FIXED INCOME SECURITIES

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Yield Curve Modeling

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Topics

- **Static Modeling:**
  - Spline Fitting
  - Nelson-Siegel Parameterization

- **Dynamics:**
  - Macroeconomic Factors
  - Principal Components
  - Hedging
  - Affine models

Static Yield Curve Modeling

- We define the yield curve as the relationship between zero-coupon yields and time to maturity.
- Non-parametric methods: We use the Spline method to fit market data with a curve composed of many segments. Constraints are imposed to ensure that the overall curve is "smooth".
- Parametric methods: We choose a parameterized family of functions that spans the space of yield curves. The parameters are then optimized to fit the data.

Static Yield Curve Modeling : Cubic Splines

Each polynomial has its own parameters. The set of splines, form a continuous curve. We have n points (xi , yi), and we are going to find (n-1) cubic polynomials:

$$S(x) = \begin{cases} s_1(x) & \text{if} \quad x_1 \le x < x_2 \\ s_2(x) & \text{if} \quad x_2 \le x < x_3 \\ & \vdots \\ s_{n-1}(x) & \text{if} \quad x_{n-1} \le x < x_n \end{cases}$$

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

$$s_i'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i$$
$$s_i''(x) = 6a_i(x - x_i) + 2b_i$$

Cubic Spline

- Smoothness Conditions

1. The piecewise function $S(x)$ will interpolate all data points.
2. $S(x)$ will be continuous on the interval $[x_1, x_n]$
3. $S'(x)$ will be continuous on the interval $[x_1, x_n]$
4. $S''(x)$ will be continuous on the interval $[x_1, x_n]$

For condition 1:

$$S(x_i) = y_i$$

$$y_i = s_i(x_i)$$

$$y_i = a_i(x_i - x_i)^3 + b_i(x_i - x_i)^2 + c_i(x_i - x_i) + d_i$$

$$y_i = d_i$$

$$i = 1, 2, \ldots, n-1.$$

Cubic Spline

For condition 2 : $s_i(x_i) = s_{i-1}(x_i)$

$$s_i(x_i) = d_i$$

$$s_{i-1}(x_i) = a_{i-1}(x_i - x_{i-1})^3 + b_{i-1}(x_i - x_{i-1})^2 + c_{i-1}(x_i - x_{i-1}) + d_{i-1}$$

So:

$$h = x_i - x_{i-1}$$

$$\boxed{d_i = a_{i-1}h^3 + b_{i-1}h^2 + c_{i-1}h + d_{i-1}}$$

For condition 3 : $s_i'(x_i) = s_{i-1}'(x_i)$

$$s_i'(x_i) = c_i$$

$$s_{i-1}'(x_i) = 3a_{i-1}(x_i - x_{i-1})^2 + 2b_{i-1}(x_i - x_{i-1}) + c_{i-1}$$

$$\boxed{c_i = 3a_{i-1}h^2 + 2b_{i-1}h + c_{i-1}}$$

Cubic Spline

For condition 4 :

$$s''_{i+1}(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i$$

$$2b_{i+1} = 6a_i h + 2b_i$$

The system of linear equations defined is underdetermined. A unique solution can be obtained by imposing additional constraints such as requiring the second derivatives at the end points $\{s''_i(x_i)$, for i=1,n$\}$ to equal zero.

In conclusion:

- Pros:
  - The yield curve is guaranteed to pass through every data point.
  - Construction delivers a smooth (differentiable) yield curve.
- Cons:
  - Too many variables need to be specified.
  - Splines are not well behaved outside interval – may diverge.
  - Sensitive to the location of the knot points between different segments.
  - Overfitting.

Parametric Modeling: Nelson-Siegel

- Second Order Differential Equation(2^{nd} ODE) :
  - $f"(t) - 2\lambda f'(t) + \lambda^2 f(t) = 0$
- Has a Characteristic Equation :
  - $r^2 - 2\lambda r + \lambda^2 = 0$
  - $r = \lambda$
- Solution of :
  - $f(t) = B_1 + B_2 e^{\lambda t} + B_3 \lambda e^{\lambda t}$
- Assume the forward curve follows the above 2^{nd} ODE, with solution:
  - $f(t) = \beta_1 + \beta_2 e^{\lambda t} + \beta_3 \lambda e^{\lambda t}$

Parametric Modeling: Nelson-Siegel

- Yield curves are typically "monotonic", "humped" or "S-shaped". The family of solutions to second order differential equations contain these shapes.
- Suppose the instantaneous forward rates were given by
  - $f(t) = \beta_1 + \beta_2 e^{\lambda t} + \beta_3 \lambda e^{\lambda t}$
- Then, since the yield is related to the forward rate by :

$$y_t(\tau) = \frac{1}{\tau}\int_0^{\tau} f_t(u)du$$

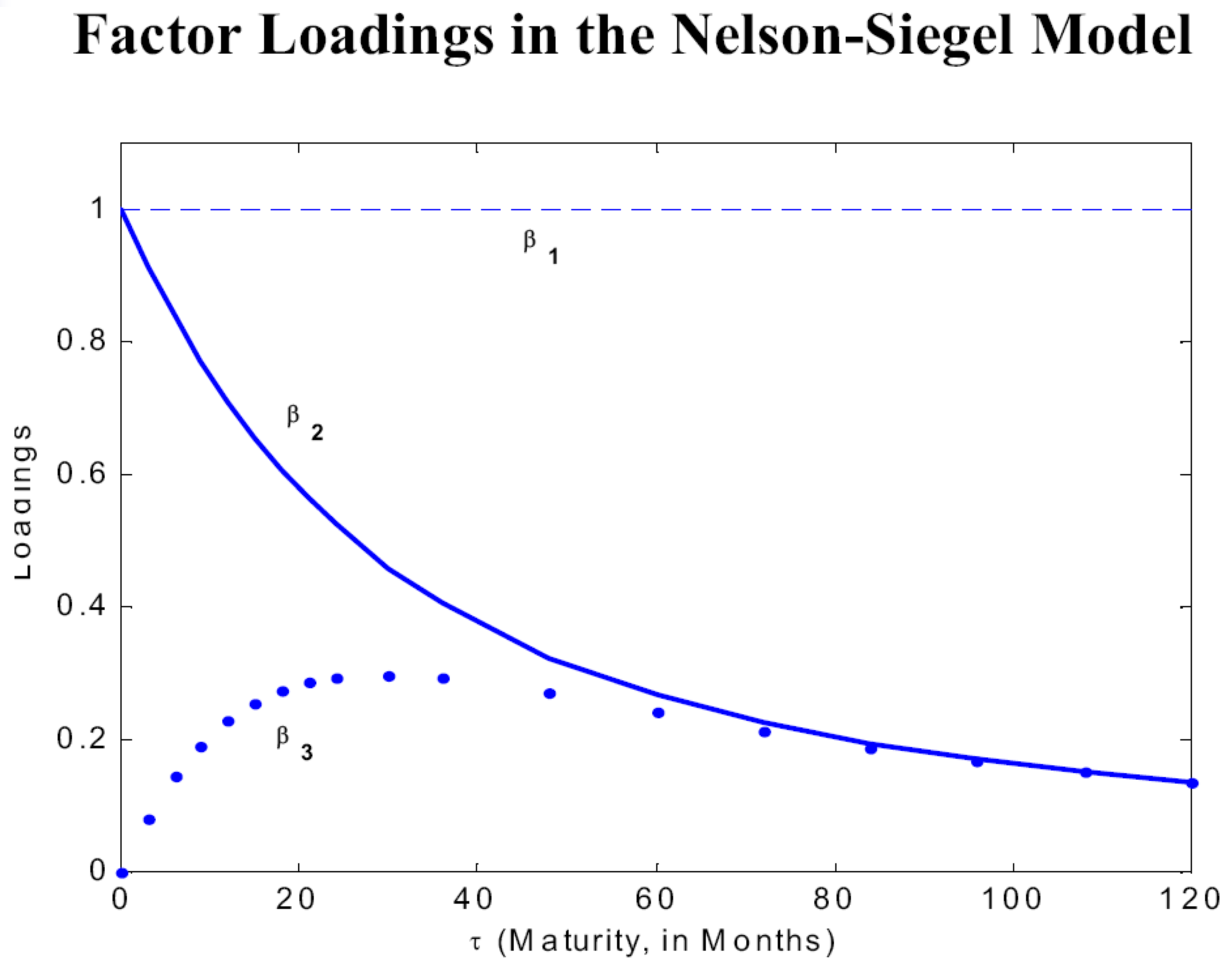
- The yield curve is then given by

$$y(\tau) = \beta_1 + \beta_2\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_3\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right)$$

Parametric Modeling: Nelson-Siegel

- The entire Forward and Yield curve can now be described by only 4 parameters.
- Let us take a closer look at these parameters:
  - Level : $\lim_{t\to\infty} f(t) = \beta_1$. Asymptomatic value of the forward curve
  - Slope : $\lim_{t\to 0} f(t) = \beta_1 + \beta_2$ initial value of the forward curve or the instantaneous spot rate. ($\beta_2 > 0$) will produce downward and vs. versa.
  - Curvature : $\beta_3 \lambda e^{\lambda t}$ will produce a hump ($\beta_3 > 0$) or trough ($\beta_3 < 0$)
  - Constant λ control both the decay factor of the slope and the maximum of the curvature

Nelson-Siegel (cont.)



Family of yield curves generated by the Nelson-Siegel Model

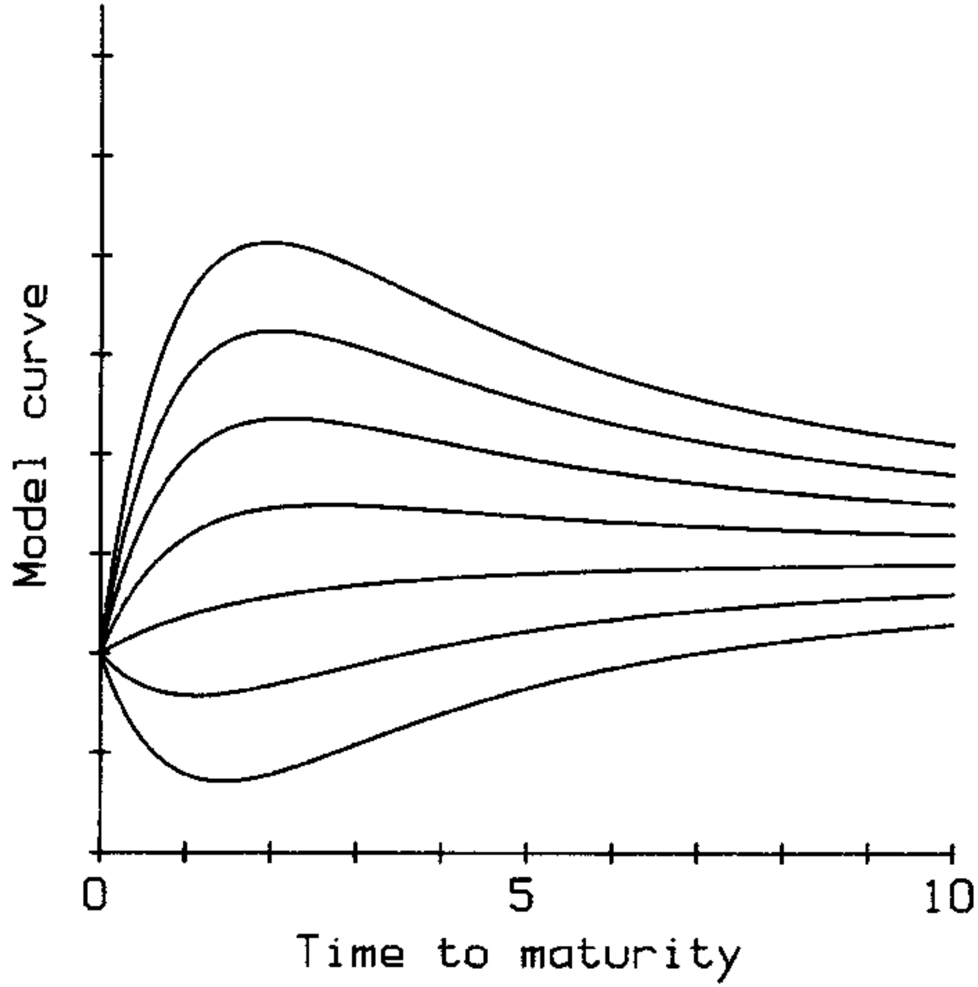


FIG. 1.—Yield curve shapes

Estimation Static Nelson-Siegel

- The N-S model is not just a local approximation. It captures the basic shape of the yield curve.
- Estimation
  - Given a set of bond prices on a given day estimate the parameters $(\lambda, \beta_1, \beta_2, \beta_3)$ To minimize MSE between the actual price and fitted price .
  - Given a set of zero coupon yield estimate $(\lambda, \beta_1, \beta_2, \beta_3)$ to minimize MSE the actual yield and fitted yield.
  - Given a set of zero coupon yield fix λ and then estimate $(\beta_1, \beta_2, \beta_3)$ to minimize MSE the actual yield and fitted yield.

Estimation Static Nelson-Siegel

- Assume we have $i = 1, \dots, m$ US government T-bonds and T-note each paying semi-annual coupon $C_j/2$
- For Bond j assume :
  - $P_j^{t_0}(y_j) = \sum_{i=1}^{N} \frac{C_j}{2} \left[\frac{1}{1+y_{T_i}}\right]^{\frac{T_i - t_0}{365}} + 100 \left[\frac{1}{1+y_{T_N}}\right]^{\frac{T_N - t_0}{365}}$
  - $C_j/2$ is the coupon payment and y_j is YTM for Bond j
  - N is the number of coupons in the life of the bond
  - t_0 and T_i are todays date and date the i^{th} coupon
  - $T_i - t_0$ is the number of calendar days between today t_0 and T_i
  - y_{T_i} is unobservable rate for T_i

Estimation Static Nelson-Siegel

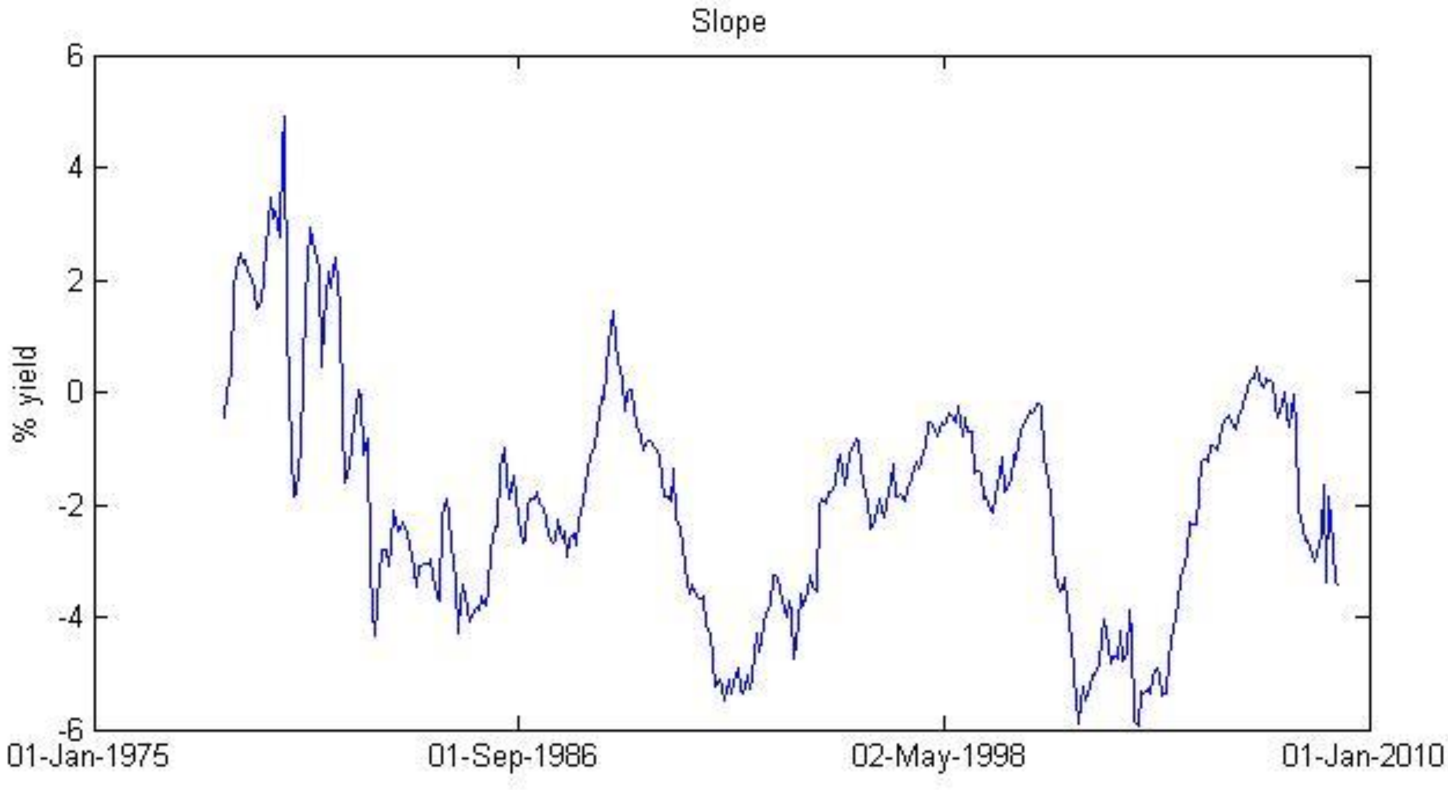
- For Bond j assume :
  - $P_j^{t_0}(y_j) = \sum_{i=1}^{N} \frac{C_j}{2}\left[\frac{1}{1+y_{T_i}}\right]^{\frac{T_i - t_0}{365}} + 100\left[\frac{1}{1+y_{T_N}}\right]^{\frac{T_N - t_0}{365}}$
  - $y_{T_i} \approx y(y_{T_i}; \lambda, \beta_1, \beta_2, \beta_3) = \beta_1 + \beta_2\left(\frac{1-e^{-\lambda T_i}}{\lambda T_i}\right) + \beta_3\left(\frac{1-e^{-\lambda T_i}}{\lambda T_i} - e^{-\lambda T_i}\right)$
  - $\tilde{P}_j^{t_0}(\tilde{y}_j) = \sum_{i=1}^{N} \frac{C_j}{2}\left[\frac{1}{1+y(y_{T_i};\lambda,\beta_1,\beta_2,\beta_3)}\right]^{\frac{T_i - t_0}{365}} + 100\left[\frac{1}{1+y(y_{T_N};\lambda,\beta_1,\beta_2,\beta_3)}\right]^{\frac{T_N - t_0}{365}}$
  - $\min_{\lambda,\beta_1,\beta_2,\beta_3} \sum_{i=1}^{m}\left[P_j^{t_0}(y_j) - \tilde{P}_j^{t_0}(\tilde{y}_j)\right]^2$
- This is best estimation but difficult and may get unstable answer and possible local rather than global min

Estimation Static Nelson-Siegel

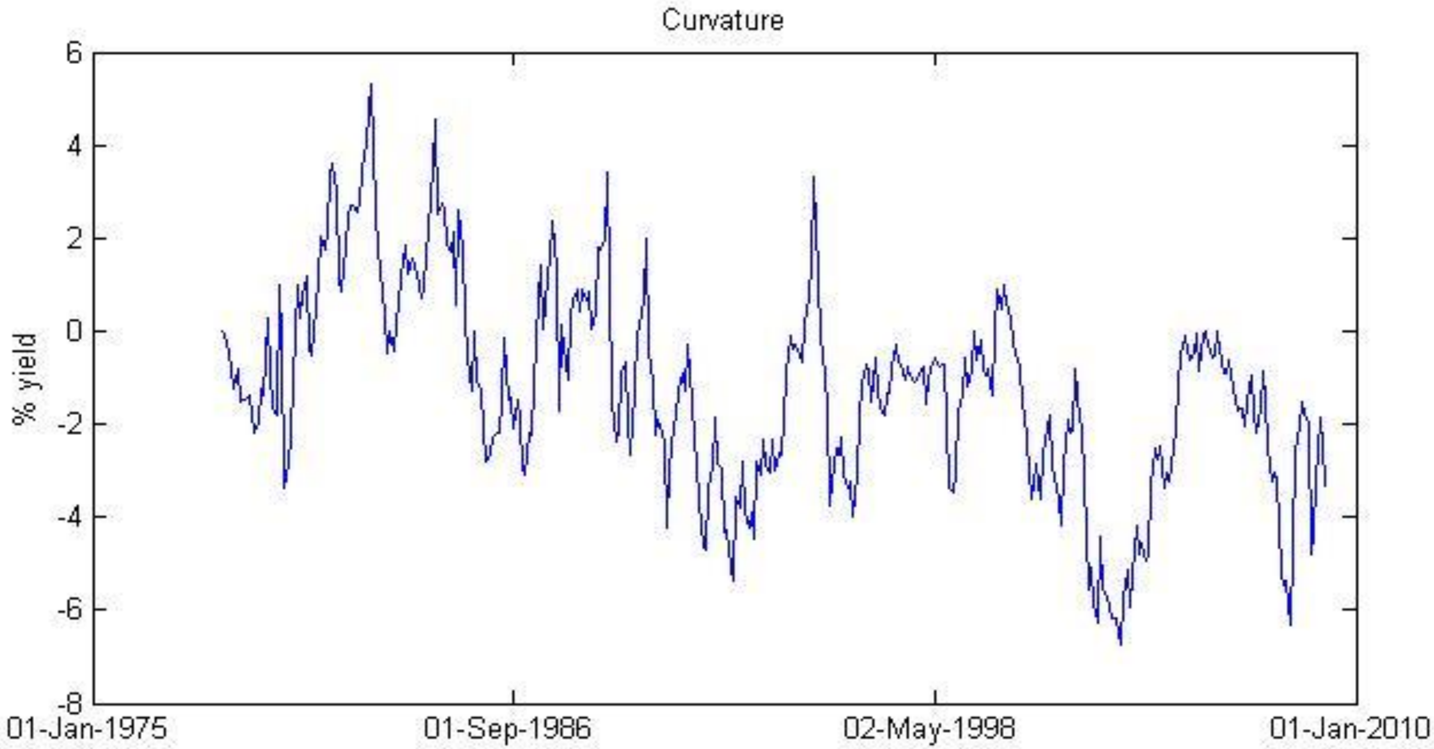
- Assume we have a set of zero coupon y_{T_i} *for i = 1,…,m*
  - $y_{T_i} = y(y_{T_i}; \lambda, \beta_1, \beta_2, \beta_3) + \varepsilon_{T_i} = \beta_1 + \beta_2\left(\frac{1-e^{-\lambda T_i}}{\lambda T_i}\right) + \beta_3\left(\frac{1-e^{-\lambda T_i}}{\lambda T_i} - e^{-\lambda T_i}\right) + \varepsilon_{T_i}$
  - $y_{T_i} = \beta_1 + \beta_2\, x_1(\lambda, T_i) + \beta_3\, x_2(\lambda, T_i) + \varepsilon_{T_i}$
- Method 1 : use a non-linear optimization and simultaneously minimize the MSE
  - $\min_{\lambda,\beta_1,\beta_2,\beta_3} \sum_{i=1}^{m}\left[y_{T_i} - \beta_1 + \beta_2\, x_1(\lambda, T_i) + \beta_3\, x_2(\lambda, T_i)\right]^2 = \min_{\lambda,\beta_1,\beta_2,\beta_3} \sum_{i=1}^{m} \varepsilon_{T_i}{}^2$
- Method 2 : Fix $\lambda = \bar{\lambda}$ and then run a OLS or GLS regression and then solve for optimal λ by minimizing the MSE
  - $\min_{\beta_1,\beta_2,\beta_3} \sum_{i=1}^{m} \varepsilon_{T_i}{}^2 = MSE(\bar{\lambda})$
  - $\hat{\lambda} = \min_{\bar{\lambda}} MSE(\bar{\lambda})$
  - $\hat{\lambda}, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3 = \min_{\bar{\lambda}}\{\min_{\beta_1,\beta_2,\beta_3} \sum_{i=1}^{m} \varepsilon(\bar{\lambda})_{T_i}{}^2\}$

Factor Evolution - Level

Factor Evolution - Slope



Factor Evolution - Curvature



Observable vs. Latent Factors

Macroeconomic Factors:

- Level and inflation (see Rudebusch and Wu)
  - Fisher Equation
  - Unit Root Process
- Slope and the business cycle (see Diebold, Aruoba and Rudebusch)
  - Capacity Utilization
  - Cyclical dynamics at the business cycle frequency
- Curvature and interest rate volatility (see Litterman, Scheinkman and Weiss)
  - Implied volatilities from options on treasury bond futures

Principal Component Analysis (see Tsay)

- Reduction in Dimensionality: Orthogonal Factors
- K assets with log-returns given by $\boldsymbol{r} = (r_1, \ldots, r_k)'$ and covariance matrix $\boldsymbol{\Sigma}_r$.
- A portfolio $\boldsymbol{c}_i$ has return given by $y_i = \boldsymbol{c}_i'\boldsymbol{r} = \sum_{j=1}^{k} c_{ij} r_j$
- Construct a sequence of portfolios :

1. the first principal component of $\boldsymbol{r}$ is the linear combination $y_1 = \boldsymbol{c}_1'\boldsymbol{r}$ that maximizes $\text{Var}(y_1)$ subject to the constraint $\boldsymbol{c}_1'\boldsymbol{c}_1 = 1$,

2. the second principal component of $\boldsymbol{r}$ is the linear combination $y_2 = \boldsymbol{c}_2'\boldsymbol{r}$ that maximizes $\text{Var}(y_2)$ subject to the constraints $\boldsymbol{c}_2'\boldsymbol{c}_2 = 1$ and $\text{Cov}(y_2, y_1) = 0$, and

3. the ith principal component of $\boldsymbol{r}$ is the linear combination $y_i = \boldsymbol{c}_i'\boldsymbol{r}$ that maximizes $\text{Var}(y_i)$ subject to the constraints $\boldsymbol{c}_i'\boldsymbol{c}_i = 1$ and $\text{Cov}(y_i, y_j) = 0$ for $j = 1, \ldots, i-1$.

Principal Component Analysis

Main Result:

- Let $(\lambda_1, \boldsymbol{e}_1), \ldots, (\lambda_k, \boldsymbol{e}_k)$ represent the eigenvalue –eigenvector pairs of Σ_r such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$.
- Then, the *i*-th principal component of r is

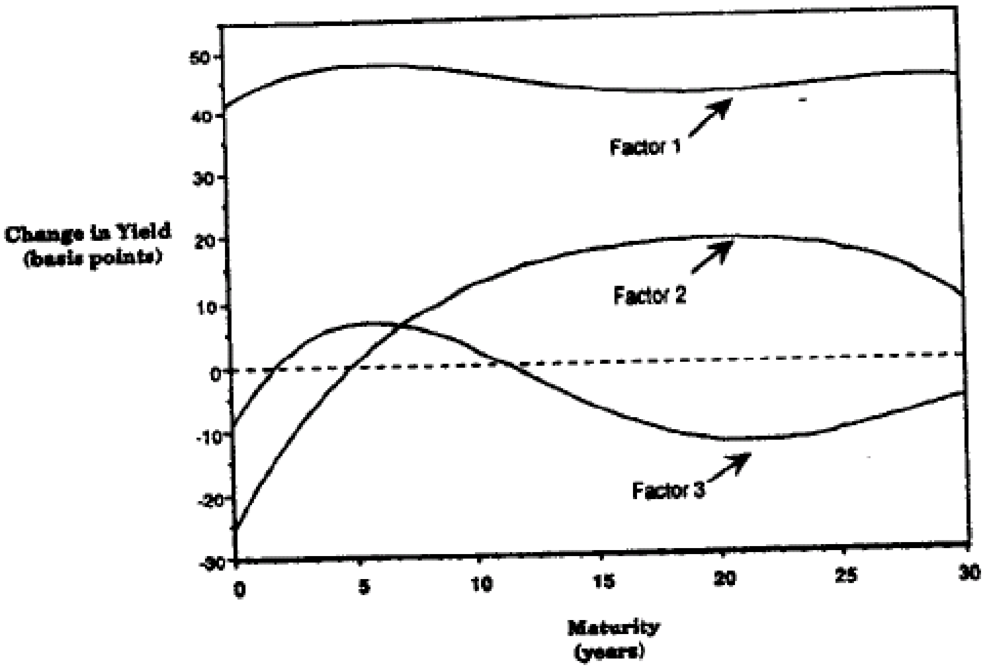
$$y_i = \boldsymbol{e}_i' \boldsymbol{r} = \sum_{j=1}^{k} e_{ij} r_j$$

- And,

$$\frac{\mathrm{Var}(y_i)}{\sum_{i=1}^{k} \mathrm{Var}(r_i)} = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_k}.$$

Common Factors Affecting Bond Returns (Litterman, Scheinkman 1991)

- Perform PCA on the excess returns of the zero coupon bonds of varying maturities.
- The three most significant factors explain 96% of the variance of excess returns of any bond.
- The impact of these factors on yields at different maturities is shown below:



Yield curves - Fits

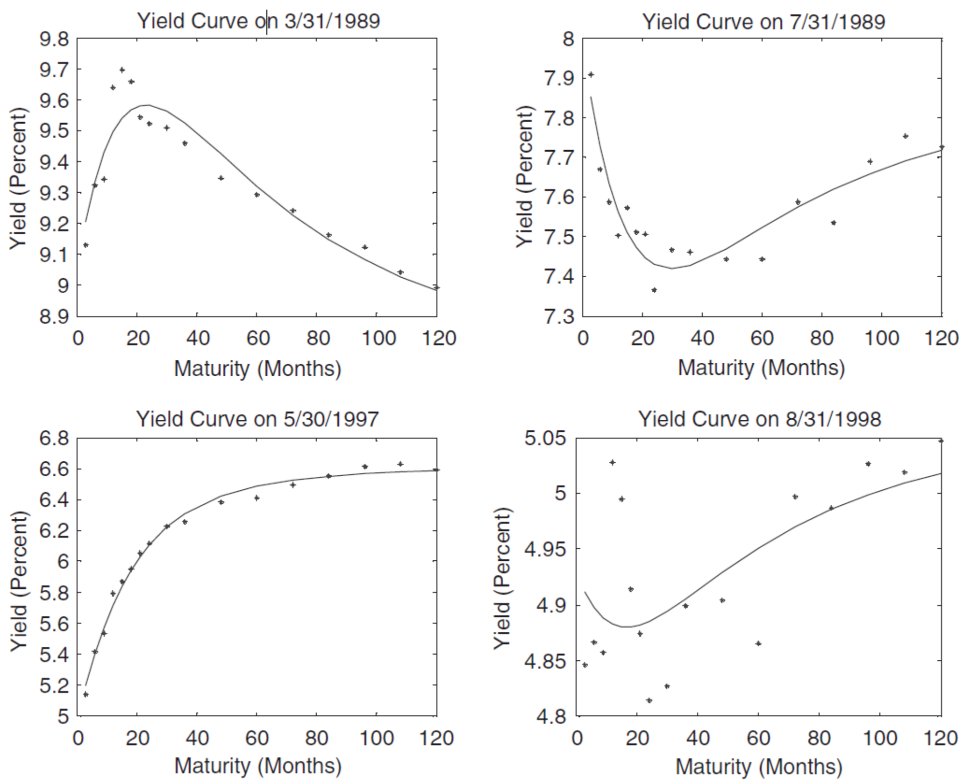


Fig. 5. Selected fitted (model-based) yield curves. We plot fitted yield curves for selected dates, together with actual yields. See text for details.

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Yield curves – Descriptive statistics

Descriptive statistics, yield curves

Maturity (Months)	Mean	Std. dev.	Minimum	Maximum	$\hat{\rho}(1)$	$\hat{\rho}(12)$	$\hat{\rho}(30)$
3	5.630	1.488	2.732	9.131	0.978	0.569	−0.079
6	5.785	1.482	2.891	9.324	0.976	0.555	−0.042
9	5.907	1.492	2.984	9.343	0.973	0.545	−0.005
12	6.067	1.501	3.107	9.683	0.969	0.539	0.021
15	6.225	1.504	3.288	9.988	0.968	0.527	0.060
18	6.308	1.496	3.482	10.188	0.965	0.513	0.089
21	6.375	1.484	3.638	10.274	0.963	0.502	0.115
24	6.401	1.464	3.777	10.413	0.960	0.481	0.133
30	6.550	1.462	4.043	10.748	0.957	0.479	0.190
36	6.644	1.439	4.204	10.787	0.956	0.471	0.226
48	6.838	1.439	4.308	11.269	0.951	0.457	0.294
60	6.928	1.430	4.347	11.313	0.951	0.464	0.336
72	7.082	1.457	4.384	11.653	0.953	0.454	0.372
84	7.142	1.425	4.352	11.841	0.948	0.448	0.391
96	7.226	1.413	4.433	11.512	0.954	0.468	0.417
108	7.270	1.428	4.429	11.664	0.953	0.475	0.426
120 (level)	7.254	1.432	4.443	11.663	0.953	0.467	0.428
Slope	1.624	1.213	−0.752	4.060	0.961	0.405	−0.049
Curvature	−0.081	0.648	−1.837	1.602	0.896	0.337	−0.015

Note: We present descriptive statistics for monthly yields at different maturities, and for the yield curve level, slope and curvature, where we define the level as the 10-year yield, the slope as the difference between the 10-year and 3-month yields, and the curvature as the twice the 2-year yield minus the sum of the 3-month and 10-year yields. The last three columns contain sample autocorrelations at displacements of 1, 12, and 30 months. The sample period is 1985:01–2000:12.

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Estimated Factors– Descriptive statistics

Descriptive statistics, estimated factors

Factor	Mean	Std. Dev.	Minimum	Maximum	$\hat{\rho}(1)$	$\hat{\rho}(12)$	$\hat{\rho}(30)$	ADF
$\hat{\beta}_{1t}$	7.579	1.524	4.427	12.088	0.957	0.511	0.454	−2.410
$\hat{\beta}_{2t}$	−2.098	1.608	−5.616	0.919	0.969	0.452	−0.082	−1.205
$\hat{\beta}_{3t}$	−0.162	1.687	−5.249	4.234	0.901	0.353	−0.006	−3.516

Note: We fit the three-factor Nelson–Siegel model using monthly yield data 1985:01–2000:12, with λ_t fixed at 0.0609, and we present descriptive statistics for the three estimated factors $\hat{\beta}_{1t}$, $\hat{\beta}_{2t}$, and $\hat{\beta}_{3t}$. The last column contains augmented Dickey–Fuller (ADF) unit root test statistics, and the three columns to its left contain sample autocorrelations at displacements of 1, 12, and 30 months.

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Stylized facts of the yield curve

- The average yield curve is increasing and concave
- Yield dynamics are persistent, and spread dynamics are much less persistent.
- Persistent yield dynamics would correspond to strong persistence of level factor, and less persistent spread dynamics would correspond to weaker persistence of slope factor.
- The short end of the yield curve is more volatile than the long end.
- Long rates are more persistent than short rates.

Downloading Benchmark bonds

Once selected the country go to “Member Weightings”

- YCGT0007 Canada
- YCGT0001 Australia
- YCGT0025 USA
- YCGT0016 Germany

	Tenor	Description	Price	Yield	Source	Update
11)	3M	CTB 0 05/30/19 Corp	99.575	1.672	BGN	12:30
12)	6M	CTB 0 08/08/19 Corp	99.231	1.734	BGN	12:31
13)	1Y	CTB 0 02/06/20 Corp	98.343	1.783	BGN	12:31
14)	2Y	CAN 2 ¼ 02/01/21 Corp	100.921	1.761	BGN	12:31
15)	3Y	CAN 0 ½ 03/01/22 Corp	96.299	1.771	BGN	12:31
16)	4Y	CAN 1 ¾ 03/01/23 Corp	99.896	1.777	BGN	12:31
17)	5Y	CAN 2 ¼ 03/01/24 Corp	102.252	1.778	BGN	12:31
18)	6Y	CAN 2 ½ 06/01/24 Corp	103.569	1.785	BGN	12:31
19)	7Y	CAN 2 ¼ 06/01/25 Corp	102.569	1.814	BGN	12:31
20)	8Y	CAN 1 ½ 06/01/26 Corp	97.706	1.839	BGN	12:31
21)	9Y	CAN 1 06/01/27 Corp	93.465	1.857	BGN	12:31
22)	10Y	CAN 2 06/01/28 Corp	101.051	1.876	BGN	12:31
23)	20Y	CAN 5 06/01/37 Corp	144.042	2.086	BGN	12:31
24)	30Y	CAN 2 ¾ 12/01/48 Corp	113.662	2.128	BGN	12:31

Affine Term Structure Models

- Short Rate Process:

$$r = R(x) = \delta_0 + \delta_1^\top x.$$

- Evolution of Factors:

$$dx_t = \kappa(\overline{x} - x_t)dt + \Sigma dz_t.$$

- Bond Pricing:

$$P_t^{(\tau)} = E_t^* \left[\exp\left(- \int_t^{t+\tau} r_u \, du \right) \right].$$

$$P_t^{(\tau)} = F(x_t, \tau). \quad F(x, \tau) = \exp\left(a\left(\tau\right) + b\left(\tau\right)^\top x \right)$$

- Yield Curve:

$$y_t^{(\tau)} = -\frac{\log F\left(x_t, \tau\right)}{\tau} = A\left(\tau\right) + B\left(\tau\right)^\top x_t$$

Affine Models:
Returning to Nelson-Siegel:

$$y_t^{(\tau)} = -\frac{\log F(x_t, \tau)}{\tau} = A(\tau) + B(\tau)^\top x_t$$

$$y_t(\tau) = L_t + S_t\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) + C_t\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right),$$

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1 - e^{-\tau_1\lambda}}{\tau_1\lambda} & \frac{1 - e^{-\tau_1\lambda}}{\tau_1\lambda} - e^{-\tau_1\lambda} \\ 1 & \frac{1 - e^{-\tau_2\lambda}}{\tau_2\lambda} & \frac{1 - e^{-\tau_2\lambda}}{\tau_2\lambda} - e^{-\tau_2\lambda} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1 - e^{-\tau_N\lambda}}{\tau_N\lambda} & \frac{1 - e^{-\tau_N\lambda}}{\tau_N\lambda} - e^{-\tau_N\lambda} \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix}$$

Affine Models:

- Can one construct an affine term structure model for which the yields are related to the factors by the Nelson-Siegel factor loadings ?
- Indeed, one can approximate the Nelson-Siegel yield curve structure by an Affine Term structure model by imposing additional constraints on
  - the structure of the evolution process of the underlying factors and
  - the relationship between the short rate and the underlying factors
  - (For more details, see Christensen et al.)

- The Nelson Siegel yield structure does NOT by itself rule out the possibility of arbitrage.
- However, Christensen *et. al* make the necessary modifications to the Nelson-Siegel model to make it arbitrage-free.

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