Clarke's Dual Action Principle

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Symplectic Geometry Seminar

Outline

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Symplectic Geometry Refresh

Standard Setting

- 4-dimensional symplectic vector space $\mathbb{R}^4$, coordinates $x = (q_1, q_2, p_1, p_2)$.
- Symplectic form

$$\omega_0 = \sum_{i=1}^{2} dq_i \wedge dp_i, \quad \omega_0(u, v) = \langle Ju, v \rangle,$$

- Almost complex structure

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}, \quad J^2 = -I_4, \quad J((q, p)) = (-p, q).$$

- Liouville 1-form

$$\lambda_0 = \tfrac{1}{2}\langle Jx, dx \rangle = \tfrac{1}{2}\sum_{i=1}^{2}(p_i \, dq_i - q_i \, dp_i), \quad d\lambda_0 = \omega_0.$$

- Higher dimensions $\mathbb{R}^{2n}$ work analogously.

Loops, Action, Hamiltonians

- Hamiltonian function $H : \mathbb{R}^4 \to \mathbb{R}$.
- Hamiltonian vector field $X_H = J\nabla H$.
- Hamiltonian orbit: periodic solution $\gamma : [0, T] \to \mathbb{R}^4$ of

$$\dot{\gamma}(t) = X_H(\gamma(t)) = J\nabla H(\gamma(t)).$$

- Preserves energy levels: $H(\gamma(t)) \equiv \text{const}.$
- Symplectic action of a loop $\gamma : [0, T] \to \mathbb{R}^4$:

$$A(\gamma) = \int_\gamma \lambda_0 = \int_0^T \lambda_0(\dot{\gamma}(t)) \, dt = \tfrac{1}{2} \int_0^T \langle -J\dot{\gamma}(t), \gamma(t) \rangle \, dt.$$

- Per Stokes' theorem, if γ bounds a disk $u : D^2 \to \mathbb{R}^4$, then

$$A(\gamma) = \int_{D^2} u^* \omega_0 = \int_{D^2} \omega_0(\partial_x u, \partial_y u) \, dx \, dy.$$

Reeb flow

- Smooth star-shaped hypersurface $\Sigma \subset \mathbb{R}^4$. E.g., energy surface $\Sigma = \{H = 1\}$ for certain H.
- Contact form: $\alpha = \lambda_0|_\Sigma$.
- Reeb vector field R defined by

$$\iota_R d\alpha = 0, \qquad \alpha(R) = 1.$$

- A Reeb orbit is a periodic solution $\gamma : [0, T] \to \Sigma$ of

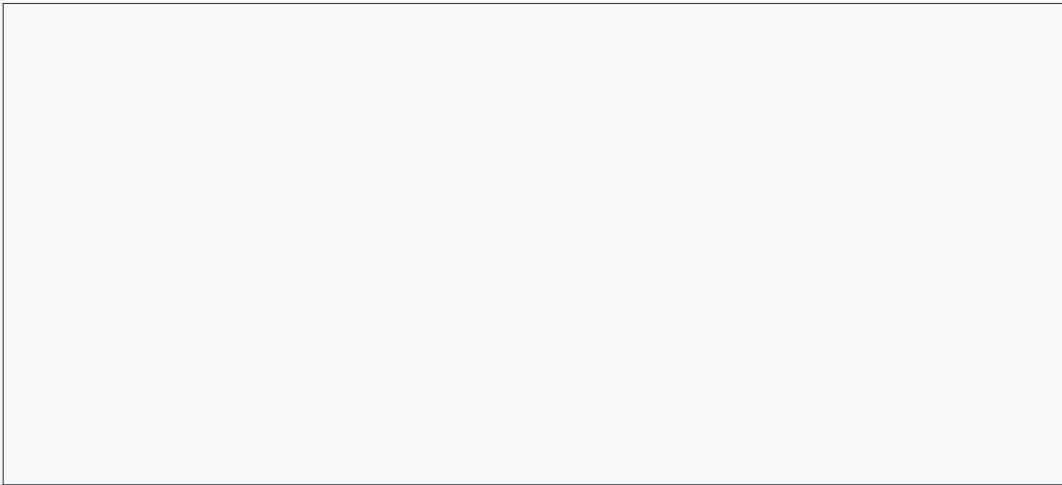$$\dot{\gamma} = R(\gamma).$$

- Useful fact: $A(\gamma) = \int_0^T \alpha(R(\gamma(t))) \, dt = T$.

Closed Characteristics

Hamiltonian orbit	Reeb orbit	Closed characteristic
$\gamma : [0, T] \to \mathbb{R}^4$	$\gamma : [0, T] \to \Sigma$	$\gamma : [0, 1] \to \Sigma$
$\dot{\gamma}(t) = J\nabla H(\gamma(t))$	$\dot{\gamma}(t) = R(\gamma(t))$	$\dot{\gamma}(t) \in \mathbb{R}_+ R(\gamma(t))$
$H(\gamma) \equiv \mathrm{const}$	$\gamma(t) \in \Sigma$	$\gamma(t) \in \Sigma$
$A(\gamma) = T$	$A(\gamma) = T$	-

- We exclude constant loops with $T = 0, A(\gamma) = 0$ from all three definitions.

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Viterbo's Conjecture

EHZ capacity / Minimum action

- Fix a convex body $K \subset \mathbb{R}^4$ (compact, convex, $0 \in \mathrm{int}(K)$).
- So far: smooth case, but we will generalize later.
- Define the Ekeland–Hofer–Zehnder capacity as

$$c_{\mathrm{EHZ}}(K) = A_{\mathsf{min}}(K) := \min\{A(\gamma) \colon \gamma \text{ closed characteristic on } \partial K\}.$$

- Note: we do not take the minimum over all loops, only over closed characteristics.
- This minimum is attained.
- Note: $c_{\mathrm{EHZ}}(K) > 0$, since we exclude constant loops, and if γ is a closed characteristic, then $A(\gamma) = T > 0$.

Symplectic Capacity
The map $c_{\mathrm{EHZ}} : K \mapsto \mathbb{R}_+$ is a symplectic capacity, i.e.:

1. (Monotonicity) If there exists a symplectic embedding $K_1 \to K_2$, then
   $c_{\mathrm{EHZ}}(K_1) \leq c_{\mathrm{EHZ}}(K_2)$.

2. (Conformality) For $\alpha \neq 0$, $c_{\mathrm{EHZ}}(\alpha K) = |\alpha|^2 c_{\mathrm{EHZ}}(K)$.

3. (Normalization) $c_{\mathrm{EHZ}}(B^4(1)) = c_{\mathrm{EHZ}}(Z^4(1)) = \pi$, where

$$B^4(1) = \{x \in \mathbb{R}^4 : |x| \leq 1\}, \qquad Z^4(1) = \{(q_1, q_2, p_1, p_2) : q_1^2 + p_1^2 \leq 1\}.$$

Viterbo's Conjecture

Viterbo's Conjecture
For any convex body $K \subset \mathbb{R}^{2n}$:

$$\mathrm{sys}(K) := \frac{c_{\mathrm{EHZ}}(K)^n}{n!\,\mathrm{vol}(K)} \leq 1.$$

Strong Viterbo Conjecture
All normalized symplectic capacities agree on convex domains. In particular, the EHZ capacity, the Gromov width, and the cylindrical capacity coincide.

$$c_{\mathrm{Gromov}}(K) := \sup\{\pi r^2 : B^{2n}(r) \xrightarrow{\text{symp.}} K\}$$

$$c_{\mathrm{cylindrical}}(K) := \inf\{\pi r^2 : K \xrightarrow{\text{symp.}} Z^{2n}(r)\}$$

$$c_{\mathrm{EHZ}}(K) := \min\{A(\gamma) : \gamma \text{ closed characteristic on } \partial K\}$$
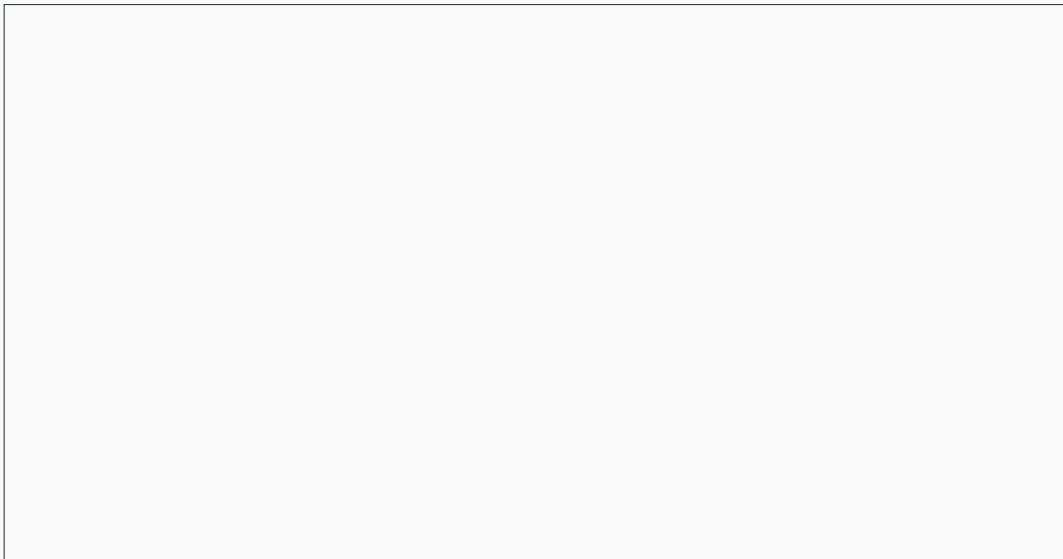
Viterbo's Conjecture – Counterexample

- Viterbo's Conjecture has been open since 2000.
- Recently (2024), a counterexample in $\mathbb{R}^4$ was found

Theorem (Haim–Kislev 2024)
For the lagrangian product polytope $K = P_5 \times_L 90° P_5$ where P_5 is the regular pentagon, and $90° P_5$ is its rotation by 90°, we have

$$\mathsf{sys}(K) = \frac{c_{\mathrm{EHZ}}(K)^2}{2! \operatorname{vol}(K)} = \frac{\sqrt{5} + 3}{5} > 1.$$

Thesis Topic

- Surprisingly simple counterexample to a longstanding conjecture!
- Previous searches for counterexamples stopped too early.
- What more can we learn from computational approaches?
- Thesis topic: probe Viterbo's conjecture **computationally**, make observations and conjectures.

Computing the EHZ capacity

- Milestone: compute $c_{\mathrm{EHZ}}(K)$ for convex polytopes $K \subset \mathbb{R}^4$.
- Why polytopes?
    - Dense in convex bodies (approximation).
    - Finite combinatorial structure (facets, normals).
    - Definitions can be generalized from the smooth setting (generalized closed characteristics).

Generalized Closed Characteristics on Polytopes

Polytopes

- A polytope is a compact intersection of finitely many half-spaces.

$$K = \bigcap_{i=1}^{N} \{x \in \mathbb{R}^4 : \langle x, n_i \rangle \leq h_i\}.$$

- Facet normals n_i are outward unit normals.
- Facet heights/supports h_i are positive, so that $0 \in \mathrm{int}(K)$.
- The boundary ∂K is **not smooth**
  - Flat facets $F_i = \{x : \langle x, n_i \rangle = h_i\} \cap \partial K$.
  - Facets intersect into 0,1,2-dimensional faces.
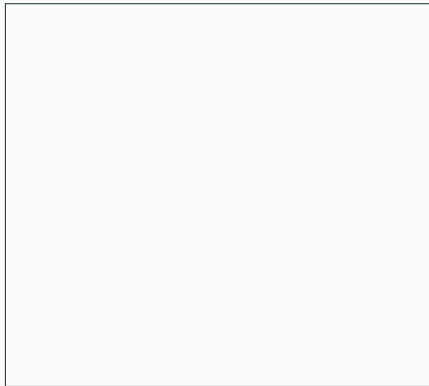
Polytopes

- Question: Polytopes are limits of smooth convex bodies. What are the limits of Hamiltonian orbits / Reeb orbits / closed characteristics?
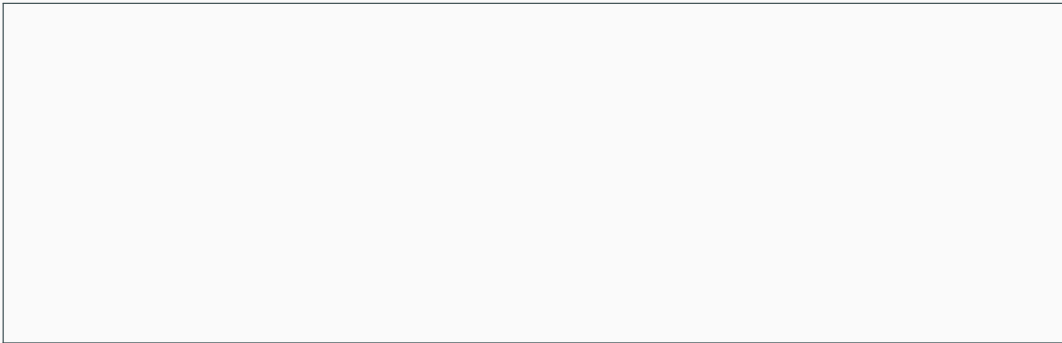
$$\dot{\gamma}(t) = R(\gamma(t)), \quad \iota_R d\alpha = 0, \quad \alpha(R) = 1$$

- Answer: Generalized Hamiltonian orbits / Reeb orbits / closed characteristics.

$$\dot{\gamma}(t) \in \text{conv}\{R_i : \gamma(t) \in F_i\}$$

where the R_i are the constant Reeb vectors on each facet F_i.

$$R_i = \frac{2}{h_i} J n_i$$

Generalized Closed Characteristics

Hamiltonian orbit	Reeb orbit	Closed characteristic
$\gamma \in W^{1,2}([0, T], \mathbb{R}^4)$	$\gamma \in W^{1,2}([0, T], \partial K)$	$\gamma \in W^{1,2}([0, 1], \partial K)$
$\int_0^T \dot{\gamma}(t)\, dt = 0$	$\int_0^T \dot{\gamma}(t)\, dt = 0$	$\int_0^1 \dot{\gamma}(t)\, dt = 0$
$\dot{\gamma}(t) \in J\partial H(\gamma(t))$ a.e.	$\dot{\gamma}(t) \in \mathrm{conv}\{R_i : \ \gamma(t) \in F_i\}$ a.e.	$\dot{\gamma}(t) \in JN_+(\gamma(t))$ a.e.
$A(\gamma) = T$	$A(\gamma) = T$	-

with

- Subdifferential $\partial H(x) = \{y : H(z) \geq H(x) + \langle y, z - x \rangle \ \forall z\}$, for a convex function H.
- Normal cone $N_+(x) = \mathbb{R}_+ \mathrm{conv}\{n_i : \ x \in F_i\}$.

EHZ capacity for polytopes

Definition
The Ekeland–Hofer–Zehnder capacity of a convex polytope $K \subset \mathbb{R}^{2n}$ is

$$c_{\mathrm{EHZ}}(K) = \min\{A(\gamma) : \ \gamma \text{ generalized closed characteristic on } \partial K\}.$$

- This minimum is attained.
- $c_{\mathrm{EHZ}}(K)$ is continuous with respect to the Hausdorff metric on convex bodies.

The Primal Optimization Problem

	Primal Problem (Closed Characteristics)
Minimize	$A(\gamma) = \int_\gamma \lambda_0$
Function Space	$\gamma \in W^{1,2}([0,1], \mathbb{R}^4)$
Constraints	$\int_0^1 \dot{\gamma}(t)\,dt = 0$
	$\gamma(t) \in \partial K$ for all t
	$\dot{\gamma}(t) \in JN_+(\gamma(t))$ a.e.

- Infinite-dimensional search space.
- Constraints are hard to handle (especially $\gamma(t) \in \partial K$).

The Primal Optimization Problem

$$H(x) = g_K^2(x), \quad g_K(x) = \inf\{r > 0 : x \in rK\} \text{ (gauge function)}.$$

$$\nabla H(x) = \frac{2}{h_i} n_i \text{ for } x \in \mathrm{int}(F_i), \quad J\partial H(x) = \mathrm{conv}\left\{\frac{2}{h_i} J n_i : x \in F_i\right\} = \mathrm{conv}\{R_i : x \in F_i\}.$$

	Primal Problem (Hamiltonian)
Minimize	$A(\gamma) = \int_\gamma \lambda_0$
Function Space	$\gamma \in W^{1,2}([0, T], \mathbb{R}^4)$
Constraints	$\int_0^T \dot{\gamma}(t)\, dt = 0$
	$H(\gamma) \equiv 1$
	$\dot{\gamma}(t) \in J\partial H(\gamma(t))$ a.e.

Break

Clarke dual action principle

Support and Gauge

- Convex body $K \subset \mathbb{R}^4$ with $0 \in \mathrm{int}(K)$.
- Support function:
$$h_K(y) = \sup_{x \in K} \langle x, y \rangle.$$
- Gauge (Minkowski functional):
$$g_K(x) = \inf\{r > 0 : \ x \in rK\}.$$
- Note: $g_K(x) \equiv 1$ on ∂K.
- Note: $h_K(n_i) = h_i$ for facet normals n_i.

Lemma (Legendre-Fenchel duality)
For a convex body $K \subset \mathbb{R}^4$ with $0 \in \text{int}(K)$:

$$g_K^2(x) = \sup_{y \in \mathbb{R}^4} \left(\langle x, y \rangle - \tfrac{1}{4} h_K^2(y) \right),$$

$$\tfrac{1}{4} h_K^2(y) = \sup_{x \in \mathbb{R}^4} \left(\langle x, y \rangle - g_K^2(x) \right).$$

We get an inequality:

$$g_K^2(x) + \tfrac{1}{4} h_K^2(y) \geq \langle x, y \rangle.$$

Equality holds iff

$$g_K^2(x) + \tfrac{1}{4} h_K^2(y) = \langle x, y \rangle \iff y \in \partial g_K^2(x) \iff x \in \partial(\tfrac{1}{4} h_K^2)(y).$$

Dual Problem

- For a Hamiltonian orbit $-J\dot{\gamma} \in \partial H(\gamma)$, the equality condition holds with $x = \gamma$, $y = -J\dot{\gamma}$:

$$g_K^2(\gamma) + \tfrac{1}{4} h_K^2(-J\dot{\gamma}) = \langle \gamma, -J\dot{\gamma} \rangle.$$

- Idea: integrate the Fenchel inequality over time, to switch from the minimization target $A(\gamma)$ to some other functional I_K that depends on $-J\dot{\gamma}$ only.

$$I_K(z) = \tfrac{1}{4} \int_0^T h_K^2(-J\dot{z}(t)) \, dt,$$

- For a Hamiltonian orbit with $g_K^2(\gamma) \equiv 1$ we get:

$$T + I_K(\gamma) = 2A(\gamma) = 2T \implies I_K(\gamma) = T = A(\gamma).$$

- Idea: show that this gives us a dual optimization problem in a variable z, such that the critical points of I_K correspond 1:1 to Hamiltonian orbits.

Primal and Dual Problems

	Primal Problem	**Dual Problem**
Minimize	$A(\gamma) = \int_\gamma \lambda_0$	$I_K(z) = \frac{1}{4} \int_0^T h_K^2(-J\dot{z}(t))\, dt$
Function Space	$\gamma \in W^{1,2}([0, T], \mathbb{R}^4)$	$z \in W^{1,2}([0, T], \mathbb{R}^4)$
Constraints	$\int_0^T \dot{\gamma}(t)\, dt = 0$	$\int_0^T z(t)\, dt = 0$
	$H(\gamma) \equiv 1$	$\int_0^T \langle -J\dot{z}(t), z(t) \rangle\, dt = 2T$
	$\dot{\gamma}(t) \in J\partial H(\gamma(t))$ a.e.	-
	-	$\int_0^T z(t) = 0$

Theorem (Clarke dual action principle)
The minimizers of the primal and dual problems correspond 1:1, with

$$z = \gamma - \mathrm{center}(\gamma), \quad I_K(z) = T = A(\gamma).$$

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Proof: Clarke Dual Action Principle

- From variational calculus, critical points of A are given by

$$-J\dot{\gamma}(t) \in \partial g_K^2(\gamma(t)) \text{ a.e.}, \quad g_K^2(\gamma(t)) \equiv 1.$$

- From variational calculus, critical points of I_K under the constraints are given by

$$z(t) + \text{const} \in \partial(\tfrac{1}{4}h_K^2)(-J\dot{z}(t)) \text{ a.e.}, \quad \int_0^T \langle -J\dot{z}(t), z(t) \rangle \, dt = 2T.$$

- By the equality condition of Fenchel inequality, the first two differential inclusions are equivalent under the transformation $z = \gamma - \text{center}(\gamma)$.

- The second two constraints then are also equivalent.

- Finally we get by integrating the Fenchel equality over time that

$$T + I_K(z) = 2A(\gamma) \implies I_K(z) = T = A(\gamma).$$

Dual Problem Is Nicer

|	**Primal Problem**	**Dual Problem**
Minimize	$A(\gamma) = \int_\gamma \lambda_0$	$I_K(z) = \frac{1}{4} \int_0^T h_K^2(-J\dot{z}(t)) \, dt$
Function Space	$\gamma \in W^{1,2}([0, T], \mathbb{R}^4)$	$z \in W^{1,2}([0, T], \mathbb{R}^4)$
Constraints	$\int_0^T \dot{\gamma}(t) \, dt = 0$	$\int_0^T z(t) \, dt = 0$
	$H(\gamma) \equiv 1$	$\int_0^T \langle -J\dot{z}(t), z(t) \rangle \, dt = 2T$
	$\dot{\gamma}(t) \in J\partial H(\gamma(t))$ a.e.	-
	-	$\int_0^T z(t) = 0$

- Minimization target I_K depends only on velocity $\dot{z}$.
- Function space has an action constraint, but no direct position constraint.
- Function space has no differential inclusion constraint.

Existence of Simple Minimizers

Theorem (Haim-Kislev 2019)

Let $K \subset \mathbb{R}^{2n}$ be a convex polytope. Then there exists a generalized closed characteristic / Reeb orbit / Hamiltonian orbit $\gamma^* \in W^{1,2}([0, T], \partial K)$ with minimal action $c_{\mathrm{EHZ}}(K)$, such that

- γ^* is piecewise affine (breakpoints need not be on facet intersections).
- Velocities $\dot{\gamma}^*(t)$ are pure facet Reeb vectors, not convex combinations.
- Each facet Reeb vector appears at most once, i.e.

$$\{t : \dot{\gamma}^*(t) = R_i\} \text{ is an interval or empty, for each } i.$$
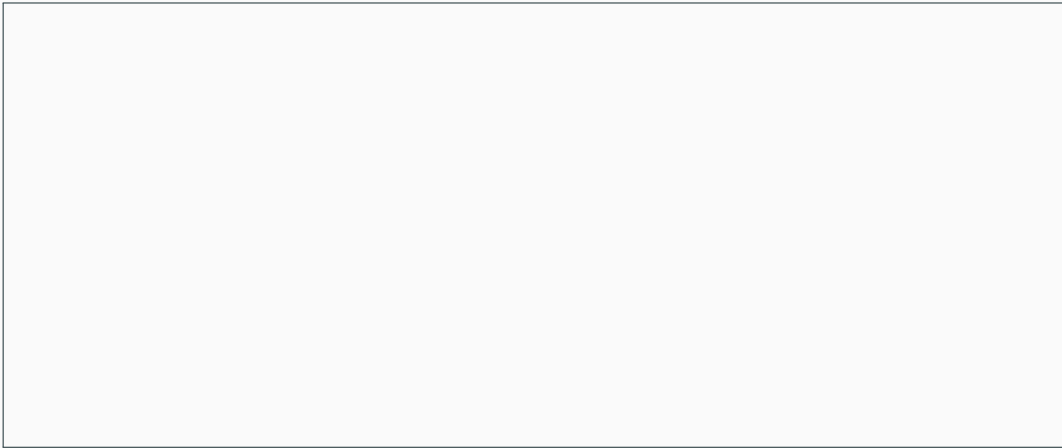
Proof outline: existence of a simple minimizer

1. **Approximate:** replace z by piecewise affine z_N.
2. **Split:** convex-combination velocities $\rightarrow$ pure facet velocities.
3. **Rearrange:** group equal velocities ("grow+shrink").
4. **Renormalize:** restore the action constraint by rescaling.
5. **Compactness:** take a limit in a finite-dimensional parameter space.

Step 1: Approximation by piecewise affine loops

- Start with a minimizer z of the dual problem.
- Approximate z in $W^{1,2}$ by piecewise affine loops z_N.
- Ensure $\dot{z}_N(t)$ stays in the same allowed cone:

$$\dot{z}_N(t) \in \mathrm{conv}\{R_i\} \quad \text{a.e.}$$

- Then $A(z_N) \to A(z)$ and $I_K(z_N) \to I_K(z)$ by continuity of the integrals.

Step 2: Split convex combinations into pure facet velocities

- We may have mixed velocities (note that in the dual problem we do not track what facets z lies on).

$$\dot{z}_N(t) \in \operatorname{conv}\{R_1, \ldots, R_F\}.$$

- Replace each mixed segment by a concatenation of segments with *pure* velocities R_{i_j}.

- Choose the time order so that the action/area constraint does not decrease:

$$A(z'_N) \geq A(z_N).$$

- Possible since reversing the time order reverses the change in area.

- For pure paths we have

$$I_K(z'_N) = T$$

So the functional changed only slightly.

Step 3: Rearrange equal velocities ("grow+shrink")

- After splitting, $\dot{z}$ takes values in a finite set $\{R_1, \ldots, R_F\}$.
- If some R_i appears in disjoint time intervals, we merge them by a rearrangement step.
- One chooses the rearrangement direction so the action constraint does not decrease:

$$A(z_N'') \geq A(z_N').$$

- Again possible since reversing the pair reverses the change in area.
- The functional I_K is unchanged by this rearrangement (it depends only on the velocity / time spent).

Step 4: Renormalize to restore the action constraint

- After Steps 2–3, we may have increased the action: $\frac{A(z_N'')}{T} \geq A(z_N)$
- Scale the time spent on each sigment by factor β to restore the constraint:

$$A(z_N''') = T_N''' = T \cdot \beta = A(z_N'')\beta^2$$

$$\beta = T/A(z_N'')$$

- Then $A(z_N''') = T'''$, and $I_K(z_N''') = \beta I_K(z_N'') = \beta T = T_N'''$

Step 5: Compactness and taking a limit

- A simple loop is encoded by finite data:

$$(\text{facet order } \sigma, \text{ segment lengths } (|l_i|)).$$

- The order set is finite, and the set of segment lengths that fulfills our constraints is compact.

- Take an approximation sequence, take their simple loops and extract a convergent subsequence.

- We get by $A(z_N) \to A(z)$ and $I_K(z_N) \to I_K(z)$ and minimality of $I_K(z)$ and by $I_K(z_N''') = T^2/A_{(}z_N'')$ that $T^* = T$, $I_K(z^*) = I_K(z)$, so we have found a simple minimum.

Remark: the manipulations are homotopies

Remark
Each modification step (approximation, splitting, grow+shrink, rescale) can be done as a homotopy through minimizers of the dual problem. Thus, every minimizer is homotopic to a simple minimizer, though we don't claim all minimizers are connected via such homotopies.

Dual Optimization Problem – Finite Dimensional Reduction

	Dual Problem (Simple Piecewise Affine)
Minimize	$I_K = \sum |I_i|$
Function Space	Facet order $\sigma \in S_N$
	Segment times $|I_i| \geq 0$
Constraints	$\sum |I_i| R_{\sigma(i)} = 0$
	$\sum_{j<i} |I_i||I_j|\omega(R_{\sigma(i)}, R_{\sigma(j)}) = 2\sum |I_i|$

- The problem can be further turned into a Quadratic Programming problem.
- There's additional structure to exploit, e.g. facet adjacency constraints on what changes of velocities are possible.

References

- P. Haim-Kislev, *On the Symplectic Size of Convex Polytopes*, Geom. Funct. Anal. 29(2):440–463 (2019). doi: 10.1007/s00039-019-00486-4. (arXiv: 1712.03494)

- P. Haim-Kislev, Y. Ostrover, *A Counterexample to Viterbo's Conjecture*, arXiv: 2405.16513v3 (2025). doi: 10.48550/arXiv.2405.16513.

- J. Chaidez, M. Hutchings, *Computing Reeb dynamics on four-dimensional convex polytopes*, J. Comput. Dyn. 8(4):403–445 (2021). doi: 10.3934/jcd.2021016. (arXiv: 2008.10111)

Summary

1. Viterbo's conjecture requires solving an optimization problem over closed characteristics.
2. Clarke's Dual Action Principle switches to a dual problem over arbitrary loops.
3. In the dual problem, we can show existence of minimizers that are piecewise affine and use only pure facet Reeb vectors.
4. AND we can show that the existence of minimizers that use each facet Reeb vector at most once!
5. This finally yields a finite-dimensional optimization problem.