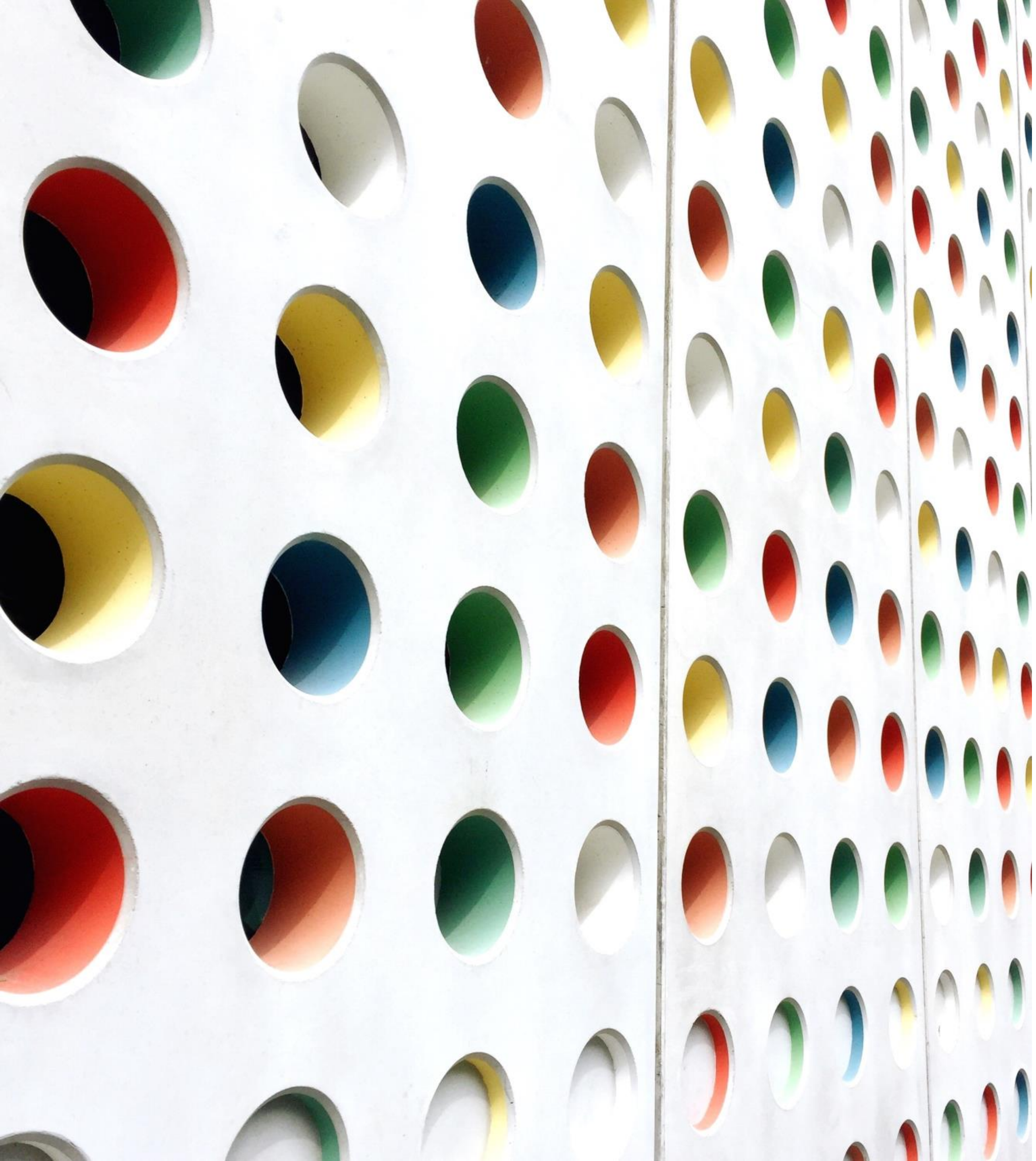


VL10

Multiperiod Risky Cash Flows

VALUATION 6103

PROF DAVID SHIMKO

CAPM as generally implemented

Use historical returns to estimate one-period beta and expected returns of cash flows

Or, calibrate the cash flows to traded assets, which have an implied beta

Apply the single period discount rate to all future expected cash flows

Implicit assumptions

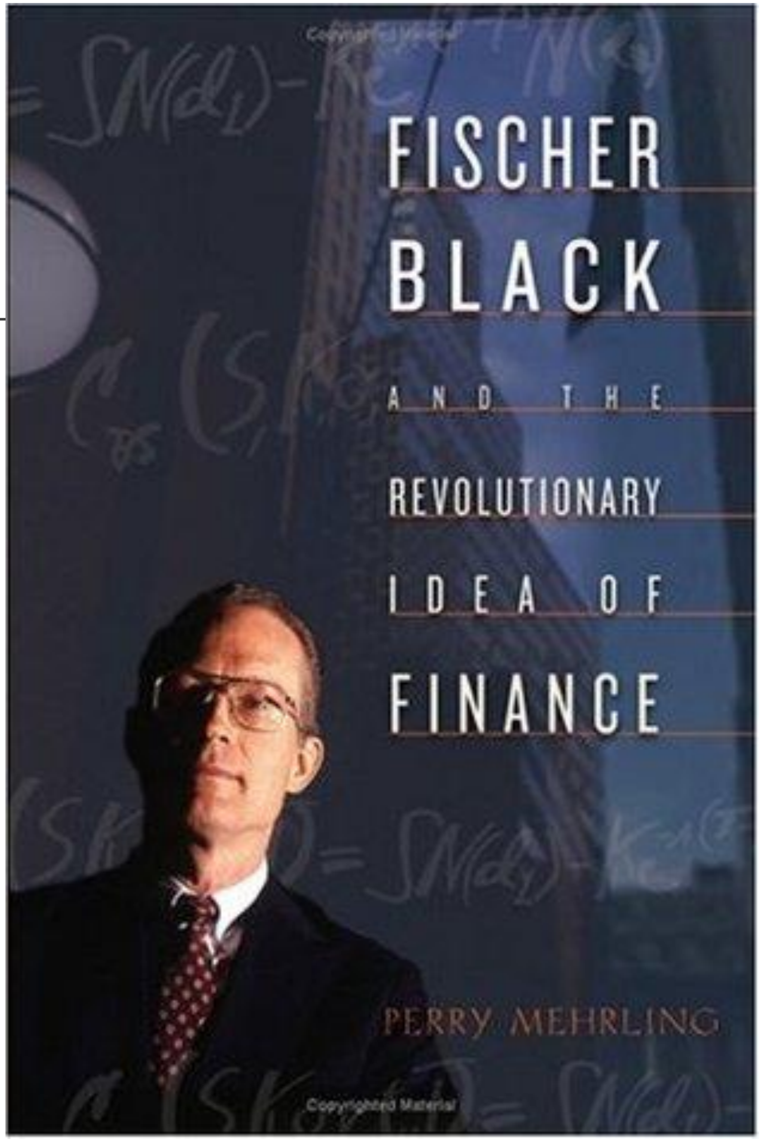
- Risk is a constant proportion of asset value (allowing a constant risk premium)
  - Implies total risk grows exponentially with time
  - For most projects and companies, risks decrease over time
- Cash flows are uncorrelated over time
  - CAPM **ignores** intertemporal cash flows – they have no effect on value

Fischer Black on MARKETS

“[An investor] needs a strategy for changing his position over time. One of the elements of this strategy is likely to be that he will want diversification across time, as well as diversification across assets.

"Time diversification" is often just as important as diversification across stocks.

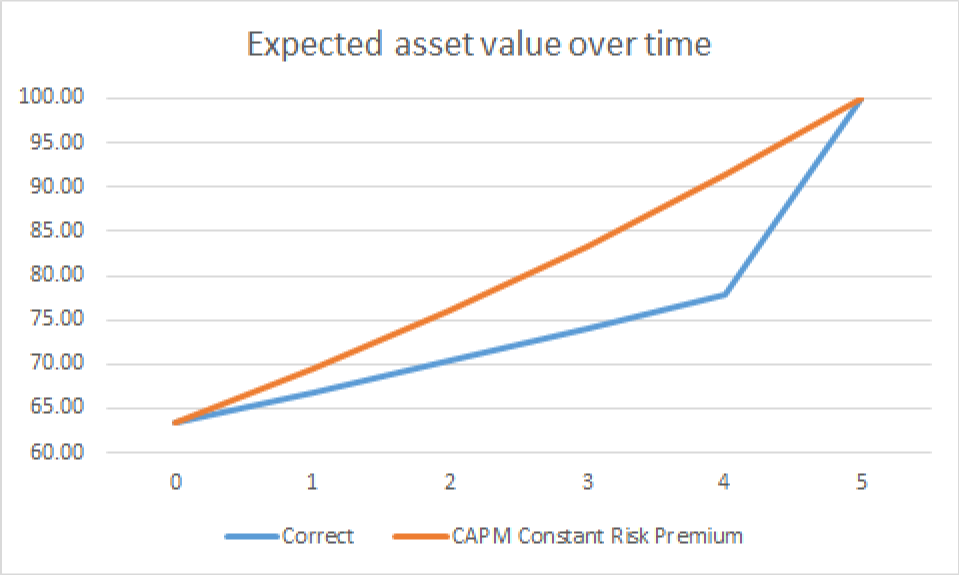
It helps because what happens to the market in one period is largely independent of what happens to the market in some other period. In fact, adding a stock to a portfolio may not help to diversify the portfolio much, if there are other stocks like it already in the portfolio. But adding investment in a different time period always helps, because with the market, every time period is different from every other.”

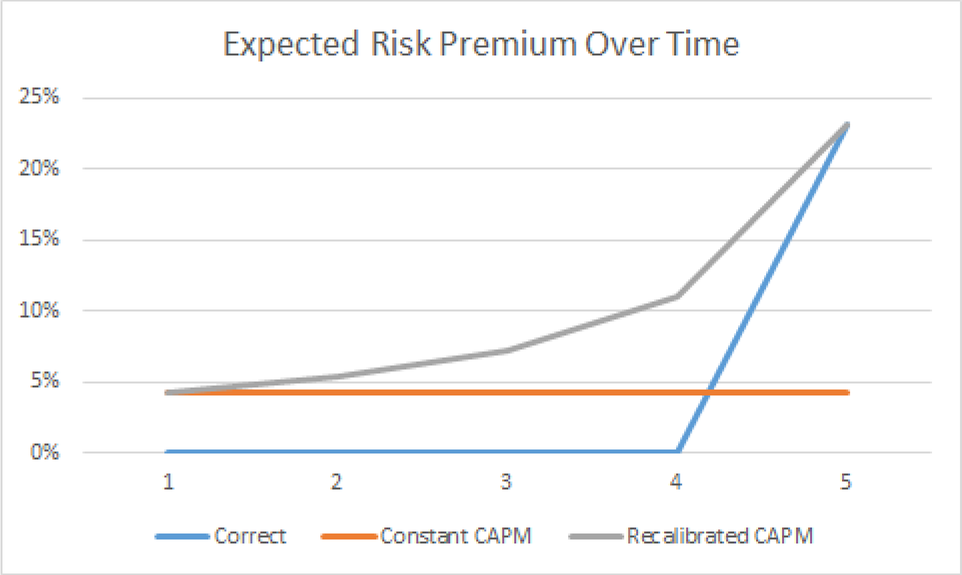


Fischer Black on MARKETS Vol. 1, No. 5
April 12, 1976

Example of intertemporal risk reduction

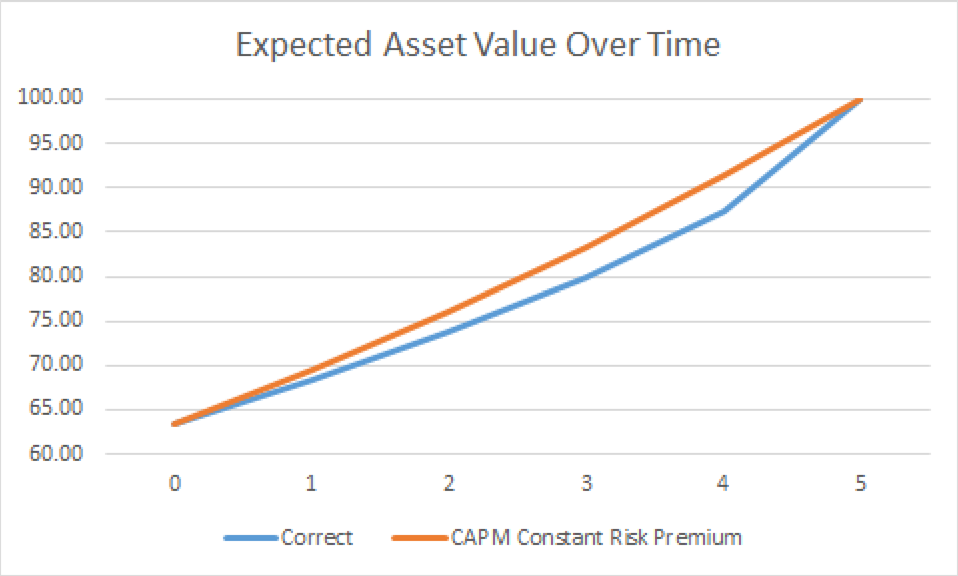
- Use the standard deviation model
- A single cash flow in year 5 has mean \$100 and s.d. \$30
- No risk resolution before year 5
- One period r.f. discount factor is 0.95, and k = 0.60 (Sharpe Ratio)

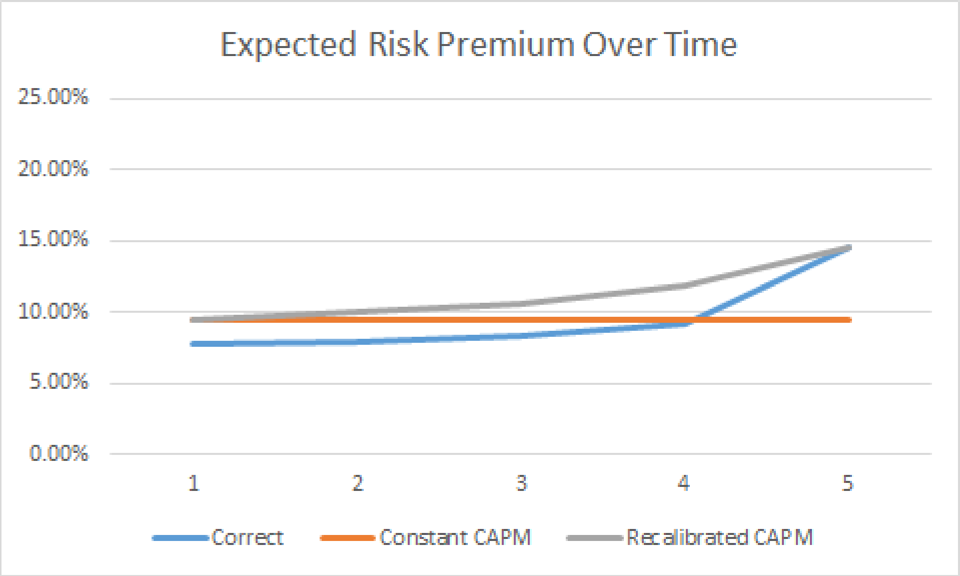




Random walk example

- Use the standard deviation model, Same parameters as #1, C is paid only at time 5
- Now, C follows an Arithmetic Random Walk with zero drift so the payment parameters are the same
- This implies some risk resolution before year 5
- One period r.f. discount factor is 0.95, and k = 0.60 (Sharpe Ratio)



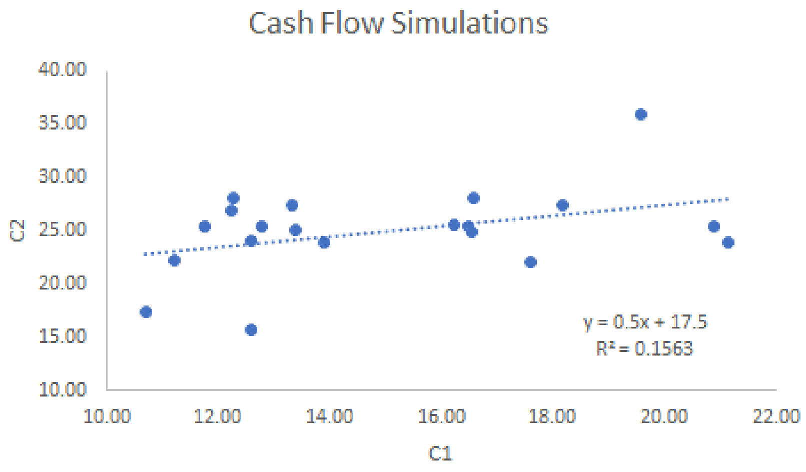


Numerical example: Two-period risk allocation

We will receive two correlated cash flows in the next two periods.

Cash Flow	Mean	Covariance		Cholesky		Corr	
C1	15	10	5	3.16	0.00	100.00%	39.53%
C2	25	5	16	1.58	3.67	39.53%	100.00%

The variance of their sum is 36.



LINEST	Slope	Const	
Coeff	0.5000	17.5000	
StErr	0.2739	4.1982	Pop se
R2/se	0.1563	3.8730	3.67

1. How should their combined risk be allocated to the two cash flows individually?
2. How do we value the cash flows?
3. How does the correlation change the answer?

Solution to numerical example, Part 1

Cash Flow	Mean	Covariance		Cholesky		Corr		Schur	
C1	15	10	5	3.16	0.00	100.00%	39.53%	22.50	
C2	25	5	16	1.58	3.67	39.53%	100.00%		13.50

Work backwards.

At time 1, after C1 has been paid, the conditional variance of C2 is 3.67^2.

- See the bottom right entry in the Cholesky matrix, the residual standard error of the regression

The exact calculation can easily be taken from the covariance matrix: $16 - \frac{5\times 5}{10} = 13.50$

- This is called the *Schur complement* of the covariance matrix

The variance of the sum is 36, so the remainder belongs to the first cash flow. $36 - 13.50 = 22.50$

Solution to numerical example, Part 2

Cash Flow	Mean	Covariance		Cholesky		Corr		Schur		Risk-Neutral
C1	15	10	5	3.16	0.00	100.00%	39.53%	22.50		12.75
C2	25	5	16	1.58	3.67	39.53%	100.00%		13.50	23.65
		Risk-free	5.00%		Cost of variance risk		10.00%			$33.59

We know how much risk to assign each cash flow, so we can construct the risk-neutral values.

Risk-neutral value of each cash flow = Expected cash flow minus cost of risk × allocated risk.

Risk-neutral values can be discounted at the risk-free rate.

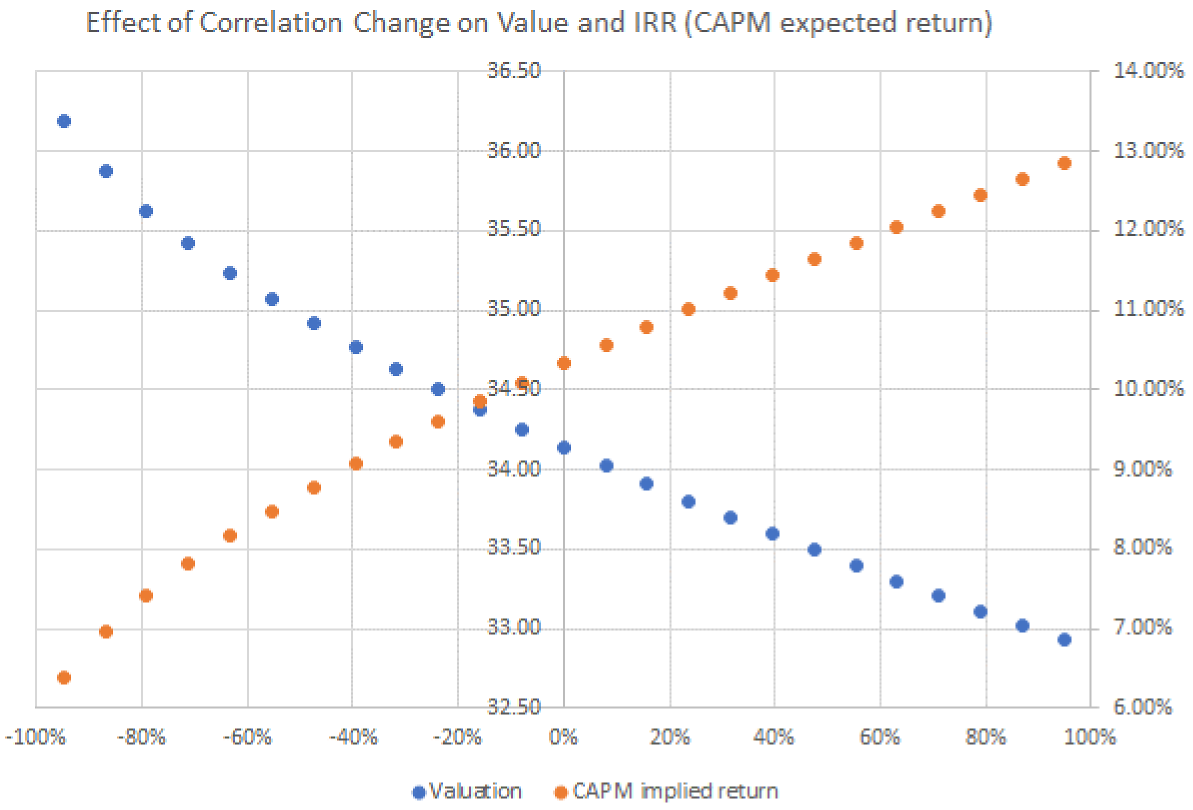
Sensitivity analysis for numerical example

Lower correlations lead to higher valuations and therefore lower expected returns.

Depending on the correlation between the cash flows, valuations can vary from $33-$36.

The implied CAPM returns range from 6-13%.

Even in this simple example, correlation has a great impact on value and expected return.



General formula:

Valuation of single asset over many periods

Assumptions

- The asset has N future cash flows given by random column vector C
- The C vector is multivariate normal, with a full rank covariance matrix
- Risk charge proportional to variance

Partition the cash flows into known $C_j = \{C_1..C_j\}$ and unknown $C_N = \{C_{j+1}..C_N\}$ cash flows at any point in time j

Then the **unconditional** moments of the cash flows (i.e. at time 0) are given by

$$C \equiv \begin{pmatrix} C_j \\ C_N \end{pmatrix} \qquad \mu \equiv \begin{pmatrix} \mu_j \\ \mu_N \end{pmatrix} \qquad \Sigma \equiv \begin{pmatrix} \Sigma_{jj} & \Sigma_{jN} \\ \Sigma_{Nj} & \Sigma_{NN} \end{pmatrix}$$

Conditional covariance matrix

This gives us a kind of multiple regression formula together with the Schur decomposition of the covariance matrix.

$$E(C_N|C_j) = \mu_N + \Sigma_{Nj}\Sigma_{jj}^{-1}(C_j - \mu_j)$$

$$\Sigma_j = Var(C_N|C_j) = \Sigma_{NN} - \Sigma_{Nj}\Sigma_{jj}^{-1}\Sigma_{jN} \; for \; j = 1..N-1 \; with \; \Sigma_0 = \Sigma$$

with the single subscript, Σ_j is the covariance matrix for the unknown cash flows after time j, given the knowledge of the cash flows up to and including time j

Important note: The conditional covariance matrix does not depend on cash flow outcomes.

Let’s try a three-period example

At time 0	100	10	20
	10	144	-30
	20	-30	196
	Grand sum		440
After C1		143	-32
		-32	192
			271
After C2			184.84

After C1 we have		MINUS				EQUALS COND COV	
Unconditional cov		Outer product		Divided by 100			
144	-30	100	200	1	2	143	-32
-30	196	200	400	2	4	-32	192
		MINUS					
Unconditional cov		Outer product		Divided by	143		
	192		1024		7.16		184.84

The risk allocations to each period are then (440-271), (271-184.84), and (184.84)

We can deduct the risk charges from the mean cash flow, and discount at the risk-free rate

The Schur Decomposition and Cholesky

Each successive conditional covariance matrix has the same corresponding entries in the Cholesky decomposition

You can verify this by multiplying each Cholesky matrix by its transpose

	CONDITIONAL COVARIANCE			CHOLESKY DECOMPOSITION		
At time 0	100	10	20	10	0	0
	10	144	-30	1	11.96	0
	20	-30	196	2	-2.68	13.60
	Grand sum		440			
After C1		143	-32		11.96	0
		-32	192		-2.68	13.60
			271			
After C2			184.84			13.60

Valuation formulae – finite case

Assumptions

- Finitely many cash flows, jointly normally distributed with known covariance matrix
- Risk charge may be variance or standard deviation
- B is a vector of present values of the appropriate length, starting with the value of the 1 period ZCB that pays $1 in 1 period

Variance formula

$$V_j = B'E[C_N|C_j] - kB'(1'\Sigma_j 1 - 1'\Sigma_{j+1} 1)$$

Standard deviation formula

$$V_j = B'E[C_N|C_j] - kB'(sqrt(1'\Sigma_j 1) - sqrt(1'\Sigma_{j+1} 1))$$

Infinite cash flow sequence - ARW

Compute the conditional covariance of the remaining cash flows after the first payment is received.

Ans:________________________________

What is the conditional covariance of the 9th and 10th payment after the first 8 payments have been made?

Ans:________________________________

Unconditional covariance for C_1-C_{10}*

1	1	1	1	1	1	1	1	1	1	
1	2	2	2	2	2	2	2	2	2	
1	2	3	3	3	3	3	3	3	3	
1	2	3	4	4	4	4	4	4	4	
1	2	3	4	5	5	5	5	5	5	
1	2	3	4	5	6	6	6	6	6	
1	2	3	4	5	6	7	7	7	7	
1	2	3	4	5	6	7	8	8	8	
1	2	3	4	5	6	7	8	9	9	
1	2	3	4	5	6	7	8	9	10	
									385	Grand sum

*Divided by σ^2

Valuing a company's free cash flow (ARW)

Assume RW corp has annual cash flows that follow an arithmetic random walk (ARW), with drift μ and volatility σ

The cost of variance risk is k, the starting free cash flow (FCF) per share at time 0 is C_0, and the risk-free rate is r. Cash flows have no fixed termination date.

What is the risk-adjusted present value of the share, discounting the FCF?

SINGLE RANDOM WALK

C0	10
Mu	3
Sigma	5
k	60.00%
r	5.00%

RFPV	1460.00
SD	105.00
V0	200.00
IRR	15.92%

In this perpetual case, easier to use the GVE

$$rV_0 + k\,SD(V_1 + C_1) = E[V_1 - V_0 + C_1]$$

$V_1 = V_0 + \frac{C_1 - C_0}{r}$ (since PV risk charge is unaffected by cash flow outcomes)

$$E[V_1 - V_0 + C_1] = C_0 + \mu + \frac{\mu}{r}$$

$$SD(V_1 + C_1) = (1 + \frac{1}{r})\sigma$$

$$V_0 = \frac{C_0 + \mu - k\sigma}{r} + \frac{\mu - k\sigma}{r^2}$$

Discussion questions: ARW valuation

Can the PV become negative? If yes, does this make sense?

How might you use this model in practice?

How does the expected market price of this asset change over time?

What stochastic process does the valuation itself follow (not the cash flow)?

Risk-neutrality

Recall our one period pricing model (using standard deviation as a risk measure)

$$V_0 = \frac{\mu - k\sigma}{(1+r)}$$

And remember our ARW cash flow pricing model

$$V_0 = \frac{C_0 + \mu - k\sigma}{r} + \frac{\mu - k\sigma}{r^2}$$

Do these formulae suggest any simplification to you?

Suppose we "tranche" the ARW cash flows

Class A investor receives min(C_t,15) each period for 20 periods

Residual investor receives the remainder

How do we value their respective claims on the cash flows?

Major problem: We don't know what discount rates to use!

POSSIBLE SOLUTION

If risk-neutrality applies, simulate the ARW using the risk-neutral process

Divide the risk-neutral cash flow into the A and residual components

Find the expected present value **at the risk-free rate**

SINGLE RANDOM WALK

C0	10
Mu	3
Sigma	5
k	60.00%
r	5.00%
RFPV	1460.00
SD	105.00
V0	200.00
IRR	15.92%

In-class exercises

1. Value the tranched ARW from the previous page

2. Use the spreadsheet (CLIENTSIMS) that your client provided to you to value the next 5 years of free cash flow. Your client wants to price shares in the project to yield a Sharpe Ratio of 60%. What is your recommended share price? The risk-free discount rate is 8%.