# Option Pricing*

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**Abstract**

European option pricing and greeks

European *option valuation* involves calculating the expected value of the *option payoff* using the *underlying* at *expiration*. *Greeks* are derivatives of the option value with respect to *model parameters*. This short note derives formulas for these that can be used for any positive underlying.

## Black-Scholes/Merton

The classic Black-Scholes/Merton formula for the spot value of a call option is

$$v_0 = sN(d_1) - ke^{-rt}N(d_2)$$

where $N$ is the standard normal cumluative distribution function, $s$ is the spot price, $k$ is the call strike, $r$ is the risk-free continuously compounded interest rate, $t$ is the time in years to expiration, $d_1 = (\log(s/k) + (r + \sigma^2/2)t)/\sigma\sqrt{t}$, and $d_2 = d_1 - \sigma\sqrt{t}$.

Option *delta* is the derivative of value with respect to the underlying. It is true that $\partial_s v_0 = \partial v_0/\partial s = N(d_1)$, but $d_1$ and $d_2$ involve $s$ so one needs to show $s\partial_s N(d_1) - ke^{-rt}\partial_s N(d_2) = 0$. Plowing through the calculations involved is a ritual we all perform when first learning the theory.

Their Nobel Prize winning work showed how to replicate the payoff of an option by dynamically hedging it with the underlying. The value of an option is the cost of setting up the initial hedge. It is not trivial to show the value is the expectation of the option payoff under some probability measure. This is why Nobel Prizes are awarded.

Fischer Black simplified this formula by expressing it in terms of *forward values*.

$$v_t = fN(d_1) - kN(d_2),$$

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where $f = se^{rt}$ is forward price and $v_t = v_0 e^{rt}$ is the forward value of the option. In this case $d_1 = (\log(f/k) + \sigma^2 t/2)/\sigma\sqrt{t}$ and $d_2 = d_1 - \sigma\sqrt{t}$ which eliminates the parameter $r$. Letting $s = \sigma\sqrt{t}$ eliminates $t$.

We will skip the theory of stochastic differential equations, Ito's lemma, self-financing portfolios, and other dainty mathematical machinery required to prove their result. Let's fast-forward to calculating expected values and derivatives with respect to model parameters.

## Black Model

In Black's model, the forward at expiration is $F_t = fe^{\sigma B_t - \sigma^2 t/2}$, where $B_t$ is standard Brownian motion. The forward value of a call option is the expected value of the call payoff at expiration

$$v_t = E[\max\{F_t - k, 0\}]$$

The expiration $t$ can be subsumed into the *vol* $s = \sigma\sqrt{t}$ so $F_t = F = fe^{sX - s^2/2}$ where $X$ is standard normal. The only fact we use about Brownian motion is $B_t$ is normal with mean 0 and variance $t$.

Since $(F - k)^+ = \max\{F - k, 0\} = (F - k)1(F \geq k)$,

$$\begin{aligned}
v &= E[\max\{F - k, 0\}] \\
&= E[(F - k)1(F \geq k)] \\
&= E[F1(F \geq k)] - kE[1(F \geq k)] \\
&= fE[e^{sX - s^2/2}1(F \geq k)] - kP(F \geq k)
\end{aligned}$$

**Exercise**. *Show $F \geq k$ if and only if $-X \leq d_2$.*

**Exercise**. *Use $E[e^{sX - s^2/2}g(X)] = E[g(X + s)]$ to show $E[e^{sX - s^2/2}1(F \geq k)] = P(Fe^{s^2} \geq k)$.*

**Exercise**. *Show $Fe^{s^2} \geq k$ if and only if $-X \leq d_1$.*

This establishes the Black formula for the forward value of an option since $-X$ has the same distribution as $X$.

For any differentiable function $\nu$, $\partial_f E[\nu(F)] = E[\nu'(F)\partial_f F] = E[\nu'(F)e^{sX - s^2/2}] = E[\nu'(Fe^{s^2})]$ so

$$\partial_f v = E[1(Fe^{s^2} \geq k)] = P(Fe^{s^2} \geq k) = N(d_1).$$

This establishes the formula for option delta without any turmoil. Option values and greeks for any positive underlying can be calculated in a similar fashion.

## Share Measure

Let $F$ be the price of the *underlying* instrument at option expiration. The forward value of an option paying $\nu(F)$ in some currency at expiration is $E[\nu(F)]$. If $F$ is positive we can also consider the payoff in terms of shares of $F$, $\nu_s(F) = \nu(F)F/E[F]$; if we receive $\nu_s(F)$ shares of $F$ at expiration we can convert those at price $F$ to $\nu(F)$ in the currency.

*Share measure* for positive underlyings is defined by $E_s[\nu(F)] = E[\nu_s(F)] = E[\nu(F)F/E[F]]$. Note $F > 0$ and $E_s[1] = 1$ so share measure is a probability measure. It shows up in the formula for valuing a call

$$
\begin{aligned}
E[(F-k)^+] &= E[(F-k)1(F \geq k)] \\
&= E[F1(F \geq k)] - kP(F \geq k) \\
&= fP_s(F \geq k) - kP(F \geq k).
\end{aligned}
$$

Every positive random variables $F$ can be written $F = fe^{sX-\kappa(s)}$ where $X$ is a random variable with mean 0 and variance 1 and $\kappa(s) = \log E[e^{sX}]$ is the cumulant of $X$. Note $f = E[F]$ and $s^2 = \mathrm{Var}(\log F)$.

**Exercise**. *Clearly* $\log(F/E[F]) = m + sX$ *for some random variable* $X$ *with mean 0 and variance 1. Show* $E[F] = f$ *implies* $m = -\kappa(s)$.

If we let $\varepsilon_s(x) = e^{sx-\kappa(s)}$, so $F = f\varepsilon_s(X)$, this can be written $E_s[\nu(F)] = E[\nu(F)\varepsilon_s(X)]$ and we see share measure is the Esscher transform. The cumulative distribution of $F$ under this measure is

$$
P_s(F \leq y) = P_s(X \leq x) = E[1(X \leq x)e^{sX-\kappa(s)}]
$$

where $x = x(y) = \varepsilon_s^{-1}(y/f) = (\log y/f + \kappa(s))/s$ is the *moneyness* of $y$. Note $\partial_x \varepsilon_s(x) = \varepsilon_s(x)s$, $\partial_s \varepsilon_s(x) = \varepsilon_s(x)(x - \kappa'(s))$, and $\varepsilon_s(x(y)) = y/f$.

### Greeks

Let $\nu(F)$ be the option payoff at expiration. The forward value of the option is $v = E[\nu(F)]$. Option *delta* is the derivative of value with respect to the forward

$$
\partial_f v = E[\nu'(F)\partial_f F] = E[\nu'(F)\varepsilon_s(X)] = E_s[v'(F)]
$$

using $\partial_f F = \varepsilon_s(X)$.

*Gamma* is the second derivative with respect to the forward

$$
\partial_f^2 v = E[\nu''(F)\varepsilon_s^2(X)].
$$

*Vega* is the derivative with respect to vol

$$
\partial_s v = E[\nu'(F)\partial_s F] = E[\nu'(F)F(X - \kappa'(s))] = fE_s[\nu'(F)(X - \kappa'(s))]
$$

using $\partial_s F = F(X - \kappa'(s))$.

The inverse of option value as a function of vol is the *implied vol*.

## Put and Call

A *put option* pays $\nu(F) = (k - F)^+ = \max\{k - F, 0\}$ at expiration and has value $p = E[(k - F)^+]$. A *call option* pays $\nu(F) = (F - k)^+$ at expiration and has value $c = E[(F - k)^+]$. Note $(F - k)^+ - (k - F)^+ = F - k$ is a *forward* with *strike* $k$ so all models satisfy *put-call parity*: $c - p = f - k$. Call delta is $\partial_f c = \partial_f p + 1$ and call gamma equals put gamma $\partial_f^2 c = \partial_f^2 p$. We also have $\partial_s c - \partial_s p = 0$ because forwards are independent of vol so call vega equals put vega.

The value of a put is

$$p = E[(k - F)^+] = kP(F \le k) - fP_s(F \le k).$$

Put delta is

$$\partial_f p = E[-1(F \le k)\varepsilon_s(X)] = -P_s(F \le k).$$

Gamma for either a put or call is

$$\partial_f^2 p = E[\delta_k(F)\varepsilon_s(X)^2] = E_s[\delta_k(F)\varepsilon_s(X)]$$

where $\delta_k$ is a point mass at $k$.

Vega for a put or call is

$$\partial_s p = E[-1(F \le k)F(X - \kappa'(s))] = -fE_s[1(F \le k)(X - \kappa'(s))].$$

### Distribution

Let $\Phi(x) = P(X \le x)$ be the cumulative distribution functions of $X$ and $\Phi_s(x) = P_s(X \le x) = E[1(X \le x)\varepsilon_s(X)]$ be the *share* cdf. Of course $\Phi(x) = \Phi_0(x)$. Let $\Psi_s(y) = P_s(F \le y) = \Phi_s(x)$ be the share cumulative distribution function of $F$ where $y = f\varepsilon_s(x)$. The share density function is

$$\psi_s(y) = \varphi_s(x)\partial x/\partial y = \varphi_s(x)/ys$$

since $\partial y/\partial x = ys$. We also have

$$\partial_s \Phi_s(x) = E[1(X \le x)\varepsilon_s(X)(X - \kappa'(s))] = E_s[1(X \le x)(X - \kappa'(s))].$$

In terms of the distribution function for $X$, the value is

$$p = k\Phi(x(k)) - f\Phi_s(x(k),$$

put delta is

$$\partial_f p = -\Phi_s(x(k)),$$

put gamma is

$$\partial_f^2 p = E_s[\delta_k(F)\varepsilon_s(X)] = \psi_s(k)\varepsilon_s(x(k)) = (\varphi_s(x(k))/ks)(k/f) = \varphi_s(x(k))/fs,$$

and put vega is

$$\partial_s p = -fE_s[1(F \le k)(X - \kappa'(s))] = -f\partial_s \Phi_s(x(k)).$$

## Black Model

In the Black modes $F = fe^{sX - s^2/2}$ where $X$ is standard normal. Recall if $X$ is standard normal then $E[g(X)e^{sX}] = e^{s^2/2}E[g(X+s)]$. Using $g(x) = 1$ we see $\kappa(s) = s^2/2$. Using $g(X) = 1(X \leq x)$ we get $\Phi_s(x) = P(X + s \leq x) = \Phi(x - s)$ and $\partial\Phi_s(x)/\partial s = -\varphi(x - s) = -\varphi_s(x)$.

Put value is

$$p = k\Phi(x(k)) - f\Phi(x(k) - s),$$

where $x(k) = \log(k/f)/s + s/2$.

Put delta is

$$\partial_f p = -\Phi_s(x(k)) = -\Phi(x(k) - s).$$

Gamma is

$$\partial_f^2 p = \varphi_s(x(k))/fs = \varphi(x(k) - s)/fs$$

Vega is

$$\partial_s v = -f\partial_s\Phi_s(x(k)) = f\varphi_s(x(k)) = f\varphi(x(k) - s).$$

## Digital

A *digital put* has payoff $\nu(F) = 1(F \leq k)$ and a *digital call* has payoff $\nu(F) = 1(F > k)$. Since $1(F \leq k) + 1(F > k) = 1$ we have digital put-call parity $p + c = 1$ where $p$ is the digital put value and $c$ is the digital call value:

$$p = P(F \leq k), c = P(F > k) = 1 - p.$$

Digital put delta is

$$\partial_f p = -E[\delta_k(F)\varepsilon_s(X)] = -E_s[\delta_k(F)]$$

Digital gamma is

$$\partial_f^2 p = E[\delta_k'(F)\varepsilon_s(X)^2] = E_s[\delta_k'(F)\varepsilon_s(X)].$$

Digital vega is

$$\partial_s p = -E[\delta_k(F)F(X - s)] = -fE_s[\delta_k(F)(X - \kappa(s))].$$

## Parameters

The Black-Scholes/Merton values and greeks can be calculated in terms of the parameters $f$ and $s$ using the chain rule. For example, the Black model takes $X$ to be standard normal and vol $s = \sigma\sqrt{t}$ where $\sigma$ is the standard Black volatilty and $t$ is time in years to expiration. In this case standard vega is $\partial_\sigma E[\nu(F)] = \partial_s E[\nu(F)]\partial_\sigma s = \partial_s E[\nu(F)]\sqrt{t}$.

The Black-Merton/Scholes model uses *spot* prices instead of forward. If a risk-free bond has realized return $R = e^{rt}$ over the period, the value of the underlying at expiration is $U = Rue^{sX-\kappa(s)}$. Since $F = U$ we have $f = Ru$. The *spot* value of the option is $v_0 = E[\nu(U)]/R$. We have

$$\partial_u v_0 = E[\nu'(U)\partial_u U]/R = E[\nu'(F)\partial_f F \partial_u f]/R = E[\nu'(F)\partial_f F R]/R = \partial_f v.$$

Spot and forward delta are equal but the spot gamma is

$$\partial_u^2 v_0 = \partial_u \partial_f v = \partial_f^2 v \partial_u f = \partial_f^2 v R.$$

Spot vega is

$$\partial_s v_0 = E[\nu'(U)\partial_s U]/R = E[\nu'(F)\partial_s F/R = \partial_s v/R.$$