Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios

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Abstract: In this paper we examine the statistical properties of the profit to be made from hedging vanilla options that are mispriced by the market and/or hedged using a delta based on different volatilities. We derive formulas for the expected profit and the variance of profit for single options and for portfolios of options on the same underlying. We suggest several ways to choose optimal portfolios.

1 Introduction

There are many thousands of papers on forecasting volatility using a host of increasingly sophisticated, even Nobel-Prize-winning, statistical techniques. A possible goal of these is, presumably, to help one exploit mispricings in derivatives, and so profit from volatility arbitrage. There is a similar order of magnitude of papers on various, and again, increasingly complicated, volatility models calibrated to vanilla option prices and used to price fancy exotic contracts. The former set of papers usually stop short of the application of the volatility forecasts to options and how to actually make that volatility arbitrage profit via delta hedging. And the second set blind us with science without ever checking the accuracy of the volatility models against data for the underlying. (A marvelous exception to this is the paper by Schoutens, Simons & Tistaert (2004) which rather exposes the Emperor for his lack of attire.) In this paper we are going to do none of this clever stuff. No, we are going right back to basics.

In this paper we address the obvious question of how to make money from volatility arbitrage. We are going to keep the model and analysis very simple, hardly straying from the Black–Scholes world at all. We are going to analyze different delta-hedging strategies in a world of constant volatility. Or more accurately, three constant volatilities: Implied, actual and hedging.

Much of what we will examine is the profit to be made hedging options that are mispriced by the market. This is the subject of how to delta hedge when your estimate of future actual volatility differs from that of the market as measured by the implied volatility (Natenberg, 1994). Since there are two volatilities in this problem, implied and actual, we have to study the effects of using each of these in the classical delta formula (Black & Scholes, 1973). But why stop there? Why not use a volatility between implied and actual, or higher than both or lower? We will look at the profit or loss to be made hedging vanilla options and portfolios of options with different 'hedging volatilities.'

We will see how you can hedge using a delta based on actual volatility or on implied volatility, or on something different. Whichever hedging volatility you use you will get different risk/return profiles. Part of what follows repeats the excellent work of Carr (2005) and Henrard (2003). Carr derived the expression for profit from hedging using different volatilities. Henrard independently derived these results and also performed simulations to examine the statistical properties of the possibly path-dependent profit. He also made important observations on portfolios of options and on the role of the asset's growth rate in determining the profit. Our paper extends their analyses in several directions.

Other relevant literature in this area includes the paper by Carr & Verma (2005) which expands on the problem of hedging using the implied volatility but with implied volatility varying stochastically. Dupire (2005) discusses the advantages of hedging using volatility based on the realized quadratic variation in the stock price. Related ideas applied to the hedging of barrier options and Asians can be found in Forde (2005).

In Section 2 we set up the problem by explaining the role that volatility plays in hedging. In Section 3 we look at the mark-to-market profit and the final profit when hedging using actual volatility. In Section 4 we then examine the mark-to-market and total profit made when hedging using implied volatility. This profit is path dependent. Sections 3 and 4 repeat the analyses of Carr (2005) and Henrard (2003). Because the final profit depends on the path taken by the asset when we hedge with implied volatility we look at simple statistical properties of this profit. In Section 4.1 we derive a closed-form formula for the expected total profit and in Section 4.2 we find a closed-form formula for the variance of this profit.

In Section 5 we look at hedging with volatilities other than just implied or actual, examining the expected profit, the standard deviation of profit as well as minimum and maximum profits. In Section 6 we look at the advantages and disadvantages of hedging using different volatilities. For the remainder of the paper we focus on the case of hedging using implied volatility, which is the more common market practice.

Portfolios of options are considered in Section 7, and again we find closed-form formulas for the expectation and variance of profit. To find the full probability distribution of total profit we could perform simulations (Henrard, 2003) or solve a three-dimensional differential equation. We outline the latter approach in Section 8. This is to be preferred generally since it will be faster than simulations, therefore making portfolio optimizations more practical. In Section 9 we outline a portfolio selection method based on exponential utility. In Section 10 we briefly mention portfolios of options on many underlyings, and draw conclusions and comment on further work in Section 11. Technical details are contained in an appendix.

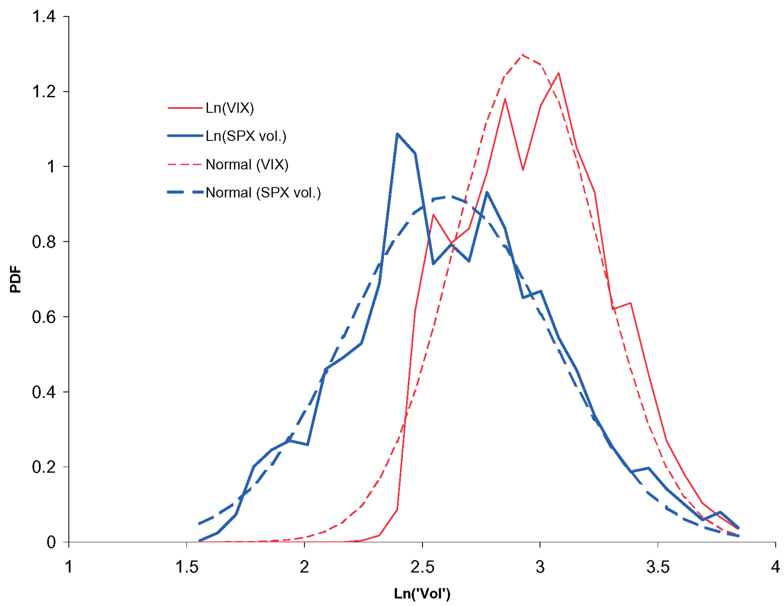

Figure 1: Distributions of the logarithms of the VIX and the rolling realized SPX volatility, and the normal distributions for comparison.

Some of the work in this paper has been used successfully by a volatility arbitrage hedge fund.

Before we start, we present a simple plot of the distributions of implied and realized volatilities, Figure 1. This is a plot of the distributions of the logarithms of the VIX and of the rolling 30-day realized SPX volatility using data from 1990 to mid 2005. The VIX is an implied volatility measure based on the SPX index and so you would expect it and the realized SPX volatility to bear close resemblance. However, as can be seen in the figure, the implied volatility VIX seems to be higher than the realized volatility. Both of these volatilities are approximately lognormally distributed (since their logarithms appear to be Gaussian), especially the realized volatility. The VIX distribution is somewhat truncated on the left. The mean of the realized volatility, about 15%, is significantly lower than the mean of the VIX, about 20%, but its standard deviation is greater.

2 Implied versus Actual, Delta Hedging but Using which Volatility?

Actual volatility is the amount of 'noise' in the stock price, it is the coefficient of the Wiener process in the stock returns model, it is the amount of randomness that 'actually' transpires. Implied volatility is how the market is pricing the option currently. Since the market does not have perfect knowledge about the future these two numbers can and will be different.

Imagine that we have a forecast for volatility over the remaining life of an option, this volatility is forecast to be constant, and further assume that our forecast turns out to be correct.

We shall buy an underpriced option and delta hedge to expiry. *But which delta do you choose?* Delta based on actual or implied volatility?

Scenario: Implied volatility for an option is 20%, but we believe that actual volatility is 30%. Question: How can we make money if our forecast is correct? Answer: Buy the option and delta hedge. But which delta do we use? We know that

$$\Delta = N(d_1)$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$$

and

$$d_1 = \frac{\ln(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

We can all agree on S, E, $T-t$ and r (almost), but not on σ. So should we use $\sigma = 0.2$ or 0.3?

In what follows we use σ to denote actual volatility and $\tilde{\sigma}$ to represent implied volatility, both assumed constant.

3 Case 1: Hedge with Actual Volatility, σ

By hedging with actual volatility we are replicating a short position in a *correctly priced* option. The payoffs for our long option and our short replicated option will exactly cancel. The profit we make will be exactly the difference in the Black–Scholes prices of an option with 30% volatility and one with 20% volatility. (Assuming that the Black–Scholes assumptions hold.) If $V(S, t; \sigma)$ is the Black–Scholes formula then the guaranteed profit is

$$V(S, t; \sigma) - V(S, t; \tilde{\sigma}).$$

But how is this guaranteed profit realized? Let us do the analysis on a mark-to-market basis.

In the following, superscript 'a' means actual and 'i' denotes implied, these can be applied to deltas and option values. For example, Δ^a is the delta using the actual volatility in the formula. V^i is the theoretical option value using the implied volatility in the formula. Note also that V, Δ, Γ and Θ are all simple, known, Black–Scholes formulas.

The model is the classical

$$dS = \mu S\, dt + \sigma S\, dX.$$

TABLE 1: PORTFOLIO COMPOSITION AND VALUES, TODAY.

Component	Value
Option	V^i
Stock	$-\Delta^a S$
Cash	$-V^i + \Delta^a S$

TABLE 2: PORTFOLIO COMPOSITION AND VALUES, TOMORROW.

Component	Value
Option	$V^i + dV^i$
Stock	$-\Delta^a S - \Delta^a dS$
Cash	$(-V^i + \Delta^a S)(1 + r\, dt) - \Delta^a DS\, dt$

Now, set up a portfolio by buying the option for V^i and hedge with Δ^a of the stock. The values of each of the components of our portfolio are shown in the following table, Table 1.

Leave this hedged portfolio 'overnight,' and come back to it the next 'day.' The new values are shown in Table 2. (We have included a continuous dividend yield in this.)

Therefore we have made, mark to market,

$$dV^i - \Delta^a\, dS - r(V^i - \Delta^a S)\, dt - \Delta^a DS\, dt.$$

Because the option would be correctly valued at V^a we have

$$dV^a - \Delta^a\, dS - r(V^a - \Delta^a S)\, dt - \Delta^a DS\, dt = 0.$$

So we can write the mark-to-market profit over one time step as

$$\begin{aligned} &dV^i - dV^a + r(V^a - \Delta^a S)\, dt - r(V^i - \Delta^a S)\, dt \\ &= dV^i - dV^a - r(V^i - V^a)\, dt = e^{rt}\, d\left(e^{-rt}(V^i - V^a)\right). \end{aligned}$$

That is the profit from time t to $t + dt$. The present value of this profit at time t_0 is

$$e^{-r(t-t_0)} e^{rt}\, d\left(e^{-rt}(V^i - V^a)\right) = e^{rt_0}\, d\left(e^{-rt}(V^i - V^a)\right).$$

So the total profit from t_0 to expiration is

$$e^{rt_0} \int_{t_0}^{T} d\left(e^{-rt}(V^i - V^a)\right) = V^a - V^i.$$

This confirms what we said earlier about the guaranteed total profit by expiration.

We can also write that one time step mark-to-market profit (using Itô's lemma) as

$$
\begin{aligned}
&\Theta^i\,dt + \Delta^i\,dS + \tfrac{1}{2}\sigma^2 S^2\Gamma^i\,dt - \Delta^a\,dS - r(V^i - \Delta^a S)\,dt - \Delta^a DS\,dt\\
&\quad = \Theta^i\,dt + \mu S(\Delta^i - \Delta^a)\,dt + \tfrac{1}{2}\sigma^2 S^2\Gamma^i\,dt - r(V^i - \Delta^a S)\,dt\\
&\quad + (\Delta^i - \Delta^a)\sigma S\,dX - \Delta^a DS\,dt\\
&\quad = (\Delta^i - \Delta^a)\sigma S\,dX + (\mu - r + D)S(\Delta^i - \Delta^a)\,dt + \tfrac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)S^2\Gamma^i\,dt
\end{aligned}
$$

(using Black–Scholes with $\sigma = \tilde{\sigma}$)

$$= \tfrac{1}{2}(\sigma^2 - \tilde{\sigma}^2)S^2\Gamma^i\,dt + (\Delta^i - \Delta^a)((\mu - r + D)S\,dt + \sigma S\,dX).$$

The conclusion is that the final profit is guaranteed (the difference between the theoretical option values with the two volatilities) but how that is achieved is random, because of the dX term in the above. On a mark-to-market basis you could lose before you gain. Moreover, the mark-to-market profit depends on the real drift of the stock, μ. This is illustrated in Figure 2. The figure shows several simulations of the same delta-hedged position. Note that the final P&L is not *exactly* the same in each case because of the effect of hedging discretely, we hedged 'only' 1000 times for each realization. The option is a one-year European call, with a strike of 100, at the money initially, actual volatility is 30%, implied is 20%, the growth rate is 10% and interest rate 5%.

When S changes, so will V. But these changes do not cancel each other out. This leaves us with a dX in our mark-to-market P&L and from a risk management point of view this is not ideal.

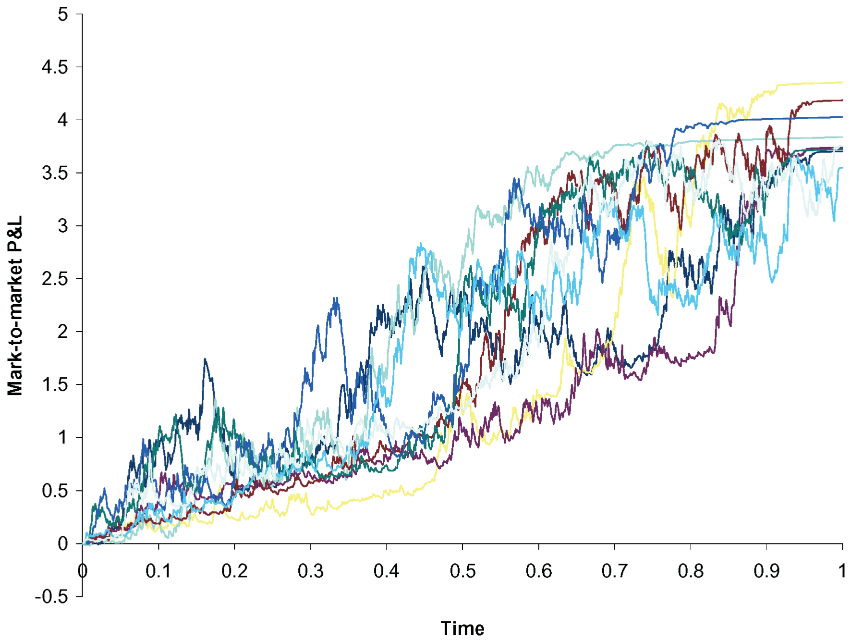

Figure 2: P&L for a delta-hedged option on a mark-to-market basis, hedged using actual volatility.

There is a simple analogy for this behavior. It is similar to owning a bond. For a bond there is a guaranteed outcome, but we may lose on a mark-to-market basis in the meantime.

4 Case 2: Hedge with Implied Volatility, $\tilde{\sigma}$

Compare and contrast now with the case of hedging using a delta based on implied volatility. By hedging with implied volatility we are balancing the random fluctuations in the mark-to-market option value with the fluctuations in the stock price. The evolution of the portfolio value is then 'deterministic' as we shall see.

Buy the option today, hedge using the implied delta, and put any cash in the bank earning r. The mark-to-market profit from today to tomorrow is

$$
\begin{aligned}
&dV^i - \Delta^i\,dS - r(V^i - \Delta^i S)\,dt - \Delta^i DS\,dt\\
&\quad = \Theta^i\,dt + \tfrac{1}{2}\sigma^2 S^2\Gamma^i\,dt - r(V^i - \Delta^i S)\,dt - \Delta^i DS\,dt \qquad (1)\\
&\quad = \tfrac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)S^2\Gamma^i\,dt.
\end{aligned}
$$

Observe how the profit is deterministic, there are no dX terms. From a risk management perspective this is much better behaved. There is another advantage of hedging using implied volatility, we do not even need to know what actual volatility is. To make a profit all we need to know is that actual is always going to be greater than implied (if we are buying) or always less (if we are selling). This takes some of the pressure off forecasting volatility accurately in the first place.

Integrate the present value of all of these profits over the life of the option to get a total profit of

$$\frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)\int_{t_0}^{T} e^{-r(t-t_0)}S^2\Gamma^i\,dt.$$

This is always positive, but highly path dependent. Being path dependent it will depend on the drift μ. If we start off at the money and the drift is very large (positive or negative) we will find ourselves quickly moving into territory where gamma and hence expression (1) is small, so that there will be not much profit to be made. The best that could happen would be for the stock to end up close to the strike at expiration, this would maximize the total profit. This path dependency is shown in Figure 3. The figure shows several realizations of the same delta-hedged position. Note that the lines are not perfectly smooth, again because of the effect of hedging discretely. The option and parameters are the same as in the previous example.

The simple analogy is now just putting money in the bank. The P&L is always increasing in value but the end result is random.

Carr (2005) and Henrard (2003) show the more general result that if you hedge using a delta based on a volatility σ_h then the PV of the total profit is given by

$$V(S,t;\sigma_h) - V(S,t;\tilde{\sigma}) + \frac{1}{2}\left(\sigma^2 - \sigma_h^2\right)\int_{t_0}^{T} e^{-r(t-t_0)}S^2\Gamma^h\,dt, \qquad (2)$$

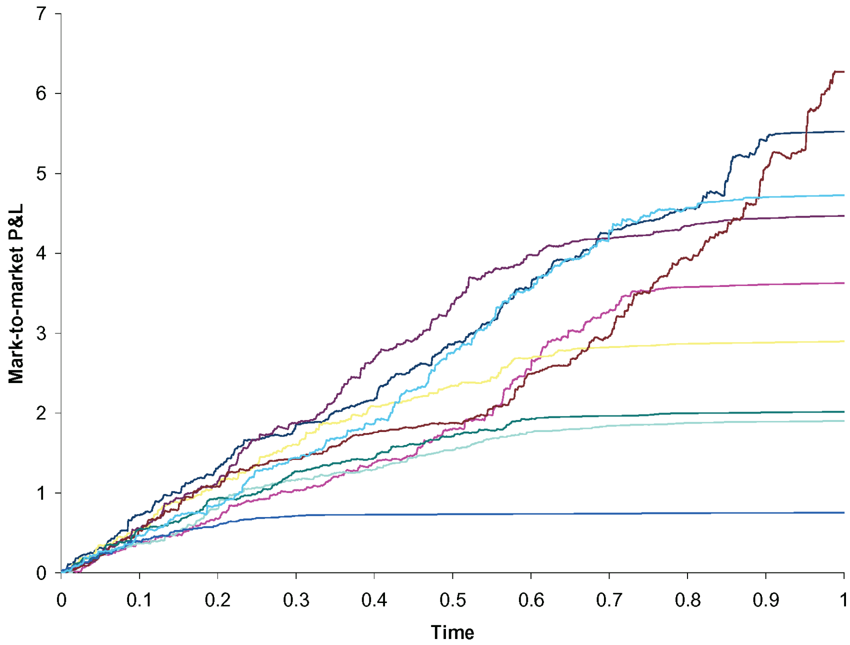

Figure 3: P&L for a delta-hedged option on a mark-to-market basis, hedged using implied volatility.

where the superscript on the gamma means that it uses the Black–Scholes formula with a volatility of σ_h.

From this equation it is easy to see that the final profit is bounded by

$$V(S,t;\sigma_h) - V(S,t;\tilde{\sigma})$$

and

$$V(S,t;\sigma_h) - V(S,t;\tilde{\sigma}) + \frac{E\left(\sigma^2 - \sigma_h^2\right) e^{-r(T-t_0)}\sqrt{T-t_0}}{\sigma_h\sqrt{2\pi}}.$$

The right-hand side of the latter expression above comes from maximizing $S^2\Gamma^h$. This maximum occurs along the path $\ln(S/E) + (r - D - \sigma_h^2/2)(T-t) = 0$, that is

$$S = E\exp\left(-\left(r - D - \sigma_h^2/2\right)(T-t)\right).$$

4.1 The expected profit after hedging using implied volatility

When you hedge using delta based on implied volatility the profit each 'day' is deterministic but the present value of total profit by expiration is path dependent, and given by

$$\frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)\int_{t_0}^{T} e^{-r(s-t_0)} S^2 \Gamma^i \, ds.$$

Introduce

$$I = \frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)\int_{t_0}^{t} e^{-r(s-t_0)} S^2 \Gamma^i \, ds.$$

Since therefore

$$dI = \tfrac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(t-t_0)} S^2 \Gamma^i \, dt$$

we can write down the following partial differential equation for the *real* expected value, $P(S,I,t)$, of I

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + \mu S \frac{\partial P}{\partial S} + \frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(t-t_0)} S^2 \Gamma^i \frac{\partial P}{\partial I} = 0,$$

with

$$P(S,I,T) = I.$$

Look for a solution of this equation of the form

$$P(S,I,t) = I + F(S,t)$$

so that

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(t-t_0)} S^2 \Gamma^i = 0.$$

The source term can be simplified to

$$\frac{E\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(T-t_0)} e^{-d_2^2/2}}{2\tilde{\sigma}\sqrt{2\pi(T-t)}},$$

where

$$d_2 = \frac{\ln(S/E) + (r - D - \frac{1}{2}\tilde{\sigma}^2)(T-t)}{\tilde{\sigma}\sqrt{T-t}}.$$

Change variables to

$$x = \ln(S/E) + \frac{2}{\sigma^2}\left(\mu - \frac{1}{2}\sigma^2\right)\tau \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T-t),$$

where E is the strike and T is expiration, and write

$$F(S,t) = w(x,\tau).$$

The resulting partial differential equation is a then simpler.

Result 1: After some manipulations we end up with the expected profit initially ($t = t_0$, $S = S_0$, $I = 0$) being the single integral

$$F(S_0, t_0) = \frac{Ee^{-r(T-t_0)}(\sigma^2 - \tilde{\sigma}^2)}{2\sqrt{2\pi}} \int_{t_0}^{T} \frac{1}{\sqrt{\sigma^2(s-t_0) + \tilde{\sigma}^2(T-s)}}$$

$$\exp\left(- \frac{\left(\ln(S_0/E) + \left(\mu - \frac{1}{2}\sigma^2\right)(s-t_0) + \left(r - D - \frac{1}{2}\tilde{\sigma}^2\right)(T-s)\right)^2}{2(\sigma^2(s-t_0) + \tilde{\sigma}^2(T-s))} \right) ds.$$

Derivation: See Appendix.

Results are shown in the following figures.

In Figure 4 is shown the expected profit versus the growth rate μ. Parameters are $S = 100$, $\sigma = 0.3$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. Observe that the expected profit has a maximum. This will be at the growth rate that ensures, roughly speaking, that the stock ends up close to at the money at expiration, where gamma is largest. In the figure is also shown the profit to be made when hedging with actual volatility. For most realistic parameter regimes the maximum expected profit hedging with implied is similar to the guaranteed profit hedging with actual.

In Figure 5 is shown expected profit versus E and μ. You can see how the higher the growth rate the larger the strike price at the maximum. The contour map is shown in Figure 6.

The effect of skew is shown in Figure 7. Here we have used a linear negative skew, from 22.5% at a strike of 75, falling to 17.5% at the 125 strike. The at-the-money implied volatility is 20% which in this case is the actual volatility. This picture changes when you divide the expected profit by the price of the option (puts for lower strikes, call for higher), see Figure 8. There is no maximum, profitability increases with distance away from the money. Of course, this does not take into account the risk, the standard deviation associated with such trades.

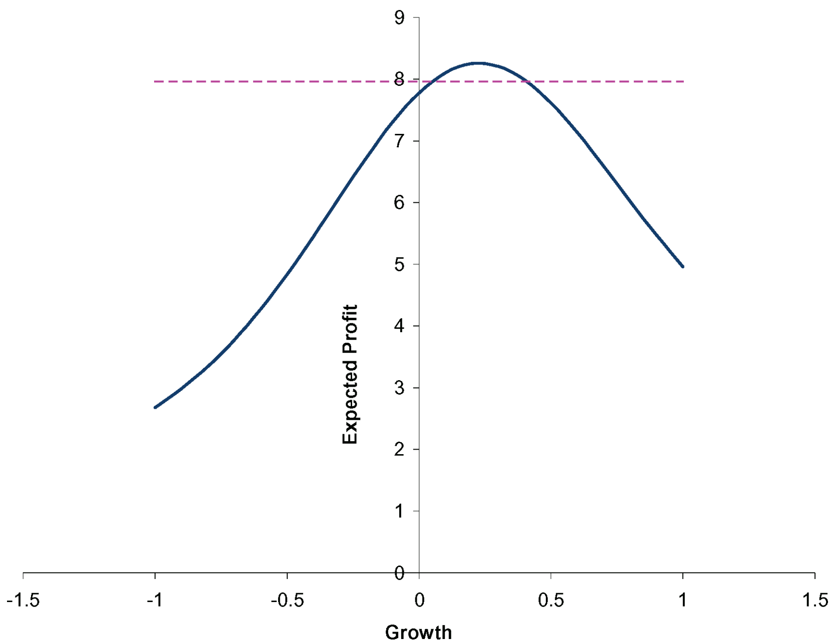

Figure 4: Expected profit, hedging using implied volatility, versus growth rate μ; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. The dashed line is the profit to be made when hedging with actual volatility.

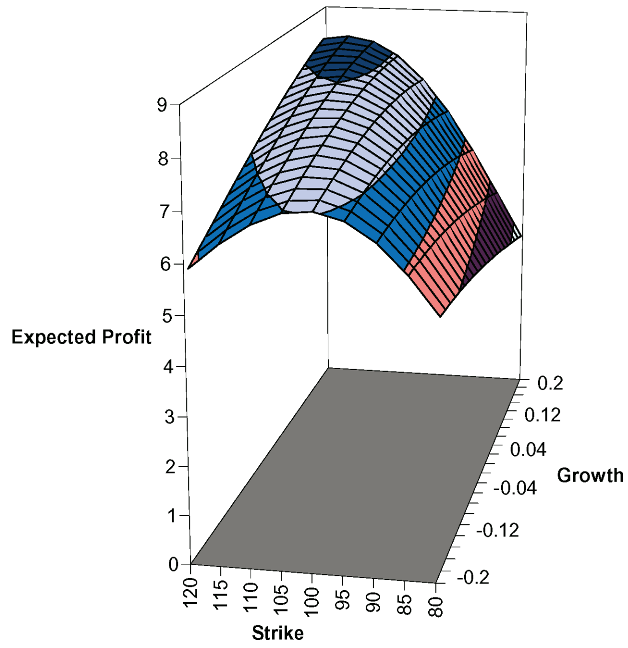

Figure 5: Expected profit, hedging using implied volatility, versus growth rate μ and strike E; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$.

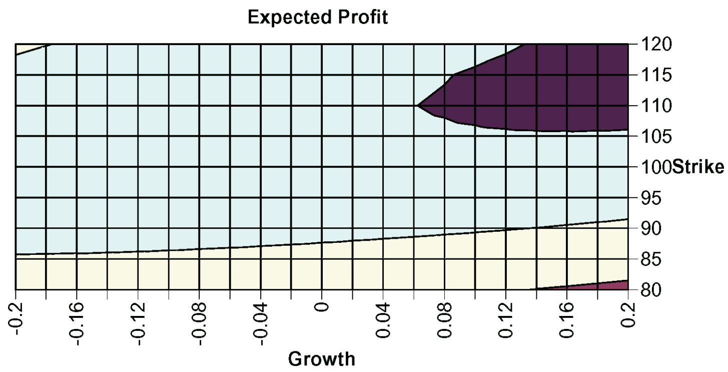

Figure 6: Contour map of expected profit, hedging using implied volatility, versus growth rate μ and strike E; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$.

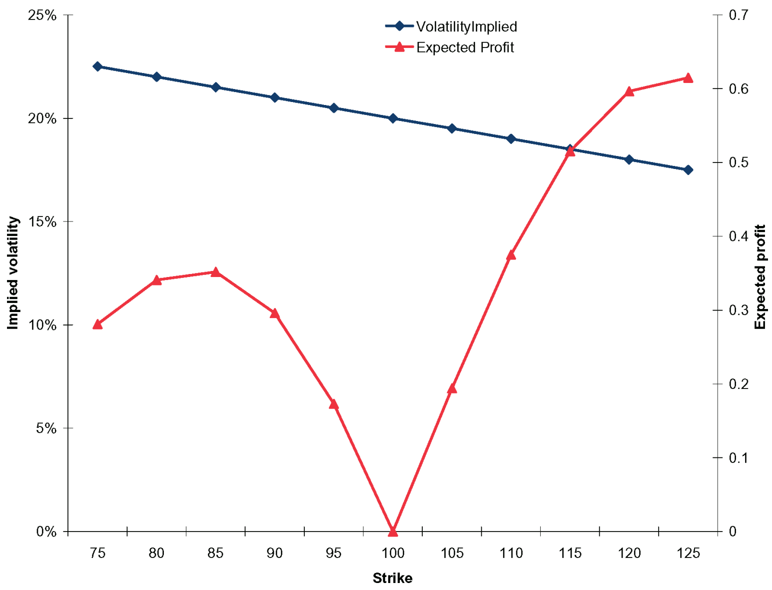

Figure 7: Effect of skew, expected profit, hedging using implied volatility, versus strike ***E***; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$.

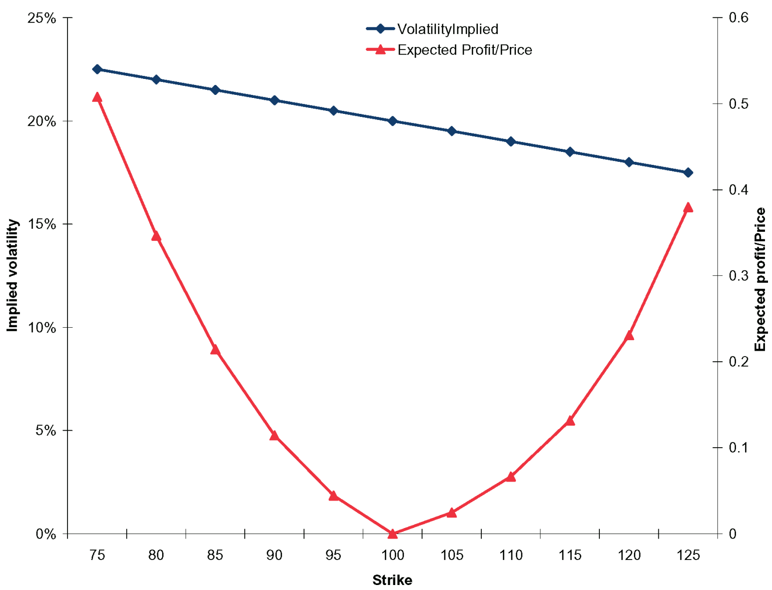

Figure 8: Effect of skew, ratio of expected profit to price, hedging using implied volatility, versus strike ***E***; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$.

4.2 The variance of profit after hedging using implied volatility

Once we have calculated the expected profit from hedging using implied volatility we can calculate the variance in the final profit. Using the above notation, the variance will be the expected value of I^2 less the square of the average of I. So we will need to calculate $v(S, I, t)$ where

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \mu S \frac{\partial v}{\partial S} + \frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(t-t_0)} S^2 \Gamma^i \frac{\partial v}{\partial I} = 0,$$

with

$$v(S, I, T) = I^2.$$

The details of finding this function v are rather messy, but a solution can be found of the form

$$v(S, I, t) = I^2 + 2I\, H(S, t) + G(S, t).$$

Result 2: The initial variance is $G(S_0, t_0) - F(S_0, t_0)^2$, where

$$G(S_0, t_0) = \frac{E^2(\sigma^2 - \tilde{\sigma}^2)^2 e^{-2r(T-t_0)}}{4\pi\sigma\tilde{\sigma}} \int_{t_0}^{T}\int_{s}^{T} \frac{e^{p(u,s;S_0,t_0)}}{\sqrt{s-t_0}\sqrt{T-s}\sqrt{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)} \times \sqrt{\frac{1}{\sigma^2(s-t_0)} + \frac{1}{\tilde{\sigma}^2(T-s)} + \frac{1}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}}}\, du\, ds \qquad (3)$$

where

$$p(u, s; S_0, t_0) = -\frac{1}{2}\frac{(x-\alpha(T-s))^2}{\tilde{\sigma}^2(T-s)} - \frac{1}{2}\frac{(x-\alpha(T-u))^2}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)} + \frac{1}{2}\frac{\left(\frac{x-\alpha(T-s)}{\tilde{\sigma}^2(T-s)} + \frac{x-\alpha(T-u)}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}\right)^2}{\frac{1}{\sigma^2(s-t_0)} + \frac{1}{\tilde{\sigma}^2(T-s)} + \frac{1}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}}$$

and

$$x = \ln(S_0/E) + \left(\mu - \tfrac{1}{2}\sigma^2\right)(T - t_0), \quad \text{and} \quad \alpha = \mu - \tfrac{1}{2}\sigma^2 - r + D + \tfrac{1}{2}\tilde{\sigma}^2.$$

Derivation: See Appendix.

In Figure 9 is shown the standard deviation of profit versus growth rate, $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. Figure 10 shows the standard deviation of profit versus strike, $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0.1$, $T = 1$, $\tilde{\sigma} = 0.2$.

Note that in these plots the expectations and standard deviations have not been scaled with the cost of the options.

In Figure 11 are shown expected profit divided by cost versus standard deviation divided by cost, as both strike and expiration vary. In these $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0.1$, $\tilde{\sigma} = 0.2$. To some extent, although we emphasize only *some*, these diagrams can be interpreted in a classical mean-variance manner. The main criticism is, of course, that we are not working with normal distributions, and, furthermore, there is no downside, no possibility of any losses.

Figure 12 completes the earlier picture for the skew, since it now contains the standard deviation.

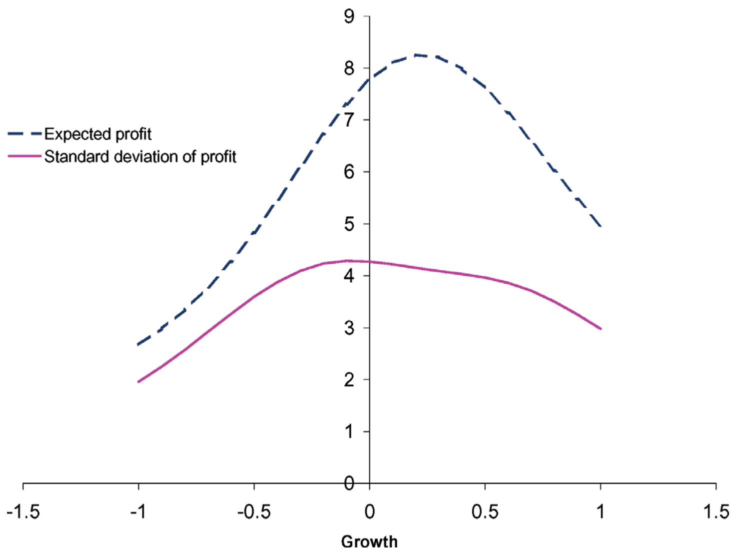

Figure 9: Standard deviation of profit, hedging using implied volatility, versus growth rate μ; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. The expected profit is also shown.

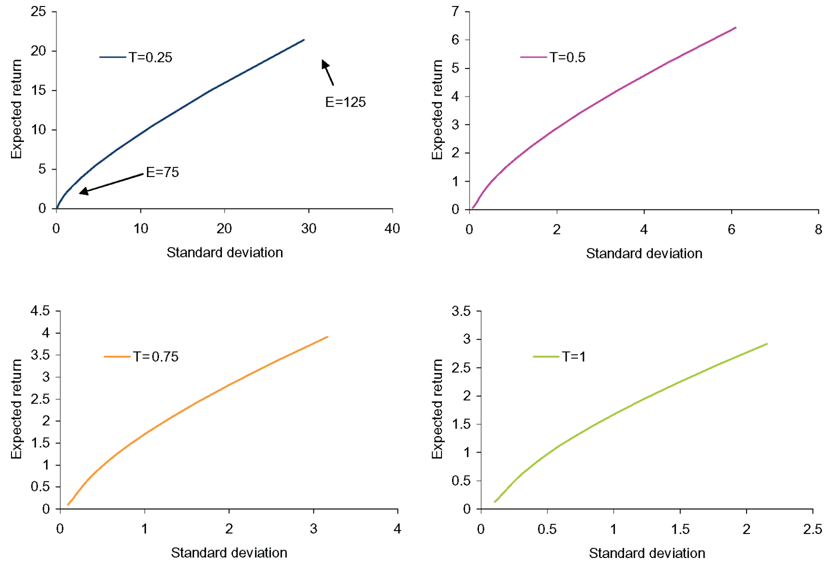

Figure 11: Scaled expected profit versus scaled standard deviation; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0.1$, $\tilde{\sigma} = 0.2$. Four different expirations, varying strike.

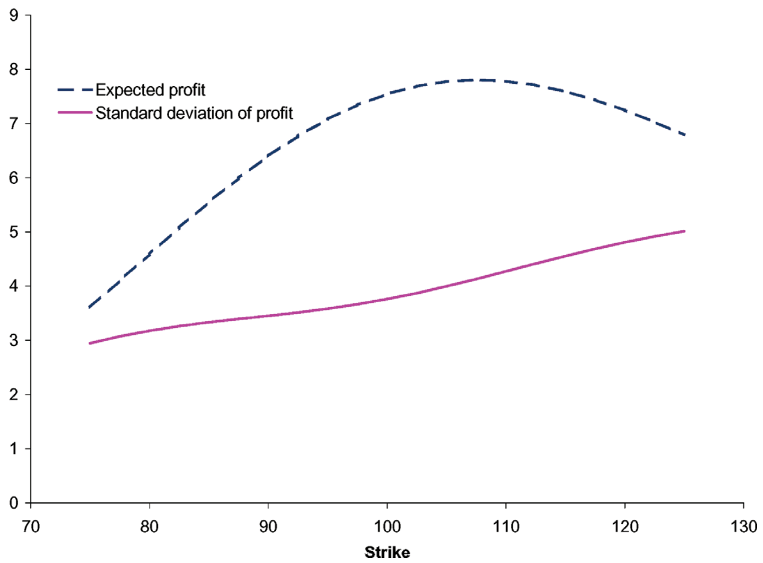

Figure 10: Standard deviation of profit, hedging using implied volatility, versus strike E; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0$, $T = 1$, $\tilde{\sigma} = 0.2$. The expected profit is also shown.

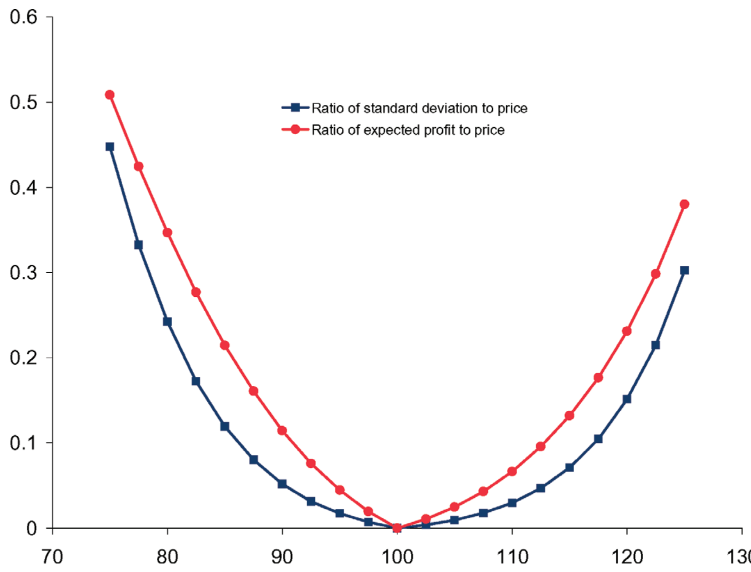

Figure 12: Effect of skew, ratio of expected profit to price, and ratio of standard deviation to price, versus strike E; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$.

5 Hedging with Different Volatilities

We will briefly examine hedging using volatilities other than actual or implied, using the general expression for profit given by (2).

The expressions for the expected profit and standard deviations now must allow for the $V(S, t; \sigma_h) - V(S, t; \tilde{\sigma})$, since the integral of gamma term can be treated as before if one replaces $\tilde{\sigma}$ with σ_h in this term. Results are presented in the next sections.

5.1 Actual volatility = Implied volatility

For the first example let's look at hedging a long position in a correctly priced option, so that $\sigma = \tilde{\sigma}$. We will hedge using different volatilities, σ^h. Results are shown in Figure 13. The figure shows the expected profit and standard deviation of profit when hedging with various volatilities. The chart also shows minimum and maximum profit. Parameters are $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.1$, $D = 0$, $T = 1$, and $\tilde{\sigma} = 0.2$.

With these parameters the expected profit is small as a fraction of the market price of the option ($13.3) regardless of the hedging volatility. The standard deviation of profit is zero when the option is hedged at the actual volatility. The upside, the maximum profit is much greater than the downside. Crucially all of the curves have zero value at the actual/implied volatility.

5.2 Actual volatility > Implied volatility

In Figure 14 is shown the expected profit and standard deviation of profit when hedging with various volatilities when actual volatility is greater than implied. The chart again also shows minimum and maximum profit.

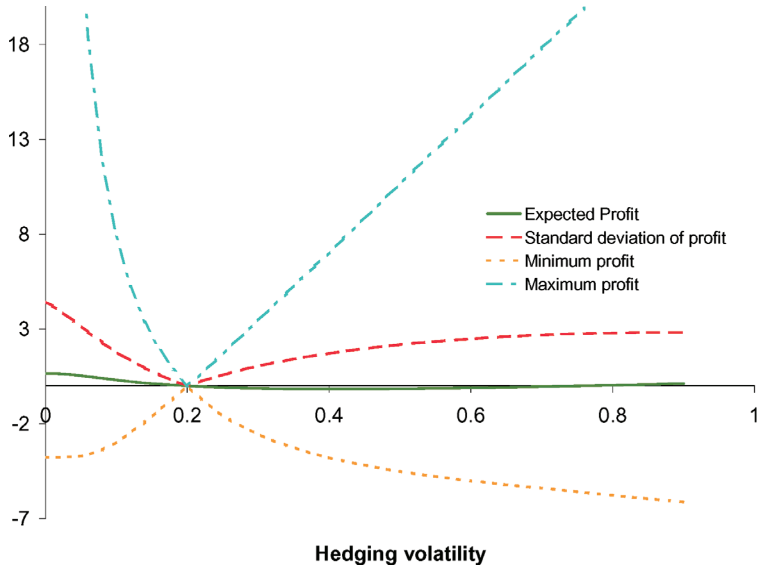

Figure 13: Expected profit, standard deviation of profit, minimum and maximum, hedging with various volatilities. $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$.

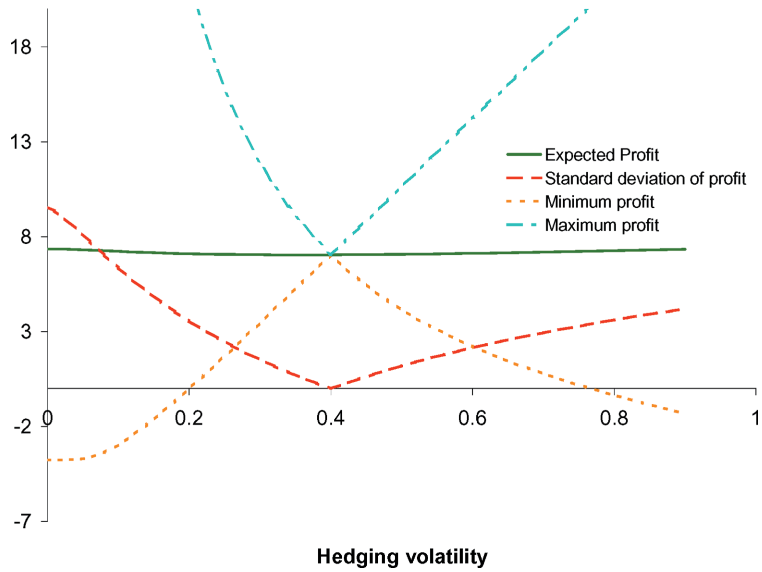

Figure 14: Expected profit, standard deviation of profit, minimum and maximum, hedging with various volatilities. $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$.

Parameters are $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, and $\tilde{\sigma} = 0.2$. Note that it is possible to lose money if you hedge at below implied, but hedging with a higher volatility you will not be able to lose until hedging with a volatility of approximately 75%. The expected profit is again insensitive to hedging volatility.

5.3 Actual volatility < Implied volatility

In Figure 15 is shown properties of the profit when hedging with various volatilities when actual volatility is less than implied. We are now selling the option and delta hedging it. Parameters are $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.1$, $D = 0$, $T = 1$, and $\tilde{\sigma} = 0.4$. Now it is possible to lose money if you hedge at above implied, but hedging with a lower volatility you will not be able to lose until hedging with a volatility of approximately 10%. The expected profit is again insensitive to hedging volatility. The downside is now more dramatic than the upside.

6 Pros and Cons of Hedging with each Volatility

Given that we seem to have a choice in how to delta hedge it is instructive to summarize the advantages and disadvantages of the possibilities.

6.1 Hedging with actual volatility

Pros: The main advantage of hedging with actual volatility is that you know exactly what profit you will get at expiration. So in a classical risk/reward sense this seems to be the best choice, given that the expected

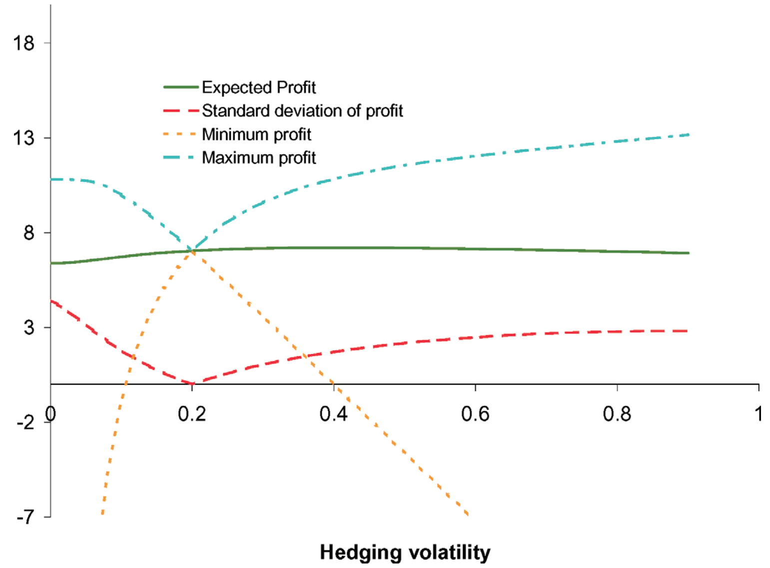

Figure 15: Expected profit, standard deviation of profit, minimum and maximum, hedging with various volatilities. $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.4$.

profit can often be insensitive to which volatility you choose to hedge with whereas the standard deviation is always going to be positive away from hedging with actual volatility.

Cons: The P&L fluctuations during the life of the option can be daunting, and so less appealing from a 'local' as opposed to 'global' risk management perspective. Also, you are unlikely to be totally confident in your volatility forecast, the number you are putting into your delta formula. However, you can interpret the previous two figures in terms of what happens if you intend to hedge with actual but don't quite get it right. You can see from those that you do have quite a lot of leeway before you risk losing money.

6.2 Hedging with implied volatility

Pros: There are three main advantages to hedging with implied volatility. The first is that there are no local fluctuations in P&L, you are continually making a profit. The second advantage is that you only need to be on the right side of the trade to profit. Buy when actual is going to be higher than implied and sell if lower. Finally, the number that goes into the delta is implied volatility, and therefore easy to observe.

Cons: You don't know how much money you will make, only that it is positive.

6.3 Hedging with another volatility

You can obviously balance the pros and cons of hedging with actual and implied by hedging with another volatility altogether. See Dupire (2005) for work in this area.

In practice which volatility one uses is often determined by whether one is constrained to mark to market or mark to model. If one is able to mark to model then one is not necessarily concerned with the day-to-day fluctuations in the mark-to-market profit and loss and so it is natural to hedge using actual volatility. This is usually not far from optimal in the sense of possible expected total profit, and it has no standard deviation of final profit. However, it is common to have to report profit and loss based on market values. This constraint may be imposed by a risk management department, by prime brokers, or by investors who may monitor the mark-to-market profit on a regular basis. In this case it is more usual to hedge based on implied volatility to avoid the daily fluctuations in the profit and loss.

We can begin to quantify the 'local' risk, the daily fluctuations in P&L, by looking at the random component in a portfolio hedged using a volatility of σ^h. The standard deviation of this risk is

$$\sigma S \mid \Delta^i - \Delta^h \mid \sqrt{dt}. \tag{4}$$

Note that this expression depends on all three volatilities.

Figure 16 shows the two deltas (for a call option), one using implied volatility and the other the hedging volatility, six months before expiration. If the stock is far in or out of the money the two deltas are similar and so the local risk is small. The local risk is also small where the two deltas cross over. This 'sweet spot' is at

$$\frac{\ln(S/E) + (r - D + \tilde{\sigma}^2/2)(T-t)}{\tilde{\sigma}\sqrt{T-t}} = \frac{\ln(S/E) + (r - D + \sigma^{h^2}/2)(T-t)}{\sigma^h\sqrt{T-t}},$$

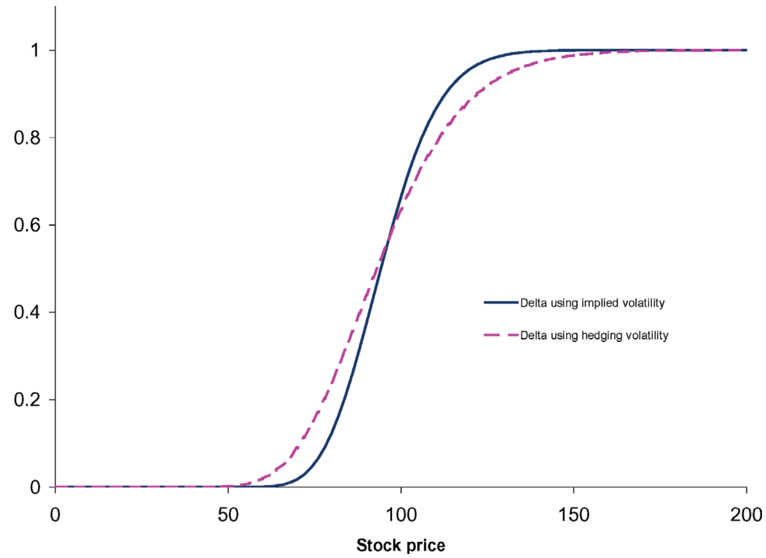

Figure 16: Deltas based on implied volatility and hedging volatility. $E = 100$, $S = 100$, $r = 0.1$, $D = 0$, $T = 0.5$, $\tilde{\sigma} = 0.2$, $\sigma^h = 0.3$.

that is,

$$S = E \exp\left(-\frac{T-t}{\tilde{\sigma}-\sigma^h}\left(\tilde{\sigma}(r-D+\sigma^{h^2}/2)-\sigma^h(r-D+\tilde{\sigma}^2/2)\right)\right).$$

Figure 17 shows a three-dimensional plot of expression (4), without the $\sqrt{dt}$ factor, as a function of stock price and time. Figure 18 is a contour map of the same. Parameters are $E = 100$, $S = 100$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$, $\sigma^h = 0.3$.

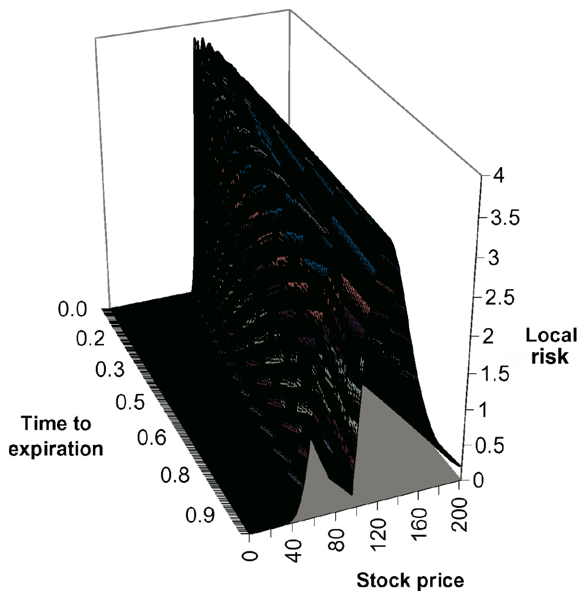

Figure 17: Local risk as a function of stock price and time to expiration. $E = 100$, $S = 100$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$, $\sigma^h = 0.3$.

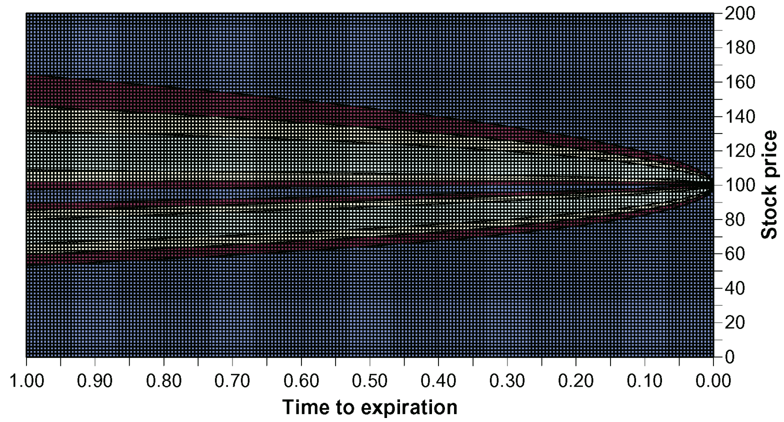

Figure 18: Contour map of local risk as a function of stock price and time to expiration. $E = 100$, $S = 100$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$, $\sigma^h = 0.3$.

In the spirit of the earlier analyses and formulas we would ideally like to be able to quantify various statistical properties of the local mark-to-market fluctuations. This will be the subject of future work.

For the remainder of this paper we will only consider the case of hedging using a delta based on implied volatility, although the ideas can be easily extended to the more general case.

7 Portfolios when Hedging with Implied Volatility

A natural extension to the above analysis is to look at portfolios of options, options with different strikes and expirations. Since only an option's gamma matters when we are hedging using implied volatility, calls and puts are effectively the same since they have the same gamma.

The profit from a portfolio is now

$$\frac{1}{2}\sum_k q_k\left(\sigma^2-\tilde{\sigma}_k^2\right)\int_{t_0}^{T_k} e^{-r(s-t_0)}S^2\Gamma_k^i\, ds,$$

where k is the index for an option, and q_k is the quantity of that option. Introduce

$$I = \frac{1}{2}\sum_k q_k\left(\sigma^2-\tilde{\sigma}_k^2\right)\int_{t_0}^{t} e^{-r(s-t_0)}S^2\Gamma_k^i\, ds, \tag{5}$$

as a new state variable, and the analysis can proceed as before. Note that since there may be more than one expiration date since we have several different options, it must be understood in Equation (5) that Γ_k^i is zero for times beyond the expiration of the option.

The governing differential operator for expectation, variance, etc. is then

$$\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2}{\partial S^2}+\mu S\frac{\partial}{\partial S}+\frac{1}{2}\sum_k\left(\sigma^2-\tilde{\sigma}_k^2\right)\, e^{-r(t-t_0)}S^2\Gamma_k^i\,\frac{\partial}{\partial I}=0,$$

with final condition representing expectation, variance, etc.

7.1 Expectation

Result 3: The solution for the present value of the expected profit ($t = t_0$, $S = S_0$, $I = 0$) is simply the sum of individual profits for each option,

$$F(S_0,t_0)=\sum_k q_k\frac{E_k e^{-r(T_k-t_0)}(\sigma^2-\tilde{\sigma}_k^2)}{2\sqrt{2\pi}}\int_{t_0}^{T_k}\frac{1}{\sqrt{\sigma^2(s-t_0)+\tilde{\sigma}_k^2(T_k-s)}}$$

$$\exp\left(-\frac{\left(\ln(S_0/E_k)+\left(\mu-\frac{1}{2}\sigma^2\right)(s-t_0)+\left(r-D-\frac{1}{2}\tilde{\sigma}_k^2\right)(T_k-s)\right)^2}{2(\sigma^2(s-t_0)+\tilde{\sigma}_k^2(T_k-s))}\right)ds.$$

Derivation: See Appendix.

7.2 Variance

Result 4: The variance is more complicated, obviously, because of the correlation between all of the options in the portfolio. Nevertheless, we can find an expression for the initial variance as $G(S_0, t_0) - F(S_0, t_0)^2$ where

$$G(S_0, t_0) = \sum_j \sum_k q_j q_k G_{jk}(S_0, t_0)$$

where

$$G_{jk}(S_0, t_0) = \frac{E_j E_k(\sigma^2 - \tilde{\sigma}_j^2)(\sigma^2 - \tilde{\sigma}_k^2)e^{-r(T_j - t_0) - r(T_k - t_0)}}{4\pi\sigma\tilde{\sigma}_k} \int_{t_0}^{\min(T_j, T_k)} \int_s^{T_j}$$

$$\frac{e^{p(u,s;S_0,t_0)}}{\sqrt{s-t_0}\sqrt{T_k - s}\sqrt{\sigma^2(u-s) + \tilde{\sigma}_j^2(T_j - u)} \sqrt{\frac{1}{\sigma^2(s-t_0)} + \frac{1}{\tilde{\sigma}_k^2(T_k - s)} + \frac{1}{\sigma^2(u-s) + \tilde{\sigma}_j^2(T_j - u)}}}\, du\, ds \qquad (6)$$

where

$$p(u, s; S_0, t_0) = -\frac{1}{2}\frac{(\ln(S_0/E_k) + \bar{\mu}(s - t_0) + \bar{r}_k(T_k - s))^2}{\tilde{\sigma}_k^2(T_k - s)}$$

$$- \frac{1}{2}\frac{(\ln(S_0/E_j) + \bar{\mu}(u - t_0) + \bar{r}_j(T_j - u))^2}{\sigma^2(u-s) + \tilde{\sigma}_j^2(T_j - u)}$$

$$+ \frac{1}{2}\frac{\left(\frac{\ln(S_0/E_k) + \bar{\mu}(s-t_0) + \bar{r}_k(T_k - s)}{\tilde{\sigma}_k^2(T_k - s)} + \frac{\ln(S_0/E_j) + \bar{\mu}(u-t_0) + \bar{r}_j(T_j - u)}{\sigma^2(u-s) + \tilde{\sigma}_j^2(T_j - u)}\right)^2}{\frac{1}{\sigma^2(s-t_0)} + \frac{1}{\tilde{\sigma}_k^2(T_k - s)} + \frac{1}{\sigma^2(u-s) + \tilde{\sigma}_j^2(T_j - u)}}$$

and

$$\bar{\mu} = \mu - \tfrac{1}{2}\sigma^2, \quad \bar{r}_j = r - D - \tfrac{1}{2}\tilde{\sigma}_j^2 \quad \text{and} \quad \bar{r}_k = r - D - \tfrac{1}{2}\tilde{\sigma}_k^2.$$

Derivation: See Appendix.

7.3 Portfolio optimization possibilities

There is clearly plenty of scope for using the above formulas in portfolio optimization problems. Here we give one example.

The stock is currently at 100. The growth rate is zero, actual volatility is 20%, zero dividend yield and the interest rate is 5%. Table 3 shows the available options, and associated parameters. Observe the negative skew. The out-of-the-money puts are overvalued and the out-of-the-money calls are undervalued. (The 'Profit Total Expected' row assumes that we buy a single one of that option.)

Using the above formulas we can find the portfolio that maximizes or minimizes target quantities (expected profit, standard deviation, ratio of profit to standard deviation). Let us consider the simple case of maximizing the expected return, while constraining the standard deviation to be one. This is a very natural strategy when trying to make a profit from volatility arbitrage while meeting constraints imposed by regulators, brokers, investors etc. The result is given in Table 4.

The payoff function (with its initial delta hedge) is shown in Figure 19. This optimization has effectively found an ideal risk reversal trade. This portfolio would cost −$0.46 to set up, i.e. it would bring in premium. The expected profit is $6.83.

TABLE 3: AVAILABLE OPTIONS.

	A	B	C	D	E
Type	Put	Put	Call	Call	Call
Strike	80	90	100	110	120
Expiration	1	1	1	1	1
Volatility, Implied	0.250	0.225	0.200	0.175	0.150
Option Price, Market	1.511	3.012	10.451	5.054	1.660
Option Value, Theory	0.687	2.310	10.451	6.040	3.247
Profit Total Expected	−0.933	−0.752	0.000	0.936	1.410

TABLE 4: AN OPTIMAL PORTFOLIO.

	A	B	C	D	E
Type	Put	Put	Call	Call	Call
Strike	80	90	100	110	120
Quantity	−2.10	−2.25	0	1.46	1.28

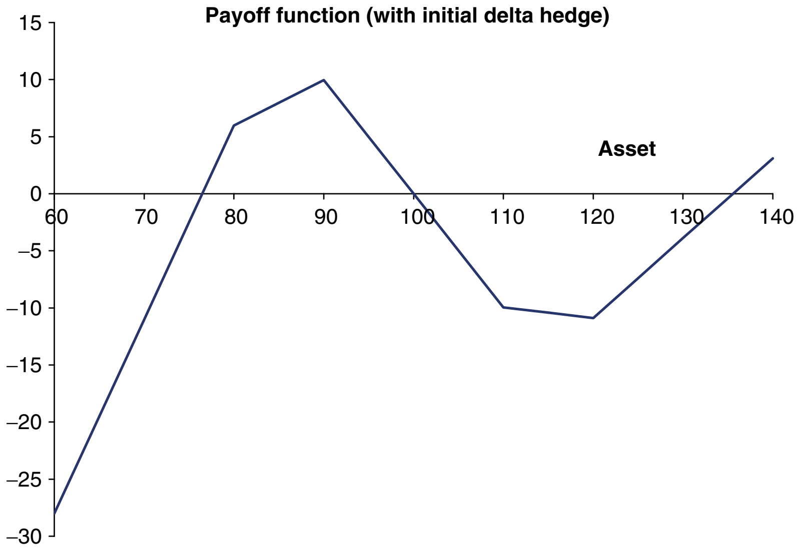

Figure 19: Payoff with initial delta hedge for optimal portfolio; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$. See text for additional parameters and information.

Because the state variable representing the profit, I, is not normally distributed a portfolio analysis based on mean and variance is open to criticism. So now we shall look at other ways of choosing or valuing a portfolio.

8 Other Optimization Strategies

Rather than choose an option or a portfolio based on mean and variance it might be preferable to examine the probability density function for I. The main reason for this is the observation that I is not normally distributed. Mathematically the problem for the cumulative distribution function for the final profit I' can be written as $C(S_0, 0, t_0; I')$ where $C(S, I, t; I')$ is the solution of

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sum_k \left(\sigma^2 - \tilde{\sigma}_k^2\right) e^{-r(t-t_0)} S^2 \Gamma_k^i \frac{\partial C}{\partial I} = 0,$$

subject to the final condition

$$C(S, I, T_{\max}; I') = \mathcal{H}(\mathcal{I}' - \mathcal{I}),$$

where $T_{\max}$ is the expiration of the longest maturity option and $\mathcal{H}(\cdot)$ is the Heaviside function. The same equation, with suitable final conditions, can be used to choose or optimize a portfolio of options based on criteria such as the following.

- Maximize probability of making a profit greater than a specified amount or, equivalently, minimize the probability of making less than a specified amount
- Maximize profit at a certain probability threshold, such as 95% (a Value-at-Risk type of optimization, albeit one with no possibility of a loss)

Constraints would typically need to be imposed on these optimization problems, such as having a set budget and/or a limit on number of positions that can be held.

In the spirit of maximizing expected growth rate (Kelly Criterion) we could also examine solutions of the above three-dimensional partial differential equation having final condition being a logarithmic function of I.

9 Exponential Utility Approach

Rather than relying on means and variances, which could be criticized because we are not working with a Gaussian distribution for I, or solving a differential equation in three dimensions, which may be slow, there is another possibility, and one that has neither of these disadvantages. This is to work within a utility theory framework, in particular using constant absolute risk aversion with utility function

$$-\frac{1}{\eta}\, e^{-\eta I}.$$

The parameter η is then a person's absolute risk aversion.

The governing equation for the expected utility, U, is then

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + \mu S \frac{\partial U}{\partial S} + \frac{1}{2}\sum_k \left(\sigma^2 - \tilde{\sigma}_k^2\right) e^{-r(t-t_0)} S^2 \Gamma_k^i \frac{\partial U}{\partial I} = 0,$$

with final condition

$$U(S, I, T_{\max}) = -\frac{1}{\eta}\, e^{-\eta I}.$$

where $T_{\max}$ is the expiration of the longest maturity option.

We can look for a solution of the form

$$U(S, I, t) = -\frac{1}{\eta} e^{-\eta I} Q(S, t),$$

so that

$$\frac{\partial Q}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Q}{\partial S^2} + \mu S \frac{\partial Q}{\partial S} - \frac{\eta Q}{2}\sum_k \left(\sigma^2 - \tilde{\sigma}_k^2\right) e^{-r(t-t_0)} S^2 \Gamma_k^i = 0,$$

with final condition

$$Q(S, T_{\max}) = 1.$$

Being only a two-dimensional equation this will be very quick to solve numerically. One can then pose and solve various optimal portfolio problems. We shall not pursue this in this paper.

10 Many Underlyings and Portfolios of Options

Although methodologies based on formulas and/or partial differential equations are the most efficient when the portfolio only has a small number of underlyings, one must use simulation techniques when there are many underlyings in the portfolio.

Because we know the instantaneous profit at each time, for a given stock price, via simple option pricing formulas for each option's gamma, we can very easily simulate the P&L for an arbitrary portfolio. All that is required is a simulation of the real paths for each of the underlyings.

10.1 Dynamics linked via drift rates

Although option prices are independent of real drift rates, only depending on risk-neutral rates, i.e. the risk-free interest adjusted for dividends, the profit from a hedged mispriced option is not. As we have seen above the profit depends crucially on the growth rate because of the path dependence. As already mentioned, ideally we would like to see the stock following a path along which gamma is large since this gives us the highest profit. When we have many underlyings we really ought to model the drift rates of all the stocks as accurately as possible, something that is not usually considered when simply valuing options in complete markets. In the above example we assumed constant growth rates for each stock, but in reality there may be more interesting, interacting dynamics at work. Traditionally, in complete markets the sole interaction between stocks that need concern us is via correlations in the dX_i terms, that is, a relationship at the infinitesimal timescale. In reality, and in the context of the present work, there will also be longer timescale relationships operating, that is, a coupling in the growth rates. This can be represented by

$$dS_i = \mu_i(S_1, \dots, S_n)\, dt + \sigma_i S_i\, dX_i.$$

11 Conclusions and Further Work

This paper has expanded on the work of Carr and Henrard in terms of final formulas for the statistical properties of the profit to be made hedging mispriced options. We have also indicated how more sophisticated portfolio construction techniques can be applied to this problem relatively straightforwardly. We have concentrated on the case of hedging using deltas based on implied volatilities because this is the most common in practice, giving mark-to-market profit the smoothest behavior.

Appendix: Derivation of Results

Preliminary results

In the following derivations we often require the following simple results.

First,

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}. \tag{7}$$

Second, the solution of

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2} + f(x, \tau)$$

that is initially zero and is zero at plus and minus infinity is

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \int_0^{\tau} \frac{f(x', \tau')}{\sqrt{\tau - \tau'}} e^{-(x-x')^2/4(\tau-\tau')}\,d\tau'\,dx'. \tag{8}$$

Finally, the transformations

$$x = \ln(S/E) + \frac{2}{\sigma^2}\left(\mu - \frac{1}{2}\sigma^2\right)\tau \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T - t),$$

turn the operator

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \mu S \frac{\partial}{\partial S}$$

into

$$\frac{1}{2}\sigma^2\left(-\frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2}\right). \tag{9}$$

Result 1: Expectation, single option

The equation to be solved for $F(S, t)$ is

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(t-t_0)} S^2 \Gamma^i = 0,$$

with zero final and boundary conditions. Using the above changes of variables this becomes $F(S, t) = w(x, \tau)$ where

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2} + \frac{E\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(T-t_0)} e^{-d_2^2/2}}{2\sigma\tilde{\sigma}\sqrt{\pi\tau}}$$

where

$$d_2 = \frac{\sigma}{\tilde{\sigma}} \frac{x - \frac{2}{\sigma^2}\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \frac{2}{\sigma^2}\left(r - D - \frac{1}{2}\tilde{\sigma}^2\right)\tau}{\sqrt{2\tau}}.$$

The solution of this problem is, using (8),

$$\frac{1}{4\pi} \frac{E\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(T-t_0)}}{\sigma\tilde{\sigma}} \int_{-\infty}^{\infty} \int_0^{\tau} \frac{1}{\sqrt{\tau'}} \frac{1}{\sqrt{\tau - \tau'}}$$
$$\exp\left(-\frac{(x - x')^2}{4(\tau - \tau')} - \frac{\sigma^2}{4\tilde{\sigma}^2\tau'}\left(x' - \frac{2}{\sigma^2}\left(\mu - \frac{1}{2}\sigma^2\right)\tau'\right.\right.$$
$$\left.\left.+\frac{2}{\sigma^2}\left(r - D - \frac{1}{2}\tilde{\sigma}^2\right)\tau'\right)^2\right)\,d\tau'\,dx'.$$

If we write the argument of the exponential function as

$$-a(x' + b)^2 + c$$

we have the solution

$$\frac{1}{4\pi} \frac{E\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(T-t_0)}}{\sigma\tilde{\sigma}} \int_0^{\tau} \frac{1}{\sqrt{\tau'}} \frac{1}{\sqrt{\tau - \tau'}} \int_{-\infty}^{\infty} \exp\left(-a(x' + b)^2 + c\right)\,dx'\,d\tau'$$
$$= \frac{1}{4\sqrt{\pi}} \frac{E\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(T-t_0)}}{\sigma\tilde{\sigma}} \int_0^{\tau} \frac{1}{\sqrt{\tau'}} \frac{1}{\sqrt{\tau - \tau'}} \frac{1}{\sqrt{a}} \exp(c)\,d\tau'.$$

It is easy to show that

$$a = \frac{1}{4(\tau - \tau')} + \frac{\sigma^2}{4\tilde{\sigma}^2\tau'}$$

and

$$c = -\frac{\sigma^2}{4\tilde{\sigma}^2\tau'(\tau - \tau')} \frac{\left(x - \frac{2\tau'}{\sigma^2}\left(\mu - \frac{1}{2}\sigma^2 - r + D + \frac{1}{2}\tilde{\sigma}^2\right)\right)^2}{\frac{1}{\tau - \tau'} + \frac{\sigma^2}{\tilde{\sigma}^2\tau'}}.$$

With

$$T-t=\frac{2}{\sigma^2}\tau'$$

we have

$$c=-\frac{\left(\ln(S/E)+\left(\mu-\frac{1}{2}\sigma^2\right)(s-t)+\left(r-D-\frac{1}{2}\tilde{\sigma}^2\right)(T-s)\right)^2}{2(\sigma^2(s-t)+\tilde{\sigma}^2(T-s))}.$$

From this follows Result 1, that the expected profit initially ($t=t_0$, $S=S_0, I=0$) is

$$\frac{Ee^{-r(T-t_0)}(\sigma^2-\tilde{\sigma}^2)}{2\sqrt{2\pi}}\int_{t_0}^{T}\frac{1}{\sqrt{\sigma^2(s-t_0)+\tilde{\sigma}^2(T-s)}}$$
$$\exp\left(-\frac{\left(\ln(S_0/E)+\left(\mu-\frac{1}{2}\sigma^2\right)(s-t_0)+\left(r-D-\frac{1}{2}\tilde{\sigma}^2\right)(T-s)\right)^2}{2(\sigma^2(s-t_0)+\tilde{\sigma}^2(T-s))}\right)\,ds.$$

Result 2: Variance, single option

The problem for the expectation of the square of the profit is

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 v}{\partial S^2}+\mu S\frac{\partial v}{\partial S}+\frac{1}{2}\left(\sigma^2-\tilde{\sigma}^2\right)\ e^{-r(t-t_0)}S^2\Gamma^i\ \frac{\partial v}{\partial I}=0, \tag{10}$$

with

$$v(S,I,T)=I^2.$$

A solution can be found of the form

$$v(S,I,t)=I^2+2I\,H(S,t)+G(S,t).$$

Substituting this into Equation (10) leads to the following equations for H and G (both to have zero final and boundary conditions):

$$\frac{\partial H}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 H}{\partial S^2}+\mu S\frac{\partial H}{\partial S}+\frac{1}{2}\left(\sigma^2-\tilde{\sigma}^2\right)\ e^{-r(t-t_0)}S^2\Gamma^i=0;$$

$$\frac{\partial G}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 G}{\partial S^2}+\mu S\frac{\partial G}{\partial S}+\left(\sigma^2-\tilde{\sigma}^2\right)\ e^{-r(t-t_0)}S^2\Gamma^iH=0.$$

Comparing the equations for H and the earlier F we can see that

$$H=F=\frac{Ee^{-r(T-t_0)}(\sigma^2-\tilde{\sigma}^2)}{2\sqrt{2\pi}}\int_{t}^{T}\frac{1}{\sqrt{\sigma^2(s-t)+\tilde{\sigma}^2(T-s)}}$$
$$\exp\left(-\frac{\left(\ln(S/E)+\left(\mu-\frac{1}{2}\sigma^2\right)(s-t)+\left(r-D-\frac{1}{2}\tilde{\sigma}^2\right)(T-s)\right)^2}{2(\sigma^2(s-t)+\tilde{\sigma}^2(T-s))}\right)\,ds.$$

Notice in this that the expression is a present value at time $t=t_0$, hence the $e^{-r(T-t_0)}$ term at the front. The rest of the terms in this must be kept as the running variables S and t.

Returning to variables x and τ, the governing equation for $G(S,t)=w(x,\tau)$ is

$$\frac{\partial w}{\partial \tau}=\frac{\partial^2 w}{\partial x^2}+\frac{2}{\sigma^2}\frac{E\sigma\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}e^{-d_2^2/2}}{2\tilde{\sigma}\sqrt{\pi\tau}}\frac{E\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}}{4\sigma\tilde{\sigma}\sqrt{\pi}}$$
$$\int_0^{\tau}\frac{1}{\sqrt{\tau'}}\frac{1}{\sqrt{\tau-\tau'}}\frac{1}{\sqrt{a}}\exp(c)\ d\tau' \tag{11}$$

where

$$d_2=\frac{\sigma}{\tilde{\sigma}}\frac{x-\frac{2}{\sigma^2}\left(\mu-\frac{1}{2}\sigma^2+r-D-\frac{1}{2}\tilde{\sigma}^2\right)\tau}{\sqrt{2\tau}},$$

and a and c are as above.

The solution is therefore

$$\frac{1}{2\sqrt{\pi}}\frac{2}{\sigma^2}\frac{E\sigma\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}}{2\tilde{\sigma}\sqrt{\pi}}\frac{E\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}}{4\sigma\tilde{\sigma}\sqrt{\pi}}$$
$$\int_{-\infty}^{\infty}\int_0^{\tau}\frac{f(x',\tau')e^{-d_2^2/2}}{\sqrt{\tau-\tau'}}e^{-(x-x')^2/4(\tau-\tau')}d\tau'dx'.$$

where now

$$f(x',\tau')=\frac{1}{\sqrt{\tau'}}\int_0^{\tau'}\frac{1}{\sqrt{\tau''}}\frac{1}{\sqrt{\tau'-\tau''}}\frac{1}{\sqrt{a}}\exp(c)\ d\tau''$$

and in a and c all τs become τ''s and all τ's become τ''s, and in d_2 all τs become τ's and all x's become x's.

The coefficient in front of the integral signs simplifies to

$$\frac{1}{8\pi^{3/2}}\frac{E^2\left(\sigma^2-\tilde{\sigma}^2\right)^2e^{-2r(T-t_0)}}{\sigma^2\tilde{\sigma}^2}.$$

The integral term is of the form

$$\int_{-\infty}^{\infty}\int_0^{\tau}\int_0^{\tau'}\cdots d\tau''d\tau'dx',$$

with the integrand being the product of an algebraic term

$$\frac{1}{\sqrt{\tau'}\sqrt{\tau''}\sqrt{\tau-\tau'}\sqrt{\tau'-\tau''}\sqrt{a}}$$

and an exponential term

$$\exp\left(-\frac{1}{2}d_2^2-\frac{(x-x')^2}{4(\tau-\tau')}+c\right).$$

This exponent is, in full,

$$-\frac{1}{4\tau'}\frac{\sigma^2}{\tilde\sigma^2}\left(x'-\frac{2}{\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right)\tau'+\frac{2}{\sigma^2}\left(r-D-\frac{1}{2}\tilde\sigma^2\right)\tau'\right)^2-\frac{(x-x')^2}{4(\tau-\tau')}$$

$$-\frac{\sigma^2}{4\tilde\sigma^2\tau''(\tau'-\tau'')}\frac{\left(x'-\frac{2\tau''}{\sigma^2}\left(\mu-\frac{1}{2}\sigma^2-r+D+\frac{1}{2}\tilde\sigma^2\right)\right)^2}{\frac{1}{\tau'-\tau''}+\frac{\sigma^2}{\tilde\sigma^2\tau''}}.$$

This can be written in the form

$$-d(x'+f)^2+g,$$

where

$$d=\frac{1}{4}\frac{\sigma^2}{\tilde\sigma^2}\frac{1}{\tau'}+\frac{1}{4}\frac{1}{\tau-\tau'}+\frac{1}{4}\frac{\sigma^2}{\sigma^2(\tau'-\tau'')+\tilde\sigma^2\tau''}$$

and

$$g=-\frac{\sigma^2}{4\tilde\sigma^2\tau'}\left(x-\frac{2\alpha\tau'}{\sigma^2}\right)^2-\frac{\sigma^2}{4(\sigma^2(\tau'-\tau'')+\tilde\sigma^2\tau'')}\left(x-\frac{2\alpha\tau''}{\sigma^2}\right)^2$$

$$+\frac{1}{4}\frac{\left(\frac{\sigma^2}{\tilde\sigma^2\tau'}\left(x-\frac{2\alpha\tau'}{\sigma^2}\right)+\frac{\sigma^2}{(\sigma^2(\tau'-\tau'')+\tilde\sigma^2\tau'')}\left(x-\frac{2\alpha\tau''}{\sigma^2}\right)\right)^2}{\frac{\sigma^2}{\tilde\sigma^2}\frac{1}{\tau'}+\frac{1}{\tau-\tau'}+\frac{\sigma^2}{\sigma^2(\tau'-\tau'')+\tilde\sigma^2\tau''}},$$

where

$$\alpha=\mu-\tfrac{1}{2}\sigma^2-r+D+\tfrac{1}{2}\tilde\sigma^2.$$

Using Equation (7) we end up with

$$\frac{1}{8\pi^{3/2}}\frac{E^2\left(\sigma^2-\tilde\sigma^2\right)^2e^{-2r(T-t_0)}}{\sigma^2\tilde\sigma^2}$$
$$\int_0^{\tau}\int_0^{\tau'}\frac{1}{\sqrt{\tau'}\sqrt{\tau''}\sqrt{\tau-\tau'}\sqrt{\tau'-\tau''}\sqrt{a}}\sqrt{\frac{\pi}{d}}\exp(g)d\tau''d\tau'.$$

Changing variables to

$$\tau=\frac{\sigma^2}{2}(T-t),\quad \tau'=\frac{\sigma^2}{2}(T-s),\quad\text{and}\quad \tau''=\frac{\sigma^2}{2}(T-u),$$

and evaluating at $S=S_0$, $t=t_0$, gives the required Result 2.

Result 3: Expectation, portfolio of options

This expression follows from the additivity of expectations.

Result 4: Variance, portfolio of options

The manipulations and calculations required for the analysis of the portfolio variance are similar to that for a single contract. There is again a solution of the form

$$v(S,I,t)=I^2+2I\,H(S,t)+G(S,t).$$

The main differences are that we have to carry around two implied volatilities, $\tilde\sigma_j$ and $\tilde\sigma_k$ and two expirations, T_j and T_k. We will find that the solution for the variance is the sum of terms satisfying diffusion equations with source terms like in Equation (11). The subscript 'k' is then associated with the gamma term, and so appears outside the integral in the equivalent of (11), and the subscript 'j' is associated with the integral and so appears in the integrand.

There is one additional subtlety in the derivations and that concerns the expirations. We must consider the general case $T_j\neq T_k$. The integrations in (6) must only be taken over the intervals up until the options have expired. The easiest way to apply this is to use the convention that the gammas are zero after expiration. For this reason the s integral is over t_0 to $\min(T_j,T_k)$.

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