Thesis topics

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1 Turing machines and dynamical systems

The basis are the two articles

C. Moore, *Unpredictability and undecidability in dynamical systems*, Phys. Rev. Lett. **64** (1990), 2354–2357.

C. Moore, *Generalized shifts: unpredictability and undecidability in dynamical systems*, Nonlinearity **4** (1991), 199–230.

In these articles, Moore shows an equivalence between Turing machines (viewed as dynamical systems) and generalized shifts. By embedding generalized shifts into other systems, he obtains smooth dynamical systems that can simulate Turing machines. Now the undecidability of the halting problem for Turing machines implies that such dynamical systems are unpredictable in a stronger sense than being chaotic.

It is an interesting question which systems occuring in nature exhibit Turing unpredictability. Moore mentions in his second article several such systems: cellular automata, neural networks, gases of hard spheres, electrical circuits, and billiards. It is a topic of ongoing research whether Turing unpredictability occurs in the N-body problem of celestial mechanics, or in the Euler or Navier-Stokes equations of hydrodynamics. For the latter see

R. Cardona, E. Miranda, D. Peralta-Salas, F. Presas, *Constructing Turing complete Euler flows in dimension 3*, Proc. Natl. Acad. Sci. USA **118** (2021), no. 19, Paper No. e2026818118.

R. Cardona, E. Miranda, D. Peralta-Salas, *Rechnende Flüssigkeiten*, Spektrum der Wissenschaft, Februar 2023.

The goal is to understand Moore's articles and present them with all details and the required background on Turing machines and dynamical systems.

2 Probing Viterbo's conjecture

To each star-shaped (with respect to the origin) compact domain $X \subset \mathbb{R}^{2n}$ with smooth boundary one can associate its *systolic ratio*

$$\mathrm{sys}(X) := \frac{\mathcal{A}_{\min}(X)^n}{n!\mathrm{vol}(X)}.$$

Here $\mathrm{vol}(X)$ is the volume of X and $\mathcal{A}_{\min}(X)$ is the minimal action $\mathcal{A}(\gamma) = \int_\gamma \lambda_{\mathrm{st}}$ of a closed Reeb orbit γ on ∂X. This notion extends to the case that X has piecewise smooth boundary, see [1].

Viterbo's conjecture asserts that $\mathrm{sys}(X) \leq 1$ if X is convex. It was disproved in [2] by producing an explicit polytope $P \subset \mathbb{R}^4$ with $\mathrm{sys}(P) > 1$. Nonetheless, it remains an interesting question to understand better how the systolic ratio behaves on the space of convex polytopes. For example, Viterbo's conjecture for Lagrangian products of a centrally symmetric convex body in $\mathbb{R}^n$ with its dual is equivalent to the famous *Mahler conjecture* in convex geometry, which is proved for $n \leq 3$ and open for $n \geq 4$.

The task in this project is to write an efficient computer program for computing the systolic ratio of polytopes in $\mathbb{R}^4$, and use it to probe more precisely where and how Viterbo's conjecture fails. One should also use machine learning to determine the systolic ratio more efficiently and predict regions in the landscape of polytopes where Viterbo's conjecture is likely to hold or fail.

Literature:

[1] Julian Chaidez and Michael Hutchings, *Computing Reeb dynamics on 4d convex polytopes*, arXiv:2008.10111.

[2] Pazit Haim-Kislev and Yaron Ostrover, *A Counterexample to Viterbo's Conjecture*, arXiv:2405.16513.