the most beautiful equation in finance

Video Lecture VL01

Valuation for Financial Engineers

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Video Lecture VL01 Coverage

- Present value is a modeling exercise, not merely a calculation. False valuation assumptions beget false conclusions.

- When we measure investment returns in dollars instead of percentages, cash flows, securities and derivative contracts fall into the same valuation framework.

- The *most beautiful equation in finance* (the *General Valuation Equation*) adapts to all financial assets and contracts.

- This first GVE application derives present value formulas for cash flows with known constant discount rates.

  - single payments, fixed and growing perpetuities, and fixed and growing annuities.
  - includes exponential, linear, and mean-reverting patterns.

- Later videos apply the GVE to other asset and contract types. These derivations are simpler and more intuitive than explanations given in traditional finance textbooks.

What is the value of an asset?

Valuation is a modelling exercise, not a calculation.

"All models are wrong, some are useful."

Box and Draper

There are many useful valuation models (What are their names?)

Accountants

1. What the highest available bidder will pay for an asset now — Liquidation value - too low
2. What it would cost to recreate the asset — Replacement value - too high
3. The value of similar assets, with valuation adjustments for imperfect matches (e.g. similar properties in real estate) — Comparables - good if available
4. The current value of the net appreciation and cash flows that an asset or contract creates — Net present value - adjust for time and risk - NPV is "A MODEL"
5. The increase in the value of a portfolio as a result of the addition of the asset. — Portfolio value

Financial

What is *present value*?

- The *present value* (PV) **model** compares cash flow streams of different types, frequencies and maturities in order to evaluate a financial decision

- Decisions to invest, divest, rescale, restructure, borrow, refinance, and manage risk can be informed by present value calculations
  - *Example*: Should I get a 5-year loan at 10% interest, or an 8-year loan at 7% interest?

- This may seem like a very restrictive model, but cash flows describe many assets
  - What is the value of a corporation? All corporations produce the same thing – cash!
  - What is the value of a cow? Cow = Sales of milk + sales of beef – production costs

- The PV procedure discounts expected cash flows at a given discount rate to find an equivalent value that can be easily compared across many alternatives

- If an investment is held to maturity, and the expected cash flows are realized, the discount rate is also the yield to maturity for fully capitalized investments.

- Present value works best as a model if the cash flows can be traded in a *capital market*.

How should we define *returns* on an asset?

- If you bought a stock for \$50 and sold for \$60, what was the return? How did you calculate it?
  - DOLLARS \$10 PERCENT 20% = 10/50 = (P/L)/max loss
- If you sold the stock short at \$50 and later repurchased it for \$40, what was the return?
  - DOLLARS \$10 PERCENT undefined
- What is the percentage return on any short sale?
  - Impossible to compute, since the maximum loss is infinite
  - The same problem with futures trades and short option trades
  - Collectively, these are all undercapitalized trades that may require more cash infusions over time
- Therefore, an integrated valuation theory should use dollar returns, not percentage returns.

Selling Short

A *short sale* requires the following sequence:

- Borrow the stock
- Sell it
- Buy it back later, hopefully at a lower price (thereby earning a profit)
- Return the stock to the owner
- There are costs involved
- Also, lender can recall the stock

"He who sells what isn't his'n, must buy it back or go to pris'n"
- Daniel Drew

Important terminology for returns

- Terms
  - The appreciation in the value of a security is also known as a *capital gain*
  - A negative capital gain is also sometimes called a *capital loss.*
  - *Dividends* are the cash payments investors receive for owning shares of stocks.

    Coupons bonds
- Suppose you purchased a share of Walmart for $70 one year ago, expecting it would pay a $1.00 dividend and appreciate in value by $6.00 within a year.
- Then, it actually paid $0.50 in dividends and appreciated by $10 in value.
- Then we could say any and all of the following (answer in $):
  - The expected one-year total dollar return was _$7__ one year ago
  - The expected one-year capital gain was __6__ one year ago
  - The expected dividend was __1___ one year ago
  - The realized one-year total dollar return was __$10.50
  - The realized one-year capital gain was _10___
  - The realized one-year dividend was _0.50_

Future returns are "expected" and past returns are "realized"

The theory of valuation is based on expected returns.

- These can all be expressed as dollars or percentages because this position is fully capitalized

Expected and Required Total Returns

- Investors expect to earn…
  - Capital gains
    - This is the change in the value of the investment, (V_1-V_0), whether they sell it or not

      (V1 : after cash flow is paid)
  - Cash flows
    - This refers to cash income (C) earned from the investment, e.g. stock dividends, bond coupons or other payments

- …but investors require a minimum
  - Foregone time value
    - For giving up cash, investors must be compensated the interest (r) they would have earned in a risk-free investment of equal initial value (V_0)
  - Incurred risk cost
    - For taking the risk of loss, investors must earn additional compensation
    - This is typically computed by assuming a risk measure (σ) and multiplying by a cost per unit risk (k)

$$ETR = ECG + ECF$$

$$RTR = TV + RISK$$

This key relationship determines pricing

- What is the *expected* total dollar return on an asset?

ETR = ECG + ECF

- What is the *required* dollar return on an asset?

RTR = TV + RISK

- What is the relationship between expected and required return for a buyer of the asset?

Buyer

ETR ≥ RTR

- What is the relationship between expected and required return for a seller of the asset?

Seller

ETR ≤ RTR

- How can these both be true at the same time?

if same info…

ETR = RTR

this is the most beautiful equation in finance

Cash flows

Securities

Futures contracts

Option contracts

Real options

$$ETR = RTR$$

The GVE can be found in every financial valuation

	RTR		ETR	
	Time	**Risk**	**ECG**	**ECF**
Single r.f. cash flow	rV_0	0	$-V_0$	C
Risk-free perpetuity	rV_0	0	0	C
Forward (discrete time)	0	$E[S_T] - S_0e^{rT}$	$E[S_T] - FV(divs) - F_0$	0
Option (continuous)	rC	$(\alpha - r)SC_S$	$\alpha SC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} + C_t$	0
CAPM	r	$\beta_i(r_m - r)$	r_i	0

- These will be covered in future lectures!
- Let's do the simplest valuations now…

The GVE, or General Valuation Equation

- GVE: Required return = Expected return
  - Required return = Compensation for time plus compensation for risk
  - Expected return = Expected capital gains plus expected cash flows
- Notation
  - r = risk-free interest rate
  - C_1 = cash flow at time 1
  - V_0 = valuation at time 0
  - V_1 = valuation at time 1, **AFTER THE CASH FLOW C_1 HAS BEEN PAID**
  - σ = standard deviation (a placeholder for a risk measure)
  - k = compensation per unit risk
- Can you write the GVE using this notation?

why not $\frac{E(C_1)}{1+r+p}$?

CAPM / MBA

$$V_0 = \frac{E(V_1)+C_1 - k\sigma}{1+r}$$

$$rV_0 + k\sigma = (V_1 - V_0) + C_1$$

A note on risk

- Consider the GVE shown on the last slide

$$rV_0 + k\sigma = (V_1 - V_0) + C_1$$

- If risks are assumed to be proportional to the value of the asset, then we have

$$rV_0 + pV_0 = (V_1 - V_0) + C_1$$

- Or more simply

SPECIAL CASE

$$(r + p)V_0 = (V_1 - V_0) + C_1$$

- This means if we know the risk premium as a percentage of asset value, we can add the premium to the risk-free rate and drop the risk charge from the equation

Exercise #1

- Use the GVE to find the formula for the value of a single risk-free cash flow of C_1 in one year

- Reminder
  - r = risk-free rate
  - C_1 = cash flow at time 1
  - V_0 = valuation at time 0
  - V_1 = valuation at time 1, after cash flow C_1 has been paid
  - σ = standard deviation (an example of a risk measure)
  - k = compensation per unit risk

ex $V_0 = 100$ $r = 5\%$

$V_1 = 105$ before pmt

$= 0$ after

$100(1.05) = 105$

$100 = \frac{105}{1.05}$

$V_0 = \frac{C_1}{1+r}$

TV RISK

$$rV_0 + k\sigma = (0 - V_0) + C_1$$

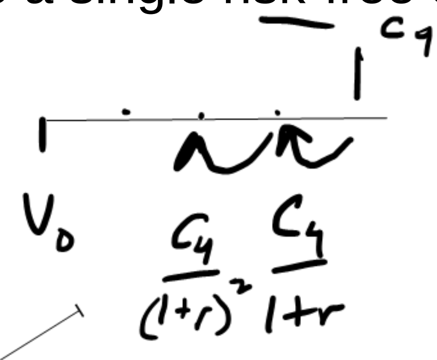
$\sigma = 0$ No RISK

$$(1+r)\, V_0 = C_1$$

$$V_0 = C_1/(1+r)$$

Exercise #2

- Use the GVE to find the formula for the value a single risk-free cash flow of "C_n" in "n" years



- $V_n = 0$ (after cash flow has been paid)
- $rV_{n-1} = (0 - V_{n-1}) + C_n \rightarrow V_{n-1} = C_n/(1+r)$
- $rV_{n-2} = (V_{n-1} - V_{n-2}) + 0 \rightarrow V_{n-2} = C_n/(1+r)^2$
- …
- $V_0 = C_n/(1+r)^n$

Using backward recursion

Sample problems

- What is the $\boxed{\text{present value}}$ of a payment of $1,000,000 in 100 years if the interest rate is 2% per year?
  - Answer: $V_0 = \dfrac{1{,}000{,}000}{(1.02)^{100}} \overset{\$}{\approx} 138K$

assuming a capital market, i.e. we can sell it

What is the value to me?

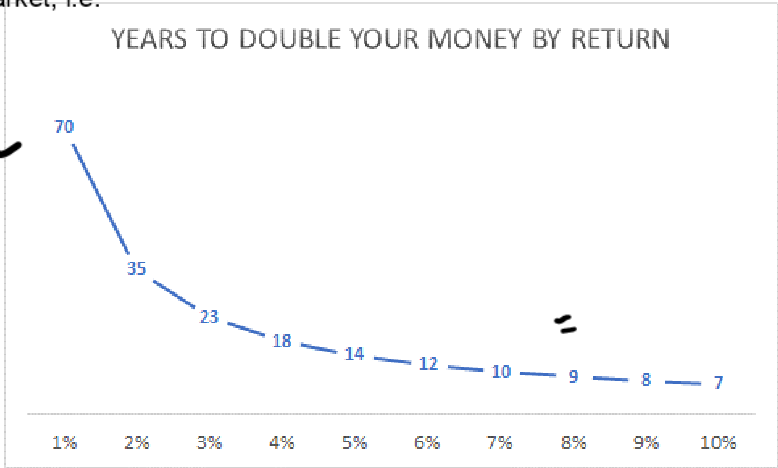
ZERO if

1. Personal circumstances (life, health)
2. Capital constraints
3. Better uses of capital

- How many years will it take for your money to double, given an annual return of r?
  - Answer: $V_0(1+r)^n = 2V_0$

Have you heard of the rule of 72s?

n



r

Common cash flow patterns

- Single fixed payment in the future ✓
- Different payments at different dates ✓
- Annuity
  - A sequence of payments starting in one period and continuing for n periods
- Perpetuity
  - A sequence of payments starting in one period and continuing forever
- Growing annuity $g \neq 0$.
  - Annuity with payments growing at a constant percentage rate through time
  - Commonly used to capture inflation or other pricing adjustments
- Growing perpetuity
  - A perpetuity with payment growing at a constant percentage rate through time
- Note for growing cash flows
  - We will also consider linear growth and mean-reverting growth later in the presentation

Finding Annuities Everywhere

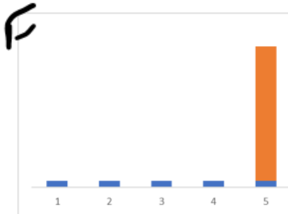
MATCH THESE TO THE DIAGRAMS

- Stocks pay (growing) dividends until they are sold C
- Bonds pay coupons until they mature
  - Corporate and sovereign bonds usually have a final principal repayment A
  - Consumer bonds (mortgages and unsecured loans --- usually have no principal payment at the end --- they are said to be *fully amortized*) B
- Commercial real estate investments receive rent and pay out expenses until they are sold C
- Annuities make regular fixed or growing payments until maturity or death B, D

Perpetuity examples:
- Preferred stock
- Consol bond
- Interest paid off an account on an annual basis

CF

A



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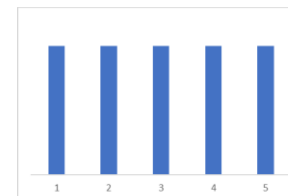
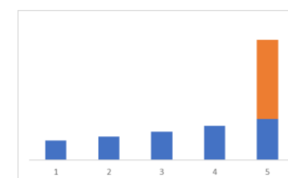
B

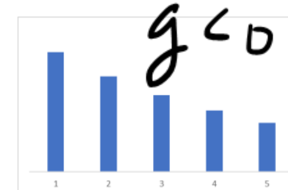


C



$g<0$

D



Did you already learn to value perpetuities the hard way?

The hard way is to notice that because of value-additivity of risk-free cash flows, we already have an equation

$$V_0 = \sum_{i=1}^{\infty} \frac{C}{(1+r)^i} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots$$

where the constant C has replaced the variable C_i. To solve this, let $d=1/(1+r)$ for convenience. Then we have

$d<1 \quad r>0$

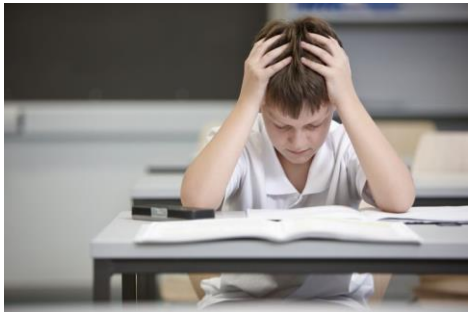
$$V_0 = Cd \times [1 + d + d^2 + d^3 + \ldots]$$

The term in brackets is an infinite geometric series, which converges as long as $d<1$, i.e. $r>0$. Therefore

$$[1 + d + d^2 + d^3 + \ldots] = 1/(1-d)$$

$$V_0 = Cd/(1-d) = C/r$$

I'm so sorry you had to suffer!



A magical perpetuity present value derivation

- At time 0, $V_0 = PV\{C, C, C, \ldots\}$
- At time 1, $V_1 = PV\{C, C, C, \ldots\}$
- Therefore, the capital gain, $V_1 - V_0 = 0$

- The GVE
  - TV + RISK = ECG + ECF
  - $rV_0 + 0 = 0 + C$
  - $V_0 = C/r$

Financial magic!

Sample perpetuity valuation

- Your inheritance trust pays $11,000 per year forever to you and your heirs starting in one year.
  - A. If the discount rate is 5%, what is the present value of the payments?
    - Answer:
  - B. If you sell the perpetuity to an investor for $165,000 cash today, what return will the investor earn annually?
    - Answer:
  - C. Should you sell it if someone offers you $165,000?
    - Answer: You should sell it if your personal discount rate exceeds 6.67%! Examples of this might be an older person (who cares little about the heirs), or a cash-constrained person such as a first-time homebuyer, or an investor who can get a higher risk-adjusted return elsewhere
  - The $220,000 PV can be justified as a valuation on an unqualified basis if we assume the perpetuity can be sold

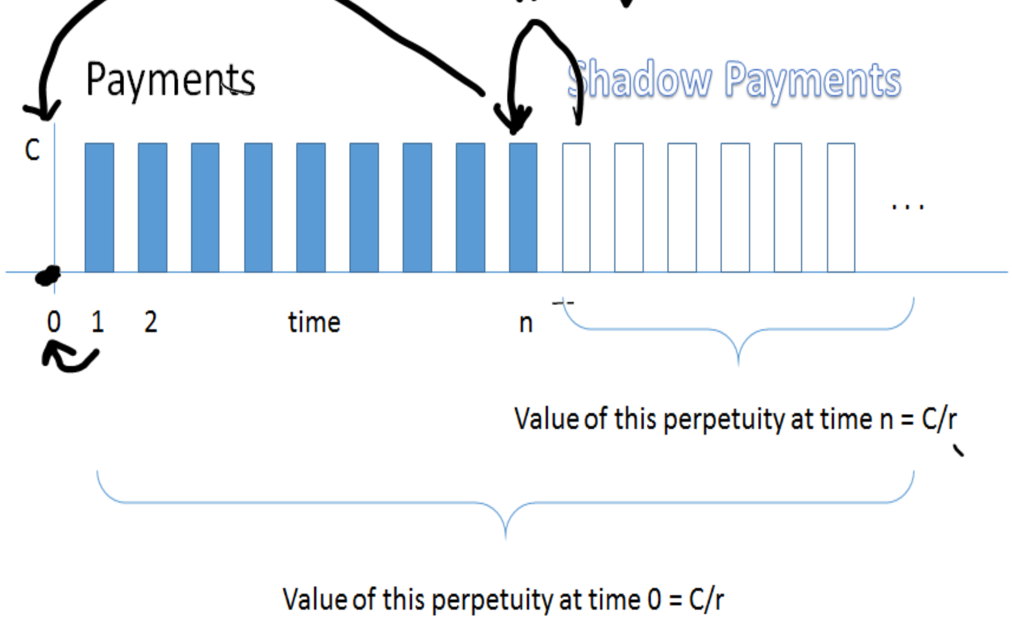
Recap two important messages

- A present value formula is a model which depends on many assumptions
  - The discount rate is constant and correct
  - There exists a market for selling the cash flows
- Otherwise, private valuations will differ due to
  - Preferences and circumstances of the recipient
  - Examples
    - Age and bequest motives
    - Cash constraints
    - Indirect benefits (e.g. the use of a house)
    - Alternative returns available (opportunity costs)
- Now let’s value annuities

Did you learn to value an annuity this way?

- As the difference between two perpetuities?

$V_0 = \frac{C}{r \cdot (1+r)^n}$ $\qquad$ $V_n = \frac{C}{r}$



PV of perpetuity

$$V_0 = \frac{C}{r}$$

PV of deferred perpetuity

$$V_0 = \frac{C/r}{(1+r)^n}$$

PV of annuity = Difference

$$V_0 = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

Or did you learn it this way?

- As a finite geometric series?

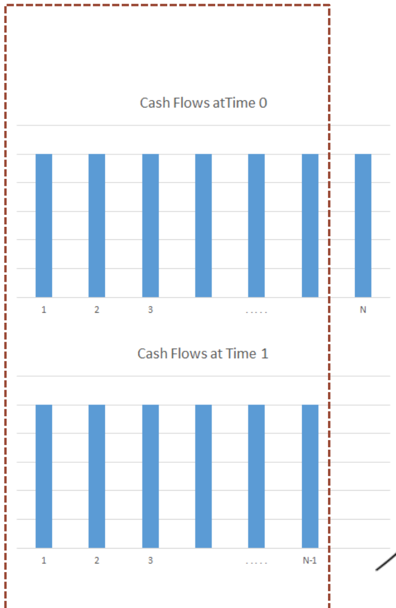
- $V_0 = \sum_{i=1}^{n} \frac{C}{(1+r)^i}; \quad let\ d = \frac{1}{1+r}$
- $V_0 = Cd \sum_{i=1}^{n} d^i$
- $\sum_{i=1}^{n} d^i = \frac{1-d^n}{1-d}$
- $V_o = \frac{Cd}{1-d}(1 - d^n) = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)$

I'm so sorry you had to suffer!

Annuity intuition with the GVE

Required return = Expected return

Time value of money + Risk compensation = Expected capital gain + Expected cash flow



Value = V_0

$$V_1 - V_0 = -\frac{C}{(1+r)^n}$$

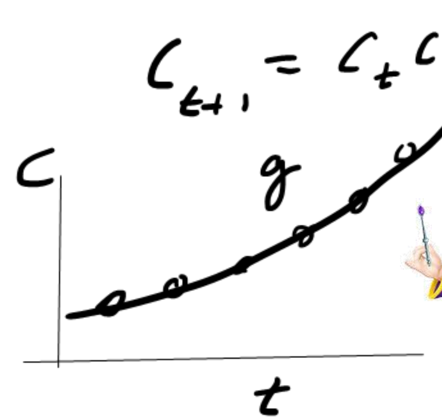
These cash flows cancel

- V_0 = value of risk-free annuity that pays C per period for N periods
  - Time value of money = rV_0
  - Risk compensation = 0
  - Expected capital gain $= V_1 - V_0 = -\frac{C}{(1+r)^n}$
  - Expected cash flow = C
  - GVE: $rV_0 = -\frac{C}{(1+r)^n} + C$

- $V_0 = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)$

How about a growing perpetuity?

- At time 0, cash flows starting one period hence are
  - $V_0 = PV$ $\{C_1, C_1(1+g), C_1(1+g)^2, \ldots\}$
- At time 1, cash flows starting one period hence are
  - $V_1 = PV$ $\{C_1(1+g), C_1(1+g)^2, C_1(1+g)^3 \ldots\}$
- Then the value at $V_1 = V_0(1+g)$
  - … since the cash flows are always proportional
- Therefore, $V_1 - V_0 = gV_0$.
- Using the GVE
  - $rV_0 + 0 = gV_0 + C_1$ or $\boxed{V_0 = C_1/(r\text{-}g)}$
  - Restricted to the case where r>g

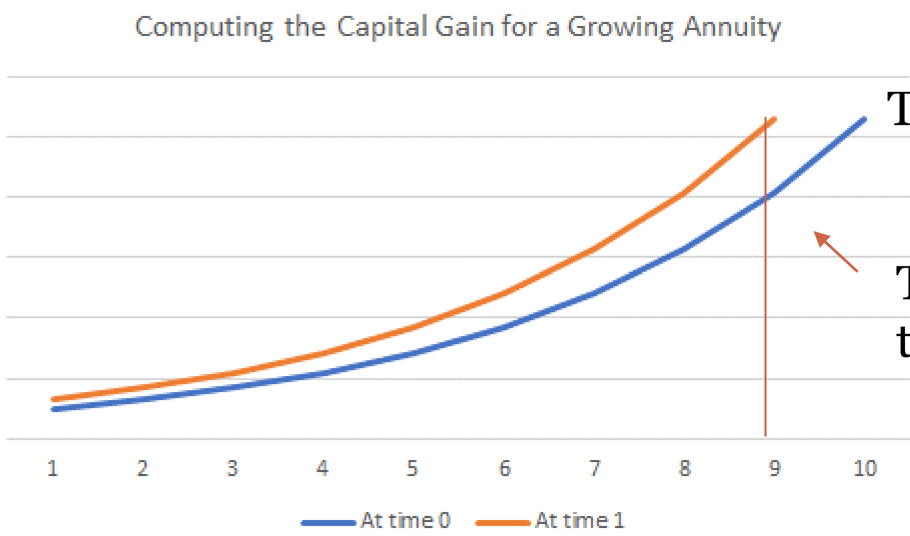


Example with a growing perpetuity

- You own a property in Brooklyn for which you will receive $80,000 a year in rent, with $10,000 a year in expenses. Both rent and expenses grow at 3% per year.
- If the value of the property is $1,300,000, what is the total annual rate of return, assuming these cash flows are in perpetuity?

A growing annuity? Just solve for the ECG

- For convenience, define C_0 to be $C_1/(1+g)$
  - It is not paid, but it simplifies the formula a little
- If we remove the last payment from V_0, we can multiply remaining payments by (1+g) to get the cash flows for V_1, therefore…
- $V_1 = (1+g)\left(V_0 - \frac{C_0(1+g)^n}{(1+r)^n}\right) \quad or \quad V_1 - V_0 = gV_0 - \frac{C_0(1+g)(1+g)^n}{(1+r)^n}$

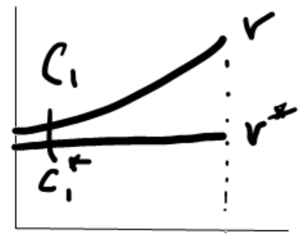


This final payment is $C_0(1+g)^n$

The V1 payments are (1+g) times the remaining V0 payments

The growing annuity GVE

1. TV + RISK = ECG + ECF — The GVE in words
2. $rV_0 + 0 = (V_1 - V_0) + C_0(1+g)$ — In symbols
3. $rV_0 = \left(gV_0 - \frac{C_0(1+g)(1+g)^n}{(1+r)^n}\right) + C_0(1+g)$ — Substituting ECG from the last page
4. $\left(\frac{r-g}{1+g}\right) V_0 = -\frac{C_0(1+g)^n}{(1+r)^n} + C_0$ — Algebraic simplification
5. Let $r^* = \frac{r-g}{1+g}$; then $\frac{1}{1+r^*} = \frac{1+g}{1+r}$



$C_1^* = C_0 = \frac{C_1}{1+g}$

$r^* = \frac{r-g}{1+g}$

- **This has the same value as a flat annuity with payment C_0 and interest rate r***

…and it is ok if g>r, i.e. r*<0

Also, a growing perpetuity has $V_0 = C_0/r^*$

What if a perpetuity grows at a linear rate?

- Value a risk-free linearly growing perpetuity that pays \$1 in one year, \$2 in two years, \$3 in 3 years and so on in perpetuity. The discount rate is r.

$rV_0 + k\sigma = V_1 - V_0 + C_1$

$V_0 = PV\{1, 2, 3, \ldots\}$

$V_1 = PV\{2, 3, 4, \ldots\}$

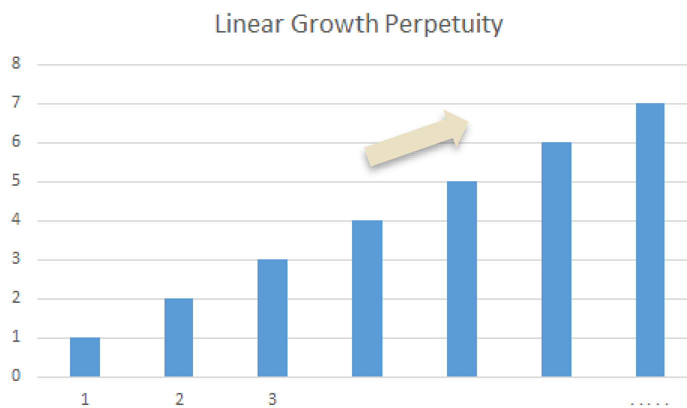
$V_1 - V_0 = PV\{1, 1, 1, \ldots\}$

$= \frac{1}{r}$

$rV_0 + 0 = \frac{1}{r} + 1$

$= \frac{1+r}{r}$

$$\boxed{V_0 = \frac{1+r}{r^2}}$$

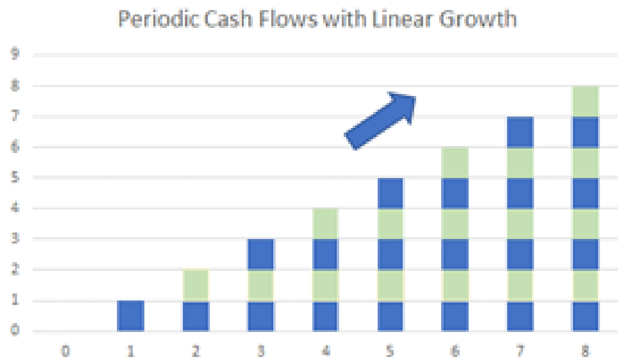


Interesting challenge:

Find the PV of the Fibonacci Sequence (Hint: use high discount rate)

Did you solve it the hard way?

- **The hard way** is to split the cash flows horizontally as with the stripes below:

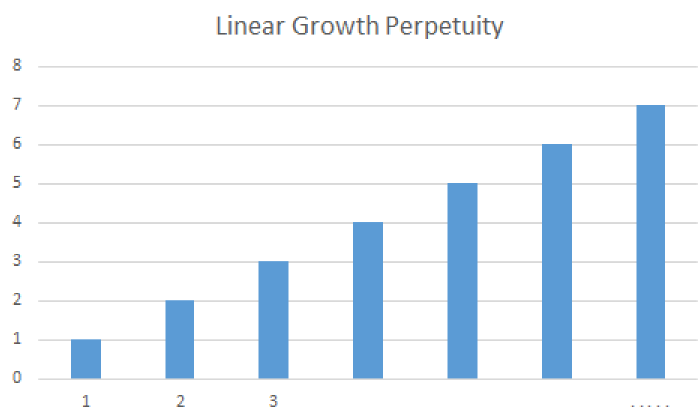


t	Value
7	1/r
6	1/r
5	1/r
4	1/r
3	1/r
2	1/r
1	1/r
0	1/r
-1	$1/r^2$
0	$(1+r)/r^2$

- Each stripe is worth 1/r one period before the cash flow begins
- Therefore this is a perpetuity of perpetuities starting at 0
- The value is therefore (1+r) times the pv of a perpetuity of perpetuities worth $1/r^2$.

I'm not really so sorry you had to suffer!

Or did you solve it the easy way?



- **The easy way**

- V_0 = PV{1,2,3,4, …}
- V_1 = PV{2,3,4,5, …}
- $V_1 - V_0$ = PV{1,1,1, …} = 1/r

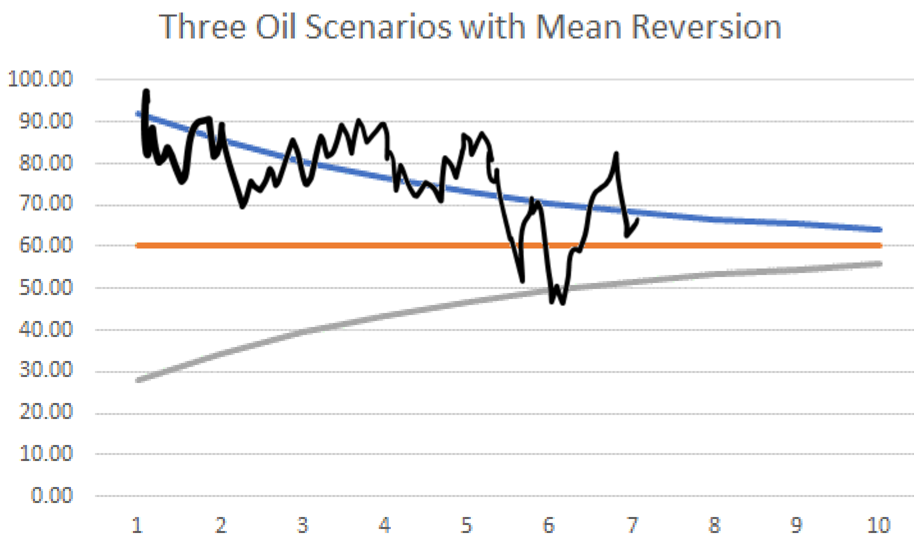
- Time value = rV_0
- Risk compensation = 0
- Capital gain = 1/r
- Cash flow = 1

- GVE: $rV_0 = \frac{1}{r} + 1$

- $V_0 = (1+r)/r^2$

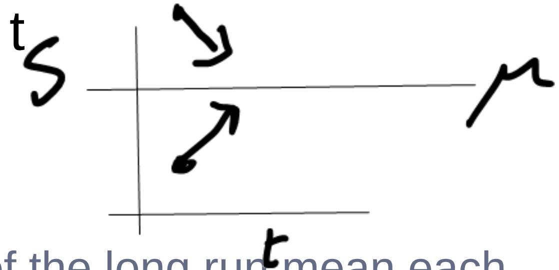
Explaining mean reversion

- Interest rates and commodity prices tend to bounce back from low or high levels over time
- For example, high oil prices increase supply, decrease demand, and tend to force oil prices to fall
- Conversely, low oil prices stimulate demand, and reduce supply over time, causing prices to rise
- This price behavior is captured using a mean-reverting process --- i.e. geometric decay towards the long run mean price



Mean reversion formulas

- S_t = Interest rate or commodity price at time t
- $\boxed{S_{t+1} = S_t + \kappa(\mu - S_t)}$
  - μ is called the long-run mean level of S
  - $\kappa>0$ is called the speed of mean reversion
    - κ is the percentage movement in the direction of the long run mean each period



- We can also solve forward for j periods
  - $S_{t+j} = \mu + (S_t - \mu)(1 - \kappa)^j$
- Therefore, we can value the mean reverting cash flow as the sum of a perpetuity (μ) and a growing perpetuity
  - The parameters of the growing perpetuity are $C_0 = S_t - \mu$, and $g = -\kappa$
- For a mean-reverting perpetuity, $V_0 = \frac{\mu}{r} + \frac{(S_t - \mu)(1-\kappa)}{r+\kappa}$ ✓

Summary

- We began with a smart discussion of value and return
- Then we valued commonly occurring cash flow patterns assuming cash flows and discount rates were known
- All were shown to be special cases of the GVE
- What you need to know
  - How to think about value, valuation models and return
  - How to apply and derive the GVE valuation model applications
  - The assumptions behind your valuations and their limitations
  - How to convert applicable problems to combinations of growing annuities (valuation or missing parameters)
    - Including linear growth and mean reversion
  - How to use Excel and Python to automate this

Key formulas – Can you identify these?

ETR = RTR

ETR = ECG + ECF

RTR = TV + RISK

$V_0 = C_n/(1+r)^n$

$V_0 = C/r$

$$V_0 = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

$$r^* = \frac{r-g}{1+g} \text{ and } C_0 = \frac{C_1}{1+g}$$

$rV_0 + k\sigma = (V_1 - V_0) + C_1$

$$V_0 = (1+r)/r^2$$

$S_{t+1} = S_t + \kappa(\mu - S_t)$

$S_{t+j} = \mu + (S_t - \mu)(1 - \kappa)^j$

$$V_0 = \frac{\mu}{r} + \frac{(S_t - \mu)(1-\kappa)}{r+\kappa}$$

Review Questions for Recitations

1. Value definitions: Liquidation, replacement, Comparables, Net present value, Portfolio valuation
2. What's the difference between valuation and valuation models?
3. Why is it important to consider dollar returns and not just percentage returns?
4. What is a capital gain?
5. What are annuities and perpetuities?
6. What is the formula for the present value of a perpetuity? An annuity?
7. What are the assumptions of the annuity and perpetuity models?
8. What is the GVE in words? As an equation?
9. What are the different common cash flow growth types?
10. How can a growing perpetuity or annuity be converted into a constant equivalent for valuation purposes?
11. How is a mean-reverting annuity valued?

Challenge Problems for Discussion

- Derive a formula to value a linear-growth annuity if at any time t, the cash flow is given by $C(t) = a + bt$, and the discount rate is r.
- Derive a formula for the present value of a mean-reverting annuity, given the discount rate (r).
- Derive the present value of a quadratic perpetuity, where $C(t) = t^2$.
- Suppose you had a continuous payment of C dt from time 0 to T. The continuously compounded annual discount rate is r. Can you write the GVE in this case?
- Continuing, can you solve the GVE?