Complex Nonlinearities for Audio Signal Processing

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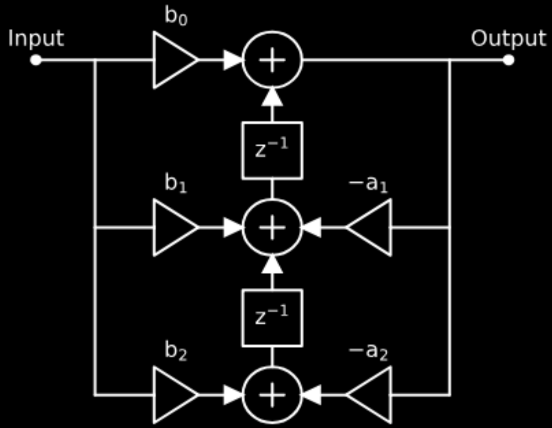
Nonlinear Filters

Biquad Filter

Transposed Direct Form II



Biquad Filter

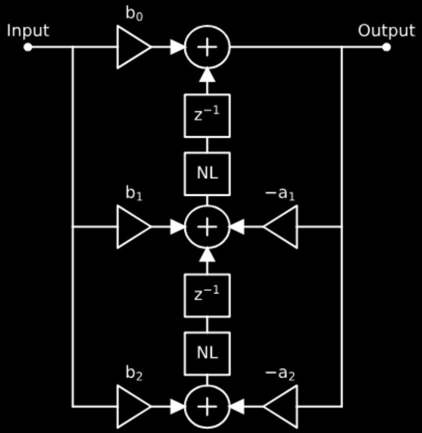
Difference equation:

$$y[n] = b_0 u[n] + b_1 u[n-1] + b_2 u[n-2] - a_1 y[n-1] - a_2 y[n-2] \quad (1)$$

State space formulation:

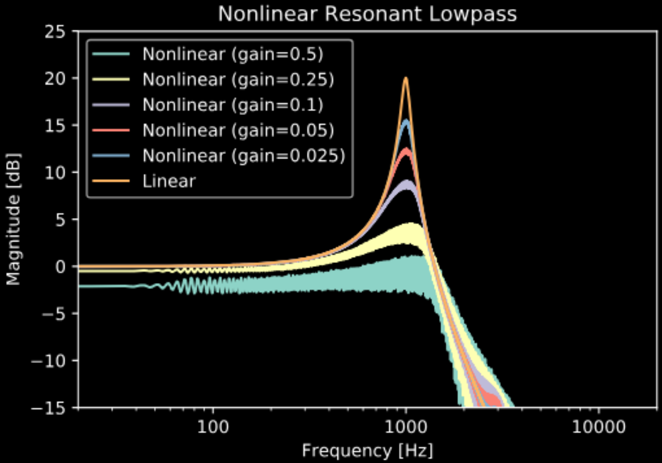
$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ y[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & -a_1 \\ 0 & 0 & -a_2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ y[n] \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_0 \end{bmatrix} u[n] \quad (2)$$

Nonlinear Biquad Filter



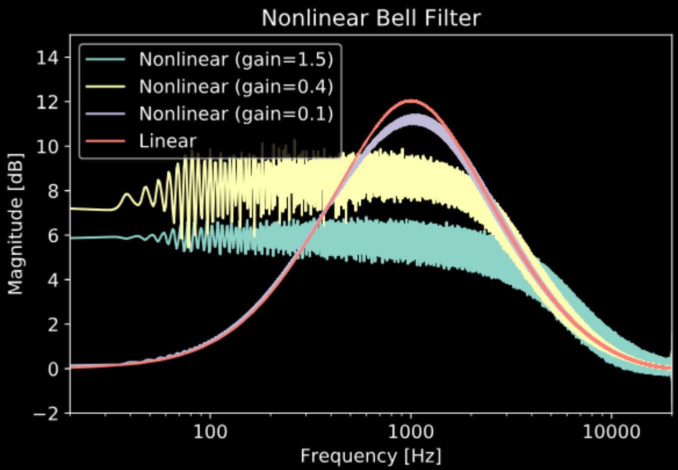
Nonlinear Biquad Filter

Saturating nonlinearities →nonlinear resonance



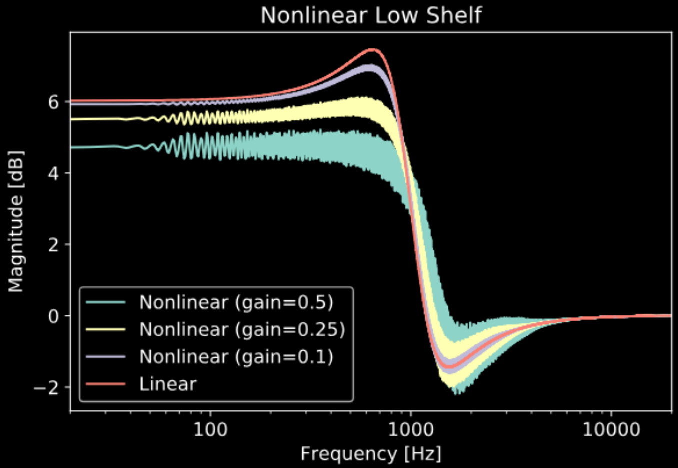
Nonlinear Biquad Filter

Saturating nonlinearities →nonlinear resonance



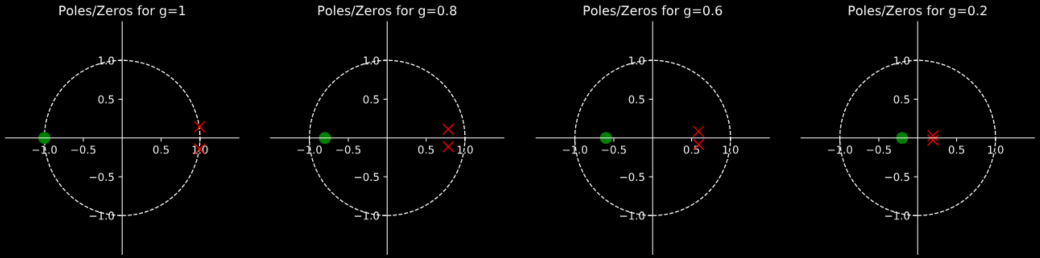
Nonlinear Biquad Filter

Saturating nonlinearities $\rightarrow$ nonlinear resonance

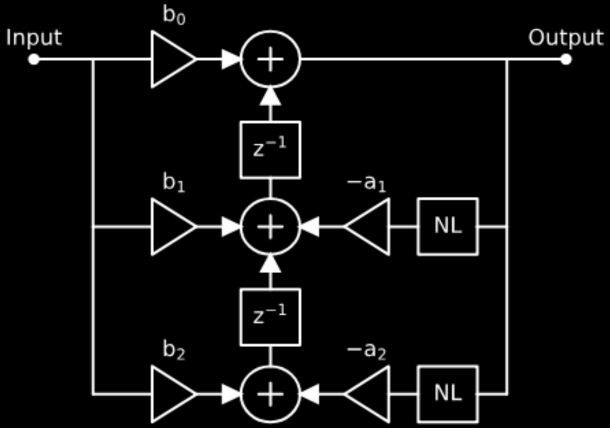


Nonlinear Biquad Filter

Pole/zero movement

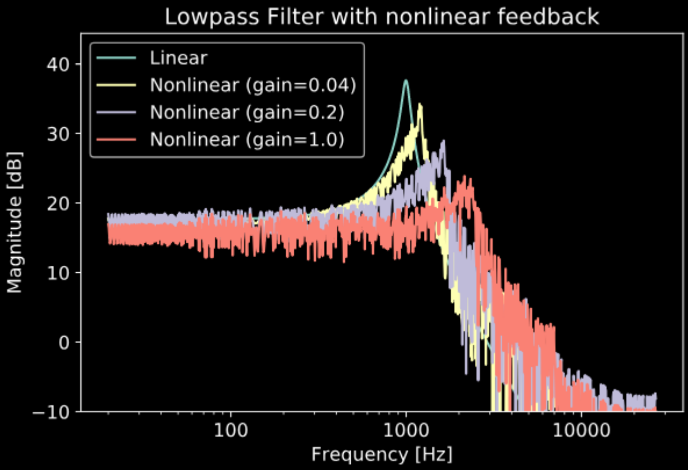


Nonlinear Feedback Filter



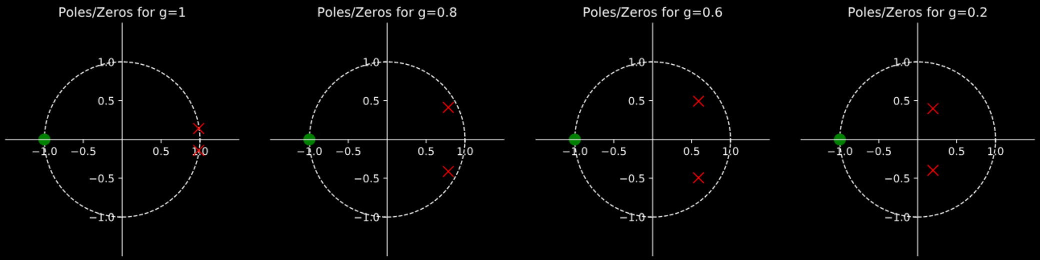
Nonlinear Feedback Filter

Saturating nonlinearity $\rightarrow$ cutoff frequency modulation



Nonlinear Feedback Filter

Pole/zero movement



Nonlinear Biquad Stability

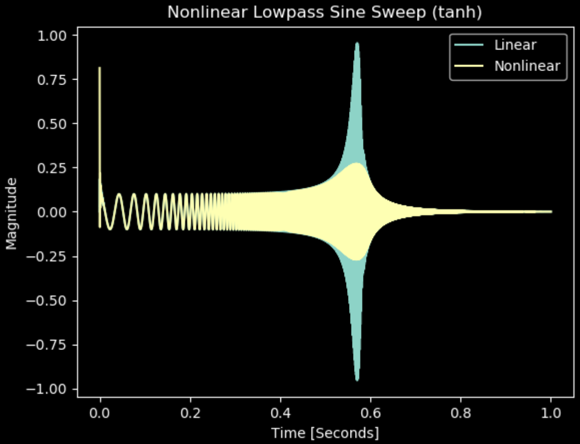
Questions:

Can we guarantee that a nonlinear filter will be stable given that its linear corrolary is stable?

For what subset of nonlinear functions is this guaranteed?

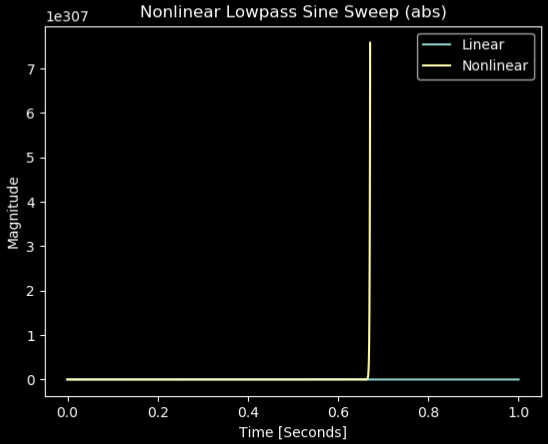
Nonlinear Biquad Stability

Test case: saturating nonlinearity, $f_{NL} = \tanh(x)$ $\rightarrow$STABLE!



Nonlinear Biquad Stability

Test case: full wave rectifier, $f_{NL} = 0.45|x|$ →UNSTABLE!



Lyapunov Stability[1]

1. Form state space equation:

$$\mathbf{x}[n+1] = \mathbf{f}(\mathbf{x}[n]) \tag{3}$$

2. Find Jacobian $\mathbf{J}$ of $\mathbf{f}$

3. If every element of $\mathbf{J}$ is less than 1 at some operating point, the system is Lyapunov stable about that point.

[1] Chen, “Stability of Nonlinear Systems”.

Nonlinear Biquad Stability

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ y[n+1] \end{bmatrix} = \mathbf{h}\left(\begin{bmatrix} x_1[n] \\ x_2[n] \\ y[n] \end{bmatrix} \right) + \begin{bmatrix} b_1 \\ b_2 \\ b_0 \end{bmatrix} u[n] \tag{4}$$

$$\begin{aligned} h_1(x_1[n], x_2[n], y[n]) =& f_{NL}(x_2[n]) - a_1 y[n] \\ h_2(x_1[n], x_2[n], y[n]) =& -a_2 y[n] \\ h_3(x_1[n], x_2[n], y[n]) =& f_{NL}(x_1[n]) \end{aligned} \tag{5}$$

Nonlinear Biquad Stability

$$\mathbf{J} = \begin{bmatrix} 0 & f'_{NL}(x_2[n]) & -a_1 \\ 0 & 0 & -a_2 \\ f'_{NL}(x_1[n]) & 0 & 0 \end{bmatrix} \tag{6}$$

Note that if f'_{NL} does not exist at some point, the system is NOT stable at that point.

Nonlinear Feedback Stability

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ y[n+1] \end{bmatrix} = \mathbf{h}\left(\begin{bmatrix} x_1[n] \\ x_2[n] \\ y[n] \end{bmatrix} \right) + \begin{bmatrix} b_1 \\ b_2 \\ b_0 \end{bmatrix} u[n] \tag{7}$$

$$\begin{aligned} h_1(x_1[n], x_2[n], y[n]) =& x_2[n] - a_1 f_{NL}(y[n]) \\ h_2(x_1[n], x_2[n], y[n]) =& - a_2 f_{NL}(y[n]) \\ h_3(x_1[n], x_2[n], y[n]) =& x_1[n] \end{aligned} \tag{8}$$

Nonlinear Feedback Stability

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & -a_1 f'_{NL}(y[n]) \\ 0 & 0 & -a_2 f'_{NL}(y[n]) \\ 1 & 0 & 0 \end{bmatrix} \tag{9}$$

Nonlinear Biquad Stability

General stability contstraint:

$$|f'_{NL}(x)| \leq 1 \tag{10}$$