FIXED INCOME SECURITIES

FRE : 6411

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2023

No-ARBITRAGE AND COMPLETE MARKET

- Given a set of Primary assets prices & their stochastic evolution of those prices, our goal is to find prices of set of secondary assets based on these information of the primary asset :

1. Set up a portfolio of primary assets
2. Dynamically re-balance the portfolio through a set of trading strategy of the primary assets to mimic exactly the cash flow of the secondary assets in every state of the world
3. Then price of the secondary assets is the value of the portfolio

One Period Two Primary Assets

- $\begin{bmatrix} p_{1u} \\ p_{2u} \end{bmatrix} w_u$

$\begin{bmatrix} p_{10} \\ p_{20} \end{bmatrix} w_0$

- $\begin{bmatrix} p_{1d} \\ p_{2d} \end{bmatrix} w_d$
- Here we have two primary assets $1 \,\&\, 2$, their prices $p_{10} \& \, p_{20}$ and price evolution $p_{1u} \,\&\, p_{2u}$ as well as $p_{1d} \,\&\, p_{2d}$.
- Price Secondary asset w such that its pay off is $w_u \,\&\, w_d$.
- Set up a portfolio of primary asset (x_{10}, x_{20}) to mimic exactly the cash flow of the secondary assets (w_u, w_d)
- $w_u = x_{10}p_{1u} + x_{20}p_{2u} \;\;\&\;\; w_d = x_{10}p_{1d} + x_{20}p_{2d}$

One Period Two Primary Assets

- Here we have two equations and two unknown solve for period 0 portfolio (x_{10}, x_{20}).
- $w_0 = x_{10}p_{1o} + x_{20}p_{20}$
- Summary :
- Construct the portfolio (x_{10}, x_{20}), at period 0, valued at w_0
- Then in period 1 state u, the value of this portfolio will be w_u
- Then in period 1 state d, the value of this portfolio will be w_d

Two Periods Two Primary Assets

- For Two period problem recursively solve the problem:
- In period 1, state u, find the portfolio (x_{1u}, x_{2u}) such that next period in states uu & ud, payoff are (w_{uu}, w_{ud}). Assume the value of this portfolio is w_u
- In period 1, state d, find the portfolio (x_{1d}, x_{2d}) such that next period in states du & dd, payoff are (w_{du}, w_{dd}). Assume the value of this portfolio is w_d
- In period 0, find the portfolio (x_{10}, x_{20}) such that period 1 payoff are (w_u, w_d). The value of the this portfolio w_0, is the price of price of the asset.

Trading Strategy in Zero-Coupon Bond

- Let our primary assets be a zero coupon bond with Maturity T_1 and rolling money market account:
- At period t state s_t the prices of our primary assets are :
- $P(t,T_1,s_t)$ & $B(t, s_{t-1})$.
- Note for MM account
- $B(t, s_{t-1}) = B(t-1, s_{t-2})\, r(t-1, s_{t-1})$
- The value of MM is going to be realized at period t, but its value is known at period $t-1$
- At period t, we are we are exiting the period by holding the portfolio of $M(t,s_t)$ & $N(t,s_t)$ of MM and zero coupon bond.
- The value of this upon *EXITING* period t is :
- $V(t,s_t) = B(t,s_{t-1})M(t,s_t) + P(t,T_1,s_t)N(t,s_t)$

Trading Strategy in Zero-Coupon Bond

- We enter period $t+1$, from period t with the portfolio $M(t,s_t)$ & $N(t,s_t)$. The value upon *ENTERING* period $t+1$ is :
- $B(t+1,s_t)M(t,s_t)+P(t+1,T_1,s_{t+1})N(t,s_t)$
- In period $t+1$, after trading the portfolio we *EXIT* the period with $M(t+1,s_{t+1})$ & $N(t+1,s_{t+1})$ holdings

We call a strategy self-financing if:

$$B(t+1,s_t)M(t,s_t)+P(t+1,T_1,s_{t+1})N(t,s_t)=$$
$$B(t+1,s_t)M(t+1,s_{t+1})+P(t+1,T_1,s_{t+1})N(t+1,s_{t+1})$$

The Value of portfolio upon *ENTERING* is the same as *EXITING*

$$\phi_1=\left\{\begin{array}{l} all \text{ self financing strategies involving only the money market account} \\ \text{and } \mathrm{T}_1 \text{ \& } \mathrm{T}_2 \text{ } maturity \text{ bond} \end{array}\right\}$$

Trading Strategy in Zero-Coupon Bond

- We can now do it with two bonds, with maturity T1 and T2 , T1<T2
  M(t;St)*B(t;St-1)+N1(t;St)*P(t,T1,St)+ N2(t;St)*P(t,T2,St)

After period T1-1 we need to set N1(t,St)=0

$$\phi_2 = \left\{ \begin{matrix} all \text{ self financing strategies involving only the money market account} \\ \text{and } \mathrm{T}_1 \ \& \ \mathrm{T}_2 \ maturity \text{ bond} \end{matrix} \right\}$$

If we have a portfolio of K bonds and define ϕ_k As above set.

and $\phi_{k-1} \subseteq \phi_k$

Arbitrage trading Strategy

- A self financing Trading strategy that cost zero today. At time the portfolio is Liquidated, T1, in every state of the world the cash flow is non-negative and one state the cash flow is strictly positive.
- It cost us nothing today , we will never loose money at liquidation and in some states we actually make money. Our expected return is positive at no risk.

$$M(0)B(0)+\sum_{j=1}^{k} n_j(0)\,p(0,T_j)=0$$

and at time of liquidation

we have:

$$\mathrm{M}(\mathrm{T}1\text{-}1;\mathrm{S}_{\mathrm{T}1\text{-}1})B(T1;\mathrm{S}_{\mathrm{T}1\text{-}1})+\sum_{j=1}^{k} n_j(T1-1;\mathrm{S}_{\mathrm{T}1\text{-}1})\,p(T1,T_j;S_{T1})\left\{\frac{\geq \forall S_{T1}}{>0\ \text{for some}\ \mathrm{T}1}\right.$$

Complete Markets

- A market is complete if any cash desired cash flow is obtainable through some self financing trading strategy. That is we replicate the payoff of any simple contingent claim.

- Note that for the market to be compete the above expression must hold for all the different contingent claims. That is we mimic its payoff using only the zero coupon bonds and the money market account. Now from the arbitrage free pricing theory the time zero price of this claim should equal the time zero price of the synthetic portfolio.

Bond Trading Strategy

- Given the stochastic Price Process of a set of zero-coupon bonds and the money market, we can price the remaining zero-coupon or any other contingent claims.
- One Factor model: We have only one T1 maturity zero-coupon and the money market prices in all states of the world from time zero to time T1 and we want to price other zero-coupons:

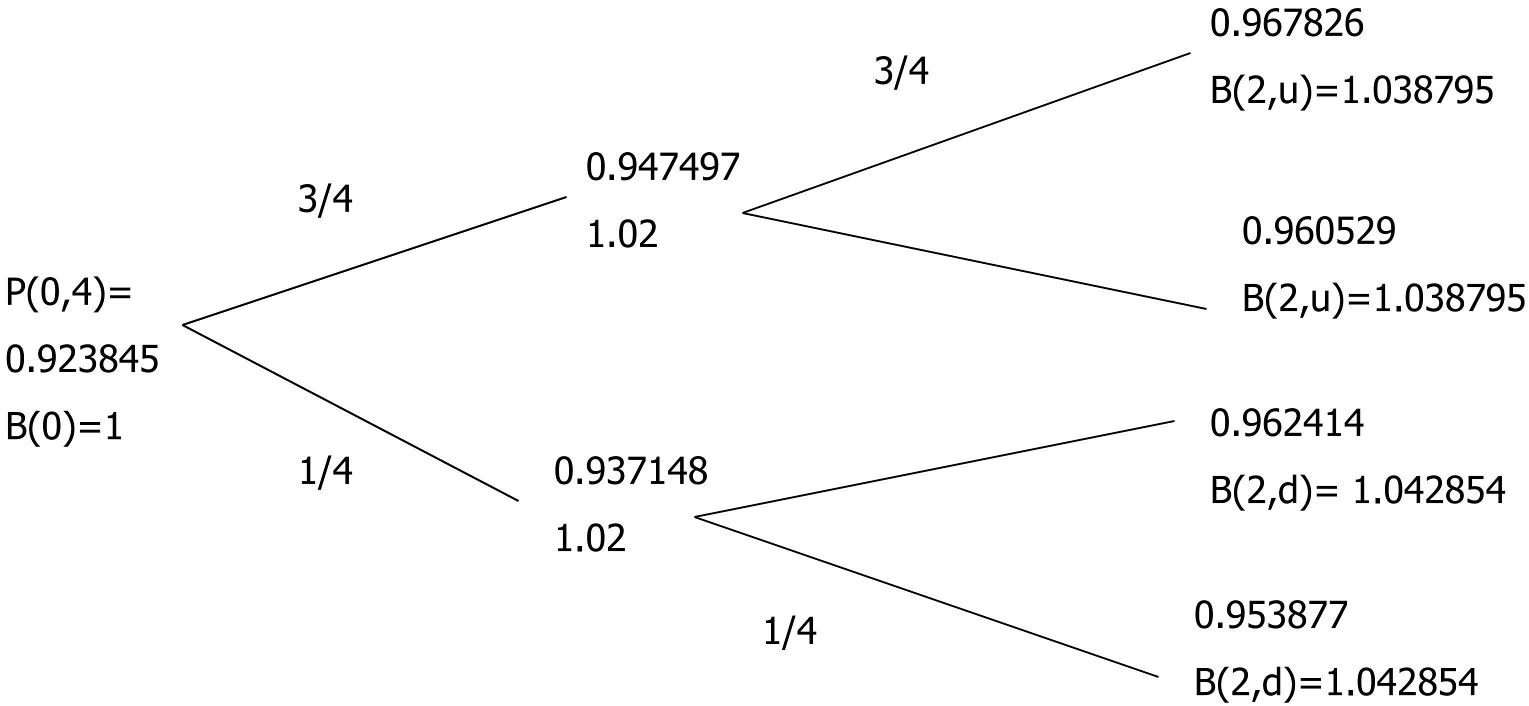
Example: Assume we have an economy with two primary assets. A four period zero coupon bond and rolling money market account. With their prices are giving below. Price a two period zero coupon bond, by constructing and trading these two primary assets.

Do dynamic programming technique. Work backwards.

Pricing-Dynamic programming

- We know the final payoff of two period zero coupon bond :
- $P(2,2,s_2) = 1\ \ for\ s_2 \in \{uu, ud, du, dd\}$
- (a) Construct portfolio of the 4-period zero coupon and money market at period $1, u$ such that payoff of this portfolio at 2, states (uu, ud) are $(P(2,2,uu\ ), P(2,2,ud\ )) = 1$
- (b) Construct portfolio of the 4-period zero coupon and money market at period $1, d$ such that payoff of this portfolio at 2, states (du, dd) are $(P(2,2,du\ ), P(2,2,dd\ )) = 1$
- Construct the portfolio of 4 period zero coupon and money market at period 0 such that in states $u\&d$, we can construct (a) and (b).
- The value of zero coupon would be the value of the portfolio that we set up at time 0

Pricing-Dynamic programming-Example



Pricing-Dynamic programming-Example

- Consider we have the price of a four-year maturity zero coupon bond for the first 2 years, we want to find the current price of the bond with 2 years maturity:

N=zero coupon ; M=Money market=MM

We should have :

- P(2,2,uu)=p(2,2,ud)=p(2,2,du)=p(2,2,dd)=1

We set a portfolio at time 1 to reach t=2 price

- M(1,u)*B(2,u)+N(1,u)*P(2,4,uu)=p(2,2,uu)=1
- M(1,u)*B(2,u)+N(1,u)*P(2,4,ud)=p(2,2,ud)=1
- M(1,d)*B(2,d)+N(1,d)*P(2,4,du)=p(2,2,du)=1
- M(1,d)*B(2,d)+N(1,d)*P(2,4,dd)=p(2,2,dd)=1

Pricing-Dynamic programming-Example

- By solving the equations we will have:

$M(1,u)=.963430$, $N(1,u)=0$

$M(1,d)=.958907$, $N(1,d)=0$

Therefore: in state u at time 1, we will only have 0.963430 of money market , so our portfolio wealth would be:

$B(1,u)*M(1,u)=0.982699$

So, in state d at time 1 , we will only have 0.958907 of money market , so our portfolio wealth would be:

$B(1,d)*M(1,d)=0.978085$

Pricing-Dynamic programming-Example

- At time zero we should construct a portfolio that have the same price in u and d states. We have:

$$M(0)*B(1,0)+N(0)*P(1,4,u)=0.982699$$
$$M(0)*B(1,0)+N(0)*P(1,4,d)= 0.978085$$

- $N(0)=0.4458$
- $M(0)=0.5493$

At time zero we need to buy 0.4458 of four period zero coupon and buy 0.5493 from money market.

$$P(0,2)=0.4458*0.923845 + 0.5493*1=0.961150$$

Pricing-Dynamic programming-Example

- Start at time zero with .961150
- With this money buy 0.5493 of $1 share of money market and .4458 of 4-period
- At time 1:

if state u happens you should have 0.982699 which consists of 4-peroid coupon and MM. The strategy is to sell all 4-period coupon and invest the proceeds in MM. so N(1,u)=0 ; M(1,u)=.963430

Similarly for state d , we should sell all zero coupon and buy MM , to have M(1,u)=.9589

- At time 2 , sell all MM and you will have $1.

Home Work

- Assume we have two Primary asset .
  - 1) Four period zero coupon bond
  - 2) Rolling Money Market Account
- Using Dynamic programming find a mimicking portfolio and the price for a three period zero coupon bond
- Assume we have a two period bond with the following pay off $p(2,uu) = p(2,du) = 1 \;\; and \; p(2,ud) = p(2,dd) = 0$
  - This bonds pay $1 at period 2 if the current state is "u"
  - Find the Mimicking portfolio and price of this Bond

Summary

- Generalization
- Assume at state s_t or "0" we have n primary assets with prices
  - $P(s_t) = \left(p_1(s_t), \dots p_n(s_t)\right)$
- Assume in next period s_{t+1} we have three possible states $(1,2,3)$
- Prices of our primary assets are :
- $P(s_t 1) = \left(p_1(s_t 1), \dots p_n(s_t 1)\right)$ in state 1 or in general $s_t 1$
- $P(s_t 2) = \left(p_1(s_t 2), \dots p_n(s_t 2)\right)$ in state 2 or in general $s_t 2$
- $P(s_t 3) = \left(p_1(s_t 3), \dots p_n(s_t 3)\right)$ in state 3 or in general $s_t 3$
- We want to find price of derivative asset, $n+1$ in period s_t (or 0), $p_{n+1}(s_t)$ which has the following payoff next period s_{t+1}
- $(p_{n+1}(s_t 1), p_{n+1}(s_t 2), p_{n+1}(s_t 3),)$ in states $(1,2,3)$

Summary

- Solution :
- Find a portfolio of primary assets $(x_1(s_t), \dots x_n(s_t))$, such that the payoff of this portfolio next period $s_{t+1}$is $(p_{n+1}(s_t1), p_{n+1}(s_t2), p_{n+1}(s_t3))$ which is payoff of derivative asset $n+1$ in state s_{t+1} .
- if we can find this mimicking portfolio, then the price of derivative asset $p_{n+1}(s_t)$, is the price of mimicking portfolio :
- $p_{n+1}(s_t) = \sum_{i=1}^{n} x_i(s_t) p_i(s_t)$
- To Find the mimicking portfolio :
  - $p_{n+1}(s_t1) = \sum_{i=1}^{n} x_i(s_t) p_i(s_t1)$
  - $p_{n+1}(s_t2) = \sum_{i=1}^{n} x_i(s_t) p_i(s_t2)$
  - $p_{n+1}(s_t3) = \sum_{i=1}^{n} x_i(s_t) p_i(s_t3)$

Summary

- Here We have three Equations and 'n' unknown :
- $\left(x_1(s_t), \ldots x_n(s_t)\right)$
- If $n < 3$: 3 Equations and less than 3 unknown, In general we can't solve for $\left(x_1(s_t), \ldots x_n(s_t)\right)$. From our primary assets, we can't find a mimicking portfolio to mimic the payoff of ALL arbitrary asset. Incomplete Market.
- If $n = 3$: 3 Equations and 3 unknown, In general we can solve for a UNIQUE $\left(x_1(s_t), \ldots x_n(s_t)\right)$. From our primary assets, we can find a UNIQUE mimicking portfolio to mimic the payoff of ALL arbitrary asset. Complete Market.
- If $n > 3$: 3 Equations and more than 3 unknown, In general we can't solve for a UNIQUE $\left(x_1(s_t), \ldots x_n(s_t)\right)$.

Summary

- We can solve for multiple mimicking portfolio $\left(x_1(s_t), \dots x_n(s_t)\right)$.
- Each of these portfolio will EXACTLY mimic the payoff of the derivate assets, but the time s_t or "0", price of each of these mimicking portfolio will be different.
- For two portfolio with exact next period payoff we found two different prices (Arbitrage exist between our primary assets, market not in equilibrium)
- Assume : $X^a = \left(x_1^a(s_t), \dots x_n^a(s_t)\right)$ and $X^b = \left(x_1^b(s_t), \dots x_1^b(s_t)\right)$
- $P_{s_t}^a = \sum_{i=}^n x_i^a(s_t) p_i(s_t) \neq \sum_{i=}^n x_i^b(s_t) p_i(s_t) = P_{s_t}^a$
- Where as $P_{s_{t+1}}^a$=$P_{s_{t+1}}^b$ in all states s_{t+1}, Arbitrage exist. Market not in Equilibrium.

Summary

- If we have T periods and n_t states in each periods we have to examine each of node of tree separately .