FIXED INCOME SECURITIES

FRE : 6411

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Term Structure of Zero Coupon

- In order to develop intuition about the shape of the yield curve, we have a standard approach : to plot the yield vs. maturity for a zero coupon or pure discount bond
- If we plot the yield curve for coupon bond , depending on the coupon rate it is not clear that the final maturity is important.
- If we plot yield Vs. Duration, depending on the coupon and payment structure same duration bonds may have different yields.
- Using the bond convention let y (t ,T) be as (1+% of yield) , it is also called holding period return ,for holding T-maturity bond from time t until T
- For pure discount bond the yield is : $y(t,T) = [\frac{1}{p(t,T)}]^{\frac{1}{T-t}}$

Forward Rate

- Time "t" forward rate for the period [T , T+1] is the rate locked in at time "t" for a bond commencing at time T & maturing at time T+1 ,.

$$f(t.T) = \frac{p(t,T)}{p(t,T+1)}$$

It is the rate contracted at time t $\leq$T for a risk-less loan over the time period $[T,T+1]$

Mimicking portfolio:

At time t buy one unit of zero coupon with maturity at time T and pay $P(t,T)$. To finance this purchase sell $\frac{P(t,T+1)}{P(t,T)}$ units of $T+1$ zero coupon bond each at price $P(t,T+1)$.

Forward Rate

- Look at the transaction and cash flow:

	Cash flow		
Time Transaction	t	T	T+1
Buy on "T" Bond	-P(t,T)	+1	
sell [P (t,T) / P (t ,T+1)] of T+1 Bond	+P(t,T)		-P (t,T) / P (t ,T+1)
net Cash Flow	0	+1	-P (t,T) / P (t ,T+1)

It is equivalent to us borrowing at time "T" one dollar and pay it back with interest of the forward rate at time "T+1". We locked this rate at time t .

Forward Rate - Example

Period	Bond Price	Yield	Forward Rate
1	0.9523	1.050	1.050
2	0.9053	1.051	1.052
3	0.8581	1.052	1.055
4	0.8119	1.053	1.057
5	0.7659	1.055	1.060
6	0.7212	1.056	1.062
7	0.6771	1.057	1.065
8	0.6358	1.058	1.065
9	0.597	1.059	1.065

- Spot Rate is defined as the rate contracted at time t for one period risk less loan commencing immediately, that is : r(t)=f(t,t)
- Money-Market Account An account starting with $1 at time 0 and investing in spot rate and is rolling it every period.
- $M(0) = 1\ ,\ M(1) = M(0) * r(0), \ldots, M(t+1) = M(t) * r(t)$

Forward Rate

- When we have zero coupon curve the forward rate are determined by arbitrage conditions:

$$P(t,t)=1$$

$$P(t,t+1)=\frac{1}{f(t,t)}$$

$$f(t,t+1)=\frac{p(t,t+1)}{P(t,t+2)}\Rightarrow P(t,t+2)=\frac{1}{f(t,t)\times f(t,t+1)}$$

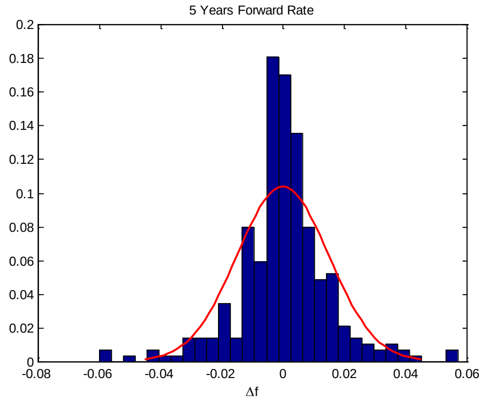
$$p(t,T)=\frac{1}{\prod_{j=1}^{T-1} f(t,j)}$$

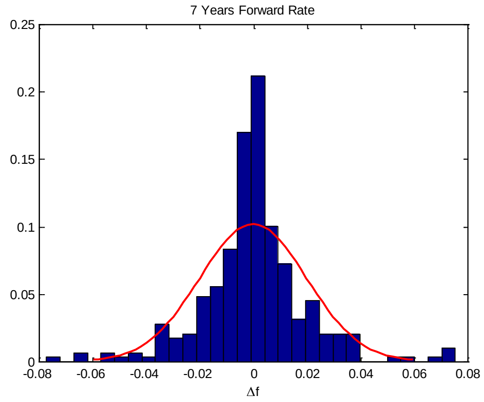
The Evolution of Term Structure of Interest Rate

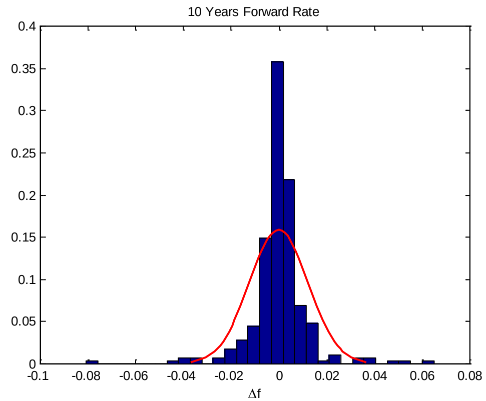
- Principal of No Arbitrage: Two perfect substitutes that are freely traded have to sell at the same price, in the absence of arbitrage
- Arbitrage Pricing or Relative Pricing Theory : Given a set primary assets, their prices and their stochastic price evolution through time. Our objective is to price secondary asset by:
- Use the primary assets construct a portfolio of these assets to mimic all cash flows of the secondary asset, by dynamically re-balancing the portfolio through time.
- Then to avoid arbitrage the price of the secondary asset MUST equal to the cost of this replicating portfolio
- In Fixed –Income the primary asset classes are the zero-coupon bonds and the money market. The secondary class could be Forward contracts , coupon bonds, other zero-coupon bond, etc.
- Note : To use Non-Arbitrage Pricing model the prices and their stochastic evolution through time of prices of the Primary assets MUST be EXOGENOUSLY given.

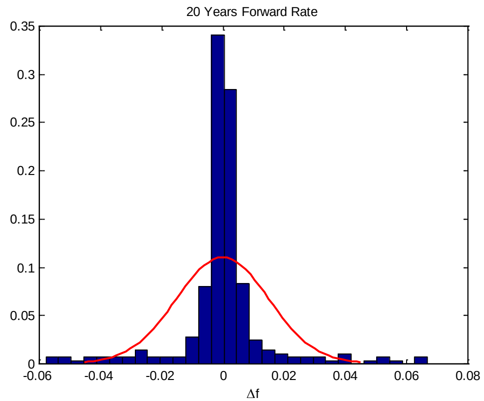
One Factor Model Economy

- Histogram of Monthly Changes in Forward Rates from January 1973 - March 1997





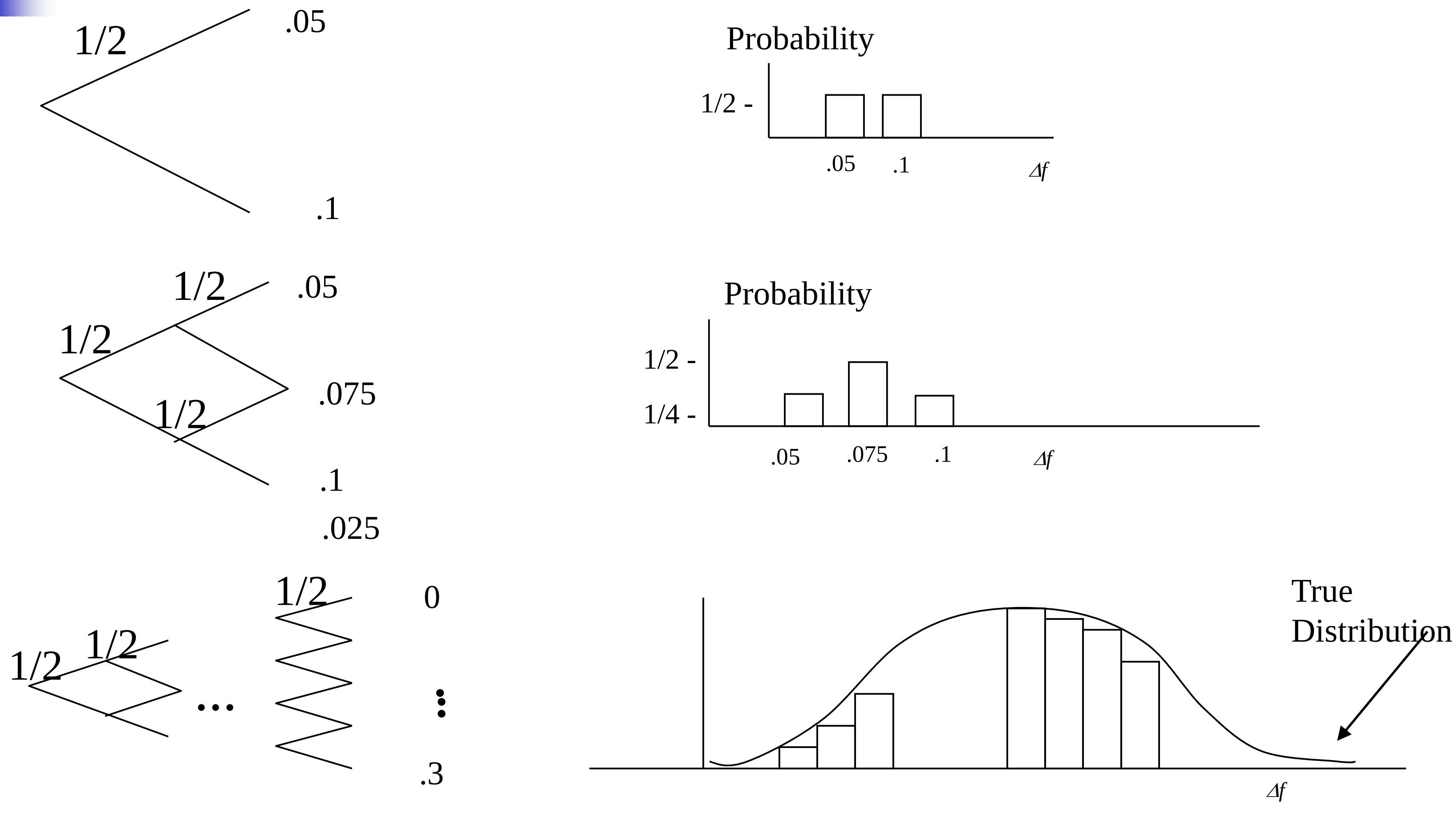




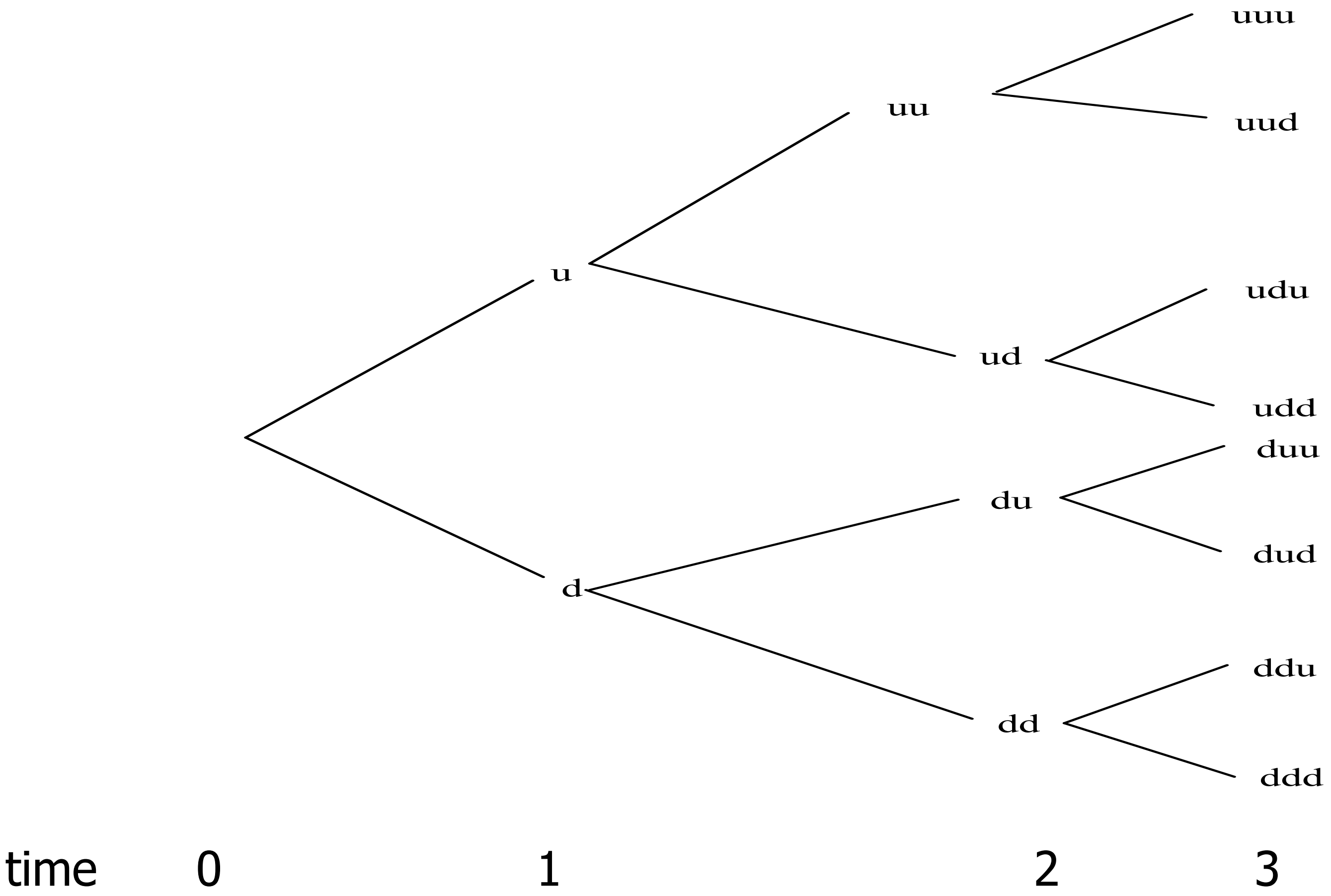
One Factor Model Economy

- build a model for the evolution of forward rates such that the model provides a reasonable approximation to the underlying distribution function
- The binomial (and its generalization – the multinomial) model provides such an approximation
- *one-factor model* : For each period "t" , only one of *two* possibilities can be realized next period "$t+1$" (up or down). Each branch also occurs with strictly positive probability. One can conceptualize the realization of next period as outcome of tossing *one coin* (one-factor).
- If, instead, at each period "t" , three possible outcome each with strictly positive probability can occur next period "$t+1$" , then it is *two-factor model.* Next period is realization is the outcome of *two coins* toss.

Example of a Binomial Approximation to the Normal Distribution



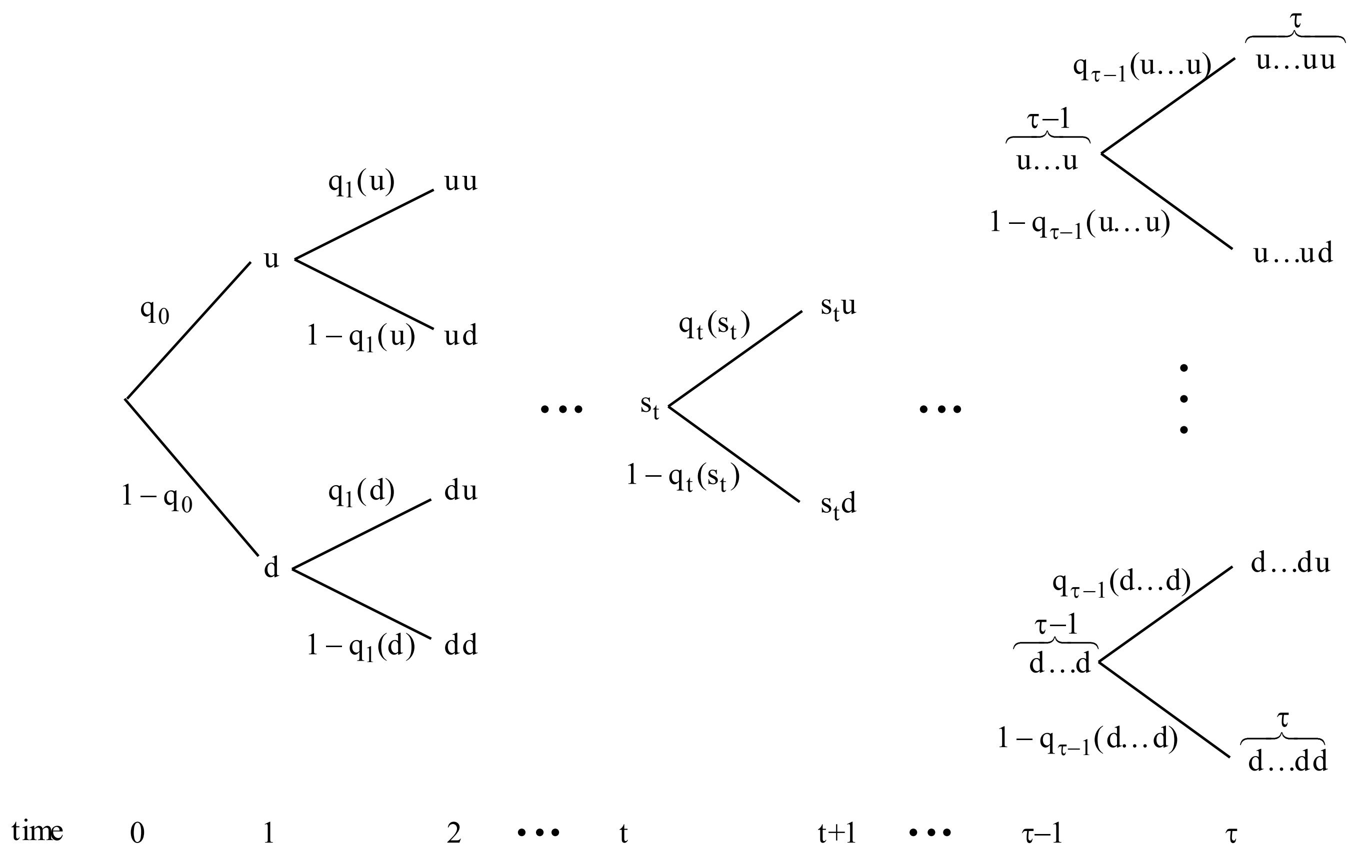
One Factor Model Economy



One Factor Model Economy

- At time 0 , one of two possible outcomes can occur. At time 1, we are either in up state 'u' with probability q0 or in down state 'd' with probability 1-q0. The realized state at time 1, s1 is a member of : $s1 \in \{u, d\}$
- From state1 the next period state, s2 is realization of two possible outcome: up 'u' with probability q1(s1) or down with probability 1-q1(s1). At time 2, $s2 \in \{s1u, s1d\}$, therefore there, or $s2 \in \{uu, ud, du, dd\}$
- $sn \in \{u \dots u, u..d, \dots, d \dots u, d \dots d\}$ or $sn \in \{s(n-1)u, s(n-1)d\}$
- The specification of state provides us with complete history of the process.
- At end of T period we could have 2^T possible outcome

One Factor Model Economy



One Factor Model Economy Bond Pricing Process

- If at the end of 3 period we are in state 'ddu' we know at time 1, time 2 and time 3, we jumped to state 'd' , 'd' , 'u' respectively
- It is called that state are path dependent and tree is busy

- Bond Pricing:

Let P(t,T,s(t) be the price at time t for T maturity zero-coupon when we are at state s(t). Since it is zero-coupon at maturity it pays $1 in all state of the world . P(T,T,S(T))=1 for all s(T)

for any time 't' <T the price depends on the state that we are at.

$$\begin{bmatrix} P(0,\tau) \\ P(0,\tau-1) \\ \vdots \\ P(0,2) \\ P(0,1) \\ P(0,0)=1 \end{bmatrix}$$

q_0

$$\begin{bmatrix} P(1,\tau;u) \\ P(1,\tau-1;u) \\ \vdots \\ P(1,2;u) \\ P(1,1;u)=1 \end{bmatrix} = \begin{bmatrix} u(0,\tau)P(0,\tau) \\ u(0,\tau-1)P(0,\tau-1) \\ \vdots \\ u(0,2)P(0,2) \\ r(0)P(0,1) \end{bmatrix}$$

$1-q_0$

$$\begin{bmatrix} P(1,\tau;d) \\ P(1,\tau-1;d) \\ \vdots \\ P(1,2;d) \\ P(1,1;d)=1 \end{bmatrix} = \begin{bmatrix} d(0,\tau)P(0,\tau) \\ d(0,\tau-1)P(0,\tau-1) \\ \vdots \\ d(0,2)P(0,2) \\ r(0)P(0,1) \end{bmatrix}$$

$q_1(u)$

$$\begin{bmatrix} P(2,\tau;uu) \\ P(2,\tau-1;uu) \\ \vdots \\ P(2,3;uu) \\ P(2,2;uu)=1 \end{bmatrix} = \begin{bmatrix} u(1,\tau;u)P(1,\tau;u) \\ u(1,\tau-1;u)P(1,\tau-1;u) \\ \vdots \\ u(1,3;u)P(1,3;u) \\ r(1;u)P(1,2;u) \end{bmatrix}$$

$1-q_1(u)$

$$\begin{bmatrix} P(2,\tau;ud) \\ P(2,\tau-1;ud) \\ \vdots \\ P(2,3;ud) \\ P(2,2;ud)=1 \end{bmatrix} = \begin{bmatrix} d(1,\tau;u)P(1,\tau;u) \\ d(1,\tau-1;u)P(1,\tau-1;u) \\ \vdots \\ d(1,3;u)P(1,3;u) \\ r(1;u)P(1,2;u) \end{bmatrix}$$

$q_1(d)$

$$\begin{bmatrix} P(2,\tau;du) \\ P(2,\tau-1;du) \\ \vdots \\ P(2,3;du) \\ P(2,2;du)=1 \end{bmatrix} = \begin{bmatrix} u(1,\tau;d)P(1,\tau;d) \\ u(1,\tau-1;d)P(1,\tau-1;d) \\ \vdots \\ u(1,3;d)P(1,3;d) \\ r(1;d)P(1,2;d) \end{bmatrix}$$

$1-q_1(d)$

$$\begin{bmatrix} P(2,\tau;dd) \\ P(2,\tau-1;dd) \\ \vdots \\ P(2,3;dd) \\ P(2,2;dd)=1 \end{bmatrix} = \begin{bmatrix} d(1,\tau;d)P(1,\tau;d) \\ d(1,\tau-1;d)P(1,\tau-1;d) \\ \vdots \\ d(1,3;d)P(1,3;d) \\ r(1;d)P(1,2;d) \end{bmatrix}$$

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$$\begin{bmatrix} P(0,\tau) \\ P(0,\tau-1) \\ \vdots \\ P(0,2) \\ P(0,1) \\ P(0,0)=1 \end{bmatrix}$$

$$\begin{bmatrix} P(1,\tau;u) \\ P(1,\tau-1;u) \\ \vdots \\ P(1,2;u) \\ P(1,1;u)=1 \end{bmatrix} = \begin{bmatrix} u(0,\tau)P(0,\tau) \\ u(0,\tau-1)P(0,\tau-1) \\ \vdots \\ u(0,2)P(0,2) \\ r(0)P(0,1) \end{bmatrix}$$

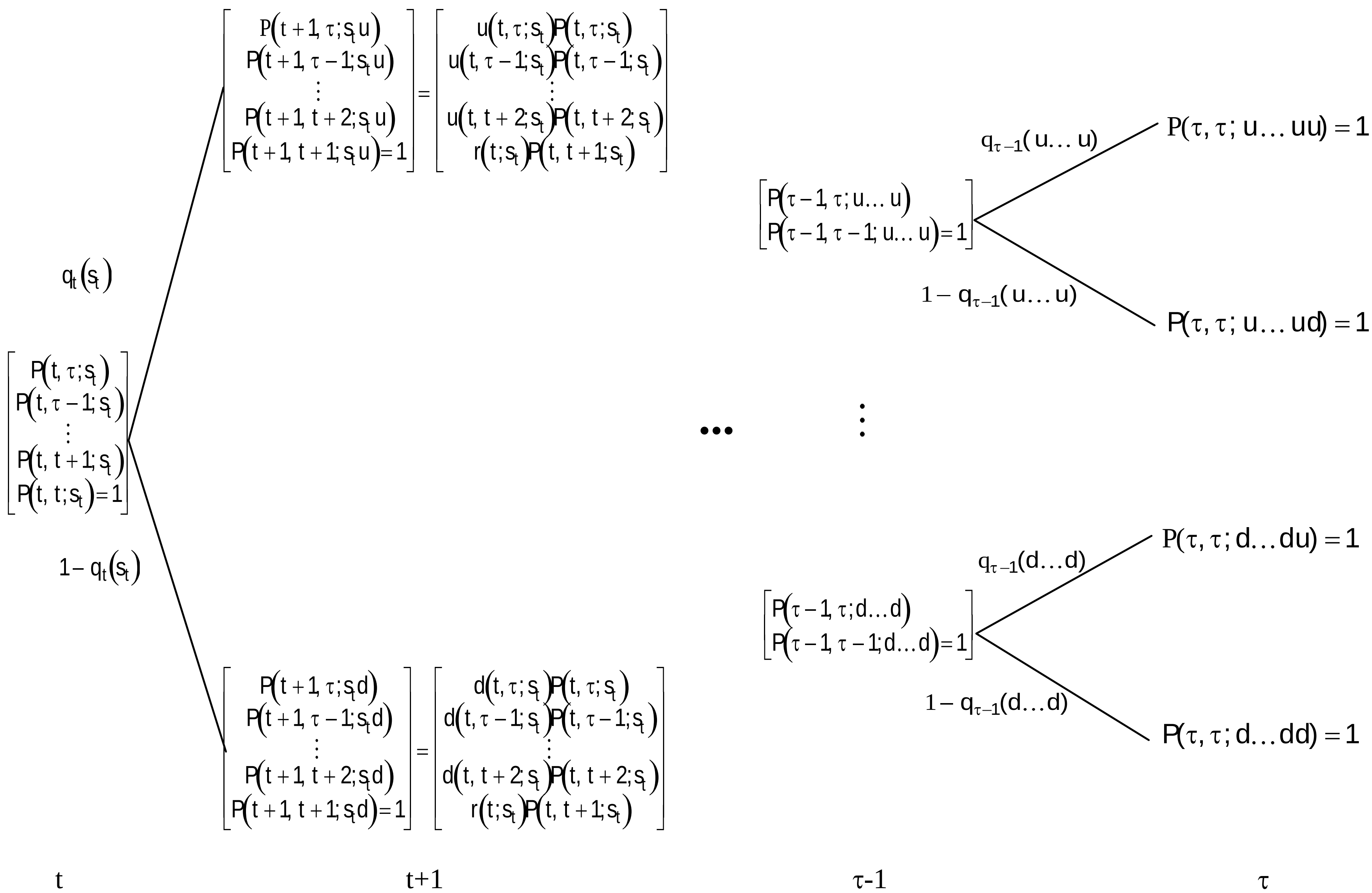
$$\begin{bmatrix} P(1,\tau;d) \\ P(1,\tau-1;d) \\ \vdots \\ P(1,2;d) \\ P(1,1;d)=1 \end{bmatrix} = \begin{bmatrix} d(0,\tau)P(0,\tau) \\ d(0,\tau-1)P(0,\tau-1) \\ \vdots \\ d(0,2)P(0,2) \\ r(0)P(0,1) \end{bmatrix}$$

$$\begin{bmatrix} P(2,\tau;uu) \\ P(2,\tau-1;uu) \\ \vdots \\ P(2,3;uu) \\ P(2,2;uu)=1 \end{bmatrix} = \begin{bmatrix} u(1,\tau;u)P(1,\tau;u) \\ u(1,\tau-1;u)P(1,\tau-1;u) \\ \vdots \\ u(1,3;u)P(1,3;u) \\ r(1;u)P(1,2;u) \end{bmatrix}$$

$$\begin{bmatrix} P(2,\tau;ud) \\ P(2,\tau-1;ud) \\ \vdots \\ P(2,3;ud) \\ P(2,2;ud)=1 \end{bmatrix} = \begin{bmatrix} d(1,\tau;u)P(1,\tau;u) \\ d(1,\tau-1;u)P(1,\tau-1;u) \\ \vdots \\ d(1,3;u)P(1,3;u) \\ r(1;u)P(1,2;u) \end{bmatrix}$$

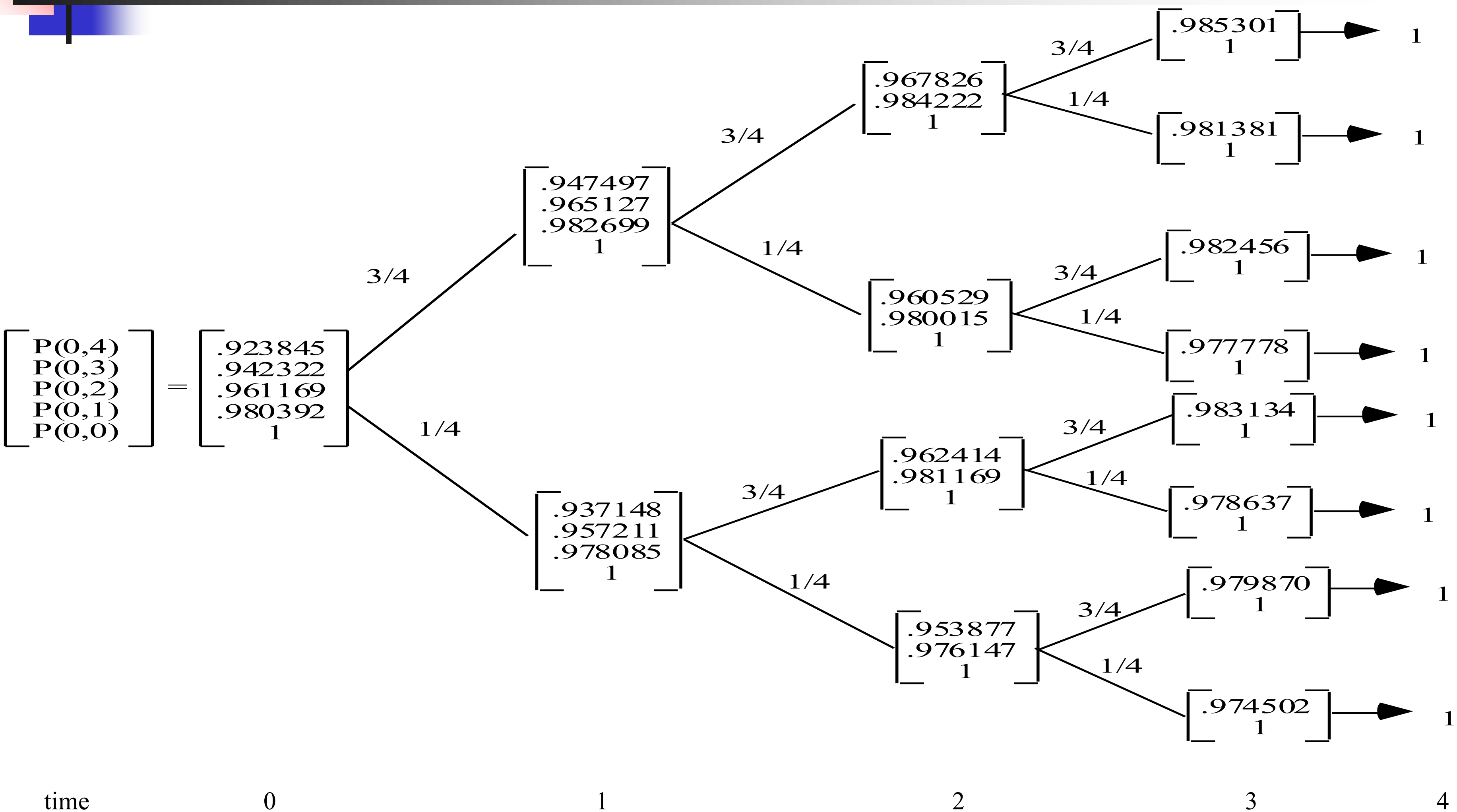
$$\begin{bmatrix} P(2,\tau;du) \\ P(2,\tau-1;du) \\ \vdots \\ P(2,3;du) \\ P(2,2;du)=1 \end{bmatrix} = \begin{bmatrix} u(1,\tau;d)P(1,\tau;d) \\ u(1,\tau-1;d)P(1,\tau-1;d) \\ \vdots \\ u(1,3;d)P(1,3;d) \\ r(1;d)P(1,2;d) \end{bmatrix}$$

$$\begin{bmatrix} P(2,\tau;dd) \\ P(2,\tau-1;dd) \\ \vdots \\ P(2,3;dd) \\ P(2,2;dd)=1 \end{bmatrix} = \begin{bmatrix} d(1,\tau;d)P(1,\tau;d) \\ d(1,\tau-1;d)P(1,\tau-1;d) \\ \vdots \\ d(1,3;d)P(1,3;d) \\ r(1;d)P(1,2;d) \end{bmatrix}$$



$\begin{bmatrix} P(t+1,\tau;s_t u) \\ P(t+1,\tau-1;s_t u) \\ \vdots \\ P(t+1,t+2;s_t u) \\ P(t+1,t+1;s_t u)=1 \end{bmatrix} = \begin{bmatrix} u(t,\tau;s_t)P(t,\tau;s_t) \\ u(t,\tau-1;s_t)P(t,\tau-1;s_t) \\ \vdots \\ u(t,t+2;s_t)P(t,t+2;s_t) \\ r(t;s_t)P(t,t+1;s_t) \end{bmatrix}$
$q_t(s_t)$
$\begin{bmatrix} P(t,\tau;s_t) \\ P(t,\tau-1;s_t) \\ \vdots \\ P(t,t+1;s_t) \\ P(t,t;s_t)=1 \end{bmatrix}$
$1-q_t(s_t)$
$\begin{bmatrix} P(t+1,\tau;s_t d) \\ P(t+1,\tau-1;s_t d) \\ \vdots \\ P(t+1,t+2;s_t d) \\ P(t+1,t+1;s_t d)=1 \end{bmatrix} = \begin{bmatrix} d(t,\tau;s_t)P(t,\tau;s_t) \\ d(t,\tau-1;s_t)P(t,\tau-1;s_t) \\ \vdots \\ d(t,t+2;s_t)P(t,t+2;s_t) \\ r(t;s_t)P(t,t+1;s_t) \end{bmatrix}$
$\cdots$
$\vdots$
$q_{\tau-1}(u\ldots u)$
$P(\tau,\tau;u\ldots uu)=1$
$\begin{bmatrix} P(\tau-1,\tau;u\ldots u) \\ P(\tau-1,\tau-1;u\ldots u)=1 \end{bmatrix}$
$1-q_{\tau-1}(u\ldots u)$
$P(\tau,\tau;u\ldots ud)=1$
$q_{\tau-1}(d\ldots d)$
$P(\tau,\tau;d\ldots du)=1$
$\begin{bmatrix} P(\tau-1,\tau;d\ldots d) \\ P(\tau-1,\tau-1;d\ldots d)=1 \end{bmatrix}$
$1-q_{\tau-1}(d\ldots d)$
$P(\tau,\tau;d\ldots dd)=1$
t
t+1
τ-1
τ

One-Factor Bond Price Curve Evolution-Example



The Forward Rate Process

- We can summarize this evolution for an arbitrary time *t* as in expression:

$$f(t+1,T;s_{t+1})=\begin{cases}\alpha(t,T;s_t)f(t,T;s_t) \text{ if } s_{t+1}=s_t u \\ \text{with } q_t(s_t)>0 \\ \beta(t,T;s_t)f(t,T;s_t) \text{ if } s_{t+1}=s_t d \\ \text{with } 1-q_t(s_t)>0\end{cases}$$

Where :

$$\tau-1\geq T\geq t+1.$$

Forward Rate Process

- Note that from bond tree we can easily get the forward rate tree since:

$$f(t+1,T;s(t+1) = \frac{P(t+1,T,s(t+1))}{p(t+1,T+1,s(t+1))}$$

$$if\ \ s(t+1) = s(t)u:$$

$$f(t+1,T;s(t+1)) = \frac{P(t,T,s(t))\times u(t,T,s(T))}{P(t,T+1,s(t))\times u(t,T+1,s(t))} = f(t,T,s(t))\times\frac{u(t,T,s(T))}{u(t,T+1,s(t))}$$

$$\Rightarrow \alpha(t,T;s(t)) = \frac{u(t,T,s(t))}{u(t,T+1,s(t))}\ and\ \beta(t,T;s(t)) = \frac{d(t,T,s(t))}{d(t,T+1,s(t))}$$

We can parameterize bond price process first and then deduct the forward rate process from it; alternatively we can parameterize the forward rate process first and then get the bond price from it

Forward Rate Process

$$\begin{bmatrix} f(0,\tau-1) \\ \vdots \\ f(0,1) \\ f(0,0)=r(0) \end{bmatrix}$$

q_0

$$\begin{bmatrix} f(1,\tau-1;u) \\ \vdots \\ f(1,1;u)=r(1;u) \end{bmatrix} = \begin{bmatrix} \alpha(0,\tau-1)f(0,\tau-1) \\ \vdots \\ \alpha(0,1)f(0,1) \end{bmatrix}$$

$1-q_0$

$$\begin{bmatrix} f(1,\tau-1;d) \\ \vdots \\ f(1,1;d)=r(1;d) \end{bmatrix} = \begin{bmatrix} \beta(0,\tau-1)f(0,\tau-1) \\ \vdots \\ \beta(0,1)f(0,1) \end{bmatrix}$$

$\cdots$

Forward Rate Process

$$\begin{bmatrix} f(t+1,\tau-1;s_t u) \\ \vdots \\ f(t+1,t+1;s_t u)=r(t+1;s_t u)\end{bmatrix} = \begin{bmatrix}\alpha(t,\tau-1;s_t)f(t,\tau-1;s_t)\\ \vdots \\ \alpha(t,t+1;s_t)f(t,t+1;s_t)\end{bmatrix}$$

$q_t(s_t)$

$$\begin{bmatrix} f(t,\tau-1;s_t) \\ \vdots \\ f(t,t;s_t)=r(t;s_t)\end{bmatrix}$$

$1-q_t(s_t)$

$$\begin{bmatrix} f(t+1,\tau-1;s_t d) \\ \vdots \\ f(t+1,t+1;s_t d)=r(t+1;s_t d)\end{bmatrix} = \begin{bmatrix}\beta(t,\tau-1;s_t)f(t,\tau-1;s_t)\\ \vdots \\ \beta(t,t+1;s_t)f(t,t+1;s_t)\end{bmatrix}$$

$\cdot\ \cdot$

$$\left[f(\tau-1,\tau-1;u\ldots u)=r(\tau-1;u\ldots u)\right]$$

$q_{\tau-1}(u\ldots u)$ N/A

$1-q_{\tau-1}(u\ldots u)$ N/A

$\vdots$

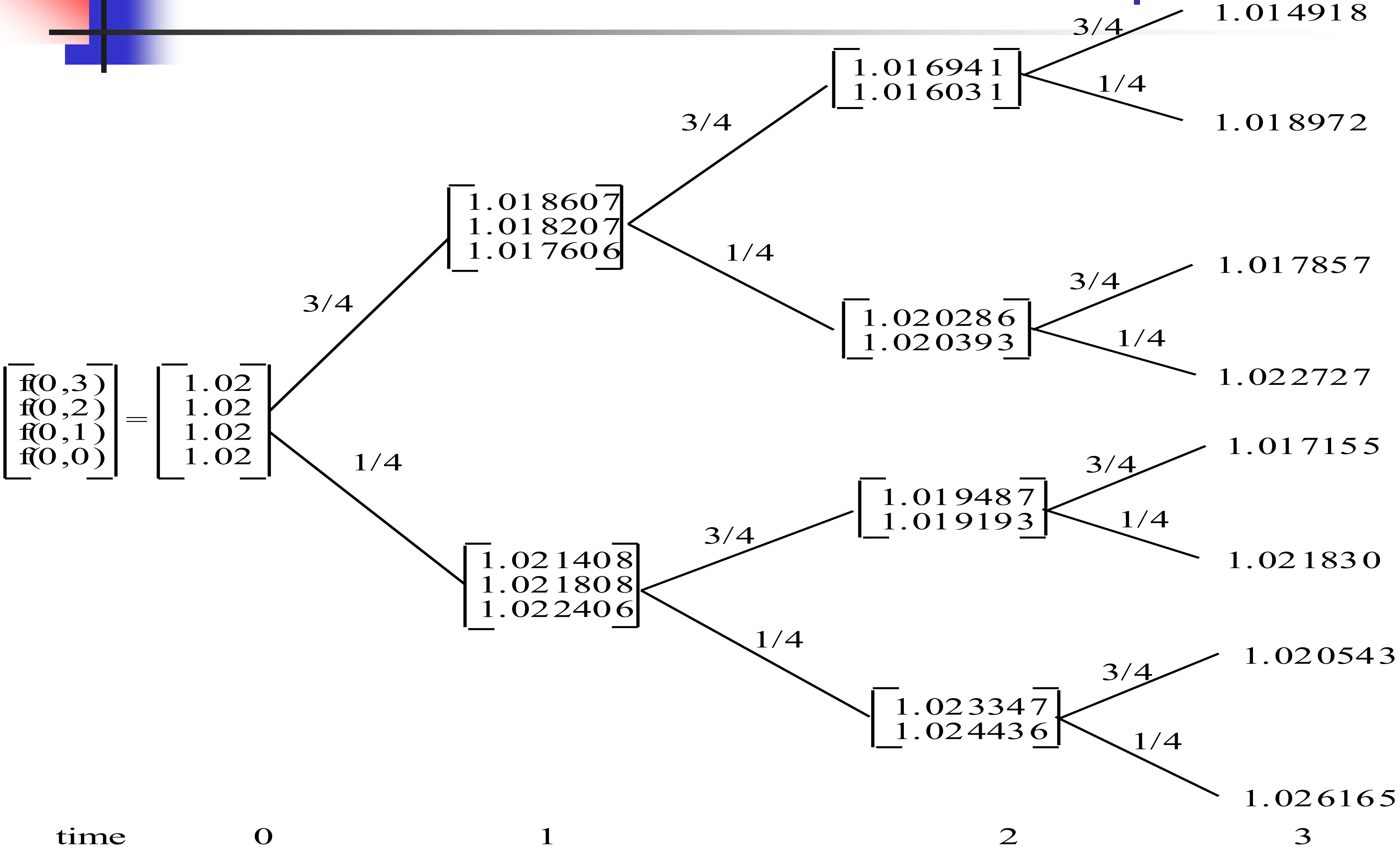
$$\left[f(\tau-1,\tau-1;d\ldots d)=r(\tau-1;d\ldots d)\right]$$

$q_{\tau-1}(d\ldots d)$ N/A

$1-q_{\tau-1}(d\ldots d)$ N/A

t $t+1$ τ-1 τ

The Forward Rate Process-Example

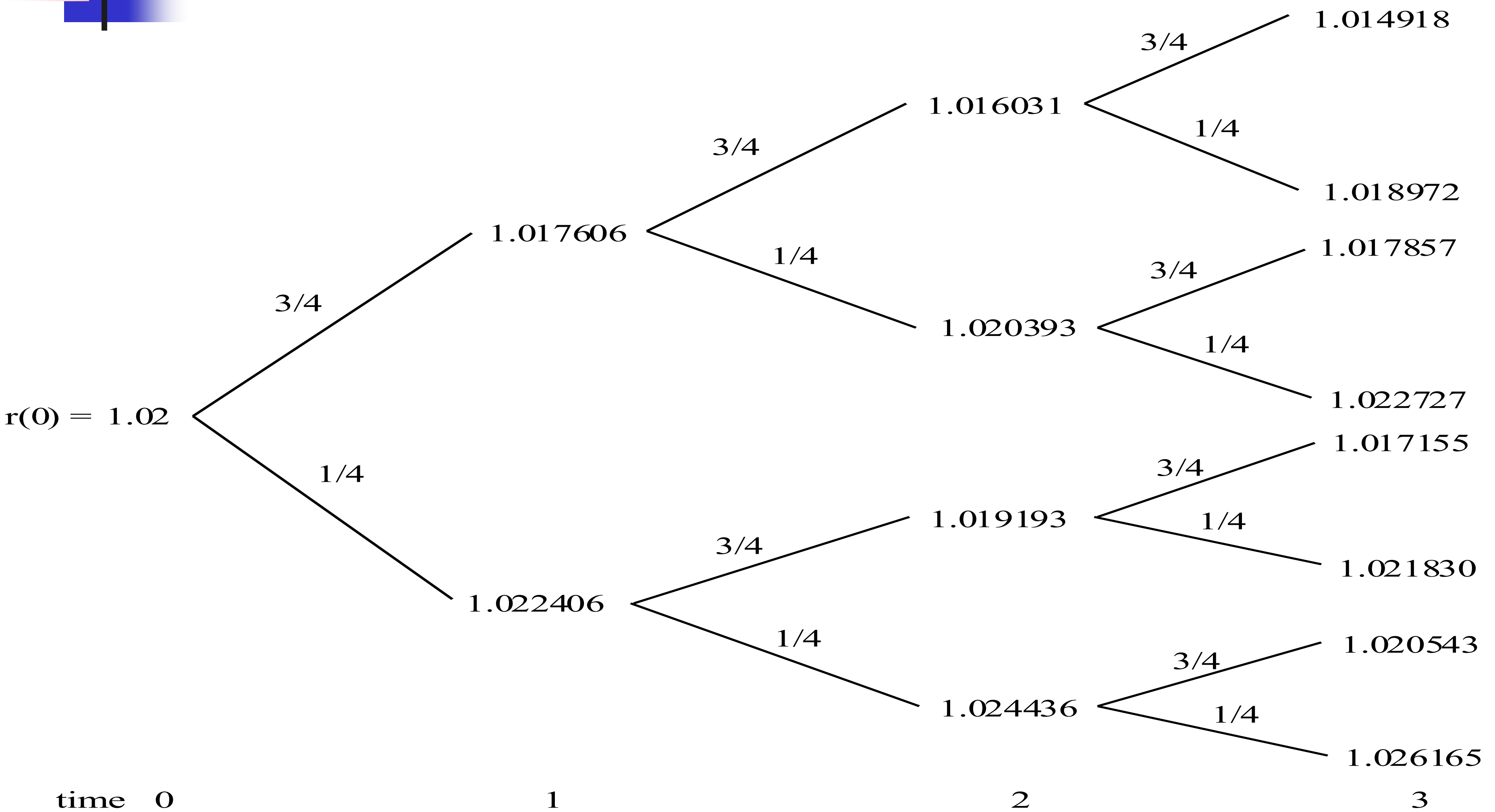


Spot Rate Process

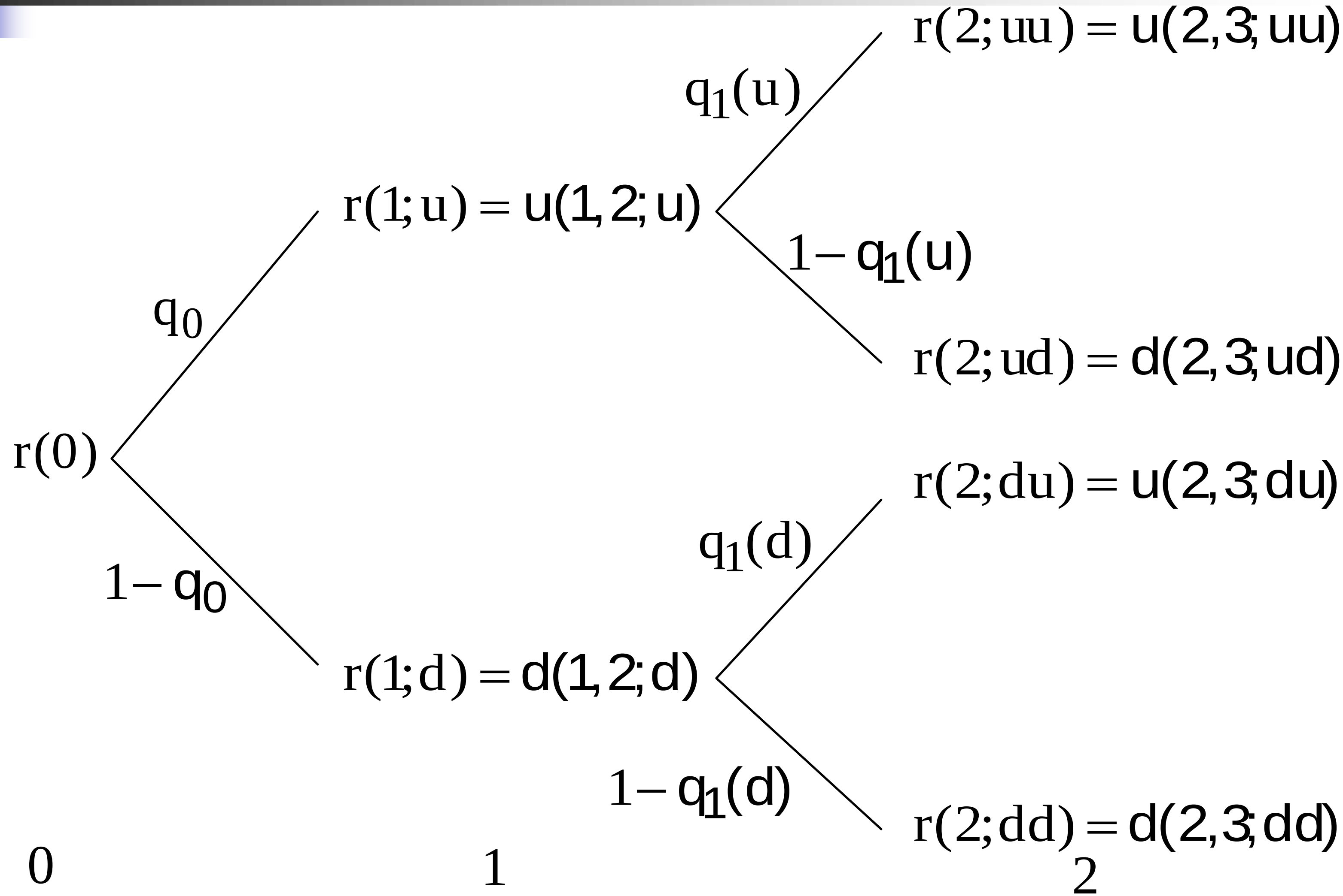
- The Spot rate process simply the price process of the bond with only one period to maturity:

$$r(t;s(t)) = \frac{1}{p(t,t+1,s(t))} = f(t,t,s(t)$$

Spot Rate Process



Spot Rate Process



Spot Rate Process

$$r(t+1; s_t u) = u(t+1, t+2; s_t u)$$

$q_t(s_t)$

$r(t; s_t)$

$1 - q_t(s_t)$

$$r(t+1; s_t d) = d(t+1, t+2; s_t d)$$

$\cdots$

$q_{\tau-2}(u \ldots u)$

$$r(\tau-1; u \ldots uu) = u(\tau-1, \tau; u \ldots uu)$$

$r(\tau-2; u \ldots u)$

$1 - q_{\tau-2}(u \ldots u)$

$$r(\tau-1; u \ldots ud) = d(\tau-1, \tau; u \ldots ud)$$

$\vdots$

$$r(\tau-1; d \ldots du) = u(\tau-1, \tau; d \ldots du)$$

$q_{\tau-2}(d \ldots d)$

$r(\tau-2; d \ldots d)$

$1 - q_{\tau-2}(d \ldots d)$

$$r(\tau-1; d \ldots dd) = d(\tau-1, \tau; d \ldots dd)$$

t | $t+1$ | τ-2 | τ-1

Summary

- At time t:
- $P(t, T, s(t))$ is time t, state s(t) price of zero coupon bond maturing in T.
- $P(t+1, T, s(t)u)$ is time t+1, price of the same bond IF we jump to state u from current state s(t)
- $P(t+1, T, s(t)d)$ is time t+1, price of the same bond IF we jump to state d from current state s(t)
- $r(t, st)$ is one risk free rate at time t.
- $P\big(T, T, s(T)\big) = 1$, price at time T, zero coupon bond maturing at time T. (immediate maturity)
- $P\big(T-1, T, s(T-1)\big) = \frac{1}{r(T-1, s(T-1))}$ price at time T-1, zero coupon bond maturing at time T. (one period to maturity)

Summary

- Money Market Account:
- At Time 0, invest $1 (M(0)=$1) in one period risk free rate $r(0)$ (known at time 0)
- At time 1, is state s1 value of this account will be :
- $M(1) = M(0) * r(0) = r(0)$
- Roll your investment and invest in one period risk free rate $r(1, s1)$
- And continue this price :
- $M(t, s(t-1)) = M(t-1, s(t-2)) * r(t-1, s(t-1))$
- Note $M(t, s(t-1))$ is realized value of rolling money market at time t, but since $r(t-1, s(t-1))$ is known at time t-1, therefore value of $M(t, s(t-1))$ is also know at time t-1