FIXED INCOME SECURITIES

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Replicating Portfolio

- If we are able to mimic the cash flow of an asset, with a portfolio of the primary assets we can then use the non-arbitrage pricing theory and price the secondary assets
- How? By using dynamic programming and self-financing principal:

A- Assume we are in the one before the last date (T1-1). Generate a portfolio on that date such that we end up with enough money in the last period to pay our obligation

B- Now assume we are in time (T1-2). Generate a portfolio such that we have enough money in portfolio of part A.

C- repeat part B, so we get to state 0.

Replicating Portfolio

- Assume a general obligation with payout of x(T1-1;S(T1-1)). Note that S(T1-1) is S(T1-2,u) or S(T1-2,d).
- Now we construct the portfolio at time T1-2 such that:

$$M = \text{number of money market asset}$$

$$M(T1-2; s_{T1-2}) \times B(T1-1; s_{T1-2}) + N(T1-2; s_{T1-2}) \times P(T1-1, T1; s_{(T1-2)}u) = x(T1-1; s_{(T1-2)}u)$$

$$M(T1-2; s_{T1-2}) \times B(T1-1; s_{T1-2}) + N(T1-2; s_{T1-2}) \times P(T1-1, T1; s_{(T1-2)}d) = x(T1-1; s_{(T1-2)}d)$$

- These equations guarantee that we have enough money to satisfy our obligation on states S(T1-2,u) and S(T1-2,d).
- We have 2 equations and 2 unknown ;which are the optimal allocation between MM and the zero coupon at time T1-2 and state S(t1-2).

Replicating Portfolio

- The cost of this portfolio at time T1-2 would be:

$$M(T1-2;s_{T1-2})\times B(T1-2;s_{T1-3})+N(T1-2;s_{T1-2})\times P(T1-2,T1;s_{T1-2})$$

So we should make sure when we are at time (T1-3), in state s(T1-3) , we have money to generate the above portfolio at T1-2 and state s(T1-2).

✓ Note S(T1-2) is either s(T1-3,u) or s(T1-3,d). and so on, until we get to time 0, and we need to have the portfolio of : **x(0)=M(0)*1 + N(0)*p(0,T1)**

✓ X(0) is the value of secondary asset at time zero

Risk –Neutral Probabilities

- In order to have no arbitrage in all states we need to have:
- $u(t, T_1, s_t) = \frac{P(t+1, T_1, s_{tu})}{P(t, T_1, s_t)} \geq r(t, s_t) \geq \frac{P(t+1, T_1, s_{td})}{P(t, T_1, s_t)} = d(t, T_1, s_t)$

For example, assume we have :

$$u(t, T_1, s_t) \geq d(t, T_1, s_t) \geq r(t, s_t)$$

1- we can do nothing in all states except s(t)

2- if s(t) happens short the MM and buy with the proceeds zero-coupon. this portfolio cost us nothing.

3- at state t+1 sell the zero-coupon and pay back the money plus interest. since the rate of return of the zero-coupon is higher than the interest rate. We definitely made money. Which is the violation of no-arbitrage model.

Risk –Neutral Probabilities

- So: risk free rate is **always** between the return in up state and return in down state. And rates are positive. Therefore, return on MM can be written as the linear combination of the bond return in the up state and down state, such that the weight sum to one.
- $r(t, s_t) = \check{\pi}\, u(t, T_1, s_t) + (1 - \check{\pi}) d(t, T_1, s_t)$
- Since $u(t, T_1, s_t) \geq r(t, s_t) \geq d(t, T_1, s_t)$
- $0 \leq \tilde{\pi} \leq 1$

so we can think of it as kind of probability; And r is the expectation of return in the next period.

✓ they are not true probabilities! They are between zero and one , and adds up to 1 and are called 'pseudo probability'

Risk –Neutral Probabilities

- $\frac{r(t,s_t)B(t,s_{t-1})}{B(t,s_{t-1})} = \tilde{\pi}\frac{P(t+1,T_1,s_{tu})}{P(t,T_1,s_t)} + (1-\tilde{\pi})\frac{P(t+1,T_1,s_{td})}{P(t,T_1,s_t)}$
- $\frac{B(t+1,s_t)}{B(t,s_{t-1})} = \widetilde{E_t}\left[\frac{P(t+1,T_1,s_{t+1})}{P(t,T_1,s_t)}\right]$
- Since $B(t+1,s_t)$ and $P(t,T_1,s_t)$ are known at time t, we can take them in and out of expectation :
- $\frac{P(t,T_1,s_t)}{B(t,s_{t-1})} = \widetilde{E_t}\left[\frac{P(t+1,T_1,s_{t+1})}{B(t+1,s_t)}\right]$
- $A(t,T_1,s_t) = \widetilde{E_t}[A(t+1,T_1,s_{t+1})]$
- Or $A(t,T_1,s_t) = \frac{P(t,T_1,s_t)}{B(t,s_{t-1})}$ is a martingale

Risk –Neutral Probabilities

- Under pseudo-probability:

$$\frac{p(t;T,s(t))}{B(t;S(t-1))} = \tilde{\tilde{\mathrm{E}}}_t\left[\frac{p(t+1;T,s(t+1))}{B(t+1;S(t))}\right]$$

✓ This is not the true expectation!

✓ T maturity zero-coupon bond price at time t , discounted by the money market account value is equal to its discounted expected value at time t+1

✓ In other words, the ratio of the bond price to the MM is a martingale under pseudo-probability

Risk –Neutral Probabilities

$$\frac{p(t;T,s(t))}{B(t;S(t-1))} = \tilde{\tilde{\mathrm{E}}}_t\left[\frac{p(t+1;T,s(t+1))}{B(t+1;S(t))}\right]$$

$$\Rightarrow p(t;T,s(t)) = \tilde{\tilde{\mathrm{E}}}_t\left[\frac{p(t+1;T,s(t+1))\times B(t;S(t-1))}{B(t+1;S(t))}\right]$$

$$= \tilde{\tilde{\mathrm{E}}}_t\left[\frac{p(t+1;T,s(t+1))}{r(t,s(t))}\right] = \frac{\tilde{\tilde{\mathrm{E}}}_t\left[p(t+1;T,s(t+1))\right]}{r(t,s(t))}$$

$$by\,\mathrm{solving}\,r(t,s(t)) = \pi(t;s(t))\times u(t;T1;s(t)) + \left(1-\pi(t;s(t))\right)\times d(t;T1;s(t))$$

$$We \text{ will have: } \pi(t;s(t)) = \frac{r(t,s(t)) - d(t;T1;s(t))}{u(t;T1;s(t)) - d(t;T1;s(t))}$$

$$1\text{-}\pi(t;s(t)) = \frac{u(t;T1;s(t)) - r(t,s(t))}{u(t;T1;s(t)) - d(t;T1;s(t))}$$

Risk –Neutral Probabilities

- No arbitrage implies the existence of pseudo probabilities and the existence of pseudo probabilities implies no arbitrage

- If we were risk-neutral, the expectation hypothesis holds, then by definition the price of bond today is the expected discounted price of bond in the next period. So when we are risk neutral true probabilities will also satisfy the expectation equation. It means true probabilities must equal pseudo probabilities . So think of pseudo probabilities as risk-neutral probabilities.

Risk –Neutral Valuation

- If we are risk neutral then the price of any contingent claim, would be its discounted cash flow at the risk free rate.

$$x(t;s(t)=\frac{\left[\pi(t;s(t))\times x(t+1;s(t)u+(1-\pi(t;s(t)))\times x(t+1;s(t)d\right]}{r(t,s(t)}$$

So, if we know the value of the claim in up and down state ,we can calculate the discounted expected value by using this pseudo probabilities and MM rate.

Risk –Neutral Valuation

$$E_t\left[A_{t+1}\right] = A_t$$

$$E_{t-1}\left[A_{t+1}\right] = E_{t-1}\left[E_t\left[A_{t+1}\right]\right] = E_{t-1}\left[A_t\right] = A_{t-1}$$

$$\Rightarrow E_0\left[A_{t+1}\right] = E_0\left[E_1\left[E_2\right]E_3..E_t\left[A_{t+1}\right]...\right] = A_0$$

By the law of iterative expectation, we can obtain the current value of the claim:

$$x(0) = \tilde{\tilde{E}}_0\left[\frac{x(T1-1; s(T1-1))}{B(t1-1; s(T1-1))}\right]B(0)$$

Risk –Neutral Valuation

$$r(t,s(t)) = \pi(t;s(t)) \times u(t;T1;s(t)) + \left(1-\pi(t;s(t))\right) \times d(t;T1;s(t))$$

$$r(t,s(t)) = \tilde{\tilde{\mathrm{E}}}\left[\frac{p(t+1;T,s(t+1))}{p(t;T,s(t))}\right]$$

$$we \text{ also have: } B(t+1;S(t)) = B(t;S(t-1)) \times r(t,s(t))$$

$$\Rightarrow \frac{B(t+1;S(t))}{B(t;S(t-1))} = \tilde{\tilde{\mathrm{E}}}_t\left[\frac{p(t+1;T,s(t+1))}{p(t;T,s(t))}\right]$$

$$\Rightarrow \frac{p(t;T,s(t))}{B(t;S(t-1))} = \tilde{\tilde{\mathrm{E}}}_t\left[\frac{p(t+1;T,s(t+1))}{B(t+1;S(t))}\right]$$

$$\mathrm{A}(\mathrm{t},\mathrm{S}(\mathrm{t})) = \tilde{\tilde{\mathrm{E}}}_t\left[\mathrm{A}(\mathrm{t}+1,\mathrm{S}(\mathrm{t}+1))\right]$$

$$\Rightarrow A(0,S(0)) = \tilde{\tilde{\mathrm{E}}}_0\left[\mathrm{A}(\mathrm{t}+1,\mathrm{S}(\mathrm{t}+1))\right]$$

$$p(0,T) = \tilde{\tilde{\mathrm{E}}}_0\left[\frac{p(t+1;T,s(t+1))}{B(t+1;S(t))}\right]$$

Contingent claim ;Risk –Neutral Valuation

at time "t" we have a portfolio of M(t) (Money market) and N(t) (zero coupon bond)

Our portfolio $= \mathrm{N(t)P(t,S_t)} + M(t)B(t)$

at time t+1:

in the up state: $\mathrm{N(t)P(t+1,S_{tu})} + M(t)B(t+1)$

in the down state: $\mathrm{N(t)P(t+1,S_{td})} + M(t)B(t+1)$

$$\frac{P(t,St)}{B(t)} = \frac{\pi(t)\times \mathrm{P(t+1,S_{tu})}}{B(t+1)} + \frac{(1-\pi(t))\times \mathrm{P(t+1,S_{td})}}{B(t+1)}$$

$$N(t)\frac{P(t,St)}{B(t)} = N(t)\frac{\pi(t)\times \mathrm{P(t+1,S_{tu})}}{B(t+1)} + N(t)\frac{(1-\pi(t))\times \mathrm{P(t+1,S_{td})}}{B(t+1)}$$

$$N(t)\frac{P(t,St)}{B(t)} + M(t)\frac{B(t)}{B(t)} = N(t)\frac{\pi(t)\times \mathrm{P(t+1,S_{tu})}}{B(t+1)} + M(t)\frac{\pi(t)\times \mathrm{B(t+1)}}{B(t+1)} +$$

$$+ N(t)\frac{(1-\pi(t))\times \mathrm{P(t+1,S_{td})}}{B(t+1)} + M(t)\frac{(1-\pi(t))\times \mathrm{B(t+1)}}{B(t+1)}$$

Contingent claim; Risk–Neutral Valuation

$$N(t)\frac{P(t,St)}{B(t)}+M(t)\frac{B(t)}{B(t)}=\pi(t)\left[\frac{N(t)\times \mathrm{P}(\mathrm{t+1},\mathrm{S}_{\mathrm{tu}})+M(t)\times B(t+1)}{B(t+1)}\right]+(1-\pi(t))\left[\frac{N(t)\times \mathrm{P}(\mathrm{t+1},\mathrm{S}_{\mathrm{td}})+M(t)\times B(t+1)}{B(t+1)}\right]$$

$$\frac{x(t;St)}{B(t)}=\pi(t)\frac{X(t+1;S_{tu})}{B(t+1)}+(1-\pi(t))\frac{X(t+1;S_{td})}{B(t+1)}$$

$$\frac{x(t;St)}{B(t)}=\overset{\approx}{E}_t\left[\frac{X(t+1;St)}{B(t+1)}\right]$$

Risk–Neutral Valuation

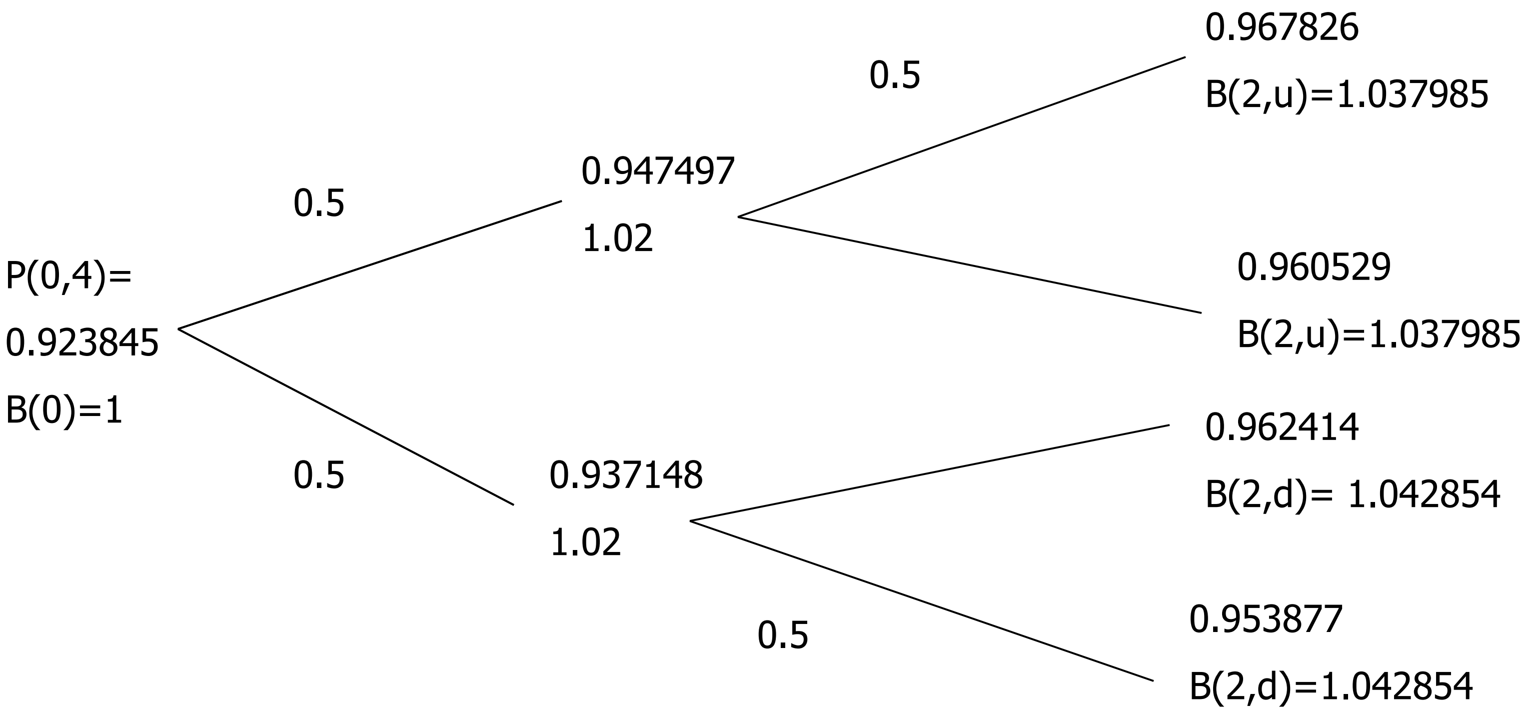
- By having the final cash flows, MM value, and pseudo probabilities, we can valuate the expectation and then find the value of the claim at the present time, t=0, by applying dynamic programming.

SUMMARY:

If there is No-arbitrage, we have these Pseudo-Probabilities

- From using price of the primary asset, find the u(t,T1,s(t)) and d(t,T1,s(t)) for all s(t) and t
- Using above equation to find pseudo-probabilities of each states
- Using the short term interest rate, find the value of MM at each of s(t1-1) state
- Find the final cash-flow of the contingent claim at every state
- Evaluate the above equation for x(0)

Example



P(0,4)=
0.923845
B(0)=1
0.5
0.5
0.947497
1.02
0.937148
1.02
0.5
0.967826
B(2,u)=1.037985
0.960529
B(2,u)=1.037985
0.962414
B(2,d)= 1.042854
0.5
0.953877
B(2,d)=1.042854

Example

Consider we have the price of a four-year maturity zero coupon bond for the first 2 years, we want to find the current price of the bond with 2 years maturity:

$$\pi(t;s(t)) = \frac{r(t,s(t)) - d(t;T1;s(t))}{u(t;T1;s(t)) - d(t;T1;s(t))}$$

d(1,4,u)=P(2,4,ud)/P(2,4,u)=1.013754

u(1,4,u)=P(2,4,uu)/P(2,4,u)= 1.021455

Pi(1,u)=(1.017606-1.013754)/(1.021455-1.013754)=0.5

Pi(1,d)=0.5

Pi(0)=0.5

$$x(0) = \overset{\approx}{E}_0\left[\frac{x(T1-1;s(T1-1))}{B(t1-1;s(T1-1))}\right]B(0)$$

Example

$$p(0,2) = \tilde{\tilde{E}}_0\left[\frac{p(2,2,S(2))}{B(2;s(1))}\right]B(0)$$

$$\frac{1}{\mathrm{B(2,u)}}\times\pi(0)\times\pi(1,u)+\frac{1}{\mathrm{B(2,u)}}\times\pi(0)\times(1-\pi(1,u))+$$

$$\frac{1}{\mathrm{B(2,d)}}\times(1-\pi(0))\times\pi(1,d)+\frac{1}{\mathrm{B(2,d)}}\times(1-\pi(0))\times(1-\pi(1,d))=$$

$$\frac{1}{1.037958}\times 0.25+\frac{1}{1.037958}\times 0.25+\frac{1}{1.042854}\times 0.25+\frac{1}{1.042854}\times 0.25=0.9611$$

Summary

- From AD lecture we know in Equilibrium for any generic asset i:
- $p_{oi} \ = \ \beta \ E_0 \left[\frac{U'(c_1)}{U'(c_0)} P_{1i}\right] = \frac{1}{r_f} E_0[Z' P_{1i}] = \frac{1}{r_f} \tilde{E}_0[P_{1i}]$
- If we have “N” assets and “I” state :
- $p_{o1} \ = \frac{1}{r_f} [\tilde{\pi}_1 p_{11} + \cdots + \tilde{\pi}_s p_{s1}]$.....
- $p_{oN} \ = \frac{1}{r_f} [\tilde{\pi}_1 p_{1N} + \cdots + \tilde{\pi}_s p_{sN}]$
- $r_f = \left[\tilde{\pi}_1 \frac{p_{11}}{p_{01}} + \cdots + \tilde{\pi}_s \frac{p_{s1}}{p_{01}}\right] = [\tilde{\pi}_1 r_{11} + \cdots + \tilde{\pi}_S r_{S1}]$
- $r_f = \left[\tilde{\pi}_1 \frac{p_{N1}}{p_{0N}} + \cdots + \tilde{\pi}_N \frac{p_{sN}}{p_{0N}}\right] = [\tilde{\pi}_1 r_{1N} + \cdots + \tilde{\pi}_S r_{SN}]$

Summary

- In General we have N equations and (S+1) unknown (r_f, $\tilde{\pi}_1$,…, $\tilde{\pi}_S$)
- These Equation must hold for ALL assets including primary assets . If we have set of "N" primary assets :
- If N=S+1, we can solve for a UNIQUE set of pseudo-probabilities and risk free rate (r_f, $\tilde{\pi}_1$,…, $\tilde{\pi}_S$). Complete Market
- If N<S+1 we can't solve for a UNIQUE set of pseudo-probabilities and risk free rate (r_f, $\tilde{\pi}_1$,…, $\tilde{\pi}_S$). In fact there are multiple sets that will stratifies these equation. Incomplete Market.
- If N>S+1, These doesn't exist ANY solution market are not in Equilibrium and Arbitrage could exist