Bachelier Model

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Abstract

Normal stock price

The Bachelier model assumes stock price has the form $F_t = f + \sigma B_t$ where f and σ are constants and B_t is standard Brownian motion, assuming zero interest rate.

The risk-neutral value of a put with strike k expiring at t is

$$E[(k - F_t)^+] = E[(k - F)1(F \leq k)] = E[k - F]P(F \leq k) + \mathrm{Cov}(k - F, 1(F \leq k)).$$

Exercise. *Show if M and N are jointly normal then* $\mathrm{Cov}(M, g(N)) = \mathrm{Cov}(M, N)E[g'(N)]$.

Hint: Use $E[e^{\alpha N} g(M)] = E[e^{\alpha N}]E[g(M + \mathrm{Cov}(\alpha N, M))]$, take a derivative with respect to α, then set it equal to 0.

This shows $\mathrm{Cov}(k - F, 1(F \leq k)) = \mathrm{Cov}(k - F, F)E[-\delta_k(F)] = \mathrm{Var}(F)E[\delta_k(F)]$.

Exercise *Show* $E[\delta_a(b + cZ)] = \phi((a - b)/c)/c$ *if ϕ is the density function of Z.*

▼ Solution

If $d_a(x) = 1/h$ for $a < x \leq a + h$ and 0 otherwise then $E[\delta_a(b + cZ)] = \lim_{h \to 0} E[d_a(b + cZ)]$. Using $y = (b + cz)/c = b/c + z$ we have

$$\begin{aligned} E[d_a(b + cZ)] &= (1/h) \int_a^{a+h} (b + cz)\phi(z)\,dz \\ &= (1/h) \int_{b/c+a}^{b/c+a+h} cy\phi(y - b/c)\,dy \end{aligned}$$

If g is differentiable at $z = g^{-1}(a)$ then $E[\delta_a(g(Z))] = \phi(z)/g'(z)$ since $g(x) \approx g(z) + g'(z)(x - z)$ for $z \approx x$.

We have $E[\delta_k(F)] = E[\delta_k(f + \sigma\sqrt{t}Z)] = \phi(z)/\sigma\sqrt{t}$, where Z is standard normal and $z = (k - f)/\sigma\sqrt{t}$ so

$$E[(k - F_t)^+] = (k - F)\Phi(z) + \sigma\sqrt{t}\phi(z)$$

where Φ is the standard normal cumulative distribution function and $\phi = \Phi'$.

Exercise. *Show* $E[(F_t - k)^+] = (f-k)\Phi(-z) + \sigma\sqrt{t}\phi(-z)$.