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B-Tech (1st year - 2-Sem)

Mathematics - II

Unit - IV

Partial Differential Equations

Syllabus :-

→ Formation of PDE by elimination of arbitrary constants and arbitrary functions

→ Method of seperation of variables

→ Solutions of One Dimensional wave equation, heat equation and

→ Solutions of two Dimensional Laplace's equation under Initial and boundary Conditions.

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(1)

Unit-4.

Partial Differential Equations.

→ An equation involving Partial derivatives is Called as Partial Differential equation.

(or)

→ An equation which Contain one (or) more Partial derivatives are Called as Partial differential equations

→ The PDE involves atleast two Independent variables and One dependent variable.

→ Whenever we Consider the Case of two Independent variables we shall usually take them to be x and y and take Z to be the dependent variable.

→ The Partial differential Co-efficients $\frac{\partial Z}{\partial x}, \frac{\partial Z}{\partial y}$ will be denoted by p and q respectively.

ie :- $\frac{\partial Z}{\partial x} = p$, $\frac{\partial Z}{\partial y} = q$

→ The Second Order Partial derivatives

$\frac{\partial^2 Z}{\partial x^2}, \frac{\partial^2 Z}{\partial y^2}, \frac{\partial^2 Z}{\partial x \partial y}$ are denoted by r, s, t respectively.

ie :- $\frac{\partial^2 Z}{\partial x^2} = r$, $\frac{\partial^2 Z}{\partial x \partial y} = s$ and $\frac{\partial^2 Z}{\partial y^2} = t$

→ The Order of the PDE is the Order of the Highest derivative in the eqn.

→ The degree of an equation is the degree of the Highest Order derivative occurring in the eqn.

$$\underbrace{Z}_{\text{dependent Variable}} = f\underbrace{(x, y)}_{\text{Independent Variable}}$$

For example :-

(1) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$ is of the first Order and first degree.

(2) $\left(\dfrac{\partial^2 z}{\partial x^2}\right)^1 + \left(\dfrac{\partial^2 z}{\partial x \partial y}\right) + \left(\dfrac{\partial^2 z}{\partial y^2}\right) = 0$ is of the Second Order and 1st degree.

Formation of Partial Differential Equations

Generally the Partial differential equation can be formed either by elimination of arbitrary Constants (or) by the elimination of arbitrary functions from a relation involving three (or) more variables

(1) Elimination of arbitrary Constants

(2) Elimination of arbitrary functions.

(2)

1) Elimination of Arbitrary Constants :-

Let the eqn be $f(x, y, z, a, b) = 0 \longrightarrow (1)$

Here x, y are any two Independent variables and z be a dependent variable.

where a, b are two arbitrary Constants

we know that to eliminate two Constants we need atleast three equations.

From these (3) eqns we can eliminate the two Constants a and b.

now differentiating (1) Partially w.r.to x and y we get.

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \longrightarrow (2)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \longrightarrow (3)$$

now eliminating a and b from (1) (2) and (3) we get the Partial differential eqn of first Order in the Form

$$F(x, y, z, p, q) = 0$$

→ Finally we can eliminate arbitrary Constants a and b from the given eqn.

→ For eliminating three Constants we need 4 eqns.

Problems

1) Form the Partial differential equation by eliminating the arbitrary Constants a and b from

$$Z = ax + by + a^2 + b^2$$

Sol: given $Z = ax + by + a^2 + b^2 \longrightarrow (1)$

(Here we are having two arbitrary Constants a and b
$\therefore$ we need two more equations to eliminate a and b)

differentiating eqn (1) Partially w.r.to x and y we have

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a \longrightarrow (2)$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \longrightarrow (3)$$

now substitute (2) and (3) in (1) we get

$$Z = px + qy + p^2 + q^2$$ is the required Partial Differential equation

(2) Form the Partial differential equation by eliminating the arbitrary Constants a and b from

$$Z = ax + by + \frac{a}{b} - b$$

Sol: given $Z = ax + by + \frac{a}{b} - b \longrightarrow (1)$

differentiating eqn (1) Partially w.r.to x and y we have

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a \longrightarrow (2)$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \longrightarrow (3)$$

now Substitute a and b values in eqn (1') we have (3')

$$Z = px + qy + \frac{p}{q} - q$$ is the required PDE.

(3) Form the PDE by eliminating arbitrary Constants a and b from $Z = (x-a)^2 + (y-b)^2 + 1$

Sol :- given $Z = (x-a)^2 + (y-b)^2 + 1 \longrightarrow (1)$

differentiating eqn (1) Partially w.r.to x and y we have

$$\frac{\partial Z}{\partial x} = 2(x-a) \Rightarrow P = 2(x-a) \quad (\text{or}) \quad (x-a) = \frac{P}{2} \longrightarrow (2)$$

$$\frac{\partial Z}{\partial y} = 2(y-b) \Rightarrow q = 2(y-b) \quad (\text{or}) \quad (y-b) = \frac{q}{2} \longrightarrow (3)$$

now on Substituting these values of $(x-a)$ and $(y-b)$ in (1) we get

$$Z = \left(\frac{P}{2}\right)^2 + \left(\frac{q}{2}\right)^2 + 1 \quad (\text{or}) \quad Z = \frac{P^2}{4} + \frac{q^2}{4} + 1.$$

$$(\text{or}) \quad 4Z = P^2 + q^2 + 4 //.$$

(4) Eliminate h, K from $(x-h)^2 + (y-K)^2 + Z^2 = a^2$

Sol :- given $(x-h)^2 + (y-K)^2 + Z^2 = a^2 \longrightarrow (1)$

differentiate eqn (1) Partially w.r.to x and y we have

$$2(x-h) + 2z\frac{\partial Z}{\partial x} = 0$$

$$\Rightarrow (x-h) + PZ = 0 \longrightarrow (2)$$

$$2(y-k)+2z\frac{\partial z}{\partial y}=0$$

$$\Rightarrow (y-k)+qz=0 \longrightarrow (3)$$

from (2) we have $x-h=-pz$ and

from (3) we have $y-k=-qz$

now substituting these values of $(x-h)$ and $(y-k)$ in (1) we get

$$(-pz)^2+(-qz)^2+z^2=a^2$$

$$\Rightarrow p^2z^2+q^2z^2+z^2=a^2$$

$$\Rightarrow z^2(p^2+q^2+1)=a^2$$ is the required P.D.E.

(5) Form the PDE by eliminating arbitrary Constants a, b from

$$2z=(x+a)^{1/2}+(y-a)^{1/2}+b.$$ $\left(\frac{\partial}{\partial x}\sqrt{x}=\frac{1}{2\sqrt{x}}\right)$

Sol :- given $2z=(x+a)^{1/2}+(y-a)^{1/2}+b$

$$2z=\sqrt{x+a}+\sqrt{y-a}+b \longrightarrow (1)$$

Differentiate (1) Partially w.r.to x and y we get

$$2\frac{\partial z}{\partial x}=2p=\frac{1}{2\sqrt{x+a}} \Rightarrow \sqrt{x+a}=\frac{1}{4p} \longrightarrow (2)$$

$$2\frac{\partial z}{\partial y}=2q=\frac{1}{2\sqrt{y-a}} \Rightarrow \sqrt{y-a}=\frac{1}{4q} \longrightarrow (3)$$

from (2) $\Rightarrow \sqrt{x+a}=\frac{1}{4p} \Rightarrow x+a=\frac{1}{16p^2}$ (SQ.B.S)

$$\Rightarrow a=\frac{1}{16p^2}-x \longrightarrow (4)$$

from ③ $\Rightarrow \sqrt{y-a} = \frac{1}{4q} \Rightarrow y-a = \frac{1}{16q^2}$ ④

$$\Rightarrow a = y - \frac{1}{16q^2} \longrightarrow ⑤$$

from ④ & ⑤

$$\frac{1}{16p^2} - x = y - \frac{1}{16q^2}$$

$\Rightarrow \frac{1}{p^2} + \frac{1}{q^2} = 16(x+y)$ which is the required PDE.

(6) Eliminate a, b from $(x-a)^2 + (y-b)^2 + z^2 = r^2$ where r be Parameter.

Sol :- given $(x-a)^2 + (y-b)^2 + z^2 = r^2 \longrightarrow ①$

differentiate eqn ① Partially w.r.to 'x' we get

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow (x-a) + zp = 0 \Rightarrow (x-a) = -zp \longrightarrow ②$$

differentiate eqn ① Partially w.r.to 'y' we get

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow (y-b) + zq = 0 \Rightarrow (y-b) = -zq \longrightarrow ③$$

now substitute ② and ③ in ① we get.

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$

$$\Rightarrow z^2p^2 + z^2q^2 + z^2 = r^2$$

$$\Rightarrow z^2[p^2+q^2+1] = r^2 \Rightarrow z^2 = \frac{r^2}{p^2+q^2+1}$$

(7) Eliminate a, b from $Z = ax^3 + by^3$

Sol :- given $Z = ax^3 + by^3 \longrightarrow (1)$

diff eqn (1) Partially w.r.to 'x' we get

$$\frac{\partial Z}{\partial x} = 3ax^2 \Rightarrow p = 3ax^2 \Rightarrow a = \frac{p}{3x^2} \longrightarrow (2)$$

diff eqn (1) Partially w.r.to 'y' we get

$$\frac{\partial Z}{\partial y} = 3by^2 \Rightarrow q = 3by^2 \Rightarrow b = \frac{q}{3y^2} \longrightarrow (3)$$

now substitute (2) and (3) in (1) we get

$$Z = \left(\frac{p}{3x^2}\right)x^3 + \left(\frac{q}{3y^2}\right)y^3$$

$Z = \frac{1}{3}\left[px + qy\right]$ is the required PDE

(8) Eliminate a, b from

$$(x-a)^2 + (y-b)^2 = Z^2 \cot^2\alpha$$

where α is a Parameter.

Sol :- given $(x-a)^2 + (y-b)^2 = Z^2 \cot^2\alpha \longrightarrow (1)$

P.D. w.r.to 'x' on b.s

$$2(x-a) = 2Z\frac{\partial Z}{\partial x}\cot^2\alpha$$

$$(x-a) = Zp\cot^2\alpha \longrightarrow (2)$$

P.D. w.r.to 'y' on b.s

$$2(y-b) = 2Z\frac{\partial Z}{\partial y}\cot^2\alpha$$

$$(y-b) = Zq\cot^2\alpha \longrightarrow (3)$$

Substitute (2) & (3) in (1) we get (5)

$$(Z\,p\cot^2\alpha)^2 + (Z\,q\cot^2\alpha)^2 = Z^2\cot^2\alpha$$

$$\Rightarrow Z^2p^2\cot^4\alpha + Z^2q^2\cot^4\alpha = Z^2\cot^2\alpha$$

$$\Rightarrow Z^2\cot^4\alpha\,(p^2+q^2) = Z^2\cot^2\alpha$$

$$p^2+q^2 = \frac{1}{\cot^2\alpha} \quad (\text{or}) \quad p^2+q^2 = \tan^2\alpha$$

9) Eliminate a, b from

$$Z = axe^{y} + \frac{1}{2}a^2e^{2y} + b.$$

Sol :- given $Z = axe^{y} + \frac{1}{2}a^2e^{2y} + b \longrightarrow (1)$

Partially diff (1) w.r.to 'x' we get.

$$\frac{\partial Z}{\partial x} = p = ae^{y}$$

$$\Rightarrow e^{y} = \frac{p}{a} \longrightarrow (2)$$

Partially diff (1) w.r.to 'y' we get.

$$\frac{\partial Z}{\partial y} = q = axe^{y} + \frac{a^2}{\cancel{2}}\cdot\cancel{2}e^{2y}$$

$$q = axe^{y} + a^2e^{2y} \longrightarrow (3)$$

now substitute (2) in (3) we get.

$$q = \cancel{a}x\left(\frac{p}{\cancel{a}}\right) + \cancel{a^2}\left(\frac{p^2}{\cancel{a^2}}\right) \quad \left(\because (e^{y})^2 = e^{2y}\right)$$

$\boxed{q = px + p^2}$ is the required PDE

***10) Form the Differential eqn by eliminating a & b from $\log(az-1) = x + ay + b$

Sol: given $\log(az-1) = x + ay + b \longrightarrow (1)$ $\quad \left(\log x = \frac{1}{x}\right)$

diff Partially w.r.to x on b.s

$$\frac{1}{az-1} \cdot a\frac{\partial z}{\partial x} = 1$$

$$\Rightarrow \frac{1}{az-1} \cdot ap = 1 \Rightarrow ap = (az-1) \longrightarrow (2)$$

diff Partially w.r.to y on b.s

$$\Rightarrow \frac{1}{az-1} \cdot a\frac{\partial z}{\partial y} = a$$

$$\frac{1}{az-1} \cdot aq = a \Rightarrow aq = a(az-1)$$

$$q = (az-1) \longrightarrow (3)$$

$$\frac{(3)}{(2)} \Rightarrow \frac{q}{ap} = \frac{\cancel{(az-1)}}{\cancel{(az-1)}} \quad \left(\text{or } (2) = (3) \Rightarrow ap = q\right)$$

$$\frac{q}{ap} = 1 \Rightarrow \frac{q}{p} = a \Rightarrow ap = q \longrightarrow (4)$$

$$ap = q$$
$$a = \frac{q}{p}$$

now substitute (4) in (2) we get.

$$q = (az-1) \quad \left(a = \frac{q}{p}\right)$$

$$q = \frac{q}{p} \cdot z - 1$$

$$\Rightarrow pq = qz - p \quad \text{(or)} \quad p(q+1) = qz \,//.$$

$$\Rightarrow \cancel{pq = qz + p}$$

$$pq + p = qz$$

$$p(q+1) = qz \,//$$

(6)

11) Find the differential equation of all planes passing through the Origin

Sol :- The required eqn of the plane passing through the Origin is $ax + by + cz = 0 \longrightarrow (1)$

$ax+by+cz+d=0$. Since it passes through origin. eqn should satisfy $d=0$.

where a, b, c are arbitrary Constants

P.D w.r.to 'x' on b.s

$$a + c\frac{\partial z}{\partial x} = 0 \Rightarrow a + cp = 0 \Rightarrow a = -cp \longrightarrow (2)$$

P.D w.r.to 'y' on b.s

$$b + c\frac{\partial z}{\partial y} = 0 \Rightarrow b + cq = 0 \Rightarrow b = -cq \longrightarrow (3)$$

Substitute (2) & (3) in (1) we get

$$-cpx - cqy + cz = 0$$

$$cpx + cqy - cz = 0 \Rightarrow \cancel{c}px + \cancel{c}qy = \cancel{c}z$$

$$\Rightarrow px + qy = z$$

$\therefore z = px + qy$ which is the required PDE

12) Form the differential eqn of all planes, having equal intercepts on x and y axis (~~passing through~~)

Sol :- The required eqn of the planes having equal intercepts on x and y axis is

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \longrightarrow (1)$$

Diff part w.r.to 'x' we get

$$\frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{1}{a} + \frac{1}{c}p = 0 \Rightarrow \frac{1}{a} = -\frac{1}{c}p \rightarrow (2)$$

Diff part w.r.to 'y' we get

$$\frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{1}{a} + \frac{1}{c}q = 0 \Rightarrow \frac{1}{a} = -\frac{1}{c}q \rightarrow (3)$$

now from (2) and (3) we have

$$-\frac{1}{c}p = -\frac{1}{c}q \Rightarrow p = q$$

$\therefore p = q$ is the required PDE

13) Find the differential eqn of all spheres of radius 5 having their centres in the xy plane.

Sol:- The eqn of the family of spheres having their centres in the xy plane and having radius 5 is

$$(x-a)^2+(y-b)^2+z^2=25 \longrightarrow (1)$$

now Diff Partially w.r.to 'x' we get.

$$2(x-a)+2z\frac{\partial z}{\partial x}=0 \Rightarrow 2(x-a)+2zp=0.$$

$$\Rightarrow (x-a)+zp=0.$$

$\Rightarrow \cancel{(x-a)+zp=0}$

$$\Rightarrow (x-a)=-zp \longrightarrow (2)$$

again Diff Partially w.r.to 'y' we get.

$$2(y-b)+2z\frac{\partial z}{\partial y}=0 \Rightarrow 2(y-b)+2zq=0$$

$$\Rightarrow (y-b)+zq=0$$

$$\Rightarrow (y-b)=-zq \longrightarrow (3).$$

now substituting these values of $(x-a)$ & $(y-b)$ in (1) we get.

$$z^2p^2+z^2q^2+z^2=25 \quad (or)$$

$$z^2(p^2+q^2+1)=25.$$ This is the required DE.

14) Form the PDE by eliminating the arbitrary constants from $2z=\sqrt{x+a}+\sqrt{y+b}$

Sol:- given $2z=\sqrt{x+a}+\sqrt{y+b} \longrightarrow (1).$ $\left(\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}\right)$

Diff Partially (1) w.r.to 'x' we get.

$$2\frac{\partial z}{\partial x}=2p=\frac{1}{2\sqrt{x+a}} \Rightarrow \sqrt{x+a}=\frac{1}{4p} \longrightarrow (2).$$

Diff Partially (1) w.r.to 'y' we get

$$2\frac{\partial z}{\partial y}=2q=\frac{1}{2\sqrt{y+b}} \rightarrow \sqrt{y+b}=\frac{1}{4q} \longrightarrow (3).$$

now substituting (2) and (3) in (1) we get. (7)

$$2Z = \frac{1}{4p} + \frac{1}{4q}$$

$$\Rightarrow 8Z = \frac{1}{p} + \frac{1}{q}.$$

$$8Z = \frac{p+q}{pq} \Rightarrow 8pqZ = p+q$$

which is the required Partial differential eqn

*** 15) Form the Partial Differential eqn by eliminating the arbitrary Constants

$$Z = (x^2+a)(y^2+b)$$

Sol :- given $Z = (x^2+a)(y^2+b) \longrightarrow (1)$

P.D. w.r.to 'x' we get.

$$p = \frac{\partial z}{\partial x} = 2x(y^2+b) \Rightarrow p = 2x(y^2+b)$$

$$\Rightarrow \frac{p}{2x} = y^2+b \longrightarrow (2).$$

P.D w.r.to 'y' we get.

$$q = \frac{\partial z}{\partial y} = 2y(x^2+a) \Rightarrow q = 2y(x^2+a)$$

$$\Rightarrow \frac{q}{2y} = x^2+a \longrightarrow (3)$$

Substituting these values of (y^2+b) and (x^2+a) from (2) & (3) in (1) we get.

Substitute (2) and (3) values in (1) we get.

$$Z = \frac{p}{2x} \cdot \frac{q}{2y} \Rightarrow pq = 4xyz.$$

which is the required Partial DE

**16) Form a Partial differential eqn by eliminating the arbitrary Constants

$$x^2+y^2+(z-c)^2=a^2$$

(or)

Find the Partial differential eqn of all spheres whose Centres lie on Z-axis with a given radius r.

Sol :- The eqn of the family of spheres having their Centres on Z-axis and having radius r is

$$(x-0)^2+(y-0)^2+(z-c)^2=r^2$$

ie :- $x^2+y^2+(z-c)^2=r^2 \longrightarrow (1)$

where c and r are arbitrary Constants

Diff (1) Partially w.r.to 'x' we get

$$2x+2(z-c)\frac{\partial z}{\partial x}=0$$

(or) $x+(z-c)p=0 \longrightarrow (2)$

Diff (1) Partially w.r.to 'y' we get

$$2y+2(z-c)\frac{\partial z}{\partial y}=0$$

(or) $y+(z-c)q=0 \longrightarrow (3)$

from (2) $z-c=-\dfrac{x}{p} \longrightarrow (4)$

from (3) $z-c=-\dfrac{y}{q} \longrightarrow (5)$

(4) = (5) $-\dfrac{x}{p}=-\dfrac{y}{q} \Rightarrow \dfrac{x}{p}=\dfrac{y}{q}$ (or) $xq-yp=0$.

This is the required differential eqn

(8)

2) Elimination of Arbitrary functions :-

~~Let $U = (x, y, z)$ and~~

~~$V = f(x$~~

Let $U = U(x, y, z)$ and $V = V(x, y, z)$ be Independent functions of the variables x, y, z and

Let $\phi(u, v) = 0 \longrightarrow (1)$ be an arbitrary relation between them.

we shall obtain a Partial differential eqn by eliminating the functions u, v regarding Z as the dependent variable and differentiating (1) Partially w.r.to x and y then we get the solution which is the required PDE.

***

1) Form a PDE by eliminating arbitrary functions from

$$Z = f(x^2 + y^2).$$

Sol :- given $Z = f(x^2 + y^2)$

Put $u = x^2 + y^2$ then $Z = f(u)$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = 2y$$

$$Z = f(u) \longrightarrow (1)$$

now diff eqn (1) Partially w.r.to 'x' we get

$$\frac{\partial Z}{\partial x} = f'(u)\frac{\partial u}{\partial x} \quad \left(\because \frac{\partial u}{\partial x} = 2x, \ \frac{\partial Z}{\partial x} = p\right)$$

$$p = f'(u)\, 2x \longrightarrow (2)$$

Illy diff eqn (1) Partially w.r.to 'y' we get

$$\frac{\partial Z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

$$q = f'(u)(2y) \longrightarrow (3)$$

now $\frac{(2)}{(3)}$ $\frac{p}{q} \times \frac{2x\, \cancel{f'(u)}}{2y\, (\cancel{f'(u)})}$ [CM] $\cancel{2}yp = \cancel{2}xq$

$py - qx = 0$ which is required PDE

(ii) $Z = f(x^2 - y^2)$

Put $x^2 - y^2 = u$ then $Z = f(u)$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2y$$

$$Z = f(u) \longrightarrow (1)$$

Diff eqn (1) Partially w.r.to 'x' we get

$$\frac{\partial Z}{\partial x} = f'(u)\frac{\partial u}{\partial x}$$

$$p = f'(u)(2x) \longrightarrow (2)$$

Diff eqn (1) Partially w.r.to 'y' we get

$$\frac{\partial Z}{\partial y} = f'(u)(-2y)$$

$$q = f'(u)(-2y) \longrightarrow (3)$$

now (2) ÷ (3) $\quad \frac{p}{q} \times \frac{2x(\cancel{f'(u)})}{-2y(\cancel{f'(u)})}$ [on CM]

$$-\cancel{2}yp = \cancel{2}xq$$

$py + qx = 0$ which is the required PDE

(2) Form a PDE by eliminating arbitrary function from

$$Z = f(x) + e^y g(x)$$

Sol :- given $Z = f(x) + e^y g(x) \longrightarrow (1)$

Diff (1) Partially w.r.to 'x' we get

$$\frac{\partial Z}{\partial x} = f'(x) + e^y g'(x) \longrightarrow (2) \quad \left(e^y \text{ is Constant w.r.to } x\right)$$

Ily Diff (1) Partially w.r.to 'y' we get

$$\frac{\partial Z}{\partial y} = e^y g(x) \longrightarrow (3)$$

(9)

Again Diff eqn (3) Partially w.r.to 'y' we get

$$\frac{\partial^2 z}{\partial y^2} = e^y g(x) = \frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = 0$$

$\Rightarrow \underline{\underline{t - q = 0}}$ $\underline{\underline{\text{which is}}}$ the required PDE

3) Form a PDE by eliminating the arbitrary functions ϕ from $lx + my + nz = \phi(x^2+y^2+z^2)$

Sol:- given $lx + my + nz = \phi(x^2+y^2+z^2) \longrightarrow (1)$

Diff (1) Partially w.r.to 'x' we get

$$l + n\frac{\partial z}{\partial x} = \phi'(x^2+y^2+z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right) \rightarrow (2)$$

Diff (1) Partially w.r.to 'y' we get

$$m + n\frac{\partial z}{\partial y} = \phi'(x^2+y^2+z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right) \rightarrow (3)$$

now (2) $\div$ (3)

$$\frac{l+np}{m+nq} = \frac{\cancel{\phi'(x^2+y^2+z^2)}(\cancel{2}x + \cancel{2}zp)}{\cancel{\phi'(x^2+y^2+z^2)}(\cancel{2}y + \cancel{2}zq)}$$

$$\frac{l+np}{m+nq} = \frac{x+pz}{y+qz} \quad [\text{on CM}]$$

$$\Rightarrow (l+np)(y+qz) = (m+nq)(x+pz)$$

$$\Rightarrow (l+np)y + \underline{\underline{z(lq - mp)}} = (m+nq)x$$

which $\underline{\underline{\text{is the required}}}$ PDE

*** (4) Form the PDE by eliminating arbitrary function f from $xyz = f(x^2+y^2+z^2)$

Sol: given $xyz = f(x^2+y^2+z^2) \longrightarrow (1)$

PD w.r.to 'x' we get

$$yz + xy\frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow yz + xyp = f'(x^2+y^2+z^2)(2x+2zp) \longrightarrow (2)$$

PD w.r.to 'y' we get

$$xz + xy\frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right)$$

$$\Rightarrow xz + xyq = f'(x^2+y^2+z^2)(2y+2zq) \longrightarrow (3)$$

now (2) ÷ (3)

$$\frac{(2)}{(3)} \Rightarrow \frac{yz+xyp}{xz+xyq} = \frac{\cancel{f'(x^2+y^2+z^2)}(2x+2zp)}{\cancel{f'(x^2+y^2+z^2)}(2y+2zq)}$$

$$\Rightarrow \frac{yz+xyp}{xz+xyq} = \frac{x+zp}{y+zq} \quad [\text{C.M}]$$

$$\Rightarrow (yz+xyp)(y+zq) = (x+zp)(xz+xyq)$$

which is the required PDE

(10)

5) a) Form the PDE by eliminating arbitrary function f from $Z = xy + f(x^2+y^2)$

Sol :- given $Z = xy + f(x^2+y^2) \longrightarrow (1)$

Diff Partially (1) w.r.to 'x' we get

$$\frac{\partial z}{\partial x} = p = y + f'(x^2+y^2)(2x)$$

$$\Rightarrow p - y = 2x\, f'(x^2+y^2) \longrightarrow (2)$$

Diff Partially (1) w.r.to 'y' we get

$$\frac{\partial z}{\partial y} = q = x + f'(x^2+y^2)(2y)$$

$$\Rightarrow q - x = 2y\, f'(x^2+y^2) \longrightarrow (3)$$

(2) ÷ (3)

$$\frac{(2)}{(3)} \Rightarrow \frac{p-y}{q-x} = \frac{\cancel{2}x\, \cancel{f'(x^2+y^2)}}{\cancel{2}y\, \cancel{f'(x^2+y^2)}}$$

$$\Rightarrow \frac{p-y}{q-x} = \frac{x}{y} \quad [\text{C.M}]$$

$$\Rightarrow (p-y)y = x(q-x)$$

$$\Rightarrow py - y^2 = qx - x^2$$

$$\Rightarrow x^2 - y^2 = qx - py$$

which is the required solution

⑥ Form the PDE by eliminating arbitrary function from

$$Z = f(y) + \phi(x+y)$$

Sol :- given $Z = f(y) + \phi(x+y) \longrightarrow (1)$

Diff (1) Partially w.r.to 'x' we get

$$\frac{\partial z}{\partial x} = p = \phi'(x+y) \longrightarrow (2)$$

Diff (1) Partially w.r.to 'y' we get

$$\frac{\partial z}{\partial y} = q = f'(y) + \phi'(x+y) \longrightarrow (3)$$

now again diff (2) Partially w.r.to 'x' we get

$$\frac{\partial^2 z}{\partial x^2} = r = \phi''(x+y) \longrightarrow (4)$$

$$\frac{\partial^2 z}{\partial x \partial y} = s = \phi''(x+y) \longrightarrow (5)$$

now from (4) & (5) we have $r = s$ (or) $r - s = 0$

$\therefore$ $r - s = 0$ is the required PDE //.

Here we note that two arbitrary functions f and ϕ are given
$\therefore$ after elimination of these two functions we get second order PDE
Hence we have to find PDE upto Second Order.
If elimination is not Possible we should find 3rd Order PDE and so on.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

(11)

**(7) Form the PDE by eliminating the arbitrary function $z = f(x^2+y^2+z^2)$

Sol: given $z = f(x^2+y^2+z^2) \longrightarrow (1)$

diff (1) partially w.r.to 'x' we get

$$\frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right)$$

$$p = f'(x^2+y^2+z^2)(2x+2zp) \longrightarrow (2)$$

diff (1) partially w.r to 'y' we get

$$\frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right)$$

$$q = f'(x^2+y^2+z^2)(2y+2zq) \longrightarrow (3)$$

now (2) ÷ (3) we have

$$\frac{(2)}{(3)} = \frac{p}{q} = \frac{\cancel{f'(x^2+y^2+z^2)}(\cancel{2}x+\cancel{2}zp)}{\cancel{f'(x^2+y^2+z^2)}(\cancel{2}y+\cancel{2}zq)}$$

$$\frac{p}{q} = \frac{x+zp}{y+zq} \quad [CM]$$

$$p(y+zq) = q(x+zp)$$

$$\Rightarrow py + \cancel{pzq} = qx + \cancel{qzp}$$

(or)

$$y+q = x+p$$

from (2) we have

$$f'(x^2+y^2+z^2) = \frac{p}{2x+2zp} \rightarrow (4)$$

now substitute (4) in (3) we get

$$q = \frac{p}{2x+2zp}(2y+2zq)$$

$$\Rightarrow 2qx + \cancel{2qpz} = 2py + \cancel{2zpq}$$

$$\cancel{2}qx = \cancel{2}py$$

$$py - qx = 0$$

This is the req. PDE of 1st Order

(8) Form a PDE by eliminating the arbitrary functions $f(x)$ and $g(y)$ from

$$Z = y\,f(x) + x\,g(y)$$

Sol :- given $Z = y f(x) + x g(y) \longrightarrow (1)$

Here we have to eliminate two arbitrary functions f and g

Diff (1) Partially w.r.to 'x' we get

$$\frac{\partial Z}{\partial x} = p = y f'(x) + g(y) \longrightarrow (2)$$

Diff (1) Partially w.r.to 'y' we get.

$$\frac{\partial Z}{\partial y} = q = f(x) + x g'(y) \longrightarrow (3)$$

now since the relations (1) (2) and (3) are not sufficient to eliminate f, g, f', g' so we find 2nd Order Partial derivatives

$$\frac{\partial^2 Z}{\partial x^2} = r = y f''(x) \longrightarrow (4)$$

$$\frac{\partial^2 Z}{\partial x \partial y} = s = f'(x) + g'(y) \longrightarrow (5)$$

$$\frac{\partial^2 z}{\partial y^2} = t = x g''(y) \longrightarrow (6)$$

from (2) and (3) we have

$$\left.\begin{aligned} f'(x) &= \frac{1}{y}[p - g(y)] \\ g'(y) &= \frac{1}{x}[q - f(x)] \end{aligned}\right\} \longrightarrow (7)$$

(12)

from (8) we have

$$s = f'(x) + g'(y)$$

$$s = \frac{1}{y}[p - g(y)] + \frac{1}{x}[q - f(x)] \quad (\text{using } (7))$$

ie : $xys = x[p - g(y)] + y[q - f(x)]$

$$xys = px + qy - [yf(x) + xg(y)]$$

(or)

$$xys = px + qy - Z \quad (\text{using } (1))$$

This is the required PDE of order two after ~~the~~ eliminating two arbitrary functions.

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(9) Form the PDE by eliminating arbitrary functions f & g from $Z = f(x+ct) + g(x-ct)$ (or)

$Z = f(x+at) + g(x-at)$ (JNTU(A) June 2014)

Sol : given $Z = f(x+ct) + g(x-ct) \longrightarrow (1)$

Diff (1) Partially w.r.to 'x' we get

$$\frac{\partial Z}{\partial x} = f' + g' \longrightarrow (2)$$

Diff (1) Partially w.r.to 't' we get

$$\frac{\partial Z}{\partial t} = cf' - cg' \longrightarrow (3)$$

Since the relations (1) (2) & (3) are not sufficient to eliminate f, g f' and g' so we find second order PDE

$$\frac{\partial^2 Z}{\partial x^2} = f'' + g'' \rightarrow (4) \quad \text{and} \quad \frac{\partial^2 Z}{\partial t^2} = c^2 f'' + c^2 g''$$
$$= c^2 (f'' + g'') \rightarrow (5)$$

Substitute (4) in (5) we get $\frac{\partial^2 Z}{\partial t^2} = c^2 \frac{\partial^2 Z}{\partial x^2}$

which is required PDE of second order and ~~[illegible]~~ its solution//

10) Form the PDE by eliminating the arbitrary functions ϕ_1 and ϕ_2 from

$$Z = \phi_1(x)\,\phi_2(y)$$

Sol :- given $Z = \phi_1(x)\,\phi_2(y) \longrightarrow (1)$

Diff (1) Partially w.r.to 'x' we get

$$p = \frac{\partial z}{\partial x} = \phi_1' \cdot \phi_2 \Rightarrow p = \phi_1' \cdot \phi_2 \longrightarrow (2)$$

Diff (1) Partially w.r.to 'y' we get.

$$q = \frac{\partial z}{\partial y} = \phi_1 \cdot \phi_2' \Rightarrow q = \phi_1 \phi_2' \longrightarrow (3)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \phi_1' \phi_2' \longrightarrow (4)$$

now (2) $\times$ (3) gives

$$pq = \phi_1' \phi_2 \cdot \phi_1 \phi_2'$$

$$= \phi_1 \phi_2 \cdot \phi_1' \phi_2'$$

$$= Zs \quad (\text{using } (1) \,\&\, (4))$$

$\therefore pq = Zs$

This is the required Partial differential eqn.

(13)

Method of Seperation of Variables.

1) Solve $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0$.

Sol: given $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0 \longrightarrow (1)$

Let $Z = X(x)\,Y(y) \longrightarrow (2)$ be the solution of (1)

where $X(x)$ is the function of x alone

$Y(y)$ is the function of y alone.

~~otherwise $X(x)$ is the function of x alone~~

~~$Y(y)$ is the function of y alone.~~

we get $\frac{\partial z}{\partial x} = X'Y$ $\quad \frac{\partial z}{\partial y} = XY'$

from (1) $y^3 X'Y + x^2 XY' = 0$

$\Rightarrow y^3 X'Y = -x^2 XY'$

$\Rightarrow \frac{X'}{x^2 X} = \frac{-Y'}{y^3 Y}$

In the above the LHS is a function of x and RHS is a function of y and these are equal for all values of x and y this is possible $\Longleftrightarrow$ each is equal to the same constant (say) λ

This λ is called as seperation constant.

$\therefore$ we have

$$\frac{X'(x)}{x^2 X(x)} = \frac{-Y'(y)}{y^3\, Y(y)} = \lambda \longrightarrow (3)$$

from ③ we get the two ODE (Ordinary Diff Eqns)

$$\frac{X'(x)}{x^2 X(x)} = \lambda \qquad\qquad \frac{-Y'(y)}{y^3 Y(y)} = \lambda$$

$\Rightarrow X' = x^2 X(x) \cdot \lambda$ (ODE)

let us solve.

$$X' = \lambda x^2 X(x)$$

$$\Rightarrow \frac{dX}{dx} = x^2 X \lambda$$

$$\Rightarrow \frac{dX}{X} = x^2 \lambda dx$$

on Integrating b.s

$$\Rightarrow \int \frac{dX}{X} = \int x^2 \lambda dx$$

$$\log X = \lambda \frac{x^3}{3} + \log c \text{ (or)}$$

$$\underline{\underline{X = Ae^{\lambda x^3/3}}}$$

$\Rightarrow Y' = -\lambda y^3 Y(y)$

$$\frac{dY}{dy} = -\lambda y^3 Y$$

$$\Rightarrow \frac{dY}{Y} = -\lambda y^3 dy$$

on Integrating b.s

$$\int \frac{dY}{Y} = \int -\lambda y^3 dy$$

$$\Rightarrow \log Y = -\lambda \frac{y^4}{4} + \log c$$

$$Y = Be^{-\lambda y^4/4}.$$

from ② $Z = X(x) Y(y)$

$$Z = ABe^{\lambda\left[\frac{x^3}{3} - \frac{y^4}{4}\right]}$$

$$Z = Ce^{\lambda\left[\frac{x^3}{3} - \frac{y^4}{4}\right]}$$

where C is any arbitrary Constant //

2) Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$ (14)

(or)

Solve by method of seperation of variables

$u_x = 2u_t + u$ where $u(x,0) = 6e^{-3x}$

Sol :- we have to find $u(x,t)$ such that

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \longrightarrow (1)$$

Subject to the Condition $u(x,0) = 6e^{-3x}$

Using the method of seperation of variable we seek a solution of (1) in the form

$u(x,t) = X(x)\,T(t) \longrightarrow (2)$ be a solution

where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone

$$\frac{\partial u}{\partial x} = X'T \qquad \frac{\partial u}{\partial t} = XT'$$

from (1) $X'T = 2XT' + XT$

$$X'T = X[2T' + T]$$

$$\frac{X'}{X} = \frac{2T' + T}{T}$$

Let $\frac{X'}{X} = \frac{2T'+T}{T} = \lambda$

Since LHS is a function of x and RHS is a function of t the equality is valid $\forall$ x and t

$\Longleftrightarrow$ each is equal to the same Constant λ $\forall$ x and t

$$\Rightarrow \frac{X'}{X} = \frac{2T'+T}{T} = \lambda$$

let $\frac{X'}{X} = \lambda$

$X' = \lambda X$

$\frac{dX}{dx} = \lambda X$

$\Rightarrow \frac{dX}{X} = \lambda\, dx$

Integration on b.s

$\int \frac{dX}{X} = \int \lambda\, dx$

$\Rightarrow \log X = \lambda x + \log c$

$X = Ae^{\lambda x}$

$\frac{2T'+T}{T} = \lambda$

$2T'+T = \lambda T$

$T' = \frac{T(\lambda-1)}{2}$

$\Rightarrow \frac{dT}{dt} = \frac{T(\lambda-1)}{2}$

$\Rightarrow \frac{dT}{T} = \frac{(\lambda-1)}{2}dt$

Integrating on b.s

$\int \frac{dT}{T} = \int \frac{(\lambda-1)}{2}dt$

$\Rightarrow \log T = \frac{(\lambda-1)t}{2} + \log c$

$\Rightarrow T = Be^{\frac{(\lambda-1)t}{2}}$

$\therefore u(x,t) = ABe^{\lambda x + \frac{(\lambda-1)t}{2}}$

$= Ce^{\lambda x + \frac{\lambda-1}{2}t} \longrightarrow (3)$

Given $u(x,0) = 6e^{-3x}$

from (3) $Ce^{\lambda x} = 6e^{-3x} \rightarrow c=6 \quad \lambda=-3$

Substitute in (3)

$u(x,t) = 6e^{-3x}.e^{-2t}$

$u(x,t) = 6e^{-(3x+2t)}$

$X(x)$ means terms of x
T interms of t
X interms of x.

$e^x.e^y = e^{x+y}$

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$$\frac{X'}{x^2 X} = \lambda \qquad \frac{-Y'}{y^3 Y} = \lambda$$

$$\Rightarrow \frac{X'}{X} = x^2 \lambda \qquad \Rightarrow \frac{Y'}{Y} = -y^3 \lambda$$

solving weget, solving weget,

$$\log X = \lambda \frac{x^3}{3} + \log c_1 \qquad \log Y = -\frac{y^4}{4}\lambda + \log c_2$$

$$X = e^{\frac{\lambda x^3}{3} + \log c_1}$$

$$= e^{\frac{\lambda x^3}{3}} \cdot e^{\log c_1}$$

$$X = c_1 e^{\frac{\lambda x^3}{3}}$$

$$\cancel{Y = c_2 e^{-\frac{y^4}{4}\lambda}}$$

$$Y = e^{-\frac{y^4}{4}\lambda + \log c_2}$$

$$= e^{-\frac{y^4}{4}\lambda} \cdot e^{\log c_2}$$

$$= c_2 e^{-\frac{y^4}{4}\lambda}$$

$$Z = c_1 e^{\frac{\lambda x^3}{3}} \cdot c_2 e^{-\frac{y^4}{4}\lambda}$$

$$Z = c_1 c_2 e^{\frac{\lambda x^3}{3} - \frac{y^4 \lambda}{4}}$$

① Solve $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0$.

Sol :- given $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0 \longrightarrow ①$

Let $Z = X(x)Y(y)$ be the solution of ②

where $X(x)$ is a function of x alone

$Y(y)$ is a function of y alone

$\frac{\partial Z}{\partial x} = X'Y \qquad \frac{\partial z}{\partial y} = XY'$

$\boxed{\log x = m \iff x = e^m}$

now substitute in ① we get.

$y^3 X'Y + x^2 XY' = 0$

$y^3 X'Y = -Y'x^2 X$

$\Rightarrow \frac{X'}{x^2 X} = \frac{-Y'}{y^3 Y} = \lambda.$

$\frac{X'}{x^2 X} = \lambda$

on Integrating

$\int \frac{X'}{X} = \int x^2 \lambda$

on solving we get

$\log X = \lambda \frac{x^3}{3} + \log c_1$

$X = e^{\frac{\lambda x^3}{3} + \log c_1}$

$X = e^{\frac{\lambda x^3}{3}} \cdot \cancel{e}^{\cancel{\log} c_1}$

$X = c_1 e^{\frac{\lambda x^3}{3}} \longrightarrow ③$

$\int \sin x \, dx$

$= -\cos x + C$

let $\frac{-Y'}{y^3 Y} = \lambda$

$\frac{Y'}{Y} = -y^3 \lambda$

on solving we get.

$\log Y = -\frac{y^4}{4}\lambda + \log c_2$.

$Y = e^{-\frac{y^4}{4}\lambda + \log c_2}$ (Integral Const)

$Y = e^{-\frac{y^4}{4}\lambda} \cdot \cancel{e}^{\cancel{\log} c_2}$

$Y = C_2 e^{-\frac{y^4}{4}\lambda} \longrightarrow ④$.

now sub ③ & ④ in ② we get

$Z = c_1 e^{\frac{\lambda x^3}{3}} \cdot C_2 e^{-\frac{y^4}{4}\lambda}$

$Z = c_1 C_2 e^{\frac{\lambda x^3}{3} - \frac{y^4}{4}\lambda}$

(16)

(2) Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$

where $u(x,0) = 6e^{-3x}$

Sol :- given $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \longrightarrow (1)$.

Let the solution be

$u(x,t) = X(x)T(t) \longrightarrow (2)$

where $X(x)$ is a function of x alone

$T(t)$ is a function of t alone.

$\boxed{\log x = m \Longleftrightarrow x = e^m}$

$\frac{\partial u}{\partial x} = X'T \quad \frac{\partial u}{\partial t} = XT'$

now substitute in eqn (1) we get

$X'T = 2XT' + XT$

$X'T = X[2T' + T]$

$\Rightarrow \frac{X'}{X} = \frac{2T'+T}{T} = \lambda$.

Let $\frac{X'}{X} = \lambda$

on solving we get.

$\log X = \lambda x + \log c_1$

$X = e^{\lambda x + \log c_1}$

$X = e^{\lambda x} \cdot \cancel{e}^{\cancel{\log} c_1}$

$X = c_1 e^{\lambda x} \longrightarrow (3)$

Let $\frac{2T'+T}{T} = \lambda$

$\Rightarrow \frac{2T'}{T} + 1 = \lambda$

$\Rightarrow \frac{T'}{T} = \left(\frac{\lambda-1}{2}\right)$

$\Rightarrow \log T = \left(\frac{\lambda-1}{2}\right)t + \log c_2$

$T = e^{\left(\frac{\lambda-1}{2}\right)t + \log c_2}$

$T = e^{\left(\frac{\lambda-1}{2}\right)t} \cdot \cancel{e}^{\cancel{\log} c_2}$

$T = c_2 e^{\left(\frac{\lambda-1}{2}\right)t} \longrightarrow (4)$.

Subst (3) & (4) in (2) we get.

$u(x,t) = c_1 e^{\lambda x} \cdot c_2 e^{\left(\frac{\lambda-1}{2}\right)t} \rightarrow (5)$

But $u(x,0) = 6e^{-3x}$

$$u(x,t) = C_1 e^{\lambda x} \cdot C_2 e^{\left(\frac{\lambda-1}{2}\right)t} \longrightarrow (5)$$

But $u(x,0) = 6e^{-3x}$

$$u(x,0) = C_1 e^{\lambda x} \cdot C_2 e^{\left(\frac{\lambda-1}{2}\right)(0)} = 6e^{-3x}$$

$$= C_1 C_2 e^{\lambda x} = 6e^{-3x}$$

now $C_1 C_2 = 6$

$\lambda = -3$

$\left(\frac{-3-1}{2}\right)t$

$\frac{-4^2}{2_1} = -2t$

now substitute these in (5) we get.

$$u(x,t) = 6e^{-3x} \cdot e^{-2t}$$

$$u(x,t) = 6e^{-(3x+2t)}$$

3) Solve $u_x - 4u_y = 0$ given $u(0,y) = 8e^{-3y}$.

$$\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial y} = 0.$$

Sol :- given $u_x - 4u_y = 0 \longrightarrow (1)$

Let the solution be $u(x,y) = X(x)\,Y(y) \longrightarrow (2)$

where $X(x)$ is a function of x alone

$Y(y)$ is a function of y alone

$u_x = \frac{\partial u}{\partial x} = X'Y \qquad u_y = \frac{\partial u}{\partial y} = XY'$

now substitute these in (2) we get.

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$$X'Y - 4XY' = 0.$$

$$X'Y = 4XY'$$

$\log x = m \Leftrightarrow x = e^m$

$$\Rightarrow \frac{X'}{X} = \frac{4Y'}{Y} = K.$$

$\frac{X'}{X} = K.$

Solving we get

$\log X = Kx + \log c_1$

$\cancel{X = c_1 e^{Kx}}$

$X = e^{Kx + \log c_1}$

$X = e^{Kx} \cdot \cancel{e}^{\cancel{\log} c_1}$

$X = c_1 e^{Kx} \longrightarrow (3)$

$\frac{4Y'}{Y} = K.$

Solving.

$\frac{Y'}{Y} = \frac{K}{4}.$

$\log Y = \frac{K}{4} y + \log c_2$

$Y = e^{\frac{K}{4} y + \log c_2}$

$Y = e^{\frac{K}{4} y} \cdot \cancel{e}^{\cancel{\log} c_2}$

$Y = c_2 e^{\frac{K}{4} y} \longrightarrow (4)$

Sub (3) & (4) in (2) we get.

$$u(x,y) = c_1 e^{Kx} \cdot c_2 e^{\frac{K}{4} y}.$$

$$u(x,y) = c_1 c_2 e^{Kx + \frac{K}{4} y}$$

$$u(0,y) = c_1 c_2 e^{K(0) + \frac{K}{4} y} \quad (x = 0)$$

$$8e^{-3y} = c_1 c_2 e^{\frac{K}{4} y}$$

$$c_1 c_2 = 8 \quad \frac{K}{4} = -3 \Rightarrow K = -12.$$

$$\therefore u(x,y) = 8e^{-12x - 3y} = 8e^{-3(4x+y)}$$

(4) Solve $2x\frac{\partial Z}{\partial x} - 3y\frac{\partial Z}{\partial y} = 0$ by method of seperation of Variable.

Sol:- given $2x\frac{\partial Z}{\partial x} - 3y\frac{\partial Z}{\partial y} = 0 \longrightarrow (1)$

let $Z = X(x)\cdot Y(y) \longrightarrow (2)$ be solution of (1)

Then $\frac{\partial Z}{\partial x} = X'Y$ $\quad \frac{\partial Z}{\partial y} = XY'$ substituting these in (1) we get

$2xX'Y - 3yXY' = 0 \Rightarrow 2xX'Y = 3yXY'$

$\frac{2xX'}{X} = \frac{3yY'}{Y} = \lambda$

$\frac{2xX'}{X} = \lambda$

$\Rightarrow \int \frac{X'}{X} = \int \frac{\lambda}{2x}$

$\Rightarrow \log X = \frac{\lambda}{2}\int \frac{1}{x}dx$

$\Rightarrow \log X = \frac{\lambda}{2}\log x + \log c_1$

$\Rightarrow X = e^{\frac{\lambda}{2}\log x + \log c_1}$

$X = e^{\frac{\lambda}{2}\log x} \cdot \cancel{e^{\log c_1}}$

$X = c_1 x^{\frac{\lambda}{2}} \longrightarrow (3)$

$\frac{3yY'}{Y} = \lambda$

$\Rightarrow \int \frac{Y'}{Y} = \int \frac{\lambda}{3y}$

$\Rightarrow \log Y = \frac{\lambda}{3}\int \frac{dy}{y}$

$\Rightarrow \log y = \frac{\lambda}{3}\log y + \log c_2$

$\Rightarrow$ ~~$\log y$~~ $= e^{\frac{\lambda}{3}\cdot\log y + \log c_2}$

$Y = e^{\frac{\lambda}{3}\cdot\log y + \log c_2}$

$Y = e^{\frac{\lambda}{3}\cdot\log y} \cdot \cancel{e^{\log c_2}}$

$Y = c_2 y^{\lambda/3} \longrightarrow (4)$

Sub (3) & (4) in (2)

$Z = c_1 x^{\lambda/2} \cdot c_2 y^{\lambda/3}$

~~$Z = c_1 c_2 x$~~ $\Rightarrow$ ~~$Z = c_1 c_2 xy^{\frac{\lambda}{2}+\lambda/3}$~~

$X = e^{\log x^{\frac{\lambda}{2}}} \cdot e^{\log c_1}$

$= x^{\frac{\lambda}{2}} \cdot c_1$

$X = c_1 x^{\lambda/2}$

$Y = e^{\log y^{\frac{\lambda}{3}}} \cdot \cancel{e^{\log c_2}}$

$= y^{\frac{\lambda}{3}} \cdot c_2$

$Y \Rightarrow c_2 y^{\lambda/3}$