Lecture 11

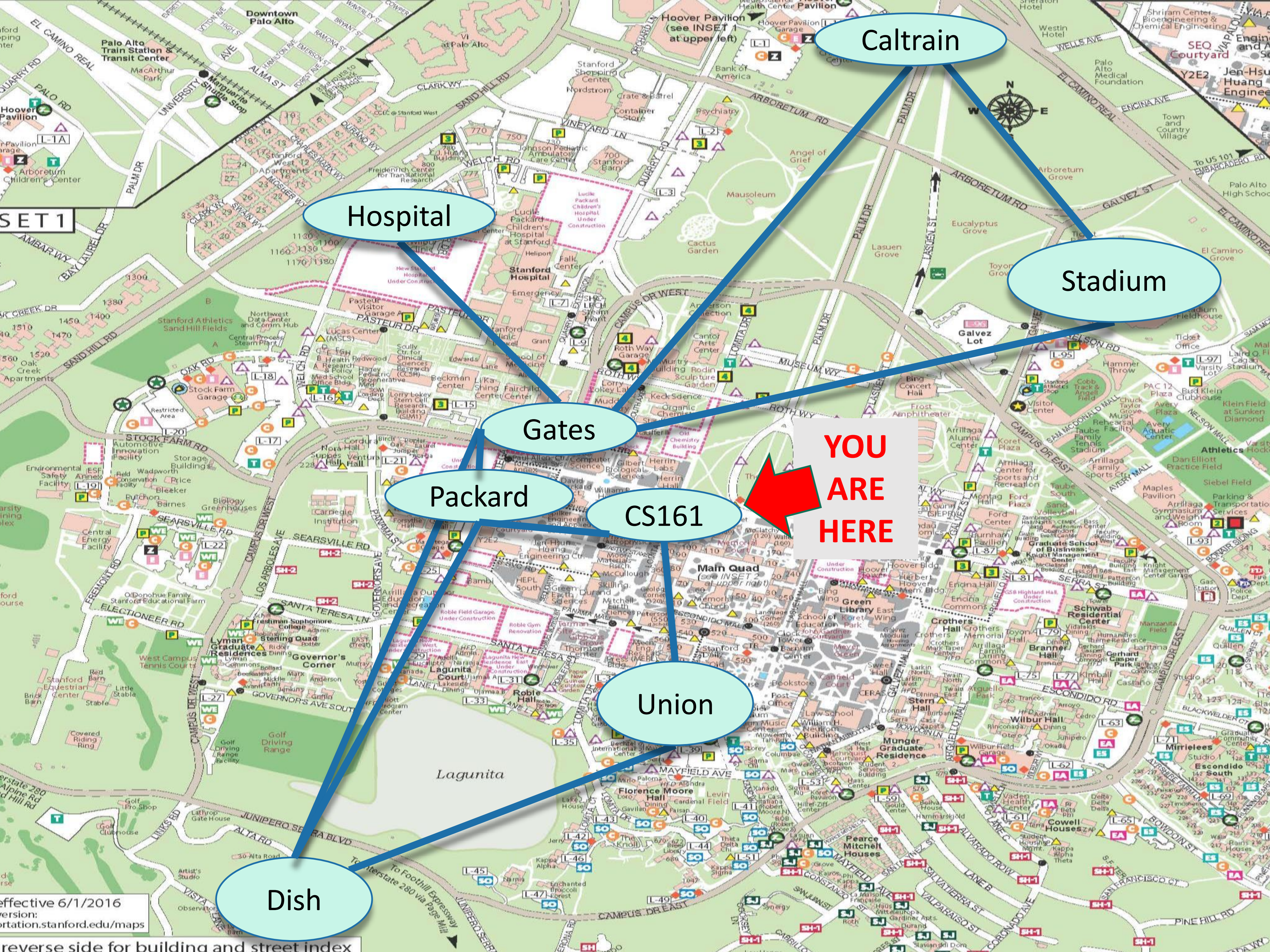
Weighted Graphs: Dijkstra and Bellman-Ford

Last Lecture

- Graphs!
- DFS
  - Topological Sorting
  - Strongly Connected Components
- BFS
  - Shortest Paths in unweighted graphs

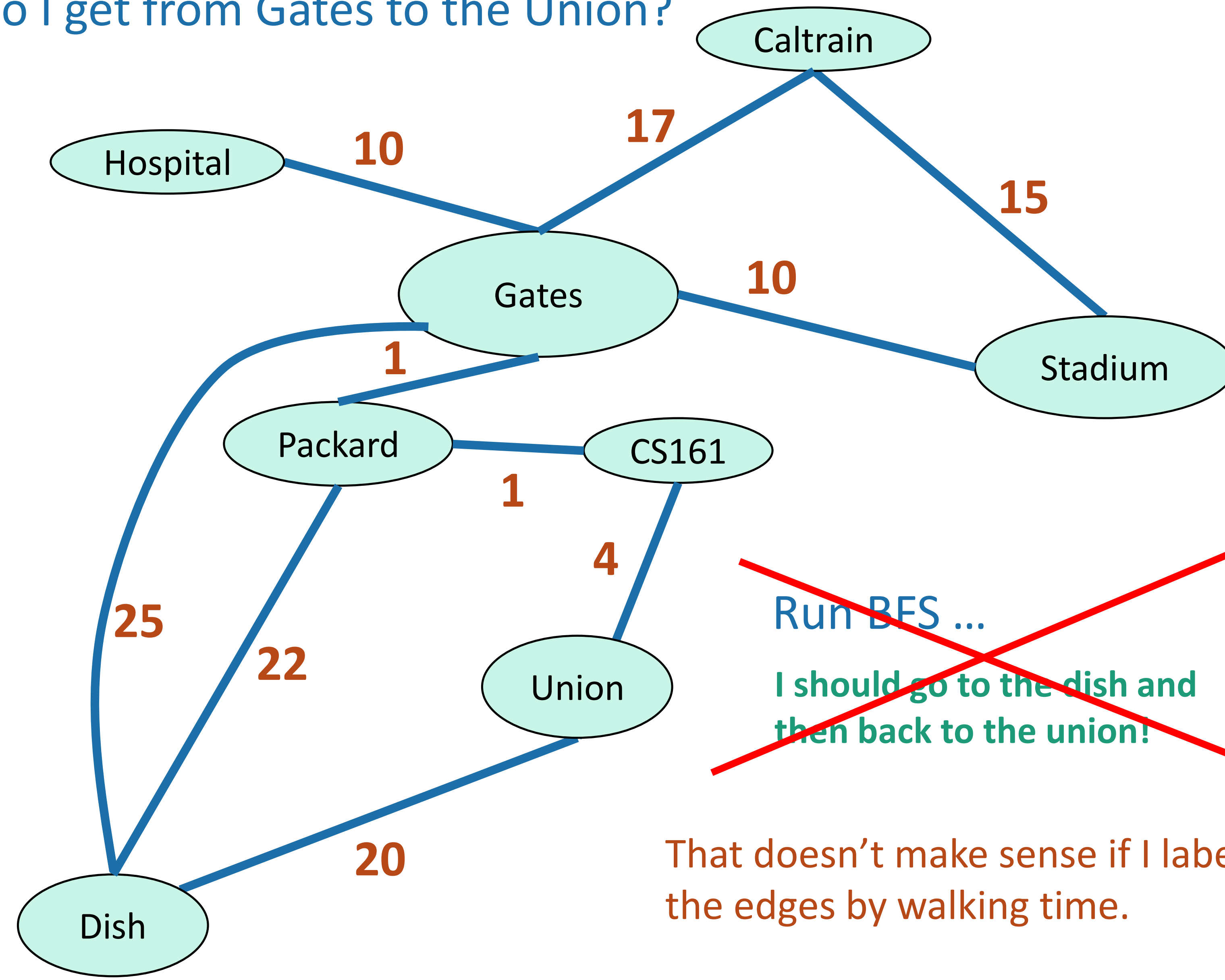
Today

- What if the graphs are weighted?
  - All nonnegative weights: Dijkstra!
  - If there are negative weights: Bellman-Ford!


Caltrain
Hospital
Stadium
Gates
Packard
CS161
YOU ARE HERE
Union
Dish
Lagunita
effective 6/1/2016
reverse side for building and street index

Just the graph

How do I get from Gates to the Union?



~~Run BFS …~~

~~**I should go to the dish and then back to the union!**~~

That doesn't make sense if I label the edges by walking time.

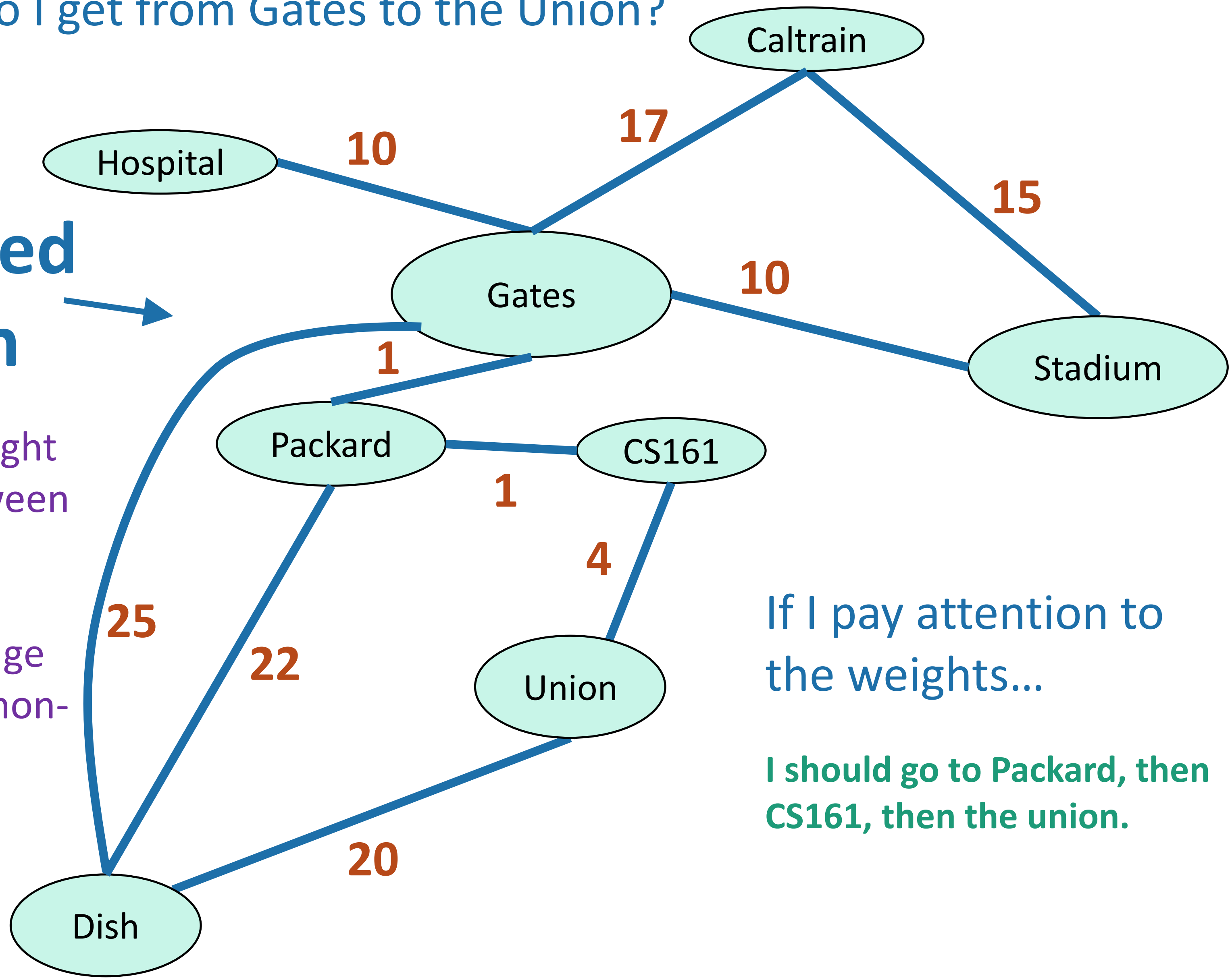
Just the graph

How do I get from Gates to the Union?

weighted graph

w(u,v) = weight of edge between u and v.

For now, edge weights are non-negative.

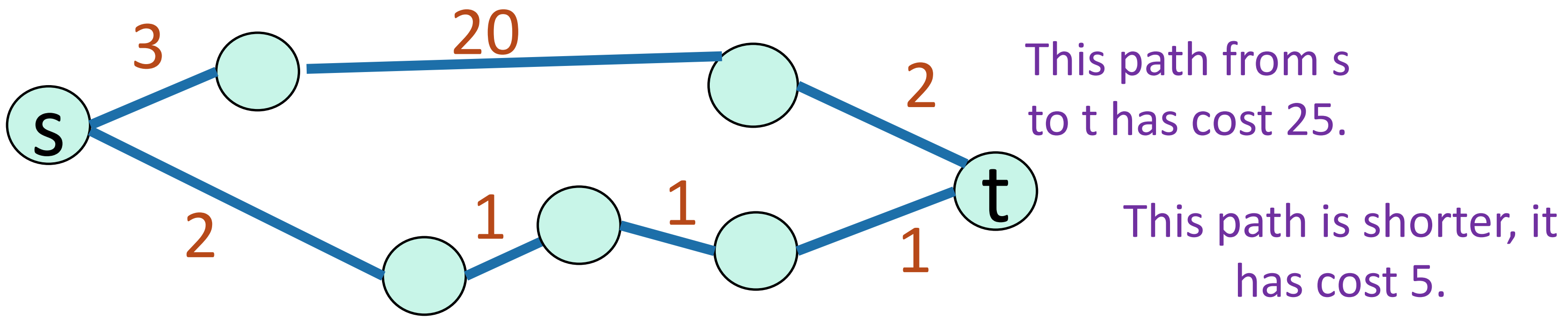


If I pay attention to the weights…

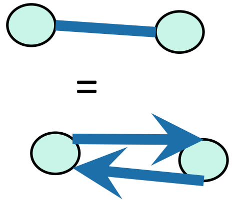
I should go to Packard, then CS161, then the union.

Shortest path problem

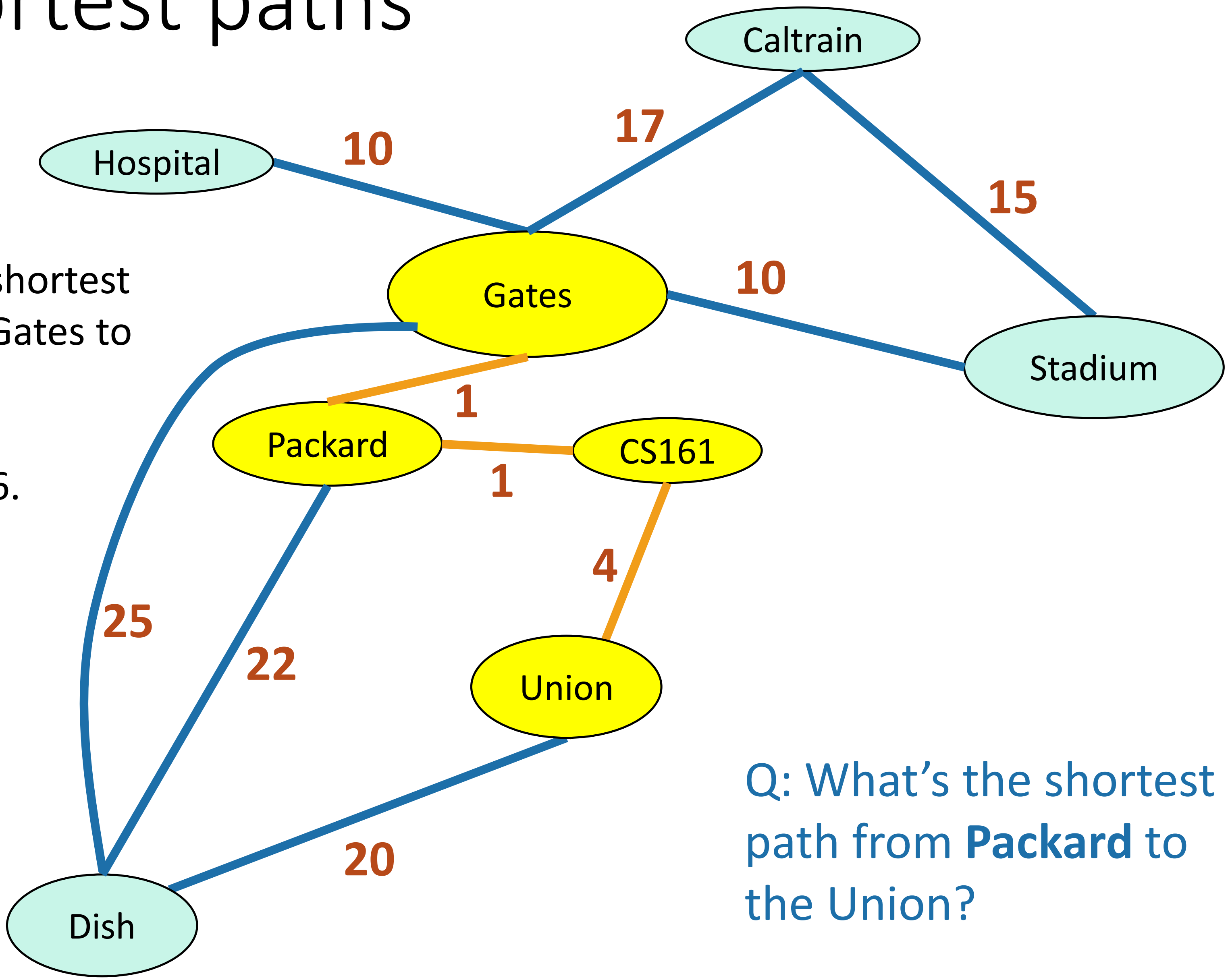
- What is the shortest path between u and v in a weighted graph?
  - the **cost** of a path is the sum of the weights along that path
  - The **shortest path** is the one with the minimum cost.



- The distance d(u,v) between two vertices u and v is the cost of the the shortest path between u and v.
- For this lecture **all graphs are directed**, but to save on notation I'm just going to draw undirected edges.



Shortest paths



This is the shortest path from Gates to the Union.

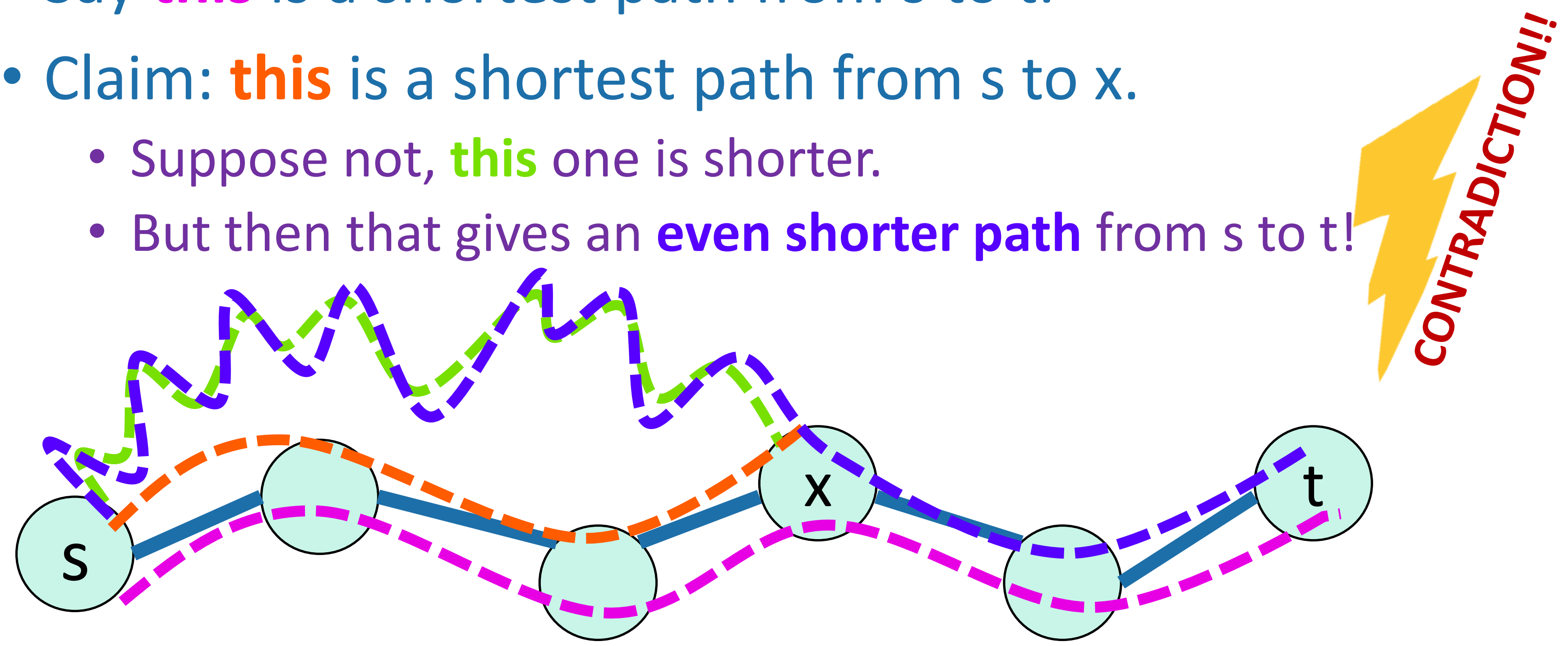
It has cost 6.

Q: What's the shortest path from **Packard** to the Union?

Warm-up

- A sub-path of a shortest path is also a shortest path.

- Say **this** is a shortest path from s to t.
- Claim: **this** is a shortest path from s to x.
  - Suppose not, **this** one is shorter.
  - But then that gives an **even shorter path** from s to t!



Single-source shortest-path problem

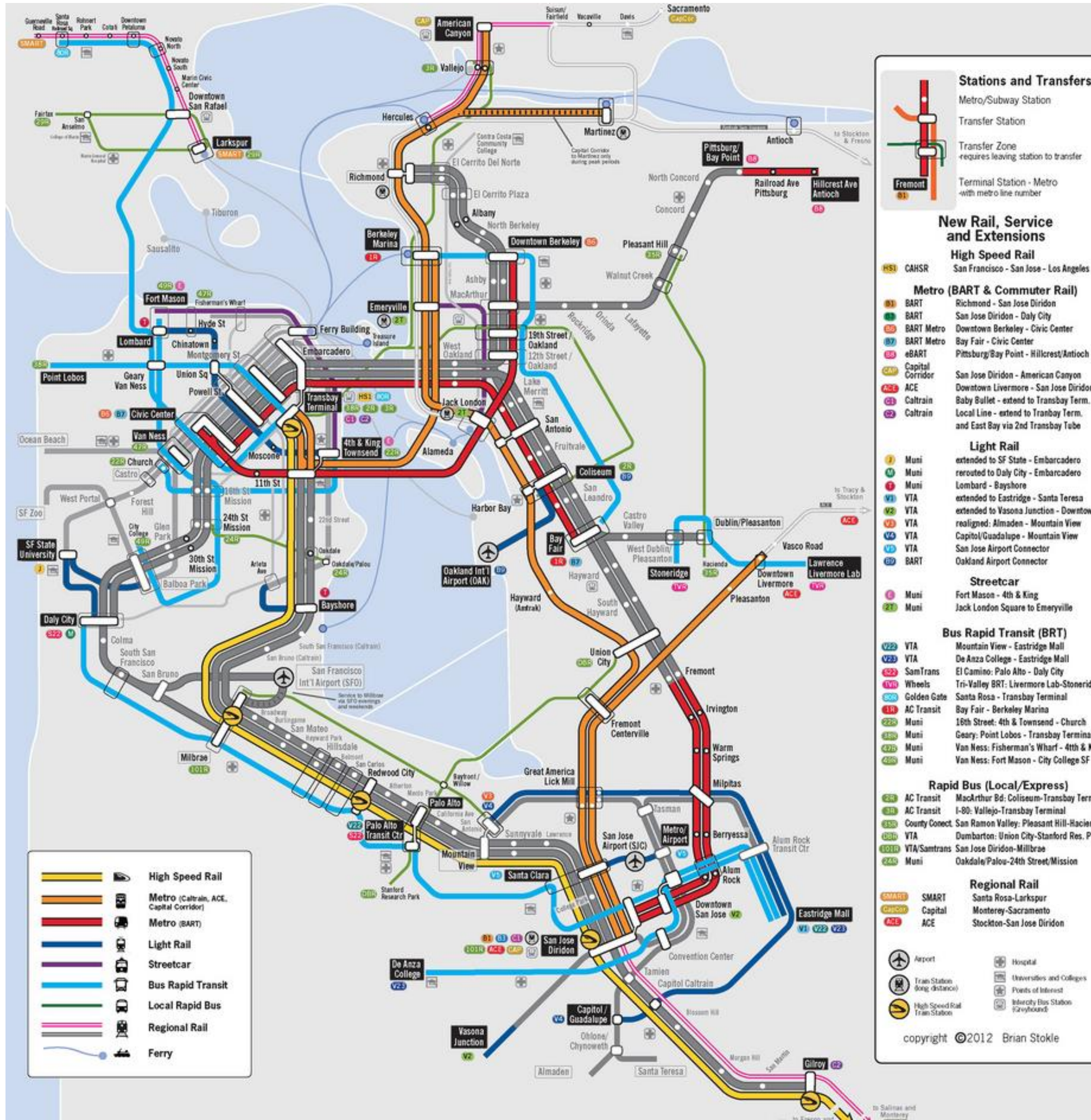
- I want to know the shortest path from one vertex (Gates) to all other vertices.

Destination	Cost	To get there
Packard	1	Packard
CS161	2	Packard-CS161
Hospital	10	Hospital
Caltrain	17	Caltrain
Union	6	Packard-CS161-Union
Stadium	10	Stadium
Dish	23	Packard-Dish

(Not necessarily stored as a table – how this information is represented will depend on the application)

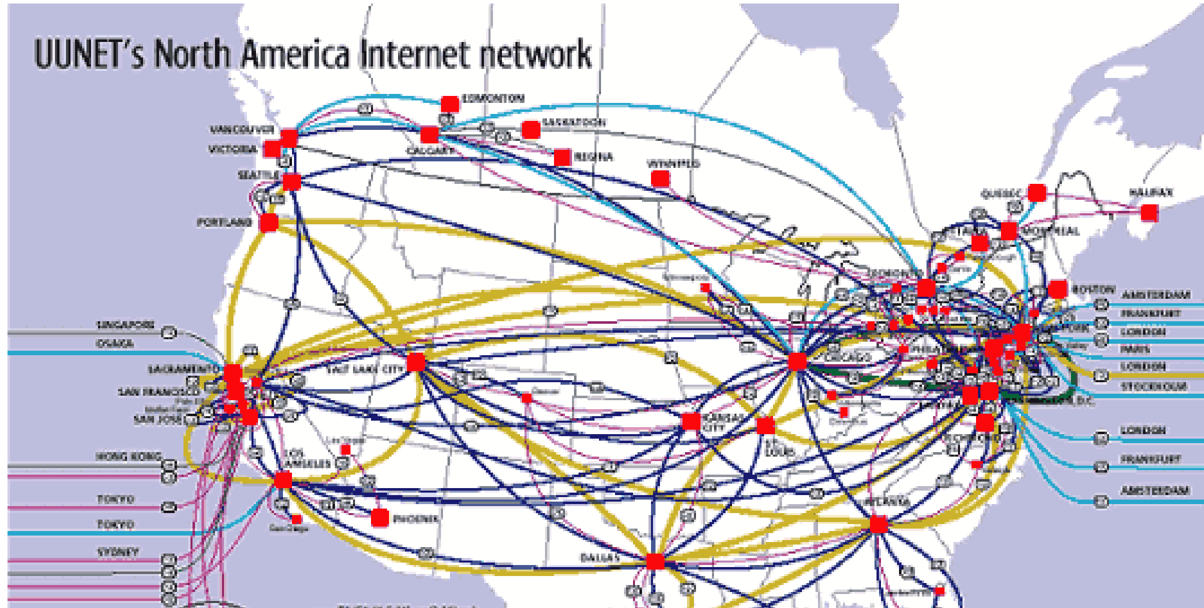
Example

- I regularly have to solve "**what is the shortest path from Palo Alto to [anywhere else]**" using BART, Caltrain, lightrail, MUNI, bus, Amtrak, bike, walking, uber/lyft.
- Edge weights have something to do with time, money, hassle. (They also change depending on my mood and traffic…).



Example

- **Network routing**
- I send information over the internet, from my computer to to all over the world.
- Each path (from a router to another router) has a cost which depends on link length, traffic, other costs, etc..
- How should we send packets?



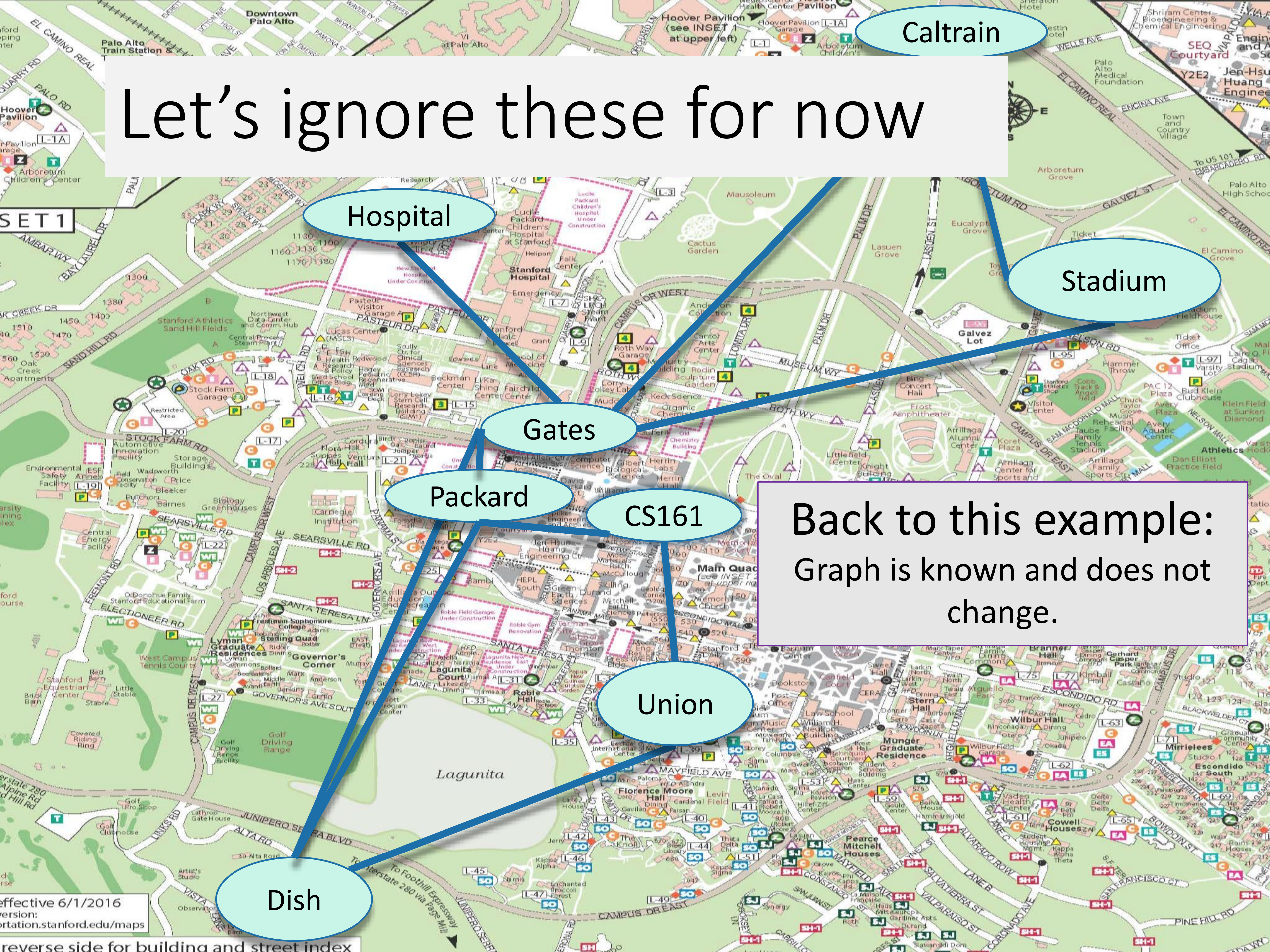

```
DN0a22a0e3:~ mary$ traceroute -a www.ethz.ch
traceroute to www.ethz.ch (129.132.19.216), 64 hops max, 52 byte packets
 1  [AS0] 10.34.160.2 (10.34.160.2)  38.168 ms  31.272 ms  28.841 ms
 2  [AS0] cwa-vrtr.sunet (10.21.196.28)  33.769 ms  28.245 ms  24.373 ms
 3  [AS32] 171.66.2.229 (171.66.2.229)  24.468 ms  20.115 ms  23.223 ms
 4  [AS32] hpr-svl-rtr-vlan8.sunet (171.64.255.235)  24.644 ms  24.962 ms  17
 5  [AS2152] hpr-svl-hpr2--stan-ge.cenic.net (137.164.27.161)  22.129 ms  4.9
 6  [AS2152] hpr-lax-hpr3--svl-hpr3-100ge.cenic.net (137.164.25.73)  12.125 m
 7  [AS2152] hpr-i2--lax-hpr2-r&e.cenic.net (137.164.26.201)  40.174 ms  38.3
 8  [AS0] et-4-0-0.4079.sdn-sw.lasv.net.internet2.edu (162.252.70.28)  46.573
 9  [AS0] et-5-1-0.4079.rtsw.salt.net.internet2.edu (162.252.70.31)  30.424 m
10  [AS0] et-4-0-0.4079.sdn-sw.denv.net.internet2.edu (162.252.70.8)  47.454
11  [AS0] et-4-1-0.4079.rtsw.kans.net.internet2.edu (162.252.70.11)  70.825 m
12  [AS0] et-4-1-0.4070.rtsw.chic.net.internet2.edu (198.71.47.206)  77.937 m
13  [AS0] et-0-1-0.4079.sdn-sw.ashb.net.internet2.edu (162.252.70.60)  77.682
14  [AS0] et-4-1-0.4079.rtsw.wash.net.internet2.edu (162.252.70.65)  71.565 m
15  [AS21320] internet2-gw.mx1.lon.uk.geant.net (62.40.124.44)  154.926 ms  1
16  [AS21320] ae0.mx1.lon2.uk.geant.net (62.40.98.79)  146.565 ms  146.604 ms
17  [AS21320] ae0.mx1.par.fr.geant.net (62.40.98.77)  153.289 ms  184.995 ms
18  [AS21320] ae2.mx1.gen.ch.geant.net (62.40.98.153)  160.283 ms  160.104 ms
19  [AS21320] swice1-100ge-0-3-0-1.switch.ch (62.40.124.22)  162.068 ms  160.
20  [AS559] swizh1-100ge-0-1-0-1.switch.ch (130.59.36.94)  165.824 ms  164.2
21  [AS559] swiez3-100ge-0-1-0-4.switch.ch (130.59.38.109)  164.269 ms  164.3
22  [AS559] rou-gw-lee-tengig-to-switch.ethz.ch (192.33.92.1)  164.082 ms  17
23  [AS559] rou-fw-rz-rz-gw.ethz.ch (192.33.92.169)  164.773 ms  165.193 ms
```

A few things that make these examples even more difficult

than trying to navigate the Stanford campus.

- Costs may change
  - If it's raining the cost of biking is higher
  - If a link is congested, the cost of routing a packet along it is higher
- The network might not be known
  - My computer doesn't store a map of the internet
- We want to do these tasks really quickly
  - More seriously, **the internet.**

Let’s ignore these for now

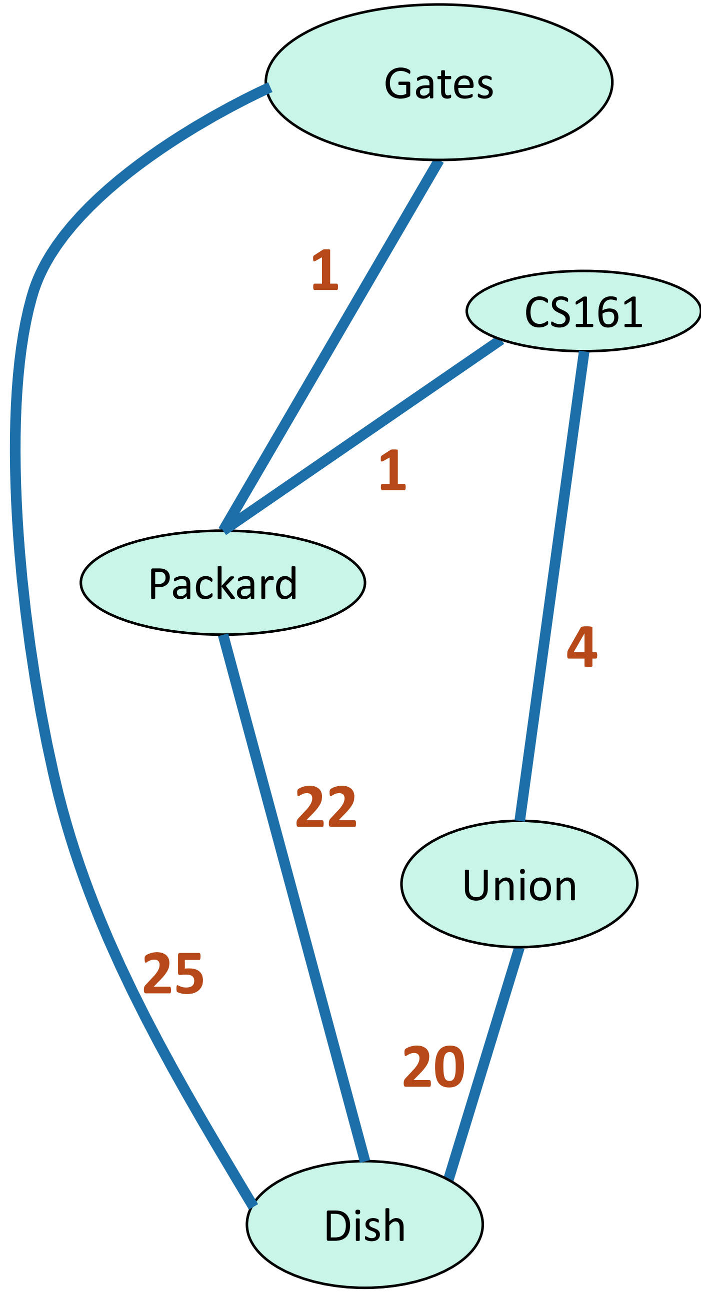


Back to this example:

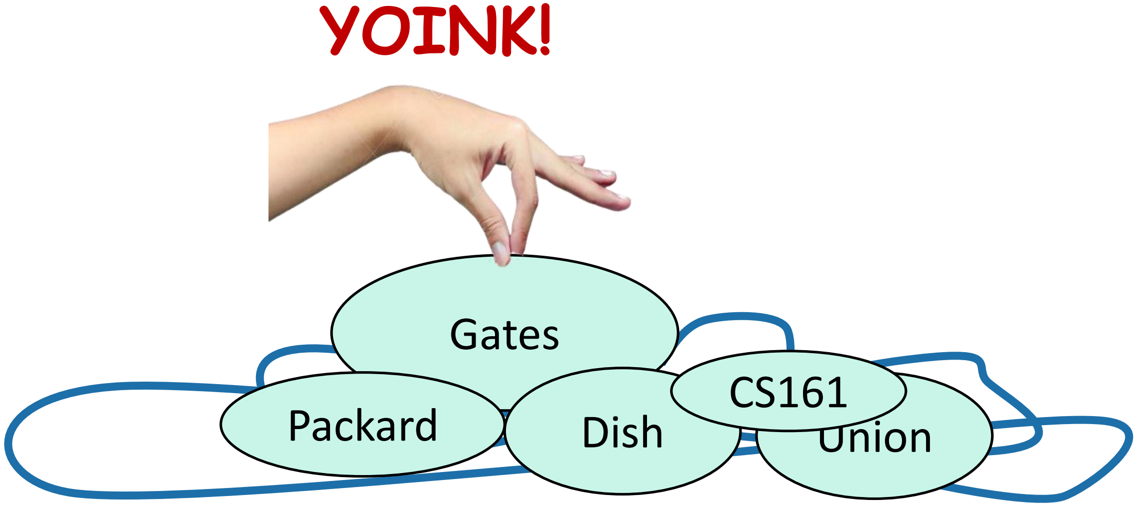
Graph is known and does not change.

Dijkstra's algorithm

- What are the shortest paths from Gates to everywhere else?

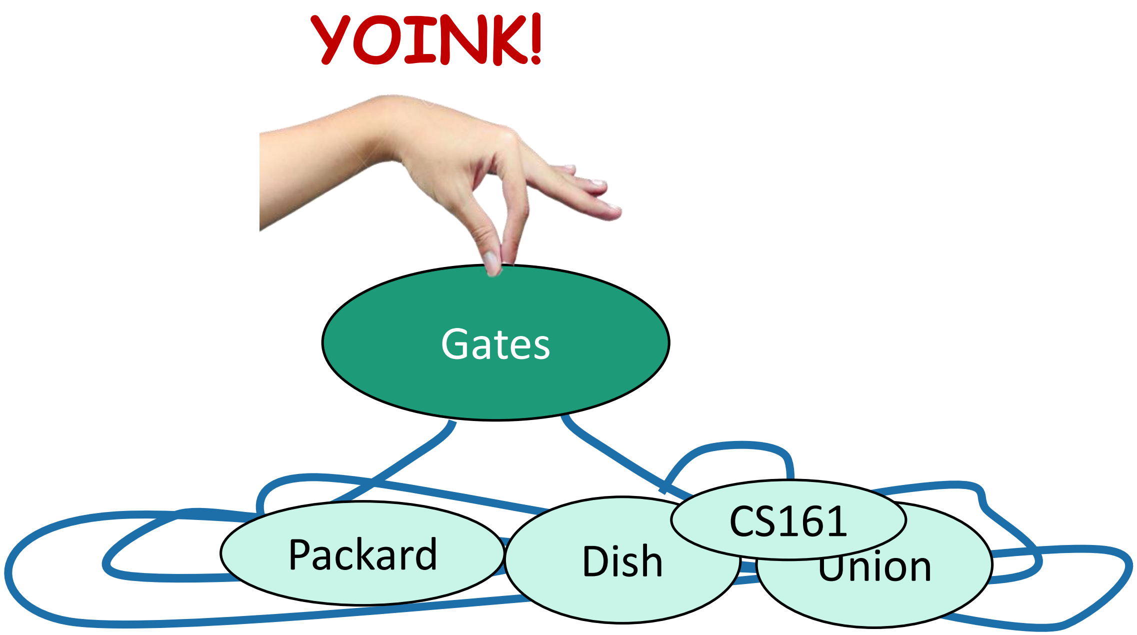


Dijkstra
intuition



Dijkstra
intuition

A vertex is done when it's not on the ground anymore.



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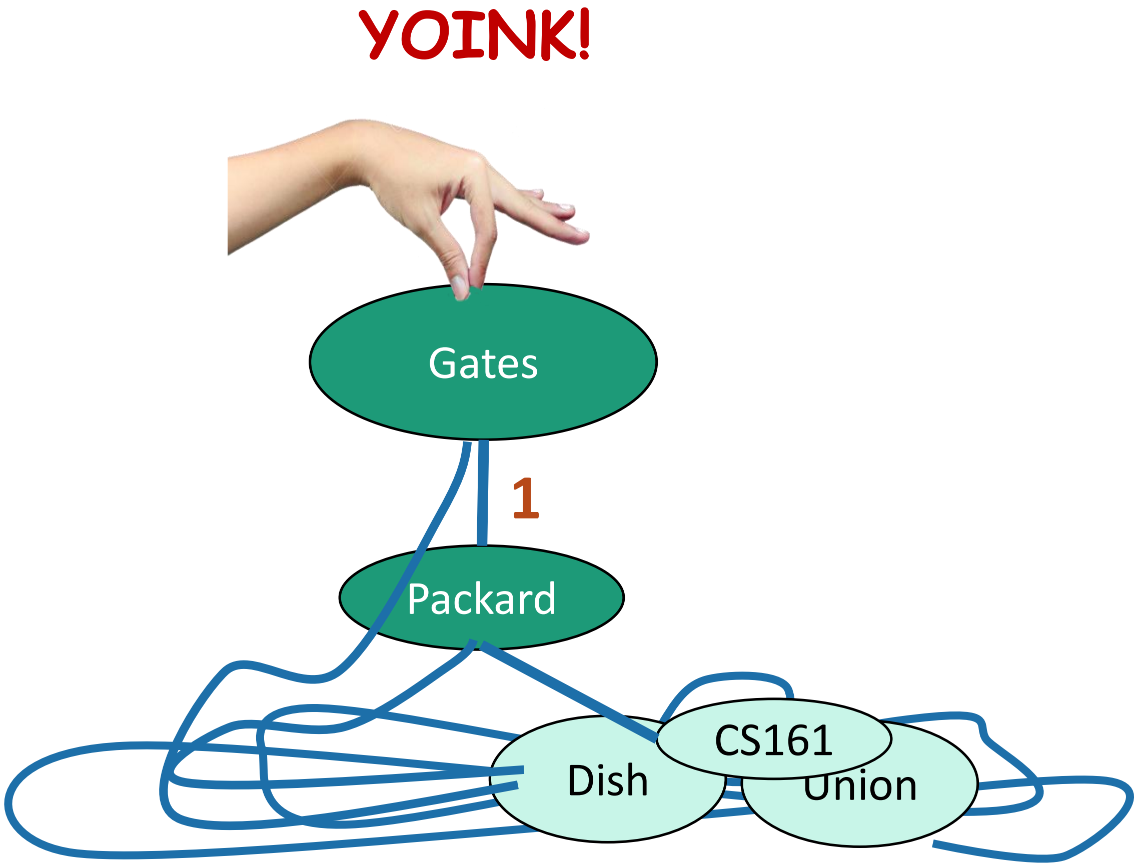

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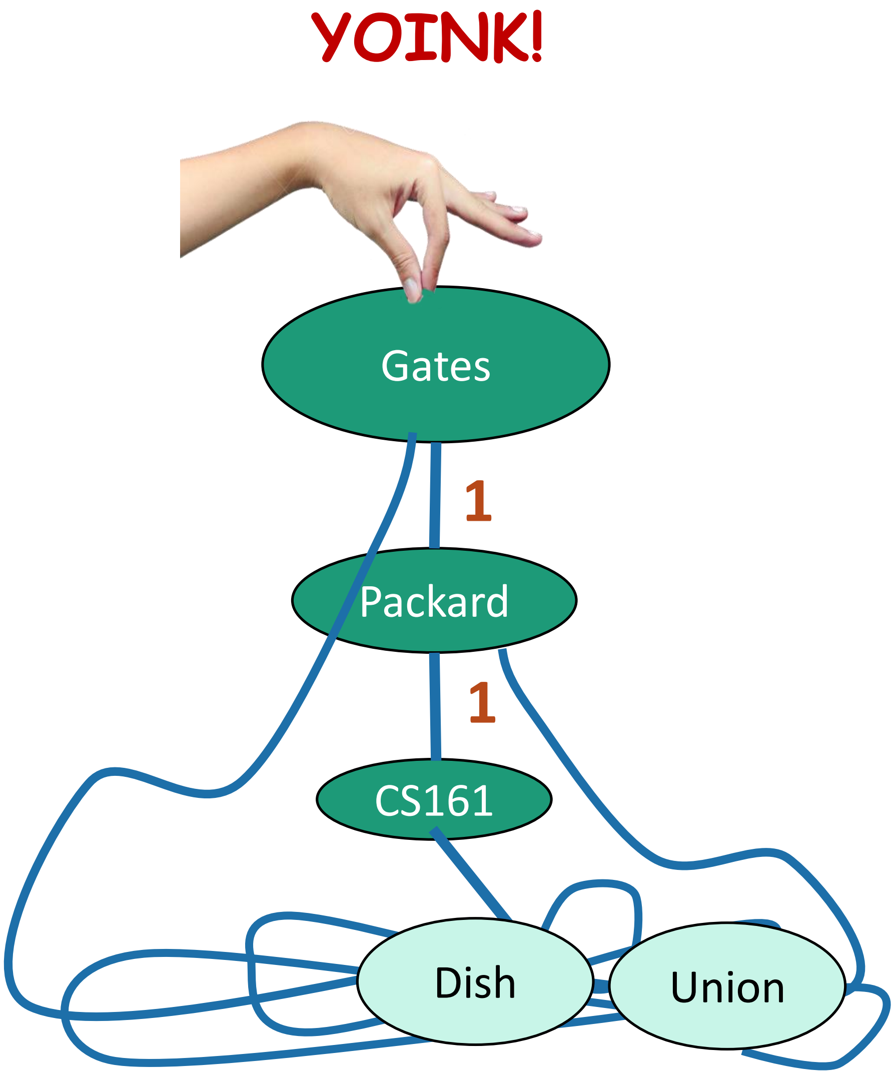

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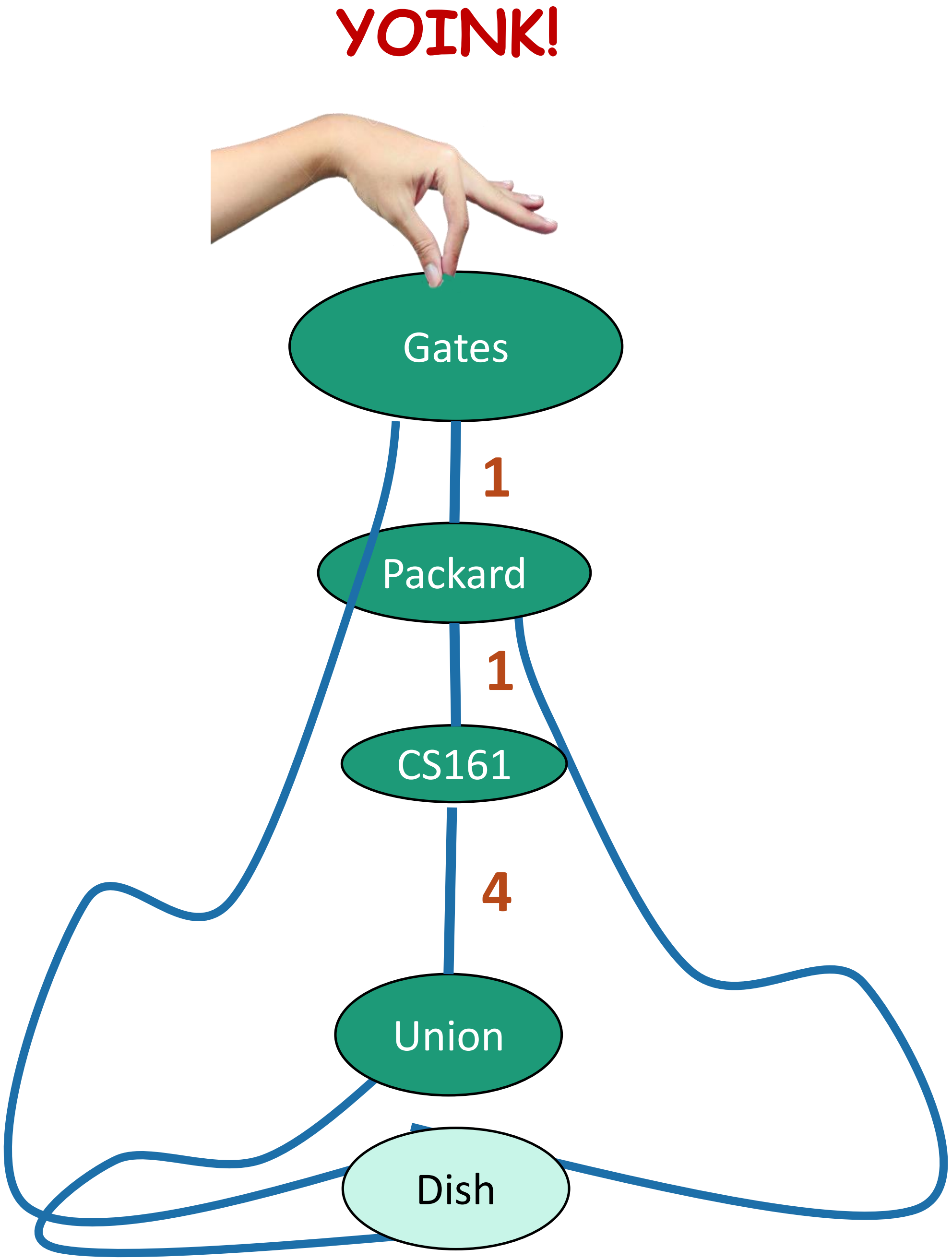

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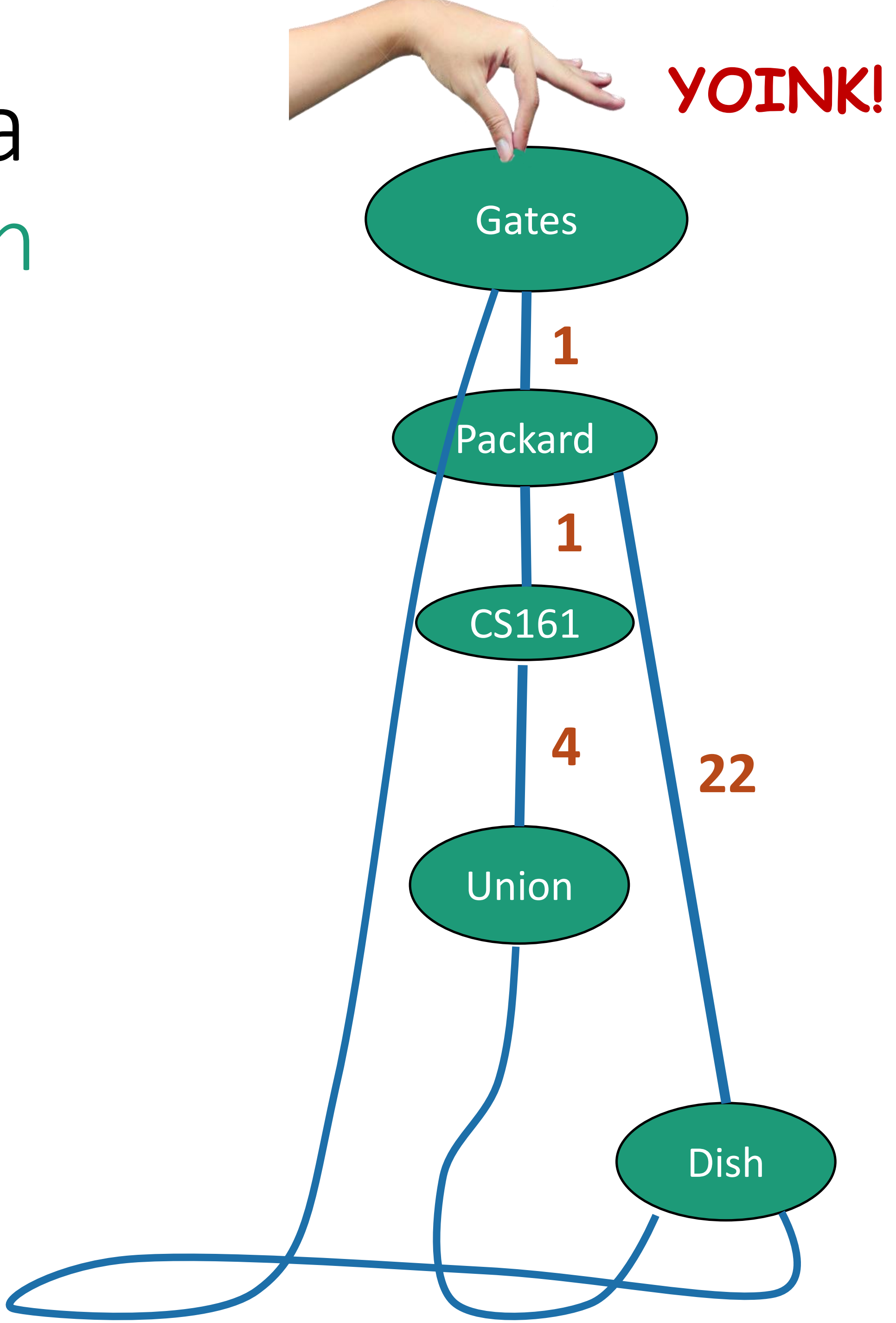

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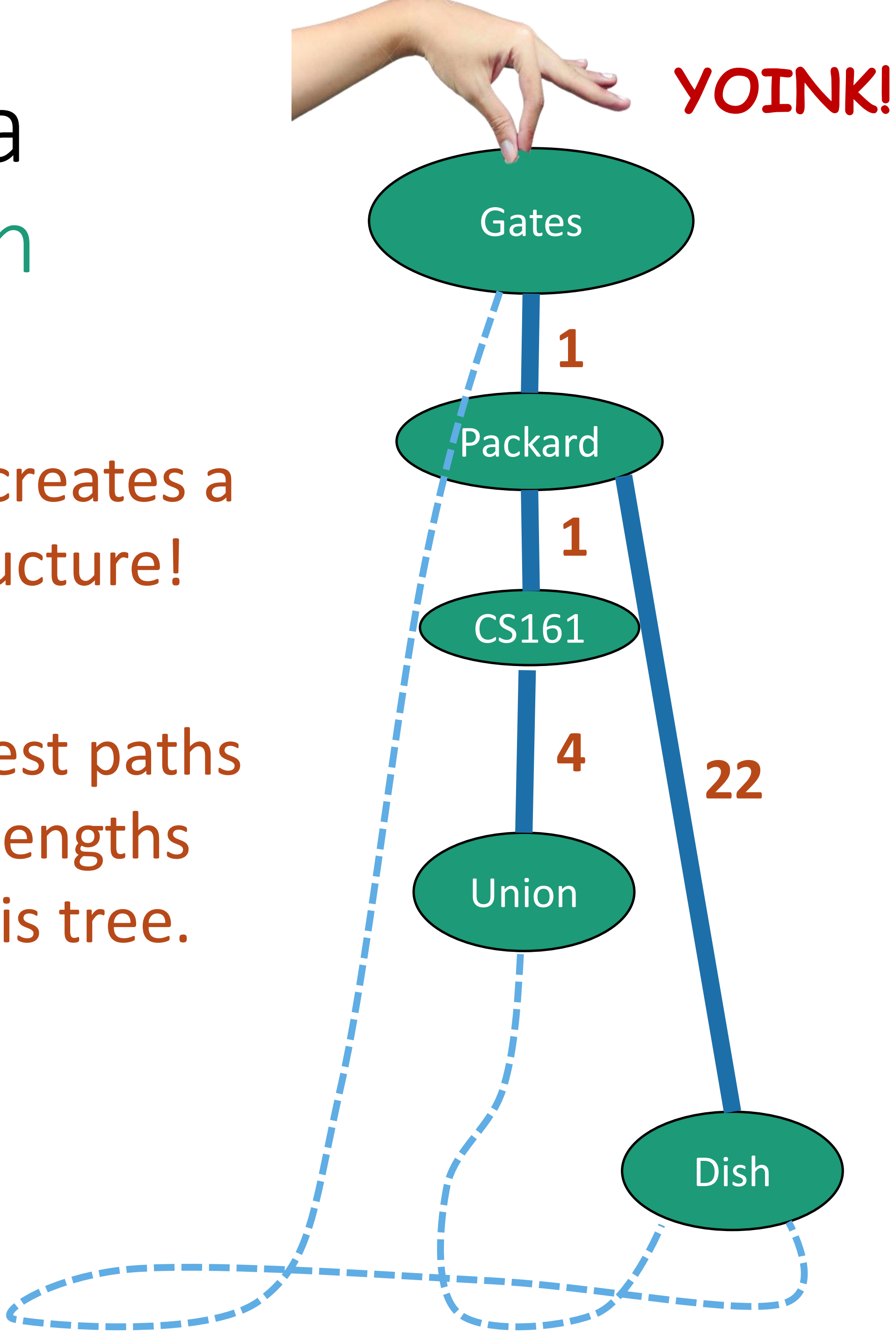
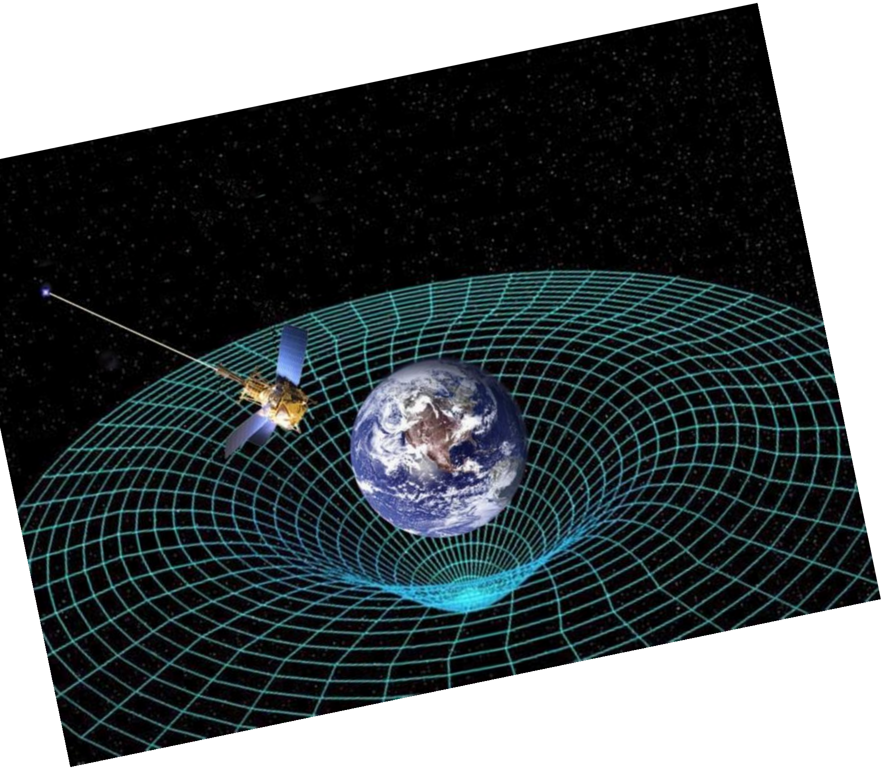


This also creates a tree structure!

The shortest paths are the lengths along this tree.

How do we actually implement this?

- **Without** string and gravity?



Dijkstra by example

How far is a node from Gates?

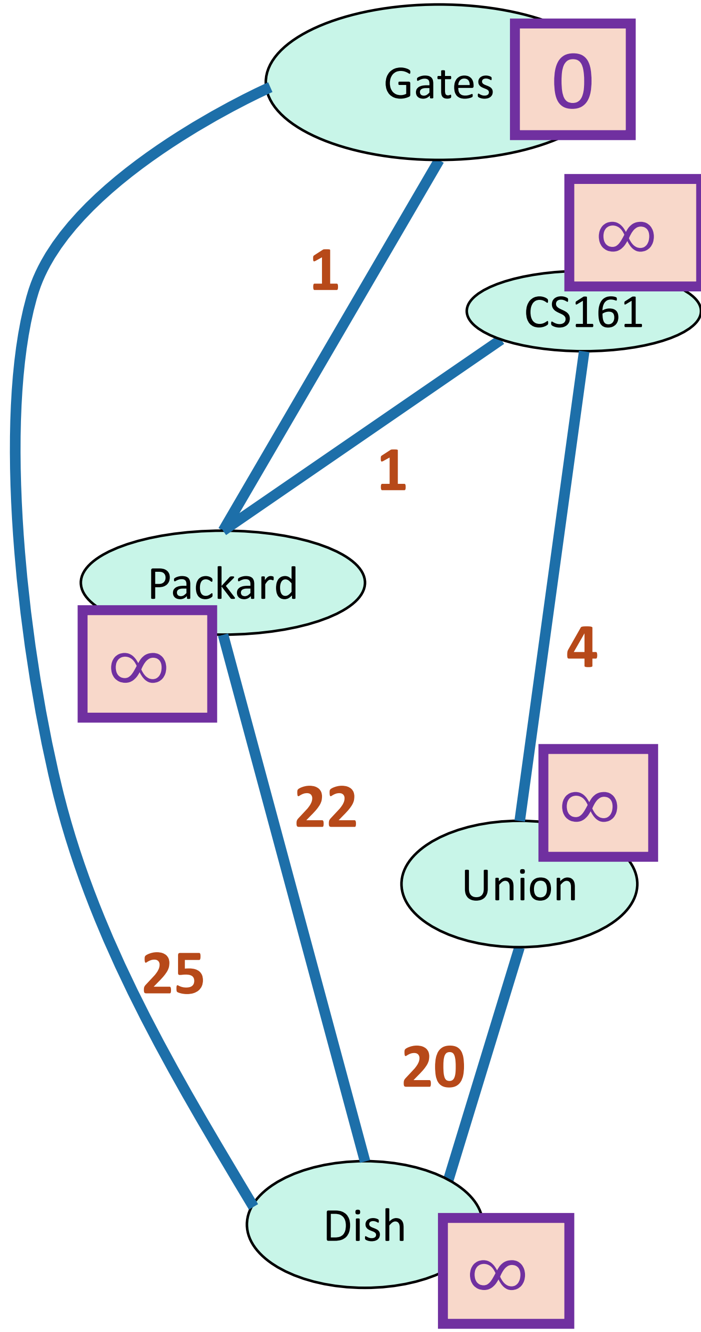
I’m not sure yet

I’m sure

x — x is my best over-estimate for a vertex v.
We’ll say d[v] = x

That is, an estimate of d(v,Gates).

Initialize d[v] = ∞
for all non-starting
vertices v, and v[Gates] = 0



- Pick the **not-sure** node u with the smallest estimate **d[u].**

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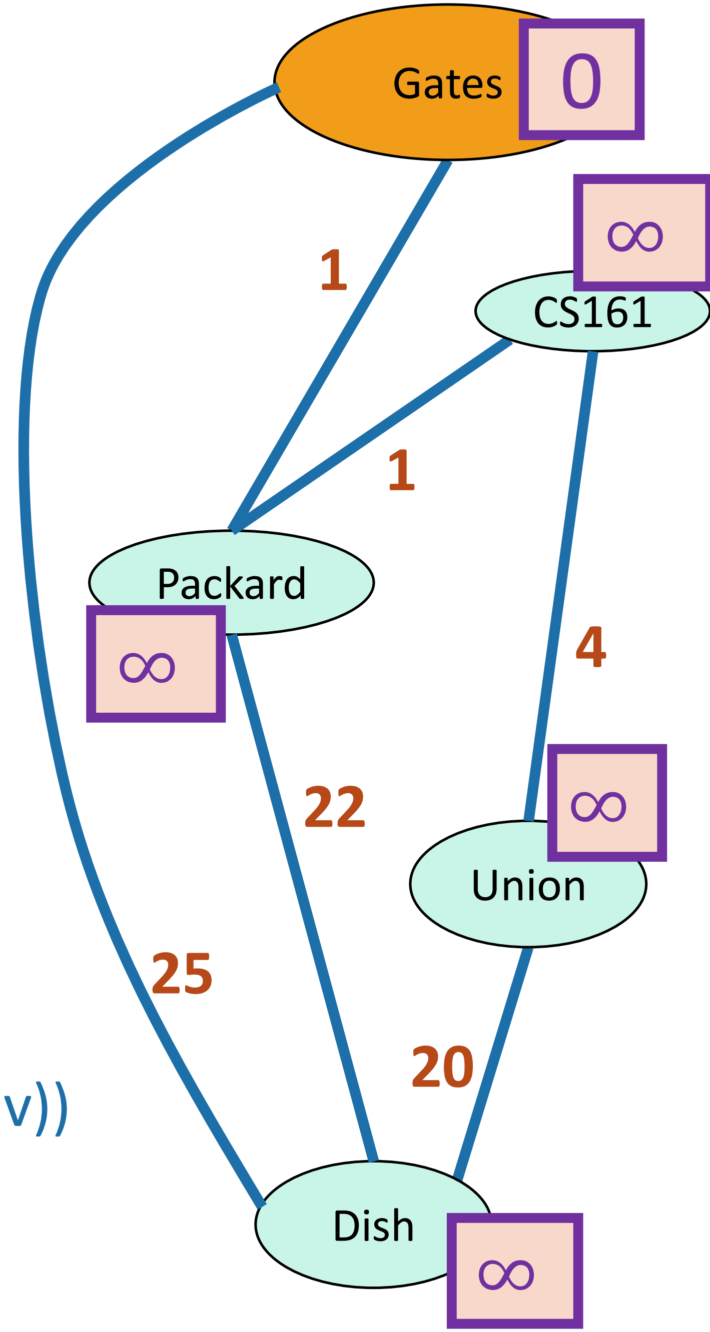

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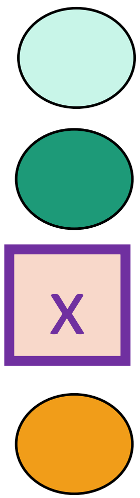

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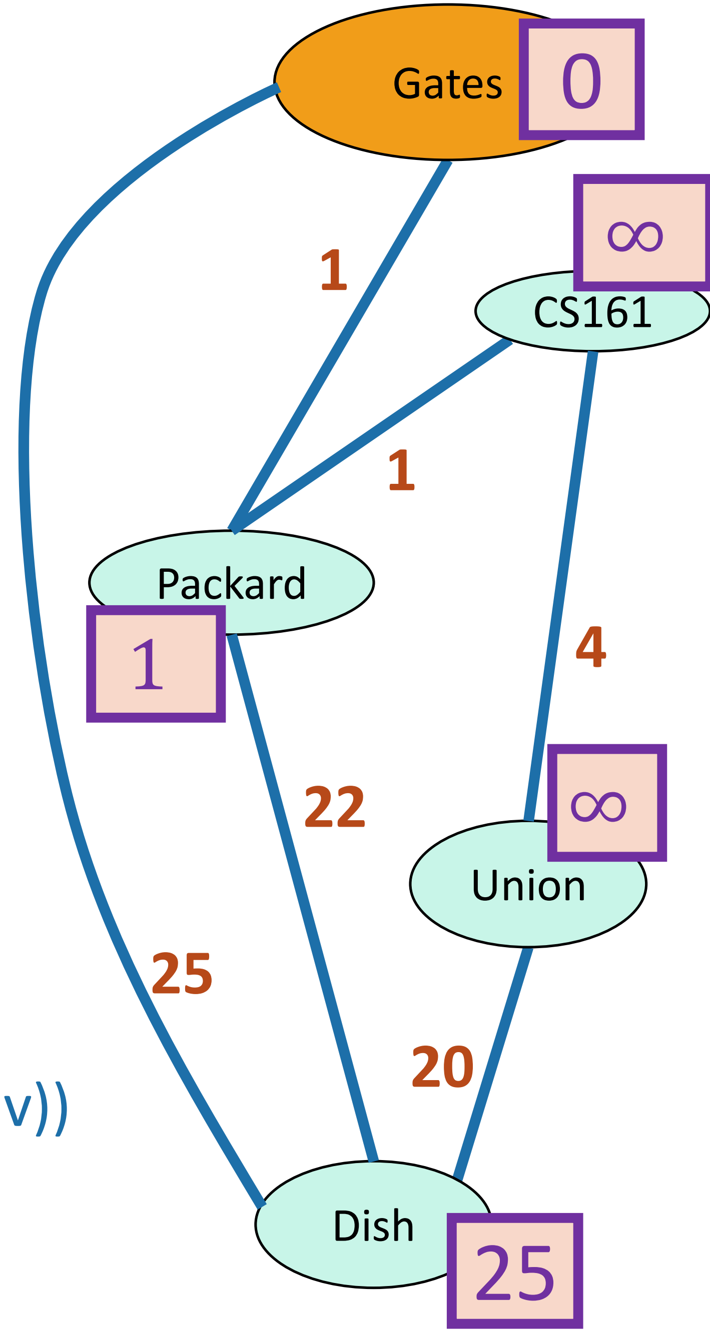

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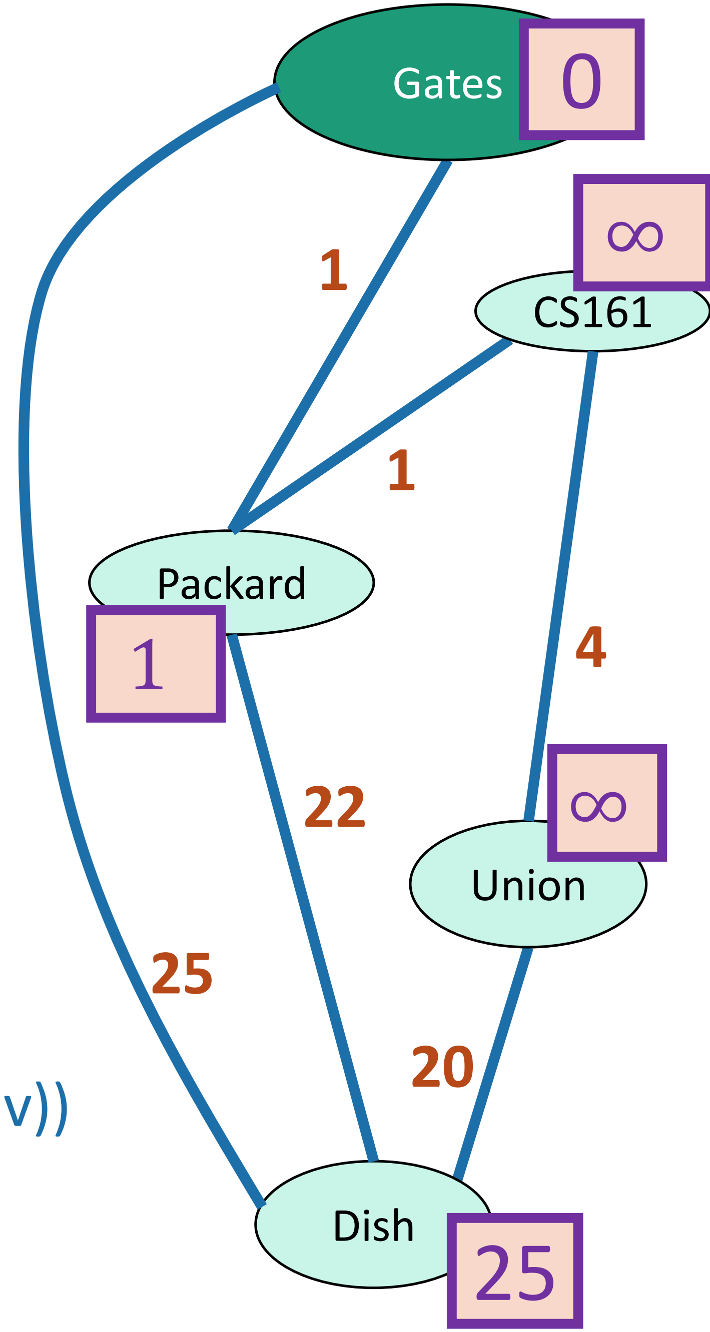

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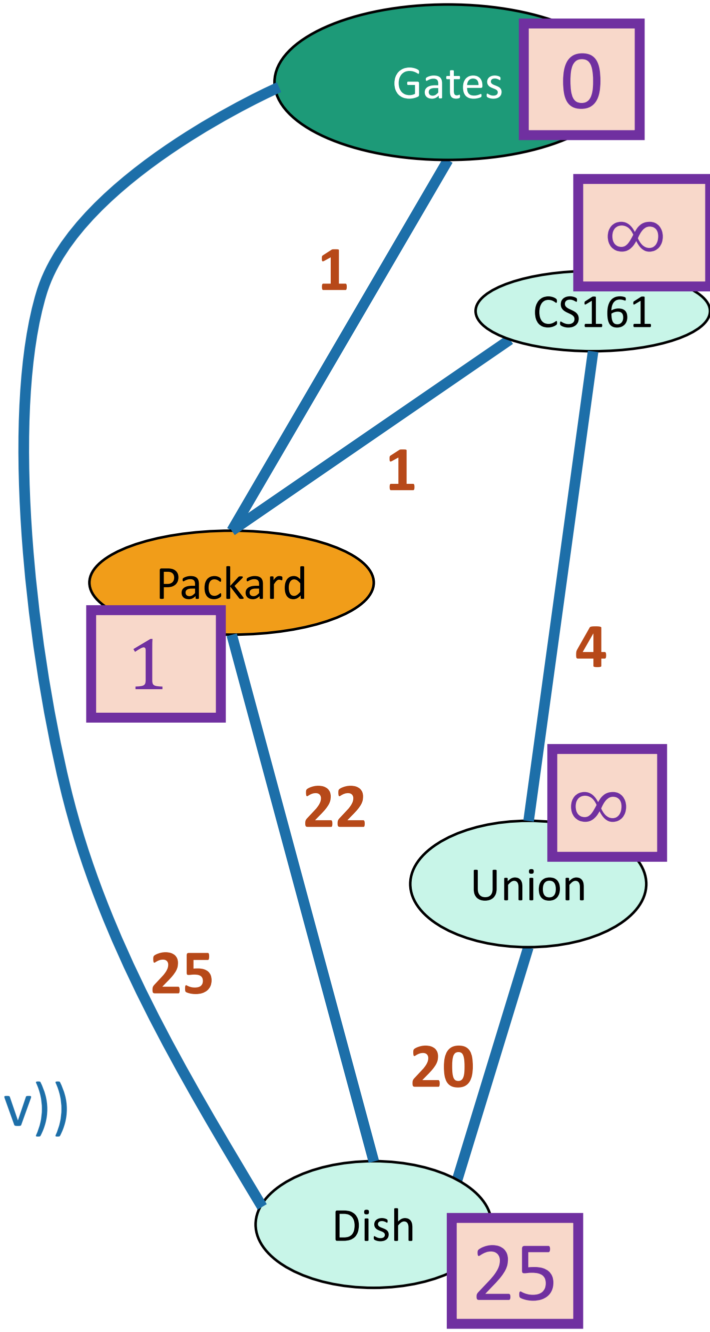

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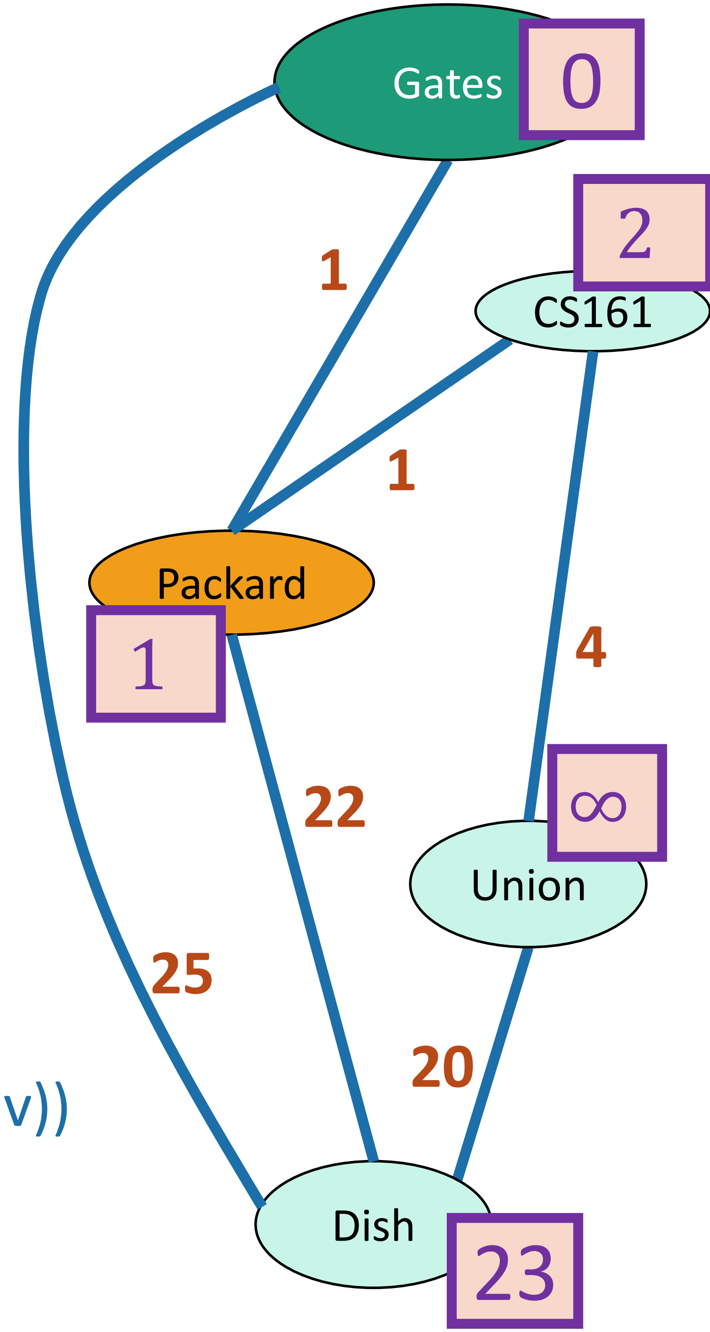

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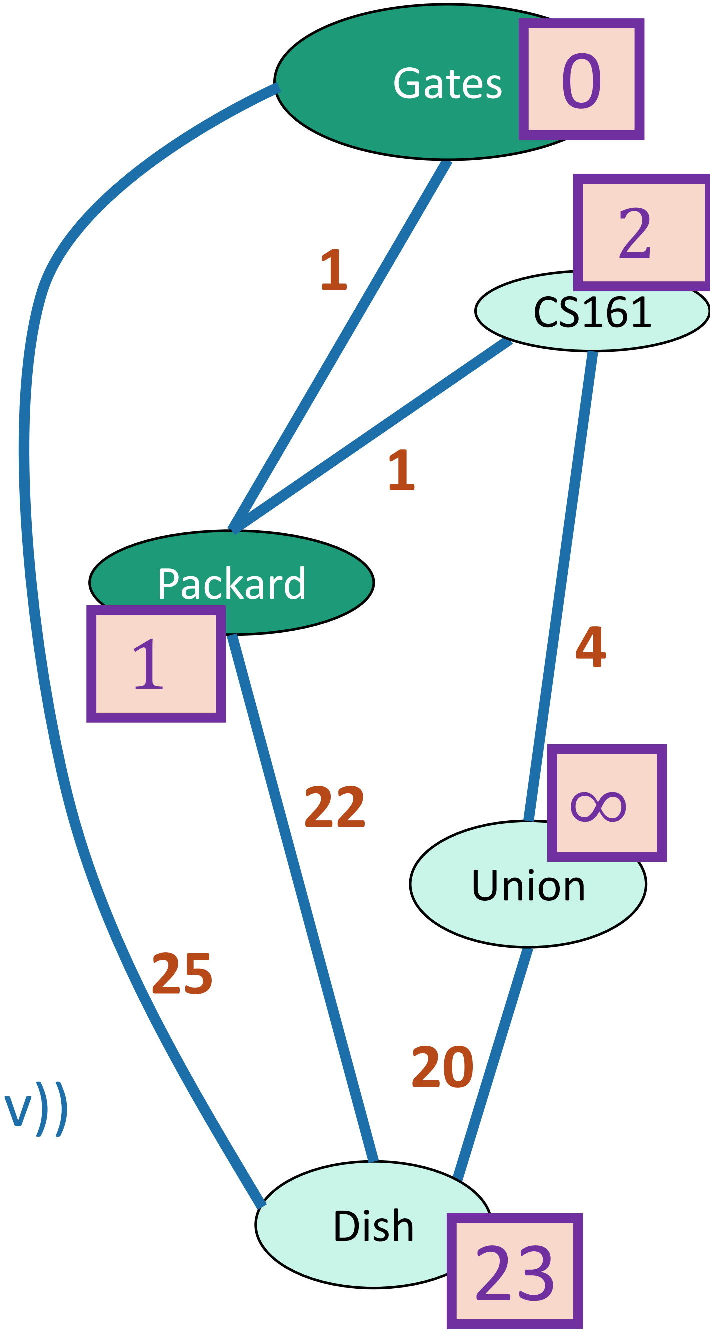

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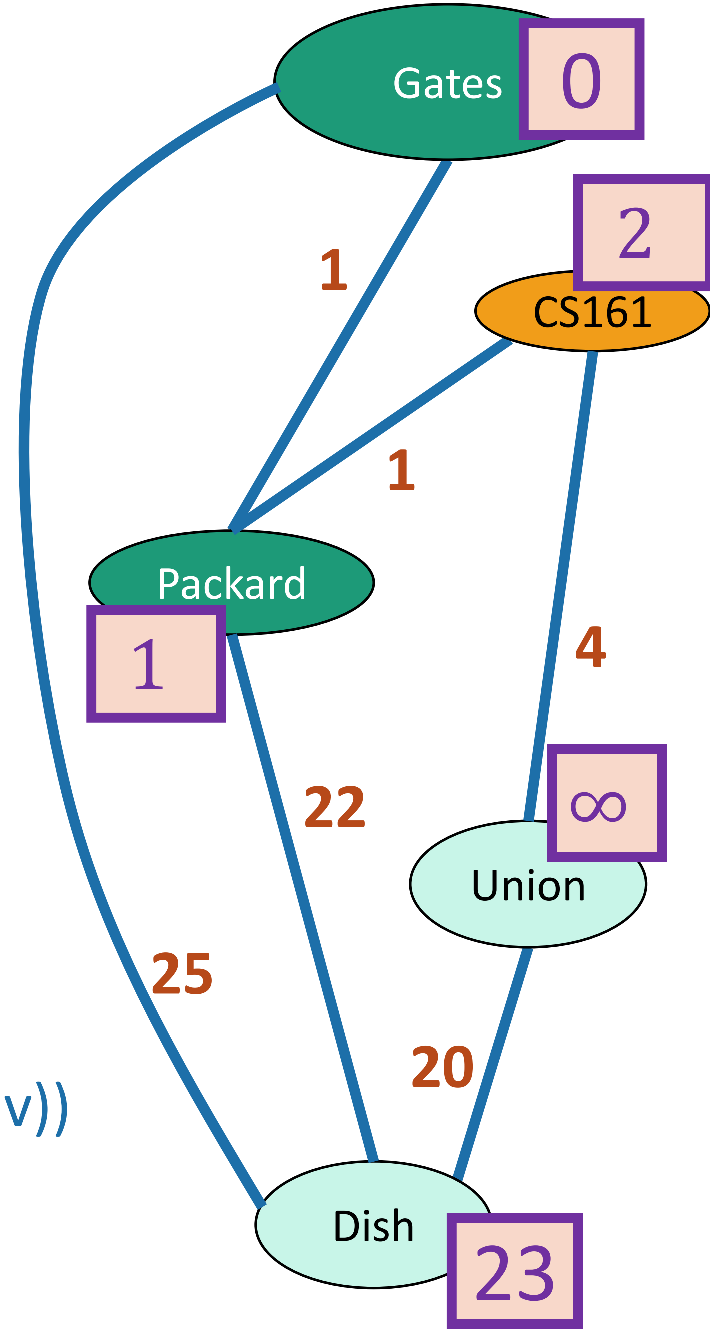

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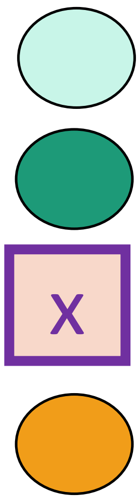

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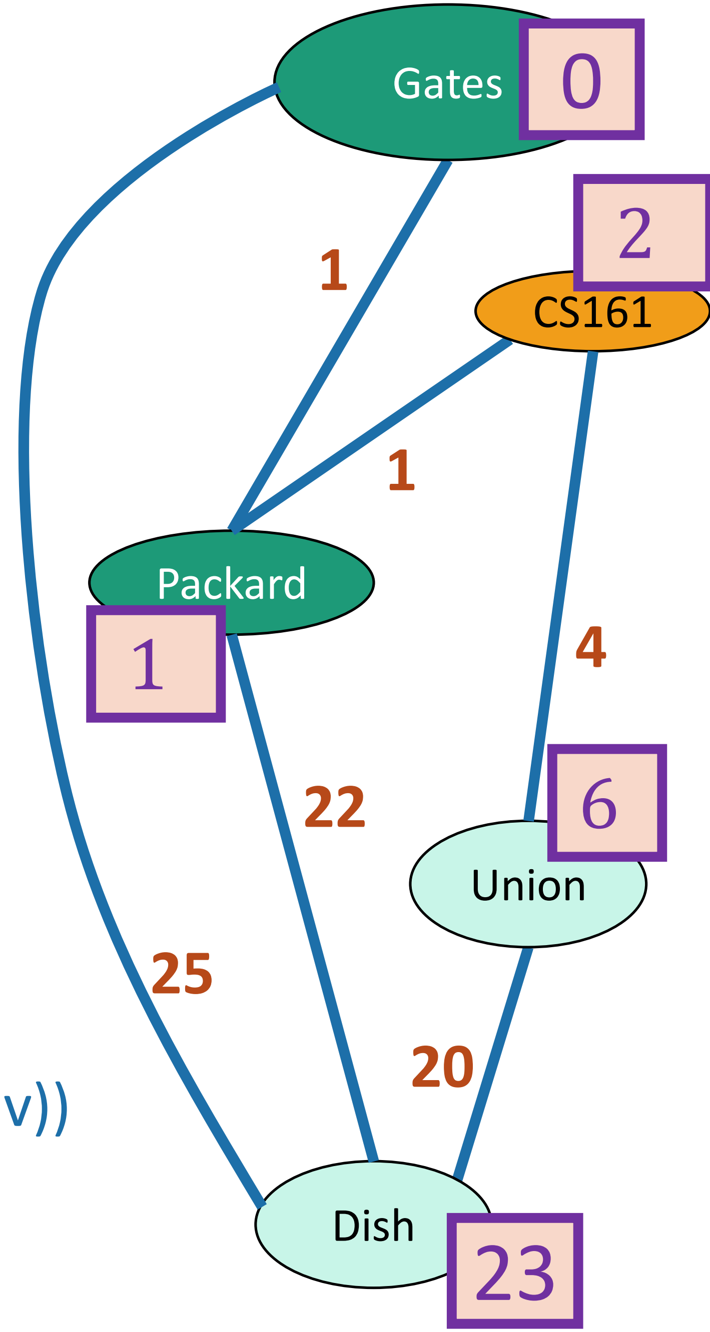

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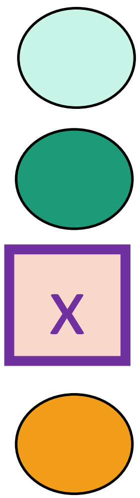

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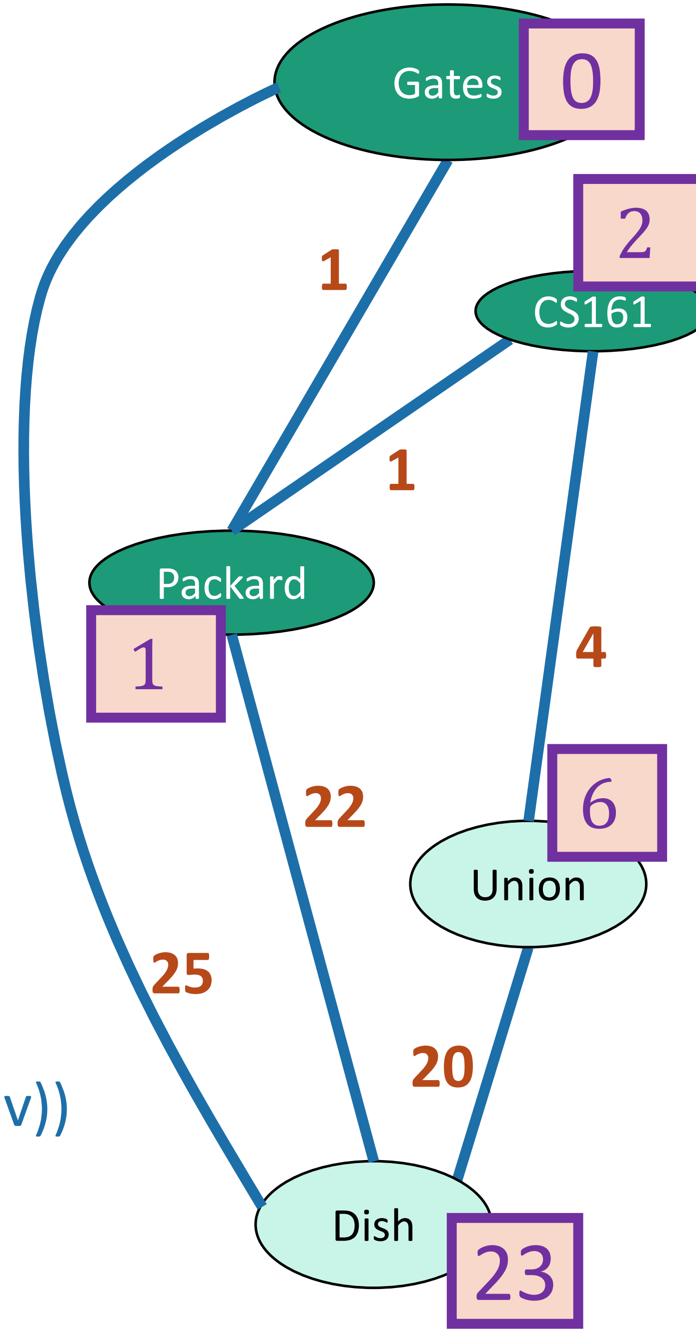

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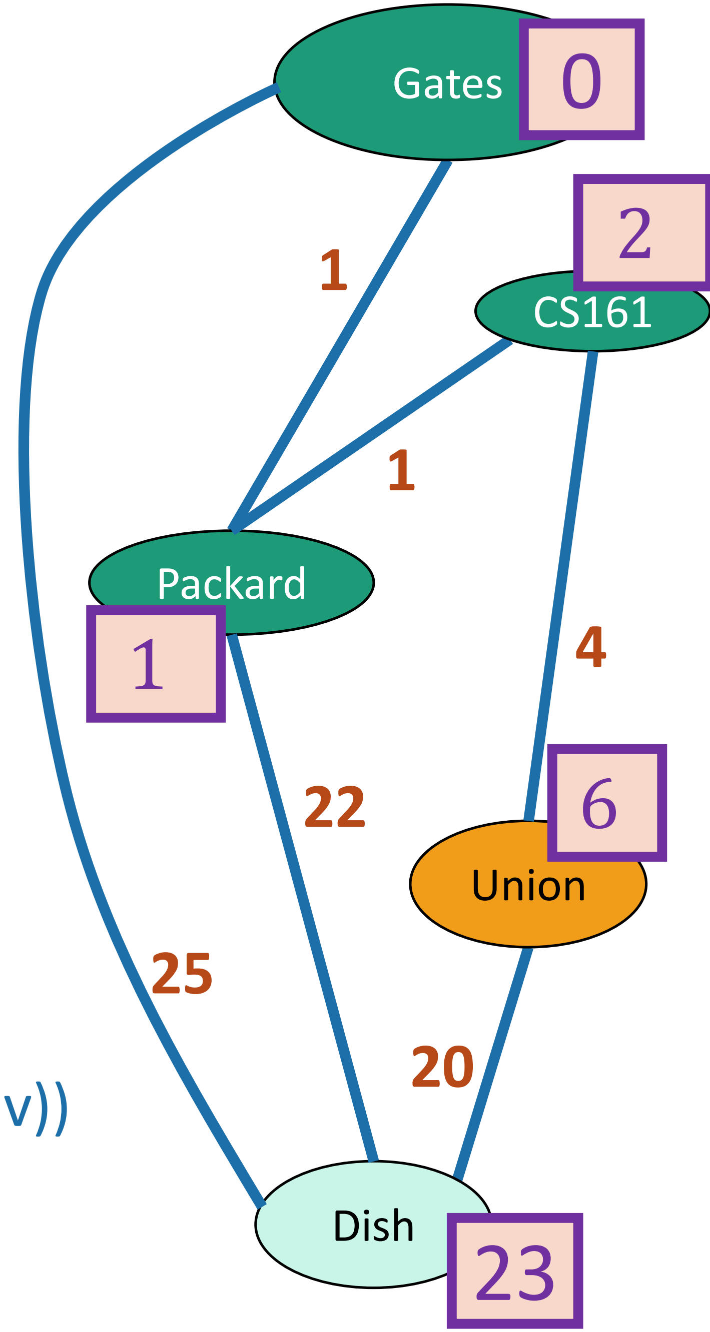

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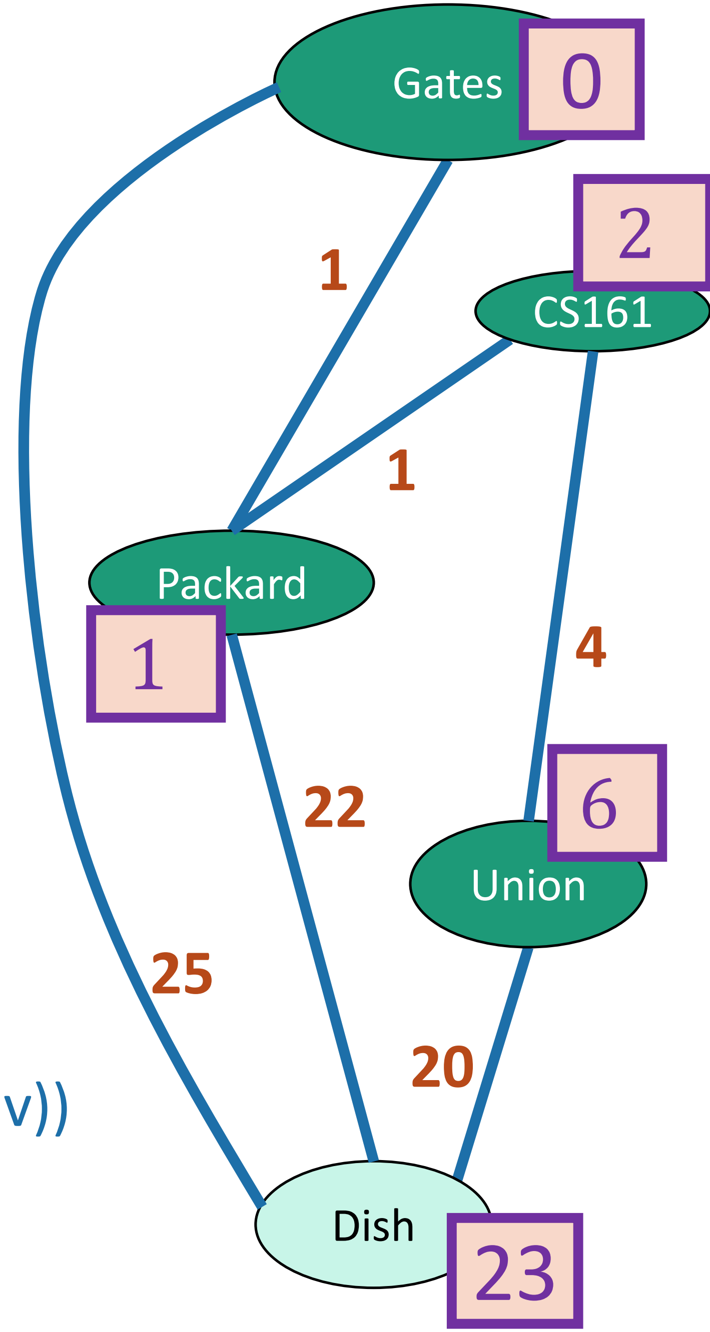

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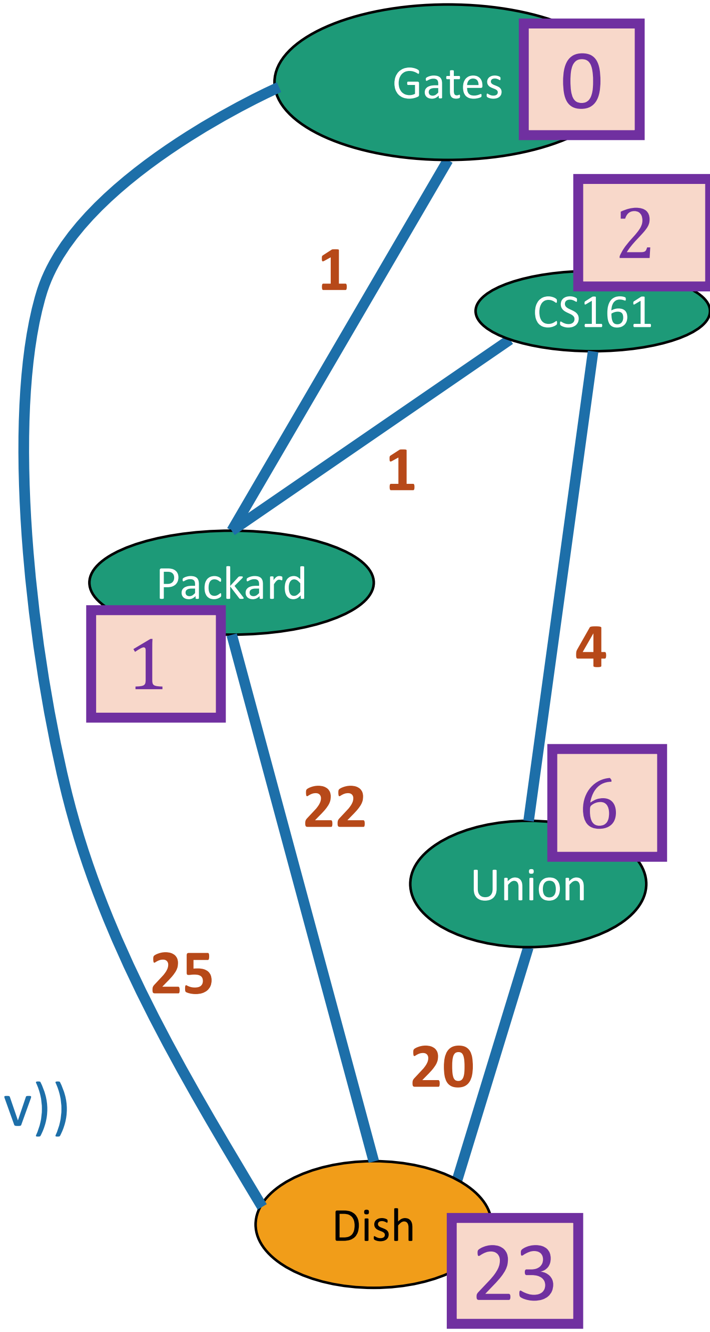

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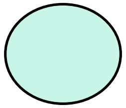

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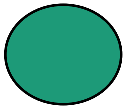
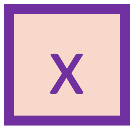


I'm sure



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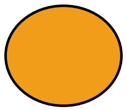

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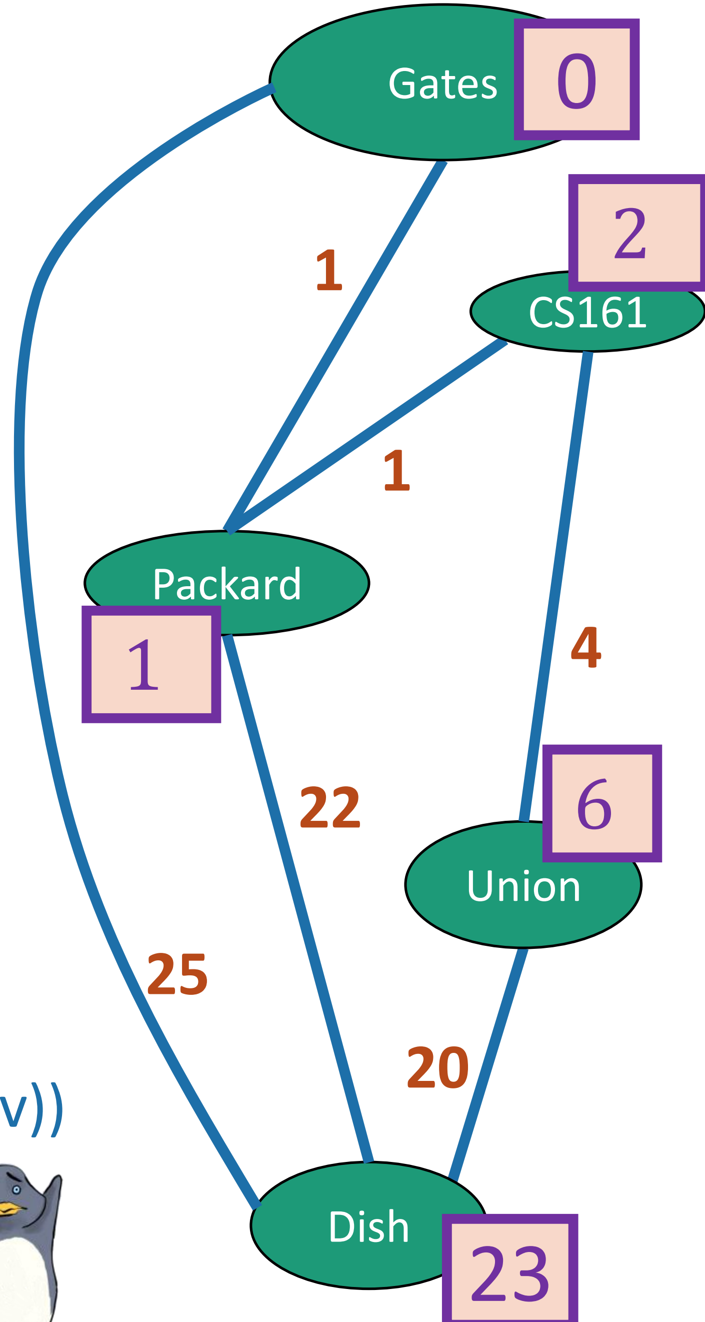
- Mark u as **sure**.
- Repeat



More formal pseudocode on board (or see CLRS)!

Why does this work?

- **Theorem**:
  - Run Dijkstra on G =(V,E).
  - At the end of the algorithm, the estimate d[v] is the actual distance d(**Gates**,v).

Let's rename "Gates" to "**s**", our starting vertex.

- Proof outline:
  - **Claim 1**: For all v, $d[v] \geq d(s,v)$.
  - **Claim 2**: When a vertex v is marked **sure**, $d[v] = d(s,v)$.

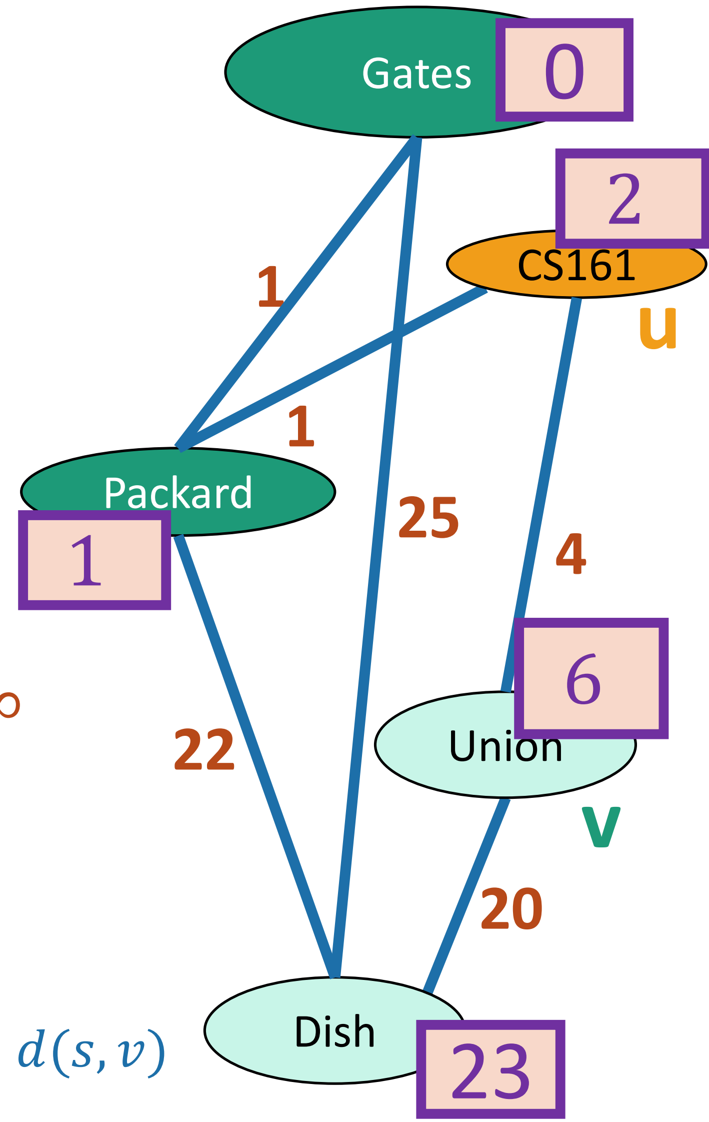
Next let's prove these!

- **Claims 1 and 2** imply the **theorem.**
  - d[v] never increases, so Claims 1 and 2 imply that **d[v] weakly decreases until d[v] = d(s,v), then never changes again.**
  - By the time we are **sure** about v, $d[v] = d(s,v)$. (Claim 1 again)
  - All vertices are eventually **sure**. (Stopping condition in algorithm)
  - So all vertices end up with $d[v] = d(s,v)$.

Claim 1

d[v] ≥ d(s,v) for all v.

- Inductive hypothesis.
  - After t iterations of Dijkstra, d[v] ≥ d(s,v) for all v.
- Base case:
  - At step 0, $\mathrm{d}(s,s) = 0$, and $d(s,v) \leq \infty$
- Inductive step: say hypothesis holds for t.
  - Then at step t+1:
    - We pick u; for each neighbor v:
    - d[v] ← min(d[v] , d[u] + w(u,v)) ≥ $d(s,v)$



(Details on board)

By induction, $d(s,v) \leq d[v]$

$\mathrm{d}(s,v) \leq d(s,u) + d(u,v)$
$\leq d[u] + w(u,v)$
using induction again for d[u]

So the inductive hypothesis holds for t+1, and Claim 1 follows.

Claim 2

When a vertex u is marked sure, d[u] = d(s,u)

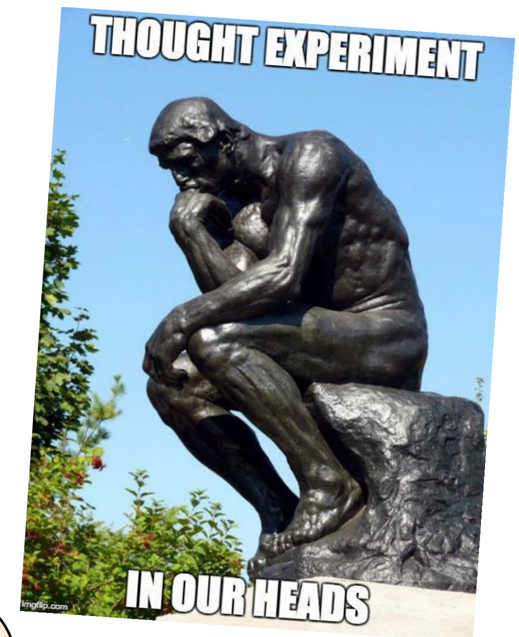
- To begin with:
  - The first vertex marked **sure** has d[s] = d(s,s) = 0.
- For t > 0:
  - Suppose that we are about to add u to the **sure** list.
  - That is, we picked u in the first line here:

- Pick the **not-sure** node u with the smallest estimate **d[u].**
- Update all u's neighbors v:
  - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**.
- Repeat

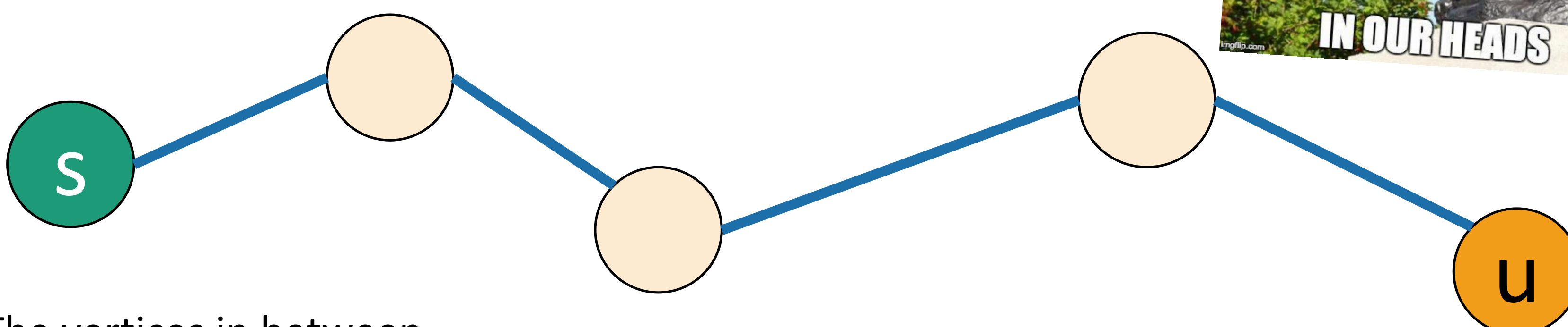
Temporary definition:
v is "good" means that d[v] = d(s,v)

Claim 2

- Want to show that u is good.



Consider a **true** shortest path from s to u:

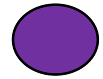
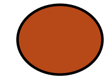


The vertices in between are beige because they may or may not be **sure.**

True shortest path.

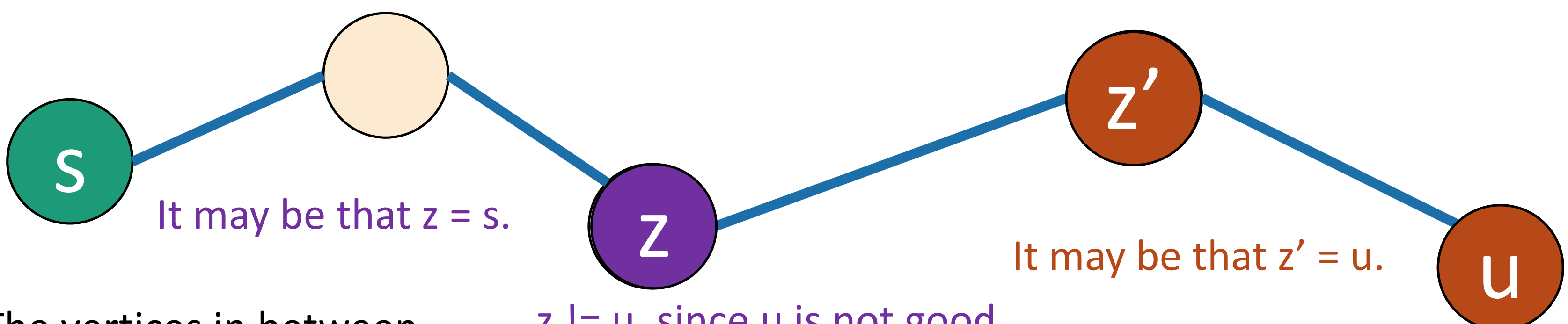
Claim 2

Temporary definition:
v is "good" means that d[v] = d(s,v)

 means good  means not good

"by way of contradiction"

- Want to show that u is good. BWOC, suppose it's not.
- Say z is the last good vertex before u.
- z' is the vertex after z.



The vertices in between are beige because they may or may not be **sure.**

True shortest path.

Claim 2

Temporary definition:
v is “good” means that d[v] = d(s,v)

means good   means not good

- Want to show that u is good. BWOC, suppose it’s not.

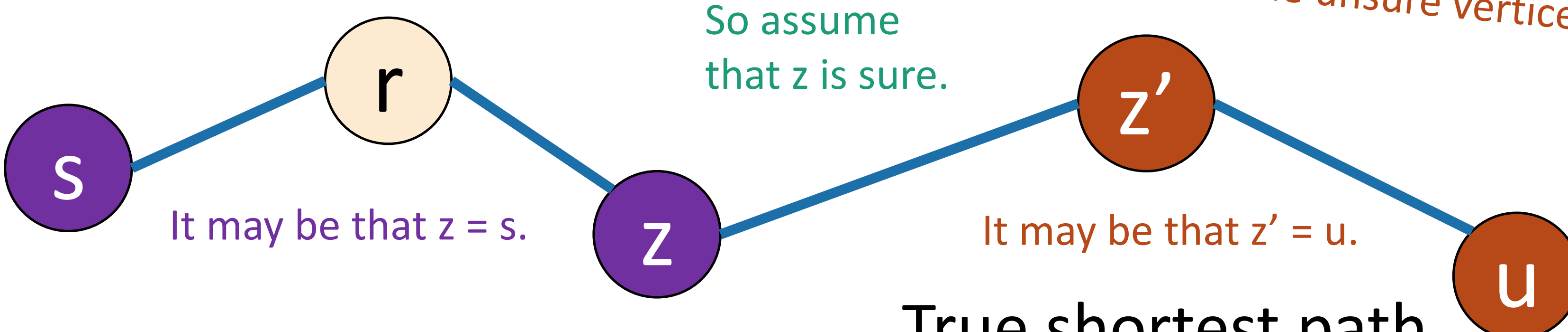
$$d[z] = d(s,z) \le d(s,u) \le d[u]$$

z is good   This is the shortest path from s to u.   Claim 1

- If $d[z] = d[u]$, then u is good.
- If $d[z] < d[u]$, then z is **sure.**

We chose u so that d[u] was smallest of the unsure vertices.

So assume that z is sure.



True shortest path.

Claim 2

Temporary definition:
v is "good" means that d[v] = d(s,v)

means good

means not good

- Want to show that u is good. BWOC, suppose it's not.
- If z is **sure** then we've already updated z':
  - $d[z'] \leftarrow \min\{\, d[z'], d[z] + w(z,z')\}$, so

$$\boldsymbol{d[z'] \leq d[z] + w(z,z') = d(s,z') \leq d[z']}$$

def of update

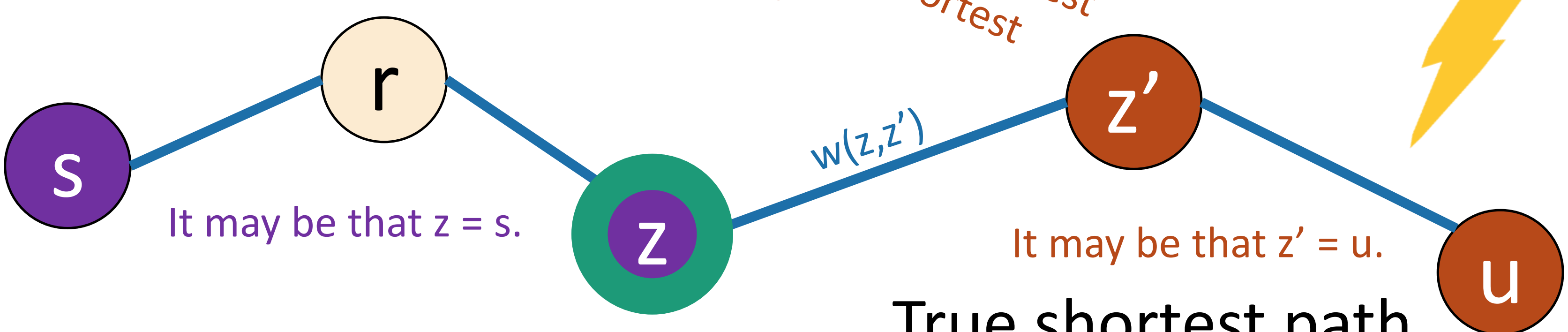
sub-paths of shortest paths are shortest paths

Claim 1

So everything is equal!

And z' is good.

CONTRADICTION!!



True shortest path.

Claim 2

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means good   means not good

- Want to show that u is good. BWOC, suppose it's not.

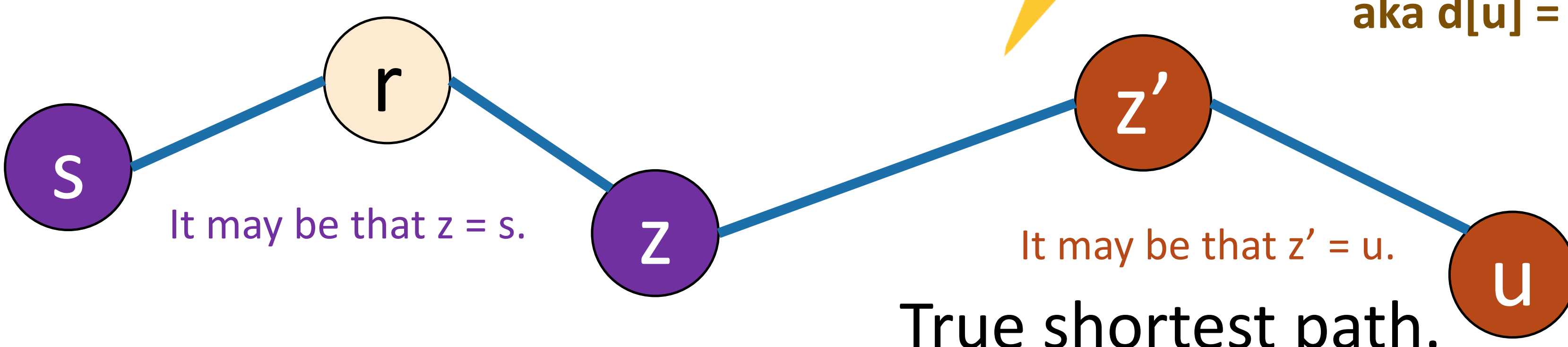
$$d[z] = d(s,z) \le d(s,u) \le d[u]$$

Def. of z   This is the shortest path from s to x   Claim 1

- If $d[z] = d[u]$, then u is good.
- If $d[z] < d[u]$, then z is **sure**.

So u is good!

aka d[u] = d(s,v)



Claim 2

Back to this slide

When a vertex is marked sure, d[u] = d(s,u)

- To begin with:
  - The first vertex marked **sure** has d[s] = d(s,s) = 0.
- For t > 0:
  - Suppose that we are about to add u to the **sure** list.
  - That is, we picked u in the first line here:

Then u is good!

aka d[u] = d(s,u)

- Pick the **not-sure** node u with the smallest estimate **d[u].**
- Update all u's neighbors v:
  - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**.
- Repeat

Why does this work?

Now back to this slide

- **Theorem**: At the end of the algorithm, the estimate d[v] is the actual distance d(**s**,v).

- Proof outline:
  - **Claim 1**: For all v, d[v] $\geq$ d(s,v).
  - **Claim 2**: When a vertex is marked **sure**, d[v] = d(s,v).

- **Claims 1 and 2** imply the **theorem.**
  - We will never mess up d[v] after v is marked **sure**, because d[v] is a decreasing over-estimate.

Why does this work?

Now back to this slide

- **Theorem**:
  - Run Dijkstra on G =(V,E).
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    the estimate d[v] is the actual distance d(**s**,v).

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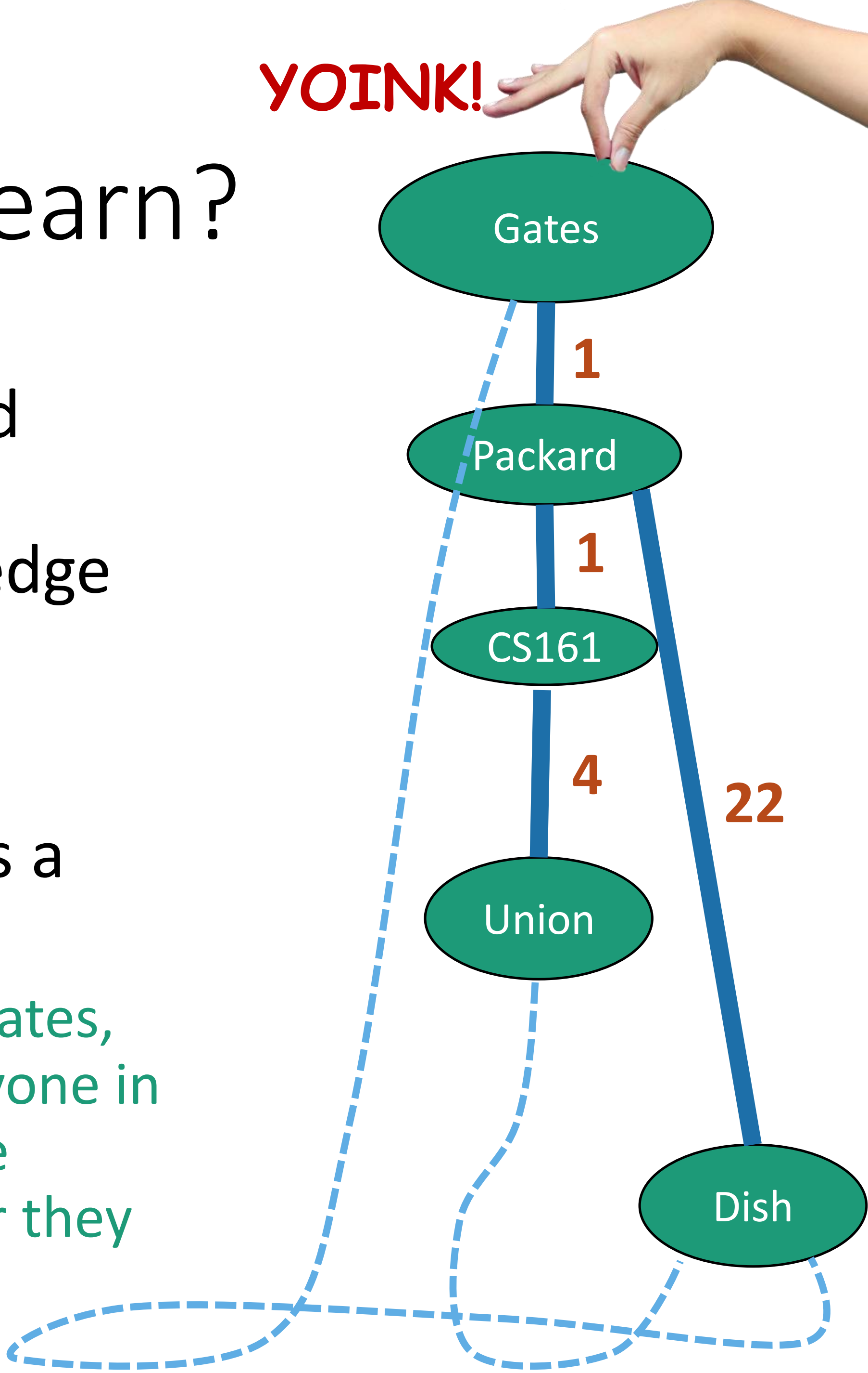
  - All vertices are eventually **sure**. (Stopping condition in algorithm)
  - So all vertices end up with d[v] = d(s,v).

What did we just learn?

- Dijkstra's algorithm can find shortest paths in weighted graphs with non-negative edge weights.
- Along the way, it constructs a nice tree.
  - We could post this tree in Gates, and it would be easy for anyone in Gates to figure out what the shortest path is to wherever they want to go.



Running time?

This is not very precise pseudocode (eg, initialization step is missing)…but it's good enough for this reasoning.

- Pick the **not-sure** node u with the smallest estimate **d[u].**
- Update all u's neighbors v:
  - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**.
- Repeat

This will run for n iterations, since there's one iteration per vertex.

How long does an iteration take?

Depends on how we implement it…

We need a data structure that:

- Stores unsure vertices v
- Keeps track of d[v]
- Can find v with minimum d[v]
  - `findMin()`
- Can remove that v
  - `removeMin(v)`
- Can update the d[v]
  - `updateKey(v,d)`

- Pick the **not-sure** node u with the smallest estimate **d[u].**
- Update all u's neighbors v:
  - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**.
- Repeat

Total running time is big-oh of:

$$\sum_{u \in V} \left(T(findMin) + \left(\sum_{v \in u.neighbors} T(updateKey) \right) + T(removeMin) \right)$$

$$n(\, T(\texttt{findMin}) + T(\texttt{removeMin})\,) + m\, T(\texttt{updateKey})$$

O(n(T(findMin) + T(removeMin)) + m T(updateKey))

If we use an array

- T(findMin) = O(n)
- T(removeMin) = O(n)
- T(updateKey) = O(1)

- Running time of Dijkstra

  =O(n(T(findMin) + T(removeMin)) + m T(updateKey))

  =O(n^2) + O(m)

  =O(n^2)

$O(n(T(\texttt{findMin}) + T(\texttt{removeMin})) + m\,T(\texttt{updateKey}))$

If we use a red-black tree

- $T(\text{findMin}) = O(\log(n))$
- $T(\text{removeMin}) = O(\log(n))$
- $T(\text{updateKey}) = O(\log(n))$

- Running time of Dijkstra

$$= O(n(T(\texttt{findMin}) + T(\texttt{removeMin})) + m\,T(\texttt{updateKey}))$$
$$= O(n\log(n)) + O(m\log(n))$$
$$= O((n + m)\log(n))$$

Better than an array if the graph is sparse!

aka m is much smaller than n^2

$O(n(T(\texttt{findMin}) + T(\texttt{removeMin})) + m\ T(\texttt{updateKey}))$

Is a hash table a good idea here?

- **Not really**:
  - `Search`(v) is fast (in expectation)
  - But `findMin`() will still take time O(n) without more structure.

O(n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`))

Can also use a Fibonacci Heap

- This can do all operations in amortized time* O(1).
- Except `deleteMin` which takes amortized time* O(log(n)).
- See CLRS for more!

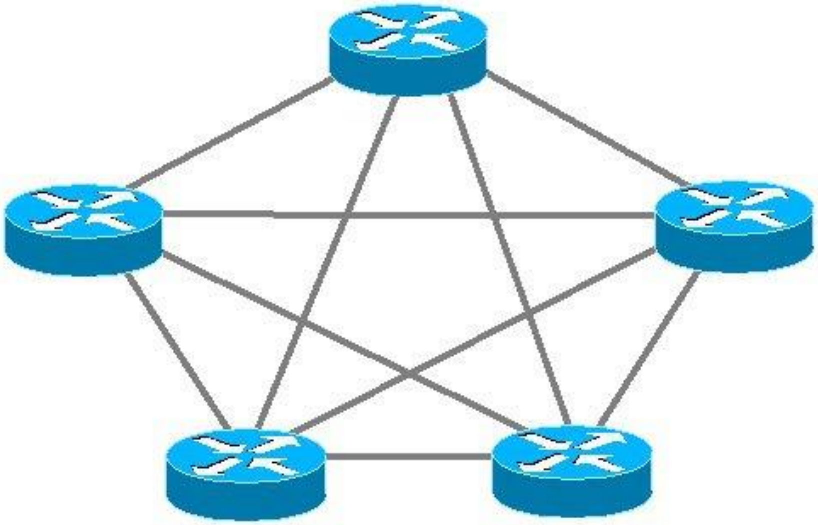
- This gives (amortized) runtime O(m + nlog(n)) for Dijkstra's algorithm.

*Any sequence of d `deleteMin` calls takes time at most O(d log(n)). But some of the d may take longer and some may take less time.

Dijkstra is used in practice

- O(nlog(n) + m) is really fast!
- eg, OSPF (Open Shortest Path First), a routing protocol for IP networks, uses Dijkstra.

But there are some things it's not so good at.



Dijkstra Drawbacks

- Needs non-negative edge weights.
- If the weights change, we need to re-run the whole thing.
  - in OSPF, a vertex broadcasts any changes to the network, and then every vertex re-runs Dijkstra's algorithm from scratch.

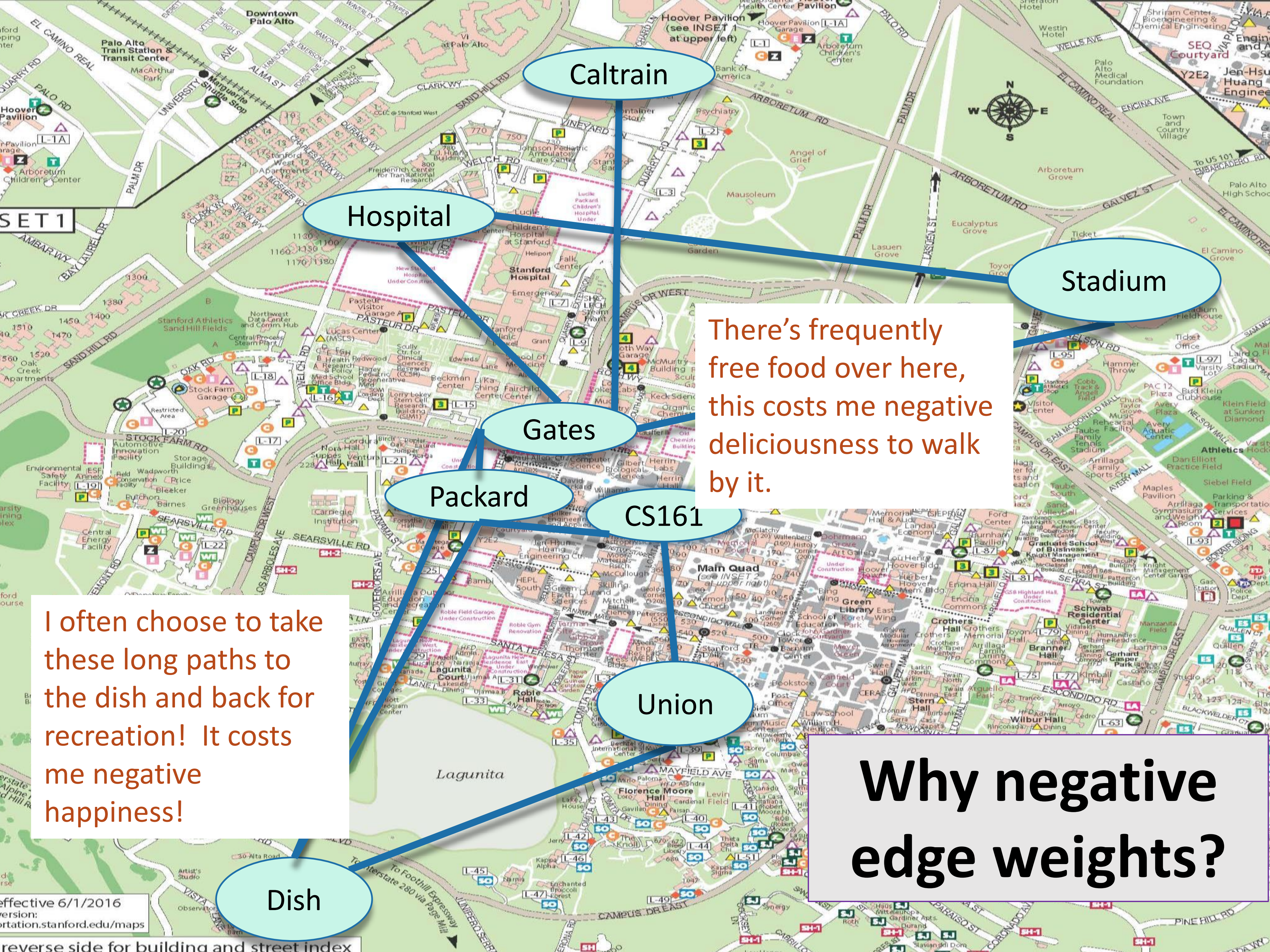
WE STOPPED HERE IN LECTURE

- Bonus slides follow, but material on the Bellman-Ford algorithm is also in slides for lecture 12.

- The slides below are different than those in lecture 12 (in order to maintain internal consistency within lectures), so they might be interesting for a different perspective.

Bellman-Ford algorithm

- Slower than Dijkstra's algorithm
- Can handle negative edge weights.
- Allows for some flexibility if the weights change.
  - We'll see what this means later



There's frequently free food over here, this costs me negative deliciousness to walk by it.

I often choose to take these long paths to the dish and back for recreation! It costs me negative happiness!

Why negative edge weights?

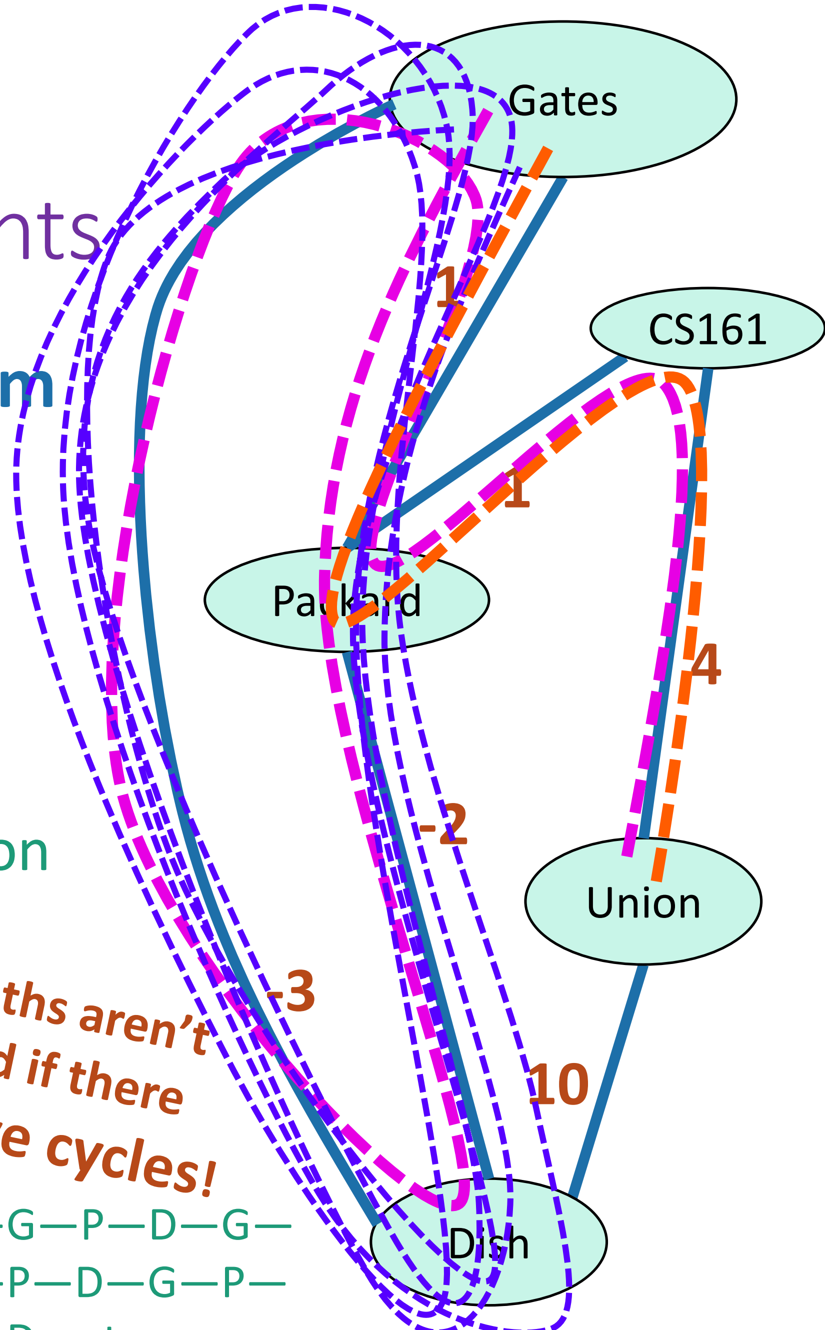
Problem

with negative edge weights

- **What is the shortest path from Gates to the Union?**
- Should still be

  Gates—Packard—CS161—Union
- But what about
  - G—P—D—G—P—CS161—Union
- That costs
  - 1-2-3+1+1+4 = 2.
- And why not

  G—P—D—G—P—D—G—P—D—G—P—D—G—P—D—G—P—D—G—P—D—G—P—D—G—P—D—G—P—D—G—P—D—G—P—D—G—P—D—G—P—D—etc….



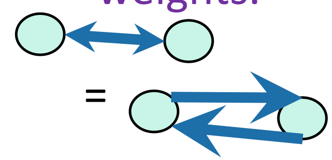
Let's put that aside for a moment

Onwards!

To the Bellman-Ford algorithm!

Bellman-Ford

Start with the same graph, no negative weights.



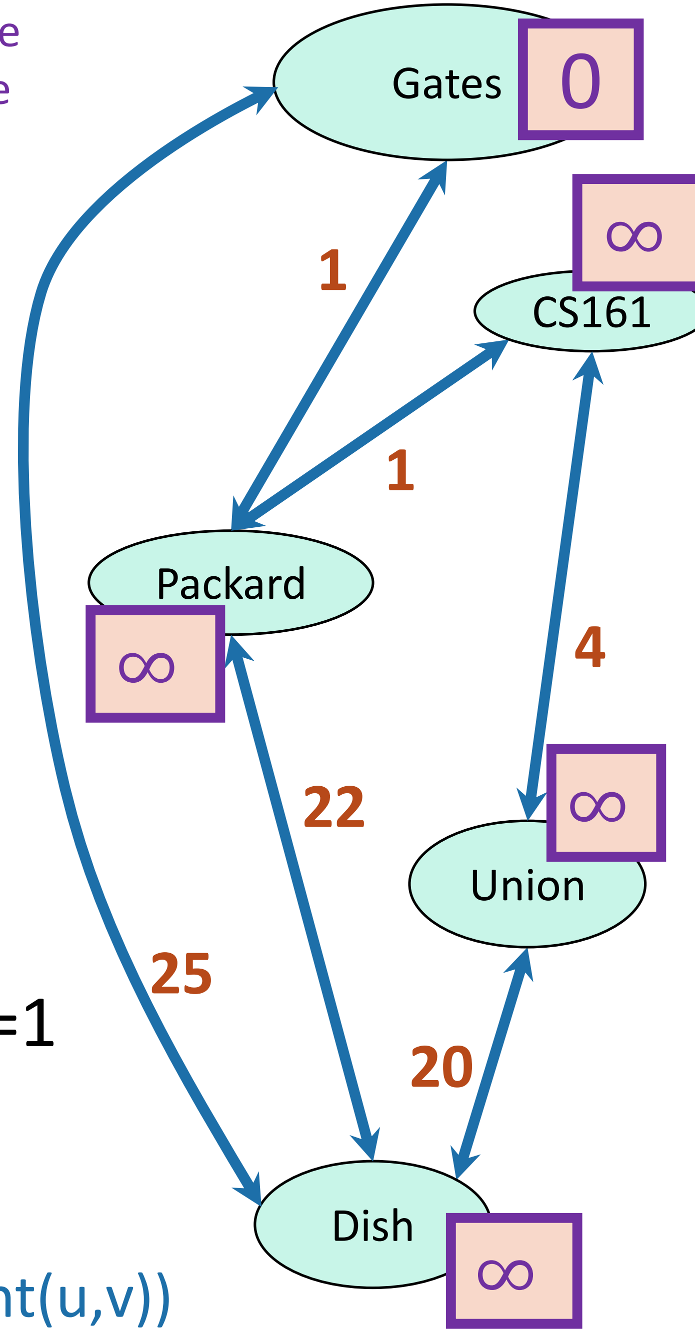
How far is a node from Gates?

Current edge

$\boxed{x}$ x is my best over-estimate for a vertex v.
We'll say d[v] = x

- For v in V:
  - d[v] = ∞
- d[s] = 0
- **For** i = 1,..,n-1:
  - **For** each edge e = (u,v) in E:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))

i=1



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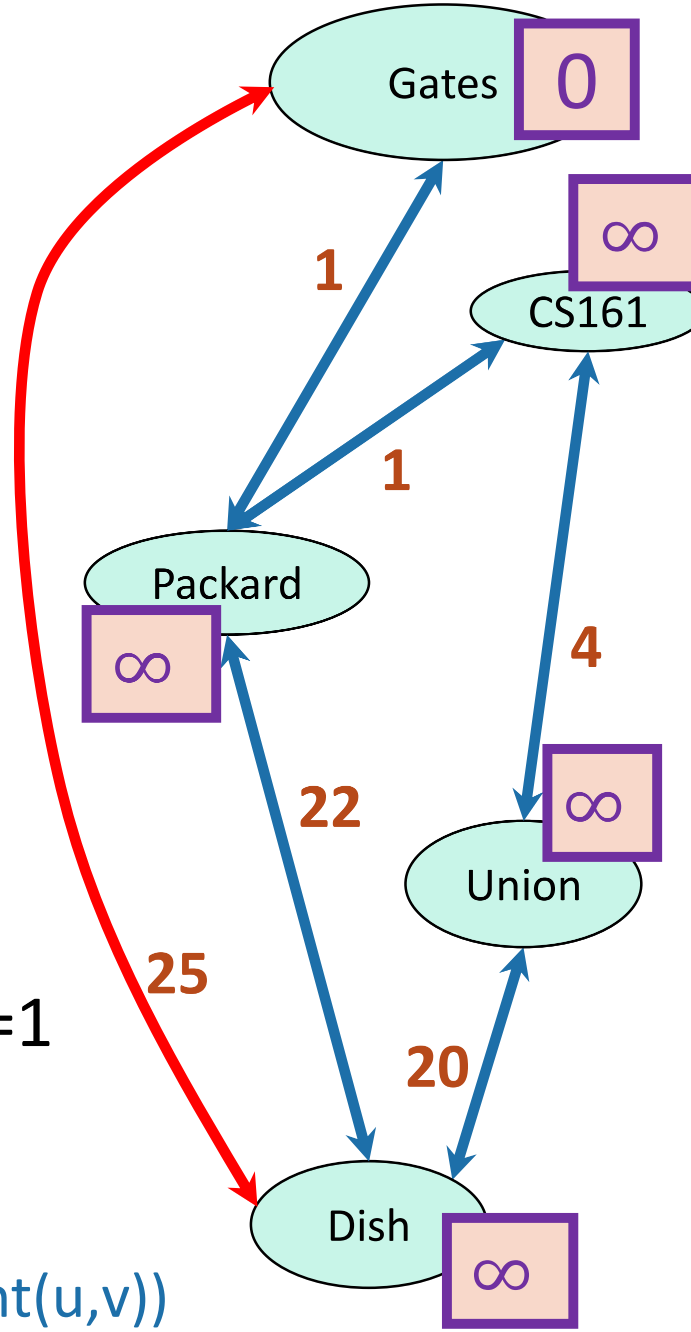

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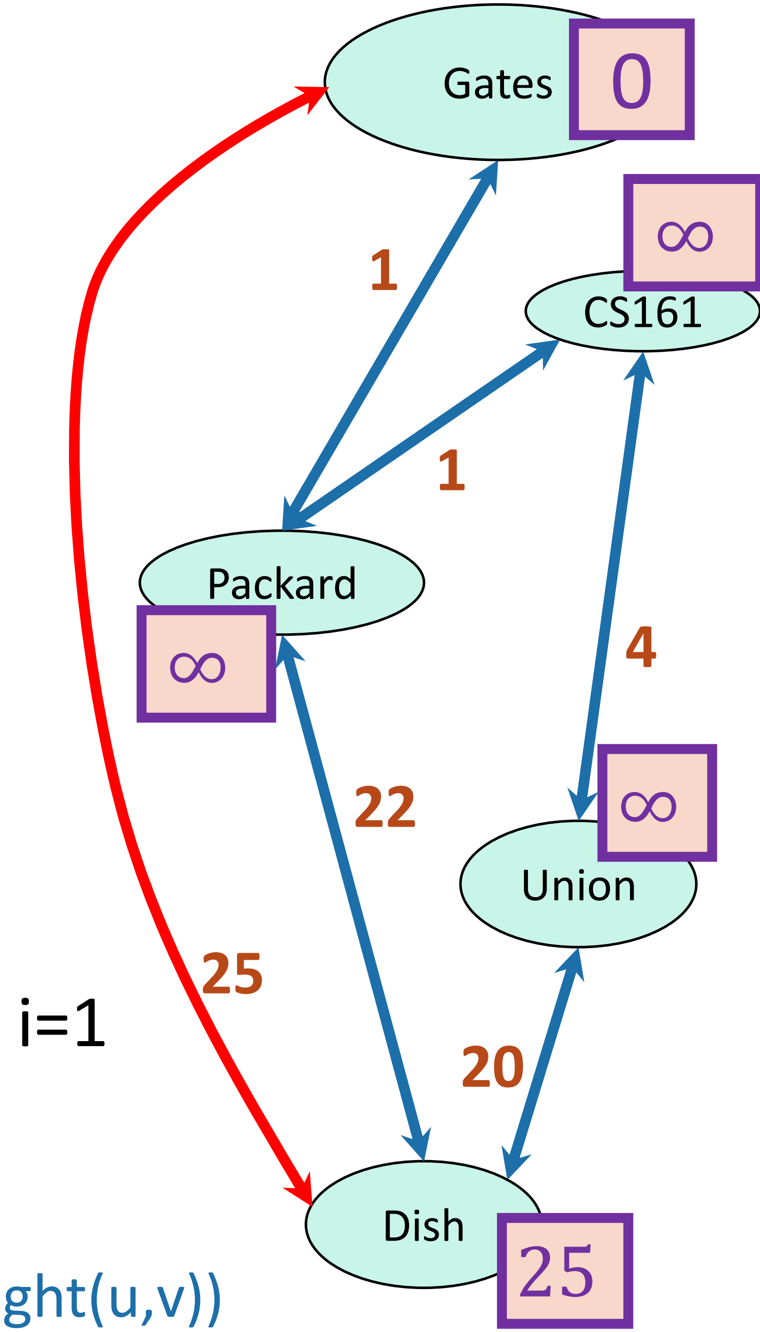

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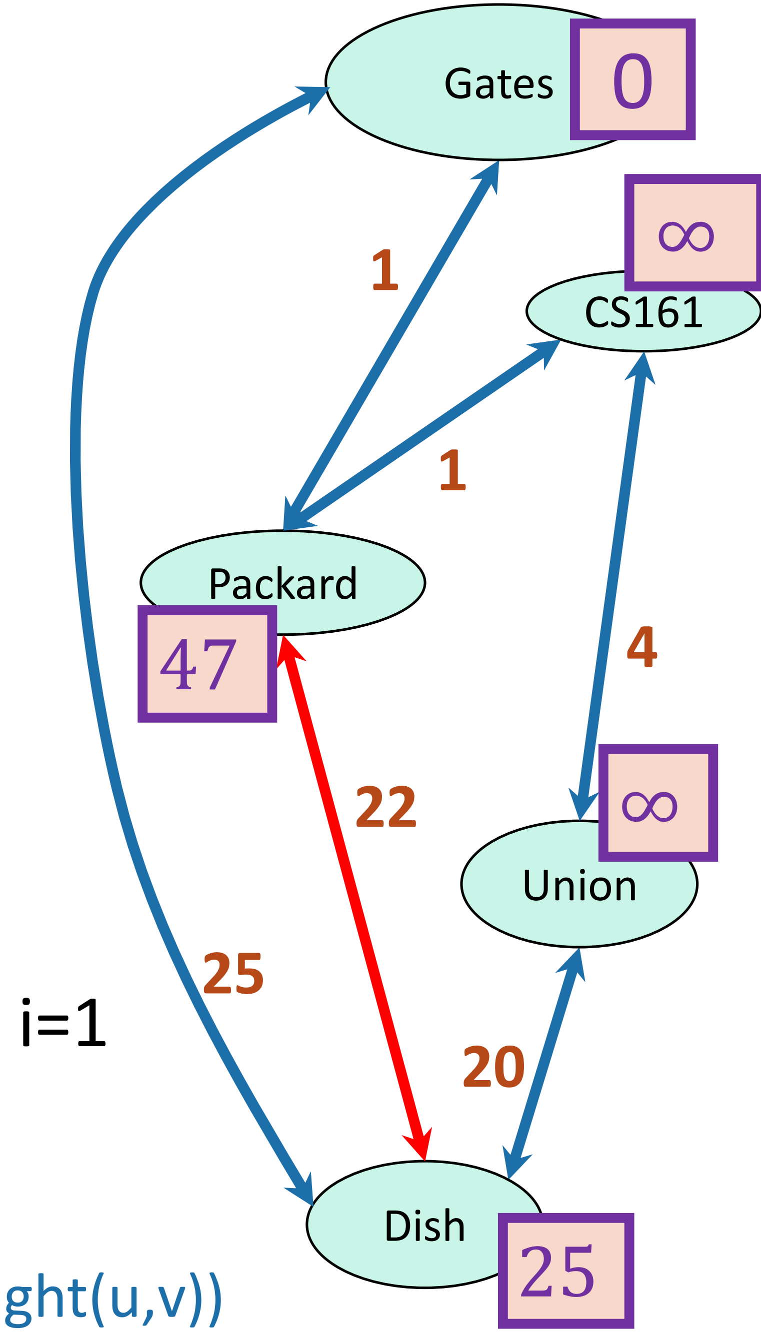

Bellman-Ford

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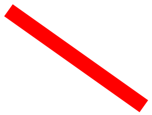
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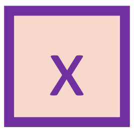

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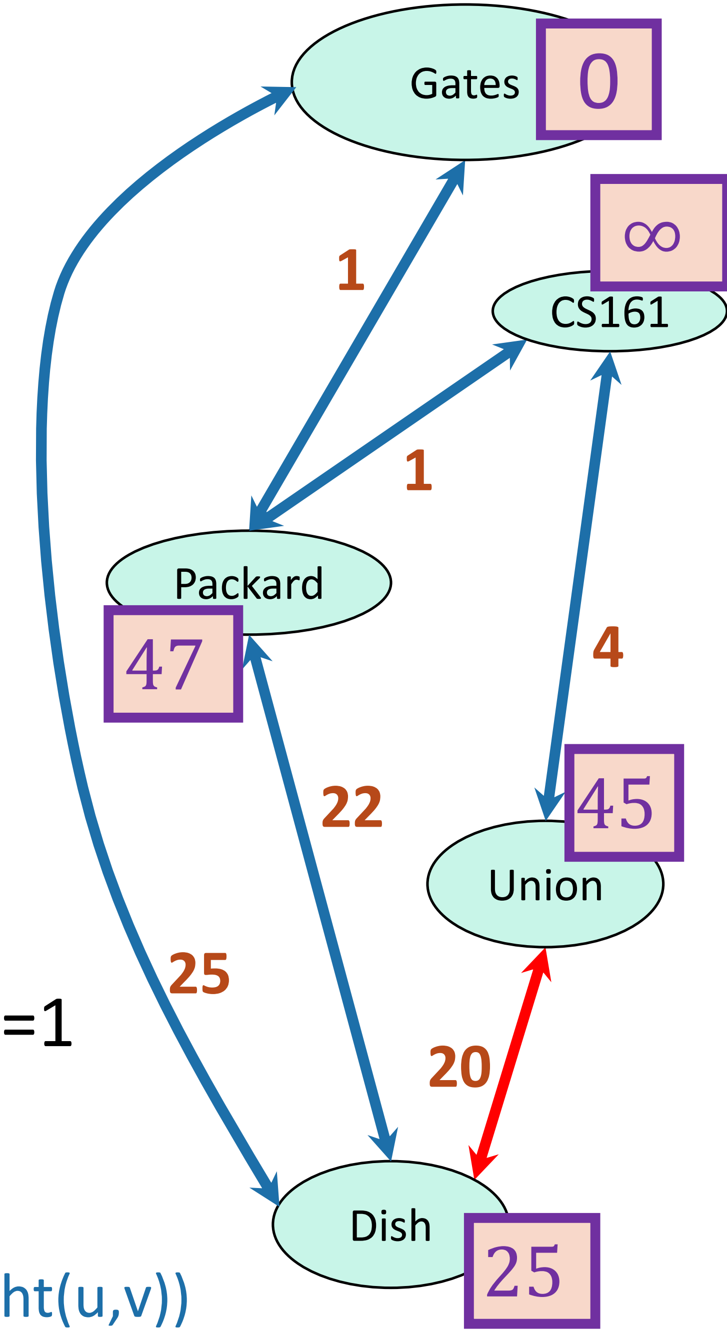

Bellman-Ford

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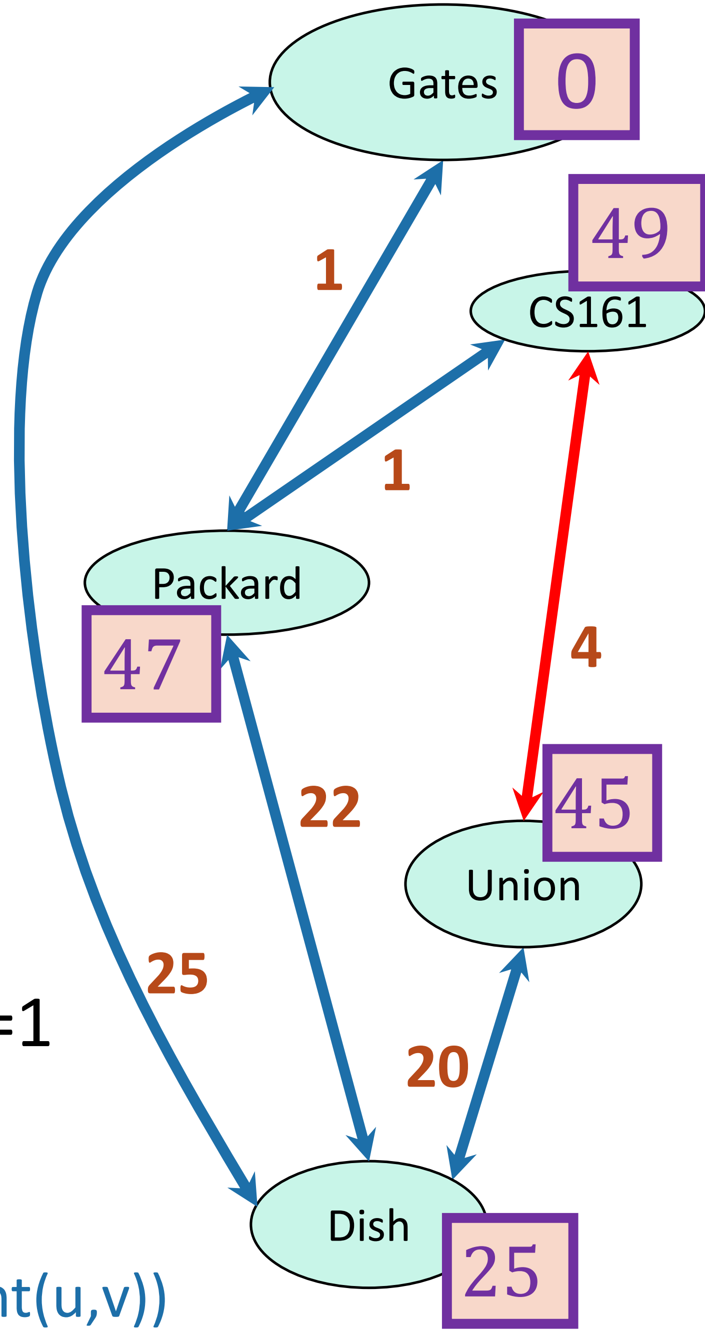

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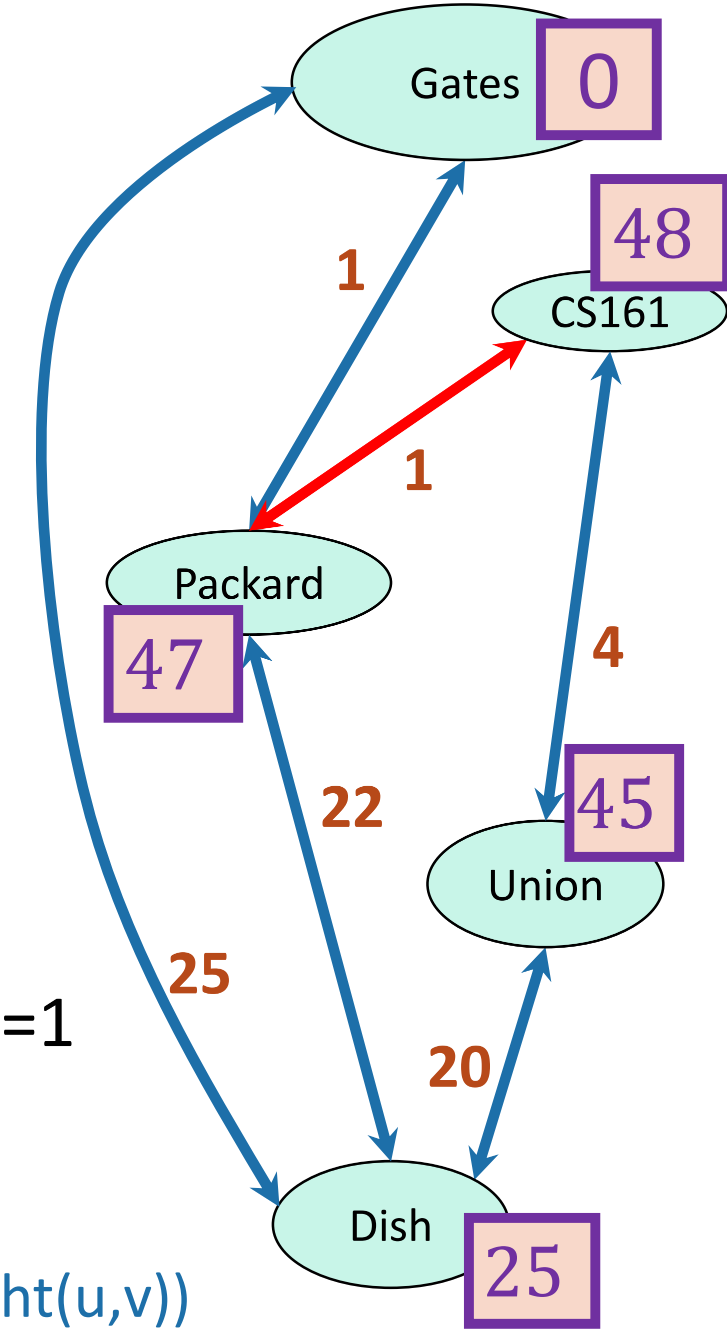

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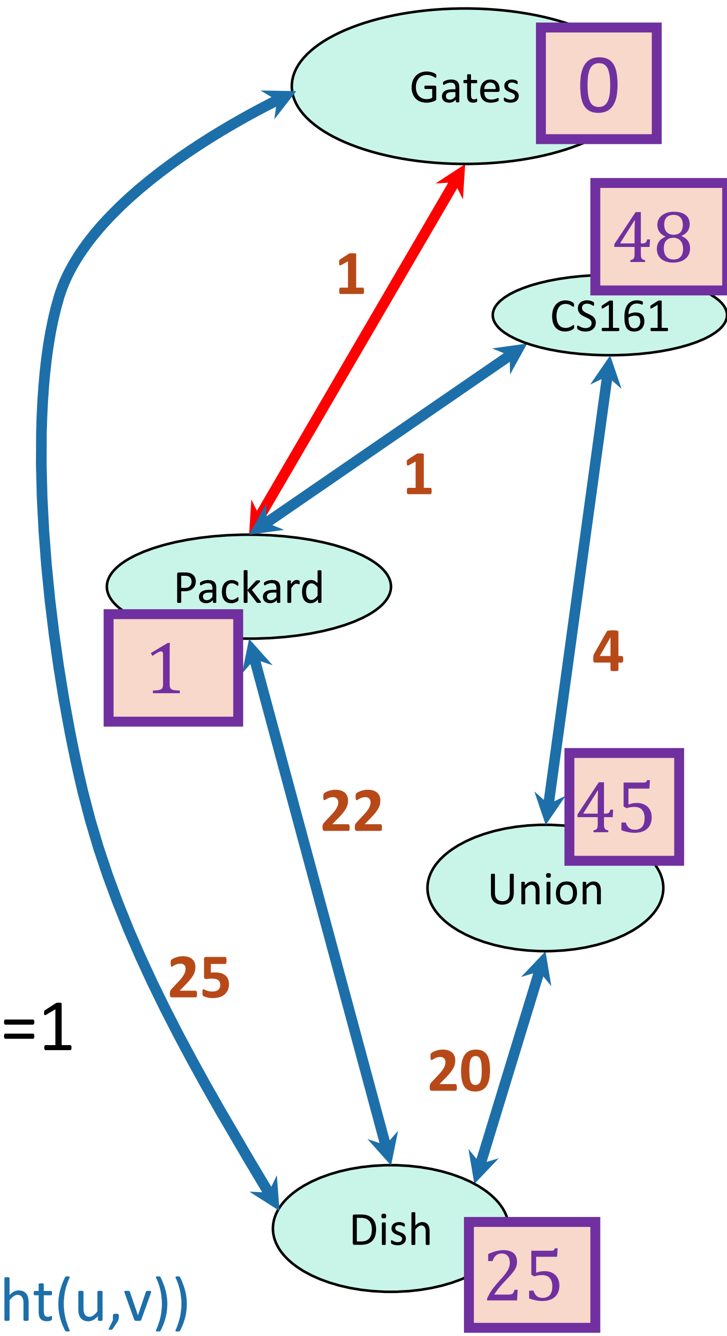

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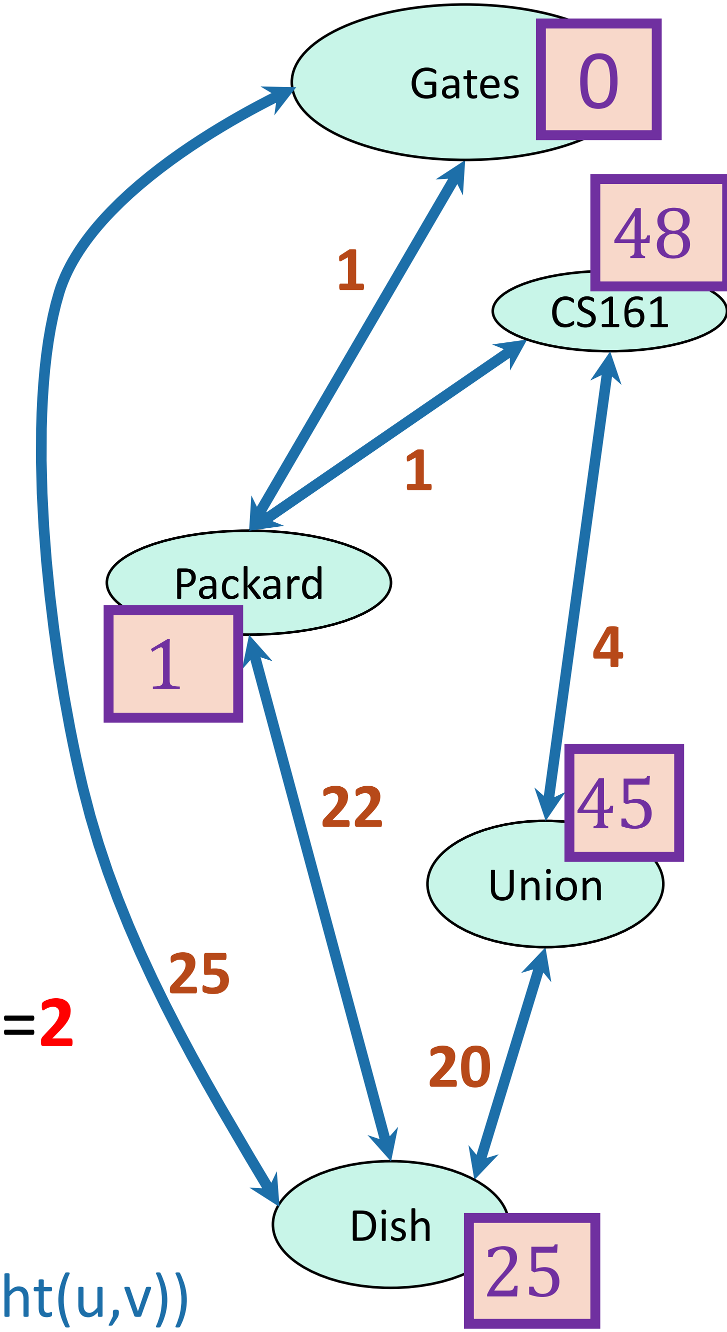
i=2



Bellman-Ford

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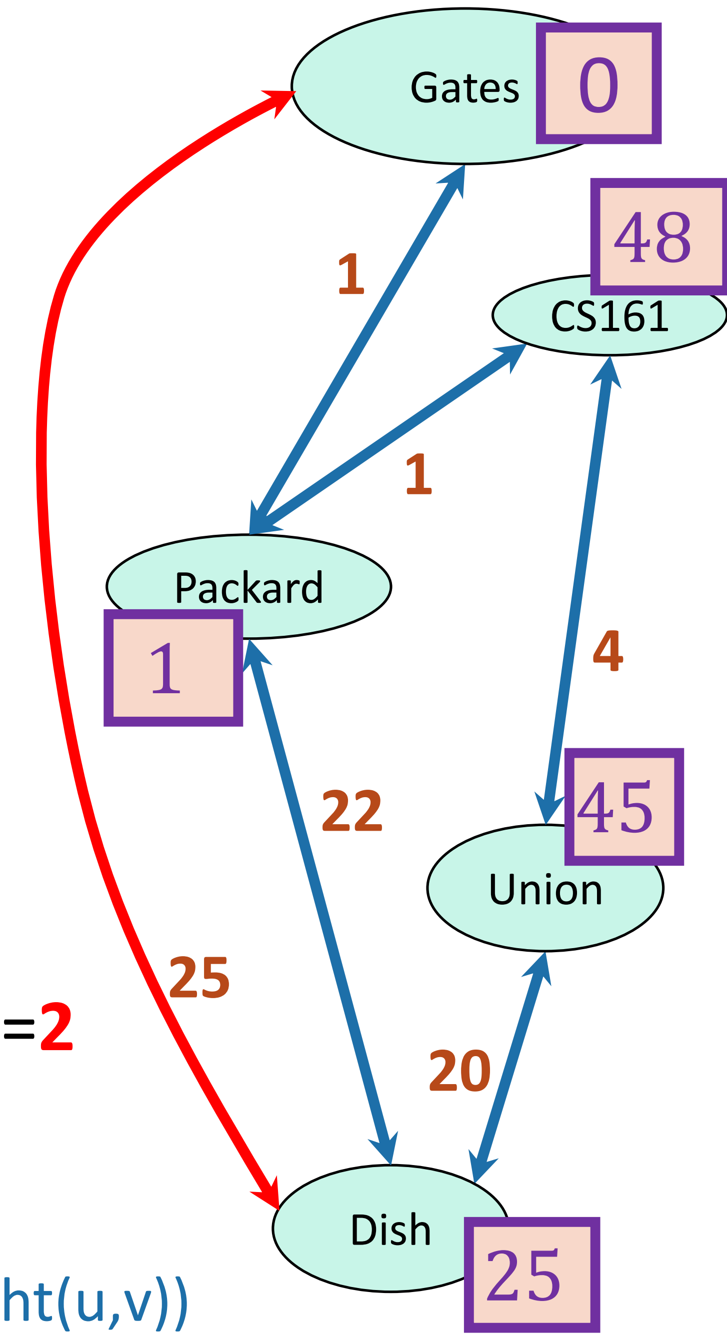

i=**2**

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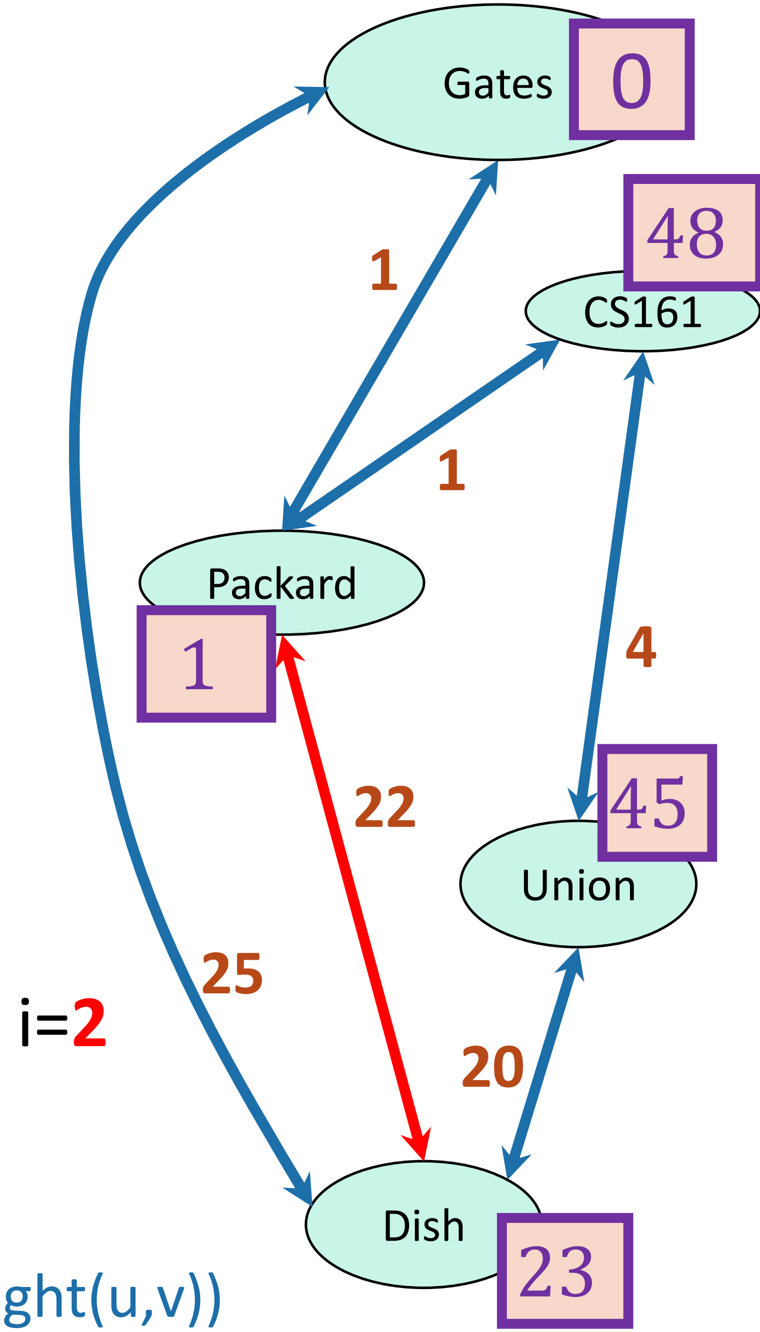

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Bellman-Ford

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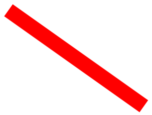

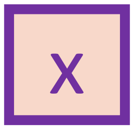

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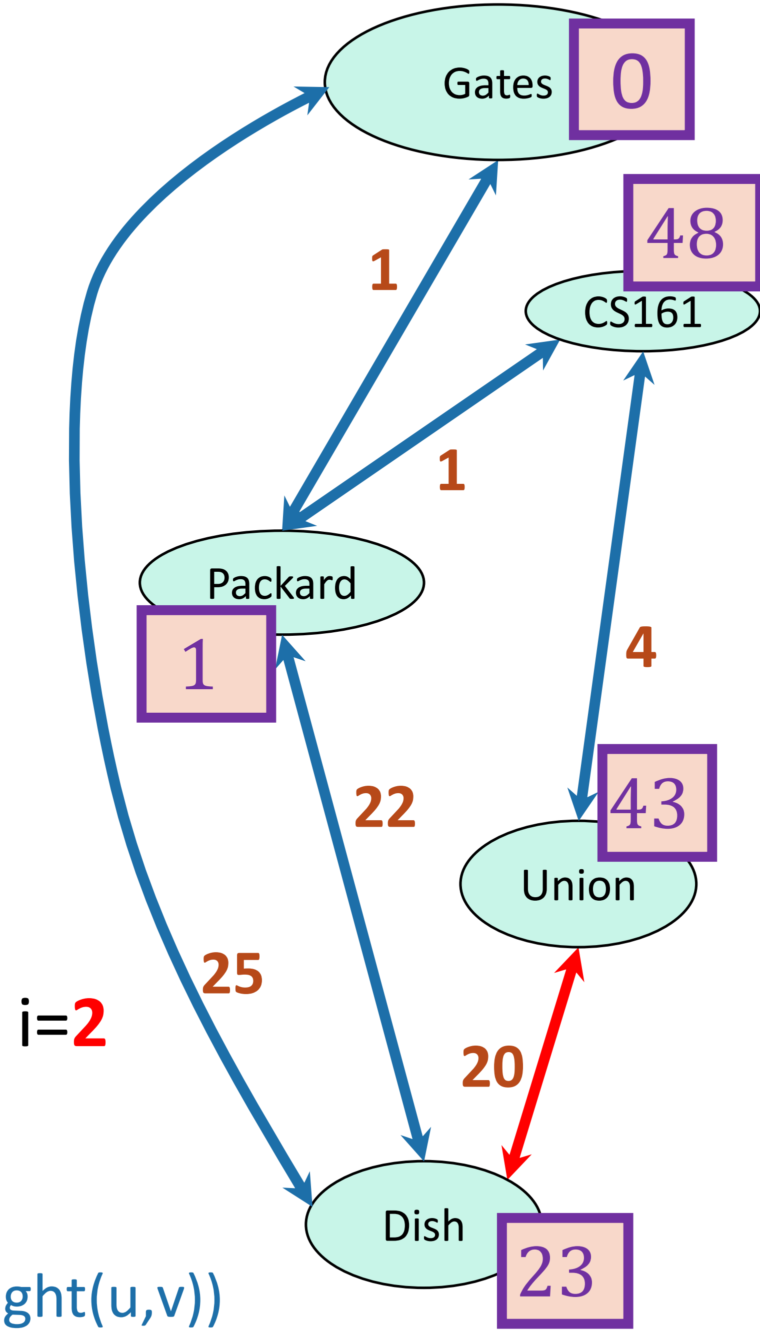

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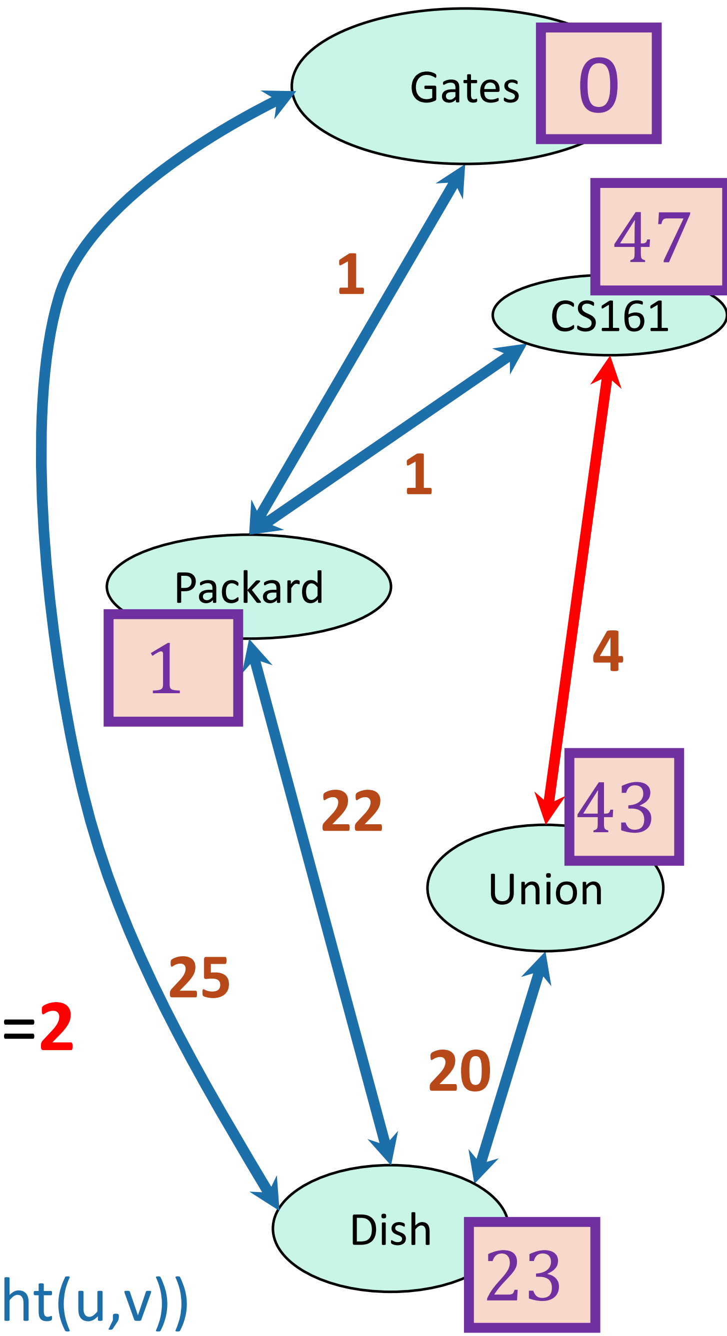

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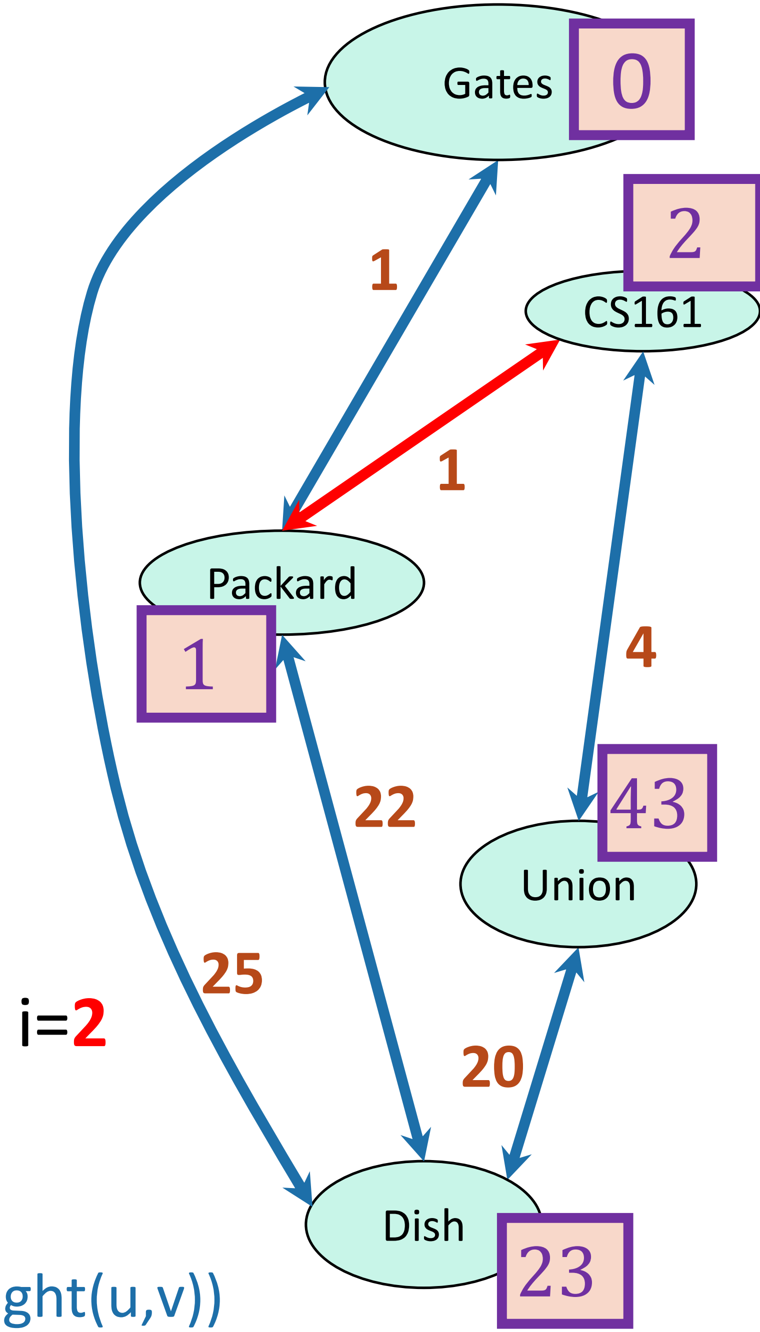

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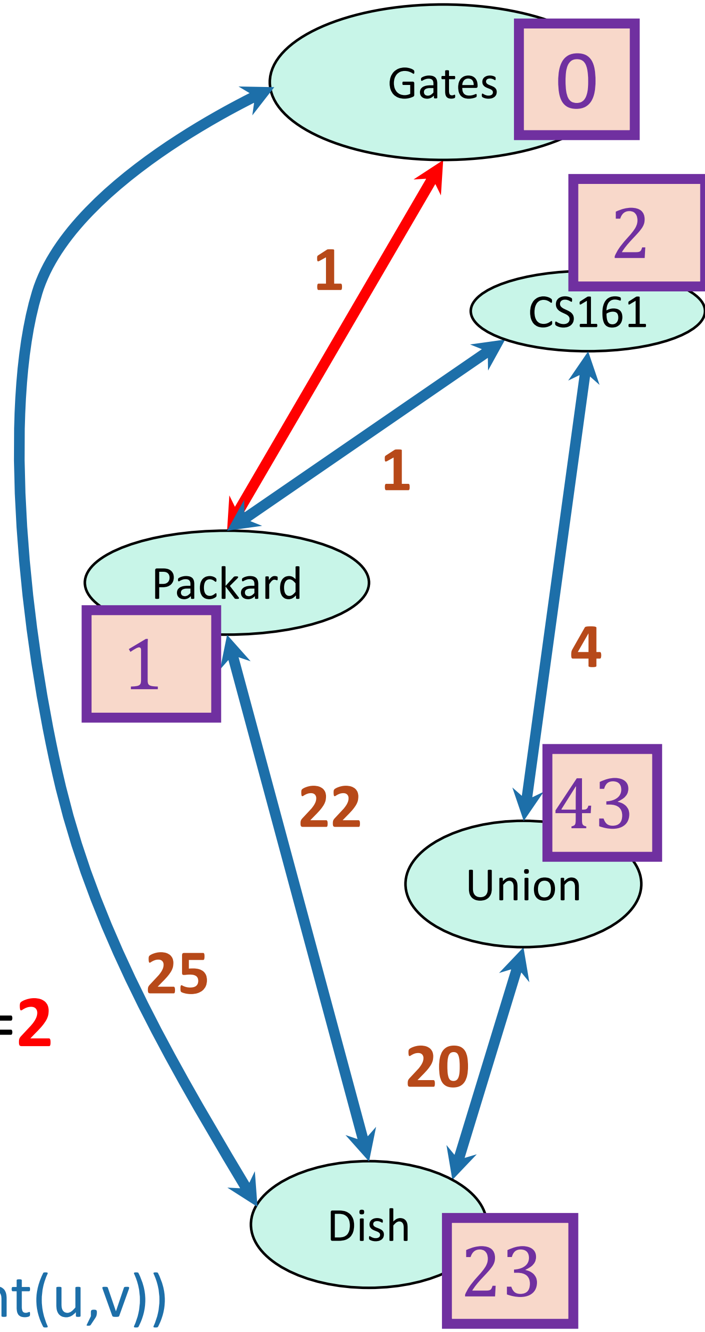

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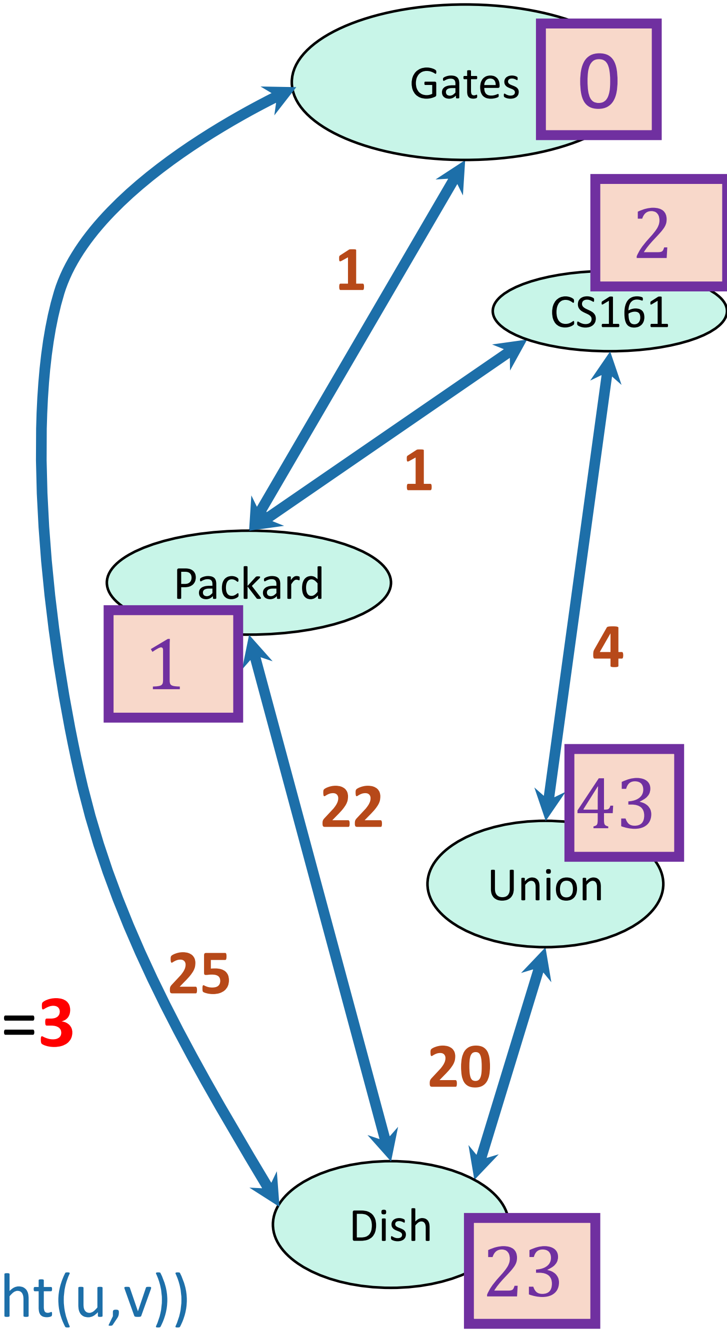

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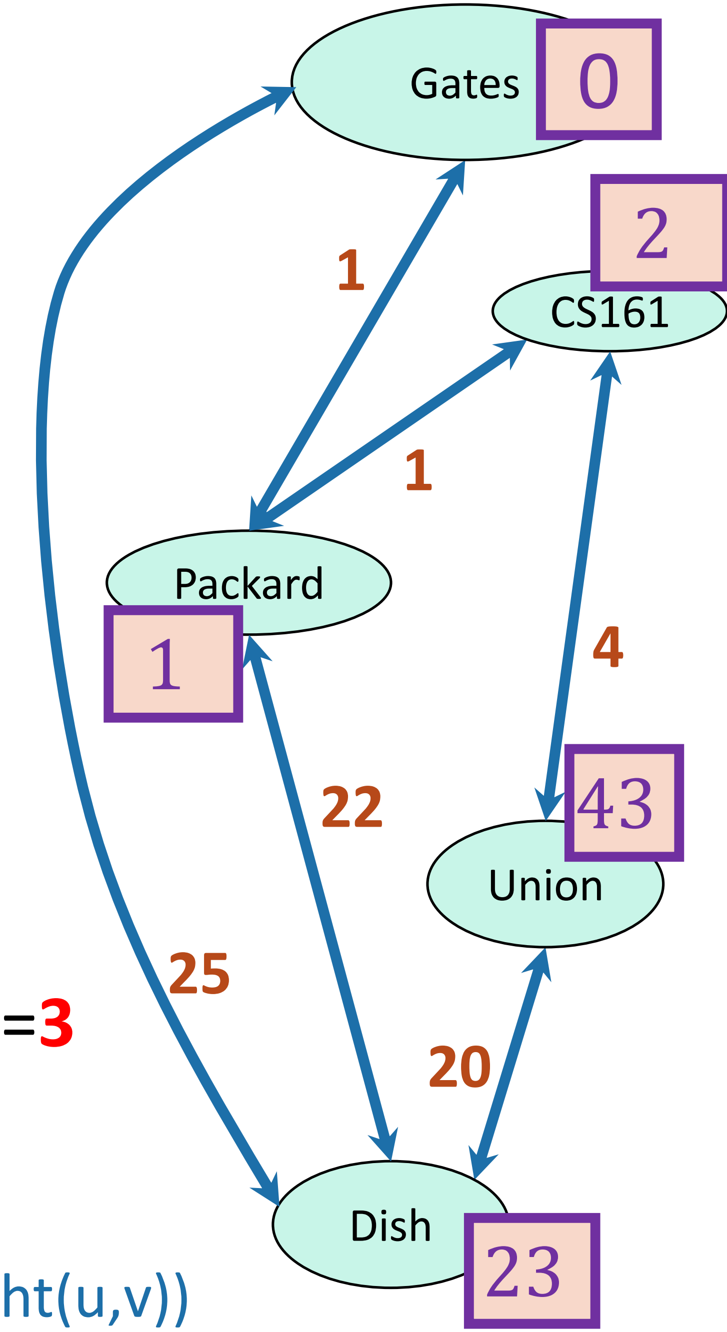

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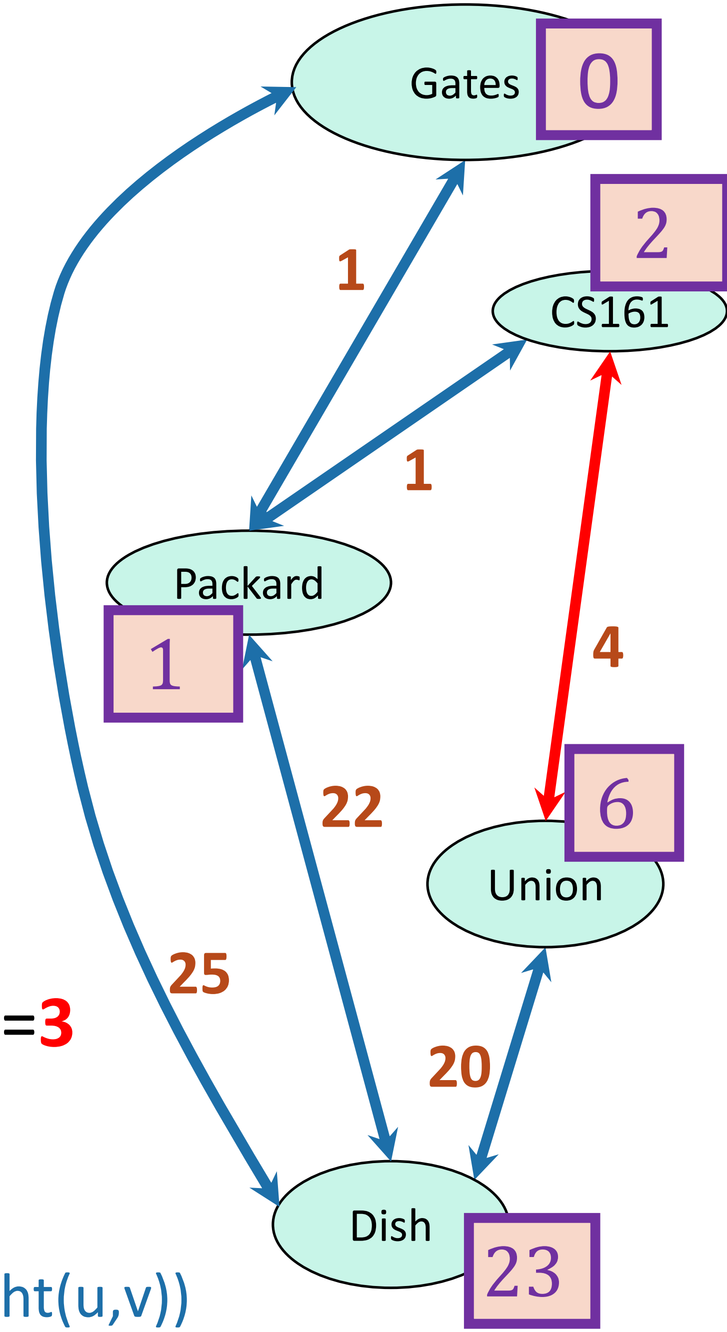

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This will keep on running until i=4,
but nothing more will happen.

we say it's **converged.**

- For v in V:
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- **For** i = 1,..,n-1:
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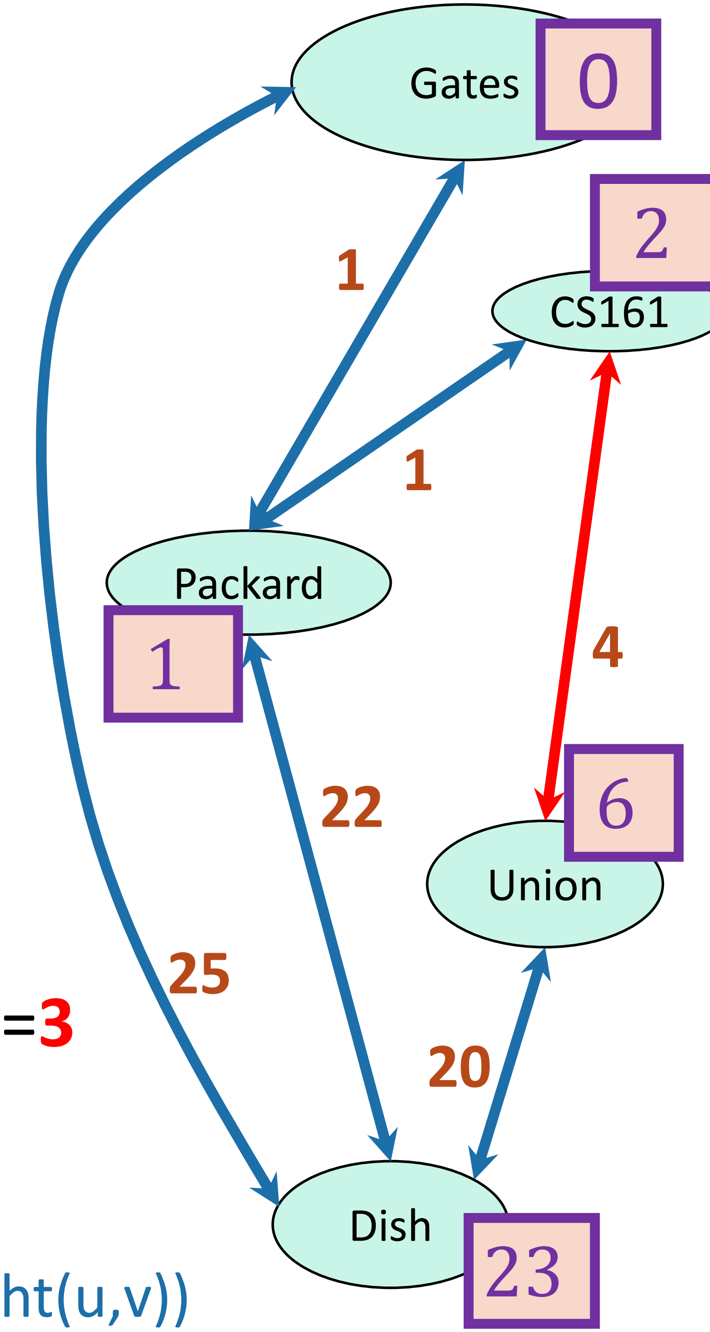


This seems much slower than Dijkstra

- And it is:

  Running time $O(mn)$

- However, it's also more flexible in a few ways.
  - Can handle negative edges
  - If we keep on doing these iterations, then changes in the network will propagate through.

- **For** $i = 1,..,n-1$:
  - **For** each edge $e = (u,v)$ in E:
    - $d[v] \leftarrow \min(d[v]\ ,\ d[u] + \mathrm{edgeWeight}(u,v))$

But first

- Why does it work as is?

We will show:

- After iteration i, for each v,
  - d[v] is equal to the shortest path between s and v…
  - **…with at most i edges**.

This is what we want.

In particular:

- After iteration n-1, for each v,
  - d[v] is equal to the shortest path between s and v …
  - ~~**…with at most n-1 edges**~~.

All paths in a graph with n vertices have at most n-1 edges.

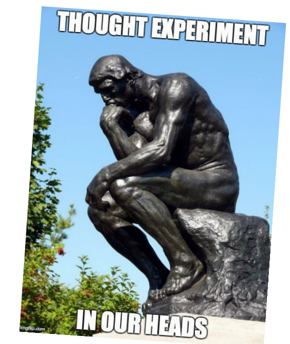
Proof by induction

- **Inductive Hypothesis:**
  - After iteration i, for each v, d[v] is equal to the cost of the shortest path between s and v with at most i edges.
- **Base case:**
  - After iteration 0… ✓
- **Inductive step:**

Inductive step

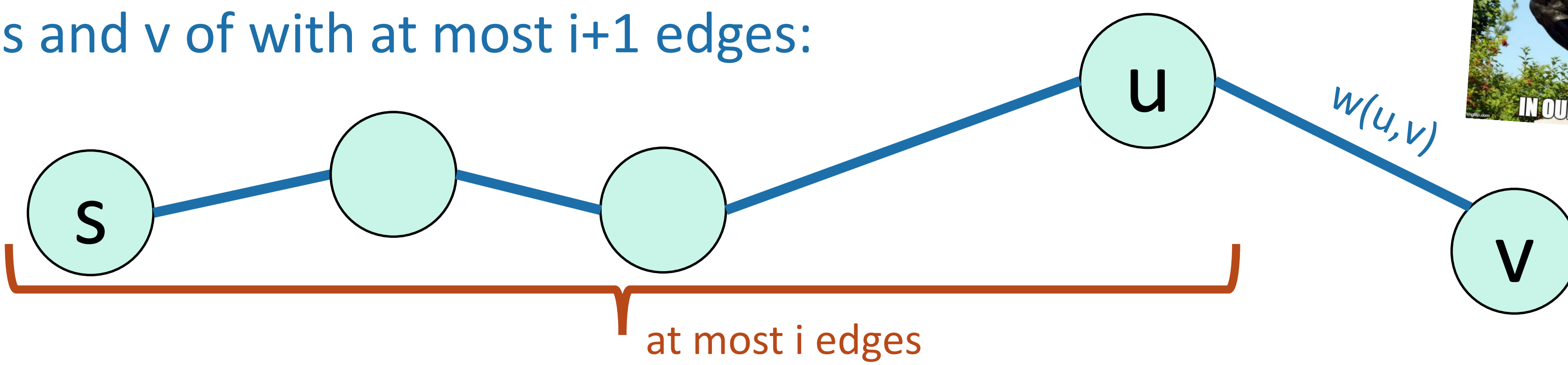
Hypothesis: After iteration i, for each v, d[v] is equal to the cost of the shortest path between s and v with at most i edges.

- Suppose the inductive hypothesis holds for i.
- We want to establish it for i+1.



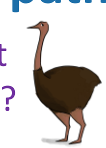
Say this is the shortest path between s and v of with at most i+1 edges:

Let u be the vertex right before v in this path.



- By induction, d[u] is the cost of a shortest path between s and u of i edges.
- By setup, d[u] + w(u,v) is the cost of a shortest path between s and v of i+1 edges.
- In the i+1'st iteration, when (u,v) is active, we ensure **d[v] <= d[u] + w(u,v).**
- **So d[v] <= cost of shortest path between s and v with i+1 edges.**
- **But d[v] = cost of a particular path of at most i+1 edges >= cost of shortest path.**
- So d[v] = cost of shortest path with at most i+1 edges.

Why is d[v] the cost of a particular path?



Proof by induction

- **Inductive Hypothesis:**
  - After iteration i, for each v, d[v] is equal to the cost of the shortest path between s and v of length at most i edges.
- **Base case:**
  - After iteration 0…
- **Inductive step:**
- **Conclusion:**
  - After iteration n-1, for each v, d[v] is equal to the cost of the shortest path between s and v of length at most n-1 edges.
  - **Aka, d[v] = d(s,v) for all v.**

Something is wrong

- We never used that there weren't any negative cycles!!

Proof by induction

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  - After iteration i, for each v, d[v] is equal to the cost of the shortest path between s and v of length at most i edges.
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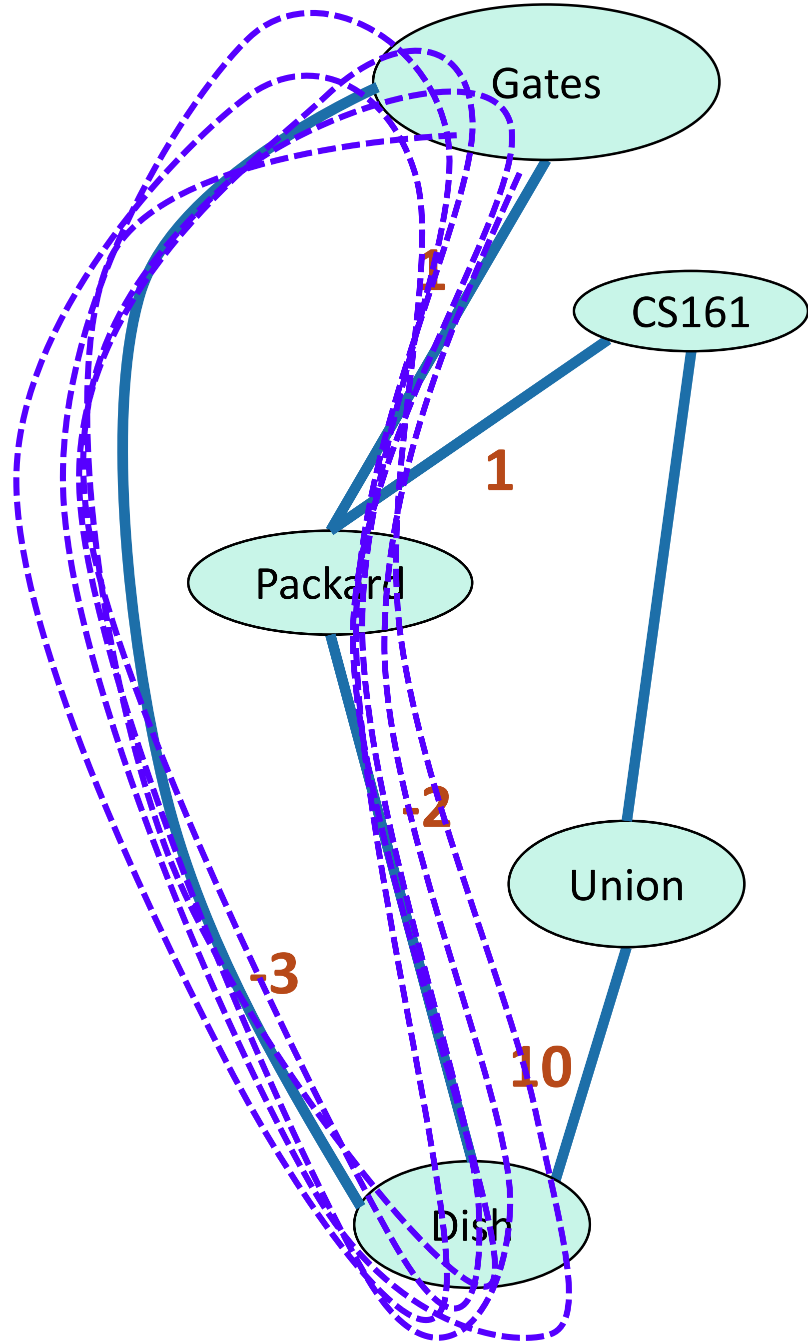
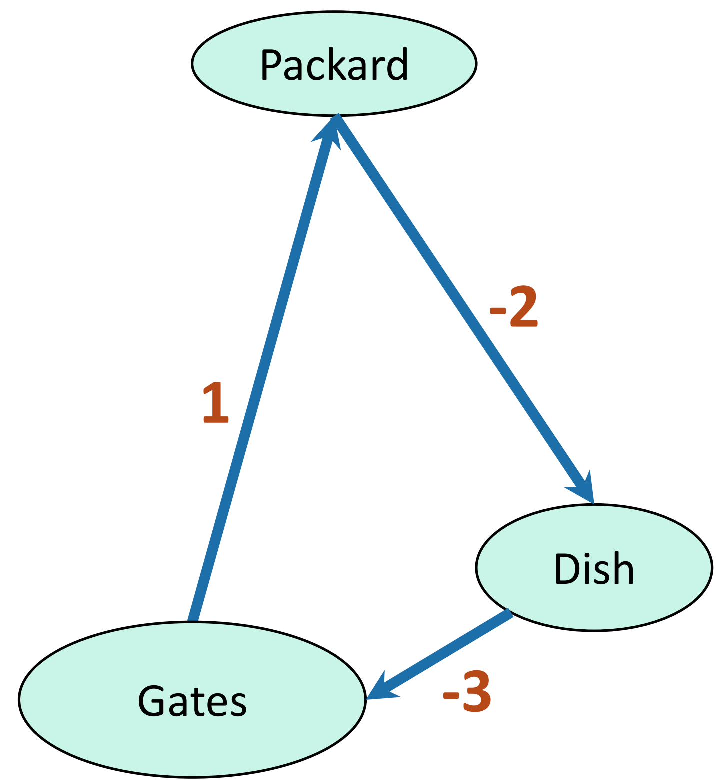
Some paths have more than n-1 edges.

- So we've correctly concluded:
  - After iteration n-1, for each v, d[v] is equal to the cost of the shortest path between s and v of length at most n-1 edges.
- But that's not what we wanted to show.



This is a problem if there are negative cycles.

- A **negative cycle** is a cycle so that the sum of the edges is negative:



- If there is a negative cycle in G, then there are always shorter paths of length >n
  - Because we can always make a path shorter by going around the cycle.
- We kind of want to ignore this case, though, because "shortest path" doesn't even make sense…

Suppose there are no negative cycles.

- Then all shortest paths are simple paths.
  - A simple path has no cycles.

- It's true that all **simple** paths on n vertices have length at most n-1.

- So then we can make the conclusion that we want.

Proof by induction

Suppose there are no negative cycles

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  - **Aka, the d[v] = d(s,v).**

Theorem

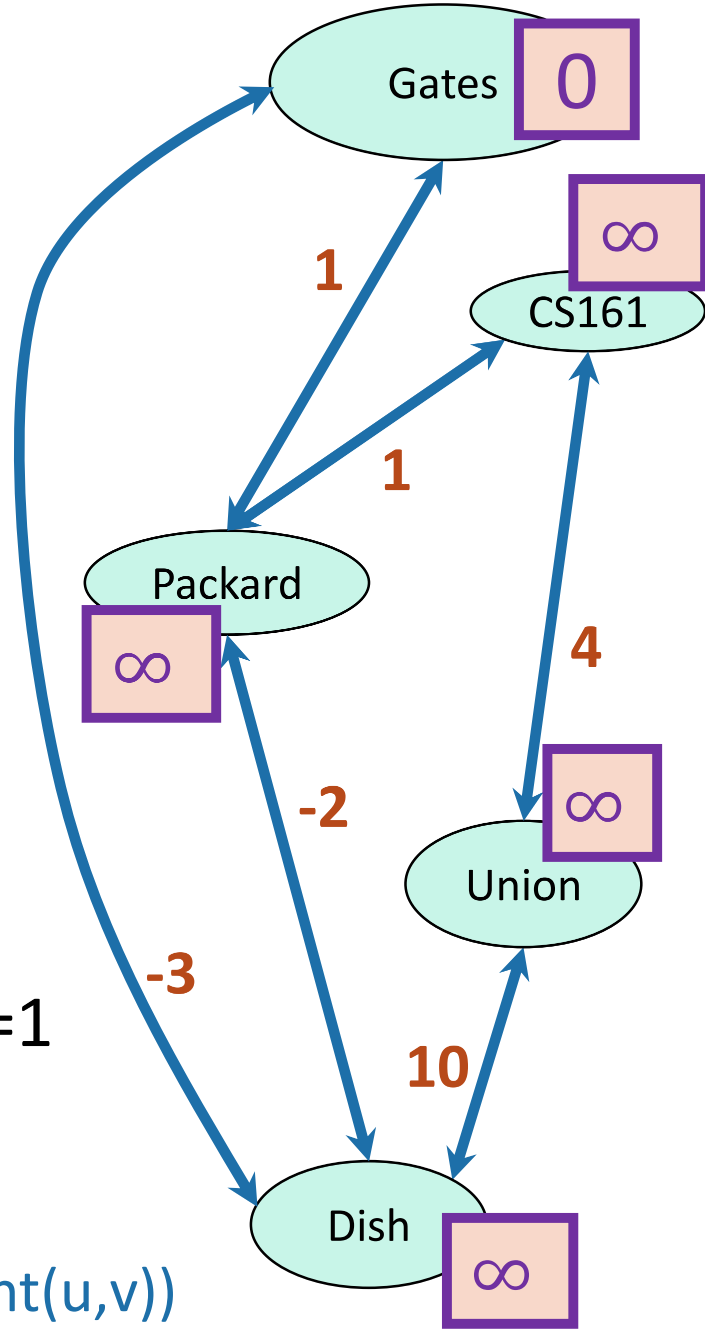
- The Bellman-Ford algorithm runs in time O(nm) on a graph G with n vertices and m edges.
- If there are no negative cycles in G, then the BF algorithm terminates with d[v] = d(s,v).
- Notice, negative **weights** are okay.

Okay, so what if there are negative cycles?

What does B-F do?

Current edge

$\boxed{x}$ x is my best over-estimate for a vertex v.
We'll say d[v] = x



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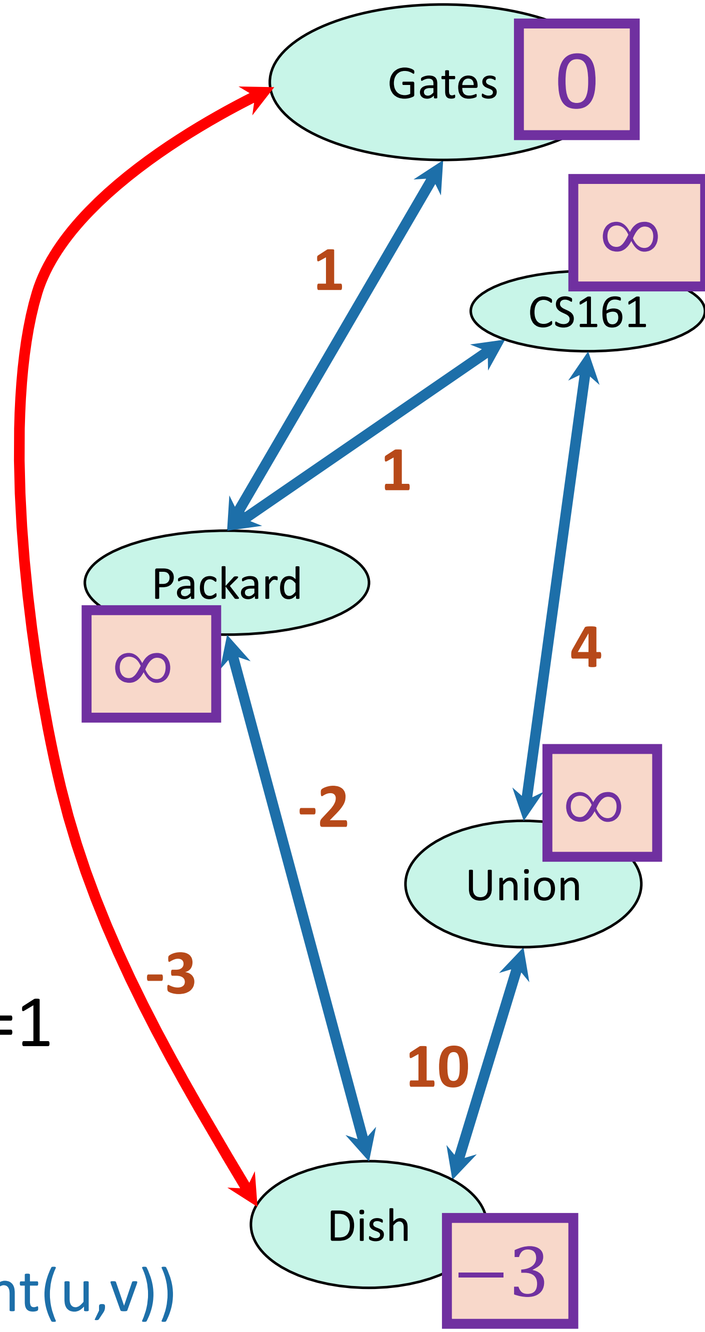

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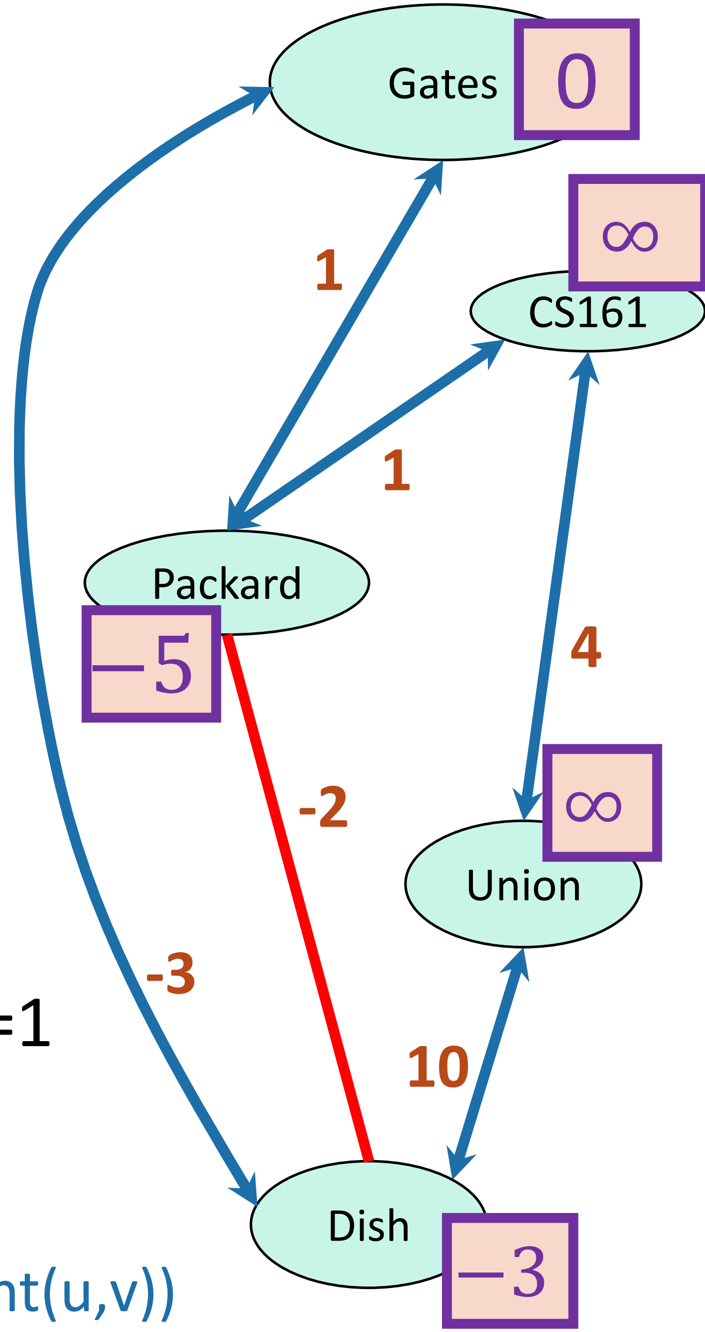

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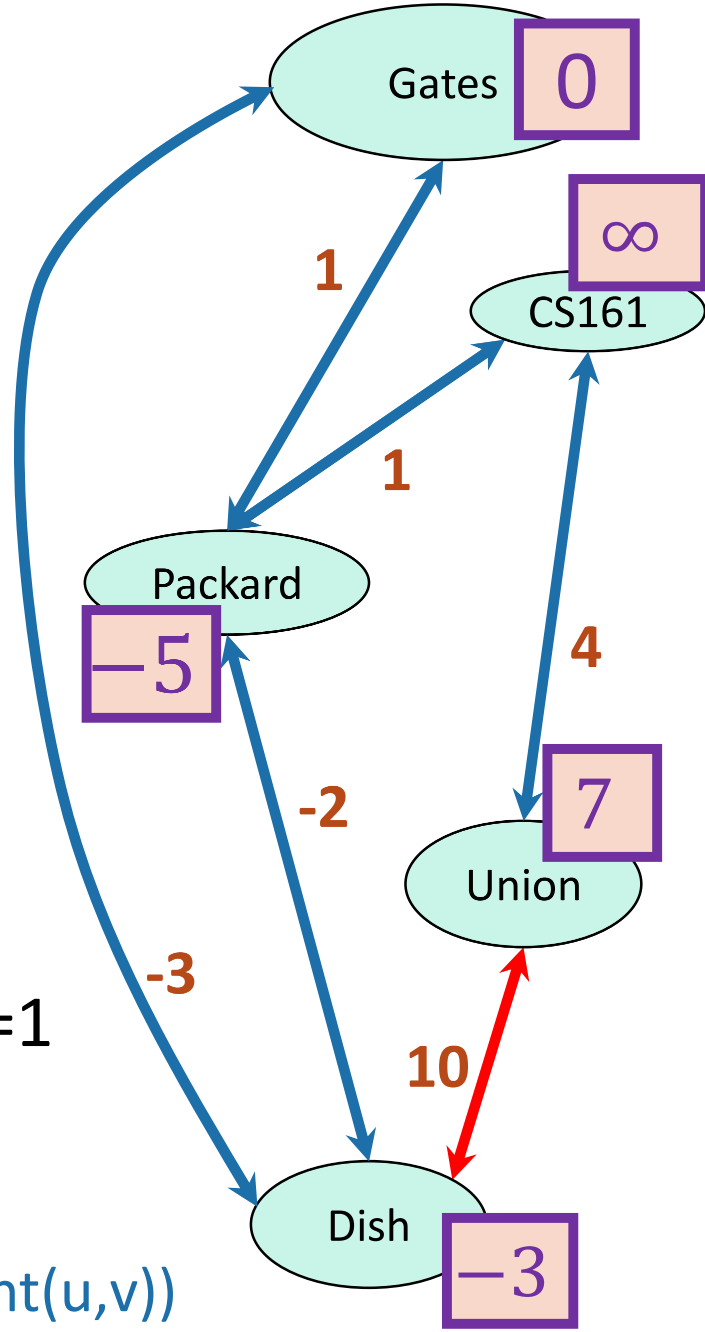

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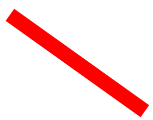
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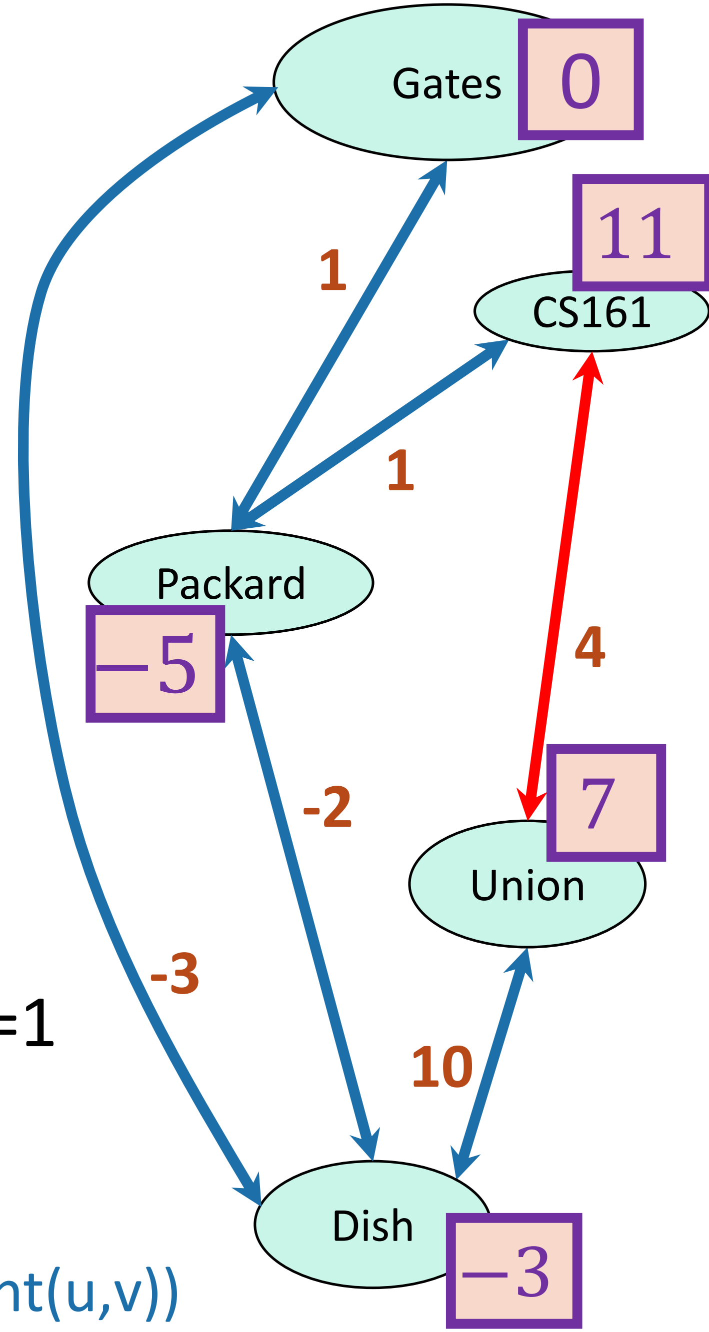

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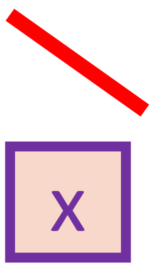

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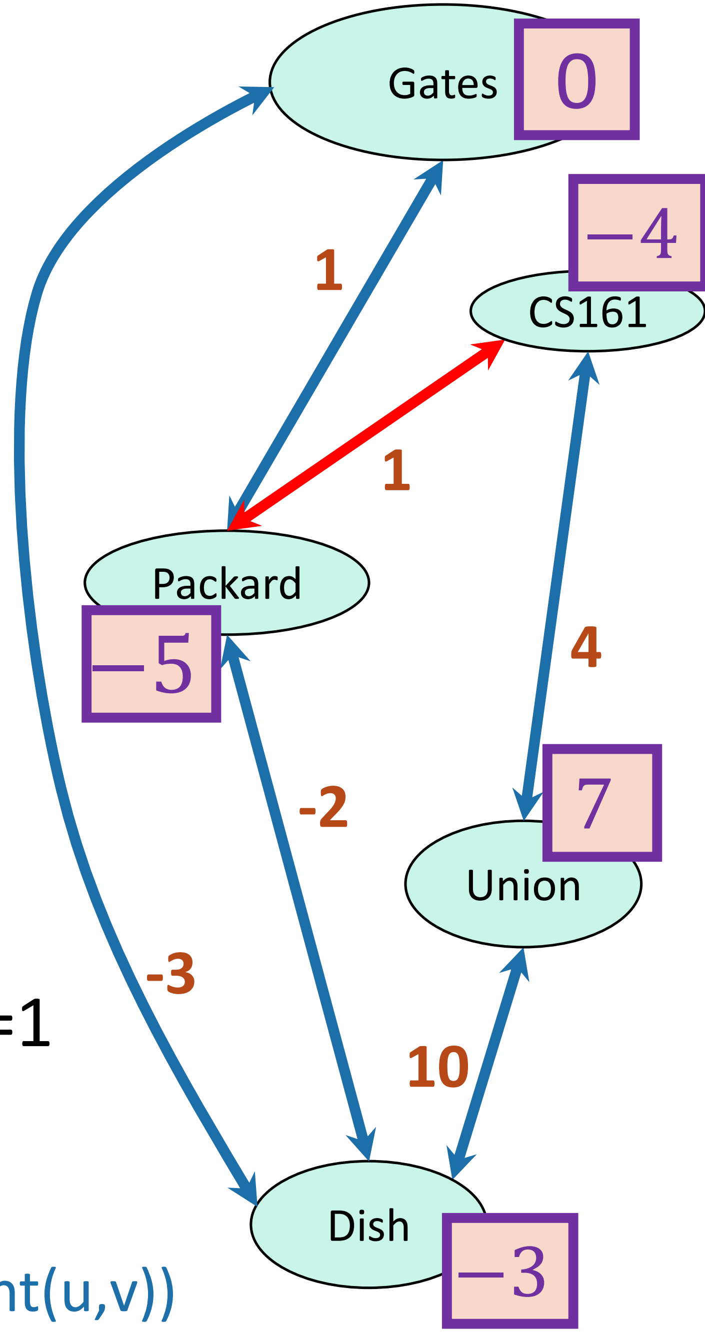

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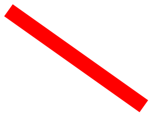

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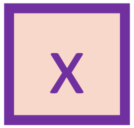

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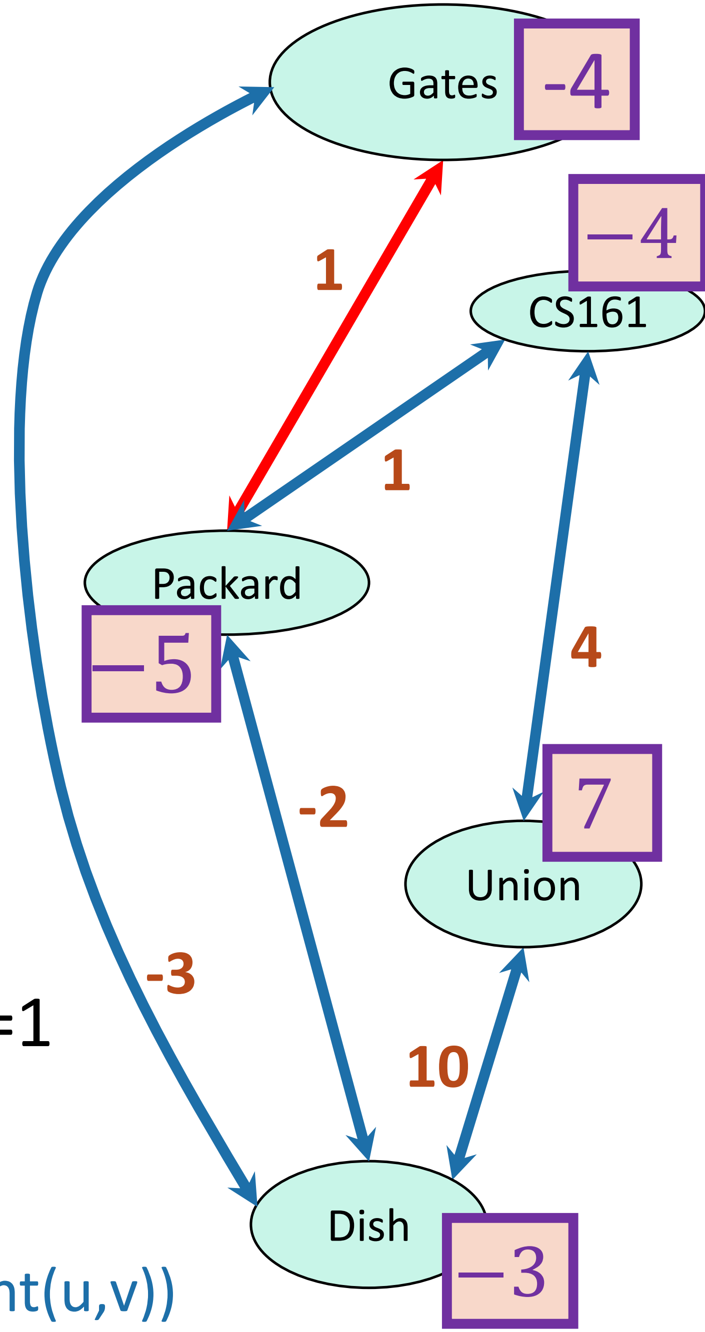

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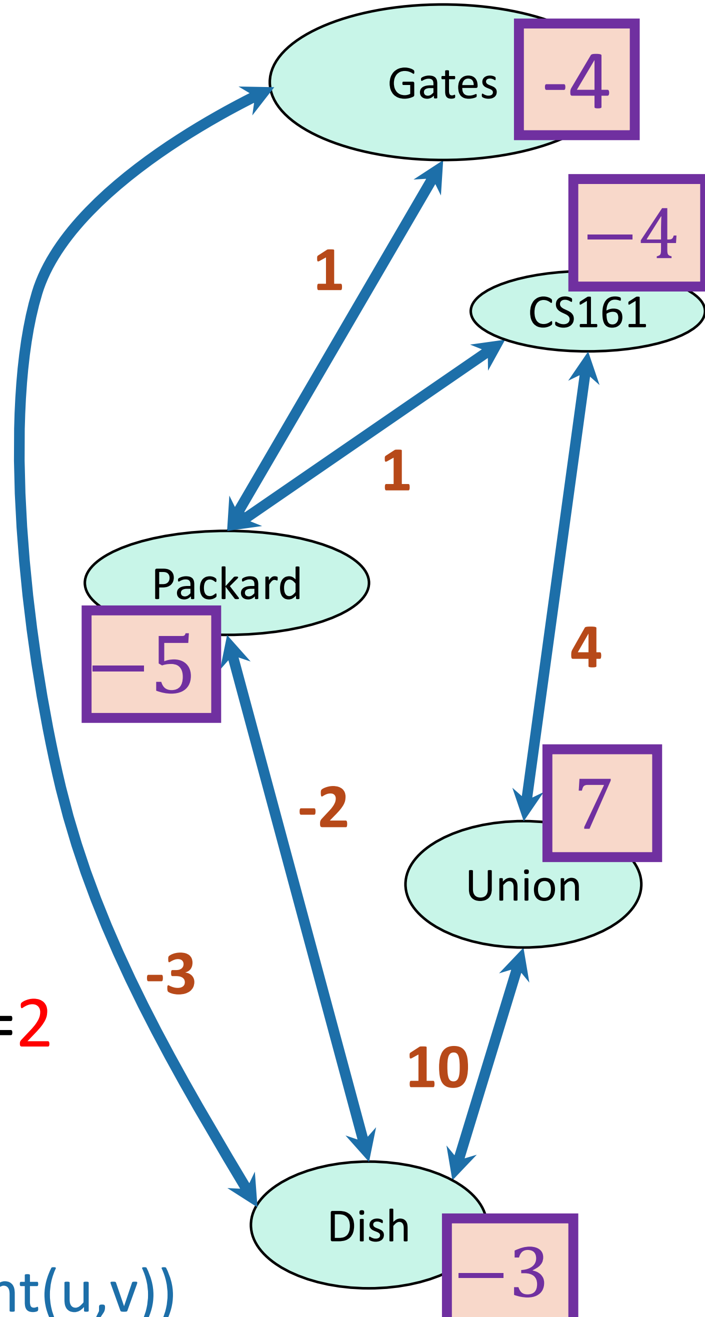

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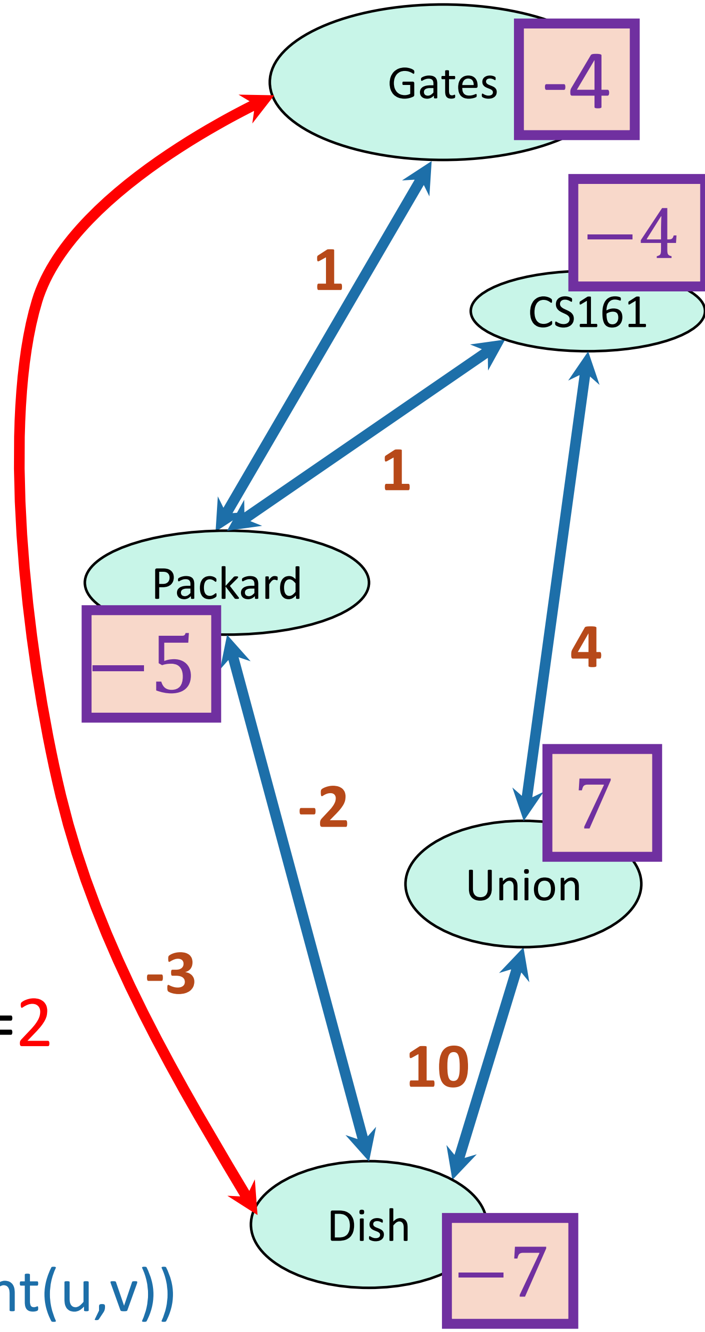

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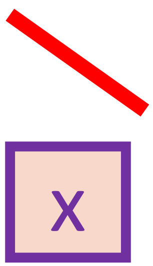

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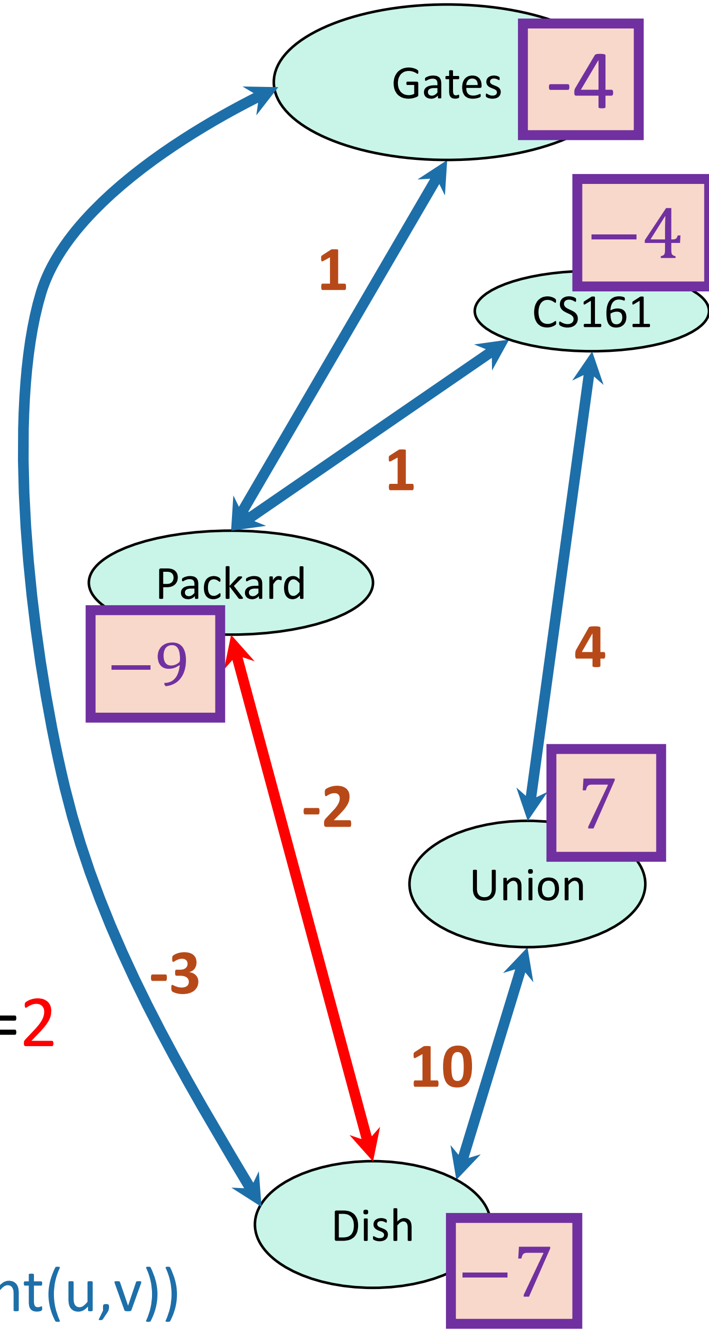

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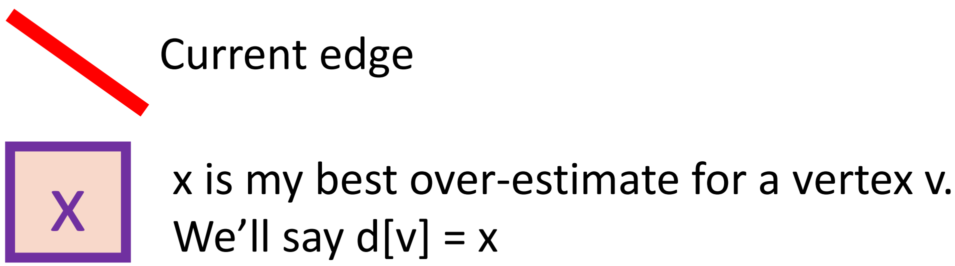

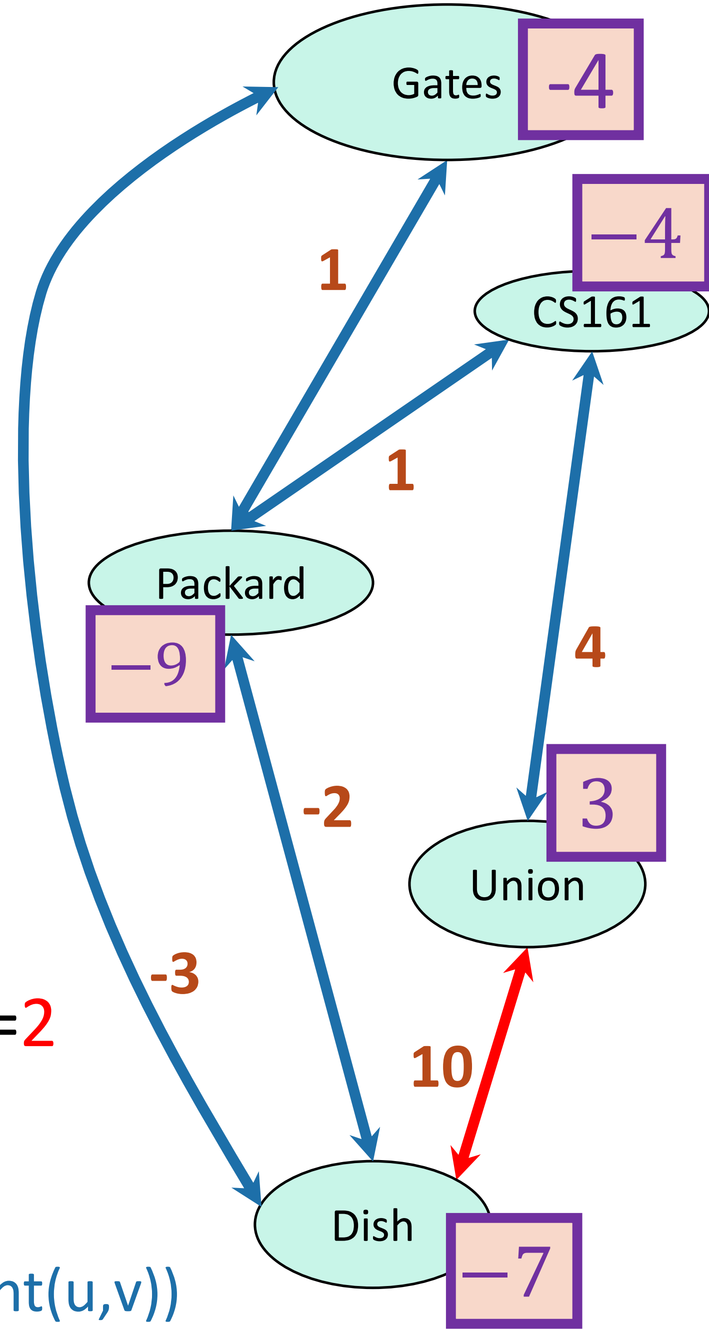

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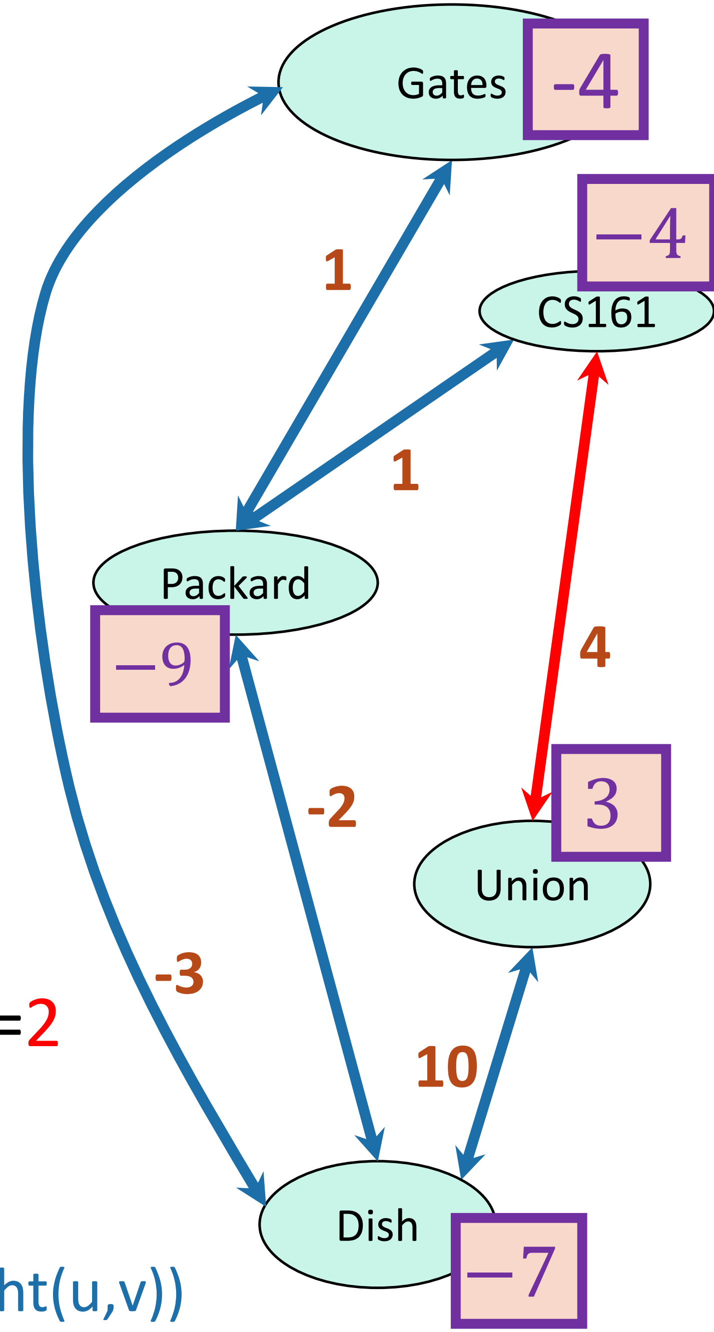

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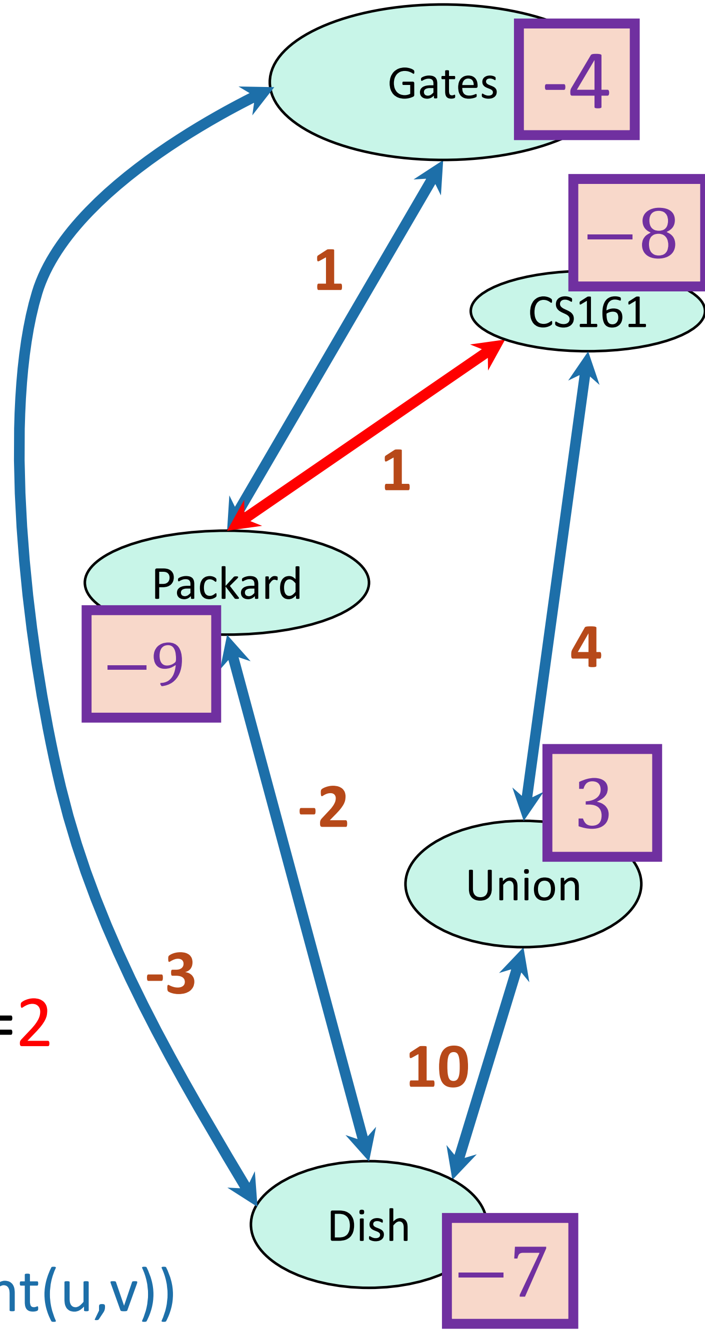

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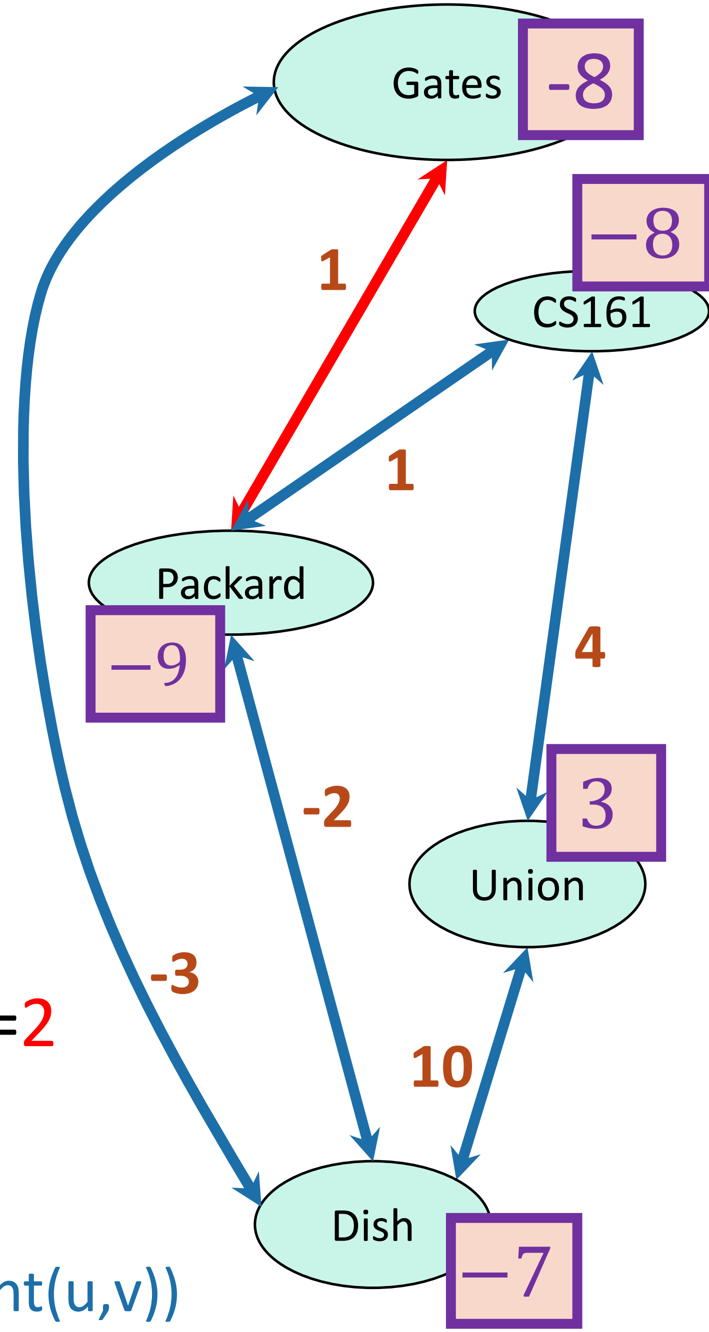

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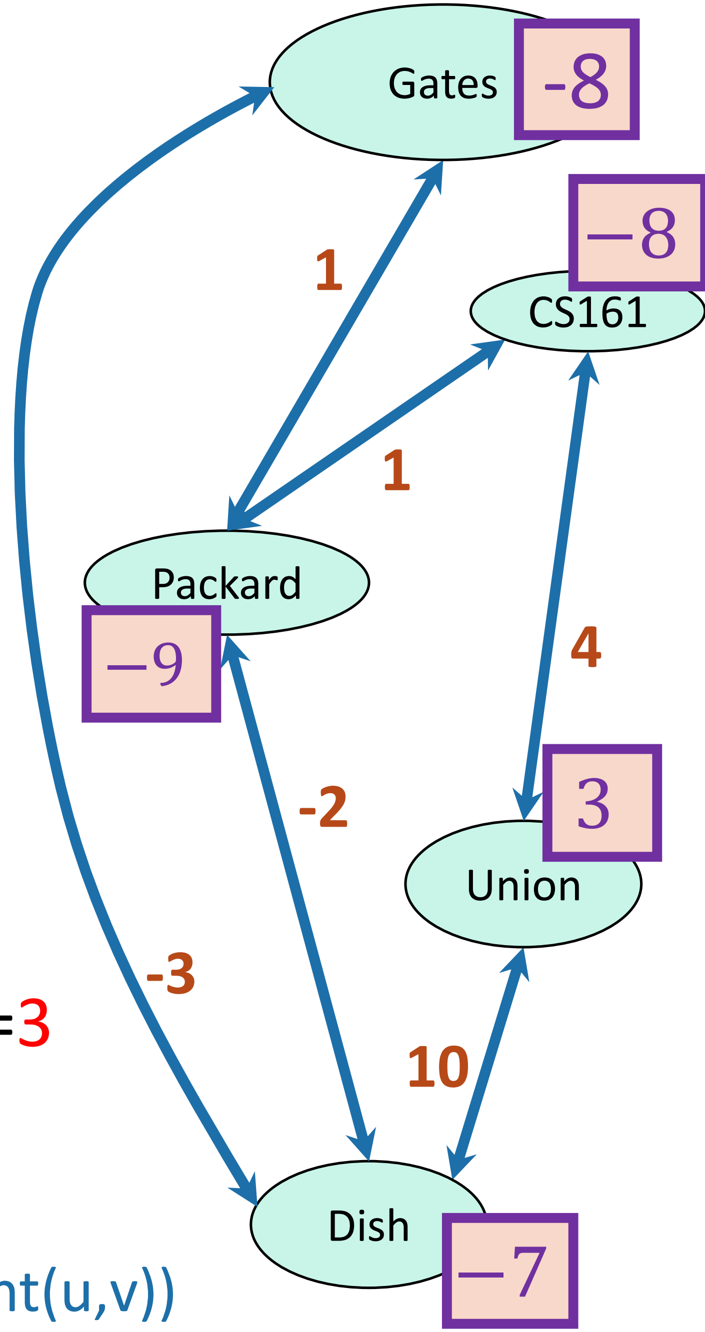
You can see where this is going: this will never converge.



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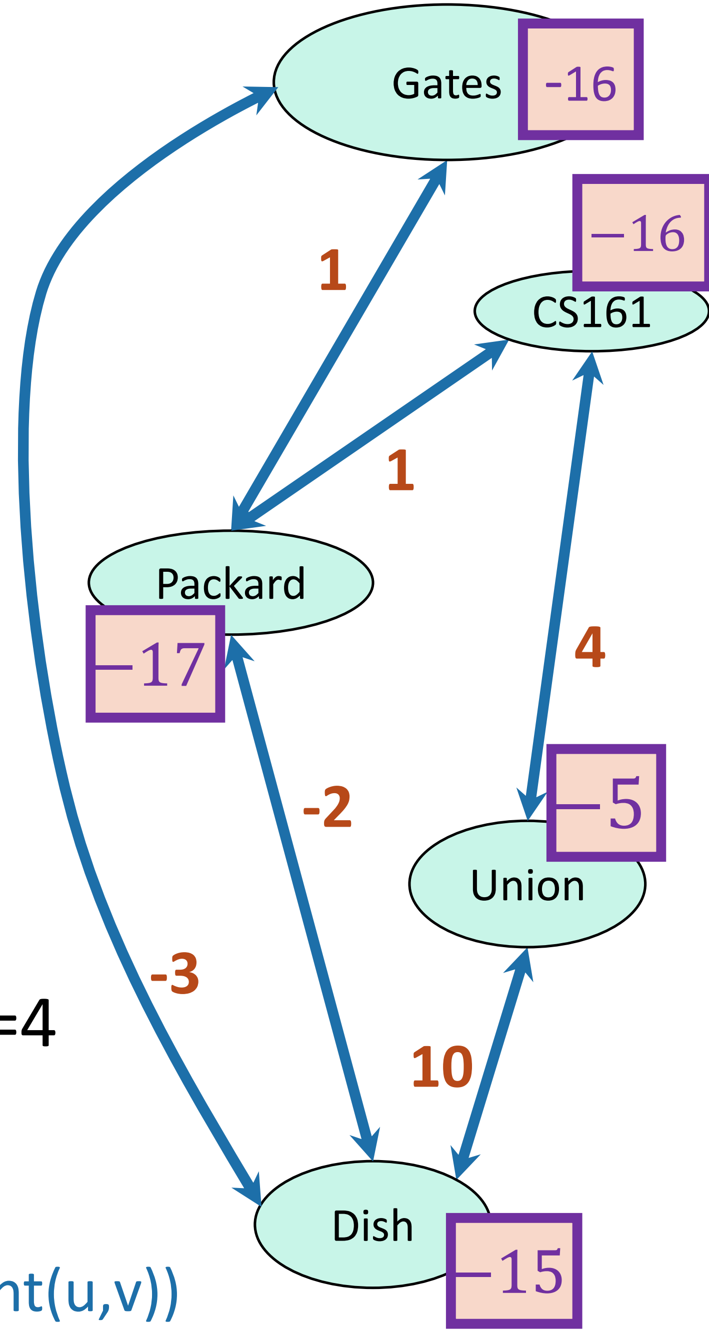
this will never converge.

After n-1 iterations, we stop and get something like this.



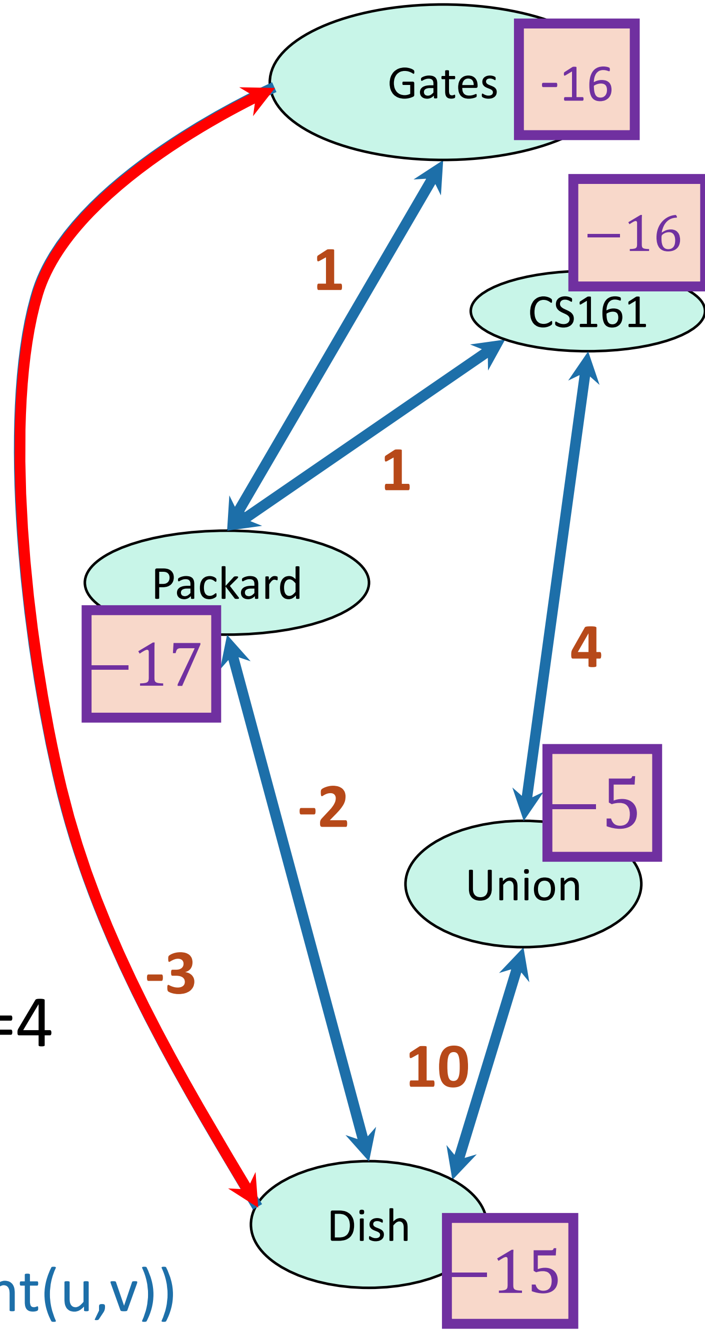
i=4

- **For** i = 1,..,n-1:
  - **For** each edge e = (u,v) in E:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))

How can we tell that this didn't work?

- If we had converged and the algorithm had worked, if we kept going to i=n, **nothing would happen**
- But if we keep going, then **something does happen.**

i=4



- **For** i = 1,..,n-1:
  - **For** each edge e = (u,v) in E:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))

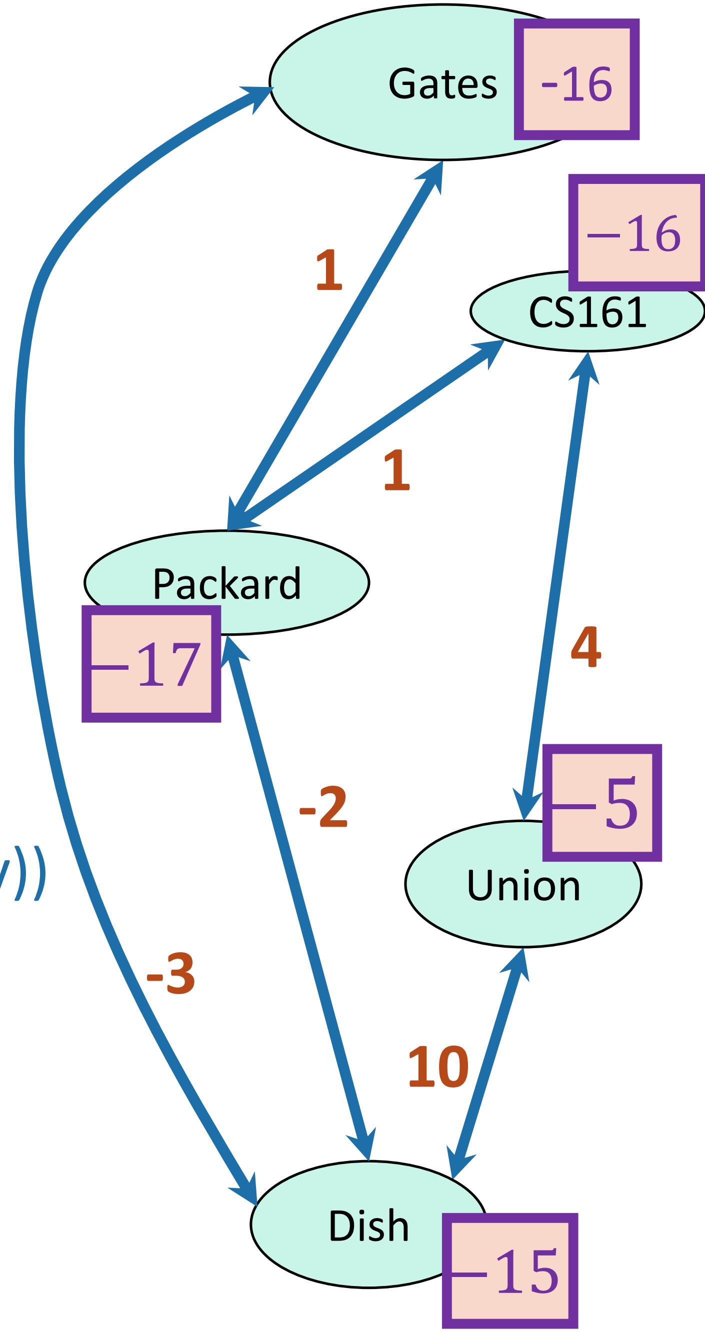
This suggests:

Bellman-Ford Algorithm:

- **For** v in V:
  - d[v] = ∞
- d[s] = 0

- **For** i = 1,..,n-1:
  - **For** each edge e = (u,v) in E:
    - d[v] ← min(d[v] , d[u] + weight(u,v))

- **For** each edge e = (u,v) in E:
  - **if** d[v] < d[u] + weight(u,v)):
    - return **negative cycle**



What have we just learned?

Theorem

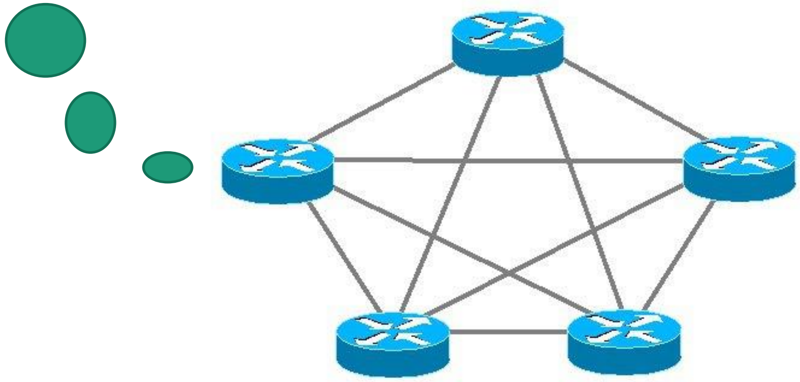
- The Bellman-Ford algorithm runs in time O(nm) on a graph G with n vertices and m edges.
- If there are no negative cycles in G, then the BF algorithm terminates with d[v] = d(s,v).
- If there are negative cycles in G, then the BF algorithm returns `negative cycle`.

Bellman-Ford is also used in practice.

- eg, Routing Information Protocol (RIP) uses something like Bellman-Ford.
  - Older protocol, not used as much anymore.

- Each router keeps a **table** of distances to every other router.
- Periodically we do a Bellman-Ford update.
- This means that if there are changes in the network, this will propagate. (maybe slowly…)

Destination	Cost to get there	Send to whom?
172.16.1.0	34	172.16.1.1
10.20.40.1	10	192.168.1.2
10.155.120.1	9	10.13.50.0



Recap: shortest paths

- BFS can do it in unweighted graphs
- In weighted graphs:
  - **Dijkstra's algorithm** is real fast but:
    - doesn't work with negative edge weights
    - is very "centralized"
  - **The Bellman-Ford algorithm** is slower but:
    - works with negative edge weights
    - can be done in a distributed fashion, every vertex using only information from its neighbors.

Mini-topic (if time)

Amortized analysis!

- We mentioned this when we talked about implementing Dijkstra.

*Any sequence of d `deleteMin` calls takes time at most O(d log(n)). But some of the d may take longer and some may take less time.

- What's the difference between this notion and expected runtime?

Example

- Incrementing a binary counter n times.

0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111
	1	2	1	3	1	2	1	4	1	2	1	3	1	2	1

- Say that flipping a bit is costly.
  - Above, we’ve noted the cost in terms of bit-flips.

Example

- Incrementing a binary counter n times.

0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111
	1	2	1	3	1	2	1	4	1	2	1	3	1	2	1

- Say that flipping a bit is costly.
  - Some steps are very expensive.
  - Many are very cheap.
- ***Amortized*** over all the inputs, it turns out to be pretty cheap.
  - O(n) for all n increments.

This is different from expected runtime.

- The statement is deterministic, no randomness here.

- But it is still weaker than worst-case runtime.
  - We may need to wait for a while to start making it worth it.

Recap

- BFS can do it in unweighted graphs
- In weighted graphs:
  - **Dijkstra's algorithm**
  - **The Bellman-Ford algorithm**

- One can implement Dijkstra's algorithm using a fancy data structure (a Fibonacci heap) so that it has good amortized time, **O(m + nlog(n))**.
  - And now we have a slightly better idea what amortized time means.