# Clarke's Dual Action Principle

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Symplectic Geometry Seminar

# Outline

# Symplectic Geometry Refresh

# Standard Setting

- 4-dimensional symplectic vector space $\mathbb{R}^4$, coordinates $x = (q_1, q_2, p_1, p_2)$.
- Symplectic form

$$\omega_0 = \sum_{i=1}^{2} dq_i \wedge dp_i, \quad \omega_0(u, v) = \langle Ju, v \rangle,$$

- Almost complex structure

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}, \quad J^2 = -I_4, \quad J((q, p)) = (-p, q).$$

- Liouville 1-form

$$\lambda_0 = \tfrac{1}{2}\langle Jx, dx \rangle = \tfrac{1}{2} \sum_{i=1}^{2} (p_i \, dq_i - q_i \, dp_i), \quad d\lambda_0 = \omega_0.$$

- Higher dimensions $\mathbb{R}^{2n}$ work analogously.

# Loops, Action, Hamiltonians

- Hamiltonian function $H : \mathbb{R}^4 \to \mathbb{R}$.
- Hamiltonian vector field $X_H = J\nabla H$.
- Hamiltonian orbit: periodic solution $\gamma : [0, T] \to \mathbb{R}^4$ of

$$\dot{\gamma}(t) = X_H(\gamma(t)) = J\nabla H(\gamma(t)).$$

- Preserves energy levels: $H(\gamma(t)) \equiv \mathrm{const}$.
- Symplectic action of a loop $\gamma : [0, T] \to \mathbb{R}^4$:

$$A(\gamma) = \int_\gamma \lambda_0 = \int_0^T \lambda_0(\dot{\gamma}(t))\, dt = \tfrac{1}{2}\int_0^T \langle -J\dot{\gamma}(t), \gamma(t)\rangle\, dt.$$

- Per Stokes' theorem, if $\gamma$ bounds a disk $u : D^2 \to \mathbb{R}^4$, then

$$A(\gamma) = \int_{D^2} u^*\omega_0 = \int_{D^2} \omega_0(\partial_x u, \partial_y u)\, dx\, dy.$$

# Reeb flow

- Smooth star-shaped hypersurface $\Sigma \subset \mathbb{R}^4$. E.g., energy surface $\Sigma = \{H = 1\}$ for certain $H$.
- Contact form: $\alpha = \lambda_0|_\Sigma$.
- Reeb vector field $R$ defined by

$$\iota_R d\alpha = 0, \qquad \alpha(R) = 1.$$

- A Reeb orbit is a periodic solution $\gamma : [0, T] \to \Sigma$ of

$$\dot{\gamma} = R(\gamma).$$

- Useful fact: $A(\gamma) = \int_0^T \alpha(R(\gamma(t)))\, dt = T$.

# Closed Characteristics

| Hamiltonian orbit | Reeb orbit | Closed characteristic |
|---|---|---|
| $\gamma : [0, T] \to \mathbb{R}^4$ | $\gamma : [0, T] \to \Sigma$ | $\gamma : [0, 1] \to \Sigma$ |
| $\dot{\gamma}(t) = J\nabla H(\gamma(t))$ | $\dot{\gamma}(t) = R(\gamma(t))$ | $\dot{\gamma}(t) \in \mathbb{R}_+ R(\gamma(t))$ |
| $H(\gamma) \equiv \mathrm{const}$ | $\gamma(t) \in \Sigma$ | $\gamma(t) \in \Sigma$ |
| $A(\gamma) = T$ | $A(\gamma) = T$ | - |

- We exclude constant loops with $T = 0, A(\gamma) = 0$ from all three definitions.

# Viterbo's Conjecture

# EHZ capacity / Minimum action

- Fix a convex body $K \subset \mathbb{R}^4$ (compact, convex, $0 \in \mathrm{int}(K)$).
- So far: smooth case, but we will generalize later.
- Define the Ekeland–Hofer–Zehnder capacity as

$$c_{\mathrm{EHZ}}(K) = A_{\mathsf{min}}(K) := \min\{A(\gamma) \colon \gamma \text{ closed characteristic on } \partial K\}.$$

- Note: we do not take the minimum over all loops, only over closed characteristics.
- This minimum is attained.
- Note: $c_{\mathrm{EHZ}}(K) > 0$, since we exclude constant loops, and if $\gamma$ is a closed characteristic, then $A(\gamma) = T > 0$.

# EHZ capacity / Minimum action

**Symplectic Capacity**
The map $c_{\mathrm{EHZ}} : K \mapsto \mathbb{R}_+$ is a symplectic capacity, i.e.:

1. (Monotonicity) If there exists a symplectic embedding $K_1 \to K_2$, then $c_{\mathrm{EHZ}}(K_1) \leq c_{\mathrm{EHZ}}(K_2)$.
2. (Conformality) For $\alpha \neq 0$, $c_{\mathrm{EHZ}}(\alpha K) = |\alpha|^2 c_{\mathrm{EHZ}}(K)$.
3. (Normalization) $c_{\mathrm{EHZ}}(B^4(1)) = c_{\mathrm{EHZ}}(Z^4(1)) = \pi$, where

$$B^4(1) = \{x \in \mathbb{R}^4 : \ |x| \leq 1\}, \qquad Z^4(1) = \{(q_1, q_2, p_1, p_2) : \ q_1^2 + p_1^2 \leq 1\}.$$

**Viterbo's Conjecture**
For any convex body $K \subset \mathbb{R}^{2n}$:

$$\mathrm{sys}(K) := \frac{c_{\mathrm{EHZ}}(K)^n}{n!\,\mathrm{vol}(K)} \leq 1.$$

**Strong Viterbo Conjecture**
All normalized symplectic capacities agree on convex domains. In particular, the EHZ capacity, the Gromov width, and the cylindrical capacity coincide.

$$c_{\mathrm{Gromov}}(K) := \sup\{\pi r^2 : \; B^{2n}(r) \xhookrightarrow{\text{symp.}} K\}$$

$$c_{\mathrm{cylindrical}}(K) := \inf\{\pi r^2 : \; K \xhookrightarrow{\text{symp.}} Z^{2n}(r)\}$$

$$c_{\mathrm{EHZ}}(K) := \min\{A(\gamma) : \; \gamma \text{ closed characteristic on } \partial K\}$$

## Viterbo's Conjecture – Counterexample

- Viterbo's Conjecture has been open since 2000.
- Recently (2024), a counterexample in $\mathbb{R}^4$ was found

**Theorem (Haim–Kislev 2024)**
For the lagrangian product polytope $K = P_5 \times_L 90^\circ P_5$ where $P_5$ is the regular pentagon, and $90^\circ P_5$ is its rotation by $90^\circ$, we have

$$\mathsf{sys}(K) = \frac{c_{\mathrm{EHZ}}(K)^2}{2!\,\mathrm{vol}(K)} = \frac{\sqrt{5}+3}{5} > 1.$$

# Thesis Topic

- Surprisingly simple counterexample to a longstanding conjecture!
- Previous searches for counterexamples stopped too early.
- What more can we learn from computational approaches?
- Thesis topic: probe Viterbo's conjecture **computationally**, make observations and conjectures.

# Computing the EHZ capacity

- Milestone: compute $c_{\mathrm{EHZ}}(K)$ for convex polytopes $K \subset \mathbb{R}^4$.
- Why polytopes?
    - Dense in convex bodies (approximation).
    - Finite combinatorial structure (facets, normals).
    - Definitions can be generalized from the smooth setting (generalized closed characteristics).

# Generalized Closed Characteristics on Polytopes

# Polytopes

- A polytope is a compact intersection of finitely many half-spaces.

$$K = \bigcap_{i=1}^{N} \{x \in \mathbb{R}^4 : \ \langle x, n_i \rangle \leq h_i\}.$$

- Facet normals $n_i$ are outward unit normals.
- Facet heights/supports $h_i$ are positive, so that $0 \in \mathrm{int}(K)$.
- The boundary $\partial K$ is **not smooth**
  - Flat facets $F_i = \{x : \ \langle x, n_i \rangle = h_i\} \cap \partial K$.
  - Facets intersect into 0,1,2-dimensional faces.

# Polytopes

- Question: Polytopes are limits of smooth convex bodies. What are the limits of Hamiltonian orbits / Reeb orbits / closed characteristics?

$$\dot{\gamma}(t) = R(\gamma(t)), \quad \iota_R d\alpha = 0, \quad \alpha(R) = 1$$

- Answer: Generalized Hamiltonian orbits / Reeb orbits / closed characteristics.

$$\dot{\gamma}(t) \in \mathrm{conv}\{R_i : \gamma(t) \in F_i\}$$

where the $R_i$ are the constant Reeb vectors on each facet $F_i$.

$$R_i = \frac{2}{h_i} J n_i$$

# Generalized Closed Characteristics

| **Hamiltonian orbit** | **Reeb orbit** | **Closed characteristic** |
|---|---|---|
| $\gamma \in W^{1,2}([0,T],\mathbb{R}^4)$<br>$\int_0^T \dot\gamma(t)\,dt = 0$<br>$\dot\gamma(t) \in J\partial H(\gamma(t))$ a.e.<br>$A(\gamma) = T$ | $\gamma \in W^{1,2}([0,T],\partial K)$<br>$\int_0^T \dot\gamma(t)\,dt = 0$<br>$\dot\gamma(t) \in \mathrm{conv}\{R_i :\ \gamma(t) \in F_i\}$ a.e.<br>$A(\gamma) = T$ | $\gamma \in W^{1,2}([0,1],\partial K)$<br>$\int_0^1 \dot\gamma(t)\,dt = 0$<br>$\dot\gamma(t) \in JN_+(\gamma(t))$ a.e.<br>- |

with

- Subdifferential $\partial H(x) = \{y : H(z) \geq H(x) + \langle y, z - x\rangle\ \forall z\}$, for a convex function $H$.
- Normal cone $N_+(x) = \mathbb{R}_+\mathrm{conv}\{n_i :\ x \in F_i\}$.

# EHZ capacity for polytopes

**Definition**
The Ekeland–Hofer–Zehnder capacity of a convex polytope $K \subset \mathbb{R}^{2n}$ is

$$c_{\mathrm{EHZ}}(K) = \min\{A(\gamma) : \ \gamma \text{ generalized closed characteristic on } \partial K\}.$$

- This minimum is attained.
- $c_{\mathrm{EHZ}}(K)$ is continuous with respect to the Hausdorff metric on convex bodies.

# The Primal Optimization Problem

| | **Primal Problem (Closed Characteristics)** |
|---|---|
| Minimize | $A(\gamma) = \int_\gamma \lambda_0$ |
| Function Space | $\gamma \in W^{1,2}([0,1], \mathbb{R}^4)$ |
| Constraints | $\int_0^1 \dot{\gamma}(t)\, dt = 0$<br>$\gamma(t) \in \partial K$ for all $t$<br>$\dot{\gamma}(t) \in JN_+(\gamma(t))$ a.e. |

- Infinite-dimensional search space.
- Constraints are hard to handle (especially $\gamma(t) \in \partial K$).

# The Primal Optimization Problem

$$H(x) = g_K^2(x), \quad g_K(x) = \inf\{r > 0 : \; x \in rK\} \text{ (gauge function)}.$$

$$\nabla H(x) = \frac{2}{h_i} n_i \text{ for } x \in \mathrm{int}(F_i), \quad J\partial H(x) = \mathrm{conv}\left\{\frac{2}{h_i} J n_i : \; x \in F_i\right\} = \mathrm{conv}\{R_i : \; x \in F_i\}.$$

| | **Primal Problem (Hamiltonian)** |
|---|---|
| Minimize | $A(\gamma) = \int_\gamma \lambda_0$ |
| Function Space | $\gamma \in W^{1,2}([0, T], \mathbb{R}^4)$ |
| Constraints | $\int_0^T \dot{\gamma}(t)\, dt = 0$<br>$H(\gamma) \equiv 1$<br>$\dot{\gamma}(t) \in J\partial H(\gamma(t))$ a.e. |

# Break

# Clarke dual action principle

## Support and Gauge

- Convex body $K \subset \mathbb{R}^4$ with $0 \in \mathrm{int}(K)$.
- Support function:

$$h_K(y) = \sup_{x \in K} \langle x, y \rangle.$$

- Gauge (Minkowski functional):

$$g_K(x) = \inf\{r > 0 : \ x \in rK\}.$$

- Note: $g_K(x) \equiv 1$ on $\partial K$.
- Note: $h_K(n_i) = h_i$ for facet normals $n_i$.

# Legendre-Fenchel

**Lemma (Legendre-Fenchel duality)**
For a convex body $K \subset \mathbb{R}^4$ with $0 \in \mathrm{int}(K)$:

$$g_K^2(x) = \sup_{y \in \mathbb{R}^4} \Big( \langle x, y \rangle - \tfrac{1}{4} h_K^2(y) \Big),$$

$$\tfrac{1}{4} h_K^2(y) = \sup_{x \in \mathbb{R}^4} \Big( \langle x, y \rangle - g_K^2(x) \Big).$$

We get an inequality:

$$g_K^2(x) + \tfrac{1}{4} h_K^2(y) \geq \langle x, y \rangle.$$

Equality holds iff

$$g_K^2(x) + \tfrac{1}{4} h_K^2(y) = \langle x, y \rangle \iff y \in \partial g_K^2(x) \iff x \in \partial (\tfrac{1}{4} h_K^2)(y).$$

# Dual Problem

- For a Hamiltonian orbit $-J\dot{\gamma} \in \partial H(\gamma)$, the equality condition holds with $x = \gamma$, $y = -J\dot{\gamma}$:

$$g_K^2(\gamma) + \tfrac{1}{4} h_K^2(-J\dot{\gamma}) = \langle \gamma, -J\dot{\gamma} \rangle.$$

- Idea: integrate the Fenchel inequality over time, to switch from the minimization target $A(\gamma)$ to some other functional $I_K$ that depends on $-J\dot{\gamma}$ only.

$$I_K(z) = \tfrac{1}{4} \int_0^T h_K^2(-J\dot{z}(t))\, dt,$$

- For a Hamiltonian orbit with $g_K^2(\gamma) \equiv 1$ we get:

$$T + I_K(\gamma) = 2A(\gamma) = 2T \implies I_K(\gamma) = T = A(\gamma).$$

- Idea: show that this gives us a dual optimization problem in a variable $z$, such that the critical points of $I_K$ correspond 1:1 to Hamiltonian orbits.

# Primal and Dual Problems

| | Primal Problem | Dual Problem |
|---|---|---|
| Minimize | $A(\gamma) = \int_\gamma \lambda_0$ | $I_K(z) = \frac{1}{4}\int_0^T h_K^2(-J\dot{z}(t))\,dt$ |
| Function Space | $\gamma \in W^{1,2}([0,T],\mathbb{R}^4)$ | $z \in W^{1,2}([0,T],\mathbb{R}^4)$ |
| Constraints | $\int_0^T \dot{\gamma}(t)\,dt = 0$<br>$H(\gamma) \equiv 1$<br>$\dot{\gamma}(t) \in J\partial H(\gamma(t))$ a.e.<br>- | $\int_0^T z(t)\,dt = 0$<br>$\int_0^T \langle -J\dot{z}(t), z(t)\rangle\,dt = 2T$<br>-<br>$\int_0^T z(t) = 0$ |

**Theorem (Clarke dual action principle)**
The minimizers of the primal and dual problems correspond 1:1, with

$$z = \gamma - \mathrm{center}(\gamma)\,, \quad I_K(z) = T = A(\gamma).$$

# Proof: Clarke Dual Action Principle

- From variational calculus, critical points of $A$ are given by

$$-J\dot{\gamma}(t) \in \partial g_K^2(\gamma(t)) \text{ a.e.}\,, \quad g_K^2(\gamma(t)) \equiv 1.$$

- From variational calculus, critical points of $I_K$ under the constraints are given by

$$z(t) + \mathrm{const} \in \partial(\tfrac{1}{4}h_K^2)(-J\dot{z}(t)) \text{ a.e.}\,, \quad \int_0^T \langle -J\dot{z}(t), z(t)\rangle\, dt = 2T.$$

- By the equality condition of Fenchel inequality, the first two differential inclusions are equivalent under the transformation $z = \gamma - \mathrm{center}(\gamma)$.
- The second two constraints then are also equivalent.
- Finally we get by integrating the Fenchel equality over time that

$$T + I_K(z) = 2A(\gamma) \implies I_K(z) = T = A(\gamma).$$

# Dual Problem Is Nicer

| | **Primal Problem** | **Dual Problem** |
|---|---|---|
| Minimize | $A(\gamma) = \int_\gamma \lambda_0$ | $I_K(z) = \frac{1}{4}\int_0^T h_K^2(-J\dot{z}(t))\,dt$ |
| Function Space | $\gamma \in W^{1,2}([0,T],\mathbb{R}^4)$ | $z \in W^{1,2}([0,T],\mathbb{R}^4)$ |
| Constraints | $\int_0^T \dot{\gamma}(t)\,dt = 0$<br>$H(\gamma) \equiv 1$<br>$\dot{\gamma}(t) \in J\partial H(\gamma(t))$ a.e.<br>- | $\int_0^T z(t)\,dt = 0$<br>$\int_0^T \langle -J\dot{z}(t), z(t)\rangle\,dt = 2T$<br>-<br>$\int_0^T z(t) = 0$ |

- Minimization target $I_K$ depends only on velocity $\dot{z}$.
- Function space has an action constraint, but no direct position constraint.
- Function space has no differential inclusion constraint.

# Existence of Simple Minimizers

**Theorem (Haim-Kislev 2019)**
Let $K \subset \mathbb{R}^{2n}$ be a convex polytope. Then there exists a generalized closed characteristic / Reeb orbit / Hamiltonian orbit $\gamma^* \in W^{1,2}([0,T], \partial K)$ with minimal action $c_{\mathrm{EHZ}}(K)$, such that

- $\gamma^*$ is piecewise affine (breakpoints need not be on facet intersections).
- Velocities $\dot{\gamma}^*(t)$ are pure facet Reeb vectors, not convex combinations.
- Each facet Reeb vector appears at most once, i.e.

$$\{t : \dot{\gamma}^*(t) = R_i\} \text{ is an interval or empty, for each } i.$$

# Proof outline: existence of a simple minimizer

1. **Approximate:** replace $z$ by piecewise affine $z_N$.
2. **Split:** convex-combination velocities $\to$ pure facet velocities.
3. **Rearrange:** group equal velocities ("grow+shrink").
4. **Renormalize:** restore the action constraint by rescaling.
5. **Compactness:** take a limit in a finite-dimensional parameter space.

# Step 1: Approximation by piecewise affine loops

- Start with a minimizer $z$ of the dual problem.
- Approximate $z$ in $W^{1,2}$ by piecewise affine loops $z_N$ (polygonal in time).
- Ensure $\dot{z}_N(t)$ stays in the same allowed cone:

$$\dot{z}_N(t) \in \mathrm{conv}\{R_i\} \quad \text{a.e.}$$

  (so we never leave the polytope-type dynamics).
- Then $A(z_N) \to A(z)$ and $I_K(z_N) \to I_K(z)$ by continuity of the integrals.

# Step 2: Split convex combinations into pure facet velocities

- On lower-dimensional faces, we may have mixed velocities:

$$\dot{z}_N(t) \in \mathrm{conv}\{R_{i_1}, \dots, R_{i_k}\}.$$

- Replace each mixed segment by a concatenation of segments with *pure* velocities $R_{i_j}$.
- Choose the time order so that the action/area constraint does not decrease:

$$A(z_N') \geq A(z_N).$$

- Key input: on a face-cone, $h_K(-J\dot{z})$ depends only on the scale, so $I_K$ does not get worse under splitting.

# Step 3: Rearrange equal velocities (“grow+shrink”)

- After splitting, $\dot{z}$ takes values in a finite set $\{R_1, \ldots, R_F\}$.
- If some $R_i$ appears in disjoint time intervals, we merge them by a rearrangement step.
- One chooses the rearrangement direction so the action constraint does not decrease:

$$A(z_N'') \geq A(z_N').$$

- The functional $I_K$ is unchanged by this rearrangement (it depends only on the velocity scale / time spent).

## Step 4: Renormalize to restore the action constraint

- After Steps 2–3, we may have increased the action: $\alpha := \frac{A(z_N'')}{T} \geq 1$.
- Rescale to restore the constraint:
$$z_N^* := \frac{1}{\sqrt{\alpha}} z_N''.$$
- Then $A(z_N^*) = T$, and $I_K(z_N^*) = \frac{1}{\alpha} I_K(z_N'') \leq I_K(z_N'')$.
- By minimality, this shows we did not "pay" for simplification.

## Step 5: Compactness and taking a limit

- A simple loop is encoded by finite data:

$$(\text{facet order } \sigma,\ \text{segment lengths } (|I_i|)).$$

- The order set is finite, and the normalized length simplex is compact.
- Take a minimizing sequence of simple loops and extract a convergent subsequence.
- The limit is still feasible and minimizes $I_K$, hence gives a simple minimizer.

# Remark: the manipulations are homotopies

**Remark**
Each modification step (approximation, splitting, grow+shrink, rescale) can be done as a homotopy through minimizers of the dual problem. Thus, every minimizer is homotopic to a simple minimizer, though we don't claim all minimizers are connected via such homotopies.

# Dual Optimization Problem – Finite Dimensional Reduction

| | **Dual Problem (Simple Piecewise Affine)** |
|---|---|
| Minimize | $I_K = \sum \lvert I_i \rvert$ |
| Function Space | Facet order $\sigma \in S_N$<br>Segment times $\lvert I_i \rvert \geq 0$ |
| Constraints | $\sum \lvert I_i \rvert R_{\sigma(i)} = 0$<br>$\sum_{j<i} \lvert I_i \rvert \lvert I_j \rvert \omega(R_{\sigma(i)}, R_{\sigma(j)}) = 2 \sum \lvert I_i \rvert$ |

- The problem can be further turned into a Quadratic Programming problem.
- There's additional structure to exploit, e.g. facet adjacency constraints on what changes of velocities are possible.

# References


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# Summary

1. Viterbo's conjecture requires solving an optimization problem over closed characteristics.
2. Clarke's Dual Action Principle switches to a dual problem over arbitrary loops.
3. In the dual problem, we can show existence of minimizers that are piecewise affine and use only pure facet Reeb vectors.
4. AND we can show that the existence of minimizers that use each facet Reeb vector at most once!
5. This finally yields a finite-dimensional optimization problem.