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Optimal Trading with Stochastic Liquidity and Volatility*

Robert Almgren†

Abstract. We consider the problem of mean-variance optimal agency execution strategies, when the market liquidity and volatility vary randomly in time. Under specific assumptions for the stochastic processes satisfied by these parameters, we construct a Hamilton–Jacobi–Bellman equation for the optimal cost and strategy. We solve this equation numerically and illustrate optimal strategies for varying risk aversion. These strategies adapt optimally to the instantaneous variations of market quality.

Key words. optimal trading, dynamic programming, mean-variance optimization

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A fundamental part of agency algorithmic trading in equities and other asset classes is trade scheduling. Given a trade target, that is, a number of shares that must be bought or sold before a fixed time horizon, trade scheduling means planning how many shares will be bought or sold by each time instant between the beginning of trading and the horizon. This is done so as to optimize some measure of execution quality, usually measured as the final average execution price relative to some benchmark price.

One of the most popular benchmarks is the "arrival price," that is, the price prevailing in the market at the time the order was received into the trading system. The difference between the execution price and this pretrade price is the "implementation shortfall" [12] or "slippage." For example, if we are buying, then if the price drops in the course of execution, the trade cost will be negative; if the price rises, whether because of the impact of this trade or random market fluctuation, then the trade cost will be positive. The realized execution cost is thus inevitably a random variable whose properties are to be tailored.

Grinold and Kahn [8] and Almgren and Chriss [1] suggested that the optimal trajectory could be determined by balancing market impact cost, which leads toward slow trading so as to reduce the expected value of execution cost, against market volatility which pushes toward rapid completion of the order so as to reduce the variance of the execution cost. Optimal trade schedules are typically "front-loaded": They execute as much as possible early in the program to reduce risk relative to the benchmark price. The degree of front-loading depends on a risk-aversion parameter that must be specified by the trading client. The exact shape of the schedule depends on the form of the assumed market impact model. Despite its approximations, this "arrival price" framework has proven remarkably robust and useful in designing practical trading systems.

The largest alternative category of benchmarks is composed of some form of average

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†Courant Institute, New York University, New York, NY 10012, and Quantitative Brokers LLC, New York, NY 10036 (almgren@cims.nyu.edu).

market price during the trading interval: usually either time-weighted average price (TWAP) or volume-weighted average price (VWAP). For those benchmarks, optimal strategy follows the benchmark profile closely, since deviation from that profile generally increases both risk relative to the benchmark and impact costs. Determining optimal response to short-term price signals is an interesting topic for optimization, but it is not the subject of this paper.

A fundamental assumption of most of this work has been that the market parameters, such as liquidity and volatility, are constant or at least have known predictable profiles. This assumption is reasonably accurate for large-cap US stocks. Under that assumption, and assuming that mean and variance are reevaluated as execution proceeds (see [11, 14]), optimal strategies are *static*; that is, the trade schedule can be determined before trading starts and is not modified by the new information revealed by price moves during trading. (Almgren and Chriss [1] did consider a model in which the market parameters updated at a single time to one of a known set of possible new values.)

The genesis of this work was a problem posed to the author when he worked at a major investment bank several years ago: figure out how to extend effective algorithmic trade execution from large capitalization stocks down to smaller cap stocks that are widely recognized as being more "difficult" to trade. A distinguishing feature of less liquid assets is that their liquidity and volatility vary randomly in time. That is, there will be times during the trading day when trading is very expensive and times when trading is cheap; similarly there will be times when delaying trading introduces large amounts of volatility risk and other times when delay is relatively costless. The modeling challenge is to determine optimal strategies that adapt to the instantaneous market state, while retaining the mean-variance tradeoff inherent in the arrival price framework. Walia [15] solved this problem in a discrete-time discrete-state model. This paper provides a systematic mathematical solution to this problem in continuous time and continuous state.

In section 1 we present the basic price and impact models that we use, and we present the optimal trading problem. We present the special case of "coordinated variation" in which liquidity and volatility vary together, which is very realistic and which greatly simplifies the mathematical problem. We also present the small-impact approximation in which risk is simply accumulated via volatility; this approximation is necessary for a dynamic programming solution. In section 2 we use dynamic programming to construct a Hamilton–Jacobi–Bellman (HJB) partial differential equation (PDE) describing the optimal cost function and trade rate. In section 3 we describe some aspects of the numerical solution of this PDE and present example solutions.

1. The liquidation problem.

The trader begins at time $t = 0$ with a purchase target of X shares, which must be completed by time $t = T$. The number of shares remaining to purchase at time t is the trajectory $x(t)$, with $x(0) = X$ and $x(T) = 0$. The rate of buying is $v(t) = -dx/dt$. Thus for a buy program, $X > 0$, and we expect $x(t) \geq 0$ decreasing and $v(t) \geq 0$ (a sell program may be modeled similarly). In general, the trajectory $x(t)$ may be determined depending on price motions and on market conditions that are discovered during trading, so it is a random variable.

The price $S(t)$ follows arithmetic Brownian motion

$$dS(t) = \sigma(t)\, dB(t), \qquad S(0) = S_0,$$

where $B(t)$ is a standard Brownian process, and the instantaneous volatility $\sigma(t)$ depends on time either deterministically or randomly. It is possible to include permanent market impact in the price equation, but it is not central to our problem. The total time T is usually one day or less, so the difference between arithmetic and geometric Brownian motion is negligible. But note that σ is an absolute volatility rather than fractional, and hence contains an implicit factor of some reference price S_0.

The price actually received on each trade is

$$\tilde{S}(t) = S(t) + \eta(t)\, v(t),$$

where $\eta(t)$ is the coefficient of temporary market impact, also time-varying. Again, η is an absolute coefficient rather than fractional. Much richer market impact models have been considered in the literature [5], but this simple one is adequate to highlight the response to stochastic liquidity, which is our main goal.

We assume that both $\sigma(t)$ and $\eta(t)$ are observable in real time with some reasonable degree of confidence. There is a variety of available techniques for doing this estimation:

- For volatility $\sigma(t)$, there is an extensive literature on estimation using high-frequency market data (for example, [6]). The primary focus of this work is to find an effective means to filter out noise associated with market details such as bid and offer prices, so as to obtain reliable estimates on time intervals that are as short as possible. Thus, for example, one could estimate $\sigma(t)$ by using market data from the preceding 5 minutes, which typically would contain hundreds of trades and potentially thousands of quote updates.
- Instantaneous liquidity, the inverse of $\eta(t)$, is more difficult to estimate, since it is the estimation of what *would* happen if one were to submit trades to the market rather than a quantity observable in itself. One proxy for instantaneous liquidity would be the realized trade volume over the last few minutes: if more people are trading actively in the market, then we will be able to move a given number of shares with lower slippage. A refined version of this would be to measure trade volume occurring at or near the bid price if we are a buyer (at the ask price if we are a seller): large volume there might indicate the presence of a motivated seller and a good opportunity for us to go in as a buyer with low impact. Although these measures are not quantitatively very precise, they are often adequate to distinguish "good" market conditions from "bad."

Both of these estimators rely on the persistence of market properties (volatility and liquidity), so that information about the past provides reasonable forecasts for the future. Such persistence, at least across short horizons, is well documented (see [3] for a recent review).

We may address two broad classes of problems:

1. First is the case in which $\sigma(t)$ and $\eta(t)$ are known nonrandom functions of time. This would accommodate the well-known intraday profiles of liquidity and volatility: generally markets are more active in the morning and near the close than in the middle of the day. This case is not our primary focus.
2. The second case is when volatility and liquidity vary randomly through the day, so that $\sigma(t)$ and $\eta(t)$ follow some stochastic processes. This effect is very important in algorithmic trading of small- and medium-capitalization stocks and other assets that are less heavily traded than large-cap US stocks.

1.1. Cost of trading. The *cost of trading*, relative to the arrival price benchmark, is the difference between the total dollars paid to purchase X shares and the initial market value:

$$\begin{aligned}\mathcal{C} \;&=\; \int_0^T \tilde{S}(t)\, v(t)\, dt \;-\; X\, S_0 \\ &=\; \int_0^T \sigma(t)\, x(t)\, dB(t) \;+\; \int_0^T \eta(t)\, v(t)^2\, dt\end{aligned}$$

after integrating by parts and using $x(t) = \int_t^T v(s)\, ds$. The cost $\mathcal{C}$ is a random variable, both because of the price uncertainty $B(t)$ in the first term and because of the liquidity uncertainty. We determine the strategy $x(t)$ to tailor the properties of this random variable to meet some optimality criterion.

More generally, starting at time $t \geq 0$ with $x(t)$ shares remaining to purchase, the cost of a strategy $x(s)$ on $t \leq s \leq T$ is

$$\mathcal{C} \;=\; \int_t^T \sigma(s)\, x(s)\, dB(s) \;+\; \int_t^T \eta(s)\, v(s)^2\, ds. \tag{1.1}$$

We determine the optimal trajectory by the mean-variance criterion

$$\min_{x(s): t \leq s \leq T} \mathrm{E}(\mathcal{C}) \;+ \lambda\, \mathrm{Var}(\mathcal{C}),$$

where $\lambda \geq 0$ is a risk-aversion coefficient, and expectation E and variance Var are taken at time t, assuming that the history from time 0 to t is no longer random. This is the appropriate point of view for a trader who wants to optimize the mean and variance of the remaining trade profile; it is not necessarily the appropriate point of view for a trade desk that wants to optimize sample mean and variance across a collection of trades (see Note at the end of section 1.2 below).

Note that

$$\mathrm{E}(\mathcal{C}) \;=\; \mathrm{E} \int_t^T \eta(s)\, v(s)^2\, ds,$$

since the first term is an Itô integral, and

$$\begin{aligned}\mathrm{Var}(\mathcal{C}) = \mathrm{E} &\int_t^T \sigma(s)^2\, x(s)^2\, ds \\ &+ \big\{\text{terms arising from uncertainty in } \eta(s),\ \sigma(s), \text{ and } v(s)\big\}.\end{aligned} \tag{1.2}$$

The first term in (1.2) contains the largest source of uncertainty, which is price changes during execution. The other terms arise from uncertainty in the market impact $\eta(s)$ that will be paid on a transaction in the future, in the volatility $\sigma(s)$ that will be experienced at a later time, and in the trade strategy $v(s)$ itself if it is determined in response to uncertain market conditions. We shall argue below that the first term in (1.2) dominates the other terms.

The risk-aversion coefficient λ is rarely determined in terms of fundamental investment preferences (but see [4]). Rather, it is used as a parameter to adjust the trajectories to a form that seems reasonable by other criteria such as representing a desired fraction of market volume.

It is not entirely obvious that variance is the best measure of risk, nor that risk is the primary driver for accelerating trading and incurring the resulting higher impact costs. Other motivations for rapid trading might be a short-term price signal ("alpha") which is expected to dissipate rapidly. Such signals can be included as a drift term in the stochastic differential equation (SDE), but that is not our focus here. One might also interpret risk as exposure to events (Greek default, say); each minute that passes before the trade is completed incurs such risk, with an exposure level perhaps proportional to the open position $x(s)$ or its square. One might also measure risk in terms of Value-at-Risk (VaR), semivariance, or other risk measures. Despite all these possibilities, the simple mean-variance tradeoff expressed above is commonly used to capture the essential features of trade execution.

1.2. Constant coefficients. The "classic" problem [1] takes σ and η to be constant. Then for a strategy $x(t)$ that is fixed in advance and does not adapt to price motions,

$$\mathrm{E}(\mathcal{C}) + \lambda \operatorname{Var}(\mathcal{C}) \;=\; \int_t^T \Big(\eta\, v(s)^2 \;+\; \lambda\sigma^2 x(s)^2 \Big)\, ds. \tag{1.3}$$

No expectation appears on the right-hand side because the trajectory $x(t)$ is assumed fixed, the market parameters are constant, and the variance of price motion has been incorporated into σ^2. Again, the mean and variance are assumed to be reevaluated at each time t for the remaining part of the trajectory to T.

Using the calculus of variations to minimize this over trajectories $x(s)$ gives the second-order ordinary differential equation (ODE)

$$\frac{d^2x}{ds^2} \;=\; \kappa^2\, x(s), \qquad \text{with} \quad \kappa^2 = \frac{\lambda\sigma^2}{\eta}.$$

The solution is a combination of exponentials $\exp(\pm\kappa s)$

$$x(s) \;=\; x(t)\, \frac{\sinh\big(\kappa(T-s)\big)}{\sinh\big(\kappa(T-t)\big)}, \qquad v(s) \;=\; \kappa\, x(t)\, \frac{\cosh\big(\kappa(T-s)\big)}{\sinh\big(\kappa(T-t)\big)}.$$

Thus $1/\kappa$ is the characteristic time scale of liquidation. With linear impact (see [2] for the nonlinear case), this time scale does not depend on portfolio size.

This strategy may also be expressed as the rule for $v(t)$

$$v(t) \;=\; \kappa\, x(t)\, \coth\big(\kappa(T-t)\big). \tag{1.4}$$

The cost function is

$$\mathcal{C}(x,t,\eta,\sigma) \;=\; \eta\,\kappa\, x^2 \coth\big(\kappa(T-t)\big) \;=\; \eta\, v(t)\, x. \tag{1.5}$$

The total cost is equal to the impact cost component (neglecting the volatility term) incurred by trading x shares at a price concession given by the instantaneous velocity v. The actual trajectory slows down as the position size decreases, thus reducing market impact costs, but the total cost includes volatility risk as well as impact cost, giving the above value.

The shape of the solution is governed by the nondimensional quantity $\kappa(T-t)$, the ratio of time remaining to the intrinsic time scale determined by the market parameters and the trader's risk aversion.

In the infinite-horizon limit $\kappa(T-t) \gg 1$, the strategy has the limit $x(s) = x(t)\exp\bigl(-\kappa(s-t)\bigr)$ independently of T, with $v(t) = \kappa x(t)$ and execution cost $\mathcal{C} \to \eta\kappa x^2$. That is, when the horizon T is much longer than the intrinsic execution time $1/\kappa$, the value of T has almost no effect on the trade trajectory. This is also the solution that would be obtained if the problem truly had an infinite horizon, that is, no time limit on execution. Note that we have not included discounting, so the only motivation for rapid execution is risk aversion. In the limit of complete risk neutrality ($\lambda \to 0$), minimization of market impact costs would lead the trader to use all available time, and no infinite-horizon limit would exist; since $\kappa \to 0$ we would never achieve the regime $\kappa T \gg 1$.

In the short-horizon limit $\kappa(T-t) \ll 1$, the optimal strategy has the linear form $v(s) = x(t)/(T-t)$ and the cost function has local behavior

$$\mathcal{C} \;\sim\; \frac{\eta x^2}{T-t} \;+\; \frac{\lambda\sigma^2 x^2}{3}\,(T-t) \;+\; \mathcal{O}\bigl((T-t)^3\bigr), \qquad \kappa(T-t) \to 0, \tag{1.6}$$

where $\sim$ denotes asymptotic equivalence, that is, equal up to terms that are asymptotically smaller than the displayed expressions in the given limit. The first term in this expression is the transaction costs associated with selling x shares at a price concession of $\eta v = \eta x/(T-t)$. The second term is the risk penalty for holding an average of $x^2/3$ shares across time $T-t$.

Note. Whether adaptive strategies are better than fixed ones is a subtle question. Almgren and Chriss [1] showed that if the strategy is reevaluated at an intermediate time, using mean and variance measured at that time, then the optimal strategy is the remaining part of the initial strategy, and hence the optimal strategy is fixed. That is the context in which we work here, since it is appropriate for execution of a single transaction. In contrast, Lorenz and Almgren [11] showed that adaptive strategies are optimal if mean and variance are measured at the initial time, for portfolios that are large enough so that their impact is a substantial fraction of volatility, and Tse et al. [14] have given a fuller discussion of optimal solutions in that framework. Measuring mean and variance at the initial time is appropriate for a trading desk that wants to optimize the reported ex post measured sample mean and variance across a large collection of similar trades. Schied, Schöneborn, and Tehranchi [13] showed that improvement from adaptivity depends on the risk-aversion profile; for example, it vanishes for a utility function with constant absolute risk aversion (CARA).

1.3. Coordinated variation. Suppose $\sigma(t)$ and $\eta(t)$ vary perfectly inversely, so that

$$\sigma(t)^2\,\eta(t) \;=\; \text{constant} \;=\; \bar\sigma^2\,\bar\eta,$$

where $\bar\sigma$ and $\bar\eta$ are constant reference values. For example, this relationship would be a natural consequence of a "trading time" model [7, 9] in which the single source of uncertainty is the arrival rate of trade events. If each trade event brings both a fixed amount of price variance and the opportunity to trade a fixed number of shares for a particular cost, then we obtain the above relation. This assumption is reasonable for markets during “normal” periods, though it can be severely violated during exceptional events when volatility sharply increases but liquidity is also withdrawn. Since the largest fraction of algorithmic execution occurs during normal periods, we are reasonably comfortable with this approximation.

We may then change time to an artificial variable $\hat{t}(t)$ defined by

$$d\hat{t} \;=\; \sigma(t)^2\, dt.$$

In this time frame, we have a modified Brownian motion $\hat{B}(\hat{t})$ with

$$d\hat{B}(\hat{t}) \;=\; \sigma(t)\, dB(t).$$

The share holdings are the same trajectory at different times, so $\hat{x}(\hat{t}) = x(t)$. The trade rate is modified to

$$\hat{v}(\hat{t}) \;=\; -\frac{d\hat{x}}{d\hat{t}} \;=\; \frac{v(t)}{\sigma(t)^2}.$$

In terms of these new variables, the trading cost is

$$\mathcal{C} \;=\; \int_0^{\hat{T}} \hat{x}(\hat{t})\, d\hat{B}(\hat{t}) \;+\; \bar{\sigma}^2\bar{\eta} \int_0^{\hat{T}} \hat{v}(\hat{t})^2\, d\hat{t},$$

where $\hat{T} = \hat{t}(T)$ is the time horizon in the changed variable. This is easy to solve in two cases:

1. If the time-varying liquidity and volatility have known nonrandom profiles, then we can compute the upper bound $\hat{T}$ explicitly. The problem reduces exactly to the constant-coefficient problem (1.3), and the solution is the exponential functions computed there. The rule (1.4) becomes

   $$v(t) \;=\; \kappa(t)\, x(t)\, \coth\!\left(\frac{\kappa(t)}{\sigma(t)^2} \int_t^T \sigma(s)^2\, ds\right),$$

   where $\kappa(t) = \sqrt{\lambda\sigma(t)^2/\eta(t)}$ is the time scale formed with the instantaneous values of the parameters. This change to “volume time” is the first and easiest way to accommodate intraday seasonality.
2. If the coefficients vary randomly, then the problem is not the same as the constant-coefficient problem, because of the uncertainty in the end time. But in the infinite-horizon limit $T \to \infty$, we also have $\hat{T} \to \infty$ (under very mild assumptions on $\sigma(t)$, and also assuming nonzero risk aversion). The trade rate is $v(t) = \kappa(t)x(t)$ and the cost is $C = \eta(t)\kappa(t)x^2 = x^2\sqrt{\lambda\bar{\sigma}^2\bar{\eta}}$. We thus use the static solution formula, though the actual rate of execution at time t is not fixed until we observe $\sigma(t)$ and $\eta(t)$ and hence determine $\kappa(t)$.

Somewhat surprisingly, the optimal cost in the coordinated-variation random-market infinite-horizon case does not depend on the instantaneous market state $\eta(t)$ and $\sigma(t)$, though the instantaneous rate of trading certainly does depend on the market state. In effect, since volatility is low whenever impact is high, the strategy is always able to wait for favorable market conditions without incurring very much risk from the delay.

Thus the interesting problems come from two sources:

- With nonrandom coefficients, the variation of the profiles of $\sigma(t)$ and $\eta(t)$ away from the "base case" $\sigma(t)^2\eta(t) =$ constant. For example, even within the "trading time" framework, intraday profiles may vary away from $\sigma^2\eta =$ constant because different market participants are active at different times of the day. This leads to problems in ODEs that are not the focus of this paper.
- With random coefficients, the proper handling of uncertainty as the end time is approached, even in the coordinated variation case. For example, if liquidity is temporarily low, is it worthwhile to wait for a better opportunity, or is there a large risk that this opportunity will not come before expiration? Addressing this tradeoff is the major subject of this paper.

1.4. Rolling horizon approximate strategy. One way to determine a plausible strategy is to use the formula (1.4) to compute $v(t)$, using the instantaneous values of $\eta(t)$ and $\sigma(t)$ and hence $\kappa(t)$. That is, we assume that the values observed at each instant will remain constant through the end of the liquidation period, and we determine the statically optimal strategy using those values. When the values change, we recompute a stationary solution. This strategy is strictly optimal only in the infinite-horizon case, and only when the market parameters covary in the appropriate way. In general it is not optimal but provides a reasonable solution that is easy to implement. Walia [15] carefully compared the rolling horizon strategy with the true optimal strategy in a discrete framework.

1.5. Small-impact approximation. To do dynamic programming when $\eta(t)$ and $\sigma(t)$ vary randomly, from now on we approximate the variance by keeping only the first term in (1.2). We call this the "small-impact approximation": When market impact is small in absolute terms, it is nonetheless significant because it is always positive, but the uncertainty in market impact is negligible compared to price volatility. This is equivalent to the "small-portfolio" approximation used in [11] to neglect uncertainty in impact cost when the portfolio is small enough. That approximation relies on small values of the "market power" parameter $\mu = (\eta X/T)/(\sigma\sqrt{T})$, the amount by which our trading moves the market compared to its intrinsic motion due to volatility, across time T.

Then we take as the value function

$$c(t,x,\eta,\sigma) \;=\; \min_{v(s),t\le s\le T} \mathrm{E}\int_t^T \Big(\lambda\sigma(s)^2x(s)^2 \;+\; \eta(s)\,v(s)^2 \Big)\,ds \tag{1.7}$$

to which dynamic programming can easily be applied.

2. Dynamic programming. It is simplest to derive the PDE for the coordinated variation case directly, since that is the only one we attempt to solve numerically. After this derivation, we briefly indicate the extensions that are necessary to handle the general case.

Since $\eta(t)$ and $\sigma(t)$ are positive, it is convenient to write

$$\eta(t) \;=\; \bar\eta\exp\xi(t) \qquad\text{and}\qquad \sigma(t) \;=\; \bar\sigma\exp\left(-\frac{\xi(t)}{2}\right),$$

where $\bar\eta$ and $\bar\sigma$ are typical values of σ and η, and $\xi(t)$ is a single nondimensional variable indicating the "market state": When $\xi(t)$ is large positive, the market is nonvolatile and illiquid and we should trade slowly; when $\xi(t)$ is large negative, the market is volatile and liquid and we should trade fast. We write $\bar\kappa = \sqrt{\lambda\bar\sigma^2/\bar\eta}$ for the intrinsic time scale in the mean market state and $\kappa = \sqrt{\lambda\sigma^2/\eta} = \bar\kappa e^{-\xi}$ for the instantaneous value. The value function $c(t,x,\xi)$ then depends on only three variables.

We assume that $\xi(t)$ solves an SDE of the form

$$d\xi \;=\; a(\xi)\,dt \;+\; b(\xi)\,dB_{\mathrm{L}}(t),$$

where $B_{\mathrm{L}}(t)$ is a Brownian motion independent of that driving the price motion. Then by standard dynamic programming applied to (1.7) with the instantaneous trade rate $v = -dx/dt$ as the control parameter, we write

$$c(t,x,\xi) = \min_v \Big[\; \lambda\sigma^2 x^2 dt \;+\; \eta v^2 dt \;+\; \mathrm{E}c\big(t+dt, x+dx, \xi+d\xi\big) \;\Big],$$

giving the HJB PDE

$$0 \;=\; c_t \;+\; \lambda\sigma^2x^2 \;+\; \min_v\Big[\,\eta v^2 \;-\; vc_x\,\Big] \;+\; ac_\xi \;+\; \frac{1}{2}b^2c_{\xi\xi}.$$

The minimum is clearly

$$v \;=\; \frac{c_x}{2\eta}, \tag{2.1}$$

and then the PDE for c is

$$-c_t \;=\; \lambda\sigma^2x^2 \;-\; \frac{c_x^2}{4\eta} \;+\; ac_\xi \;+\; \frac{1}{2}b^2c_{\xi\xi}. \tag{2.2}$$

The initial data for the PDE (2.2) is in fact a local asymptotic condition and must be treated with some care. Near expiration, we must liquidate on a linear trajectory $v = x/(T-t)$. As with constant coefficients (1.6), the cost comes primarily from market impact. To accurately approximate the cost, we must account for expected changes in the impact coefficient during the time $t \le s \le T$. A simple application of Itô's lemma shows that

$$\mathrm{E}\eta(s) \;\sim\; \eta(t)\left(\;1 \;+\; \left(a+\frac{1}{2}b^2\right)(s-t)\;\right), \qquad s-t\to 0,$$

and hence the average value of $\eta(s)$ for s between t and T is

$$\eta \;\sim \eta(t)\left(\;1 \;+\; \frac{1}{2}\left(a+\frac{1}{2}b^2\right)(T-t)\;\right), \qquad T-t\to 0.$$

Using this value in (1.6), with $\eta(t) = \bar\eta e^{\xi}$, we have the cost expansion

$$c(t,x,\xi) \;\sim\; \frac{\bar\eta e^{\xi}x^2}{T-t} \;+\; \frac{1}{2}\left(a+\frac{1}{2}b^2\right)\bar\eta e^{\xi}x^2 \;+\; \mathcal{O}(T-t), \qquad T-t\to 0. \tag{2.3}$$

In the $\mathcal{O}(T-t)$ terms there would appear both the risk contribution and a further expansion of the impact cost.

2.1. Lognormal model and nondimensionalization. We now assume that $\xi(t)$ evolves according to an Ornstein–Uhlenbeck mean-reverting process of mean zero. Thus we set

$$a(\xi) = -\frac{\xi}{\delta} \qquad \text{and} \qquad b(\xi) = \frac{\beta}{\sqrt{\delta}}.$$

Here δ is a market relaxation time, and β is a nondimensional "burstiness" parameter describing the dispersion of liquidity and volatility around their average levels. In steady state, $\xi(t)$ is normal, with unconditional moments

$$\mathrm{E}\big(\xi(t)\big) = 0, \qquad \mathrm{Var}\big(\xi(t)\big) = \frac{1}{2}\beta^2.$$

We also observe that the value function $c(t,x,\xi)$ is strictly proportional to x^2, the square of the number of shares remaining to execute. This is a consequence of our linear market impact model, since both variance and expected cost are quadratic in quantity.

We may now nondimensionalize using δ as the time scale and incorporating the factor x^2. We define $\tau = (T-t)/\delta$, the time remaining to expiration as a multiple of the market relaxation time, and set

$$c(t,x,\xi) \;=\; \frac{\bar{\eta}\, x^2}{\delta}\, u\!\left(\frac{T-t}{\delta},\, \xi\right),$$

where $u(\tau,\xi)$ is a nondimensional function of nondimensional variables. Then (2.2) becomes

$$u_\tau \,+\, \xi\, u_\xi \;=\; e^{-\xi}\big(K^2 - u^2\big) \,+\, \frac{1}{2}\,\beta^2 u_{\xi\xi}, \tag{2.4}$$

in which the nondimensional risk-aversion parameter is

$$K \;=\; \bar{\kappa}\,\delta \;=\; \frac{\text{market relaxation time}}{\text{trade time scale in mean market state}}.$$

From (2.1), the dimensional trade velocity is

$$v \;=\; \frac{x}{\delta}\, e^{-\xi}\, u\big(\tau,\xi\big). \tag{2.5}$$

Substituting the above expressions into (2.3), we determine the initial condition

$$u(\tau,\xi) \;\sim\; \frac{e^{\xi}}{\tau} \,-\, \frac{1}{2}\left(\xi - \frac{1}{2}\beta^2\right) e^{\xi} \,+\, \mathcal{O}(\tau) \qquad \text{as } \tau \to 0 \text{ for each fixed } \xi. \tag{2.6}$$

For $\xi < 0$ when trading is fast, the region of approximate validity of this expression is limited by the rate of trading itself, and this expression should be replaced by (2.7) for $\tau > \mathcal{O}(e^{-}\xi)$. For $\xi > 0$ when trading is slow, the region of validity is $\tau \ll \mathcal{O}(1)$, since the market state itself changes on times of scale $O(1)$. Figure 1 summarizes the various asymptotic behaviors of solutions to (2.4).

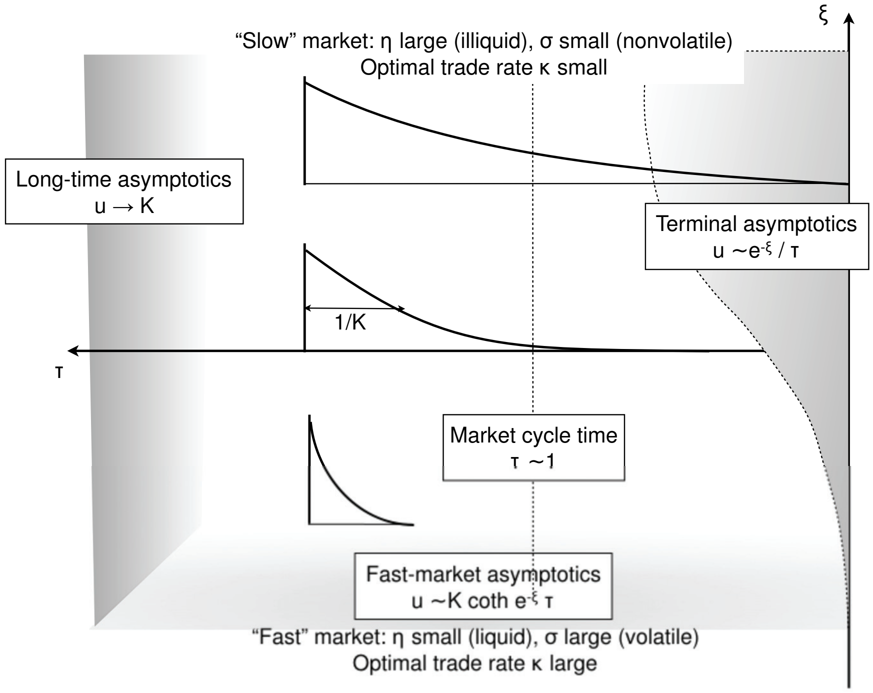

Figure 1. *Asymptotic regions for solutions to PDE (2.4). Example trade trajectories are illustrated for $\xi > 0$, for $\xi < 0$, and for $\xi = 0$. The risk parameter K is the time scale of the optimal trajectory in the "base state" $\xi = 0$ as a fraction of the market cycle time δ.*

Constant market. The steady-state market takes $\beta = 0$. Along the line $\xi = 0$, the PDE (2.4) reduces to the ODE

$$u_\tau \;=\; K^2 - u^2, \qquad \text{with} \qquad u(\tau) \sim \frac{1}{\tau} + \mathcal{O}(\tau) \quad \text{as } \tau \to 0,$$

whose solution is

$$u(\tau) \;=\; K \coth K\tau.$$

On undoing the changes of variables, this reduces exactly to (1.5).

To generalize the above solution, we consider the limit $\xi \to -\infty$. That is, market impact is temporarily very small, and volatility is very large: the optimal strategy is to trade very quickly. Since the market relaxation time scales are fixed, fast trading means that the program is completed before the market parameters have time to change. Thus the cost is the static cost (1.5) using the instantaneous market parameters, which in the transformed functions becomes

$$u(\tau,\xi) \;\sim\; K \coth\Big(K\, e^{-\xi}\tau\Big), \qquad \xi \to -\infty. \tag{2.7}$$

The corresponding trade rate is the "rolling horizon" strategy of section 1.4, which is always an admissible though suboptimal strategy. The expression (2.7) accurately describes the optimal cost only in the indicated limit when indeed the market coefficients do not change substantially before trading is completed.

Long time. As noted in section 1.3, with coordinated variation when time is far from expiration, the value function is $C = x^2\sqrt{\lambda\bar{\sigma}^2\bar{\eta}}$, or $u(\xi,\tau) \to K$ as $\tau \to \infty$ in nondimensional terms. Certainly $u = K$ is a steady-state solution of the PDE. And since the value function must be decreasing in τ, we know that $u(\tau,\xi) \geq K$ for all $\tau > 0$. As a consequence, we know that the initial expansion (2.6) can be valid only when $e^\xi/\tau \geq K$ or $\tau \leq \mathcal{O}(e^\xi)$, a very thin region when ξ is negative.

Provided that a unique solution $u(\tau,\xi)$ to (2.4) exists, a standard verification argument tells us that this function indeed gives us the optimal control to the original control problem.

2.2. Initial behavior of PDE. First, note that (2.4) is a nondegenerate diffusion equation with lower-order terms and thus certainly has smooth unique solutions locally in time if the solution at some positive time satisfies $u(\tau,\xi) < C\exp(a\xi^2)$.

To understand the singular behavior near the initial time, we write

$$u(\tau,\xi) \;=\; \frac{e^\xi}{\tau}\Big(1 \,+\, w(\tau,\xi)\Big),$$

where $w(\tau,\xi)$ for $\tau > 0$ satisfies the PDE

$$\begin{aligned} w_\tau \,+\, \frac{w}{\tau}(1+w) = K^2\,\tau\, e^{-2\xi} \,&-\, \Big(\xi - \frac{1}{2}\beta^2\Big) \\ &-\, \Big(\xi - \frac{1}{2}\beta^2\Big)\, w \,-\, \Big(\xi - \beta^2\Big)\, w_\xi \,+\, \frac{1}{2}\,\beta^2\, w_{\xi\xi} \end{aligned}$$

and has $\lim_{\tau\to 0} w(\tau,\xi) = 0$ for each ξ. It is in this sense that $u(\tau,\xi)$ satisfies its PDE (2.4) and singular boundary condition. We are not in a position to formally prove existence and uniqueness of the function $w(\tau,\xi)$ and hence of $u(\tau,\xi)$, but there appear to be no obstacles.

We look for a regular perturbation expansion of the form

$$w(\tau,\xi) \;\sim\; \tau\, w_1(\xi) \,+\, \tau^2\, w_2(\xi) \,+\, \cdots, \qquad \tau \to 0,$$

and readily determine

$$\begin{aligned} w_1(\xi) \;&=\; -\frac{1}{2}\left(\xi - \frac{1}{2}\beta^2\right), \\ w_2(\xi) \;&=\; \frac{1}{3}K^2 e^{-2\xi} + \frac{1}{12}\left(\xi^2 + \big(2-\beta^2\big)\xi + \frac{1}{4}\beta^4 - 2\beta^2\right). \end{aligned}$$

The construction of this asymptotic behavior is strong evidence that the solution exists and has the associated local behavior. Thus, a description of the local behavior of $u(\tau,\xi)$ slightly fuller than (2.6) is

$$\begin{aligned} u(\tau,\xi) \;\sim\; &\frac{e^\xi}{\tau} \,-\, \frac{1}{2}\left(\xi - \frac{1}{2}\beta^2\right) e^\xi \\ &+\, \tau\left\{\frac{1}{3}K^2 e^{-\xi} + \frac{1}{12}e^\xi\left[\xi^2 + \big(2-\beta^2\big)\xi + \frac{1}{4}\beta^4 - 2\beta^2\right]\right\} \,+\, \mathcal{O}(\tau^2), \qquad \tau \to 0. \end{aligned}$$

The volatility and risk aversion K appear only at the third order in τ.

2.3. Two-variable model. We briefly indicate the extensions that would be necessary to handle the case of noncoordinated variation. We let $\xi(t)$ and $\zeta(t)$ be two separate stochastic processes, defining the instantaneous market state by

$$\eta(t) \;=\; \bar{\eta}\exp\xi(t) \qquad \text{and} \qquad \sigma(t) \;=\; \bar{\sigma}\exp\zeta(t)$$

so that $\kappa(t) = \bar{\kappa}\exp\bigl(\zeta - \frac{1}{2}\xi\bigr)$. We take $\xi(t)$ and $\zeta(t)$ to evolve according to SDEs of the forms

$$d\xi = a_\xi\, dt + b_\xi\, dB_{\mathrm{L}} \qquad \text{and} \qquad d\zeta = a_\zeta\, dt + b_\zeta\, dB_{\mathrm{V}}, \tag{2.8}$$

where a_ξ, b_ξ, a_ζ, and b_ζ are coefficients whose values may depend on ξ and ζ. $B_{\mathrm{L}}(t)$ and $B_{\mathrm{V}}(t)$ are Brownian motions, independent from the process $B(t)$ driving the price process but possibly correlated with each other, with $\mathrm{E}\bigl(dB_{\mathrm{L}}\, dB_{\mathrm{V}}\bigr) = \rho\, dt$. The same procedure as above gives the PDE for $c(t,x,\xi,\zeta)$

$$-c_t \;=\; \lambda\sigma^2x^2 \;-\; \frac{c_x^2}{4\eta} \;+\; a_\xi c_\xi + a_\zeta c_\zeta \;+\; \frac{1}{2}b_\xi^2 c_{\xi\xi} + \rho b_\xi b_\zeta c_{\xi\zeta} + \frac{1}{2}b_\zeta^2 c_{\zeta\zeta}.$$

The initial condition is still (2.3), since the volatility dynamics do not enter.

We take specific functional forms

$$a_\xi(\xi) = -\frac{\xi}{\delta_{\mathrm{L}}}, \qquad b_\xi = \frac{\beta_{\mathrm{L}}}{\sqrt{\delta_{\mathrm{L}}}},$$
$$a_\zeta(\zeta) = -\frac{\zeta}{\delta_{\mathrm{V}}}, \qquad b_\zeta = \frac{\beta_{\mathrm{V}}}{\sqrt{\delta_{\mathrm{V}}}}.$$

Here δ_{L} and δ_{V} are potentially different relaxation times for liquidity and volatility, and β_{L} and β_{V} are independent burstiness parameters.

We again nondimensionalize using $\delta = \delta_{\mathrm{L}}$ as the time scale, and we have an additional nondimensional parameter $\mu = \delta_{\mathrm{L}}/\delta_{\mathrm{V}}$. Then the PDE for $u(\tau,\xi,\zeta)$ is

$$u_\tau + \xi u_\xi + \mu\zeta u_\zeta \;=\; K^2e^{2\zeta} \;-\; e^{-\xi}u^2 \;+\; \frac{1}{2}\beta_{\mathrm{L}}^2 u_{\xi\xi} \;+\; \rho\sqrt{\mu}\,\beta_{\mathrm{L}}\beta_{\mathrm{V}}u_{\xi\zeta} \;+\; \frac{1}{2}\mu^2\beta_{\mathrm{L}}^2 u_{\zeta\zeta}. \tag{2.9}$$

The coordinated variation case is recovered by assuming the following:

- The time scales of liquidity and volatility are equal: $\delta_{\mathrm{L}} = \delta_{\mathrm{V}} = \delta$, so $\mu = 1$.
- The Brownian motions $B_{\mathrm{L}}, B_{\mathrm{V}}$ have perfect positive correlation $\rho = 1$.
- The fluctuation magnitudes $\beta_{\mathrm{L}} = \beta$ and β_{V} satisfy $\beta_{\mathrm{L}} + 2\beta_{\mathrm{V}} = 0$.
- At some point in the past, $\xi(t)$ and $\zeta(t)$ have satisfied $\xi + 2\zeta = 0$. With the assumed conditions on $\delta_{\mathrm{L,V}}$, $\beta_{\mathrm{L,V}}$, and ρ, this relation is maintained for all t.

Then in the plane $\xi + 2\zeta = 0$, the PDE (2.9) reduces to (2.4).

3. Numerical solution. Since we are unable to give explicit analytic solutions, we resort to numerical calculations to give solutions of the PDE (2.4) for a range of relevant parameters:

- The burstiness parameter β is stock-specific. A large-cap stock will have β near zero, for a near-uniform profile. A small-cap stock will have $\beta = 1$ or larger. For our example calculations, we will fix $\beta = 1$, a moderate value that would be appropriate for a medium-sized stock.

- The risk-aversion parameter K must range across all nonnegative values, since the actual choice of trajectory will be determined by the trader's risk preference. Values of K smaller than unity are the most realistic, so that the algorithm has time to adapt to at least one market reversion time.

We briefly discuss a few technical issues with time and space discretization and then present example solutions.

3.1. Time discretization. The first obstacle is that the initial data is given as singular behavior. A simple modification to the forward Euler scheme handles this problem. Although this method seems quite straightforward, we are not aware of a reference in the existing literature. We illustrate it using an ODE for a function $u(t)$, with singular initial condition $u(t) \sim c/t$ as $t \to 0$.

We apply a forward Euler scheme to $w(t) = tu(t)$, which is regular near $u = 0$. With $w'(t) = tu'(t) + u(t)$ and denoting $u_n \approx u(t_n)$, this gives

$$u_{n+1} \;=\; u_n \;+\; \frac{t_n}{t_{n+1}} \big(t_{n+1} - t_n\big)\, u_n'. \tag{3.1}$$

Thus we apply a correction to the Euler update formula, which becomes small as we move away from the initial singular time and $t_n/t_{n+1} \to 1$.

To test this scheme, we consider the example

$$\frac{du}{dt} = -(u-a)(u-b), \qquad u(t) \sim \frac{1}{t} \quad \text{as } t \to 0, \tag{3.2}$$

whose exact solution is

$$u(t) \;=\; \frac{ae^{-bt} - be^{-at}}{e^{-bt} - e^{-at}}. \tag{3.3}$$

Either by expanding this solution or directly from the ODE, we determine the local expansion

$$u(t) \;\sim\; \frac{1}{t} \;+\; \frac{1}{2}\big(a+b\big) \;+\; \frac{t}{12}\big(a-b\big)^2 \;+\; \cdots, \qquad t \to 0. \tag{3.4}$$

We test on a regular grid with $t_n = n\,\Delta t$, starting at $n = k \geq 1$. We choose k to satisfy the stability condition for the forward Euler scheme. For an ODE $u_t = f(u)$, we require $\Delta t < 1/|f'(u)|$. In this case, $f'(u) \sim 2u \sim 2/t$, so we need $\Delta t < t/2$ or $t > 2\,\Delta t$. We thus expect the scheme should be stable for $k \geq 2$.

We explore four cases, given by all combinations of two parameters:

1. Discretization scheme:
   (a) forward Euler applied directly to u (dashed lines "u" in Figure 2), or
   (b) forward Euler applied to tu as above (solid lines "t u").
2. Data at the first time step:
   (a) $u_k = (1/t_k)$ using the given initial condition ("Order 0"), or
   (b) $u_k = (1/t_k) + \frac{1}{2}(a+b)$ using the local expansion (3.4) ("Order 1").

Figure 2 shows example solutions. The combination of improved initial data, with a time discretization that takes account of the initial singularity, yields far more accurate results than naive discretization.

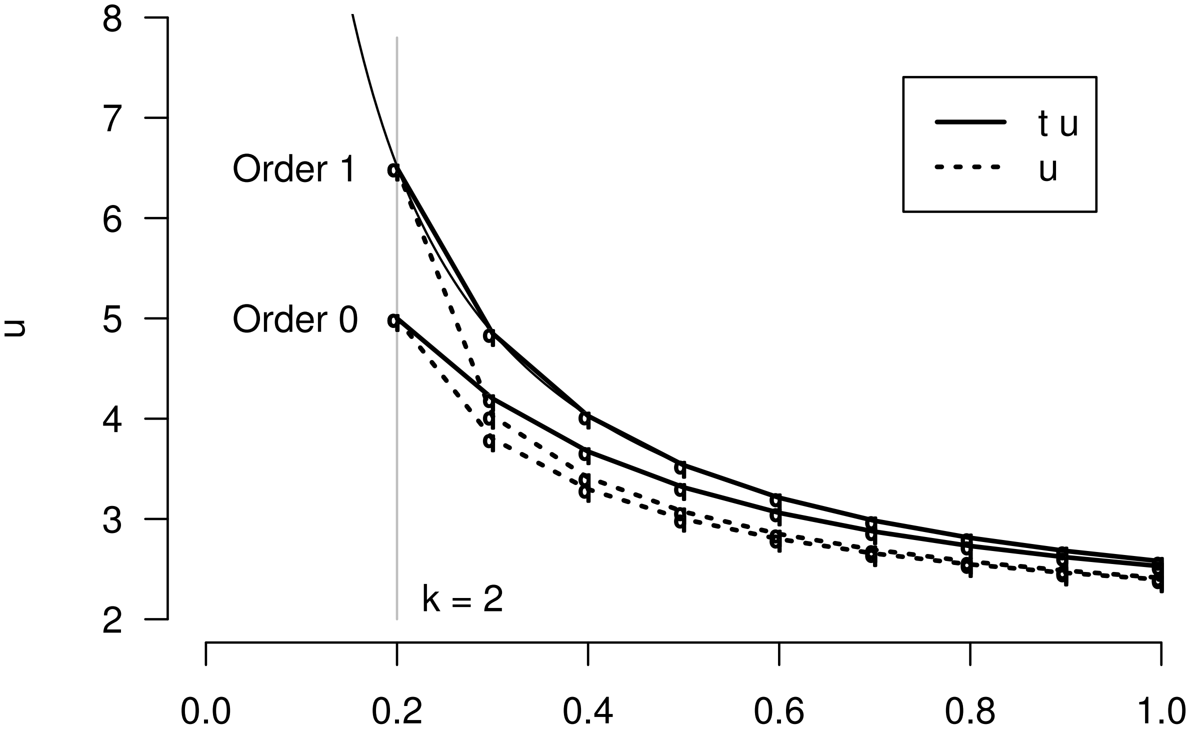

Figure 2. *Numerical solution of the ODE (3.2) with $a = 2$ and $b = 1$, discretized with $\Delta t = 0.1$ beginning at the second step $k = 2$. The light curve is the exact solution (3.3). "Order 0" initial data uses only the given initial condition $u(t) \sim 1/t$; "Order 1" initial data adds the first correction term from (3.4). The dashed curves ("u") apply a forward Euler scheme directly to $u(t)$; the solid curves ("t u") apply a forward Euler scheme to $tu(t)$ as in (3.1).*

3.2. Space discretization. We use the standard 3-point discretization for the diffusion term and upwind differencing for the convection term (see, for example, [10]). We use forward Euler time discretization with the correction above; thus we use a small time step for stability. For initial data we use the asymptotic expression (2.6) at an initial time $t = k\,\Delta t$.

It is more convenient to discretize $\tilde{v}(\tau,\xi) = e^{-\xi}u(\tau,\xi)$ rather than u directly; by (2.5), $\tilde{v}$ is the instantaneous trade rate v except for a constant dimensional factor. The PDE for $\tilde{v}$ is easily derived from (2.4).

We use a finite spatial domain $-\Xi \le \xi \le \Xi$. At the left boundary $\xi = -\Xi$, we use the far-field solution (2.7). At the right boundary $\xi = \Xi$, we use "natural" boundary conditions $v_{\xi\xi} = 0$. Since the convective term is flowing outwards, the effect of the boundary conditions is confined to a narrow boundary layer.

3.3. Example solutions. Figure 3 shows the computed solution of the PDE for $\beta = 1$ and $K = 0.1$ (the computational grid is much finer than what is plotted). As noted above, the quantity that is computed and plotted is not the value function u, but $e^{-\xi}u$, the dimensionless trade rate as a fraction of the shares remaining (we plot its natural log). As expected, when τ is small, the trade rate becomes large like $1/\tau$. When ξ is large positive, market impact is high and volatility is low, so the optimal strategy trades very slowly except near expiration.

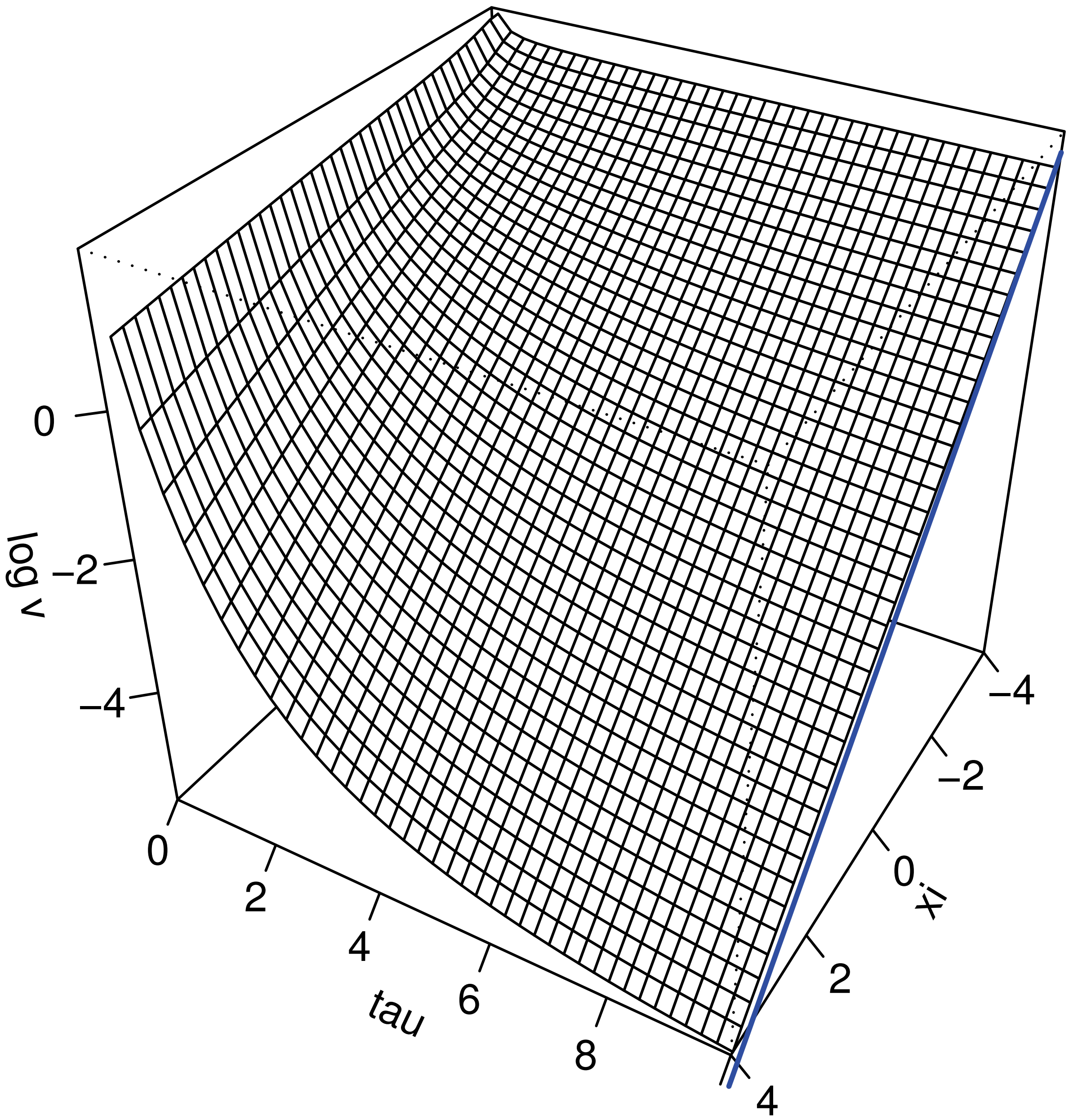

Figure 3. *Solution of the PDE for $\beta = 1$ and $K = 0.1$. The horizontal axes are the nondimensional time remaining to expiration τ and the market state ξ; the quantity plotted is the natural log of the instantaneous trade rate as a fraction of shares remaining. The thick line is the long-time asymptotic solution. When the market is liquid and volatile (ξ large negative), trading is fast; when the market is illiquid and nonvolatile (ξ large positive), trading is slow. As expiration approaches ($\tau \to 0$), trading is forced to be fast.*

When ξ is large negative, market impact is low and volatility is high, so the optimal strategy trades rapidly. As $\tau \to \infty$ the solution approaches the steady state $\log(e^{-\xi}u) = \log K - \xi$.

Figure 4 shows a realization of the market state process $\xi(t)$ used for the trajectory

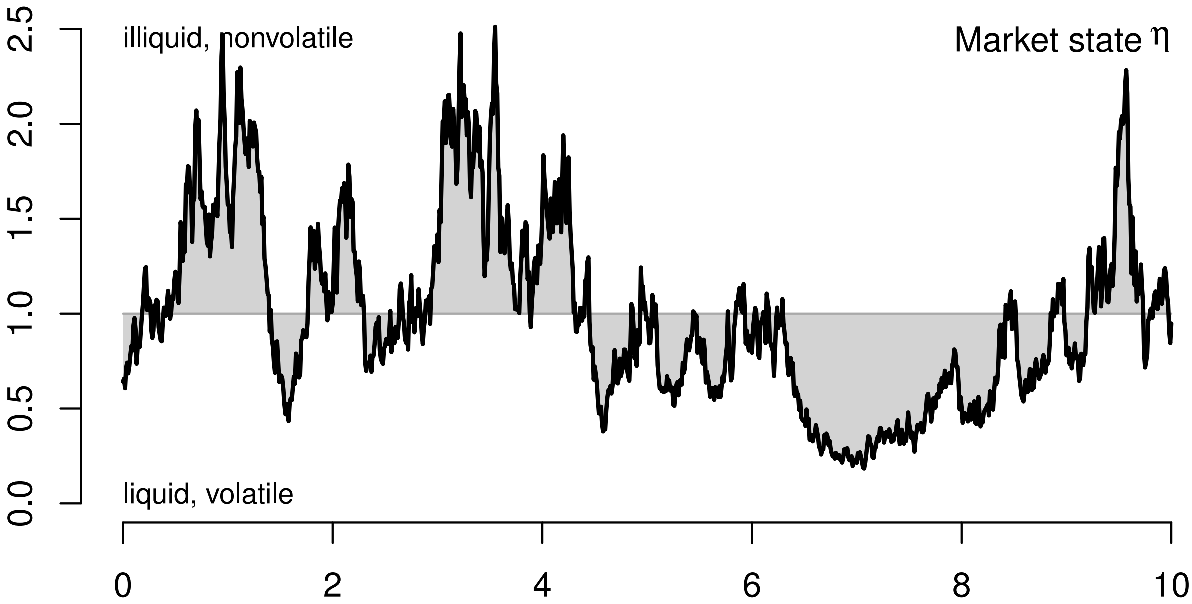

Figure 4. *Realization of the market state trajectory used for the example numerical simulations, with $\beta = 1$. The horizontal axis is time measured in units of the mean-reversion time. The value plotted is the market impact coefficient $\eta(t)$, so large values mean an illiquid market. In our simplified "coordinated variation" market assumption, the volatility $\sigma(t)$ varies inversely with $\eta(t)$: the market is either liquid and volatile, or illiquid and nonvolatile.*

simulations. With the "coordinated variation" approximation the market moves back and forth between a high-activity regime with low impact and high volatility (small ξ) and a low-activity regime with high impact and low volatility (large ξ). We have taken $\beta = 1$ so that the root-mean square fluctuation of $\xi(t)$ is $\frac{1}{2}$. The mean-reversion time $\delta = 1$, so that with a time $T = 10$ we experience several market cycles.

Figure 5 shows optimal trading trajectories for risk-aversion parameters $K = 0.1$, 0.3, and 1, compared to the nonadaptive trajectories computed in the mean market state. The dynamic response is very clear. For example, around $t = 1$ the market state is poor, so all the trajectories trade slowly and fall behind the static curves. Around $t = 1.5$ there is a brief burst of liquidity, and all the strategies accelerate in response. The trajectories with lower urgency (higher on the plots) have more shares remaining to trade, so they are able to react more than the high-urgency trajectory (lower on the plots) which has completed a substantial fraction of its goal by that time. The lowest-urgency strategy continues to adapt and is able to benefit from the large and prolonged liquidity burst around $t = 7$.

The dotted lines in Figure 5 show the "rolling horizon" strategy of section 1.4. For large risk aversion (fast trading), this approximate strategy is almost identical to the optimum. For smaller risk aversion (slow trading), the rolling horizon strategy almost rigidly follows a straight-line trajectory, while the true optimum is able to adapt to varying market state even when its profile is nearly linear. In general, the rolling horizon strategy seems to be an adequate approximation when risk aversion is relatively high.

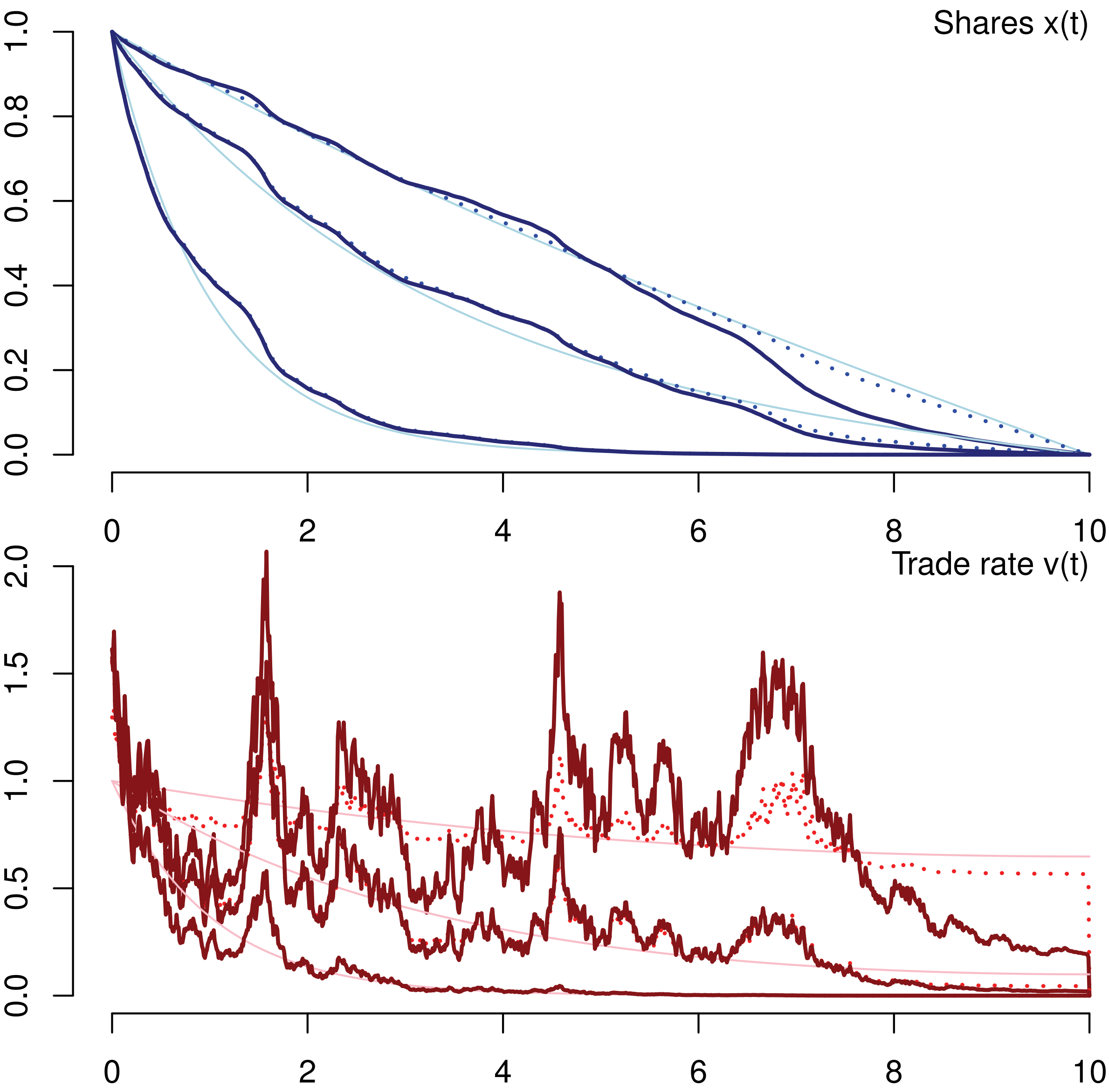

Figure 5. *Optimal trajectories for $\lambda = 0.1$ (highest curves), 0.3, and 1 (lowest curves). The horizontal axis is time t forward from the initial time, measured in units of the market mean-reversion time: the program must be completed by $t = T = 10$. The upper panel is shares remaining; the lower panel is the trade rate. Light lines are the nonadaptive solution in the mean market state; dotted lines are the "rolling horizon" approximation. In the lower panel, the trade rate is normalized so that each nonadaptive solution has the same initial trade rate.*

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