- Harry Markowitz 1927 –
- Father of Modern Portfolio Theory (1952)
- *The first Financial Engineer?*

MPT and the CAPM

Video Lecture VL07

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Valuation for Financial Engineers

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"To reduce risk it is necessary to avoid a portfolio whose securities are all highly correlated with each other. One hundred securities whose returns rise and fall in near unison afford little protection than the uncertain return of a single security."

"It's like a crapshoot in Las Vegas, except in Las Vegas the odds are with the house. As for the market, the odds are with you, because on average over the long run, the market has paid off."

"During the 1950s, I decided, as did many others, that many practical problems were beyond analytic solution and that simulation techniques were required. At RAND, I participated in the building of large logistics simulation models; at General Electric, I helped build models of manufacturing plants."

Leverage
MPT
CAPM
LCAPM - general case of CAPM

Risk as a measure of capital

- Calculate the return on
  - An overcapitalized position
    - A $100 stock that can never go below $40, vs. max loss 60
    - A $100 stock in a startup company that can lose everything 100
  - A short position in a stock
    - A theoretically unlimited loss;
    - is it really an infinite investment?
  - A forward contract no cash
- Is the way we measure returns really correct?
- What is the alternative?

VERY IMPORTANT!

$\sigma = 30 \quad 3\sigma = 90$

$3\sigma = 80$ STATISTICAL MAX LOSS.

JPM $\text{RAROC} = \dfrac{\text{Exp}(P/L) - \text{fin costs}}{\text{VALUE AT RISK (VAR)}}$

1.65σ

$\geq 25\%$

"Capital" in a financial insttution = Risk
or "Economic Capital"

An alternative to our usual measure of return

- Example:
  - The expected return of security i (traditional) is $\boxed{r_i}$
  - The risk-free rate of interest is $\boxed{r}$
  - The standard deviation of returns is $\boxed{\sigma_i}$
- Risk compensation (premium) per unit risk
- Sharpe Ratio = $\frac{r_i - r}{\sigma_i}$ = expected risk premium / risk or per unit risk

  percentages

  the price of risk or the reward for taking risk
  - Solves the problem for overcapitalized positions
- Dollarized Sharpe Ratio =
  - $\frac{\textit{Expected dollar return} - \textit{r.f.funding costs}}{\textit{Dollar standard deviation}}$
  - Solves the problem for short positions and futures!

Risk and leverage

- Example: A security has an unlevered expected return (r_u) of 10% when the risk-free rate (r) is 4%. The standard deviation is σ_u=15%. What is the Sharpe Ratio of this investment?

$$S = \frac{10-4}{15} = 40\%$$

$75

- Now do the same exercise assuming that 25% of the stock value is paid for in cash (this is called equity), with the rest being financed at the risk-free rate. What is the expected rate of return and risk on the equity?

$100 = $25 EQ + $75 DEBT

EXP RET $ = 10 − 75(.04) = 7

OPP COST for $25 = 25(.04) = 1

RISK = 15

$$S = \frac{7-1}{15} = 40\%$$

- What is the Sharpe Ratio of the levered equity?

$r_u = 10\%$

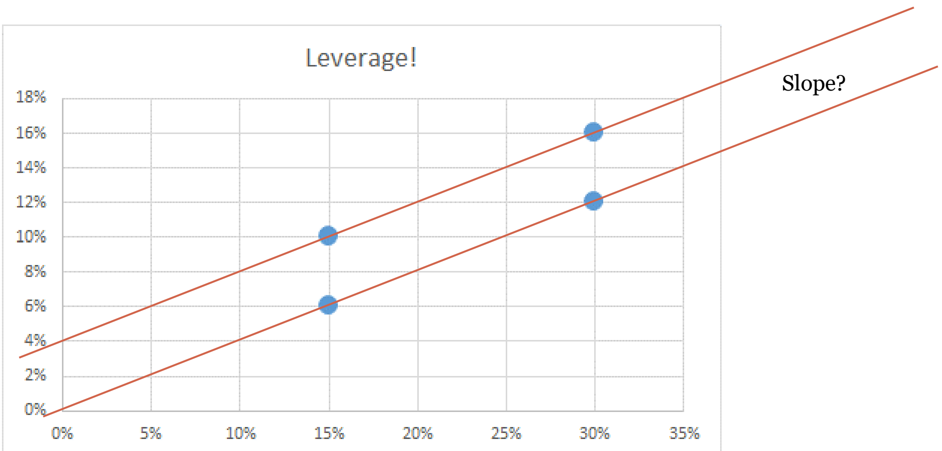
$r_L = \frac{7}{25} = 28\%$

Sharpe ratio unaffected by risk-free leverage

Leverage formula?

LEVERAGE

Modigliani and Miller (M&M)
1958
Proposition 2



r_E = return on possibly levered equity
r_U = return on unlevered position
r = risk-free return (on borrowing)
D = amount of debt
E = amount of equity
σ_E = stddev of possibly levered equity
σ_U = stddev of unlevered equity

Formulas:

$$\boxed{r_E = r_U + \frac{D}{E}(r_U - r)}$$

$$r_E = r + \left(1 + \frac{D}{E}\right)(r_U - r)$$

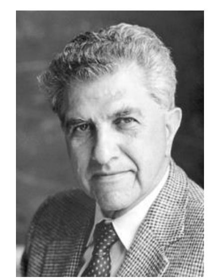
$$\boxed{\sigma_E = \sigma_U \left(1 + \frac{D}{E}\right)}$$

CDO
A+B+R=T
R=T-A-B
+: BUY, LONG
-: SELL, SHORT

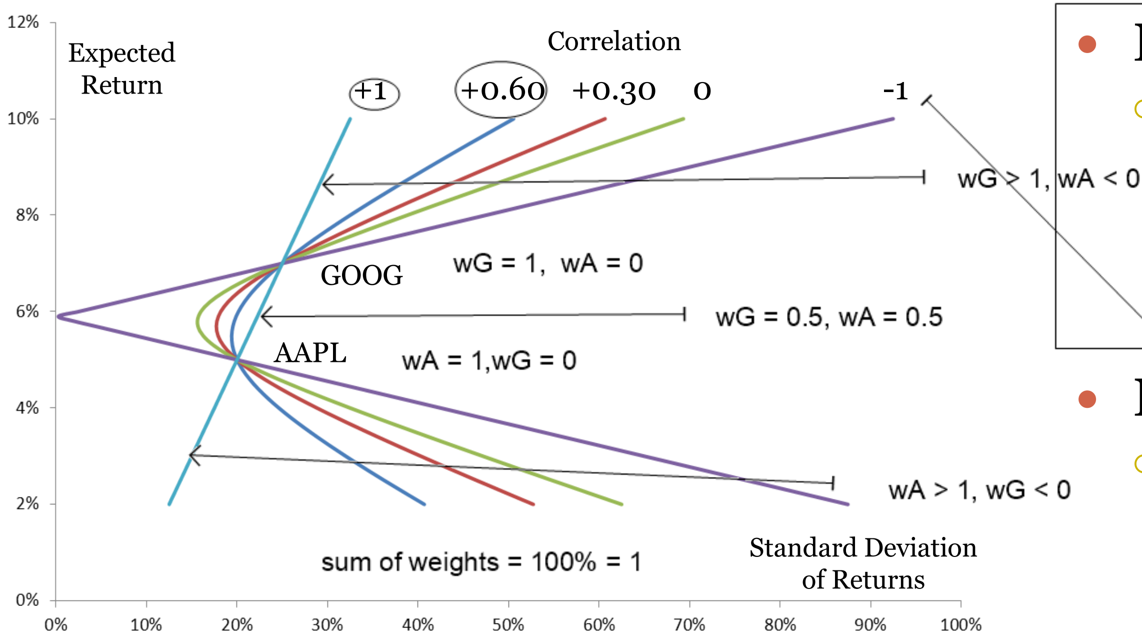
The moral of the story:

Leverage increases risk and expected return, but the Sharpe Ratio remains the same if the debt is risk-free

Position in residual is a leveraged position in the underlying assets funded by a shor position in low risk securities

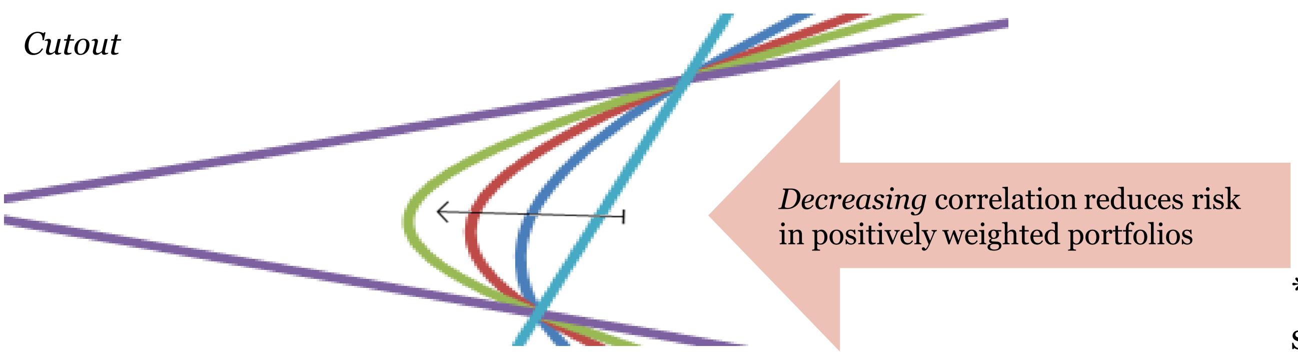


Correlation and Risk Reduction*



- **Diversification**
  - Imperfect correlation leads to risk reduction compared to perfect correlation in positively weighted portfolios of two assets
- **Hedging**
  - Perfectly negative correlation leads to the elimination of risk for the right combination of two assets

Cutout

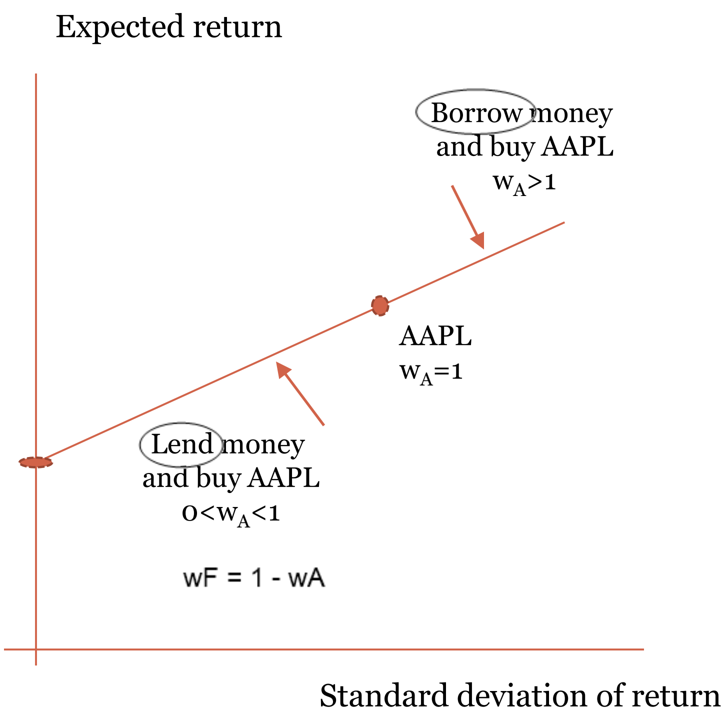


Assumptions:
We know stock returns are normal.
We know expected returns and covariance matrix
There may exist a risk-free asset
We can go long/shor, hold any proportion of any asset

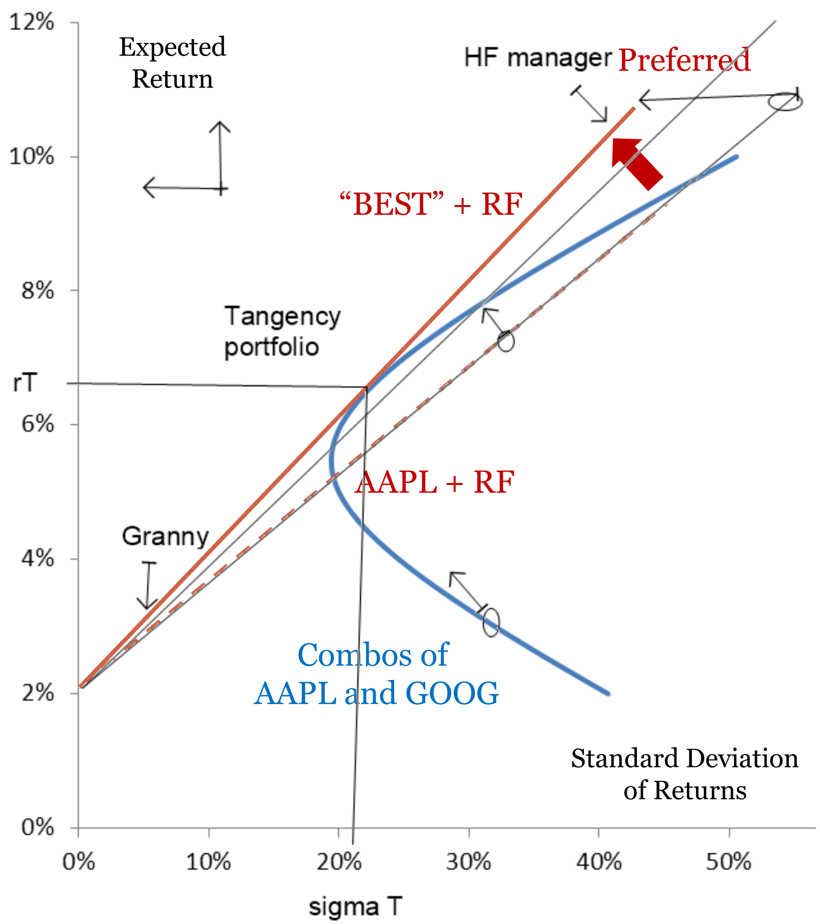
*Holding expected returns and standard deviations constant

One risky and one risk-free asset

- The formulas for the case of two risky assets:
  - $r_p = w_A r_A + w_B r_B$ $(a\ straight\ line)$
  - $\sigma_p^2 = w_A^2\sigma_A^2 + w_G^2\sigma_G^2 + 2\rho w_A w_G \sigma_A \sigma_G$ $(a\ parabola)$
- If one asset (G) is risk-free, then $\sigma_G=0$
  - $\sigma_p^2 = w_A^2\sigma_A^2 + \cancel{w_G^2\sigma_G^2 + 2\rho w_A w_G \sigma_A \sigma_G}$ $(a\ straight\ line)$



Two risky assets and a risk-free asset



- The combination of any asset with the risk-free asset produces a straight line set of feasible portfolios
- The tangency portfolio provides the most efficient use of leverage
- What mathematical objective does the tangency portfolio achieve?

Max (rT - rF)/sigma T

s.t. sum of weights = 1

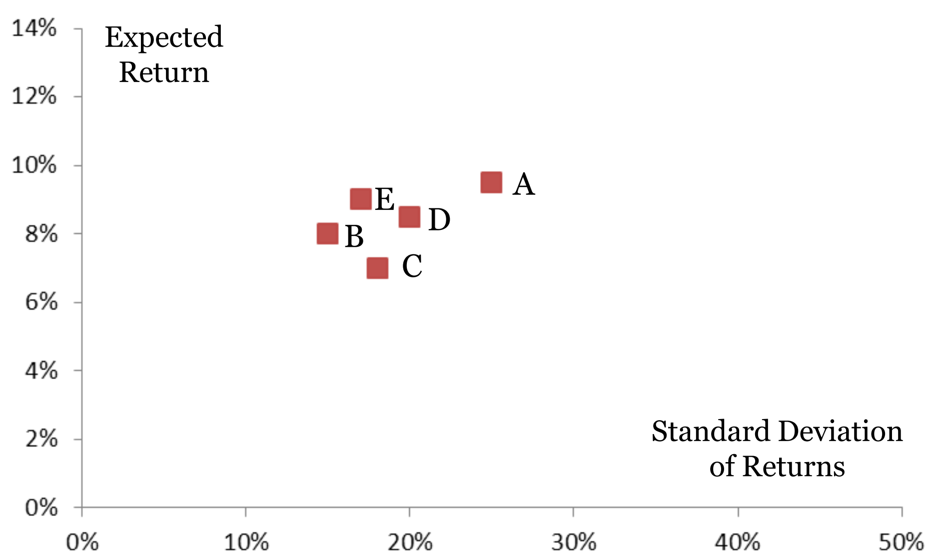
"Max Sharpe Ratio"

quadratic programming problem
with constraint

Now move to multiple assets

Assume returns are jointly normally distributed with these parameters:

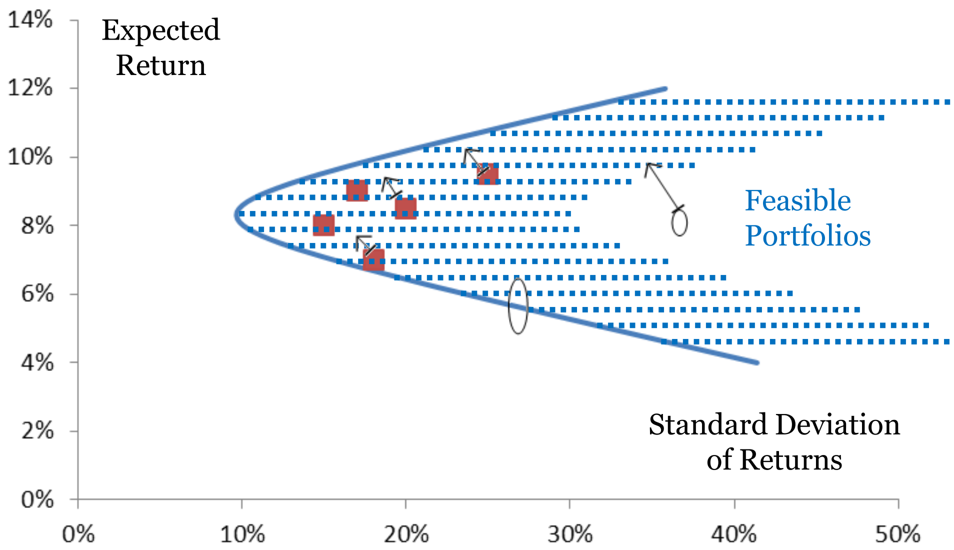
	E[r]	σ	Correlation matrix				
A	9.5%	25%	1.00	0.10	0.05	0.20	0.00
B	8.0%	15%	0.10	1.00	0.15	0.15	0.05
C	7.0%	18%	0.05	0.15	1.00	0.10	0.20
D	8.5%	20%	0.20	0.15	0.10	1.00	0.05
E	9.0%	17%	0.00	0.05	0.20	0.05	1.00



- Notation
  - **1** = vector of 1's
  - w = vector of weights
    - with $w'1 = 1$
  - r = expected return vector
  - Σ = covariance matrix
- What are the mean and standard deviation of the portfolio defined by w?
  - Mean = r′w
  - StdDev = $\sqrt{w'\Sigma w}$

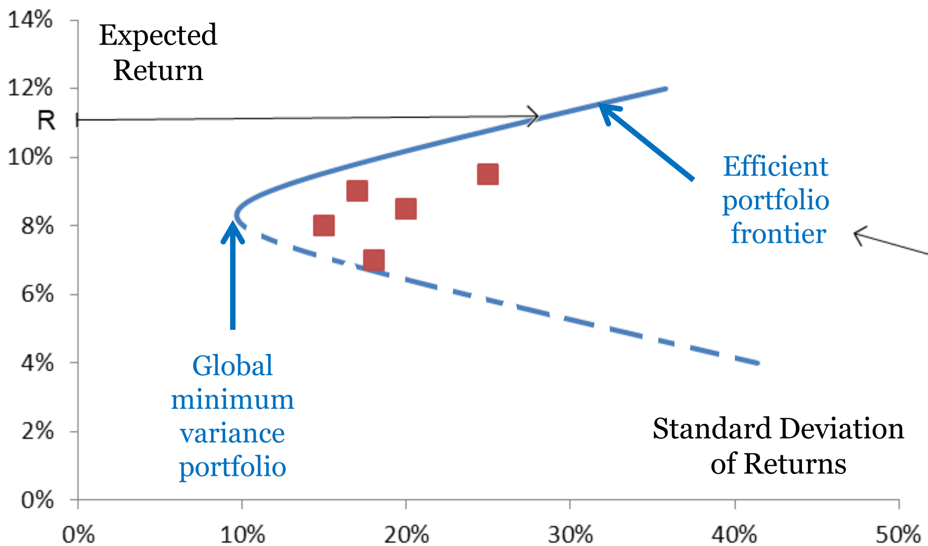
Constructing *feasible* portfolios

- Any set of portfolio weights
  - w_1, w_2, w_3, w_4, w_5
  - May be positive or negative
    - Negative for short position
  - *Must sum to 100%*



Efficient portfolios

- For a given target expected return
  - Investors would only choose the lowest risk portfolio for a given return
  - The *global minimum variance portfolio* refers to the feasible portfolio with the lowest standard deviation
- Minimum variance portfolios trace the set of *Efficient Portfolios*
- Dotted line shows minimum variance portfolios that are dominated
  - Investors can take the same risk and earn higher return



Formula for efficient frontier for a given R:

- "1" is a vector of ones
- r is the expected return vector

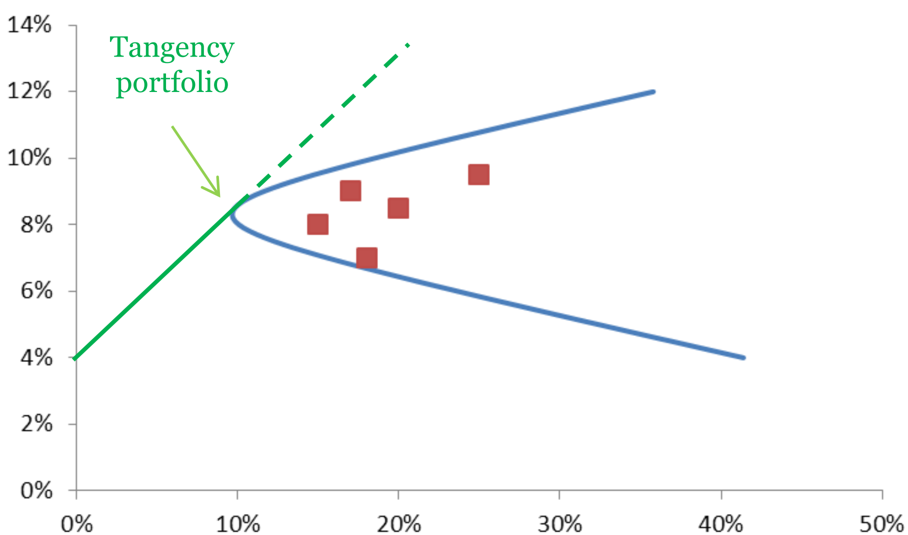
$$A = 1'\Sigma^{-1}1$$
$$B = 1'\Sigma^{-1}r$$
$$C = r'\Sigma^{-1}r$$
$$D = AC - B^2$$

$$\boxed{\sigma^2 = \frac{AR^2 - 2BR + C}{D}}$$

graph of efficient frontier

Adding a risk-free asset

- Notice that the 5-asset efficient frontier has the same shape as the 2-asset curve
- Adding a risk-free asset has the same effect as in the 2 risky asset case
- More risk averse investors will allocate more weight to the risk-free asset
- Less risk-averse investors will allocate more weight to the risky asset
  - Dashed green line shows investors borrowing to invest in the tangency portfolio
- Assume the risk-free rate is 4%



Useful formulae:

$$r_T = \frac{C - Br_f}{B - Ar_f}$$

$$w_T = \frac{\Sigma^{-1}(r - r_f 1)}{B - Ar_f}$$

weights of 5 assets for tangency portfolio

Now you try with historical data

- We collected price data on 10 tech stocks in 2016-2017 (monthly)
  - Compute historical monthly average returns, standard deviations and covariance
  - Find the MPT parameters A, B, C and D
  - Determine the "optimal portfolio"
  - Show a graph of the diversification benefits
  - Comment on your asset allocations

Filename: Lecture 3 in-class exercises.xlsx

MEET	Meet Group Inc.
BIDU	Baidu Inc.
MLNX	Mellanox Tech Ltd
CRM	Salesforce.com Inc.
NTES	NetEase Inc.
OLED	Universal Display Corporation
WEB	WebJet Ltd
SOHU	Sohu.com Ltd.
NTCT	NetScout Systems, Inc.
AAPL	Apple Inc.

Closing Price	MEET	BIDU	MLNX	CRM	NTES	OLED	WEB	SOHU	NTCT	AAPL
29-Feb-16	3.11	173.42	50.81	67.75	134.61	47.78	18.15	43.47	20.67	96.69
31-Mar-16	2.84	190.88	54.33	73.83	143.58	54.1	19.82	49.54	22.97	108.99
29-Apr-16	3.42	194.3	43.25	75.8	140.7	58.31	19.99	44.93	22.26	93.74
31-May-16	3.78	178.54	47.4	83.71	177.84	67.15	16.97	41.77	24.26	99.86
30-Jun-16	5.33	165.15	47.96	79.41	193.22	67.8	18.18	37.86	22.25	95.6
29-Jul-16	6.43	159.6	44.18	81.8	204.27	70.84	18.86	38.68	27.98	104.21
31-Aug-16	5.76	171.07	43.84	79.42	211.97	57.59	17.46	42.54	29.58	106.1
30-Sep-16	6.2	182.07	43.25	71.33	240.78	55.51	17.27	44.25	29.25	113.05
31-Oct-16	4.89	176.86	43.4	75.16	256.99	51.7	16.1	37.43	27.45	113.54
30-Nov-16	4.82	166.95	41.45	72	224.1	54.65	15.95	34.63	31.2	110.52
30-Dec-16	4.93	164.41	40.9	68.46	215.34	56.3	21.15	33.89	31.5	115.82
31-Jan-17	4.92	175.07	47.35	79.1	253.9	66	18.95	39.67	33.3	121.35
22-Feb-17	4.94	186.01	48.8	82.08	304.07	71.25	20.4	42.16	37.75	137.11

Results

A	12,124	Riskfree rate (monthly)			0.30%
B	230			rT	3.19%
C	7				
D	30,340	σ	r	(r-rf)/B-Arf)	wT
MEET		2.21%	5.09%	0.02%	6.51%
BIDU		1.15%	0.77%	0.00%	87.70%
MLNX		1.47%	0.06%	0.00%	51.82%
CRM		0.91%	1.87%	0.01%	-6.27%
NTES		3.68%	7.54%	0.04%	19.70%
OLED		1.54%	3.86%	0.02%	-12.30%
WEB		0.92%	1.62%	0.01%	18.91%
SOHU		1.39%	0.23%	0.00%	-59.95%
NTCT		2.49%	5.55%	0.03%	55.25%
AAPL		1.24%	3.22%	0.02%	-61.37%
Tangency		1.22%	3.19%		

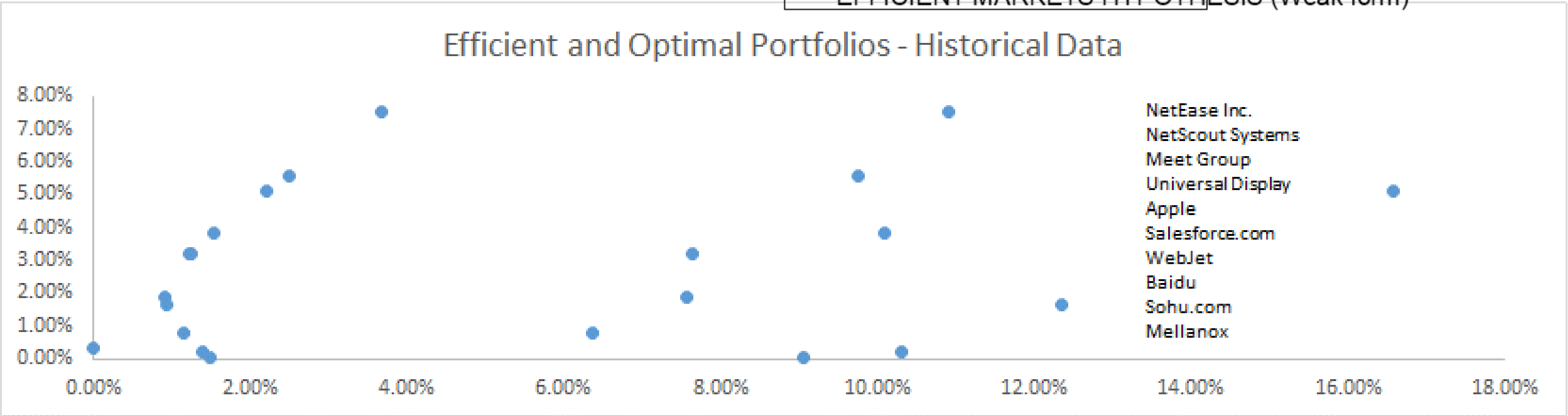
What is wrong with these weights?

Recommended portfolio weights

MEET	Meet Group Inc.
BIDU	Baidu Inc.
MLNX	Mellanox Tech Ltd
CRM	Salesforce.com Inc.
NTES	NetEase Inc.
OLED	Universal Display Corporation
WEB	WebJet Ltd
SOHU	Sohu.com Ltd.
NTCT	NetScout Systems, Inc.
AAPL	Apple Inc.

Never use past returns to predict future returns.

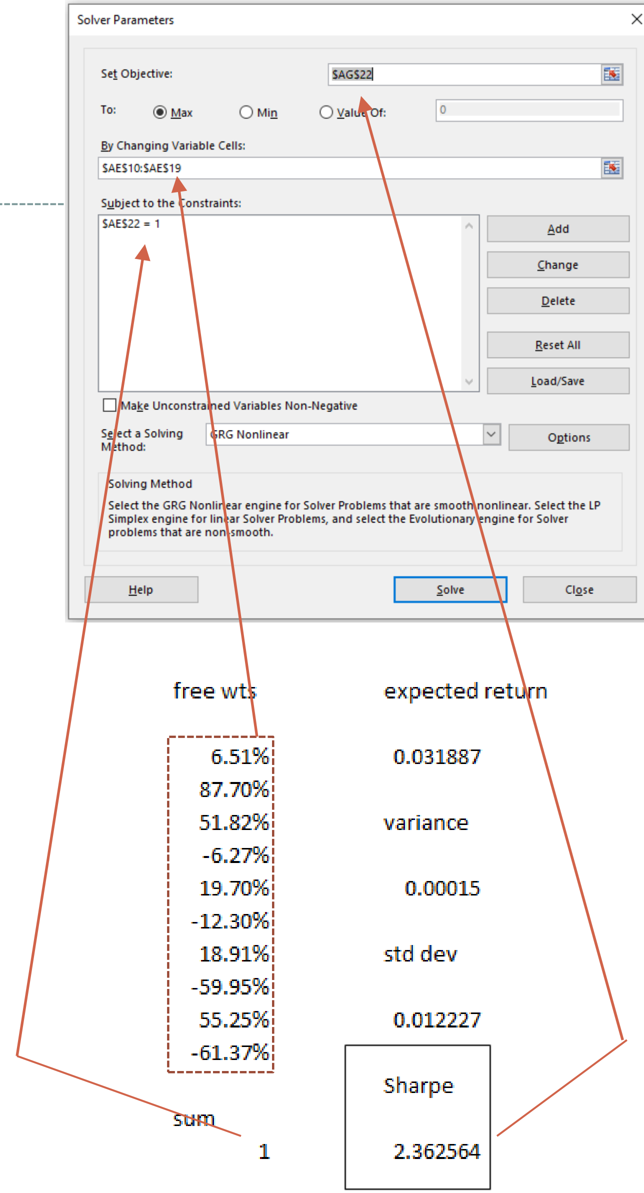
EFFICIENT MARKETS HYPOTHESIS (Weak form)



Same result, less matrix calculation!

- Demonstration using Solver
  - The "weights" are any 10 numbers in a column that add to one
  - The expected return is the sum of the products of the weights and the individual average returns
  - The variance is the matrix product using this formula
    - =mmult(transpose(w),mmult(cov,w))
    - See classroom demo for detail on this
    - Shade in result box, type formula, hit *Ctl* and *Shift* down while tapping *Enter*
  - The standard deviation is the square root of the variance
  - The Sharpe Ratio is (Expected return on portfolio minus risk-free rate)/Standard deviation

- Maximize the Sharpe Ratio
  - While ensuring the weights sum to one
  - The betas are found by taking
    - =mmult(cov,w)   (a column of 10 numbers)
    - Divide by variance of optimal portfolio



The Capital Asset Pricing Model

Sharpe

investors & portfolios

Lintner

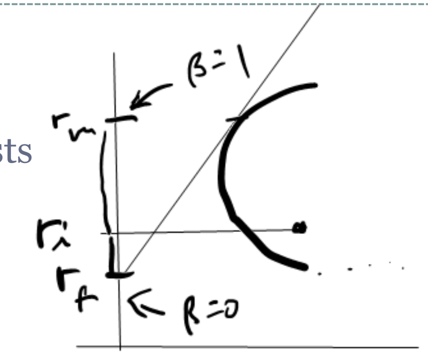
Treynor

issuers & market clearing

Mossin

CAPM logic and the market portfolio

Sharpe version



- Assume include all MPT assumptions plus...
  - If investors are identical except for risk aversion and a risk-free asset exists
- Then
  - All will invest to different degrees in the same risky portfolio
- This portfolio must be the $\boxed{\text{market portfolio}}$ tangency portfolio
  - If not, aggregate positions long would be different than issued shares
- Now consider adding a small amount of any security i to the market portfolio
  - Finance this using debt, just to keep the equation balanced
  - The incremental return contribution is $r_i - r_f$
  - The incremental variance contribution is $2\rho_{im}\sigma_i\sigma_m$
- This must also be true for all other securities, including a market index security
  - The incremental return contribution is $r_m - r_f$
  - The incremental variance contribution is $2\rho_{mm}\sigma_m\sigma_m = 2\sigma_m^2$
- Setting the ratio of return to risk the same we obtain the CAPM equation
  - $r_i - r_f = \beta_i(r_m - r_f)$ for all securities

$$\beta_i = \frac{\rho_{im}\sigma_i}{\sigma_m} = \frac{\rho_{im}\sigma_i\sigma_m}{\sigma_m^2} = \frac{cov}{var}$$

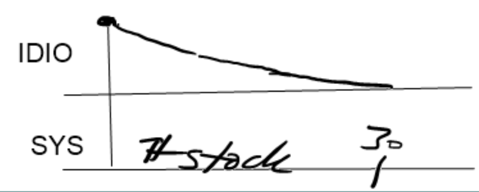
Key conclusions of the CAPM

time value measure of risk cost of risk

- Expected returns are determined by asset characteristics
- Expected return of any security is given by
  - $r_i = r_f + \beta_i(r_m - r_f)$ where $\beta_i = cov(r_i, r_m)/\sigma_m^2$
  - r_i = "wait" and "worry" since it contains a term for the time value of money and a risk premium
  - This can be interpreted as the GVE where covariance with the market is the measure of risk
- β_i can also be interpreted as the elasticity in security i's returns with respect to the market

  if $\beta_i = 2$ then mkt ↑1% => stock ↑2%

  ↓1% => ↓2%
  - i.e. if the market goes up 1%, we expect a security to rise β_i%
- The total risk of a security return measured in variance
  - $\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_\varepsilon^2$ where the last term is idiosyncratic variance
- Idiosyncratic risk has no price

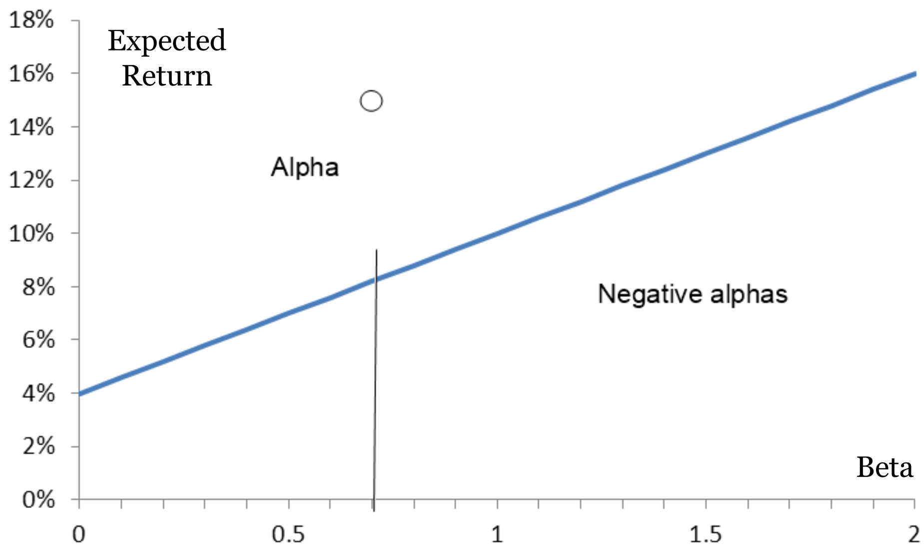


measured in % returns

systematic + unsystematic
market + nonmarket
systematic + idiosyncratic

The security market line (SML)

- Expresses expected return as a function of a security's β
  - In this example, r_f=4% and r_m=10%
- Securities above the line are said to have a positive α
  - In the CAPM, α=0 in theory (ex ante)
- Don't confuse this with the capital market line (CML)
  - CML is the line corresponding to the tangency portfolio

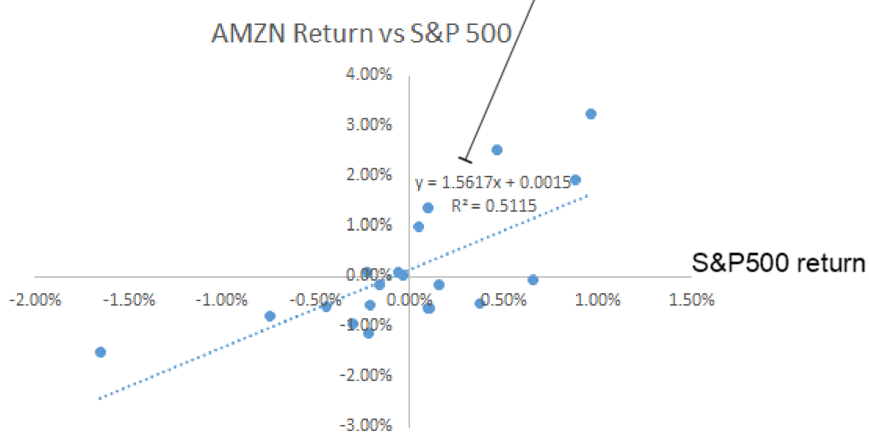


Contributions of the CAPM

- Explicit relationship between risk and expected return
  - Covariance risk of diversified portfolios with the market portfolio
- Explicit pricing of systematic risk and idiosyncratic (diversifiable) risk
- Explanation of the source of risk premia

How is beta calculated in practice?

- Betas are usually calculated from % returns
- Alpha is the excess return over that predicted by the CAPM
- Historical method
  - Find the historical price levels of a given stock (y) and also an appropriate index (x)
  - Compute returns
- Do a scatterplot or use a function
  - Excel graph feature: "Trend" line
  - =slope(yrange, xrange)
- The slope of the line is an estimate of β=1.56
- To find the "alpha" use
  - α=intercept(yrange,xrange)-riskfree*(1-beta)



	Prices		Returns	
Date	S&P	AMZN	X	Y
4/11/2019	2,888	1,844		
4/12/2019	2,907	1,843	0.66%	-0.05%
4/15/2019	2,906	1,845	-0.06%	0.10%
4/16/2019	2,907	1,863	0.05%	0.98%
4/17/2019	2,900	1,865	-0.23%	0.10%
4/18/2019	2,905	1,862	0.16%	-0.17%
4/22/2019	2,908	1,887	0.10%	1.38%
4/23/2019	2,934	1,924	0.88%	1.93%
4/24/2019	2,927	1,902	-0.22%	-1.14%
4/25/2019	2,926	1,902	-0.04%	0.03%
4/26/2019	2,940	1,951	0.47%	2.54%
4/29/2019	2,943	1,938	0.11%	-0.63%
4/30/2019	2,946	1,927	0.10%	-0.61%
5/1/2019	2,924	1,912	-0.75%	-0.78%
5/2/2019	2,918	1,901	-0.21%	-0.56%
5/3/2019	2,946	1,962	0.96%	3.24%
5/6/2019	2,932	1,951	-0.45%	-0.61%
5/7/2019	2,884	1,921	-1.65%	-1.51%
5/8/2019	2,879	1,918	-0.16%	-0.17%
5/9/2019	2,871	1,900	-0.30%	-0.93%
5/10/2019	2,881	1,890	0.37%	-0.52%
		Slope function		1.56

Sample beta estimates: abg-analytics.com

Ticker Symbol	Company	Beta Estimate (long-term average	Beta Estimate (current time-varying estimate)
AAPL	Apple Inc	1.07	1.09
ABBV	AbbVie Inc	1.1	1.08
ABT	Abbott Laboratories	1.03	1
ACN	Accenture Ltd	1.01	1.08
AGN	Allergan Plc	1.08	1.05
AIG	American International Group	1.12	1.09
ALL	Allstate Corporation	0.66	0.65
AMGN	Amgen Inc	1.24	1.19
AMZN	Amazon.Com Inc	1.38	1.66
AXP	American Express Co	1.12	1.12
BA	Boeing Co	1.19	1.15
BAC	Bank Of America Corp	1.28	1.27
BIIB	Biogen Inc	1.41	1.16
BK	Bank of New York Mellon Corporation	1.05	1.05
BKNG	Booking Holdings, Inc	1.36	1.26
BLK	Blackrock Incorporated	1.41	1.36
BMY	Bristol-Myers Squibb Co	0.78	0.77
BRKB	Berkshire Hathaway Cl B	0.86	0.92

Many firms provide proprietary beta estimates

How is beta used for valuation?

- Suppose we want to value a company.
- The company pays a dividend of \$0.50 per year, expected to grow indefinitely at 2% per year.
- What is the value of the company?

$$V_0 = \frac{.50}{.078 - .02} = \$8.62$$

- What is missing in the question?
- We need a discount rate, and the CAPM can provide one
  - Using historical returns of the stock, we regress on the market returns and find the beta is 0.8.
  - We take the long-term risk-free rate at 3% from government bond yield quotes
  - We estimate the historical risk premium paid on the market is 6% — 8%
  - The expected return on the stock is therefore = 0.03+0.8(.06) = 7.8%
  - The value of the growing dividend perpetuity is therefore
    - 0.50/(0.078-0.02)= \$8.62 per share
  - The dividend multiplier is 1/(0.078-0.02)=17.24
    - This can be applied to comparable stocks

$$r_i = r + \beta_i(r_m - r)$$

Extensions of the CAPM

- Black's Zero Beta Model
  - No riskless asset
  - Replace r_f with r_Z, the return of the efficient zero-beta portfolio
- Brennan's tax model
  - Replace returns with after-tax returns
- Merton's ICAPM (Intertemporal CAPM)
  - Add risk premia from other nondiversifiable risk factors, e.g. interest rate risk
  - Implies multiple factors
- Breeden CCAPM (Consumption CAPM)
  - Measure consumption risk as opposed to investment risk, since people ultimately care about consumption
  - Scale this measure and use it as the risk premium
- International CAPM
  - Same as CAPM but use international benchmarks

Example of a multifactor CAPM

There are several refinements to the CAPM, including:

- Fama-French
  - Three factor model – based on their observations that company size is an important driving factor behind returns

$$\boxed{r_i - r_f = b_{1i}\left(r_M - r_f\right)} + b_{2i}\left(r_{Small} - r_{Big}\right) + b_{3i}\left(r_{highBTM} - r_{lowBTM}\right) + \varepsilon_i$$

Return on a Small Cap portfolio – Return on a Large Cap portfolio

Return on a portfolio with high Book-to-Market ratio versus low BTM

- Carhart 4-Factor
  - Adds one factor onto Fama-French to allow for momentum (persistence of returns)

The Arbitrage Pricing Theory

- Based on arbitrage, not equilibrium
- However, assumes returns are generated by linear combinations of factors
- In this model, every factor risk has a price
- Every security has factor loadings, which are like factor betas
- The resulting expected return equation is the same

factors not observable

  - $r_i = r_f + \lambda_{i1}\pi_1 + \lambda_{i2}\pi_2 + \ldots$
  - π_j = risk premium for factor j
  - λ_{ij} = factor loading of security i on factor j

Approaches to estimating the APT

- Exogenous/explicit factors
  - Specified in advance
  - Correspond to either macro-type variables, or firm-type variables
- Endogenous/implicit factors
  - Determined by factor analysis (maximum likelihood)
    - Attempts to explain all data
  - Or by Principal Components (maximum variance)
    - Finds linear combinations of the data to maximize variance
    - First principal component corresponds to first eigenvalue of covariance matrix decomposition
  - Challenge in both of these methods is interpreting the derived numerical factors

Factor Models and APT (Cont'd)

The BARRA Model

- Two categories of factors
  - Industry groupings
    - Ex: for the US there are 52 industrial categories
  - 13 Broad risk indices
    - Volatility
    - Momentum
    - Size (Capitalization)
    - Size, 2nd order (non-linear)
    - Trading Activity
    - Growth
    - Earnings Yield
    - Value
    - Earnings volatility
    - Leverage
    - FX sensitivity
    - Dividend Yield
    - Non-estimation Universe Indicator (for companies outside the database universe)
  - Factors constructed by countries/region
  - Factors normalized to a null mean and unit standard deviation
  - Weights determined by regression

Matching Quiz

Assume the CAPM assumptions hold for the purposes of this quiz.

Click on the left box to reveal match on the right.

The price of idiosyncratic risk	Breeden
The elasticity of a stock price with respect to the market	Alpha
The expected return of a negative beta stock	Less than risk-free
Security market line	Negative
Measure of return on capital	Return vs sigma
Consumption CAPM	Black and Litterman
	Sharpe Ratio
	Zero
	Black and Scholes
	Beta
	Return vs beta
	Merton

Takeaways

- Risk is used as a measure of capital by financial institutions
- Risk-free leverage increases expected return and risk, but has no effect on the Sharpe Ratio
- MPT assumes investors know expected returns, standard deviations and correlations --- and care about nothing else
- Reduction in correlation increases asset diversification benefits in positively weighted portfolios
- The multiple asset case simplifies to the two risky asset case
- Efficient portfolios are those with the lowest variance for a given expected return
- When a risk-free asset is present, the efficient portfolios form a tangency line to the feasible portfolio set
- The most bothersome assumption is that investors know expected returns --- this should be determined from the asset investment characteristics
- One interpretation of the CAPM is that it rationalizes expected returns

Takeaways (2)

- The CAPM assumptions are many, but they boil down to homogeneous investors and perfect markets (in addition to MPT assumptions)
- The most famous CAPM equations
  - $r_i = r_f + \beta_i(r_m - r_f)$ --- the Security Market Line
  - $\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_\varepsilon^2$
- Most important CAPM implications
  - Explicit link between risk and return
  - Pricing of systematic risk
  - Explanation of the source of risk premium
- The APT
  - Not an equilibrium model, but derived from assumptions about factors generating returns
  - Similar expected return equation
  - Estimation relies on specifying factors, or deriving principal components or factor models
  - The latter suffers from interpretational difficulties
  - Commercial versions are available
- Keeping the two models straight
- Should you believe these models?

Some Questions with the Markowitz model

- How do we know expected returns, especially when
  - A. They must be in equilibrium
  - B. We don't know prices

prices and retuns are codetermined

%

- How do we know return standard deviations without knowing prices?
- How can we determine model sensitivities if returns and standard deviations are held constant?

- How do we deal with more than one period?
- What if the utility function is wrong?
- What is the asset valuation formula for a new asset?

Let's re-derive Markowitz & CAPM, with a twist

- One period model
- Assume n assets, producing n normally distributed **CASH FLOWS** in one period, with a constant interest rate r (assuming we know nothing about returns)
- What is the logic of the derivation?
  - Assume perfect markets, identical individuals except for risk aversion, allow risk-free borrowing & lending, and assume a utility function with risk aversion
  - Maximize investor utility over risky outcomes in one period
    - (demand equation for risky assets)
  - Force the risky securities to be priced so that all securities are bought
  - Calculate the security prices
  - Now find expected returns and return standard deviations
  - Find the relationship between expected returns across securities
  - Apply the model to pricing new securities
  - Extend to multiple periods

Value these three cash flows with the GVE

Nomally distributed future cash flows

$\mathrm{cov}(x_1, x_1+y_2+x_3)$

Asset	Mean	Covariance with each other			
1	10	25	10	5	40
2	20	10	49	10	69
3	30	5	10	64	79
	60		Grand sum	188	

Assume: one period
- Risk measure = variance
- Cost of variance risk = $\boxed{0.06}$
- Risk-free rate = 0

Value of $(x_1+x_2+x_3)$

Variance $(1\ 1\ 1)\,\Sigma\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$\cancel{r}V_0 + k\sigma = E(V_1) - V_0 + E(C_1)$$

$$V_0 = E(C_1) - k\sigma^2$$

$$V_{SUM} = 60 - 0.06(188)$$

$$V_1 = 10 - 0.06(40)$$

$$V_2 = 20 - 0.06(69)$$

$$V_3 = 30 - 0.06(79)$$

Calculate and allocate market risk charges

Asset	Mean	Covariance with each other			Cov w group	Share
1	10	25	10	5	40	21.28%
2	20	10	49	10	69	36.70%
3	30	5	10	64	79	42.02%
	60		Grand sum	188		
					unnecessary step	
Value				48.72		
Total risk charges				11.28		

Now value each security

Asset	Mean	Covariance with each other			Cov w group	Share	Alloc risk	Value	% of group
1	10	25	10	5	40	21.28%	2.40	7.60	15.60%
2	20	10	49	10	69	36.70%	4.14	15.86	32.55%
3	30	5	10	64	79	42.02%	4.74	25.26	51.85%
	60		Grand sum	188			11.28	48.72	
Value				48.72					
Total risk charges				11.28					

Expected return of first asset % = 10/7.60 - 1

StDev of first asset % = sqrt(25)/7.60

correlation betw 1 & 2 = 10/(stdev(1)*stdev(2))

What are expected returns, sigmas & correlations?

Exp ret	SD	Correlation with each other			Covariance		
31.58%	65.79%	100.00%	28.57%	12.50%	43.28%	8.30%	2.60%
26.10%	44.14%	28.57%	100.00%	17.86%	8.30%	19.48%	2.50%
18.76%	31.67%	12.50%	17.86%	100.00%	2.60%	2.50%	10.03%
23.15%							

This is where MPT begins!

What is the tangency portfolio?

Inverse			xr	/B
2.5314	-1.0266	-0.4018	0.4560	15.60%
-1.0266	5.7189	-1.1566	0.9516	32.55%
-0.4018	-1.1566	10.3620	1.5156	51.85%

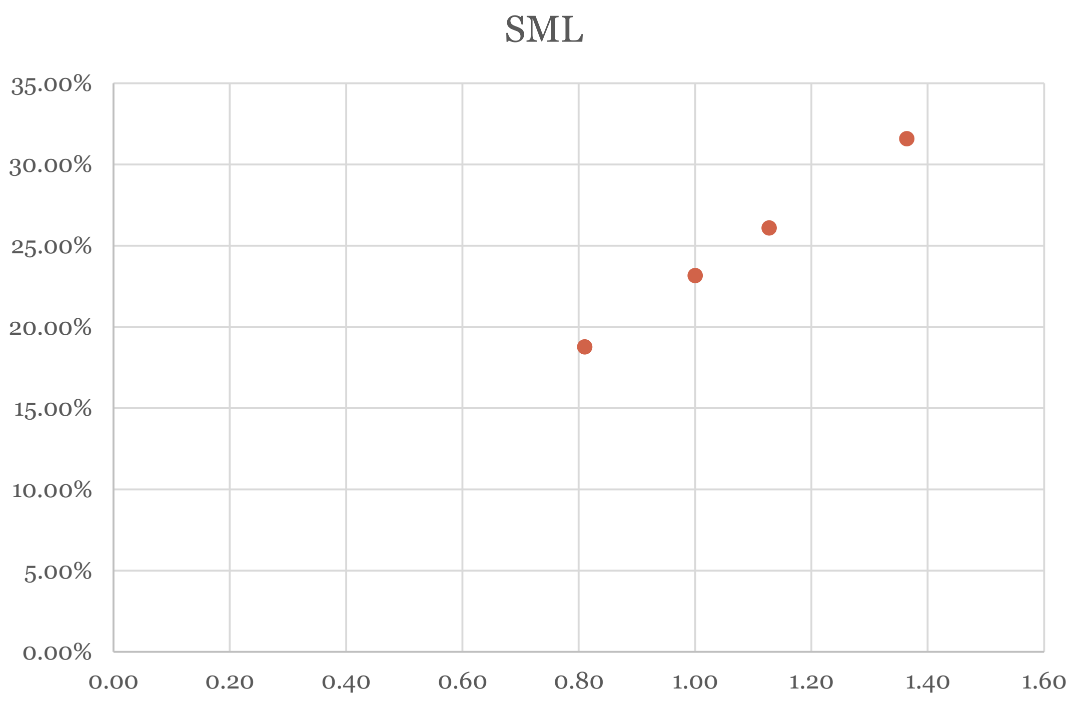
A	13.44		rT	23.15%		wT	15.60%
B	2.92		σT	28.14%			32.55%
C	0.68						51.85%
D	0.55						

Lintner CAPM yields Sharpe CAPM --- we can now DERIVE expected returns, stdevs and correlations

Does the tangency portfolio look familiar?

The Security Market Line

Beta	Exp Ret
1.36	31.58%
1.13	26.10%
0.81	18.76%
1.00	23.15%



Was this difficult?

- We pre-discounted the cash flows at the risk-free rate, meaning we could assume r = 0
- We specified the means and the covariance matrix
- We found the risk charge for the whole portfolio (our benchmark)
- We allocated risk charges to individual assets by benchmarking to the total
  - Share of total risk = Cov(asset, market)/Var(market)
- *Remember, we did not start by assuming we knew the valuations*

Can you value a CDO this way?

Bond value = scheduled payments discounted at YTM

$$\rightarrow V_T = E(PVCF@rf) - k\,Var(PVCF@rf)$$

Simulations

Solve for k

WHAT-IF TABLE		Class A	Class B	Resid
		44.89704	12.85149	68.96965
PV@RF by tranche	1	44.89704	12.85149	83.43942
10,000 sims	2	44.89704	12.85149	42.98147
	3	44.89704	12.85149	75.8361
	4	44.89704	12.85149	75.91407
Find present value of	5	44.89704	12.85149	37.02372
actual cash flows at	6	44.89704	12.85149	50.62297
risk-free rate	7	44.89704	12.85149	35.30115
	8	44.89704	12.85149	75.57285
	9	44.89704	12.85149	56.62378
	10	44.89704	12.85149	83.43942
	11	44.89704	12.85149	72.34198
	12	44.89704	12.85149	82.67842
	13	44.89704	12.85149	66.48298
	14	44.89704	12.85149	58.08189
	15	44.89704	12.85149	82.76382
	16	44.89704	12.85149	81.90012
	17	44.89704	12.85149	35.43754
	18	44.89704	12.85149	74.61391
	19	44.89704	12.85149	54.12479
	20	44.89704	12.85149	48.13511

So easy now...

Find the cost of risk from the underlying collateral (bonds PV'd at 9%)

Bonds Market Value				
		80.355		
PV Expected cash flows @ riskfree rate				
		123.58		
PV Risk charge				
		43.224		
Grand sum				
		264.75		
Cost of risk				
		0.1633		

Allocate risk charges proportional to cov/var

		Class A	Class B	Resid				
Means		44.896	12.789	65.894				
MaxErr		0.0005	0.0118	0.3219				
		Cov			w'market		Mean	Value
		0.0006	0.0099	0.0478	0.0584		44.896	44.886
		0.0099	0.348	2.5923	2.9502		12.789	12.307
		0.0478	2.5923	259.1	261.74		65.894	23.161

Let's formalize this

Lintner version of CAPM - issuer's perspective, i.e. price securities so that the market clears N = 100%

Description	Notation	Calculation
The agent's expected wealth in one year, the expected value of the risky asset holdings plus the risk-free return on residual wealth N=holdings of risky assets, μ = mean cash flow, Σ= covariance matrix, r = risk-free rate, V=Value	$E[W_1]$	$N'\mu + (W_0 - N'V)(1+r)$
The variance of wealth in one year	$Var(W_1)$	$N'\Sigma N$
The utility of wealth in one year, or a "score" used to compare different choices of N	$U[W_1]$	$N'\mu + (W_0 - N'V)(1+r) - ½A\ N'\Sigma N$
The first order condition, or solving for the optimal risky positions	$\partial U/\partial N=0$	$\mu - V(1+r) - A\Sigma N$
The choice of N that satisfies the first order condition	N*	$\Sigma^{-1}(\mu - V(1+r))/A$
The equilibrium valuation of the risky assets, determined by setting N=1, which clears the market	V*	$\boxed{\frac{\mu - A\Sigma 1}{(1+r)}}$ rowsum
A is the expected aggregate risk premium per unit variance, a measure of the reward for taking risk.	A	$\frac{1'(\mu - V(1+r))}{1'\Sigma 1}$

What do these equations imply?

- Pricing equation
  - Compute risk-adjusted cash flow by deducting a risk charge, then discount at the risk-free rate
  - The risk charge is the covariance of the asset's cash flow with the overall market portfolio, multiplied by risk aversion
  - The risk aversion coefficient equals the total market risk premium in dollars divided by the variance in $\$^2$

Now derive the security market line

Calculation	Dollars	Returns (percentages)
Total return	$\mu - V$	$(\mu - V) \oslash V$
Risk premium	$\mu - (1 + r)V = A\Sigma 1$	$[\mu - (1 + r)V] \oslash V = A\Sigma 1 \oslash V$
Covariance with market	$\Sigma 1$	$\dfrac{\Sigma 1 \oslash V}{1'V}$
Variance of market	$1'\Sigma 1$	$\dfrac{1'\Sigma 1}{(1'V)^2}$
Beta	$\dfrac{\Sigma 1}{1'\Sigma 1}$	$\dfrac{\Sigma 1 \oslash V}{1'\Sigma 1} 1'V$
Risk premium⊘Beta (constant for all securities)	$A1'\Sigma 1$	$\dfrac{A1'\Sigma 1}{1'V}$

*The symbol "⊘" is being used to compute the Hadamard quotient or the Schur quotient, which is the element-by-element quotient of two vectors or matrices

How are cash flows valued under the CAPM?

- $V_0 = \frac{\mu - \lambda \, cov(C_1, r_m)}{1+r}$
- Notice that the market portfolio is held fixed
- An uncorrelated cash flow has no risk charge
  - "Risk is diversified away, implying we can discount at the risk-free rate"
- A cash flow that is "cyclical", i.e. correlated with market returns, will have a risk charge
- The cost of (variance) risk, $\lambda = \frac{r_m - r}{\sigma_m^2}$

How would we value the asset with our derivation?

Asset	Mu	Sigma	Correl
1	100	20	0.5
2	120	30	

Covariance		x1 Vector
400	300	700
300	900	1200

Riskfree	0.02
A	0.02
W	150

Inverse		x1 Vector
0.0033	-0.0011	0.0022
-0.0011	0.0015	0.0004

		Total Return		Risk Premium		Cov with Market	
	Value	Dollars	Percent	Dollars	Percent	Dollars	Returns
1	84.31	15.69	18.60%	14.00	16.60%	700	4.65%
2	94.12	25.88	27.50%	24.00	25.50%	1200	7.15%
M	178.43	41.57	23.30%	38.00	21.30%	1900	5.97%

	Beta		Risk prem div by beta	
	Dollars	Returns	Dollars	Percent
1	0.37	0.78	38.00	0.21
2	0.63	1.20	38.00	0.21
M	1.00	1.00	38.00	0.21

	Variance		StdDev		Sharpe Ratio	
	Dollars	Returns	Dollars	Returns	Dollars	Returns
1	400	5.63%	20.00	23.72%	0.70	0.70
2	900	10.16%	30.00	31.88%	0.80	0.80
M	1900	5.97%	43.59	24.43%	0.87	0.87

Let's review the two-asset portfolio from the text…

What happens when we add a third asset, uncorrelated with the others?

$\mu = 150$
$\sigma = 40$

Adding a third uncorrelated asset

Add third uncorrelated asset

	Mu	Sigma	Correl	Covariance			
1	100	20	0.5	400	300	0	(100 - 0.02*700)/(1+r)
2	120	30		300	900	0	(120 - 0.02*1200)/(1+r)
3	150	40		0	0	1600	(150 - 0.02*1600)/(1+r)

Riskfree	0.02
A	0.02

exp return of asset 3 is a function of idio risk; higher risk implies lower value

$$V_0 = \frac{\mu - A\Sigma 1}{(1+r)}$$

	Valuations	Expected returns	Risk	Sharpe Ratio
1	84.31	18.60%	23.72%	0.70
2	94.12	27.50%	31.88%	0.80
3	115.69	29.66%	34.58%	0.80
M	294.12	25.80%	20.11%	1.18

Sharpe CAPM - Discount uncorrelated cash flows at risk-free rate

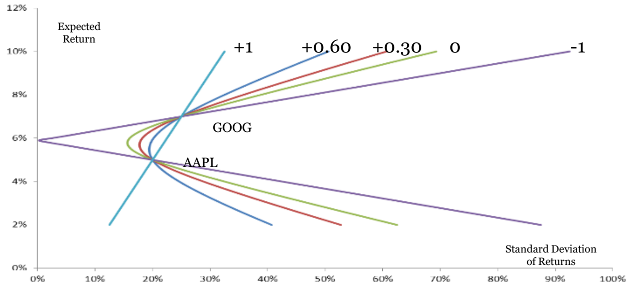
Key takeaway: In the case of an uncorrelated asset, covariance pricing is the same as variance pricing.

Effect of Variance on Price

- Traditional CAPM teaches that idiosyncratic risk is unpriced
- This derivation shows that this conclusion is true only if we ignore the equilibrium values of r and σ
- We showed:
  - For an asset outside the market portfolio, idiosyncratic risk can be unpriced
  - For an asset inside the market portfolio, expected $ return is the dollar covariance, which includes the variance of the asset
  - Beta is also a function of asset-specific variance
  - The market risk premium is also a function of asset specific variance
- Conclusion: The CAPM result does not necessarily lead to the conclusion that idiosyncratic risk is unpriced

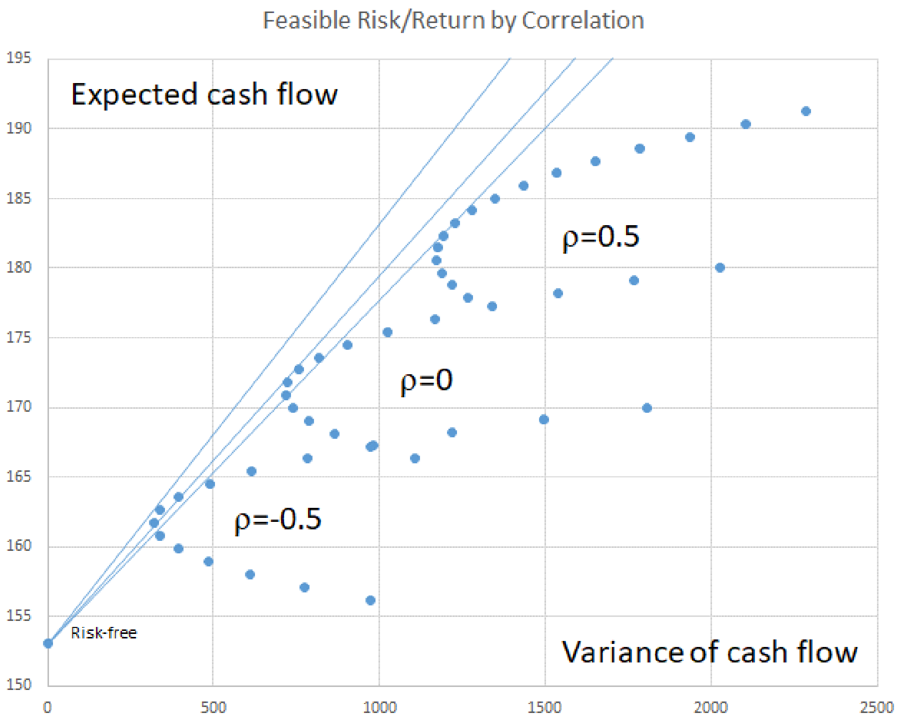
Effect of Correlation on Price

- These are efficient portfolios of two risky assets
- Falling correlation
  - Increases asset values (why?)
  - Reduces % expected returns
  - Reduces % standard deviations
- Usual impact of correlation only holds true *holding r and σ constant*



Holding r and σ constant

Correlation increases --> Risk concentration increases --> Market value falls --> Expected returns RISE --> Risks RISE



...In the cash flow model

Takeaways (3)

- In the CAPM, the market portfolio is held constant when evaluating a new asset, and $V_0 = \frac{\mu - \lambda\, cov(C_1, r_m)}{1+r}$, an example of covariance-based risk pricing
- We re-derived the CAPM by assuming that expected returns and variance of returns should be computed in equilibrium from asset prices
- The valuation equation was $\frac{\mu - A\Sigma 1}{(1+r)}$
- The CAPM equation holds in this model, but now r and σ are derived, and we can see that the risk premium is a linear function of its variance and covariance terms: $r_i - r_f = A\Sigma_i 1/V_i$
- The traditional view is that idiosyncratic risk has no impact on pricing; we showed this interpretation to be incorrect
- Including a new uncorrelated asset in a revised market portfolio leads the asset to be priced using covariance with itself, i.e. variance. The risk charge is still a multiple of the variance.
- Our understanding of correlation and risk reduction should be modified to include the effect of correlation on expected returns and standard deviation of returns

Remaining questions with the CAPM…

- How do we go to multiple periods?
- How can we guarantee expected returns are in equilibrium?
- How do we apply the CAPM to asset valuation, when it assumes asset values are known?
- How does the CAPM deal with nonlinear risk relationships?

Recitation Questions

1. Where did Harry Markowitz get his PhD? Who was his dissertation advisor?

2. What is an overcapitalized security? What can we say about its returns compared to other securities?

3. How might you calculate return on a short equity position?

4. What is the Sharpe Ratio? The dollarized Sharpe ratio? When are they each used?

5. An unlevered investment has a 6% return and 10% volatility when the risk-free rate is 2%. What is its Sharpe Ratio? What if the investment is leveraged 50%?

Recitation B

6. What is the difference between diversification and hedging?

7. What are the assumptions of MPT?

8. What is the expected return and risk of a portfolio consisting of a risk-free asset and a risky asset?

9. What is the expected return and risk of a portfolio consisting of two risky assets?

10. What is the objective function in MPT? What are the constraints?

Recitation C

11. What is the matrix notation for the multiple asset case? What are the mean returns and standard deviations of a portfolio for the multi-asset case?
12. What is the efficient frontier of risky assets? What if there is a risk-free asset?
13. Why does MPT recommend terrible portfolios from historical data?
14. What is the famous CAPM equation and what does it mean?
15. What is the security's variance in the CAPM?

Recitation D

16. What is the elasticity interpretation of β?
17. How is idiosyncratic risk treated in the CAPM?
18. What is the SML?
19. What is the source of risk premium in the CAPM?
20. How is β estimated in practice?
21. What are Fama-French and Carhart?
22. How does the APT differ from the CAPM?

Recitation E

23. What is the main difference between the Sharpe CAPM and the Lintner CAPM?
24. What is the expected return relationship for each of these models?
25. Given a set of N normally distributed cash flows, how would you find the value?
26. Same as 25, but this time, the value of the collection of cash flows is known.
27. How is idiosyncratic risk understood in the Sharpe and Lintner CAPMs?
28. How do changes in correlations affect the efficient portfolio set in the Sharpe vs the Lintner CAPM?