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# Unit - II

# Fourier Series

## Introduction :-

Fourier Series was due to French Mathematician Jean - Baptiste Joseph Fourier (1768-1830)

Suppose that a given function $f(x)$ defined in $[-\pi, \pi]$ (or) $[0, 2\pi]$ (or) in any other interval can be expressed as a Trigonometric Series

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

where a's and b's are Constants such a Series is known as Fourier Series for $f(x)$ and the Constants $a_0$, $a_n$ and $b_n$ $(n = 1, 2, 3 \ldots)$ are Called as Fourier Co-efficients of $f(x)$

## Euler's Formula :-

The Fourier Series for the function $f(x)$ in the interval $c \leq x \leq c + 2\pi$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

where

$$\boxed{a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx} \qquad \boxed{a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx}$$

$$\boxed{b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx}$$

∴ Values of $a_0$, $a_n$ and $b_n$ are known as "Euler's formula".

→ If $\boxed{c=0}$ then we have $\boxed{0 \le x \le 2\pi}$ then we have

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

where $\boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx}$ $\boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx}$

$$\boxed{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\sin nx\,dx}$$

→ If $\boxed{c=-\pi}$ then we have $\boxed{-\pi \le x \le \pi}$ then we have

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

where $\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$ $\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx}$$

**Periodic function :-**

A function $f(x)$ is said to be Periodic If $\boxed{f(x) = f(x+T)}$ $\forall\ x \in \mathbb{R}$ where $T$ is a least +ve Integer.

ie :- $f(x) = f(x+T) = f(x+2T) = f(x+3T) = \cdots\cdots$

**Ex** :- Tan x and Cot x are Periodic functions of Period $\underline{\pi}$.

→ Sin x, Cosec x, Sec x and Cos x are Periodic functions of Period $\underline{2\pi}$

## Conditions for Fourier Expansion (or) Dirchlet's Conditions :-

The German Mathematician Peter Dirchlet (1805 - 1859) has formulated certain conditions known as Dirchlet's Conditions under which certain function possess valid Fourier Series Expansion.

A function $f(x)$ has a valid Fourier Series expansion of the form.

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

where $\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$ $\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}$$



**The Conditions are :-**

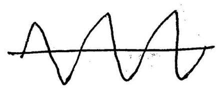
(1) $f(x)$ is well-defined, Periodic, Single valued and finite

($f(x) \to$ one variable)

answer 1 Sing sol. some only.

(2) $f(x)$ has a Finite number of finite dis-Continuities in the any interval of the definition (function not defined at only some points)

(3) $f(x)$ has atmost a finite number of Maxima and Minima in the interval of definition.



→ The above Conditions are Sufficient but not necessary

## Problems

**(1) Expand $f(x) = x$, $0 \le x \le 2\pi$ in a Fourier Series.

(JNTU 2003, 2005, 2007, 2009)

**Sol:-** given $f(x) = x \quad 0 \le x \le 2\pi$

The Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \longrightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx$$

$$= \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi}$$

$$= \frac{1}{\cancel{\pi}}\left(\frac{\cancel{4}^{2}\pi^{\cancel{2}}}{\cancel{2}}\right) = 2\pi$$

$\therefore \boxed{a_0 = 2\pi}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \cos nx\,dx$$

| | | | |
|---|---|---|---|
| D | $x$ | $1$ | $0$ |
| I | $\cos nx$ | $\frac{\sin nx}{n}$ | $-\frac{\cos nx}{n^2}$ |

**PLATE**

$\sin n\pi = 0$

$\sin 2n\pi = 0$

$\vdots$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$\cos 2n\pi = (-1)^{2n} = (-1)^2 = 1$

$$= \frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) + \left(\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} \qquad \boxed{\therefore a_n = 0}$$

$$= \frac{1}{\pi}\left[2\pi\left(\frac{\sin 2n\pi}{n}\right) + \left(\frac{\cos 2n\pi}{n^2}\right)\right] - \left[(0)\left(\frac{\sin 0}{n}\right) + \left(\frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{\cos 0}{n^2}\right] = \frac{1}{\pi n^2}\left[(-1)^{2n} - 1\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\sin nx\,dx$$

| D | $x$ | $1$ | $0$ |
|---|---|---|---|
| I | $\sin nx$ | $-\frac{\cos nx}{n}$ | $-\frac{\sin nx}{n^2}$ |

$$= \frac{1}{\pi}\int_0^{2\pi} x\,\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-x\left(\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[-2\cancel{\pi}\left(\frac{\cos 2n\pi}{n}\right) + \left(\frac{\sin 2n\pi}{n^2}\right)\right] - \left[-(0)\left(\frac{\cos 0}{n}\right) + \left(\frac{\sin 0}{n^2}\right)\right]$$

$\to 0$ $\quad\quad$ $\to 0$

$$= -\frac{2}{n}\cos 2n\pi \qquad \cos 2n\pi = 1$$

$$= -\frac{2}{n}(1) = -\frac{2}{n} \qquad \therefore \boxed{b_n = -\frac{2}{n}}$$

$\therefore$ we have $\boxed{a_0 = 2\pi}$ $\boxed{a_n = 0}$ $\boxed{b_n = -\frac{2}{n}}$

now sub $a_0$ $a_n$ $b_n$ in (1) we get

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n \sin nx\right)$$

$$x = \frac{\cancel{2}\pi}{\cancel{2}} + \sum_{n=1}^{\infty}(0)\cos nx - \frac{2}{n}\sin nx$$

$$x = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$

(2) Express $f(x) = x - \pi$ as Fourier Series in the interval $-\pi \leq x \leq \pi$.

Sol :- given $f(x) = x - \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x-\pi)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \pi x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \pi^2\right) - \left(\frac{\pi^2}{2} + \pi^2\right)\right] \qquad \therefore \boxed{a_0 = -2\pi}$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} - \pi^2 - \cancel{\frac{\pi^2}{2}} - \pi^2\right] = \frac{1}{\cancel{\pi}}\left[-2\pi^{\cancel{2}}\right] = \underline{\underline{-2\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x-\pi)\cos nx\,dx$$

$$\begin{array}{llll} D & x - \pi & 1 & 0 \\ I & \cos nx & \frac{\sin nx}{n} & -\frac{\cos nx}{n^2} \end{array}$$

$+ \quad -$

$$= \frac{1}{\pi}\left[(x-\pi)\left(\frac{\sin nx}{n}\right) + \left(\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x-\pi)\left(\frac{\sin nx}{n}\right) + \left(\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$\cos(-\theta) = \cos\theta$

$$= \frac{1}{\pi}\left[\underbrace{(\pi-\pi)}_{\to 0}\left(\frac{\sin n\pi}{n}\right) + \left(\frac{\cos n\pi}{n^2}\right)\right] - \left[\underbrace{(-\pi-\pi)\left(\frac{\sin -n\pi}{n}\right)}_{\to 0} + \left(\frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos n\pi}{n^2}\right] = 0 \qquad \therefore \boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\sin nx\,dx.$$

| | | | |
|---|---|---|---|
| D | $x-\pi$ | $1$ | $0$ |
| I | $\sin nx$ | $-\frac{\cos nx}{n}$ | $-\frac{\sin nx}{n^2}$ |

$+$ $\quad$ $-$

$$= \frac{1}{\pi}\left[-(x-\pi)\left(\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\underbrace{-(\pi-\pi)}_{\to 0}\left(\frac{\cos n\pi}{n}\right) + \underbrace{\left(\frac{\sin n\pi}{n^2}\right)}_{\to 0}\right] - \left[-(-\pi-\pi)\left(\frac{\cos n\pi}{n}\right) + \underbrace{\left(\frac{-\sin n\pi}{n^2}\right)}_{\to 0}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[-\frac{2\cancel{\pi}}{n}\cos n\pi\right] = -\frac{2}{n}\cos n\pi$$

$$= \underline{\underline{-\frac{2}{n}(-1)^n}} \text{ (or) } (-1)\frac{2}{n}(-1)^n \text{ (or) } \underline{\underline{\frac{2}{n}(-1)^{n+1}}}$$

$\therefore \boxed{a_0 = -2\pi}$ $\boxed{a_n = 0}$

$$\boxed{b_n = \frac{2}{n}(-1)^{n+1}}$$

Substitute $a_0$ $a_n$ $b_n$ values in (1) we get

$$x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$x - \pi = -\frac{\cancel{2}\pi}{\cancel{2}} + \sum_{n=1}^{\infty}(0)\cos nx + \frac{2}{n}(-1)^{n+1}\sin nx.$$

$$x - \cancel{\pi} = -\cancel{\pi} + \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx.$$

$$x = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \cdots$$

(or)

$$x - \pi = -\pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right)$$

(3) Obtain the Fourier Series for $f(x) = x - x^2$ in the interval $[-\pi, \pi]$ Hence S.T

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \quad \text{(or)}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$

(5)

Sol: $f(x) = x - x^2 \quad [-\pi, \pi]$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \longrightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} \Rightarrow \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right) - \left(\frac{(-\pi)^2}{2} - \frac{(-\pi)^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3}\right] \qquad \therefore \boxed{a_0 = -\frac{2\pi^2}{3}}$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{-2\pi^{\cancel{3}2}}{3}\right] = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos nx\,dx.$$

$$\begin{array}{llllll} D & x - x^2 & 1 - 2x & -2 & 0 & \\ I & \cos nx & \frac{\sin nx}{n} & -\frac{\cos nx}{n^2} & -\frac{\sin nx}{n^3} & \end{array}$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(\frac{\sin nx}{n}\right) + (1 - 2x)\left(\frac{\cos nx}{n^2}\right) + (2)\left(\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi - \pi^2)\left(\frac{\sin n\pi}{n}\right) + (1 - 2\pi)\left(\frac{\cos n\pi}{n^2}\right) + (2)\left(\frac{\sin n\pi}{n^3}\right)\right] -$$

$$\left[(-\pi - \pi^2)\left(-\frac{\sin n\pi}{n}\right) + (1 + 2\pi)\left(-\frac{\cos n\pi}{n^2}\right) + 2\left(-\frac{\sin n\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2}(\cancel{1} - 2\pi - \cancel{1} - 2\pi)\right] = \frac{1}{\cancel{\pi}}\left[\frac{\cos n\pi}{n^2}(-4\cancel{\pi})\right] = \frac{-4}{n^2}\cos n\pi$$

$$a_n = -\frac{4}{n^2}\cos n\pi \quad (\text{or}) \quad -\frac{4}{n^2}(-1)^n.$$

$$(-1)\frac{4}{n^2}(-1)^n = \frac{4}{n^2}(-1)^{n+1}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx \, dx$$

D $\quad x - x^2 \qquad 1 - 2x \qquad -2 \qquad 0$

$\qquad\qquad + \qquad\qquad - \qquad\qquad +$

I $\quad \sin nx \qquad -\frac{\cos nx}{n} \qquad -\frac{\sin nx}{n^2} \qquad \frac{\cos nx}{n^3}$

$$= \frac{1}{\pi}\left[-(x - x^2)\left(\frac{\cos nx}{n}\right) + (1 - 2x)\left(\frac{\sin nx}{n^2}\right) - 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi - \pi^2)\left(\frac{\cos n\pi}{n}\right) + (1 - 2\pi)\underbrace{\left(\frac{\sin n\pi}{n^2}\right)}_{0} - 2\cancel{\left(\frac{\cos n\pi}{n^3}\right)}\right] -$$

$$\left[-(-\pi - \pi^2)\left(\frac{\cos n\pi}{n}\right) + (1 + 2\pi)\underbrace{\left(-\frac{\sin n\pi}{n^2}\right)}_{0} - 2\cancel{\left(-\frac{\cos n\pi}{n^3}\right)}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n}\left(-\pi + \cancel{\pi^2} - \pi - \cancel{\pi^2}\right)\right] = \frac{1}{\cancel{\pi}}\left[\frac{\cos n\pi}{n}(-2\cancel{\pi})\right]$$

$$= -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n. \qquad \therefore \boxed{b_n = -\frac{2}{n}(-1)^n.}$$

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$$x - x^2 = \frac{-\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty} -\frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx.$$

$$x - x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right]$$
$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right].$$

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by taking $x=0$ in the above Series.

$$0 = \frac{-\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

(or)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

## 11) Fourier Series for Dis-Continuous Functions.

A function $f(x)$ is Dis-Continuous at $x_0$ in the interval defined $(c, c+2\pi)$ by

$$f(x) = \begin{cases} \phi(x) & ; \text{ If } c < x < x_0 \\ \psi(x) & ; \text{ If } x_0 < x < c+2\pi \end{cases}$$

where $x_0$ is the Point of Dis-Continuous in $(c, c+2\pi)$

Then the Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx .$$

where
$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} \phi(x)\,dx + \int_{x_0}^{c+2\pi} \psi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} \phi(x)\cos nx\,dx + \int_{x_0}^{c+2\pi} \psi(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} \phi(x)\sin nx\,dx + \int_{x_0}^{c+2\pi} \psi(x)\sin nx\,dx\right].$$

(1) Find the Fourier Series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} 0 & ; \text{ for } -\pi \le x \le 0 \\ x^2 & ; \text{ for } 0 \le x \le \pi \end{cases}$$

Sol: The Fourier Series for $f(x)$ in $[-\pi, \pi]$ is given as.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{0} 0\,dx}_{\to 0} + \int_{0}^{\pi} x^2\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left(\frac{\pi^{\cancel{3}2}}{3}\right) = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{0}(0)\cos nx\,dx}_{\to 0} + \int_0^{\pi} x^2 \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2 \cos nx\,dx.$$

D: $x^2$ $\quad 2x \quad 2 \quad 0$

I: $\cos nx \quad \frac{\sin nx}{n} \quad -\frac{\cos nx}{n^2} \quad -\frac{\sin nx}{n^3}$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + (2x)\left(\frac{\cos nx}{n^2}\right) - 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi^2\underbrace{\left(\frac{\sin n\pi}{n}\right)}_{\to 0} + 2\pi\left(\frac{\cos n\pi}{n^2}\right) - 2\underbrace{\left(\frac{\sin n\pi}{n^3}\right)}_{\to 0}\right] -$$

$$\left[(0)^2\underbrace{\left(\frac{\sin 0}{n}\right)}_{\to 0} + 2(0)\underbrace{\left(\frac{\cos 0}{n^2}\right)}_{\to 0} - 2\underbrace{\left(\frac{\sin 0}{n^3}\right)}_{\to 0}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{2\cancel{\pi}}{n^2}\cos n\pi\right] = \frac{2}{n^2}\cos n\pi = \frac{2}{n^2}(-1)^n$$

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{0} 0\,\sin nx\,dx}_{\to 0} + \int_{0}^{\pi} x^2 \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x^2 \sin nx\,dx.$$

$$\begin{array}{lccccccc} D & x^2 & \xrightarrow{+} & 2x & \xrightarrow{-} & 2 & \xrightarrow{+} & 0 \\ I & \sin nx & & -\frac{\cos nx}{n} & & -\frac{\sin nx}{n^2} & & \frac{\cos nx}{n^3} \end{array}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + (2x)\left(\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi^2\left(\frac{\cos n\pi}{n}\right) + (2\pi)\underbrace{\left(\frac{\sin n\pi}{n^2}\right)}_{\to 0} + (2)\left(\frac{\cos n\pi}{n^3}\right)\right] - $$

$$\left[\underbrace{-(0)^2\left(\frac{\cos 0}{n}\right)}_{\to 0} + 2(0)\underbrace{\left(\frac{\sin 0}{n^2}\right)}_{\to 0} + (2)\left(\frac{\cos 0}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-\pi^2\left(\frac{\cos n\pi}{n}\right) + \frac{2}{n^3}\left(\cos n\pi - \cos 0\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{n}\cos n\pi\right) + \frac{2}{n^3}\left((-1)^n - 1\right)\right]$$

$$b_n = -\frac{\pi}{n}\cos n\pi + \frac{2}{\pi n^3}\left[(-1)^n - 1\right] = -\frac{\pi}{n}(-1)^n + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]$$

Substitute $a_0$, $a_n$, $b_n$ in (1) we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left(\frac{2}{n^2}(-1)^n\cos nx\right) + \left(\frac{-\pi}{n}(-1)^n + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]\right]$$

$$f(x) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]\sin nx$$

(8) Expand $f(x) = \begin{cases} 1 & ; 0 < x < \pi \\ 0 & ; \pi < x < 2\pi \end{cases}$

**Sol** : $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$.

$\int 1\,dx = x$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} 1\,dx + \underbrace{\int_{\pi}^{2\pi} 0\,dx}_{\to 0}\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} 1\,dx = \frac{1}{\pi}[x]_0^{\pi} = \frac{1}{\cancel{\pi}}[\cancel{\pi}] = \underline{1}.$$

$\therefore \boxed{a_0 = 1}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} 1 \cos nx\,dx + \underbrace{\int_{\pi}^{2\pi} (0) \cos nx\,dx}_{\to 0}\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos nx\,dx = \frac{1}{\pi}\left(\frac{\sin nx}{n}\right)_0^{\pi} = \frac{1}{\pi}\left(\frac{\sin n\pi}{n} - \frac{\sin 0}{n}\right) = 0 //.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} 1 \sin nx\,dx + \int_{\pi}^{2\pi} (0) \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin nx\,dx = \frac{1}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi}$$

$$= \cancel{\frac{-1}{n\pi}\left\{\frac{\cos n\pi}{n}\right.} = -\frac{1}{n\pi}(\cos n\pi - \cos 0)$$

$$= -\frac{1}{n\pi}((-1)^n - 1) //.$$

$$b_n = \begin{cases} 0 & ; \text{ when } n \text{ is even.} \\ \frac{2}{n\pi} & ; \text{ when } n \text{ is odd.} \end{cases}$$

we have $a_0 = 1$, $a_n = 0$, $b_n = \begin{cases} 0 & ; \text{ when } n \text{ is even} \\ \frac{2}{n\pi} & ; \text{ when } n \text{ is odd.} \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} (0) \cos nx + \sum_{n=1,3,5\ldots}^{\infty} \frac{2}{n\pi} \sin nx.$$

$$= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{\sin nx}{n}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) //$$

(3) Find the Fourier Series to represent the function $f(x)$ (8')

given by

$$f(x) = \begin{cases} -K & ; \text{ for } -\pi < x < 0 \\ K & ; \text{ for } 0 < x < \pi \end{cases}$$

Sol :- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\,dx + \int_{0}^{\pi}(K)\,dx\right] \qquad \therefore \boxed{a_0 = 0}$$

$$= \frac{1}{\pi}\left[-K(x)_{-\pi}^{0} + K(x)_{0}^{\pi}\right] = \frac{1}{\pi}(-K\pi + K\pi) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\cos nx\,dx + \int_{0}^{\pi}(K)\cos nx\,dx\right]$$

$$= \frac{K}{\pi}\left[-\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{\sin nx}{n}\right)_{0}^{\pi}\right] = \frac{K}{\pi}[0+0] = 0. \quad \boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\sin nx\,dx + \int_{0}^{\pi}(K)\sin nx\,dx\right]$$

$$= \frac{K}{\pi}\left[\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} - \left(\frac{\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{K}{\pi}\left[\left(\frac{\cos 0}{n} - \frac{\cos n\pi}{n}\right) - \left(\frac{\cos n\pi}{n} - \frac{\cos 0}{n}\right)\right]$$

$$= \frac{K}{\pi}\left[\left(\frac{1-(-1)^n}{n}\right) - \left(\frac{(-1)^n - 1}{n}\right)\right]$$

$$= \frac{K}{\pi}\left[\frac{2 - 2(-1)^n}{n}\right] = \begin{cases} 0 & ; \text{ when } n \text{ is even} \\ \frac{4K}{n\pi} & ; \text{ when } n \text{ is odd} \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$= \frac{0}{2} + \sum_{n=1}^{\infty}(0)\cos nx + \sum_{n=1,3,5\ldots}^{\infty}\frac{4K}{n\pi}\sin nx$$

$$= \sum_{n=1,3,5\ldots}^{\infty}\frac{4K}{n\pi}\sin nx = \frac{4K}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

(4) Find the Fourier Series of the following function

$$f(x) = 0 \; ; \; -\pi \le x < 0$$
$$= \frac{\pi x}{4} \; ; \; 0 < x \le \pi.$$

Also deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Sol: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \frac{\pi x}{4}\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\pi x}{4}\,dx = \frac{1}{\pi}\cdot\frac{\pi}{4}\int_0^{\pi} x\,dx = \frac{1}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{4}\left\{\left(\frac{x^2}{2}\right)\right\}_0^{\pi}$$

$$= \frac{1}{4}\left(\frac{\pi^2}{2}\right) = \frac{\pi^2}{8}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cos nx\,dx + \int_0^{\pi} \frac{\pi x}{4}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\pi x}{4}\cos nx\,dx$$

D: $x \quad 1 \quad 0$

I: $\cos nx \quad \frac{\sin nx}{n} \quad -\frac{\cos nx}{n^2}$

$$= \frac{1}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{4}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{1}{4}\int_0^{\pi} x\cos nx\,dx = \frac{1}{4}\left[(x)\left(\frac{\sin nx}{n}\right) + \left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{4}\left[(\pi)\left(\frac{\sin n\pi}{n}\right)_{\to 0} + \left(\frac{\cos n\pi}{n^2}\right)\right] - \left\{(0)\left(\frac{\sin 0}{n}\right)_{\to 0} + \left(\frac{\cos 0}{n^2}\right)\right\}$$

$$= \frac{1}{4}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{1}{4n^2}\left[(-1)^n - 1\right] = \frac{(-1)^n - 1}{4n^2}$$

$$\therefore a_n = \begin{cases} -\frac{1}{2n^2} & ; \text{ when } n \text{ is odd} \\ 0 & ; \text{ when } n \text{ is even} \end{cases}$$

(4)

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \sin nx\,dx + \int_0^{\pi} \frac{\pi x}{4} \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\pi x}{4}\sin nx\,dx = \frac{1}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{4}\int_0^{\pi} x \sin nx\,dx = \frac{1}{4}\int_0^{\pi} x\sin nx\,dx$$

D: $x \quad 1 \quad 0$

I: $\sin nx \quad -\frac{\cos nx}{n} \quad -\frac{\sin nx}{n^2}$

$$= \frac{1}{4}\left[(-x)\left(\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\left\{(-\pi)\left(\frac{\cos n\pi}{n}\right) + \left(\frac{\sin n\pi}{n^2}\right)\right\} - \left\{(-0)\left(\frac{\cos 0}{n}\right) + \left(\frac{\sin 0}{n^2}\right)\right\}\right]$$

$$= \frac{-\pi}{4n}\cos n\pi = \frac{-\pi}{4n}(-1)^n \quad (\text{or}) \quad (-1)\frac{\pi}{4n}(-1)^n \quad (\text{or}) \quad \underline{\frac{\pi}{4n}(-1)^{n+1}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$f(x) = \frac{\pi^2}{16} + \frac{1}{4}\left[-2\cos x - \frac{2}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x - \cdots\right]$$
$$-\frac{\pi}{4}\left(-\sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \cdots\right)$$

(III) **Even and Odd Function:-**

(1) **Even function:-** A function $f(x)$ is said to be Even if

$$\boxed{f(-x) = f(x)}$$

ex:- If $f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2$

$= f(x)$

$\therefore f(-x) = f(x)$

$f(x) = \cos x$

$f(-x) = \cos(-x)$

$= \cos x$

$= f(x)$

$\therefore f(-x) = f(x)$

$\cos(-\theta) = \cos\theta$

$\therefore f(x)$ is an Even function.

Ex:- $x^2$, $x^4 + x^2 + 1$, $\cos x$, $\sec x$ are all Even functions of $x$

(2) Odd function :-

A function $f(x)$ is Said to be Odd If

$$\boxed{f(-x) = -f(x)}$$

ex : $x$, $x^3$, $x^5 + 2x^3$, $\sin x$, $\operatorname{Cosec} x$, $\tan x$ are all Odd function.

ex :- If $f(x) = \sin x$

$f(-x) = \sin(-x)$

$= -\sin x$

$= -f(x)$

$\therefore f(-x) = -f(x)$

If $f(x) = x^3$

$f(-x) = (-x)^3$

$= -x^3$

$= -f(x)$

$\therefore f(-x) = -f(x)$

$\therefore f(x)$ is an Odd function

Notes :-

→ If $f(x)$ is an Even function :-

→ $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(If Integrand is Even then 2 times integral b/w limits)

→ If $f(x)$ is an Odd function :-

→ $\int_{-a}^{a} f(x)\,dx = 0$

→ $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & ; \text{when } f(x) \text{ is an Odd function} \\ 2\int_{0}^{a} f(x)\,dx & \text{when } f(x) \text{ is an Even function.} \end{cases}$$

(10)

# Fourier Series for Even and Odd function.

A function $f(x)$ defined in $(-\pi,\pi)$ can be represented by the Fourier Series as

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$ $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

**Case-1 :- when $f(x)$ is an Even function :-**

(1) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$

even * even = even.
$x^2 * x^4 = x^6$

$$\therefore \boxed{a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx}$$

Since $\cos nx$ is an even function $f(x)\cos nx$ is also an even function

(2) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.$

even $\times$ odd = odd
$x^2 \times x^3 = x^5$

$$\therefore \boxed{a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.}$$

Since $\cos nx$ is an Even function So $f(x)\cos nx$ is also an Even function.

(3) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0.$ $\quad \therefore \boxed{b_n = 0}$

Since $\sin nx$ is an Odd function so $f(x)\sin nx$ is also an Odd function

$$\therefore \boxed{b_n = 0}$$

Thus If a function $f(x)$ is Even in $(-\pi,\pi)$ its Fourier Series expansion Contains Only Cosine terms

$$\therefore \boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$$

Case-2 :- when $f(x)$ is an Odd function :-

| | | | | |
|---|---|---|---|---|
| even | * | even | = | even |
| odd | * | odd | = | even |
| even | * | odd | = | odd |
| odd | * | even | = | odd |

Odd × even = odd

$x^3 \times x^2 = x^5$

1) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$

Since $f(x)$ is an Odd function

$$\therefore \boxed{a_0 = 0}$$

(2) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$

$$\therefore \boxed{a_n = 0}$$

Since $\cos nx$ is an even function

$f(x)$ is an Odd function

$f(x)\cos nx$ is an Odd function

(Since integrand is odd)

(3) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

Odd × odd = even

$x^3 \times x^3 = x^6$

$$\boxed{b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx}$$

Thus If a function $f(x)$ defined in $(-\pi, \pi)$ is odd its Fourier Series expansion Contains only Sine terms

$\therefore$ Fourier Series expansion is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

# Problems on Even & Odd functions (11)

**(1) Express $f(x)=x$ as a Fourier Series in $(-\pi,\pi)$

Sol :- given $f(x)=x$ $(-\pi,\pi)$

$$f(-x) = -x$$
$$= -f(x)$$

$\therefore f(-x) = -f(x)$

$\therefore f(x)$ is an Odd function in $(-\pi,\pi)$

Hence in its Fourier Series expansion, the Cosine terms are absent and only Sine terms are present.

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$x = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1).$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx \qquad \begin{array}{l} D\ x \xrightarrow{+} 1 \xrightarrow{-} 0 \\ I\ \sin nx \quad -\frac{\cos nx}{n} \quad -\frac{\sin nx}{n^2} \end{array}$$

$$= \frac{2}{\pi}\left[-x\left(\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(-\pi\left(\frac{\cos n\pi}{n}\right) + \underbrace{\left(\frac{\sin n\pi}{n^2}\right)}_{\to 0}\right) - \left[-(0)\underbrace{\left(\frac{\cos 0}{n}\right)}_{\to 0} + \underbrace{\left(\frac{\sin 0}{n^2}\right)}_{\to 0}\right]\right]$$

$$= \frac{-2\cancel{\pi}}{\cancel{\pi} n}\cos n\pi = -\frac{2}{n}\cos n\pi \text{ (or) } -\frac{2}{n}(-1)^n \text{ (or)}$$

$$(-1)\frac{2}{n}(-1)^n \text{ (or) } \frac{2}{n}(-1)^{n+1}$$

$$x = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx.$$

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right) //.$$

which is the required Fourier Series.

***

(8) Expand $f(x) = x^2$ as a Fourier series in $[-\pi, \pi]$

(or)

P.T $x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$

Hence deduce that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

Sol :- given $f(x) = x^2 \quad [-\pi, \pi]$

$f(-x) = (-x)^2 = x^2 = f(x)$.

$\therefore f(-x) = f(x)$

$\therefore f(x)$ is an Even function in $[-\pi, \pi]$

Hence in its Fourier Series expansion Cosine terms are Present and sine terms are absent

$\therefore$ The Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow (1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\left[\frac{\pi^{\cancel{3}\,2}}{3}\right] = \frac{2\pi^2}{3} \qquad \therefore \boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

| | | | | |
|---|---|---|---|---|
| D | $x^2$ | $2x$ | $2$ | $0$ |
| | $+$ | $-$ | $+$ | |
| I | $\cos nx$ | $\frac{\sin nx}{n}$ | $-\frac{\cos nx}{n^2}$ | $-\frac{\sin nx}{n^3}$ |

$$= \frac{2}{\pi}\left[ x^2\left(\frac{\sin nx}{n}\right) + 2x\left(\frac{\cos nx}{n^2}\right) - 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2\underbrace{\left(\frac{\sin n\pi}{n}\right)}_{\to 0} + 2\pi\left(\frac{\cos n\pi}{n^2}\right) - 2\underbrace{\left(\frac{\sin n\pi}{n^3}\right)}_{\to 0}\right] - \left[(0)^2\underbrace{\left(\frac{\sin 0}{n}\right)}_{\to 0} + 2(0)\underbrace{\left(\frac{\cos 0}{n^2}\right)}_{\to 0} - 2\underbrace{\left(\frac{\sin 0}{n^3}\right)}_{\to 0}\right]$$

$$= \frac{2}{\cancel{\pi}}\left[\frac{2\cancel{\pi}}{n^2}\cos n\pi\right]$$

$$= \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

$$\therefore \boxed{a_n = \frac{4}{n^2}(-1)^n}$$

Substitute $a_0$ and $a_n$ in (1) we get

$$\boxed{a_0 = \frac{2\pi^2}{3}} \quad \boxed{a_n = \frac{4}{n^2}(-1)^n}$$

Substitute $a_0$ and $a_n$ in (1) we get

$$x^2 = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx \qquad \frac{\cancel{2}\pi^2}{3}\times\frac{1}{\cancel{2}} = \frac{\pi^2}{3}$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$$

$$= \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right) \longrightarrow \text{(I)}$$

Deductions :-

(1) Putting $\boxed{x=0}$ in (I) we get

$$0 = \frac{\pi^2}{3} - 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \longrightarrow (A).$$

(2) Putting $\boxed{x=\pi}$ in (I) we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left(\cos\pi - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \frac{\cos 4\pi}{4^2} + \cdots\right)$$

$$= \frac{\pi^2}{3} - 4\left(-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \cdots\right)$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$$

$$\Rightarrow 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \longrightarrow (B)$$

now Adding (A) & (B) and dividing it by (2) we get the required result as

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8} //$$

(3) S.T for $-\pi < x < \pi$

$$\sin ax = \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2 - a^2} - \frac{2\sin 2x}{2^2 - a^2} + \frac{3\sin 3x}{3^2 - a^2} - \cdots\right]$$

(a is not an Integer)

Sol :- given $\sin ax$ is an Odd function, so its Fourier Series expansion will Consist of Sine terms only.

$$\therefore \boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

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$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx\,dx$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} 2\sin ax \sin nx\,dx$$

$$\boxed{2\sin A\sin B = \cos(A-B) - \cos(A+B)}$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(ax-nx) - \cos(ax+nx)\,dx$$

$$\int \cos ax\,dx = \frac{\sin ax}{a}$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(a-n)x - \cos(a+n)x\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(a-n)x}{a-n} - \frac{\sin(a+n)x}{a+n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(a-n)\pi}{a-n} - \frac{\sin(a+n)\pi}{a+n}\right] - [0]$$

$$= \frac{1}{\pi}\left[\frac{\sin(a\pi-n\pi)}{a-n} - \frac{\sin(a\pi+n\pi)}{a+n}\right]$$

$$\boxed{\begin{aligned}\sin(A-B) &= \sin A\cos B - \cos A\sin B\\ \sin(A+B) &= \sin A\cos B + \cos A\sin B.\end{aligned}}$$

$$= \frac{1}{\pi}\left[\frac{\sin a\pi\cos n\pi - \underbrace{\cos a\pi\sin n\pi}_{\to 0}}{a-n} - \frac{\sin a\pi\cos n\pi + \underbrace{\cos a\pi\sin n\pi}_{\to 0}}{a+n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin a\pi\cos n\pi}{a-n} - \frac{\sin a\pi\cos n\pi}{a+n}\right]$$

$$(a-b)(a+b) = a^2-b^2$$

$$= \frac{1}{\pi}\left[\sin a\pi\cos n\pi\left[\frac{1}{a-n} - \frac{1}{a+n}\right]\right]$$

$$= \frac{1}{\pi}\sin a\pi\cos n\pi\left[\frac{\cancel{a}+n-\cancel{a}+n}{a^2-n^2}\right]$$

$$= \frac{1}{\pi}\left(\sin a\pi \cos n\pi\right)\left[\frac{2n}{a^2-n^2}\right]$$

$$= \frac{1}{\pi}\sin a\pi \,(-1)^n\left[\frac{2n}{a^2-n^2}\right]$$

$$b_n = \frac{(-1)^n\, 2n}{\pi\,(a^2-n^2)}\sin a\pi$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\sin ax = \sum_{n=1}^{\infty}\left(\frac{(-1)^n\, 2n}{\pi\,(a^2-n^2)}\sin a\pi\right)\sin nx \qquad (-1) \text{ Common down}$$

$$= \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n . n}{a^2-n^2}\sin nx = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1} . n}{n^2-a^2}\sin nx$$

$$\sin ax = \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \cdots\right]$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\sin ax = \sum_{n=1}^{\infty}\left(\frac{(-1)^n\, 2n}{\pi(a^2-n^2)}\sin a\pi\right)\sin nx$$

$$\sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n . n}{a^2-n^2} . \sin nx$$

$$= \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1} . n}{n^2-a^2}\sin nx \qquad (-1) \text{ Common down}$$

$$= \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \cdots\right]$$

$\rightarrow$ we note here that since $a$ is not an Integer, $1^2-a^2,\ 2^2-a^2,\ 3^2-a^2 \ldots$ etc will not become zero.

(14')

(4) Express $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ as a Fourier Series in the Interval $-\pi < x < \pi$

Sol :- given $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ $\quad -\pi < x < \pi$

$$f(-x) = \frac{\pi^2}{12} - \frac{(-x)^2}{4} = \frac{\pi^2}{12} - \frac{x^2}{4} = f(x) \qquad (-x)^2$$

$$\therefore f(-x) = f(x)$$

$\therefore$ $f(x)$ is an Even function.

The Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow (1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right) dx$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{12}\cdot x - \frac{x^3}{12}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{12}\cdot\pi - \frac{\pi^3}{12}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{12} - \frac{\pi^3}{12}\right] = 0 \qquad \therefore \boxed{a_0 = 0}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\cos nx\,dx$$

D: $\frac{\pi^2}{12} - \frac{x^2}{4}$ $\quad$ $-\frac{2x}{4}$ $\quad$ $-\frac{1}{2}$ $\quad$ $0$

I: $\cos nx$ $\quad$ $\frac{\sin nx}{n}$ $\quad$ $-\frac{\cos nx}{n^2}$ $\quad$ $-\frac{\sin nx}{n^3}$

(signs: $+$, $-$, $+$)

$$= \frac{2}{\pi}\left[\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{\cancel{2}x}{\cancel{4}_2}\right)\left(\frac{\cos nx}{n^2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^2}{12} - \frac{\pi^2}{4}\right)\underbrace{\left(\frac{\sin n\pi}{n}\right)}_{\to 0} - \left(\frac{\pi}{2}\right)\left(\frac{\cos n\pi}{n^2}\right) + \left(\frac{1}{2}\right)\underbrace{\left(\frac{\sin n\pi}{n^3}\right)}_{\to 0}\right] -$$

$$\left[\underbrace{\left(\frac{\pi^2}{12} - \frac{0^2}{4}\right)\left(\frac{\sin 0}{n}\right)}_{\to 0} - \underbrace{\left(\frac{0}{2}\right)}_{\to 0}\left(\frac{\cos 0}{n^2}\right) + \left(\frac{1}{2}\right)\underbrace{\left(\frac{\sin 0}{n^3}\right)}_{\to 0}\right]$$

$$= \frac{-\cancel{2\pi}}{\cancel{2\pi} n^2} \cos n\pi$$

$$= -\frac{1}{n^2} \cos n\pi = -\frac{1}{n^2} (-1)^n \quad (\text{or})$$

$$= (-1)\frac{1}{n^2} (-1)^n \quad (\text{or})$$

$$= \frac{1}{n^2} (-1)^{n+1}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{0}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$$

$$\therefore f(x) = \cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots$$

(5) Obtain the Fourier Series for the function $f(x) = e^x$ from $x = 0$ to $2\pi$

Sol :- we know that Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x\,dx = \frac{1}{\pi}\left[e^x\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[e^{2\pi} - e^0\right] = \frac{1}{\pi}\left[e^{2\pi} - 1\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\,dx$$

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

on Comparing we have

$a = 1$

$b = n$

(15)

$$= \cancel{\frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{2\pi}}$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\, dx.$$

$$\boxed{\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)}$$

$a=1 \quad b=n$

$\cos 2n\pi = 1$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left[\frac{e^{2\pi}}{1+n^2}(\cos 2n\pi + n\underbrace{\sin 2n\pi}_{\to 0})\right] - \left[\frac{1}{1+n^2}(\cos 0 + n\underbrace{\sin 0}_{\to 0})\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2} - \frac{1}{1+n^2}\right] = \frac{e^{2\pi}-1}{\pi(1+n^2)}$$

$$\therefore \boxed{a_n = \frac{e^{2\pi}-1}{\pi(1+n^2)}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\, dx$$

$$\boxed{\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}$$

$a=1, \; b=n.$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{2\pi}}{1+n^2}(\underbrace{\sin 2n\pi}_{\to 0} - n\cos 2n\pi)\right) - \frac{1}{1+n^2}(\underbrace{\sin 0}_{\to 0} - n\cos 0)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}(-n) - \frac{1}{1+n^2}(-n)\right] = \frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}$$

$$\therefore e^x = \frac{e^{2\pi}-1}{2\pi} + \sum_{n=1}^{\infty}\frac{e^{2\pi}-1}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}\sin nx.$$

$$\therefore e^x = \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} - \sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}\right]$$

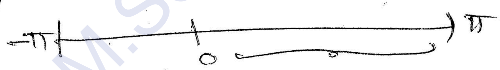
is the required Fourier series.

# Half Range Fourier Series :

Sometimes it is necessary to expand a function $f(x)$ as a Fourier Series in a Half range $(0, \pi)$ (or) $(0, l)$ not in the Full range $(-\pi, \pi)$ (or) $(-l, l)$ such a Series is Known as "Half Range Fourier Series"

It is often required to obtain Fourier Series of a function $f(x)$ in the interval $(0, \pi)$

If $f(x)$ is a Function defined in the interval $0 < x < \pi$ then the Half Range Series is given below.



## Half Range Sine Series :-

The Half range Sine Series in $(0, \pi)$ is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

where $\boxed{b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx}$

## Half Range Cosine Series :-

The Half range Cosine Series in $(0, \pi)$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$$

where $\boxed{a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx}$ & $\boxed{a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx}$

$\longrightarrow$ 1) Suppose $f(x) = x$ in $[0, \pi]$ it can have Fourier Cosine Series expansion as well as Fourier Sine Series expansion in $[0, \pi]$

$\longrightarrow$ 2) Suppose $f(x) = x^2$ in $[0, \pi]$ it can have Fourier Cosine Series expansion as well as Fourier Sine Series expansion in $[0, \pi]$

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*** 1) Find the Half range Cosine and Sine Series for the function $f(x)=x$ in the range $0<x<\pi$

(or)

Prove that the function $f(x)=x$ can be expanded in a Series of Cosines in $0\le x\le\pi$ as

$$x=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right]$$

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$.

Sol :- The Cosine Series :-

The Half range Cosine Series expansion of $f(x)$ is given by

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx}$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}x\,dx=\frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}=\frac{\cancel{2}}{\cancel{\pi}}\left(\frac{\pi^{\cancel{2}}}{\cancel{2}}\right)=\pi.$$

$$\therefore \boxed{a_0=\pi}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$$

D: $x$, $1$, $0$

I: $\cos nx$, $\frac{\sin nx}{n}$, $-\frac{\cos nx}{n^2}$

$$=\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right)+\left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}=\frac{2}{\pi}\left[\pi\underbrace{\left(\frac{\sin n\pi}{n}\right)}_{\to 0}+\left(\frac{\cos n\pi}{n^2}\right)\right]-\left[\underbrace{(0)\left(\frac{\sin 0}{n}\right)}_{0}+\left(\frac{\cos 0}{n^2}\right)\right]$$

$$=\frac{2}{\pi}\left[\frac{\cos n\pi}{n^2}-\frac{\cos 0}{n^2}\right]=\frac{2}{\pi n^2}\left[\cos n\pi-\cos 0\right]$$

$$\boxed{a_n=\frac{2}{\pi n^2}\left[(-1)^n-1\right]}$$

$$a_n=\begin{cases}0 & ; \text{ If } n \text{ is even}\\ \dfrac{-4}{\pi n^2} & ; \text{ If } n \text{ is Odd}.\end{cases}$$

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$x = \frac{\pi}{2} + \sum_{n=1,3,5\ldots}^{\infty} \frac{-4}{\pi n^2} \cos nx$$

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{\cos nx}{n^2}$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right] \longrightarrow (\mathrm{I})$$

which is the required Fourier Series for $x$.

**Deduction** :- Put $\boxed{x=0}$ in (I) we have

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\frac{\pi}{2}}{\frac{4}{\pi}} = \frac{\pi}{2} \times \frac{\pi}{4} = \frac{\pi^2}{8}$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

## Half Range Sine Series :-

The Half range Sine Series expansion of $f(x)$ is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

D: $x$, $1$, $0$

I: $\sin nx$, $-\frac{\cos nx}{n}$, $-\frac{\sin nx}{n^2}$

$$= \frac{2}{\pi}\left[-x\left(\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[-\pi\left(\frac{\cos n\pi}{n}\right) + \left(\frac{\sin n\pi}{n^2}\right)\right] - \left[-(0)\left(\frac{\cos 0}{n}\right) + \left(\frac{\sin 0}{n^2}\right)\right]$$

$$= \frac{-2\cancel{\pi}}{\cancel{\pi} n} \cos n\pi = -\frac{2}{n} \cos n\pi$$

$$b_n = -\frac{2}{n}(-1)^n \text{ (or) } (-1)\frac{2}{n}(-1)^n$$

$$\text{(or) } \frac{2}{n}(-1)^{n+1}$$

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$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

is the required Fourier Series.

***

(2) Find the Half Range sine Series for $f(x) = x(\pi - x)$ in $0 < x < \pi$

Hence deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$.

Sol: given $f(x) = x(\pi - x)$ $\quad 0 < x < \pi$

Half range Sine Series :-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2) \sin nx \, dx.$$

D $\quad x\pi - x^2 \quad\quad \pi - 2x \quad\quad -2 \quad\quad 0$

I $\quad \sin nx \quad\quad -\frac{\cos nx}{n} \quad\quad -\frac{\sin nx}{n^2} \quad\quad \frac{\cos nx}{n^3}$

$$= \frac{2}{\pi}\left[-(x\pi - x^2)\left(\frac{\cos nx}{n}\right) + (\pi - 2x)\left(\frac{\sin nx}{n^2}\right) - 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\underbrace{-(\pi\pi - \pi^2)}_{\to 0}\left(\frac{\cos n\pi}{n}\right) + (\pi - 2\pi)\underbrace{\left(\frac{\sin n\pi}{n^2}\right)}_{\to 0} - 2\left(\frac{\cos n\pi}{n^3}\right)\right] - \left[\underbrace{-(0\pi - 0^2)}_{\to 0}\left(\frac{\cos 0}{n}\right) + (\pi - 2(0))\underbrace{\left(\frac{\sin 0}{n^2}\right)}_{\to 0} - 2\left(\frac{\cos 0}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-2}{n^3}\cos n\pi + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}\left(1 - (-1)^n\right)$$

$$\therefore b_n = \frac{4}{\pi n^3}\left[1 - (-1)^n\right] = \begin{cases} 0 & ; \text{when } n \text{ is even} \\ \frac{8}{\pi n^3} & ; \text{when } n \text{ is odd.} \end{cases}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$(x\pi - x^2) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8}{\pi n^3} \sin nx$$

$$x\pi - x^2 = \frac{8}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{\sin nx}{n^3}$$

$$x\pi - x^2 = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right) \longrightarrow (1).$$

which is the required Fourier Series.

**Deduction**:- Put $\boxed{x = \frac{\pi}{2}}$ in (1) we get

$$\cancel{\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\right)}$$

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3}\sin\frac{5\pi}{2} + \ldots\right)$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi}\Big[1 + \frac{1}{3^3}\sin\left(\pi + \frac{\pi}{2}\right) + \frac{1}{5^3}\sin\left(2\pi + \frac{\pi}{2}\right) + \frac{1}{7^3}\sin\left(3\pi + \frac{\pi}{2}\right) + \ldots\Big] \quad \text{(or)}$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$$

$$\therefore \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \ldots = \frac{\pi^3}{32}$$

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3) Obtain the Half-range Sine Series for $e^x$ in $(0,\pi)$

Sol :- The half range Sine Series expansion of $e^x$ in $(0,\pi)$ is given by .

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$e^x = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} e^x \sin nx \, dx$$

$$\boxed{\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}$$

on Comparing we have $a=1$, $b=n$

$$= \frac{2}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{1+n^2}(\underbrace{\sin n\pi}_{\to 0} - n\cos n\pi) - \frac{e^0}{1+n^2}(\underbrace{\sin 0}_{\to 0} - n\cos 0)\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-n)(-1)^n - \frac{1}{1+n^2}(-n)\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-n(-1)^n + \frac{n}{1+n^2}\right] = \frac{2n}{\pi(1+n^2)}\left[(-1)^{n+1}e^{\pi}+1\right]$$

$$f(x) = e^x = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{n\left[1+(-1)^{n+1}e^{\pi}\right]}{1+n^2} \sin nx$$

$$= \frac{2}{\pi}\left[\frac{1+e^{\pi}}{1^2+1}\sin x + \frac{2(1-e^{\pi})}{2^2+1}\sin 2x + \frac{3(1+e^{\pi})}{3^2+1}\sin 3x + \cdots\right]$$

(4) Find Cosine and Sine Series for

$f(x) = \pi - x$ in $[0, \pi]$

Sol :- Sine Series :-

The Half range Sine Series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx .$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx \quad = \frac{2}{\pi}\int_0^{\pi} (\pi - x) \sin nx \, dx .$$

D $\pi - x$ $\quad +$ $\quad -1$ $\quad -$ $\quad 0$

I $\sin nx$ $\quad -\frac{\cos nx}{n}$ $\quad -\frac{\sin nx}{n^2}$

$$= \frac{2}{\pi}\left[-(\pi - x)\left(\frac{\cos nx}{n}\right) - \left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[-(\pi-\pi)\underbrace{\left(\frac{\cos n\pi}{n}\right)}_{\to 0} - \underbrace{\left(\frac{\sin n\pi}{n^2}\right)}_{\to 0}\right] - \left[-(\pi - 0)\left(\frac{\cos 0}{n}\right) - \underbrace{\left(\frac{\sin 0}{n^2}\right)}_{\to 0}\right]\right\}$$

$$= \frac{2}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{n}\cos 0\right] = \frac{2}{n}\cos 0 = \frac{2}{n}(1) = \frac{2}{n} .$$

$$\therefore \boxed{b_n = \frac{2}{n}}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\pi - x = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$\pi - x = 2 \sum_{n=1,2,\dots}^{\infty} \frac{\sin nx}{n}$$

$$\pi - x = 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right) //.$$

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## Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\left[\pi\pi - \frac{\pi^2}{2}\right] - \underbrace{\left[\pi(0) - \frac{0^2}{2}\right]}_{\to 0}\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^2}{1} - \frac{\pi^2}{2}\right] = \frac{2}{\pi}\left[\frac{2\pi^2 - \pi^2}{2}\right] = \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{2}}}{\cancel{2}}\right] = \pi$$

$$\therefore \boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx$$

D $\quad \pi - x \qquad -1 \qquad 0$

I $\quad \cos nx \qquad \frac{\sin nx}{n} \qquad -\frac{\cos nx}{n^2}$

(+, −)

$$= \frac{2}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - \left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{\underbrace{(\pi-\pi)\left(\frac{\sin n\pi}{n}\right)}_{\to 0} - \left(\frac{\cos n\pi}{n^2}\right)\right\} - \left\{\underbrace{(\pi - 0)\left(\frac{\sin 0}{n}\right)}_{\to 0} - \left(\frac{\cos 0}{n^2}\right)\right\}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} + \frac{\cos 0}{n^2}\right] = \frac{2}{\pi n^2}\left\{1 - (-1)^n\right\} = \begin{cases} 0 & ; n \text{ is even.} \\ \frac{4}{\pi n^2} & ; n \text{ is odd} \end{cases}$$

$$a_n = \begin{cases} 0 & ; n \text{ is even} \\ \frac{4}{\pi n^2} & ; n \text{ is odd.} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{\pi n^2} \cos nx$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{\cos nx}{n^2} \Rightarrow \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

(5) Change of Interval.

The Fourier Series of f(x) in the interval $(c, c+2l)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx.$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx. \qquad b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

→ If $\boxed{c=-l}$ then

→ The Fourier Series of f(x) in $(-l, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx. \qquad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

If $\boxed{c=-l}$ in the interval $(c, c+2l)$ then the Fourier. Series expansion is.

→ If $\boxed{c=0}$ in $(c, c+2l)$ then $(0, 2l)$ then the Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx \qquad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

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## Fourier Series for Even and Odd functions $[-l, l]$

→ If $f(x)$ is an even function then

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)}$$

where $\boxed{a_0 = \frac{2}{l}\int_0^l f(x)\,dx}$ $\boxed{a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx}$

→ If $f(x)$ is an Odd function then

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)}$$

where $\boxed{b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx}$

## Half Range Fourier Expansion in $[0, l]$ :-

→ The Half Range Sine Series :-

~~The~~ $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

→ The Half Range Cosine Series :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ $\quad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

(1) Express $f(x) = x^2$ as a Fourier Series in $[-l, l]$

Sol :- given $f(x) = x^2$ $[-l, l]$

$$f(-x) = (-x)^2$$
$$= f(x)$$
$$\therefore f(-x) = f(x)$$

$\therefore f(x)$ is an even function

The Fourier Series of $f(x)$ in $[-l, l]$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x^2\,dx \qquad \therefore \boxed{a_0 = \frac{2l^2}{3}}$$

$$= \frac{2}{l}\left(\frac{x^3}{3}\right)_0^l = \frac{2}{\cancel{l}}\left(\frac{l^{\cancel{3}2}}{3}\right) = \frac{2l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$

| | | | | |
|---|---|---|---|---|
| D | $x^2$ | $2x$ | $2$ | $0$ |
| I | $\cos\frac{n\pi x}{l}$ | $\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}$ | $-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$ | $-\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}$ |

(signs: $+$, $-$, $+$)

$$= \frac{2}{l}\left[x^2\left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) + 2x\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

(21)

$$= \frac{2}{l}\left[ l^2\left(\frac{\sin\left(\frac{n\pi \cancel{l}}{\cancel{l}}\right)}{\frac{n\pi}{l}}\right) + 2l\left(\frac{\cos\left(\frac{n\pi \cancel{l}}{\cancel{l}}\right)}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin\left(\frac{n\pi \cancel{l}}{\cancel{l}}\right)}{\frac{n^3\pi^3}{l^3}}\right)\right] -$$

$$\left[(0)^2\left(\frac{\sin\left(\frac{n\pi 0}{l}\right)}{\frac{n\pi}{l}}\right) + 2(0)\left(\frac{\cos\left(\frac{0}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin\left(\frac{0}{l}\right)}{\frac{n^3\pi^3}{l^3}}\right)\right]$$

$\to 0 \quad \to 0 \quad \to 0$

$$= \frac{2}{\cancel{l}}\left[2\cancel{l}\left(\frac{\cos n\pi}{\frac{n^2\pi^2}{l^2}}\right)\right] = \frac{4l^2\cos n\pi}{n^2\pi^2} = \frac{4l^2(-1)^n}{n^2\pi^2}$$

$\int \cos ax\, dx = \frac{\sin ax}{a}$

$$\therefore \boxed{a_n = \frac{4l^2(-1)^n}{n^2\pi^2}}$$

$$\therefore \boxed{a_0 = \frac{2l^2}{3}} \qquad \boxed{a_n = \frac{4l^2(-1)^n}{n^2\pi^2}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

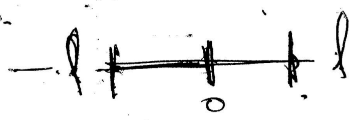
$$= \frac{\frac{2l^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2} - \frac{\cos(2\pi x/l)}{2^2} + \frac{\cos(3\pi x/l)}{3^2} - \cdots\right]$$

(2) Find the Half Range Sine Series for

$f(x)=1$ in $[0,l]$



**Sol**:- given $f(x)=1$ $[0,l]$.

Half Range Sine Series is

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \rightarrow (1) \qquad b_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

where $b_n=\frac{2}{l}\int_0^l (1)\sin\left(\frac{n\pi x}{l}\right)dx$

$$=\frac{2}{l}\left[-\frac{\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]_0^l = \frac{2}{l}\left[-\frac{\cos\left(\frac{n\pi \cancel{l}}{\cancel{l}}\right)}{\frac{n\pi}{l}} - \frac{\cos 0}{\frac{n\pi}{l}}\right]$$

$$=-\frac{2}{n\pi}\left[\cos n\pi - \cos 0\right] \qquad \cancel{l}\left[\frac{\cancel{l}}{n\pi}\right]$$

$$=-\frac{2}{n\pi}\left[(-1)^n-1\right] \text{ (or) } \quad b_n=\begin{cases} 0 & ;\text{ when } n \text{ is even} \\ \frac{4}{n\pi} & ;\text{ when } n \text{ is odd.}\end{cases}$$

$$=\frac{2}{n\pi}\left[(-1)^{n+1}+1\right] \Rightarrow$$

$$f(x)=\sum_{n=1,3,5,\ldots}^{\infty}\frac{4}{n\pi}\sin\left(\frac{n\pi x}{l}\right)$$

$$=\frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right)$$

$$1=\frac{4}{\pi}\left(\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+\cdots\right)$$

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(3) Find the Fourier Series to represent $1-x^2$ in the interval $-1 \le x \le 1$

$-\pi \le x \le \pi$
$-1 \le x \le 1$
$-l \le x \le l$

**Sol:-** given $f(x) = 1 - x^2 \quad -1 \le x \le 1$

$f(-x) = 1 - (-x)^2$
$= 1 - x^2$
$= f(x)$

$\therefore f(-x) = f(x)$

$\therefore f(x)$ is an Even function

Since the given function is even its Fourier Series expansion Contains only Cosine terms

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$a_0 = \frac{2}{l}\int_0^l 1 - x^2\,dx. \qquad l = 1$$

$$= \frac{2}{l}\left(x - \frac{x^3}{3}\right)_0^l = \frac{2}{1}\left(1 - \frac{1^3}{3}\right) = 2\left(1 - \frac{1}{3}\right)$$

$$= 2\left(\frac{3-1}{3}\right) = 2\left(\frac{2}{3}\right) = \frac{4}{3}$$

$$\therefore \boxed{a_0 = \frac{4}{3}}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx. = 2\int_0^1 (1-x^2)\cos n\pi x\,dx.$$

D: $1-x^2 \quad -2x \quad -2 \quad 0$

I: $\cos n\pi x \quad \frac{\sin n\pi x}{n\pi} \quad -\frac{\cos n\pi x}{n^2\pi^2} \quad -\frac{\sin n\pi x}{n^3\pi^3}.$

Signs: $+ \quad - \quad +$

$$= 2\left[(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - 2x\left(\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[\left[(1-1^2)\left(\frac{\sin n\pi}{n\pi}\right)_{\to 0} - 2(1)\left(\frac{\cos n\pi}{n^2\pi^2}\right) + 2\left(\frac{\sin n\pi}{n^3\pi^3}\right)_{\to 0}\right] - \left\{(1-0^2)\left(\frac{\sin 0}{n\pi}\right)_{\to} - 2(0)\left(\frac{\cos 0}{n^2\pi^2}\right)_{\to 0} + 2\left(\frac{\sin 0}{n^3\pi^3}\right)_{\to 0}\right\}\right]$$

$$= 2\left(\frac{-2}{n^2\pi^2}\cos n\pi\right)$$

$$a_n = -\frac{4}{n^2\pi^2}\cos n\pi = -\frac{4}{n^2\pi^2}(-1)^n \quad \text{(or)}$$

$$= (-1)\frac{4}{n^2\pi^2}(-1)^n \quad \text{(or)} \quad \frac{4}{n^2\pi^2}(-1)^{n+1}$$

$$a_0 = \frac{4}{3} \qquad a_n = \frac{(-1)^{n+1}\cdot 4}{n^2\pi^2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$= \frac{2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot 4}{n^2\pi^2}\cos n\pi x$$

$$= \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^2}\right)\cos n\pi x.$$

$$f(x) = \frac{2}{3} + \frac{4}{\pi^2}\left[\cos\pi x - \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x - \cdots\right]$$

which is the required Fourier Series

(4) Find the Fourier Series with Period 3 to represent

$$f(x) = x + x^2 \quad \text{in} \quad (0, 3)$$

Sol: $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\dfrac{n\pi x}{l}\right) + b_n \sin\left(\dfrac{n\pi x}{l}\right)$

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5) Find the Half range Cosine Series expansion of

$f(x) = x$ in $[0, 2]$

**Sol** :- The Half range Cosine Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Here $\boxed{l = 2}$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{2}\int_0^2 x\,dx = \int_0^2 x\,dx = \left(\frac{x^2}{2}\right)_0^2 = \frac{4}{2} = \underline{\underline{2}}$$

$$\therefore \boxed{a_0 = 2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$\begin{array}{llll} D & x & 1 & 0 \\ I & \cos\left(\frac{n\pi x}{2}\right) & \dfrac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} & -\dfrac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}} \end{array}$$

$$= \left[x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \left(\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= \left[2\left(\frac{\sin\frac{2n\pi}{2}}{\frac{n\pi}{2}}\right) + \left(\frac{\cos\frac{2n\pi}{2}}{\frac{n^2\pi^2}{4}}\right)\right] - \left[(0)\left(\frac{\sin\frac{0}{2}}{\frac{n\pi}{2}}\right) + \left(\frac{\cos\frac{0}{2}}{\frac{n^2\pi^2}{4}}\right)\right]$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1) = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & \text{; if } n \text{ is even} \\ \dfrac{-8}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases}$$

$$f(x) = \frac{2}{2} + \sum_{n=1,3,5\ldots} \frac{-8}{n^2\pi^2}\cos\frac{n\pi x}{2}$$

$$x = 1 - \frac{8}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

6) Find the Half Range Cosine Series expansion of

$f(x) = x - x^2$ in $0 < x < 1$

Sol: $f(x) = x - x^2$, $0 < x < 1$
$0 < x < l$. $(\because l = 1)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= 2\left(\frac{3-2}{6}\right) = 2\left(\frac{1}{6}\right) = \frac{1}{3}$$

$$\therefore \boxed{a_0 = \frac{1}{3}}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\int_0^1 (x - x^2)\cos n\pi x\,dx$$

D: $x - x^2$, $1 - 2x$, $-2$, $0$

I: $\cos n\pi x$, $\frac{\sin n\pi x}{n\pi}$, $-\frac{\cos n\pi x}{n^2\pi^2}$, $-\frac{\sin n\pi x}{n^3\pi^3}$

$$= 2\left[(x - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) + (1-2x)\left(\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[\underbrace{(1-1^2)\left(\frac{\sin n\pi}{n\pi}\right)}_{\to 0} + (1-2(1))\left(\frac{\cos n\pi}{n^2\pi^2}\right) + \underbrace{2\left(\frac{\sin n\pi}{n^3\pi^3}\right)}_{\to 0}\right] -$$

$$\left\{\underbrace{(0-0^2)\left(\frac{\sin 0}{n\pi}\right)}_{\to 0} + (1-2(0))\left(\frac{\cos 0}{n^2\pi^2}\right) + \underbrace{2\left(\frac{\sin 0}{n^3\pi^3}\right)}_{\to 0}\right\}\Bigg]$$

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$$= 2\left[-\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$= \frac{-2}{n^2\pi^2}\left[\cos n\pi + 1\right] = \frac{-2}{n^2\pi^2}\left[(-1)^n + 1\right]$$

(or)

$$\frac{-2}{n^2\pi^2}\left[1+(-1)^n\right] = \frac{-2\left[1+(-1)^n\right]}{n^2\pi^2}$$

$$a_n = \begin{cases} 0 \ ; & \text{when } n \text{ is odd} \\ \frac{-4}{n^2\pi^2} \ ; & \text{when } n \text{ is even.} \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$x - x^2 = \frac{1}{6} + \sum_{n=2,4,6,8}^{\infty} \frac{-4}{n^2\pi^2}\cos n\pi x \quad (l=1)$$

$$x - x^2 = \frac{1}{6} - \frac{4}{\pi^2}\left[\frac{\cos 2\pi x}{2^2} + \frac{\cos 4\pi x}{4^2} + \frac{\cos 6\pi x}{6^2} + \cdots\right]$$

is the required Fourier Series

(7) Obtain the Half range Sine Series for $e^x$ in $0<x<1$

Sol : Half Range Sine Series :-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{Here } (l=1)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = 2\int_0^1 e^x \sin n\pi x \, dx.$$

$$\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}$$

On Comparing we have.

$a = 1$

$b = n\pi$

$$= 2\left[\frac{e^x}{1+n^2\pi^2}\left(\sin n\pi x - n\pi\cos n\pi x\right)\right]_0^1$$

$$= 2\left[\left\{\frac{e^1}{1+n^2\pi^2}\left(\underset{\to 0}{\sin n\pi} - n\pi\cos n\pi\right)\right\} - \left\{\frac{e^0}{1+n^2\pi^2}\left(\underset{\to 0}{\sin 0} - n\pi\cos 0\right)\right\}\right]$$

$$= 2\left[\frac{e}{1+n^2\pi^2}\left(-n\pi\cos n\pi\right) - \frac{1}{1+n^2\pi^2}\left(-n\pi\right)\right]$$

$$= 2\left[\frac{1}{1+n^2\pi^2}\left(n\pi - n\pi e\cos n\pi\right)\right]$$

$$b_n = \frac{2n\pi}{1+n^2\pi^2}\left[1 - e(-1)^n\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$e^x = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2}\left[1-e(-1)^n\right]\sin n\pi x$$

$$e^x = \pi\left[\frac{2(1+e)}{\pi^2+1}\sin\pi x + \frac{2(1-e)}{4\pi^2+1}\sin 2\pi x + \frac{3(1+e)}{9\pi^2+1}\sin 3\pi x + \cdots\right]$$

is the required Fourier Series

(8) Express $f(x) = x^2$ as a Fourier Series in $[-l, l]$

Sol :- given $f(x) = x^2 \quad [-l, l]$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$$\therefore f(-x) = f(x)$$

$\therefore f(x)$ is an even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

(25)

$$a_0 = \frac{2}{l}\int_0^l x^2\,dx = \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{2}{\cancel{l}}\left[\frac{l^{\cancel{3}\,2}}{3}\right] = \frac{2l^2}{3}$$

$$\therefore \boxed{a_0 = \frac{2l^2}{3}}$$

$$a_n = \frac{2}{l}\int_0^l x^2 \cos\left(\frac{n\pi x}{l}\right)dx$$

$$\begin{array}{llllll} D & x^2 & 2x & 2 & & 0 \\ I & \cos\left(\frac{n\pi x}{l}\right) & \dfrac{\frac{\sin n\pi x}{l}}{\frac{n\pi}{l}} & -\dfrac{\frac{\cos n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} & & -\dfrac{\frac{\sin n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \end{array}$$

$$= \frac{2}{l}\left[x^2\left(\frac{\frac{\sin n\pi x}{l}}{\frac{n\pi}{l}}\right) + 2x\left(\frac{\frac{\cos n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\frac{\sin n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[l^2\underbrace{\left(\frac{\frac{\sin n\pi \cancel{l}}{\cancel{l}}}{\frac{n\pi}{l}}\right)}_{\to 0} + 2l\left(\frac{\frac{\cos n\pi \cancel{l}}{\cancel{l}}}{\frac{n^2\pi^2}{l^2}}\right) - 2\underbrace{\left(\frac{\frac{\sin n\pi l}{l}}{\frac{n^3\pi^3}{l^3}}\right)}_{\to 0}\right] -$$

$$\left[(0)^2\underbrace{\left(\frac{\frac{\sin 0}{l}}{\frac{n\pi}{l}}\right)}_{\to 0} + 2(0)\underbrace{\left(\frac{\frac{\cos 0}{l}}{\frac{n^2\pi^2}{l^2}}\right)}_{\to 0} - 2\underbrace{\left(\frac{\frac{\sin 0}{l}}{\frac{n^3\pi^3}{l^3}}\right)}_{\to 0}\right]$$

$$= \frac{2}{l}\left[2l\left[\frac{\cos n\pi}{\frac{n^2\pi^2}{l^2}}\right]\right] = \frac{4l^2\cos n\pi}{n^2\pi^2} = \frac{(-1)^n 4l^2}{n^2\pi^2}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) = \frac{l^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n 4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}.$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2} - \frac{\cos(2\pi x/l)}{2^2} + \frac{\cos(3\pi x/l)}{3^2} - \cdots\right]$$

9) Find the Fourier Series expansion for $f(x)$ if

$$f(x) = \begin{cases} 2 \; ; & \text{if } -2 \le x \le 0 \\ x \; ; & \text{if } 0 < x < 2. \end{cases}$$

$-2 \longleftrightarrow 2$

$l = 2$

Sol :- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \longrightarrow (1)$

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$

$= \frac{1}{2}\left[\int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx\right] = \frac{1}{2}\left[2(x)_{-2}^{0} + \left(\frac{x^2}{2}\right)_0^2\right]$

$= \frac{1}{2}\left[4 + \frac{4}{2}\right] = \frac{1}{2}\left[\frac{8+4}{2}\right]$

$= \frac{1}{\cancel{2}}\left[\frac{\cancel{12}^{\cancel{6}\,3}}{\cancel{2}}\right] = \underline{3}$ $\therefore \boxed{a_0 = 3}$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$

$= \frac{1}{2}\left[\int_{-2}^{0} (2)\cos\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2} (x)\cos\left(\frac{n\pi x}{2}\right)dx\right]$

$= \frac{1}{2}\left[2\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_{-2}^{0} + \left(x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \left(\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right)_0^2\right]$

D $\quad x \quad\quad 1 \quad\quad 0$

I $\quad \cos\frac{n\pi x}{2} \quad \frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} \quad -\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}$

$+ \quad -$

(26)

$$= \frac{1}{2}\left[2\left(\frac{\sin 0}{\frac{n\pi}{2}}\right) + \left(\frac{\sin n\pi}{\frac{n\pi}{2}}\right)\right] +$$

$$\left[(2)\left(\frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}}\right) - (0)\left(\frac{\sin \frac{0}{2}}{\frac{n\pi}{2}}\right)\right] -$$

$$+\left(\frac{\cos \frac{2n\pi}{2}}{\frac{n^2\pi^2}{4}}\right) - \left(\frac{\cos \frac{0}{2}}{\frac{n^2\pi^2}{4}}\right)\Bigg]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right]$$

$$a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & ; \text{ when } n \text{ is even} \\ \frac{-4}{n^2\pi^2} & ; \text{ when } n \text{ is odd.} \end{cases}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(2)\sin\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2}(x)\sin\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \frac{1}{2}\left[2\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_{-2}^{0} + \left\{-x\left(\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \left(\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{0}^{2}\right]$$

| | | | |
|---|---|---|---|
| D | $x$ | $1$ | $0$ |
| I | $\sin\frac{n\pi x}{2}$ | $-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}$ | $-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}$ |

$$= \frac{1}{2}\left[-2\left(\frac{\frac{\cos 0}{2}}{\frac{n\pi}{2}} + \frac{\frac{\cos 2n\pi}{2}}{\frac{n\pi}{2}}\right)\right] +$$

$$\left\{-2\left(\frac{\frac{\cos n\pi \cancel{x}}{\cancel{x}}}{\frac{n\pi}{2}}\right) + (0)\left(\frac{\frac{\cos 0}{2}}{\frac{n\pi}{2}}\right)\right\} - \left\{\left(\frac{\frac{\sin n\pi \cancel{x}}{\cancel{x}}}{\frac{n^2\pi^2}{4}}\right) - \left(\frac{\frac{\sin 0}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}$$

$\to 0$. $\to 0$. $\to 0$.

$$= \frac{1}{2}\left[-\frac{4}{n\pi} + \cancel{\frac{4}{n\pi}\cos n\pi}\right] + \frac{1}{2}\left[\cancel{-\frac{4}{n\pi}\cos n\pi}\right]$$

$$= \frac{1}{\cancel{2}}\left(-\frac{\cancel{4}^{2}}{n\pi}\right) = -\frac{2}{n\pi} \qquad \therefore b_n = -\frac{2}{n\pi}$$

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

12) Find half Range Sine Series expansion of

$f(x) = x^2$ in $[0, 4]$ $\quad b_n = \begin{cases} -32/n\pi & ; \text{ when } n \text{ is even} \\ 32\left(\frac{-4}{\pi^3 n^3} + \frac{1}{n\pi}\right) & ; \, n \text{ is odd.} \end{cases}$

13) Find half Range Cosine Series

$f(x) = x - x^2 \quad 0 < x < 1$

$a_0 = \frac{1}{3} \qquad a_n = \begin{cases} 0 & ; \text{ when } n \text{ is odd} \\ \frac{-4}{n^2\pi^2} & ; \text{ when } n \text{ is even.} \end{cases}$

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## Parseval's Formula :-

$$\int_{-l}^{l}[f(x)]^2\,dx = l\left[\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]$$

**Proof** :- we know that Fourier Series of $f(x)$ in the interval $(-l, l)$ is

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right)+b_n\sin\left(\frac{n\pi x}{l}\right)\right) \longrightarrow (1)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l}f(x)\,dx$ $\quad a_n = \frac{1}{l}\int_{-l}^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$b_n = \frac{1}{l}\int_{-l}^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx$ $\quad \rightarrow (2)$

multiplying (1) by $f(x)$ we get

$\cancel{[f(x)]^2 = \frac{a_0}{2}\int_{-l}^{l}f(x)\,dx}$

$$[f(x)]^2 = \frac{a_0}{2}f(x)+\sum_{n=1}^{\infty}a_n f(x)\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n f(x)\left(\sin\left(\frac{n\pi x}{l}\right)\right) \longrightarrow (3)$$

now integrating (3) term by term from $-l$ to $l$ we get

$$\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0}{2}\int_{-l}^{l}f(x)\,dx+\sum_{n=1}^{\infty}a_n\int_{-l}^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \sum_{n=1}^{\infty}b_n\int_{-l}^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx \longrightarrow (4)$$

Using (2) (3) becomes

$$\int_{-l}^{l}[f(x)]^2\,dx = l\frac{a_0^2}{2}+\sum_{n=1}^{\infty}l\,a_n^2+\sum_{n=1}^{\infty}l\,b_n^2$$

$$= l\left[\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]$$

This is the Parsevals formula.

**Note** :-

1) The Parseval's formula is

$$\boxed{\int_{-l}^{l} [f(x)]^2 \, dx = l \left[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]}$$

→ (1) If $0 < x < 2l$ then

$$\int_{0}^{2l} [f(x)]^2 \, dx = l \left[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

~~(2) If $0 < x < l$ then~~

$$\cancel{\int_{0}^{l} [f(x)]^2 \, dx = l \left[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]}$$

(2) If $0 < x < l$ (Half-Range Cosine Series)

$$\int_{0}^{l} [f(x)]^2 \, dx = \frac{1}{2} \left[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right]$$

(3) If $0 < x < l$ (Half-Range Sine Series)

$$\int_{0}^{l} [f(x)]^2 \, dx = \frac{1}{2} \left[ \sum_{n=1}^{\infty} b_n^2 \right]$$

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## Parseval's Formula :-

$$\int_{-l}^{l} [f(x)]^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

**Proof**:- we know that Fourier series of $f(x)$ in the interval $(-l, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right) \longrightarrow (1)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$ $\Rightarrow \int_{-l}^{l} f(x)dx = l a_0$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = l a_n$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = l b_n$ $\longrightarrow (2)$

multiply (1) by $f(x)$ we get

$$[f(x)]^2 = \frac{a_0}{2}f(x) + \sum_{n=1}^{\infty} a_n f(x)\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n f(x)\sin\left(\frac{n\pi x}{l}\right) \longrightarrow (3)$$

Integrating (3) term by term from $-l$ to $l$ we get

$$\int_{-l}^{l}[f(x)]^2 = \frac{a_0}{2}\underbrace{\int_{-l}^{l} f(x)dx}_{la_0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx}_{la_n} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx}_{lb_n} \longrightarrow (4)$$

Using (2), (4) becomes

$$\int_{-l}^{l}[f(x)]^2 dx = l\frac{a_0^2}{2} + \sum_{n=1}^{\infty} l a_n^2 + \sum_{n=1}^{\infty} l b_n^2$$

$$\int_{-l}^{l}[f(x)]^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

This is the Parsevel's formula

**Notes :-**

1) If $0 < x < 2l$ then

$$\int_0^{2l} [f(x)]^2 dx = l \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$$

2) If $0 < x < l$ then Half Range Cosine Series

$$\int_0^{l} [f(x)]^2 dx = \frac{l}{2} \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

3) If $0 < x < l$ then Half Range Sine Series

$$\int_0^{l} [f(x)]^2 dx = \frac{l}{2} \left[\sum_{n=1}^{\infty} b_n^2\right]$$

**Complex Fourier Series :-**

we know that Fourier Series for the function $f(x)$ in the interval $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx .$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx .$$

we will now write the above series in complex form as follows.

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}}$$

where $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$.

for $n = 0, \pm 1, \pm 2 \dots$

This is called the complex form of the Fourier series (or) complex Fourier series of $f(x)$

The $C_n$'s are called the complex Fourier co-efficients of $f(x)$

Complex Fourier series for Periodic function :-

If $f(x)$ is a function of period $2L$ then we get

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{L}}$$

where $C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{-in x\pi}{L}}\, dx$

$n = 0, \pm 1, \pm 2 \dots$

Q8) Expand $f(x) = x\sin x$ as a Fourier Series in $-\pi \le x \le \pi$ Hence deduce that

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots = \frac{\pi - 2}{4}.$$

Sol: Since $f(x) = x\sin x$ is an even function $b_n = 0$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{2}{\pi}\Big[x(-\cos x) - 1\cdot(-\sin x)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}(-\pi\cos\pi) = \underline{2} \quad \therefore \boxed{a_0 = 2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} x\; 2\sin x\cos nx\,dx$$

$$\boxed{2\sin A\cos B = \sin(A+B) + \sin(A-B)}$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,[\sin(n+1)x + \sin(1-n)x]\,dx$$

$$= \frac{1}{\pi}\left\{x\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(1-n)x}{1-n}\right] - 1\cdot\left[-\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(1-n)x}{(1-n)^2}\right]\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi\left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(1-n)\pi}{1-n}\right]\right] \quad (n \neq 1)$$

(Since the second term vanishes at both upper and lower limits)

$$\therefore\; a_n = -\left[\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1}\right] \quad (\because \cos(-\theta) = \cos\theta)$$

$$= -\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}\;(-1)^2}{n-1}\right]$$

$$= -\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n-1}\right]$$

$$= (-1)(-1)^{n+1}\left[\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right]$$

$$= \frac{2(-1)^{n+1}}{n^2-1}, \quad n \neq 1 \longrightarrow (A)$$

To determine $a_n$ when $n=1$

Putting $\boxed{n=1}$ in $a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx$ we get.

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\cdot\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{-\pi \cos 2\pi}{2}\right)$$

$= \frac{-1}{2}$ (Since second term vanishes at both upper and lower portions)

$$\therefore x \sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{(n-1)(n+1)}\cos nx$$

$$= 1 - \frac{1}{2}\cos x + \left(\frac{-2}{1\cdot 3}\cos 2x + \frac{2}{2\cdot 4}\cos 3x - \frac{2}{3\cdot 5}\cos 4x + \cdots\right) \rightarrow (B)$$

Putting $\boxed{x = \frac{\pi}{2}}$ in (B) we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - 0 - \frac{2}{1\cdot 3}(-1) + \frac{2}{2\cdot 4}(0) + \frac{-2}{3\cdot 5}(1) + \cdots$$

$$\Rightarrow \frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \cdots$$

$$\Rightarrow \frac{\pi}{2} - 1 = \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \cdots \Rightarrow \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots = \frac{1}{4}(\pi - 2)$$