# Complex Nonlinearities for Audio Signal Processing

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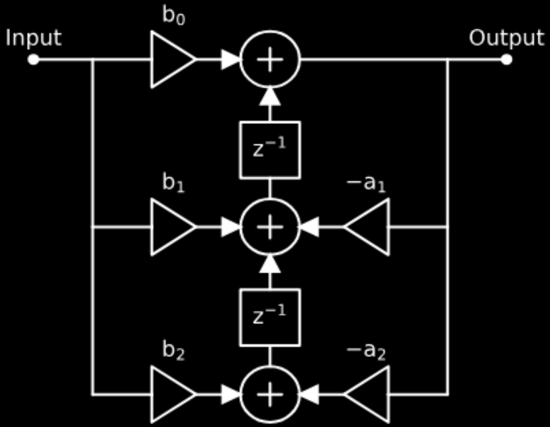
# Nonlinear Filters

# Biquad Filter

## Transposed Direct Form II



# Biquad Filter

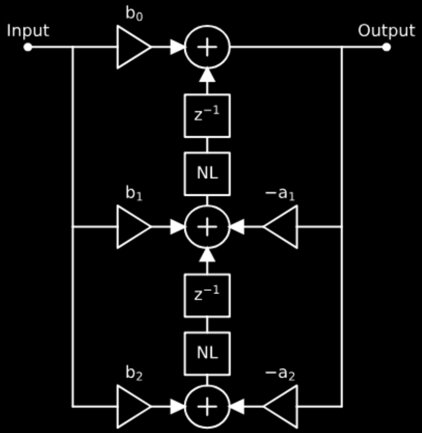
Difference equation:

$$y[n] = b_0u[n] + b_1u[n-1] + b_2u[n-2] - a_1y[n-1] - a_2y[n-2] \quad (1)$$

State space formulation:

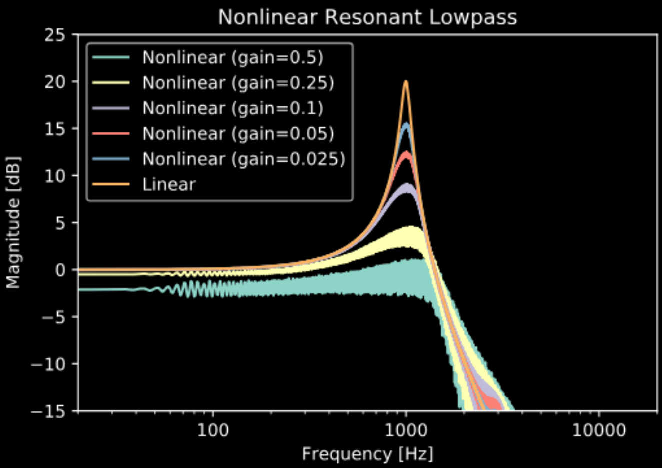
$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ y[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & -a_1 \\ 0 & 0 & -a_2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ y[n] \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_0 \end{bmatrix} u[n] \quad (2)$$

# Nonlinear Biquad Filter



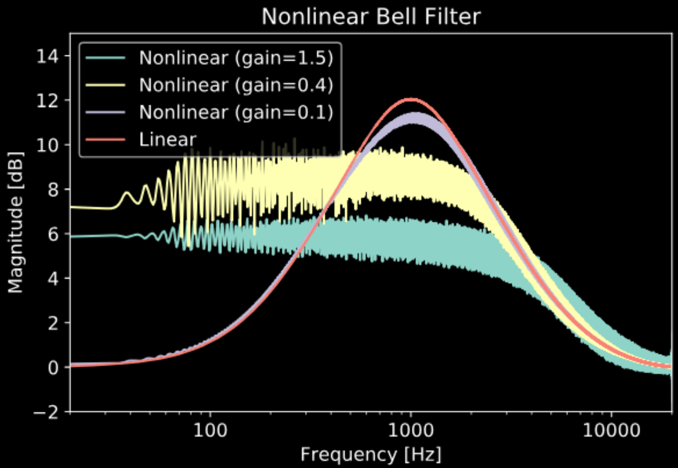
# Nonlinear Biquad Filter

Saturating nonlinearities $\rightarrow$ nonlinear resonance



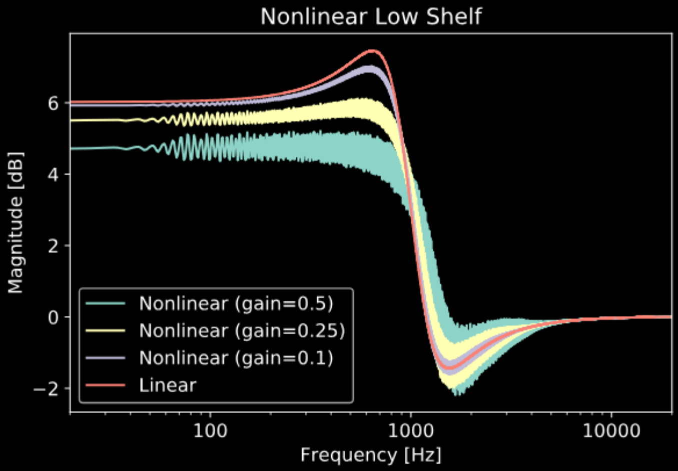
# Nonlinear Biquad Filter

Saturating nonlinearities $\rightarrow$ nonlinear resonance



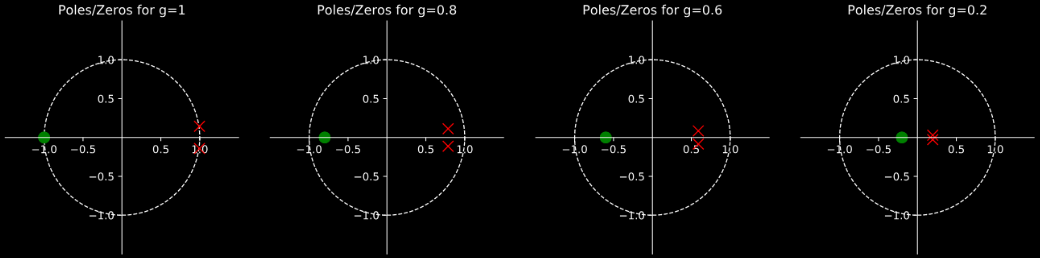
# Nonlinear Biquad Filter

Saturating nonlinearities $\rightarrow$nonlinear resonance

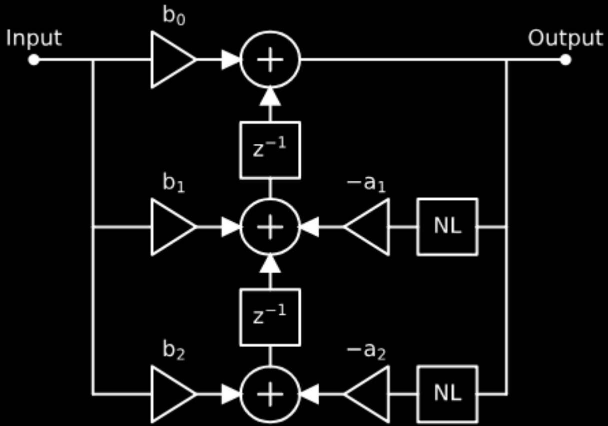


# Nonlinear Biquad Filter

## Pole/zero movement

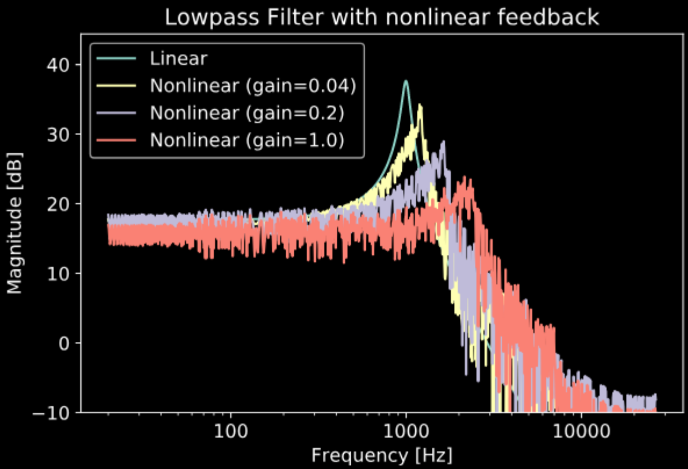


# Nonlinear Feedback Filter



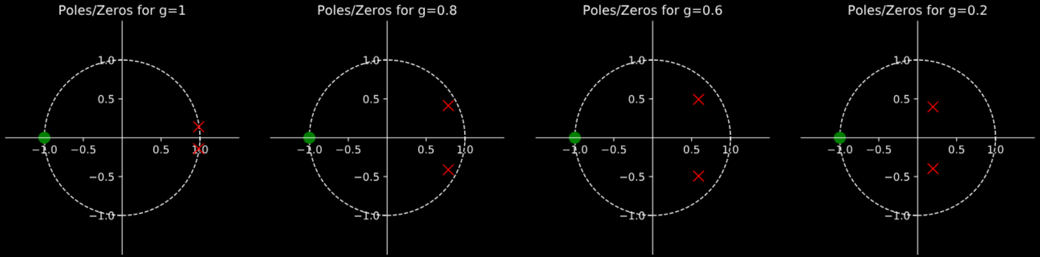
# Nonlinear Feedback Filter

## Saturating nonlinearity →cutoff frequency modulation



# Nonlinear Feedback Filter

## Pole/zero movement



# Nonlinear Biquad Stability

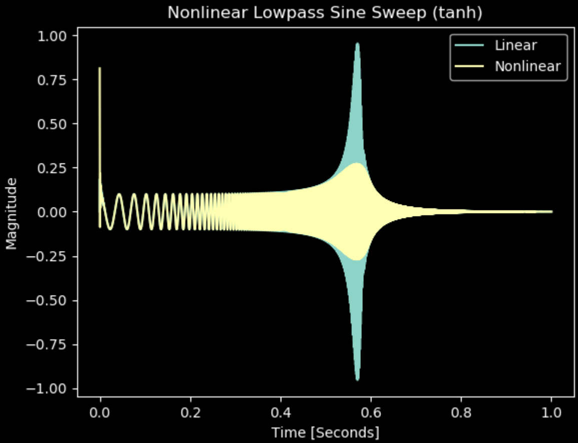
**Questions:**

Can we guarantee that a nonlinear filter will be stable given that its linear corrolary is stable?

For what subset of nonlinear functions is this guaranteed?

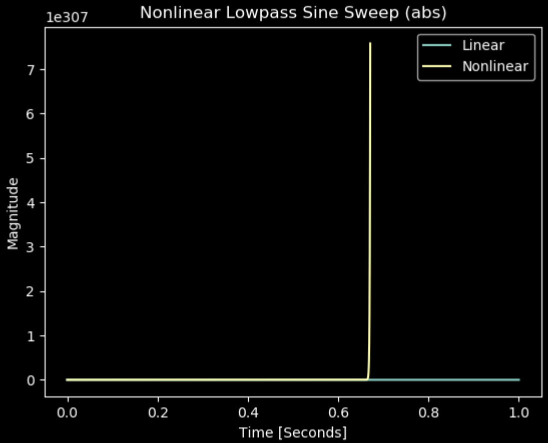
# Nonlinear Biquad Stability

Test case: saturating nonlinearity, $f_{NL} = \tanh(x)$ $\rightarrow$STABLE!



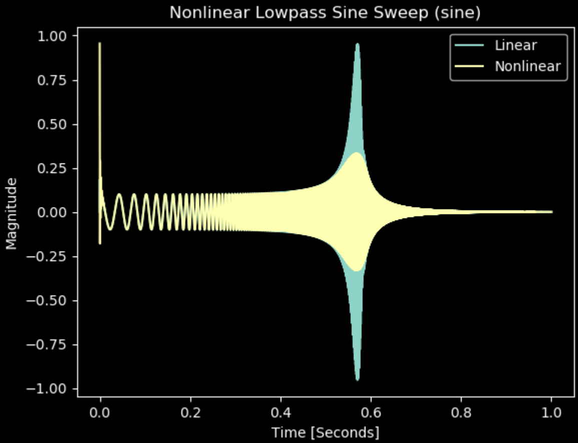
# Nonlinear Biquad Stability

Test case: full wave rectifier, $f_{NL} = 0.45|x|$ $\rightarrow$UNSTABLE!



# Nonlinear Biquad Stability

Test case: sine, $f_{NL} = \sin(x)$ →STABLE!



# Lyapunov Stability[1]

1. Form state space equation:

$$\mathbf{x}[n+1] = \mathbf{f}(\mathbf{x}[n]) \tag{3}$$

2. Find Jacobian $\mathbf{J}$ of $\mathbf{f}$

3. If every element of $\mathbf{J}$ is less than 1 at some operating point, the system is Lyapunov stable about that point.

[1] Chen, "Stability of Nonlinear Systems".

# Nonlinear Biquad Stability

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ y[n+1] \end{bmatrix} = \mathbf{h}\left(\begin{bmatrix} x_1[n] \\ x_2[n] \\ y[n] \end{bmatrix}\right) + \begin{bmatrix} b_1 \\ b_2 \\ b_0 \end{bmatrix} u[n] \tag{4}$$

$$\begin{aligned} h_1(x_1[n], x_2[n], y[n]) &= f_{NL}(x_2[n]) - a_1 y[n] \\ h_2(x_1[n], x_2[n], y[n]) &= -a_2 y[n] \\ h_3(x_1[n], x_2[n], y[n]) &= f_{NL}(x_1[n]) \end{aligned} \tag{5}$$

# Nonlinear Biquad Stability

$$\mathbf{J} = \begin{bmatrix} 0 & f'_{NL}(x_2[n]) & -a_1 \\ 0 & 0 & -a_2 \\ f'_{NL}(x_1[n]) & 0 & 0 \end{bmatrix} \tag{6}$$

Note that if $f'_{NL}$ does not exist at some point, the system is NOT stable at that point.

# Nonlinear Feedback Stability

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ y[n+1] \end{bmatrix} = \mathbf{h}\left(\begin{bmatrix} x_1[n] \\ x_2[n] \\ y[n] \end{bmatrix}\right) + \begin{bmatrix} b_1 \\ b_2 \\ b_0 \end{bmatrix} u[n] \tag{7}$$

$$\begin{aligned} h_1(x_1[n], x_2[n], y[n]) =& x_2[n] - a_1 f_{NL}(y[n]) \\ h_2(x_1[n], x_2[n], y[n]) =& -a_2 f_{NL}(y[n]) \\ h_3(x_1[n], x_2[n], y[n]) =& x_1[n] \end{aligned} \tag{8}$$

# Nonlinear Feedback Stability

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & -a_1 f'_{NL}(y[n]) \\ 0 & 0 & -a_2 f'_{NL}(y[n]) \\ 1 & 0 & 0 \end{bmatrix} \tag{9}$$

# Nonlinear Biquad Stability

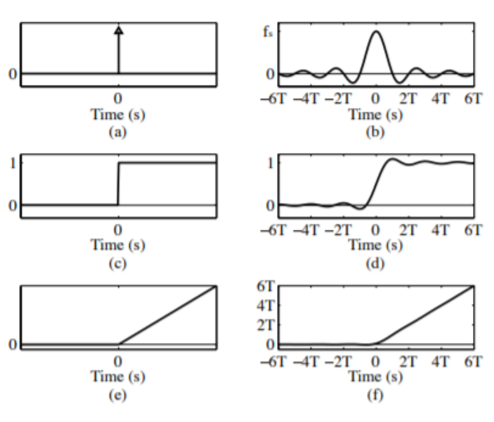
General stability contstraint:

$$|f'_{NL}(x)| \leq 1 \tag{10}$$

Note: if the $f'_{NL}(x)$ does not exist, the filter is not guaranteed stable.

# Nonlinear Biquad Stability

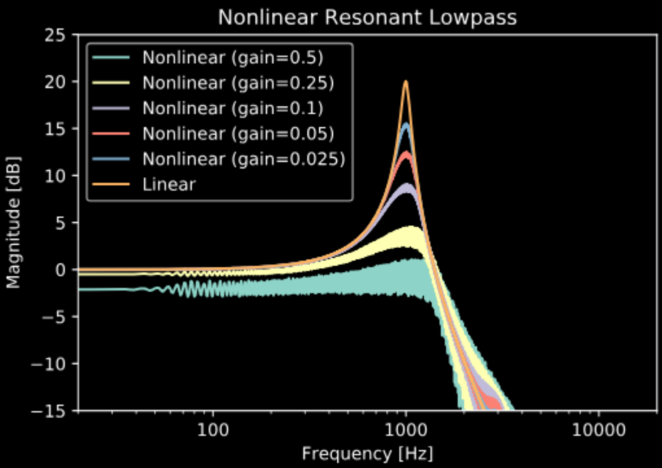
If derivative doesn't exist at every point: use BLAMP[2]!




[2] Esqueda, Valimaki, and Bilbao, "Rounding Corners with BLAMP".

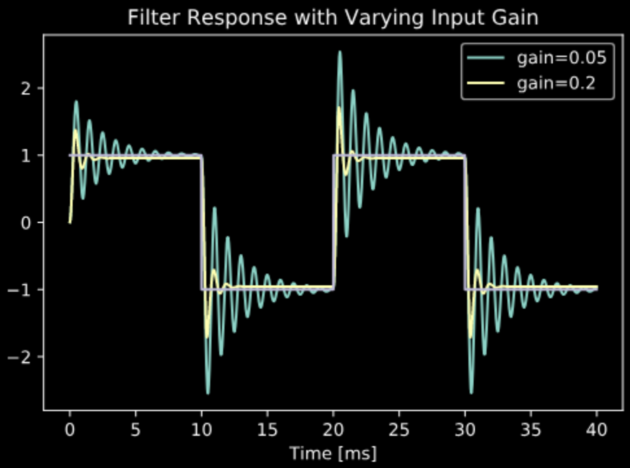
# Nonlinear Biquad Filter

Can we use this for analog modelling?



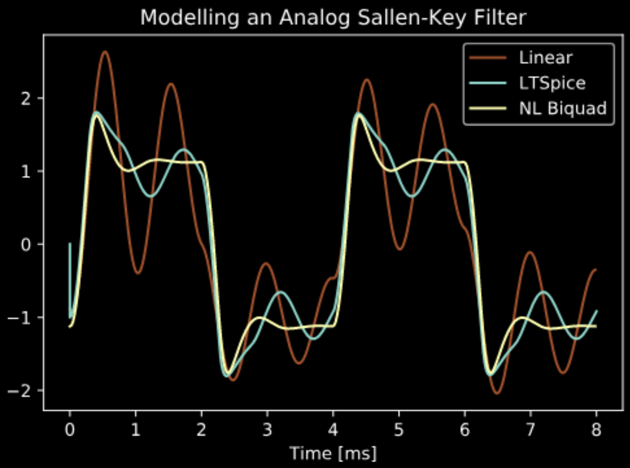
# Nonlinear Biquad Filter

Parameters: nonlinearities, input gain



# Nonlinear Biquad Filter

## Modelling an overdriven Sallen-Key lowpass filter



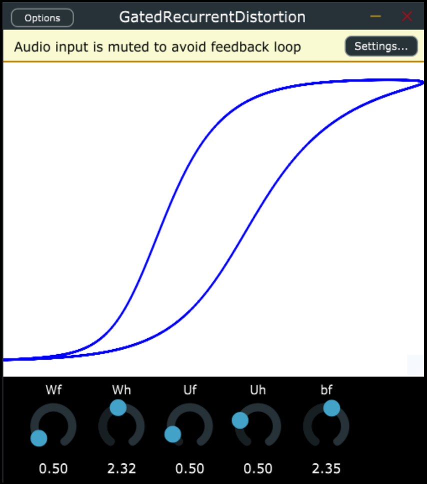
# Conclusion

# Goals

- Tools for musicians/mixing engineeers
- Inspiration/explanations for audio effect makers
- A academic paper (or two)

# Presentation

## Audio plugins (VST/AU)



# Presentation

## Medium articles

# Complex Nonlinearities Episode 4: Nonlinear Biquad Filters

Jatin Chowdhury
Oct 10, 2019 · 6 min read

For today's article, we'll be talking about filters. So far in this series I haven't spoken too much about filters, which might seem odd considering how much of signal processing in general is all about filters. The reason I've avoided filters is that most filters in audio signal processing are implemented as linear processors, and I've been focusing on nonlinear processing concepts.

# Presentation

Links:

- `https://github.com/jatinchowdhury18/ComplexNonlinearities`

- `https://medium.com/@jatinchowdhury18`

# Thank you!