-\pi}{\pi n} \cos n\pi$$

$$= -\frac{1}{n}(-1)^n$$

Inverse Finite Sine Transform :-

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin\left(\frac{n\pi x}{l}\right)$$

Inverse Finite Cosine Transform :-

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos\left(\frac{n\pi x}{l}\right)$$

(6) Find Inverse Finite Sine Transform of $f(x)$ If

$$F_s(n) = \frac{1-\cos n\pi}{n^2\pi^2} \quad \text{where } 0<x<\pi$$
$$0<x<l.$$

Sol :- we know that

Inverse Finite Sine Transform $\quad l=\pi$

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n^2\pi^2} \sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)$$

$$= \frac{2}{\pi^3} \sum_{n=1}^{\infty} \left(\frac{1-\cos n\pi}{n^2}\right) \sin nx$$

(7) Find Inverse Finite Cosine Transform of $f(x)$ If

$$F_c(n) = \frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2} \quad \text{where } 0<x<4$$
$$0<x<l.$$

Sol :- Inverse Finite Cosine Transform.

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos\left(\frac{n\pi x}{l}\right)$$

(22)

$$= \frac{1}{4}(1) + \frac{2}{4}\sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2} \cos\left(\frac{n\pi x}{4}\right)$$

$$= \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2} \cos\left(\frac{n\pi x}{4}\right)$$

(8) Find the Inverse Finite ~~Ktdn~~ Fourier Sine transform of $f(x)$ If $F_s(n) = \frac{16(-1)^{n-1}}{n^3}$

where n is a Positive Integer $0 < x < 8$
$0 < x < l$.

Sol :- we know that Inverse Fourier Sine Transform is given

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n) \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{\cancel{2}}{\cancel{8}_{4}}\sum_{n=1}^{\infty} \frac{\cancel{16}^{4}(-1)^{n-1}}{n^3} \sin\left(\frac{n\pi x}{8}\right) \quad (\because l = 8)$$

$$= 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin\left(\frac{n\pi x}{8}\right)$$

Parseval's Identity for Fourier Transforms

If $F(p)$ and $G(p)$ are Fourier Transforms of $f(x)$ and $g(x)$ respectively then

(i) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\,\overline{G(p)}\,dp = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx$$

(ii) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(p)|^2\,dp = \int_{-\infty}^{\infty} |f(x)|^2\,dx.$$

where bar denotes the Complex Conjugate.

(1) If $f(x) = \begin{cases} 1 & ; |x|<1 \\ 0 & ; |x|>1 \end{cases}$ and $F(s) = \frac{2}{s}\sin s$

Using Parseval's Identity

P.T $$\int_0^{\infty}\left[\frac{\sin t}{t}\right]^2 dt = \frac{\pi}{2}$$

Sol :- From Parseval's Identity for ~~AH~~ Fourier Transform, we have

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(s)|^2\,ds$$

$$\Rightarrow \int_{-\infty}^{-1}(0)\,dx + \int_{-1}^{1}(1)^2\,dx + \int_{1}^{\infty}(0)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^2}\sin^2 s\,ds$$

$$\Rightarrow (x)_{-1}^{1} = \frac{\cancel{4}^{2}}{\cancel{2}\pi}\int_{-\infty}^{\infty}\frac{\sin^2 s}{s^2}\,ds \qquad \left(2\int_0^{\infty} \Rightarrow \int_{-\infty}^{\infty} \text{ even}\right)$$

$\sin \to$ odd

$\sin^2 s \to$ even

$\sin^2\theta =$ even

$(\sin\theta)^2 = \sin^2\theta$

$$= \cancel{2} = \frac{\cancel{2}}{\pi}\cdot 2\int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds$$

$$\Rightarrow \int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{2} \quad (\text{or}) \quad \int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

→ IIly we can prove the following Parseval's Identities for Fourier Sine and Cosine Transforms as above.

(i) $\frac{2}{\pi}\int_0^\infty F_c(p)\, G_c(p)\, dp = \int_0^\infty f(x) g(x)\, dx$ ✓

(ii) $\frac{2}{\pi}\int_0^\infty F_s(p)\, G_s(p)\, dp = \int_0^\infty f(x) g(x)\, dx$

(iii) $\frac{2}{\pi}\int_0^\infty |F_c(p)|^2 dp = \int_0^\infty |f(x)|^2 dx$

(iv) $\frac{2}{\pi}\int_0^\infty |F_s(p)|^2 dp = \int_0^\infty |f(x)|^2 dx$.

2) Using Parseval's Identity

S.T $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$ (or)

Evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$ Using Parseval's Transform.

Sol: Let $f(x) = e^{-ax}$ Let $g(x) = e^{-bx}$

$F_c(p) = F_c(e^{-ax})$

$= \int_0^\infty e^{-ax} \cos px\, dx$

$= \frac{a}{a^2+p^2}$.

$G_c(p) = G_c(e^{-bx})$

$= \int_0^\infty e^{-bx} \cos px\, dx$

$= \frac{b}{b^2+p^2}$

Cosine $\longrightarrow$ Power Const numerator $\dfrac{a}{a^2+p^2}$

Sine $\longrightarrow$ Power Const numerator $\dfrac{p}{b^2+p^2}$

numer Const so we Use Cosine

Using Parseval's Identity for Cosine Transform

$$\frac{2}{\pi}\int_0^\infty F_c(p)\,G_c(p)\,dp = \int_0^\infty f(x)g(x)\,dx$$

$$= \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+p^2}\,\frac{b}{b^2+p^2}\,dp = \int_0^\infty e^{-ax}\cdot e^{-bx}\,dx$$

$$= \frac{2ab}{\pi}\int_0^\infty \frac{1}{(a^2+p^2)(b^2+p^2)}\,dp = \int_0^\infty e^{-(a+b)x}\,dx$$

$$= \left(\frac{e^{-(a+b)x}}{-(a+b)}\right)_0^\infty = \left(0 - \frac{1}{-(a+b)}\right) = \frac{1}{a+b}$$

(or)

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}$$

ST $\displaystyle\int_0^\infty \frac{dx}{(x^2+9)(x^2+4)} = \frac{\pi}{60}$

Put $a=3$, $b=2$

$$= \frac{\pi}{2(3)(2)(3+2)} = \frac{\pi}{(12)(5)} = \frac{\pi}{60}$$

1) $\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^2}\sin^2 s \, ds = \int_{-1}^{1} (1)^2 dx$

$= \frac{4^2}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} dx = (x)_{-1}^{1}$

$= \frac{2}{\pi}\cdot 2\int_{0}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = 2$

$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{\cancel{4}_2}\cdot\cancel{2} =$

Since Const → Cosine

$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

3) Evaluate $\int_{0}^{\infty} \frac{dx}{(a^2+x^2)^2}$ Using Parsevals Identity.

S.T $\int_{0}^{\infty} \frac{1}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3}$

Sol :- Fourier Cosine transform of $f(x) = e^{-ax}$.

$(a^m)^n = a^{mn}$

$F_c\{f(x)\} = F_c\{e^{-ax}\} = \frac{a}{p^2+a^2}$

Parsevals Identity $\frac{2}{\pi}\int_{0}^{\infty} |F_c(p)|^2 dp = \int_{0}^{\infty} |f(x)|^2 dx$

$= \frac{2}{\pi}\int_{0}^{\infty} \left|\frac{a}{p^2+a^2}\right|^2 dp = \int_{0}^{\infty} |e^{-ax}|^2 dx$

$= \frac{2a^2}{\pi}\int_{0}^{\infty}\frac{1}{(p^2+a^2)^2} dp = \int_{0}^{\infty} e^{-2ax} dx.$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{1}{(p^2+a^2)^2}\,dp = \int_0^\infty e^{-2ax}\,dx$$

$$= \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{1}{(p^2+a^2)^2}\,dp = \left[(0) \cancel{-} \frac{1}{\cancel{-}2a}\right]$$

$$= \frac{1}{2a}.$$

$$\Rightarrow \int_0^\infty \frac{1}{(p^2+a^2)^2}\,dp = \frac{\pi}{4a^3} \quad \text{replace } p \text{ by } x.$$

$$\Rightarrow \boxed{\int_0^\infty \frac{1}{(x^2+a^2)^2}\,dp = \frac{\pi}{4a^3}}$$

Problems Parsevel's Identity

Parseval's formula

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(p)|^2 \, dp = \int_{-\infty}^{\infty} |f(x)|^2 \, dx}$$

1) If $f(x) = \begin{cases} 1 & ; |x| < 1 \\ 0 & ; |x| > 1 \end{cases}$ and $F(s) = \frac{2}{s} \sin s$

Using Parseval's Identity

P.T $\int_{0}^{\infty} \left[\frac{\sin t}{t}\right]^2 dt = \frac{\pi}{2}$

Sol: Using Parsevals Identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(p)|^2 \, dp = \int_{-\infty}^{\infty} |f(x)|^2 \, dx .$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^2} \sin^2 s \, ds = \int_{-1}^{1} (1)^2 \, dx .$$

$$= \frac{\cancel{4}^2}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} \, ds = (x)_{-1}^{1}$$

sin - odd
$\sin^2 s$ - even
$\sin^2\theta$ - even

$$= \frac{2}{\pi} \cdot 2\int_{0}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = 2 .$$

$$= \int_{0}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{\cancel{4}_2} \cdot \cancel{2} \Rightarrow \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

(~~8~~) ~~Evaluate~~

2) Using Parseval's Identity S.T $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$

(or)

Evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$ Using Parseval's Transform.

Sol :- Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$.

$$F_c(p) = F_c(e^{-ax}) = \int_0^\infty e^{-ax} \cos px \, dx = \frac{a}{a^2+p^2}$$

$$G_c(p) = G_c(e^{-bx}) = \int_0^\infty e^{-bx} \cos px \, dx = \frac{b}{b^2+p^2}$$

Using Parsevals Identity for Cosine Transform we have

$$\frac{2}{\pi}\int_0^\infty F_c(p)\,G_c(p)\,dp = \int_0^\infty f(x)\,g(x)\,dx$$

numerator Const so we use Cosine

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+p^2}\,\frac{b}{b^2+p^2}\,dp = \int_0^\infty e^{-ax}\cdot e^{-bx}\,dx.$$

Variable - num sine

$$\Rightarrow \frac{2ab}{\pi}\int_0^\infty \frac{1}{(a^2+p^2)(b^2+p^2)}\,dp = \int_0^\infty e^{-(a+b)x}\,dx.$$

$$\Rightarrow \frac{2ab}{\pi}\int_0^\infty \frac{1}{(a^2+p^2)(b^2+p^2)}\,dp = \left(\frac{e^{-(a+b)x}}{-(a+b)}\right)_0^\infty$$

$$\Rightarrow \frac{2ab}{\pi} \int_0^{\infty} \frac{1}{(a^2+p^2)(b^2+p^2)}\, dp = \left(0 - \frac{1}{-(a+b)}\right)$$

$$= \frac{1}{a+b}.$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$ (p replace x)

3) S.T $\displaystyle\int_0^{\infty} \frac{dx}{(x^2+9)(x^2+4)} = \frac{\pi}{60}.$

Sol: we know that

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

Put $a=3$ $b=2$ we get

$$\frac{\pi}{2(3)(2)(3+2)} = \frac{\pi}{12(5)} = \frac{\pi}{60}.$$

$$\therefore \int_0^{\infty} \frac{dx}{(x^2+9)(x^2+4)} = \frac{\pi}{60}$$