# A STABLE STRUCTURE FOR NONLINEAR BIQUAD FILTERS

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## ABSTRACT

Biquad filters are a common tool for filter design. In this writing, we develop a structure for creating nonlinear biquad filters with guaranteed stability. We examine an example filter designed with this structure, and compare to a corresponding analog filter.

## 1. INTRODUCTION

A “biquad” filter refers to a general 2nd order IIR filter. In digital signal processing, biquad filters are often useful since any higher-order filter can be implemented using a cascade of biquad filters. While digital biquad filters are typically implemented as linear processors, for audio applications it can be useful to implement nonlinear filters. For example, in [1] the authors use a passive model of operational amplifiers to model the nonlinear behaviour of a Sallen-Key lowpass filter. In this writing, we strive to develop a more general nonlinear filter structure, one that can be used for analog modelling, but does not necessarily depend on analog modelling principles to be understood and implemented.

## 2. DEVELOPING THE STRUCTURE

### 2.1. Linear Filter

We begin with the equation for a biquad filter:

$$\begin{aligned} y[n] = b_0 u[n] + b_1 u[n-1] + b_2 u[n-2] \\ -a_1 y[n-1] - a_2 y[n-2] \end{aligned} \tag{1}$$

Where $y$ is the output signal, $u$ is the input signal, and $a_n$ and $b_n$ are the feed-back and feed-forward filter coefficients, respectively. There are several convenient "direct forms" for implementing biquad filters. In this writing we will focus on the “Transposed Direct Form II” (TDF-II), which is popular for its numerical properties [2]. Note that the poles of the filter can be described using the quadratic equation.

$$p = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \tag{2}$$

### 2.2. Nonlinear Elements

We now propose adding nonlinear elements to the above filter structure. We will refer to these nonlinear elements as “base nonlinearities” To keep the discussion as broad as possible, we consider any one-to-one nonlinear function $f_{NL}(x)$. It will often be convenient to consider the input-dependent “gain” of the nonlinear function:

$$g_{NL}(x) = \frac{f_{NL}(x)}{x} \tag{3}$$

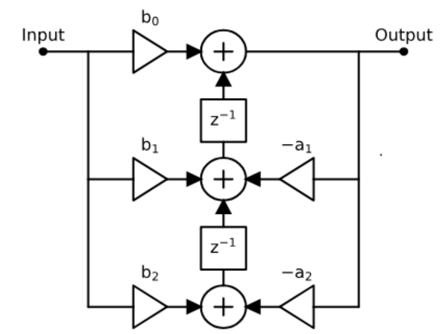


Figure 1: *Transposed Direct Form II*

In order to maintain stability, we propose the following constraint on any nonlinear functions used in these filters:

$$0 \leq g_{NL}(x) \leq 1 \tag{4}$$

Many musical nonlinearities satisfy this constraint, including most saturating, dropout, and half-wave rectifying nonlinearities. Of particular interest to us will be saturating nonlinearities, including hard-clippers, soft-clippers, and sigmoid-like functions (see fig. 2). Saturating nonlinearities satisfy the property that

$$|x| \to \infty, \; g_{sat}(x) \to 0 \tag{5}$$

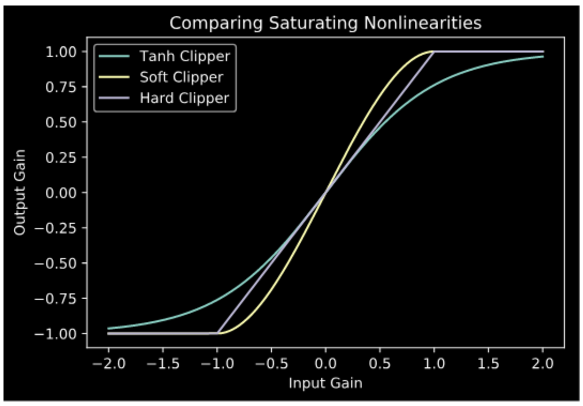


Figure 2: *Saturating Nonlinearities*

### 2.3. Nonlinear Filter Structure

We now propose adding nonlinear elements to the TDF-II strcuture in the following fashion (see fig. 3). The equation for the nonlinear biquad filter then becomes:

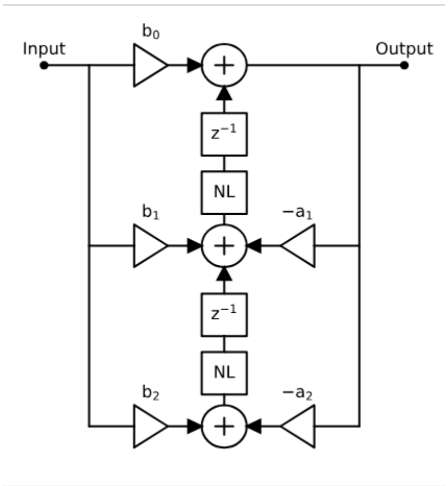


Figure 3: *Nonlinear Transposed Direct Form II. The "NL" blocks refer to a generalized nonlinear element.*

$$y[n] = b_0 u[n] + f_{NL}(b_1 u[n-1] - a_1 y[n-1] + f_{NL}(b_2 u[n-2] - a_2 y[n-2])) \tag{6}$$

Here it can be useful to define filter "state variables"

$$\begin{aligned} x_1 &= f_{NL}(x_2) + b_1 u[n-1] - a_1 y[n-1] \\ x_2 &= b_2 u[n-2] - a_2 y[n-2] \end{aligned} \tag{7}$$

Note that for saturating base nonlinearities, as the input $u$ grows large, the other terms will become negigible.

Now we can replace the nonlinear elements with their input-dependent gains:

$$\begin{aligned} g_1 &= \frac{f_{NL}(x_1)}{x_1} \\ g_2 &= \frac{f_{NL}(x_2)}{x_2} \end{aligned} \tag{8}$$

Then eq. (6) can be re-written:

$$y[n] = b_0 u[n] + g_1(b_1 u[n-1] - a_1 y[n-1] + g_2(b_2 u[n-2] - a_2 y[n-2])) \tag{9}$$

Finally, we can re-write the filter coefficients as variables dependent on state variables:

$$\begin{aligned} b_0' &= b_0 \\ b_1' &= g_1 b_1 \\ b_2' &= g_1 g_2 b_2 \\ a_1' &= g_1 a_1 \\ a_2' &= g_1 g_2 a_2 \end{aligned} \tag{10}$$

$$y'[n] = b_0' u[n] + b_1' u[n-1] - a_1' y[n-1] + b_2' u[n-2] - a_2' y[n-2] \tag{11}$$

### 2.4. Guaranteed Stability

Since the coefficients of the biquad filter will be dependent on the state of the filter, the instantaneous poles of the filter will be dependent as well. In order to calculate the instantaneous poles of the nonlinear biquad structure, we can adjust the formula from eq. (2).

$$p = \frac{-g_1 a_1 \pm \sqrt{g_l^2 a_1^2 - 4 g_1 g_2 a_2}}{2} \tag{12}$$

From the constraint in eq. (4), we can see that the magnitude of the instantaneous pole will always be less than or equal to the magnitude of the original pole from the corresponding linear filter. Therefore, we know that any nonlinear biquad filter designed with the above structure will be stable provided that the corresponding linear filter is stable, and that the constraint from eq. (4) is satisfied.

For saturating base nonlinearities, we can see from eq. (5) that as the state variables grow large, the poles will go to zero.

## 3. EXAMPLE: RESONANT LOWPASS FILTER

As an example of the nonlinear biquad structure developed in the previous section, we will now examine a resonant lowpass filter designed with the nonlinear structure, and compare to a corresponding analog filter.

### 3.1. Digital Nonlinear Filter

Our example filter will be a lowpass filter with a cutoff frequency at $f_c = 1$ kHz, and $Q = 10$. For our nonlinear elements we will use a hyperbolic tangent function $f_{NL}(x) = \tanh(x)$. Note that this nonlinear function belongs to the class of saturating nonlinearities described by eq. (5).

In fig. 4 we show the response of this filter for sine sweeps of various amplitudes, compared to the frequency response of the corresponding linear filter. In fig. 5 we show the movement of the poles and zeros of the filter for varying steady state inputs. We calculate the instantaneous poles using eq. (12), using $g_1 = g_2 = g$, as described in each figure.

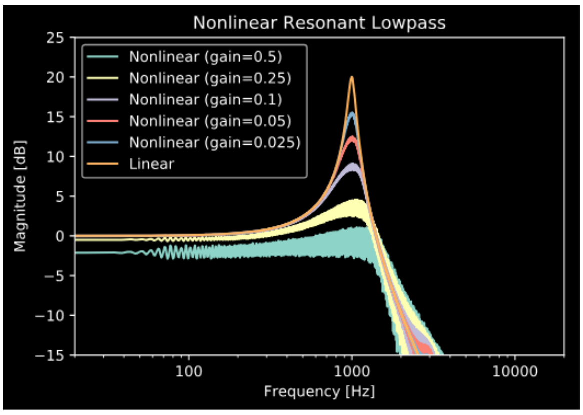


Figure 4: *Frequency responses of nonlinear lowpass filters at varying amplitudes.*

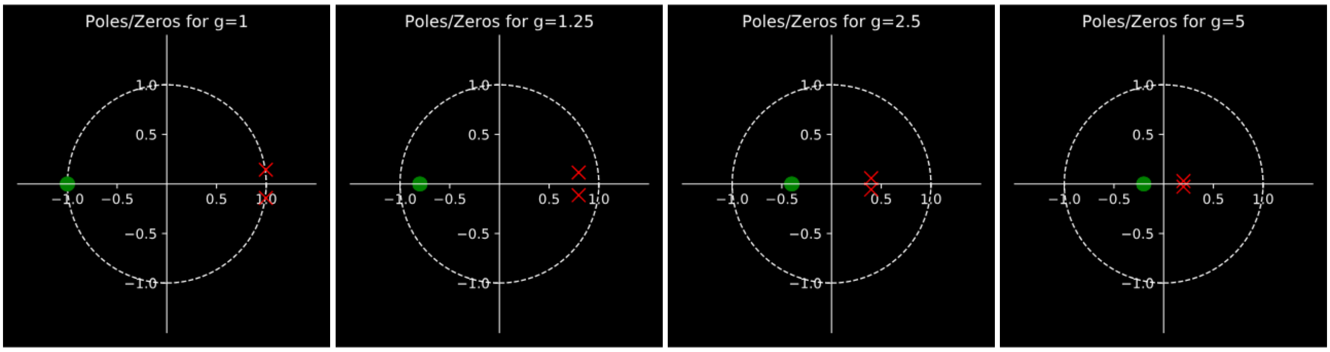


Figure 5: *Instantaneous poles for a nonlinear resonant lowpass filter at varying input levels.*

### 3.2. Comparison with Analog Filter

To show how the nonlinear filter structure can be useful for analog modelling purposes, we compare the behavior of the resonant lowpass filter with that of an analog Sallen-Key lowpass filter, similar to the comparison done in [1].

First, we note that the input gain to the nonlinear biquad can be used as a tunable parameter (see fig. 6). By tuning the input gain,

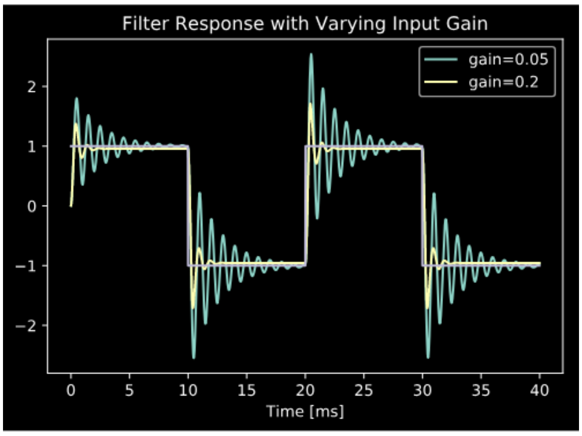


Figure 6: *Response of the nonlinear resonant lowpass filter to a 50 Hz square wave with varying input gain.*

we can attempt to match the response of an arbitrary analog filter, either by tuning the parameters by ear or using some form of numerical optimisation. Note that the choice of base nonlinearities used by the nonlinear biquad will also play a role in the accuracy of the model. For example, if the output of the analog filter being modelled is asymmetric, then to accurately model that filter, the nonlinear biquad must be constructed using an asymmetric base nonlinearities.

As an example, we can attempt to construct a naive model of Sallen-Key lowpass filter, a commonly used analog filter structure. We describe this as a naive model because we do not make any attempt to understand the physical properties of the analog filter when constructing this model. We construct a nonlinear biquad filter using $\tanh$ base nonlinearities, and design a resonant lowpass filter with cutoff frequency $f_c = 1$ kHz, and $Q = 10$, as well as a simulation of the corresponding Sallen-Key filter using LTSpice. To accentuate the nonlinear behavior of the analog filter, we choose $\pm 4$ V as the source voltages for the analog filter circuit. We then compare the outputs of the two filters for square waves at different frequencies, and use a simple staircase optimisation scheme to find the input gain for the nonlinear biquad that best matches the analog simulation. The results for the 250 Hz square wave can be seen in fig. 7. While nonlinear biquad model is not perfect, it does capture the damping effects of the analog filter much more accurately than the corresponding linear filter, and can be greatly improved with a more well-informed choice of base nonlinear functions, and a more sophisticated optimisation scheme.

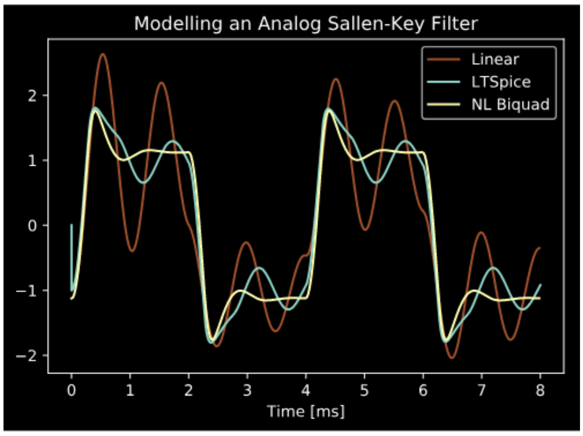


Figure 7: *Comparison between a linear resonant lowpass filter, a resonant lowpass made with a nonlinear biquad using a* $\tanh$ *clipper with input gain* $0.283$*, and a SPICE simulation of a Sallen-Key lowpass. All of the lowpass filters have* $f_c = 1$ *kHz, and* $Q = 10$*. The input signal in each case is a* $250$ *Hz square wave.*

## 4. CONCLUSION

In this paper, we have developed a structure for stable nonlinear biquad filters. We have introduced the new architecture as a modification of the Transposed Direct Form II filter form, and shown mathematically how the changed architecture affects the pole locations depending on the amplitude of the input signal. We have also derived constraints under which the structure is guaranteed stable.

As an example of the nonlinear biquad filter structure we have implemented a nonlinear resonant lowpass filter, and shown that the poles respond to the input as expected. We then show how the structure can be used to model an analog filter, using a Sallen-Key lowpass filter as an example. Note that while the nonlinear biquad structure can be used for analog modelling it can also be used purely in the digital domain, as a tool for constructing filters that sound more harmonically rich.

To demonstrate this last point, we have also developed an open-source audio plugin (VST, AU) implentation of the nonlinear biquad filter, extending to several filter shapes, and several base nonlinearities. The source code for the plugin implementation is available on GitHub[1].

Future research concerning nonlinear filtering will center around making a more informed choice of base nonlinearities, focusing on both the desired harmonic response of the filter, as well as physically meaningful base nonlinearities for use in analog modelling.

## 5. REFERENCES


[1] Remy Muller and Thomas Helie, "A minimal passive model of the operational amplifier: Application to sallen-key analog filters," in *Proc. of the 22nd Int. Conference on Digital Audio Effects*, 2019.

[2] Julius O. Smith, *Introduction to Digital Filters with Audio Applications*, W3K Publishing, http://www.w3k.org/books/, 2007.


[1] `https://github.com/jatinchowdhury18/ComplexNonlinearities`