# An Extension of the Transfer Matrix Method to Analyzing Acoustic Wave Propagation in Waveguide with Arbitrary Shape

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In this paper, based on 2-D analytical analysis and linear approximation to boundary elements, the transfer matrix method (TMM) is extended to the analysis of the sound transmission in waveguide with arbitrary shape. Based on TMM, the impedance transfer method (ITM) is also developed for calculating the resonance frequency. Moreover, finite element calculations are studied and compared with the new methods. It shows that both methods can provide pretty high accuracy within wide frequency range, and compared to mainstream numerical methods, they require much less computing time.

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## ACKNOWLEDGMENTS

This research was supported by the National Natural Science Foundation of China (11574038, 11974067), Chongqing MTPIEU Program (S201910611446).

## I. INTRODUCTION

Transfer matrix method (TMM) is widely used in the analysis of the transmission systems[2], the transfer matrix essentially describes the linear relationship between the input and the output. TMM has no dependence on the input and the output, which makes it suitable for analyzing different acoustic systems. TMM is used in the analysis of different Helmholtz resonators at first[3,4], and it is widely applied in acoustic measuring[5]. Moreover, knowing the properties of each layer, the acoustical behavior of materials stacked in series can easily be predicted using TMM[6]. For instance, N.K.Vijayasree et al. proposed an integrated TMM for analyzing a parallel assembly of multiple mufflers[7]. In contrast, for multilayered materials, the Finite Element Method (FEM) is still the most general approach to calculate the acoustic performances[6].

Until now, TMM still shows its limitation in analyzing the wave propagation in arbitrary cross sections. To tackle this problem, Min et al. extend TMM to the analysis of acoustic resonators with gradually varying cross-sectional area[8]. The transfer matrices and the resonant conditions are derived in their research for acoustic resonators with four different kinds, which are tapered, trigonometric, exponential and hyperbolic. Based on the previous study of the linear acoustic units, TMM is extended to the analysis of waveguide with arbitrary shape, which is our major work.

Similarly, the physical nature of a waveguide can be characterized by its transfer function. Obtaining the transfer function of the system will enable us to analyze the sound transmission of the system and optimize the design. Our approach is to discrete the boundary into units and take the first-order approximation. Linear approximation is usually considered effective and it allows us to make full use of the existing analytical results to achieve higher convergence speed and accuracy.

## II. GENERAL DISCUSSION OF ACOUSTIC WAVEGUIDE

Generally speaking, an acoustic waveguide can be thought of as consisting of many units in series, with each unit has its own response to the input which is called transfer function. Based on different understanding of the acoustic waveguide, the transfer function can take different forms. For instance, in order to calculate the resonance frequency, one can think of the elements as linear impedance units. Or, in order to calculate the sound transmission, each element can be regarded as a four-terminal network[9] (FIG. 1).

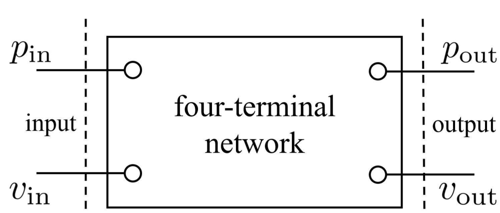


FIG. 1. Four-terminal network.

For an acoustic system, the relationship between the input and the output could be written in the matrix form

$$\begin{bmatrix} p_{\mathrm{out}} \\ v_{\mathrm{out}} \end{bmatrix} = \mathbf{T} \begin{bmatrix} p_{\mathrm{in}} \\ v_{\mathrm{in}} \end{bmatrix}. \tag{1}$$

Here $p_{out}$ and $v_{out}$ are the sound pressure and particle velocity at the output, respectively, where $p_{in}$ and $v_{in}$ are those at the input. $\mathbf{T}$ is defined as the transfer matrix of the transmission system. It is worth noting that opposite with other literature, the transfer matrix is written here to act on the input instead of the output orifice for the convenience of the derivation process.

## III. TRANSFER MATRIX METHOD (TMM)

In order to derive the transfer matrix for an acoustic system, we should make following basic assumptions:

- Uniform parameter assumption. Parameters such as static pressure, medium density, temperature, and velocity of the medium are the same along the axial direction.
- Plane wave assumption. The sound waves that propagate axially inside the muffler are plane sound waves.
- Rigid boundary assumption. The wall of the waveguide can be seen as a rigid acoustic boundary.
- Linear assumption. The sound waves follow the linear wave equation.

### A. Theoretical analysis

First, discrete the wall of the waveguide into small elements with the step length $\delta_x$ (Fig. 2). Once $\delta_x$ is small enough, the slope $\mathrm{d}R(x)/\mathrm{d}x$ can be treated as constant for each element (Fig. 2).

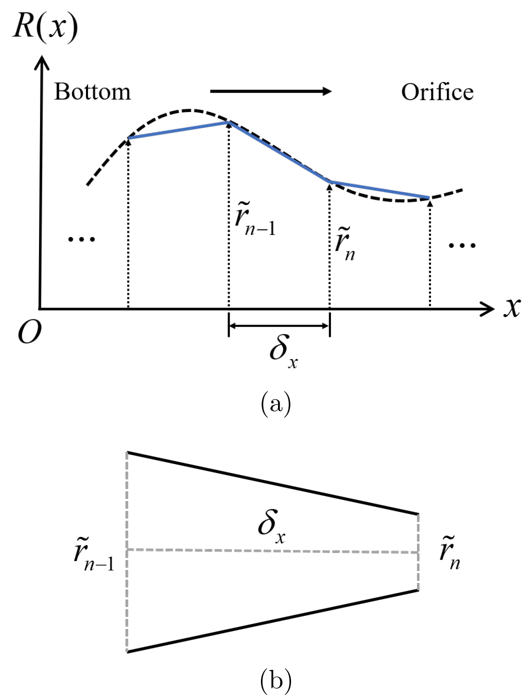


FIG. 2. (a): Finite difference. (b): Linear boundary element obtained.

For each element, the relationship between the input and the output could be written as

$$\begin{bmatrix} \tilde{p}_n \\ \tilde{v}_n \end{bmatrix} = \begin{bmatrix} \tilde{A}_n & \tilde{B}_n \\ \tilde{C}_n & \tilde{D}_n \end{bmatrix} \begin{bmatrix} \tilde{p}_{n-1} \\ \tilde{v}_{n-1} \end{bmatrix} = \widetilde{\mathbf{T}}_n \begin{bmatrix} \tilde{p}_{n-1} \\ \tilde{v}_{n-1} \end{bmatrix}. \tag{2}$$

Here $\tilde{p}_n$ and $\tilde{v}_n$ are the sound pressure and particle velocity at the output of the $n$th element, respectively, where $\tilde{p}_{n-1}$ and $\tilde{v}_{n-1}$ are those at the input. Once the initial input is given, by continuously taking the output of the previous element as the input of the next element, Eq. (2) can be iterated from the bottom to the orifice. Then, the transfer matrix of the entire system can be obtained by multiplying the transfer matrices from the bottom to the top. Therefore, for entire system

$$\begin{bmatrix} p_{\mathrm{out}} \\ v_{\mathrm{out}} \end{bmatrix} = \left( \prod_{n=1}^{N} \widetilde{\mathbf{T}}_n \right) \begin{bmatrix} p_{\mathrm{in}} \\ v_{\mathrm{in}} \end{bmatrix}, \tag{3}$$

here, $N$ is the total amount of the small elements. Now, define the shape factor $m$ of the small elements,

$$m = \frac{\tilde{r}_n - \tilde{r}_{n-1}}{\tilde{r}_n \delta_x}. \tag{4}$$

So, the matrix elements of the Eq. (2) could be written as follows[10],

$$\tilde{A}_n = \frac{1}{2}\left( \frac{\tilde{r}_{n-1}}{\tilde{r}_n} - \frac{im}{k} \right) e^{ik\delta_x} + \frac{1}{2}\left( \frac{\tilde{r}_{n-1}}{\tilde{r}_n} + \frac{im}{k} \right) e^{-ik\delta_x}, \tag{5}$$

$$\tilde{B}_n = \frac{Z_0}{2} \frac{\tilde{r}_{n-1}}{\tilde{r}_n} \left( e^{ik\delta_x} - e^{-ik\delta_x} \right), \tag{6}$$

$$\begin{aligned} \tilde{C}_n = & \frac{1}{2Z_0 k^2} \left[ (m - ik) \left( m + ik \frac{\tilde{r}_{n-1}}{\tilde{r}_n} \right) e^{ik\delta_x} - \right. \\ & \left. (m + ik) \left( m - ik \frac{\tilde{r}_{n-1}}{\tilde{r}_n} \right) e^{-ik\delta_x} \right] \end{aligned}, \tag{7}$$

$$\tilde{D}_n = \frac{1}{2k} \frac{\tilde{r}_{n-1}}{\tilde{r}_n} \left[ (k + im) e^{ik\delta_x} + (k - im) e^{-ik\delta_x} \right]. \tag{8}$$

Here, $Z_0$ is the characteristic impedance of the medium, for air, $Z_0 = \rho_0 c_0$. $k$ is the wave number, $\tilde{r}_{n-1}$ and $\tilde{r}_n$ are the radius of the input and the output of the $n$th element, respectively. $\delta_x$ is the length of each element, and they satisfy

$$\begin{cases} \tilde{r}_n = R(x)|_{x=n\delta_x} \\ \tilde{r}_{n-1} = R(x)|_{x=(n-1)\delta_x} \\ \delta_x = \frac{L}{N} \end{cases} \tag{9}$$

Here $L$ is the total length of the waveguide, and $R(x)$ is the radius (Fig. 2). Substitute Eqs. (5), (6), (7), (8) and (9) into Eq. (2), the transfer function of each element is obtained, and the transfer function of the entire system can be obtained using Eq. (3).

Assume that the length of each element is small enough, take the first order approximation and get $e^{\pm ik\delta_x} = 1 \pm ik\delta_x$. Therefore, the transfer matrix of each element $\widetilde{\mathbf{T}}_n$ is rewritten as

$$\widetilde{\mathbf{T}}_n = \begin{bmatrix} 1 & ik\frac{Z_0\tilde{r}_{n-1}}{\tilde{r}_n}\delta_x \\ ik\frac{\tilde{r}_{n-1}}{Z_0\tilde{r}_n}\delta_x & \left(\frac{\tilde{r}_{n-1}}{\tilde{r}_n}\right)^2 \end{bmatrix}. \tag{10}$$

Eq. (10) has a simple mathematical form, and it is the simplified analytical expression for waveguide with tapered cross section. For acoustic waveguide with abrupt cross section, we can take $\delta_x = 0$. For piece-wise linear waveguide, for example, trigonometric waveguide, the analytical form of system transfer matrix can be easily obtained by multiplying the transfer matrices of the linear elements from the input to the end.

The sound pressure transfer function (SPTF) of the entire waveguide is

$$\mathrm{TL} = 20\lg\left(\left|\frac{p_{\mathrm{in}}}{p_{\mathrm{out}}}\right|\right). \tag{11}$$

When it comes to acoustic waveguide which contains abrupt cross sections, significant deviation might be introduced by ignoring higher-order modes at the abrupt section (Fig. 3).


FIG. 3. (a): Acoustic waveguide with abrupt section. (b): FEM calculation of sound pressure. $\tilde{r}_{n-1} = 2.45$cm, $\tilde{r}_n = 1.25$cm, $l_1 = 3$cm, $l_2 = 5$cm, frequency: 1700Hz.

Selamet, Zhong et al.[11] suggest that the effect of high-order modes is equivalent to the impedance, and they use a two-dimensional analytical method to modify the one-dimensional model. In the calculation, one should multiply the modified matrix $\mathbf{T}'$ at the abrupt section, and the transfer matrix of the entire structure is

$$\widetilde{\mathbf{T}}_{n-1}\mathbf{T}'\widetilde{\mathbf{T}}_n = \widetilde{\mathbf{T}}_{n-1}\begin{bmatrix} 1 & 0 \\ i\frac{\tilde{r}_{n-1}^2-\tilde{r}_n^2}{\tilde{r}_n^2}\tan\left(\alpha k \tilde{r}_{n-1}\right) & \left(\frac{\tilde{r}_{n-1}}{\tilde{r}_n}\right)^2 \end{bmatrix}\widetilde{\mathbf{T}}_n. \tag{12}$$

Here, $\alpha$ is the acoustic length correction coefficient which is related to both the shape of the wall and the frequency of the sound. The numerical table is provided in[11]

### B. Calculation example of TMM

Here we use a complex shape as an example (Fig. 4). The radius of the waveguide follows $R(x) = 0.02 + 0.01\tanh(50(-x+0.1)) + 0.1(-x+0.2)^2$ and the total length of the waveguide $L$ is 20cm. The number of divisions $N$ is set to be 50, and the whole computing process is finished within 1s by a personal computer. The FEM calculation is performed by COMSOL Multiphysics, with same 2-D model set to frequency domain, the frequency scan is performed. The comparison between the results of TMM and FEM are shown in Fig. 5. It can be seen that TMM can provide very high accuracy below 1000Hz in this case. In most of the cases, ITM finishes several times faster because there is no need for iteration for convergence.

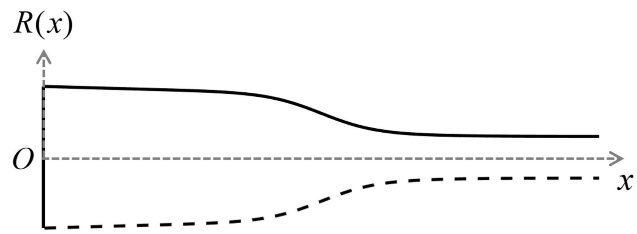


FIG. 4. Waveguide used in the calculation example of TMM.

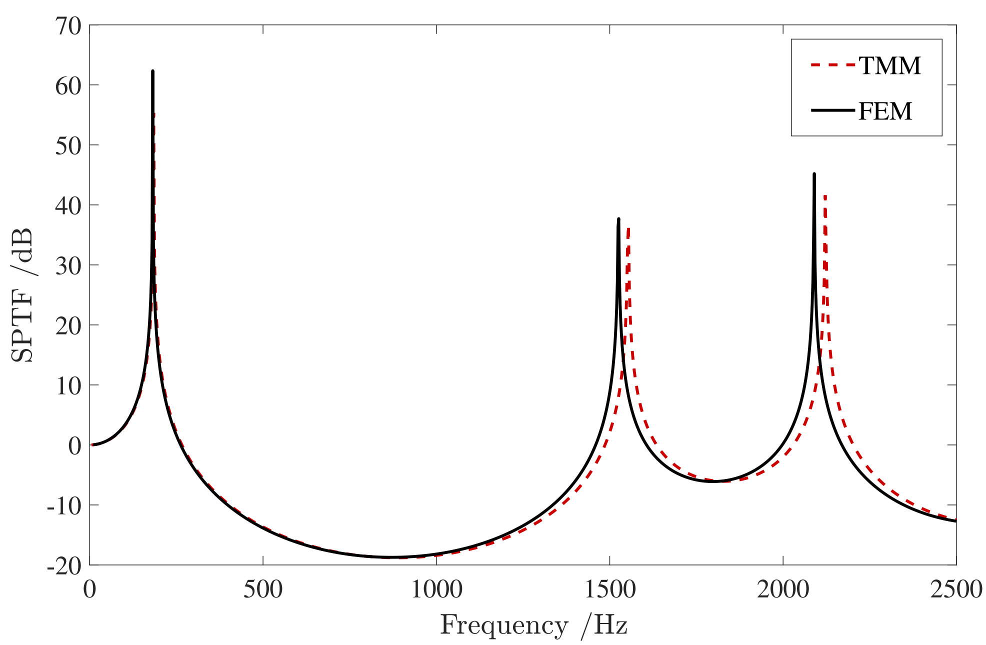


FIG. 5. Comparison between the results of TMM and FEM.

## IV. IMPEDANCE TRANSFER METHOD (ITM)

### A. Theoretical analysis

Here, we propose a method for calculating the resonance frequency of waveguide with arbitrary cross sections based on TMM. Instead of transfer matrix, we use the language of acoustic impedance. The acoustic impedance $z$ is defined as $z = p/v$, where $p$ is the sound pressure, $v$ is the particle velocity. For each element of the waveguide (Fig. 2), the output impedance $\tilde{z}_n$ is the function of the input $\tilde{z}_{n-1}$, From the simplified transfer matrix (10)

$$\tilde{z}_n = \frac{p_{out}}{v_{out}} = \frac{\tilde{A}_n \tilde{z}_{n-1} + \tilde{B}_n}{\tilde{C}_n \tilde{z}_{n-1} + \tilde{D}_n}. \tag{13}$$

Substitute the matrix elements of the matrix (10) into Eq. (13), one obtains

$$\tilde{z}_n = Z_0 \frac{\tilde{r}_n^2 \tilde{z}_{n-1} + ik\tilde{r}_n \tilde{r}_{n-1} Z_0 \delta_x}{\tilde{r}_{n-1}^2 Z_0 + ik\tilde{r}_n \tilde{r}_{n-1} \tilde{z}_{n-1} \delta_x}. \tag{14}$$

Once the input acoustic impedance $z_0$ is given, continuously taking the output impedance of the previous element as the input impedance of the next element, Eq. (14) can be iterated from the input to the end. At last, the acoustic impedance at the end of the waveguide is obtained. When the system is in resonance, it satisfies

$$\mathrm{Im}(z_{out}) = 0. \tag{15}$$

Solving the Eq. (15), the resonance frequency can be obtained. The impedance transfer method (ITM) essentially derives the impedance transfer function of the acoustic waveguide. There is no essential difference between ITM and TMM, but they are based on different understanding of acoustic systems, and are used in different situations.

For an acoustic waveguide with one open end, due to the rigid acoustic boundary, the initial acoustic impedance $z_0$ is usually infinity. Therefore, substituting $z_0$ into Eq. (14), one obtains

$$z_1 = \lim_{z_0 \to \infty} z_1 = -iZ_0 \tan k\delta_x. \tag{16}$$

Then, use Eq. (16) as a start and iterate the equation to the end, the orifice impedance can be obtained and the resonance frequency can be calculated.

Langfeldt et al. suggest that for Helmholtz resonators with multiple necks, the output impedance $Z$ follows[12]

$$Z = \frac{S_0}{\left(\frac{S_1}{Z_1}\right)} + \frac{S_0}{\sum_{i=2}^{N} \frac{S_i}{Z_i} + i\omega \frac{V_0}{\rho_0 c_0^2}}. \tag{17}$$

Here, $S_0$ is the total area of all necks, $V_0$ is cavity volume, $S_i$ is the area of the $i$th neck while $Z_i$ is its neck impedance. Combined with this method, ITM can be further extended to calculating the resonance frequency of resonators with multiple necks.

## B. Calculation example of ITM

A resonator is a waveguide with one open end. Here, we chose a resonator with tapered neck (Fig. 6) as the calculation example. Selamet suggested that significant improvement of the sound absorption capacity can be obtained by introducing neck tapering[13]. However, an effective method for calculating the resonance frequencies of the resonator is still lacking.

The classical theory about Helmholtz resonator is the Rayleigh equation, which is used to calculate the first order resonance frequency[14], and it is

$$f = \frac{c}{2\pi} \sqrt{\frac{\pi r^2}{V\left(l_3 + 2\delta_r\right)}}, \tag{18}$$

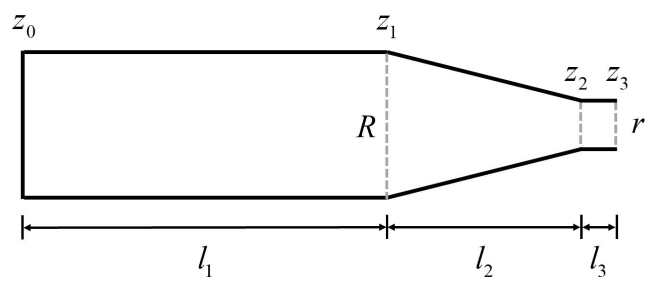


FIG. 6. Resonator with tapered section. In the calculation example we take $R = 3.0$cm, $r = 1.0$cm, $l_2 = 8.0$cm, $l_3 = 1.5$cm.

where $V = \frac{\pi h}{3}(r^2 + R^2 + rR) + \pi R^2 l_1$, $c$ is sound speed, and the Rayleigh correction $\delta_r = 0.849r$. There is also an simplified analytical method for calculating the resonance frequency of resonators with tapered neck[13], in this case, it is

$$f = \frac{c}{2\pi}\sqrt{\frac{\pi r^2}{V}\left[\frac{rl_2}{R}\left(1 + \frac{\pi R^2 l_3}{V}\right) + l_3\right]^{-1}}. \tag{19}$$

Using ITM, the resonator could be divided into two cylindrical part and one tapered part. The input acoustic impedance is $z_0$. $z1$, $z2$ and $z3$ are the output impedance of the first, second and third part, respectively (Fig. 6). Using Eqs. (14), (15), and (16), the iteration process would be

$$\begin{aligned} z_1 &= -i\rho_0 c_0 \tan kl_1, \\ z_2 &= \rho_0 c_0 \frac{r^2 z_1 + i\rho_0 c_0 kRrl_2}{\rho_0 c_0 R^2 + ikz_1 Rrl_2}, \\ z_3 &= \rho_0 c_0 \frac{z_2 + \rho_0 c_0 i \tan kl_3}{\rho_0 c_0 + iz_2 \tan kl_3}. \end{aligned} \tag{20}$$

Generally, the iteration can be done both analytically or numerically by a personal computer. Once the iteration finishes, numerically solving Eqs. (15) and (20), the relationship between the resonance frequency and cavity length $l_1$ is then obtained. The comparison of the four methods is shown in Fig. 7.

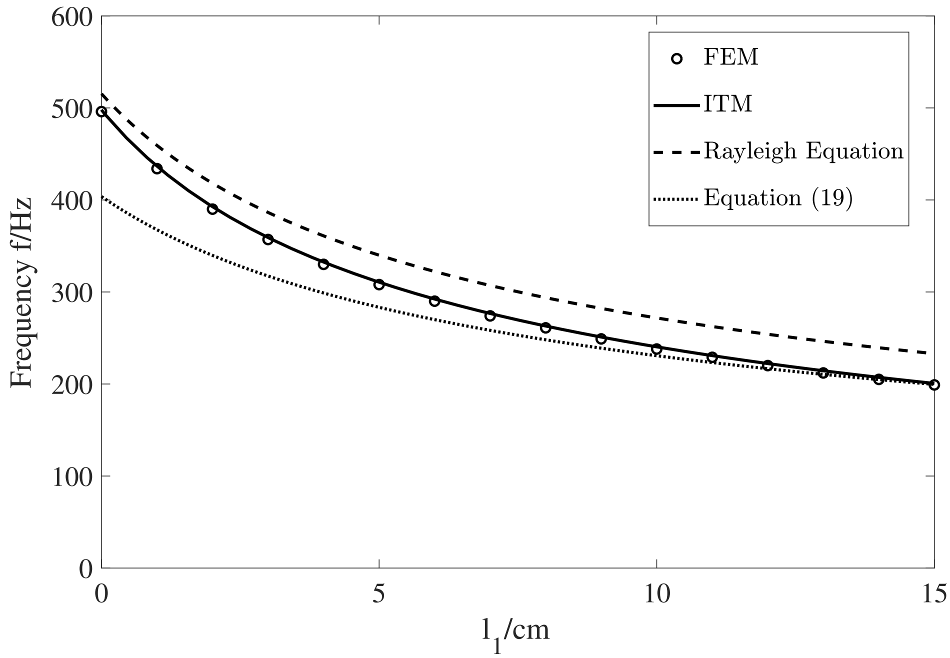


FIG. 7. First order resonance frequency of the tapered section resonator.

It can be seen in Fig. 7 that the result of ITM is very close to FEM. Eq. (19) is only accurate when $l_1$ is large, and the result of Rayleigh Eq. (18) deviates significantly in the whole figure. Moreover, the first five orders resonance frequencies are also given and compared with FEM calculations in Fig. 8. The relative error between the results from ITM and FEM is less then 1%, which proves ITM is one of the most accurate and methods up to date, and the calculation efficiency is superior.

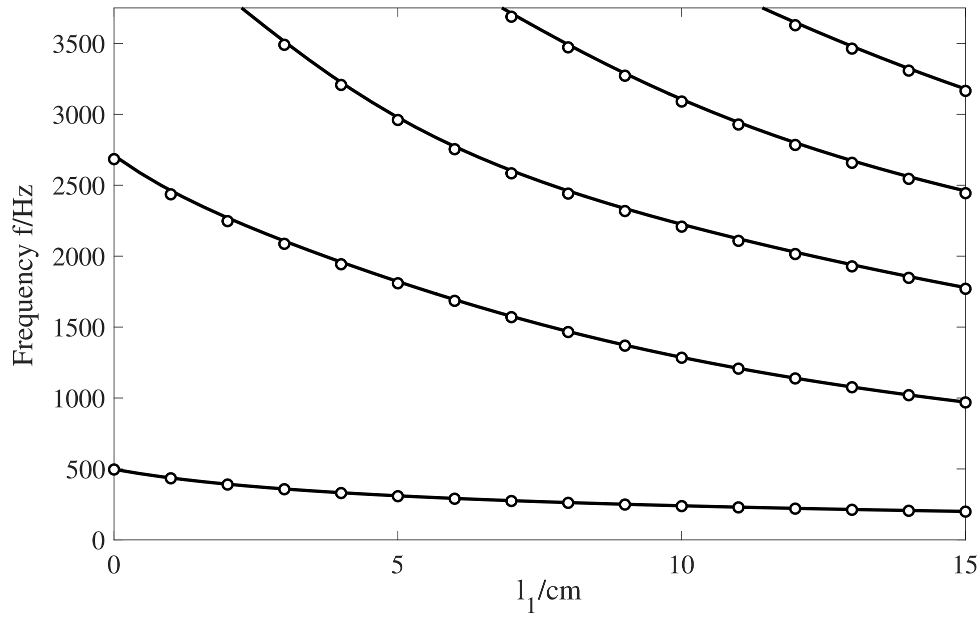


FIG. 8. First five orders resonance frequency of the tapered section resonator.

## 170 V. CONCLUSION AND DISCUSSION

In this paper, new transmission methods are developed for analyzing the wave propagation in axisymmetric waveguide with arbitrary cross section. The core idea of the new methods is to make a first-order approximation to boundary elements, based on the idea, TMM is extended to the analysis of waveguide with arbitrary cross section and ITM is developed for calculating the resonance frequency. In our research, finite element calculations are also studied and compared with the results of new methods, comparisons show that the new methods can provide good accuracy over a relatively wide frequency range. Compared to existing numerical method, much less computing time is required because they do not go over the iteration process and converging tests.

However, the new methods are not suitable for analyzing the high frequency performance of an acoustic waveguide, nor can it be extended to the analysis of a three-dimensional waveguide with arbitrary shape. This is because the transfer functions are obtained by two-dimensional analytical analysis and the plane wave assumption.

In many practical applications of analyzing the performance of waveguide with circular cross section, the new methods can be used to replace FEM calculations or to examine it. The new methods converge very quickly, and could be even quicker with a adaptive step length algorithm. The new methods basically do not encounter the singularity and convergence problem of the solution. Compared with FEM, the computing efficiency of the new methods are significantly higher.

Combining existing research, the new methods can be extended to the prediction of sound absorption and transmission of waveguide with multiple necks and extended necks. Meanwhile, this research also provide inspiration for predicting sound transmission of multilayered materials or a collection of acoustic elements in parallel.

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