Factor models via Autoencoders

A clever way of using Neural Networks to solve a familiar but important problem in Finance was proposed by [Gu, Kelly, and Xiu, 2019 (https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3335536)](https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3335536).

It is an extension of the Factor Model framework of Finance, combined with the tools of dimensionality reduction (to find the factors) of Deep Learning: the Autoencoder.

You can find code (https://github.com/stefan-jansen/machine-learning-for-trading/blob/main/20_autoencoders_for_conditional_risk_factors/06_conditional_autoenc for this model as part of the excellent book by Stefan Jansen (https://github.com/stefan-jansen/machine-learning-for-trading/blob/main/20_autoencoders_for_conditional_risk_factors/06_conditional_autoenc

- Github (https://github.com/stefan-jansen/machine-learning-for-trading)
- In order to run the code notebook, you first need to run a notebook for data prepara jansen/machine-learning-for-trading/blob/main/20_autoencoders_for_conditional_risk_factors/05_conditional_a
    - This notebook relies on files created by notebooks from earlier chapters of
    - So, if you want to run the code, you have a lot of preparatory work ahead of
    - Try to take away the ideas and the coding

Factor Model review

We will begin with a quick review/introduction to Factor Models in Finance.

First, some necessary notation:

- $\mathbf{r}_s^{(d)}$: Return of ticker s on day d.
- $\hat{\mathbf{r}}_s^{(d)}$: approximation of $\mathbf{r}_s^{(d)}$
- n_{tickers}: **large** number of tickers
- n_{dates}: number of dates
- n_{factors}: **small** number of factors: independent variables (features) in our approximation
- Matrix $\mathbf{R}$ of ticker returns, indexed by *date*
    - $\mathbf{R} : (n_{\text{dates}} \times n_{\text{tickers}})$
    - $||\mathbf{R}^{(d)}|| = n_{\text{tickers}}$
        - $\mathbf{R}^{(d)}$ is vector of returns for each of the n_{tickers} on date d
- $\mathbf{r}$ will denote a vector of single day returns: $\mathbf{R}^{(d)}$ for some date d

Notation summary

term	meaning		
s	ticker		
n_{tickers}	number of tickers		
d	date		
n_{dates}	number of dates		
n_{chars}	number of characteristics per ticker		
m	number of examples		
	$m = n_{\text{dates}}$		
i	index of example		
	There will be one example per date, so we use i and d interchangeably.		
$[\mathbf{X^{(i)}}, \mathbf{R^{(i)}}]$	example i		
	\$	\X^\ip	= (\ntickers \times \nchars)\$
	\$	\R^\ip	= \ntickers\$
$\mathbf{X}_s^{(d)}$	vector of ticker s's characteristics on day d		
	\$	\X^\dp_s	= \nchars\$

Note

The paper actually seeks to predict $\hat{\mathbf{r}}_s^{(d+1)}$ (forward return) rather than approximate the current return $\hat{\mathbf{r}}_s^{(d)}$.

We will present this as an approximation problem as opposed to a prediction problem for simplicity of presentation (i.e., to include PCA as a model).

A **factor model** seeks to approximate/explain the return of a *number* of tickers in terms of common "factors" $\mathbf{F}$

- $\mathbf{F} : (n_{\text{dates}} \times n_{\text{factors}})$

$$
\begin{aligned}
\mathbf{R}_1^{(d)} &= \beta_1^{(d)} \cdot \mathbf{F}^{(d)} + \epsilon_1 \\
\vdots \\
\mathbf{R}_{n_{\text{tickers}}}^{(d)} &= \beta_{n_{\text{tickers}}}^{(d)} \cdot \mathbf{F}^{(d)} + \epsilon_{n_{\text{tickers}}}
\end{aligned}
$$

There are several ways to create a factor model.

Pre-defined factors, solve for sensitivities

Suppose $\mathbf{F}$ is given: a matrix of returns of "factors" over a range of dates

- $\mathbf{F}^{(d)}$ includes the returns of multiple factor tickers
    - e.g., market, several industries, large/small cap indices

Solve for β_s, for each s

- n_{tickers} separate Linear Regression models
- Linear regression for ticker s:
    - r_s and $\mathbf{F}$ are time series (length n_{dates}) of returns for tickers/factors
    - Solve for β_s
        - constant over time

$$\beta_s^{(d)} = \beta_s$$

    - $\langle \mathbf{X}^{(d)}, \mathbf{y}^{(d)} \rangle = \langle \mathbf{F}^{(d)}, \mathbf{r}_s^{(d)} \rangle$

$$\mathbf{r}_s = \begin{pmatrix} \mathbf{r}_s^{(1)} \\ \mathbf{r}_s^{(2)} \\ \vdots \\ \mathbf{r}_s^{(n_{\text{dates}})} \end{pmatrix}, \; \mathbf{F} = \begin{pmatrix} \mathbf{F}_1^{(1)} & \dots & \mathbf{F}_{n_{\text{factors}}}^{(1)} \\ \mathbf{F}_1^{(2)} & \dots & \mathbf{F}_{n_{\text{factors}}}^{(2)} \\ \vdots \\ \mathbf{F}_1^{(n_{\text{dates}})} & \dots & \mathbf{F}_{n_{\text{factors}}}^{(n_{\text{dates}})} \end{pmatrix}, \; \beta_s = \begin{pmatrix} \beta_{s,1} \\ \beta_{s,2} \\ \vdots \\ \beta_{s,n_{\text{factors}}} \end{pmatrix}$$

$$\mathbf{r}_s = \mathbf{F} * \beta_s$$

Pre-defined sensitivities, solve for factors

Suppose β is given: for each ticker s, sensitivity of s to $\mathbf{F}_j$

Solve for $\mathbf{F}^{(d)}$ for each d

- n_{dates} separate Linear Regressions
- Linear regression for date d
    - $\mathbf{r}^{(d)}$ and $\beta^{(d)}$ are *cross sections* (width n_{tickers}) of one day ticker returns/sensitivities
    - Solve for $\mathbf{F}^{(d)}$
        - constant over tickers

$$\mathbf{F}_s^{(d)} = \mathbf{F}^{(d)}$$

    - $\langle \mathbf{X}^{(s)}, \mathbf{y}^{(s)} \rangle = \langle \beta^{(s)}, \mathbf{r}^{(s)} \rangle$

$$\mathbf{r}^{(d)} = \begin{pmatrix} \mathbf{r}_1^{(d)} \\ \mathbf{r}_1^{(d)} \\ \vdots \\ \mathbf{r}_{n_{\text{tickers}}}^{(d)} \end{pmatrix}, \ \mathbf{F}^{(d)} = \begin{pmatrix} \mathbf{F}_1^{(d)} \\ \mathbf{F}_2^{(d)} \\ \vdots \\ \mathbf{F}_{n_{\text{factors}}}^{(d)} \end{pmatrix}, \ \beta = \begin{pmatrix} \beta_{1,1}, & \ldots & \beta_{1,n_{\text{factors}}} \\ \beta_{2,1}, & \ldots & \beta_{2,n_{\text{factors}}} \\ \vdots \\ \beta_{n_{\text{tickers}},1}, & \ldots & \beta_{n_{\text{tickers}},n_{\text{factors}}} \end{pmatrix}$$

$$\mathbf{r}^{(d)} = \beta_s * \mathbf{F}^{(d)}$$

Solve for sensitivities and factors: PCA

Yet another possibility: solve for β and $\mathbf{F}$ *simulataneoulsy*.

Recall Principal Components

- Representing $\mathbf{X}$ (defined relative to n_{tickers} "standard" basis vectors) via an *alternate basis* $\mathbf{V}$
$$\mathbf{X} = \tilde{\mathbf{X}}\mathbf{V}^T$$

In this case, we identify $\mathbf{X}$ with the returns $\mathbf{R}$. Thus, without dimensionality reduction:
$$\mathbf{R} = \tilde{\mathbf{R}}V^T$$

where

$$\begin{aligned} \mathbf{R}, \tilde{\mathbf{R}} &: (n_{\text{dates}} \times n_{\text{tickers}}) \\ \mathbf{V}^T &: (n_{\text{tickers}} \times n_{\text{tickers}}) \end{aligned}$$

PCA seeks to *approximate* $\mathbf{R}$ with fewer than n_{tickers} basis vectors

- this is the *dimensionality reduction*
- reduce from n_{tickers} dimensions to n_{factors} dimensions

$$\mathbf{R} \approx \mathbf{F}\, \beta^T$$

- $\mathbf{F}^T$: $(n_{\text{dates}} \times n_{\text{factors}})$
  - the "alternative" basis: the "factors"
  - is $\mathbf{V}$ with columns eliminated b/c of dimensionality reduction

- β^T : $(n_{\text{factors}} \times n_{\text{tickers}})$

  - so $\beta^{(s)}$ are sensitivities of s to factors

- Solve for $\mathbf{F}$, β simultaneously

$\mathbf{r}_s$, the time series of returns of ticker s is approximated by a combination of returns of n_{factors}.

$$\mathbf{r}_s^{(d)} = \beta_s * \mathbf{F}^{(d)}$$

- $\beta_s^{(d)}$ is constant over time

$$\beta_s^{(d)} = \beta_s$$

The daily observation of n_{tickers} returns $\mathbf{R}^{(d)}$ is replaced by n_{factors} returns $\mathbf{F}^{(d)}$

This paper

This paper will create a factor model that

- Solve for $\mathbf{F}, \beta$ simultaneously
    - like PCA
- **But** where $\mathbf{F}$ and β are defined by Neural Networks

Autoencoder

The paper refers to the model as a kind of Autoencoder.

Let's review the topic.

- An Autoencoder has two parts: an Encoder and a Decoder
- The Encoder maps inputs $\mathbf{x^{(i)}}$, of length n
- Into a "latent vectors" $\mathbf{z^{(i)}}$ of length $n' \le n$
- If $n' < n$, the latent vector is a *bottleneck*
    - reduced dimension representation of $\mathbf{x^{(i)}}$
- The Decoder maps $\mathbf{z^{(i)}}$ into $\hat{\mathbf{x}}^{(\mathbf{i})}$, of length n, that is an approximation of $\mathbf{x^{(i)}}$

So, the training examples for an Autoencoder are
$$\langle \mathbf{X}^{(d)}, \mathbf{y}^{(d)} \rangle = \langle \mathbf{R}^{(d)}, \mathbf{R}^{(d)} \rangle$$

Imagine that we are given $\mathbf{R} : (n_{\text{dates}} \times n_{\text{tickers}})$

- timeseries (length n_{dates}) of returns of n_{tickers} tickers

Suppose we map a one day set of returns $\mathbf{R}^{(d)}$ into two separate values

- $\beta^{(d)} : (n_{\text{tickers}} \times n_{\text{factors}})$ -- the sensitivity of each ticker to each of n_{factors} one day "factor" returns
- $\mathbf{F}^{(d)} : (n_{\text{factors}} \times 1)$ -- the one day returns of n_{factors} factors

Our goal is to output $\hat{\mathbf{R}}^{(d)}$, an approximations of $\mathbf{R}^{(d)}$ such that

$$\begin{aligned}
\hat{\mathbf{R}}^{(d)} &= \beta^{(d)} * \mathbf{F}^{(d)} \\
\hat{\mathbf{R}}^{(d)} &\approx \mathbf{R}^{(d)}
\end{aligned}$$

This is the same goal as an Autoencoder but subject to the constraint that $\hat{\mathbf{R}}^{(d)}$

- is the product of the ticker sensitivities and factor returns

The Neural Network *simultaneously* solves for $\beta^{(d)}$ and $\mathbf{F}^{(d)}$.

This looks somewhat like PCA

- **but**, in PCA, β does not vary by day: it is constant over days
- in this model, $\beta^{(d)}$ varies by day

This paper goes one step further than the standard Autoencoder

- Inputs $\mathbf{X}$
  $: (n_{\text{dates}}$
  $\times n_{\text{tickers}}$
  $\times n_{\text{chars}})$
- rather than $\mathbf{R}$
  $: (n_{\text{dates}}$
  $\times n_{\text{tickers}})$

Each ticker s on each day d, has $n_{\text{chars}} \geq 1$ "characteristic"

- one of them may the daily return $\mathbf{R}^{(d)}$
- but may also include a number of other time varying characteristics

The proposed model is a Neural Network with two sub-networks.

The *Beta network* computes $\beta_s^{(d)} = \mathrm{NN}_\beta(\mathbf{X}_s^{(d)}; \mathbf{W}_\beta)$

- $\mathbf{X}_s^{(d)}$ as input
- parameterized by weights $\mathbf{W}_\beta$
- $\beta_s^{(d)}$ is only a function of $\mathbf{X}_s^{(d)}$, the characteristics of s
    - and **not** of any other ticker $s' \neq s$
    - $\beta_s^{(d)}$ shares $\mathbf{W}_\beta$ across **all** tickers s' and dates d'
    - contrast this with factor model with fixed factors
        - we solve for a separate β_s for each ticker s
        - via per-ticker timeseries regression
    - contrast this with PCA
        - β_s is influenced by $\mathbf{R}_{s'}$ for $s' \neq s$

The *Factor network* computes $\mathbf{F}^{(d)} = \mathrm{NN}_{\mathbf{F}}(\mathbf{R}^{(d)}, \mathbf{W}_{\mathbf{F}})$

- $\mathbf{R}^{(d)}$ as input (not $\mathbf{X}^{(d)}$ as in the Beta network)
- parameterized by weights $\mathbf{W}_{\mathbf{F}}$ $\mathbf{R}^{(d)}$ is only a function of $\mathbf{R}^{(d)}$ for date d
    - and **not** of any other date $d' \neq d$
    - $\mathbf{F}^{(d)}$ shares $\mathbf{W}_{\mathbf{F}}$ across **all** dates

This model

- has *neither* pre-defined Factors $\mathbf{F}$ or pre-defined Sensitivities β
- Simultaneously solve for $\beta_s^{(d)}$ and $\mathbf{F}^{(d)}$

Here is a picture

Summary of this paper

Approximate cross section of daily returns: $\hat{\mathbf{r}}^{(d)} \approx \mathbf{r}^{(d)}$

$$\mathbf{r}^{(d)} \approx \hat{\mathbf{r}}^{(d)} = \beta^{(d)} * \mathbf{F}^{(d)}$$

- like an Autoencoder
- subject
    - to returns as product of sensitivities and factors: $\hat{\mathbf{r}}^{(d)} = \beta^{(d)} * \mathbf{F}^{(d)}$
    - $\beta_s^{(d)} = \mathrm{NN}_\beta(\mathbf{X}_s^{(d)}; \mathbf{W}_\beta)$
    - $\mathbf{F}^{(d)} = \mathrm{NN}_\mathbf{F}(\mathbf{R}^{(d)}, \mathbf{W}_\mathbf{F})$

Shapes:

- $\mathbf{r}^{(d)} : (n_{\text{tickers}} \times 1)$
- $\beta : (n_{\text{tickers}} \times n_{\text{factors}})$
- $\mathbf{F}^{(d)} : (n_{\text{factors}} \times 1)$

Complete Neural Network

Beta (Input) side of network

Input $\mathbf{X}$

$\mathbf{X}: (n_{\text{dates}} \times n_{\text{tickers}} \times n_{\text{chars}})$

$\mathbf{X}^{(d)}: (n_{\text{tickers}} \times n_{\text{chars}})$

- Example on date d
- Consists of n_{tickers} tickers, each with n_{chars} characteristics

Sub Neural network $\mathbf{NN}_\beta$

$$\text{NN}_\beta = \texttt{Dense}\ (n_\text{factors})(\mathbf{X})$$

- Fully connected network
- $\texttt{Dense}\ (n_\text{factors})$ computes a function $(n_\text{tickers} \times n_\text{chars}) \mapsto (n_\text{tickers} \times n_\text{factors})$
- Threads over ticker dimension ([see (https://www.tensorflow.org/api_docs/python/tf/keras/layers/Dense)](https://www.tensorflow.org/api_docs/python/tf/keras/layers/Dense))
    - tickers share same weights across **all** tickers
    - single $\texttt{Dense}\ (n_\text{factors})$ **not** n_tickers copies of $\texttt{Dense}(n_\text{factors})$ with independent weights

$\mathbf{W}_\beta : (n_{\text{factors}} \times n_{\text{chars}})$

- weights shared across all d, s
  - $\mathbf{W}_{\beta,s}^{(d)} = \mathbf{W}_{\beta,s'}^{(d')}$ for all s', d'
  - the transformation of characteristics to beta *independent* of ticker
- hence, size of $\mathbf{W}_\beta$ is $(n_{\text{factors}} \times n_{\text{chars}})$

$$\beta^{(d)} = \mathrm{Dense}\,(n_{\text{factors}})(\mathbf{X}^{(d)})$$
$$\beta^{(d)} : (n_{\text{tickers}} \times n_{\text{factors}})$$

Factor side of network

Input $\mathbf{R}$

$$\mathbf{R} : (n_{\text{dates}} \times n_{\text{tickers}})$$

$$\mathbf{R}^{(d)} : (n_{\text{tickers}} \times 1)$$

- Example on date d
- Consists of returns of n_{tickers} tickers

Sub Neural network $\mathbf{NN_F}$

$$\mathrm{NN}_{\mathbf{F}} = \texttt{Dense}\ (n_{\text{factors}})$$

- Fully connected network
- `Dense(` n_{factors} `)` computes a function $n_{\text{tickers}} \mapsto n_{\text{factors}}$

$\mathbf{W_F} : (n_{\text{factors}} \times n_{\text{tickers}})$

- Weights shared across all d, s
  - $\mathbf{W}_{\mathbf{F},s}^{(d)} = \mathbf{W}_{\mathbf{F},s'}^{(d')}$ for all s', d'
  - the transformation of cross section of ticker returns to Factor returns *independent* of ticker
- hence, size of $\mathbf{W_F}$ is $(n_{\text{tickers}} \times n_{\text{factors}})$

$$\mathbf{F}^{(d)} = \mathrm{Dense}\,(n_{\mathrm{factors}})(\mathbf{R}^{(d)})$$
$$\mathbf{F}^{(d)} : n_{\mathrm{factors}}$$

Dot

$$\hat{\mathbf{r}}^{(d)} = \beta^{(d)} \cdot \mathbf{F}^{(d)}$$

Dot product threads over factor dimension

- Computes $\hat{\mathbf{r}}_s^{(d)} = \beta_s^{(d)} \cdot \mathbf{F}^{(d)}$ for each s
    - each s is a row of $\beta^{(d)}$

$$\hat{\mathbf{r}}^{(d)} : n_{\text{tickers}}$$

Loss

Let $\mathcal{L}_{(s)}^{(d)}$ denote error of ticker s on day d.

$$\mathcal{L}_{(s)}^{(d)} = \mathbf{r}_s^{(d)} - \hat{\mathbf{r}}_s^{(d)}$$

$\mathcal{L}^{(d)}$ is the loss, across tickers, on date d (one training example)

$$\mathcal{L}^{(d)} = \sum_s \mathcal{L}_{(s)}^{(d)}$$

The number of examples m equals n_{dates}

So the Total Loss is

$$\mathcal{L} = \sum_d \mathcal{L}^{(d)}$$

Predicting future returns, rather than explaining contemporaneous returns

The model is sometimes presented as predicting **day ahead** returns rather than contemporaneous returns.

In that case the objective is

$$\hat{\mathbf{r}}^{(d)} = \mathbf{r}^{(d+1)}$$

and Loss for a single ticker and date becomes

$$\mathcal{L}^{(d)}_{(s)} = \mathbf{r}^{(d+1)}_s - \hat{\mathbf{r}}^{(d)}_s$$