BONDS

Video Lecture VL03
Valuation for Financial Engineers

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Outline of today's talk

- Theory
  - Idealized bonds
  - Variable but known interest rates
  - Stochastic interest rates, but no risk premium
  - Spot and forward curves
  - Theories of the term structure
  - Pricing a bond for future delivery
  - Idealized floating rate notes (FRN)
- U.S. Government Bonds
  - Composition
  - Details of bond types
  - The ZCB term structure
  - Bootstrapping Miniproject 2
  - Inflation and government bonds

TIPS

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Idealized bonds

- Idealized bonds match the Excel model
  - First payment is in one period
  - Equally spaced payments
  - Discount rate is constant
  - YTM (Bond equivalent yield) = Periodic rate × Periods/yr
  - Principal is \$100 or \$1000
- In this case, pricing is trivial
  - Excel provides price as a function of YTM or YTM as a function of price

Bond Yield Conventions

ex. r = 10%
~~if compound once/yr --> $1 becomes $1.10~~
if compound twice/yr --> $1 becomes (1.05)(1.05) = $1.1025
...
max $1 becomes e^r or exp(r)

- Rule 1. Simple annual rates
  - Bond yields are always calculated at the same frequency as the coupon payment, e.g. a semi-annual YTM, then simply multiplied by 2 to get the *bond equivalent yield.*
- Rule 2. Effective yield
  - Effective yield can be found by compounding the periodic yield. It is used for investment analysis, **not reporting**.
  - $(1 + EAY) = (1 + r/k)^k$
- Rule 3. Continuously compounded yield
  - For zero-coupon bonds and derivatives, let $k \to \infty$, then
  - $\text{Lim}_{k\to\infty}(1 + r/k)^k \to e^r$.
  - PV of a cash flow is $V_o = Ce^{-rt}$ = C/(1+r)^t

The GVE simply combines coupons and principal

-
- $V_1 - V_0 = \frac{F}{(1+r)^{n-1}} - \frac{F+C}{(1+r)^n}$
- $rV_0 = (V_1 - V_0) + C$
- $V_0 = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{F}{(1+r)^n}$

What if rates differed each period?

- Assume the 1 period discount rate from time i to i+1 is r_i
  - Although this is in the future, assume these rates are known now

- Then how will you price an idealized bond in discrete time?
  - The payment at time t will be valued as
  - $V_0 = C_t\left(\frac{1}{1+r_{t-1}}\right)\left(\frac{1}{1+r_{t-2}}\right)\ldots\left(\frac{1}{1+r_0}\right)$

These are all one-year SPOT rates

r0 = spot rate at time 0 for 1 year debt (known)
r1 = spot rate at time 1 for 1 year debt (generally not known)
.. therefore r1 is a future spot rate
and E[r1] = expected value of unknown random variable

- How will it be priced with continuous discounting?
  - $V_0 = C_t e^{-r_{t-1}} e^{-r_{t-2}} \ldots e^{-r_0} = C_t \exp[-\sum_{i=0}^{t-1} r_i]$

What if rates were unpredictable?

- Assume 3 periods {0,1,2} years
- r_0 is known and r_1 is unknown
- r_1 will be known at time 1
- Assume r (the one-year spot rate) follows a mean-reverting process plus a random disturbance
- $r_1 = r_0 + \kappa(\mu - r_0) + \sigma z$
- $E[r_1] = \kappa\mu + (1-\kappa)r_0$
- $StDev[r_1] = \sigma$
- $E[e^{-r_1}] = \exp[-(\kappa\mu + (1-\kappa)r_0) + \frac{1}{2}\sigma^2]$

Remember!

For a normal random variable

X ~ N(μ,σ)

E[e^X] = exp(μ+0.5 σ^2)

Using backward recursion on bond values…

- $V_2 = 1$
- $V_1 = e^{-r_1}$
- $E[V_1] = E[e^{-r_1}]$
- $V_0 = e^{-r_0} E[V_1]$

No risk premium adjustment

- Let
  - $r_o = 2\%$
  - $\mu = 4\%$
  - $\kappa = 0.25$
  - $\sigma = 0.01$

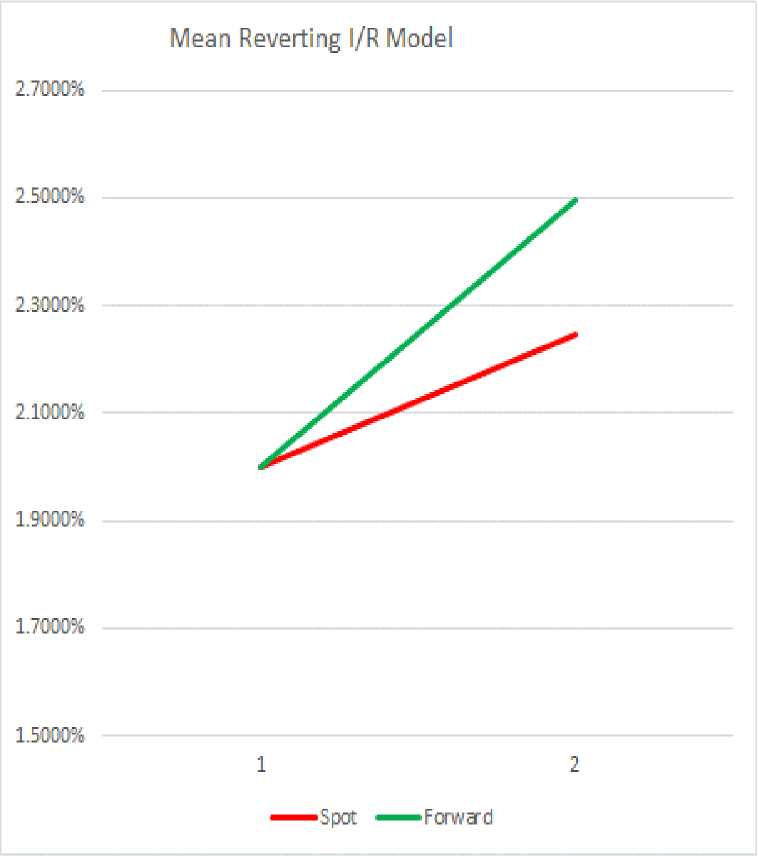
Intermediate calculations	
r1 mean	0.025
r1 sigma	0.01
E[V1]	0.975358679
V0	0.956045283
2-year spot rate	0.022475
forward rate (1,2)	0.02495

Time	Spot	Forward	Expected
1	2.0000%	2.0000%	2.0000%
2	2.2475%	2.4950%	2.5000%

Spot rates and forward rates

- The spot rate is the interest rate you lock in now for 1 or 2 year debt
- The forward rate is the implied return from year 1 to year 2
  - Obtained by buying the two-year bond and selling the one-year bond
- For continuous discounting, the spot rate is the average of the one-year forward rates

Time	Spot	Forward	Expected
1	2.0000%	2.0000%	2.0000%
2	2.2475%	2.4950%	2.5000%



Three Ways to Invest $1 for Two years

Use this calculation to compute forward rates in discrete discounting models

Rollover One-Year Spot	Two-Year Spot	One year spot plus forward lock
r_0	$r_{02} \approx 6\%$	5% r_0
$E(\tilde{r}_1)$ $\tilde{r}_1$		7% f_1
		lock in for yr $t=2$ at $t=0$.
$(1+r_0)(1+?)$	$(1+r_{02})^2$ =	$(1+r_0)(1+f_1)$
	1	1

The forward rate is determined implicitly by relative debt pricing:

$$(1+r_{02})^2 = (1+r_0)(1+f_1)$$

$$(1+f_1) = \frac{(1+r_{02})^2}{1+r_0}$$

APPROX

$$r_{02} \approx \frac{r_0 + f_1}{2}$$

Theories of the term structure

- If we know what determined forward rates, we could determine spot rates
- There are four theories of the level of forward rates

false - empirically proven

1. Expectations Hypothesis **$f = E(r_s)$**
   - Forward rates are expected future spot rates
2. Market Segmentation **$f \neq E(r_s)$**
   - The markets for short and long-term debt are different

     borrow short term lend long term - "carry trade"
   - Therefore rates are determined independently
3. Preferred Habitat **$f > E(r_s)$**
   - Corporations prefer to issue long term debt
   - Investors prefer to invest in short term debt
   - Rates must adjust to induce corporations to borrow shorter term and investors to invest longer term
4. Liquidity Preference **$f > E(r_s)$**
   - Investors value liquidity and low risk, so there is a built-in risk premium for longer term debt

Question
For you as investor, what do you expect will have a higher yield per year?

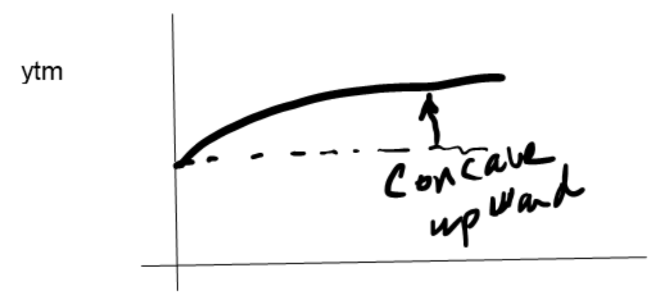
- 1 year bond
- 10 year bond

liquidity risk - risk of not having money when you need it
(also in risk management, refers to possible inability to sell securities at efficient prices)

interest rate risk - 10 year bonds fluctuate more than 1 year bonds as yields change

Why is the term structure upward sloping?

Google: Term structure movie



- Well, ok, most of the time!
- Expectations hypothesis
  - People expect rates to rise
- Market segmentation
  - Relative supply/demand for short and long-term debt
- Preferred habitat/Liquidity preference
  - Even if people expect rates to fall, the term premium can make the curve slope upward
- General equation
  - Forward rate = Expected future spot rate + Term premium

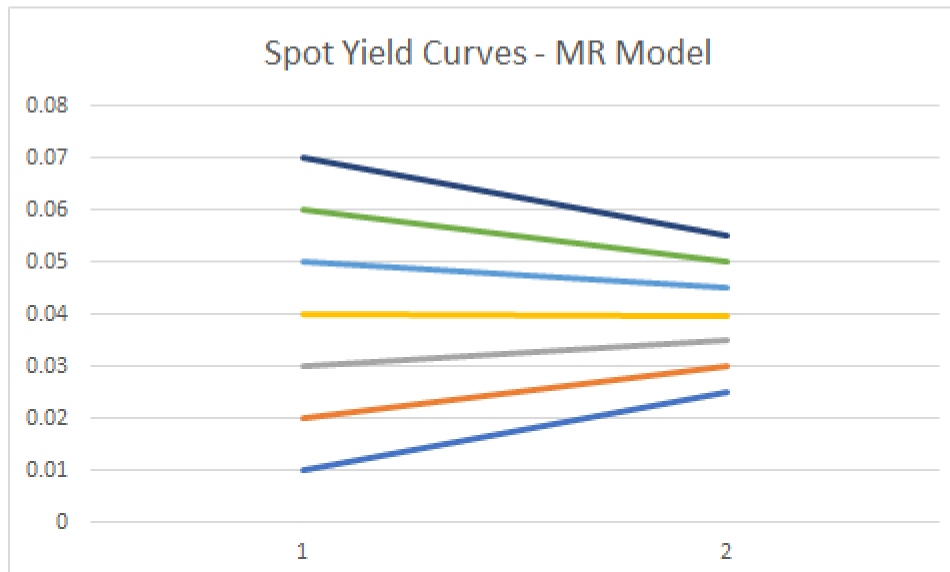
compensation for risk

What TS shapes does the stochastic model predict?

- Shorter term rates fluctuate more than long-term spot rates
  - Consistent with general TS behavior
- The yield curve is driven by a single factor
  - The term structure seems to be driven by at least 2-3 factors in reality

Two period interest rate model

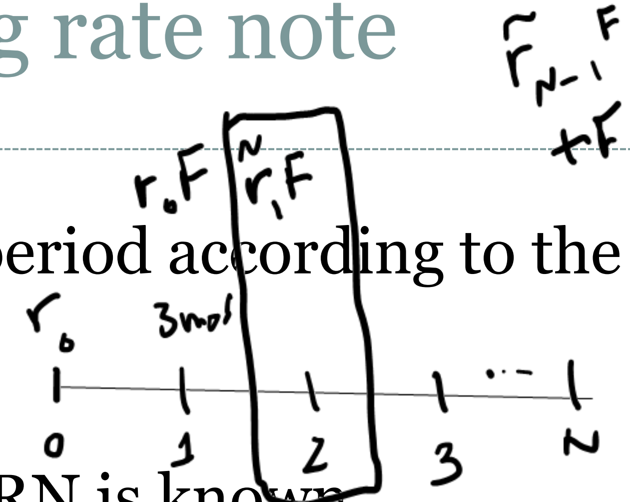
r0	0.02
mu	0.04
kappa	1
sigma	0.01



Exercises

- Using the pricing model we developed, a client has asked you to deliver a one-year bond in 1 year, that pays $100 one year later.
  - This is called a *forward bond*
- What is the fair price of that bond on delivery?
- Answer:
  - Price forward transactions by discounting with the forward discount rates
- What is an equivalent strategy to holding a 2-year bond, using this forward transaction?
- Answer:
  - Buy the one-year bond that pays for the contracted bond, and enter the contract to purchase the bond in the second year

The idealized floating rate note



- The FRN interest is different every period according to the prevailing interest rate at the time
  - Usually pays interest quarterly
- At time 0, the first payment of the FRN is known.
  - Pays principal (F) plus known interest at time 1.
  - This is called "payment in arrears"

principal = face value

PV

  - $r_0 V_0 = (F - V_0) + r_0 F, \quad so\ V_0 = F$
- The spot rate from time 1 to time 2 (r_1) is currently unknown, but can be used in the equation
  - $r_1 V_1 = (F - V_1) + r_1 F \quad or \quad V_1 = F$
  - $r_0 V_0 = (V_1 - V_0) + r_0 F \quad or \quad V_0 = F$
- Therefore, pricing an FRN is easy!

$$\frac{f_1 F}{(1+r_{02})^2}$$

A more difficult calculation

- How do you value a single FRN payment?
- Consider our stochastic interest rate model
  - What is the current value of a cash flow of r_1F paid in one year, given that r_1 is unknown?
  - F can be a principal amount or any other constant multiplier
- Answer
  - This is a second use of forward rates (in addition to pricing forward bonds)
  - The value of the cash flow is risk-adjusted by replacing it with its forward equivalent (for the previous year, i.e. in arrears)
  - The forward equivalent is discounted at the risk-free rate
  - Hence, the value is $V_0 = f_1 F e^{-r_0}$
- Proof
  - Value all the coupons of an FRN and add them up.

$$V_0 = \frac{r_0 F}{1+r_0} + \frac{f_1 F + F}{(1+r_{02})^2}$$

$$F = \frac{r_0 F}{1+r_0} + \frac{F(1+f_1)}{(1+r_{02})^2} = \frac{r_0 F}{1+r_0} + \frac{F}{1+r_0}$$

Extensions of the MR TS model

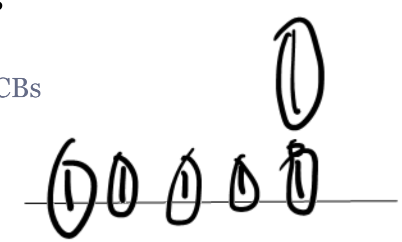
- "Discrete-Time Continuous-State Interest Rate Models", Michael A. Sullivan, OCC Economics Working paper 2000-6
  - Multiperiod, plus option pricing (easiest to read)
- "One-factor interest rate models and the valuation of interest rate derivative securities", J. Hull and A. White, **Journal of Financial and Quantitative Analysis** 28, 235-254, 1993.
  - The basis for current market standards
- "An equilibrium characterization of the term structure", **Journal of Financial Economics** 5, 177-188.
  - First continuous time stochastic model (mean-reverting, with risk premium)
- "A continuous time approach to the pricing of bonds", Brennan, M. and Eduardo S. Schwartz. (1979).
  - The first two-factor model
- "Bond pricing and the term structure of interest rates.", Heath, Jarrow and Morton (1988).
  - Takes the initial term structure as given, and constructs arbitrage-free movements

U.S. Government Bonds

Fixed Income Terms

- Fixed Income: Any security that promises a fixed payment stream over time, or a payment based on a fixed formula
- Bond: A security that promises to make fixed or floating *coupon* payments at prescribed dates in the future until the *maturity date*, when the bond's *principal* is paid in full
  - The bond *issuer* is the borrower; the bond *investor* is the lender
  - U.S. government-issued bonds pay coupons semiannually (2x per year) and repay principal at maturity
  - U.S. corporate-issued bonds pay coupons quarterly (4x per year) and repay principal at maturity
  - U.S. mortgage bonds make payments monthly (12x per year) which include both principal and interest
    - Payment of principal prior to maturity is called *amortization*
- Loan: Similar payment structures to bonds, but there are important differences
  - Loans are not usually traded in the capital markets, e.g. bank loans
  - Loans usually have a lot more restrictions on the borrower
  - The interest rates on loans tend to be lower than bonds, but there is less flexibility for borrowers

    may have flexibility to renegotiate the loan
- Floating rate bond: Payments are linked to changing interest rates, inflation rates or other indices
- Yield to maturity (YTM): Rate of return earned by a bond investor between now and the bond's maturity date (if the bond is held that long) Rate function in Excel, or implied fixed yield one would earn if they bouht the bond today and sold at maturity
- Term structure: YTM for a given fixed income instrument for different maturities
- Zero coupon bond (ZCB): pays no coupons
  - Note that all fixed payment bonds can be imagined as the combination of a number of ZCBs
  - For this reason, we will focus on ZCB term structure and use continuous discounting
- Basis point: One bp = 0.01% = 0.0001 = "One bip"



= =

ex 7,31%

Bond Yield Conventions

- Rule 1. Simple annual rates
  - Yields are always calculated at the same frequency as the coupon payment, e.g. a semi-annual YTM, then simply multiplied by 2 to get the *bond equivalent yield*.
- Rule 2. Effective yield
  - Effective yield can be found by compounding the periodic yield. It is used for investment analysis, not reporting.
  - $(1 + EAY) = (1 + r/k)^k$
- Rule 3. Continuously compounded yield
  - For zero-coupon bonds and derivatives, let $k \to \infty$, then
  - $(1 + r/k)^k \to e^r$.
  - PV of a cash flow is $V_o = Ce^{-rt}$

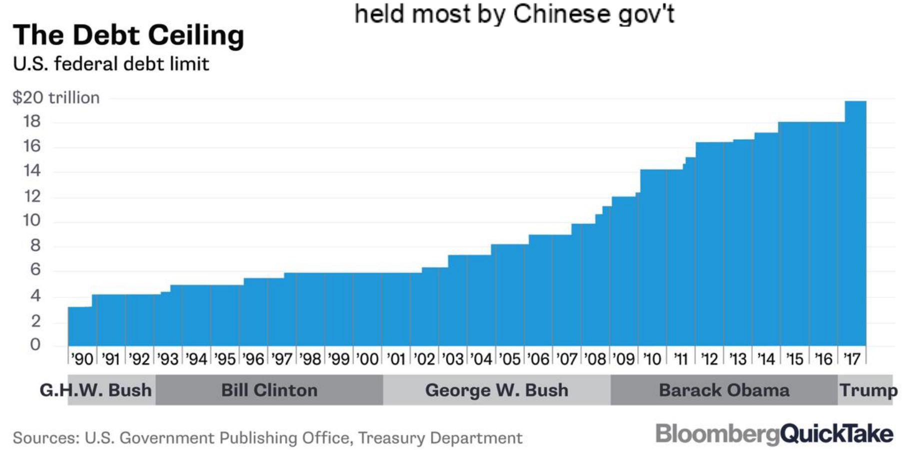
Gov't Bonds – "Risk Free"

- The most basic debt instruments are developed country debt
- The assumption behind the pricing is that these securities are not subject to any appreciable default risk
  - In reality, implicit default risk premiums are apparent when you look at the rates of the various countries in the EU (all of whom issue EUR denominated debt)
- The Government Debt market is huge. In 2020, the US Debt is on the order of $22 Trillion USD

Source: US Treasury

US Government Debt

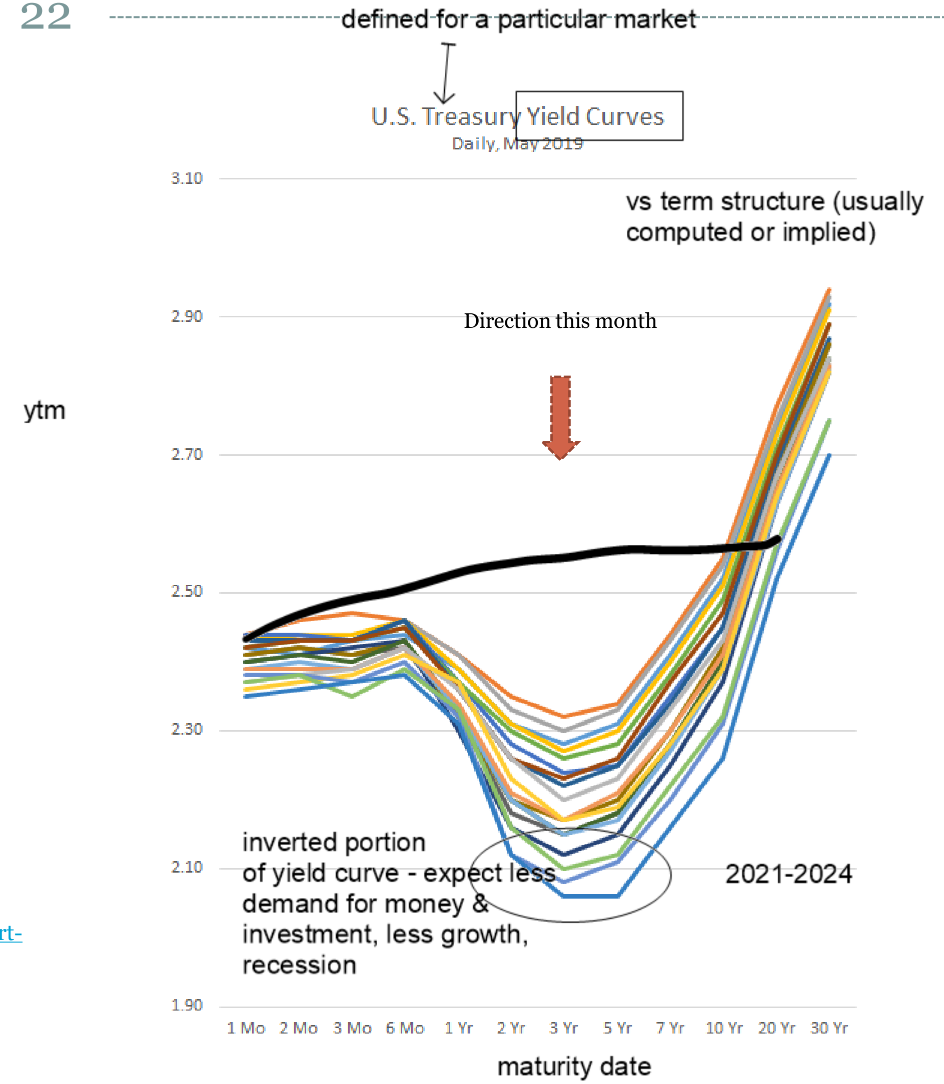
held most by Chinese gov't



The yield curve

- Yield to maturity (YTM)
  - This is the yield you will earn if you buy the bond at the offer price and hold it to maturity
  - Assumes same compounding convention as the bond
    - (i.e. semiannual yield x 2 or quarterly yield x 4)
- Yield curve definition
  - Also called the *term structure of interest rates*
  - For a given bond category, the yield curve shows how YTM depends on time
- Yield curve shape
  - Generally upward-sloping
  - U.S. Treasury Bond sample shown

Source: https://www.treasury.gov/resource-center/data-chart-center/interest-rates/pages/textview.aspx?data=yield



An interpretation, 5/29/19

“A key slice of the Treasuries yield curve became the most inverted since 2007, as growing angst over trade friction is overshadowing expectations that the Federal Reserve will cut interest rates by year-end.

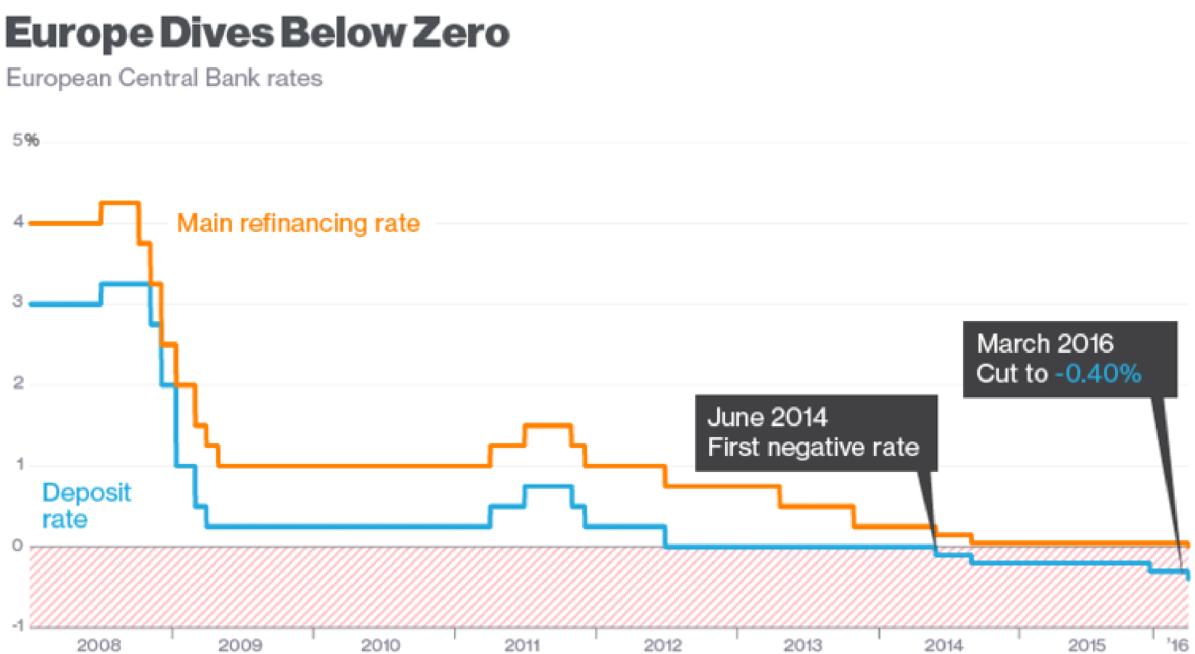
“The gap between three-month and 10-year rates dipped Wednesday to negative 12.3 basis points, breaking past a March level, when it first reached levels last seen in the global financial crisis. The spreads between most other sectors of the curve have narrowed as well. Historically, an inverted curve has been a signal that a recession is looming.

“U.S. Treasuries continued to rally Wednesday with haven buying driving the 10-year yield to the lowest since September 2017 at 2.24% as stocks in Asia fell, tracking a late sell-off on Wall Street. Global bond yields have resumed their slide after comments from President Donald Trump further dimmed the prospects of a U.S.-China trade deal and renewed tensions in Europe also damped demand for riskier assets.” - Bloomberg.com 5/29/19

Negative Interest Rates

European notional rates went negative in 2014!

Q: Why would a central bank take notional rates below 0%?
- Incentivize current consumer spending to drive recovery
- Ward off deflation (why would deflation be a concern?)
- Drive banks to take on credit risk (increase lending) to pick up yield



 Source: https://www.bloomberg.com/quicktake/negative-interest-rates

Negative Interest Rates

How would you set a floor for interest rates? [Hint: think of gold]

What are the pitfalls of negative rates?
- Banking contraction
- Retirement account investing

Source: https://www.bloomberg.com/quicktake/negative-interest-rates

U.S. Debt in June 2020

how does government finance debt?

MONTHLY STATEMENT OF THE PUBLIC DEBT OF THE UNITED STATES

JUNE 30, 2020

TABLE I -- SUMMARY OF TREASURY SECURITIES OUTSTANDING, JUNE 30, 2020

(Millions of dollars)

	Amount Outstanding			Jan 2020
	Debt Held By the Public	Intragovernmental Holdings	Totals	Debt Held By the Public
Marketable:				
Bills ZCB, $10000 ≤1 yr to maturity	5,078,833	793	5,079,626	2,403,862
Notes Fixed coupons + Prin at mat	10,306,753	7,712	10,314,465	9,990,403
Bonds	2,528,444	4,917	2,533,361	2,390,076
Treasury Inflation-Protected Securities	1,508,647	838	1,509,484	1,498,585
Floating Rate Notes [20]	461,767	0	461,767	412,991
Federal Financing Bank [1]	0	7,262	7,262	0
Total Marketable [a]	**19,884,443**	**21,522** [2]	**19,905,965**	**16,695,918**
Nonmarketable:				
Domestic Series	121,107	0	121,107	
Foreign Series	264	0	264	
State and Local Government Series	89,242	0	89,242	
United States Savings Securities	149,819	0	149,819	
Government Account Series	283,020	5,925,441	6,208,461	
Other	2,383	0	2,383	
Total Nonmarketable [b]	**645,835**	**5,925,441**	**6,571,276**	
Total Public Debt Outstanding	**20,530,278**	**5,946,963**	**26,477,241**	

https://www.treasurydirect.gov/govt/reports/pd/mspd/2020/2020_jun.htm

Treasury bond quotes

yield curve

As of 4/30/19

Treasury Yields

NAME	COUPON	PRICE	YIELD
GB3:GOV **3 Month**	0.00	2.37	2.41%
GB6:GOV **6 Month**	0.00	2.38	2.45%
GB12:GOV **12 Month**	0.00	2.31	2.38%
GT2:GOV **2 Year**	2.25	99.95	2.27%
GT5:GOV **5 Year**	2.25	99.80	2.29%
GT10:GOV **10 Year**	2.63	100.95	2.51%
GT30:GOV **30 Year**	3.00	101.05	2.95%

Problems with T-Bond YTM

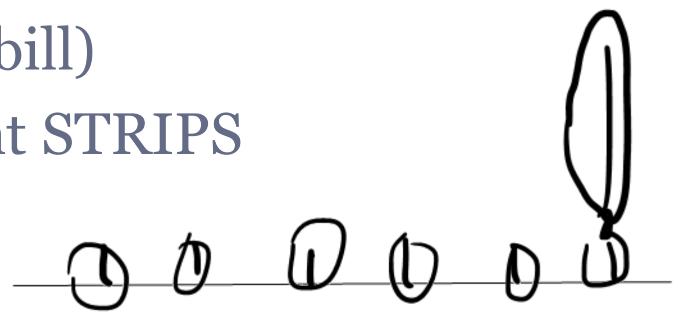
- The payment pattern is specific to U.S. government bonds
- Other pricing objectives
  - Risk-free component of corporate bonds
  - OTC transactions with corporate customers

    otc = over the counter
    - Long-term cash flow planning
  - Pricing derivatives
- We would like to know the discount rate for every single bond payment, not the bond as a whole

Why is the ZCB yield curve important?

- Any other fixed rate bond can be constructed and therefore priced from ZCBs

- Easy market references for risk-free ZCBs exist in the U.S. and most developed economies
  - T-Bill (treasury bill)
  - U.S. Government STRIPS



Sell ZCBs separately

What is a T-Bill (U.S. Treasury Bill)?

- The T-bill is a ZCB issued by the U.S. government
- Primary market
  - Sold in weekly auctions to *primary dealers*
- Secondary market
  - Identified by CUSIP numbers
  - Re-sold in over-the-counter (OTC) transactions to other investors

- Face value $10,000
- Maturity less than one year
- Considered a risk-free investment

- Use this sample T-Bill price
  - Today's date 4/26/19
  - T-bill maturity date 9/19/19
  - Days to maturity = 146
  - Asked yield = 2.378%

TREASURY BILLS

GO TO: Notes and Bonds

Friday, April 26, 2019

Treasury bill bid and ask data are representative over-the-counter quotations as of 3pm Eastern time quoted as a discount to face value. Treasury bill yields are to maturity and based on the asked quote.

Maturity	Bid	Asked	Chg	Asked yield
4/30/2019	2.388	2.378	0.003	2.411
5/2/2019	2.340	2.330	-0.002	2.363
5/7/2019	2.383	2.373	0.002	2.413
5/9/2019	2.365	2.355	0.015	2.389
5/14/2019	2.395	2.385	0.015	2.427
5/16/2019	2.368	2.358	unch.	2.393
5/21/2019	2.380	2.370	unch.	2.413
5/23/2019	2.380	2.370	0.020	2.407
5/28/2019	2.380	2.370	0.012	2.414
5/30/2019	2.353	2.343	0.005	2.380
6/4/2019	2.370	2.360	-0.010	2.405
6/6/2019	2.373	2.363	0.005	2.401
6/11/2019	2.373	2.363	-0.010	2.409
6/13/2019	2.378	2.368	0.007	2.408
6/18/2019	2.375	2.365	-0.002	2.412
6/20/2019	2.373	2.363	-0.002	2.404
6/25/2019	2.383	2.373	n.a.	2.421
6/27/2019	2.363	2.353	0.007	2.394
7/5/2019	2.383	2.373	-0.002	2.416
7/11/2019	2.373	2.363	0.002	2.407
7/18/2019	2.368	2.358	unch.	2.403
7/25/2019	2.373	2.363	-0.007	2.409
8/1/2019	2.373	2.363	-0.005	2.410
8/8/2019	2.375	2.365	-0.002	2.414
8/15/2019	2.380	2.370	unch.	2.420
8/22/2019	2.378	2.368	-0.005	2.419
8/29/2019	2.370	2.360	-0.002	2.412
9/5/2019	2.375	2.365	-0.002	2.418
9/12/2019	2.383	2.373	0.002	2.427
9/19/2019	2.388	2.378	0.003	2.440

T-Bill Conversions

- T-Bill quote to T-Bill price – "A yield is not necessarily a yield."
- T-Bill price to ZCB continuously compounded yield – helpful for comparisons to other yields
- Example: On 4/26/19, a T-Bill maturing on 9/19/19 has an asking quote of 2.378. What is the price of the bill and the continuously compounded yield?

Tbill price =

$$10{,}000\left(1 - \textit{Quoted yield} \times \frac{\textit{Days}}{360}\right)$$

$$r = -\frac{\ln\left(\frac{Price}{10{,}000}\right)}{\left(\frac{Days}{365}\right)}$$

146 days; \$9903.56; 2.42%

Why is the quotation so awkward?

STRIPs

What are they?

- Allow investors to hold interest and principal components as stand alone securities

How quoted?

- $100 face value

$$EAY = \left(\frac{100}{Price}\right)^{365/days} - 1$$

Convert to continuous ZCB?

$$r = \ln(1 + EAY)$$

Verify the yield calculation for the indicated STRIP

2161 days
EAY = 1.89%
r = 1.88%

U.S. Treasury Strips

Wednesday, September 13, 2017

U.S. zero-coupon STRIPS allow investors to hold the interest and principal components of eligible Treasury notes and bonds as separate securities. STRIPS offer no interest payment; investors receive payment only at maturity. Quotes are as of 3 p.m. Eastern time based on transactions of $1 million or more. Yields calculated on the ask quote.

Maturity	Bid	Asked	Chg	Asked yield
Treasury Bond, Stripped Principal				
2018 May 15	99.242	99.249	0.001	1.13
2018 Nov 15	98.662	98.673	-0.017	1.15
2019 Feb 15	98.331	98.345	-0.021	1.18
2023 Feb 15	90.499	90.548	-0.114	1.84
2023 Aug 15	89.429	89.482	-0.115	1.89
2024 Nov 15	86.757	86.819	-0.104	1.98
2025 Feb 15	86.180	86.243	-0.158	2.01
2025 Aug 15	[illegible]	[illegible]		

Bootstrapping T-Bills and STRIPs

- Many different ways to estimate a full ZCB yield curve.

- In this case, imagine a flexible functional form, and use regression to fit the curve

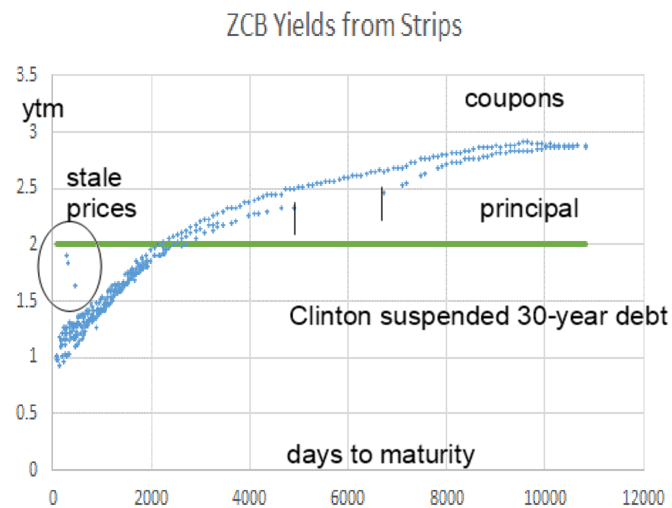
Example:

$$y(t) = a_1 + a_2 t + a_3 t^2 + a_4 e^{a_5 t} + a_6 \ln(t - a_7) + \varepsilon$$

- Shown (green): a_1=2, else =0

- Objective: Minimize sum of squared deviations between model and actual yields.

As of 9/14/17	42992	243				
	Bid	Ask	Chg	Ask yield	Date text	Datevalue
2018 May 15	99.225	99.231	-0.017	1.16	May 15, 2018	43235
2018 Nov 15	98.648	98.659	-0.014	1.16	Nov 15, 2018	43419
2019 Feb 15	98.313	98.327	-0.018	1.2	Feb 15, 2019	43511
2019 Aug 15	97.478	97.496	-0.022	1.33	Aug 15, 2019	43692
2020 Feb 15	96.783	96.806	-0.023	1.35	Feb 15, 2020	43876
2020 Feb 29	96.645	96.668	-0.031	1.38	Feb 29, 2020	43890
2020 May 15	96.33	96.355	-0.033	1.4	May 15, 2020	43966
2020 Aug 15						44058
2021 Feb 15						44242
2021 May 15						44331
2021 Aug 15						44423
2021 Nov 15						44515
2022 Aug 15						44788
2022 Nov 15						44880
2023 Feb 15						44972
2023 Aug 15						45153
2024 Nov 15						45611
2025 Feb 15						45703
2025 Aug 15						45884
2026 Feb 15						46068
2026 Aug 15						46249
2026 Nov 15	82.085	82.159	-0.004	2.16	Nov 15, 2026	46341
2027 Feb 15	81.513	81.589	-0.008	2.17	Feb 15, 2027	46433
2027 Aug 15	80.488	80.567	0.007	2.19	Aug 15, 2027	46614
2027 Nov 15	79.865	79.945	unch.	2.21	Nov 15, 2027	46706



T-Bond quotes – 9/2/20

- https://www.wsj.com/market-data/bonds/treasuries
- Treasury note and bond data are representative over-the-counter quotations as of 3pm Eastern time. For notes and bonds callable prior to maturity, yields are computed to the earliest call date for issues quoted above par and to the maturity date for issues below par.

MATURITY	COUPON	BID	ASKED	CHG	ASKED YIELD
9/15/2020	1.375	100.01	100.014	-0.004	-0.0621
9/30/2020	1.375	100.026	100.032	-0.006	-0.0017
9/30/2020	2	100.044	100.05	-0.006	-0.1168
9/30/2020	2.75	100.056	100.062	-0.01	0.101
10/15/2020	1.625	100.052	100.056	-0.004	0.059
10/31/2020	1.375	100.06	100.064	-0.006	0.086
10/31/2020	1.75	100.08	100.084	-0.006	0.064
10/31/2020	2.875	100.134	100.14	-0.01	0.098
11/15/2020	1.75	100.102	100.106	-0.008	0.056
11/15/2020	2.625	100.156	100.162	-0.008	0.064
11/30/2020	1.625	100.112	100.116	-0.008	0.097
11/30/2020	2	100.142	100.146	-0.008	0.082
11/30/2020	2.75	100.202	100.206	-0.008	0.052
12/15/2020	1.875	100.162	100.166	-0.01	0.015
12/31/2020	1.75	100.166	100.172	-0.01	0.082
12/31/2020	2.375	100.232	100.236	-0.008	0.079
12/31/2020	2.5	100.242	100.246	-0.01	0.107
1/15/2021	2	100.216	100.222	-0.014	0.09
1/31/2021	1.375	100.16	100.164	-0.01	0.109
1/31/2021	2.125	100.26	100.264	-0.012	0.092

Bootstrapping from Bonds

- Bonds provide a deeper market than STRIPs, so banks prefer to bootstrap the term structure from bonds
- There are many ways to bootstrap bonds
  - Use the smooth function approach demonstrated for STRIPs
  - Use a regression approach.
    - Each bond value is the sumproduct of its payment stream with its discount factors (i.e solve for $d(t)=e^{-rt}$ instead of r).
    - Find the discount factors that provide the best fit
      - This may require inversion of a large matrix, not suitable for Excel
  - Interpolation methods
  - Cubic splines
  - Maximizing smoothness

$$\frac{1}{(1+r)^t} \sim e^{-rt} \sim \underline{\underline{d(t)}}$$

for bond $V_0 = \sum_t C_t d(t)$

$$PV(C_t) = C_t d(t)$$

Observations on T-Bond quotes

- Prices are "clean prices" and do not include accrued interest

  Invoice price = price paid for bond = clean price + accrued interest since last coupon

  - See the ACCRINT function in Excel
- There are only two payment dates per month
  - We only need to include interest rates for those dates
- We'd like to find continuous discount factors for each of these payment dates
- These form the basis of all risk-free valuation
- Class participation points for demonstrating in class!

MINI-PROJECT #2

- Find a partner (one other person)
  - We will match you in class if you don't find someone!
- You may use Python or Excel
- For the most recent trading day, download all the US Bond prices from the Wall Street Journal or Bloomberg.
  - The TAs will provide a common dataset for all groups to use
- Develop a detailed automated algorithm to determine the continuously compounded discount rate for each bond payment date (coupons and principal).
- Write up your methodology (less than one page), and show your results in a table and graph.

What is inflation?

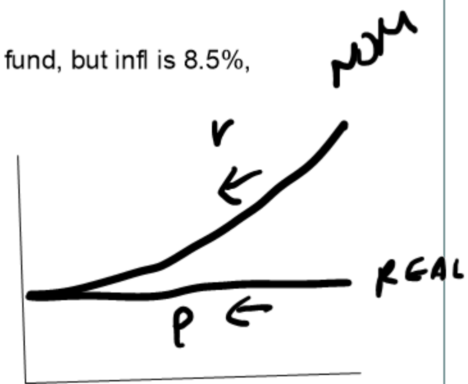
- What it means for spending power
- What it means for fixed rate liabilities
- **Nominal and real interest rates**
- Inflation equations: The *Fisher Equation*
  - Discrete time discounting
    - $\rho = \frac{r-i}{1+i}$ $r^* = \frac{r-g}{1+g}$
  - Continuous discounting (also used as a discrete time approximation)
    - RHO $\rho = r - i$
  - *Have we seen this before somewhere?*

Inflation = rate of increase in price of goods, now 8.5%

Fixed rate borrowers benefit from inflation

Nominal = contract or received cash flows ; Real = adj for inflation

if I earn 5% yield in money market fund, but infl is 8.5%, then really earn -3.5%



Key valuation conclusion:

Always discount _NOM_ cash flows at _NOM_ rates,

and _REAL_ cash flows at _REAL_ rates.

The Consumer Price Index (CPI)

cons

PPI prod

investors GDP deflator

Click and drag in the plot area to zoom in. Hover over chart to view data.
Source: U.S. Bureau of Labor Statistics.

Data extracted on: Apr 30, 2019 (10:42:21 AM)

CPI-All Urban Consumers (Current Series)

Series Title	: All items in U.S. city average, all urban consumers, not seasonally adjusted
Series ID	: CUUR0000SA0
Seasonality	: Not Seasonally Adjusted
Survey Name	: CPI-All Urban Consumers (Current Series)
Measure Data Type	: All items
Area	: U.S. city average
Item	: All items

Latest Observation:
March 2019
254.202

Jan 2017 - Mar 2019
Minimum ... 2017
242.839

Maximum Value: March 2019
254.202

Data Availability:
1913 - 2019

.XLSX .CSV

Year	Period	Label	Value
2018	M03	2018 Mar	249.554
2018	M04	2018 Apr	250.546
2018	M05	2018 May	251.588
2018	M06	2018 Jun	251.989
2018	M07	2018 Jul	252.006
2018	M08	2018 Aug	252.146
2018	M09	2018 Sep	252.439
2018	M10	2018 Oct	252.885
2018	M11	2018 Nov	252.038
2018	M12	2018 Dec	251.233
2019	M01	2019 Jan	251.712
2019	M02	2019 Feb	252.776
2019	M03	2019 Mar	254.202

CPI is the price of a standardized "basket" of consumer goods, meant to represent changes in overall price levels.

What was the rate of U.S. inflation realized over the last year, according to the CPI?

ANSWER:
=(254.202/249.554)-1
= 1.86%

U.S. 10 yrs < COVID
i ~ 2%.
& CHINA

Cost of living adjustments (COLA)

- Mingruo has just been offered an employment contract with Moody's Investor Services. The details are:
  - 5 year contract (assume end of year payments)
  - Current salary reference: $100,000 U.S. per year
  - Current CPI reference (3/19): 254.202
  - Future salary per year = Future CPI × $100,000 / 254.202
- How would you find the expected present value of these cash flows in the marketplace?
  - That is, using the market data provided in this presentation?

PV of $100,000 real per year
@ real rate

Can we find the market's expected inflation?

Features of TIPS

- Principal and interest calculations
  - Principal is increased/decreased with changes in CPI inflation; interest payments are tied to principal
- Taxation of capital gain
  - Increase of principal subject to federal tax, but exempt from state and local taxes
- Embedded option
  - Final Principal payments is at least par → insulated from deflation
- Implied inflation now?
  - Implied inflation ≈ Risk-free rate – rate on TIPS

As of 4/30/19

Treasury Inflation Protected Securities (TIPS)

NAME	BASE COUPON	PRICE	YIELD
GTII5:GOV **5 Year**	0.50	100.35	0.43%
GTII10:GOV **10 Year**	0.88	103.05	0.55%
GTII20:GOV **20 Year**	3.38	134.39	0.61%
GTII30:GOV **30 Year**	1.00	101.44	0.94%

$r_{30} = 2.95\%$

$\rho = .94\%$

$i = 2.01\%$

These are *real* yields, not nominal

Today's Lecture

- We learned how to value riskless cash flows, even in the presence of stochastic (randomly changing) interest rates

  - Idealized bonds
  - Compounding conventions
  - GVE for bonds
  - Rates differ by period
  - Add risk, solve backwards (recursively)
  - Spot Forward and Expected spot rates
  - Term structure
  - US Government bonds: Bills, Notes, Bonds, STRIPs and TIPS
  - ZCBs bootstrapping
  - Bond bootstrapping
  - Implied parameters found in bond prices

Review Questions

1. What are idealized bonds?
2. How is the effective annual yield of a bond calculated? How does it compare to a bond-equivalent yield?
3. What is the highest possible EAY one could earn for a given simple annual rate (based on compounding frequency)?
4. What is the formula for the present value of a cash payment, discounted continuously?
5. How are ZCBs priced if the discount rate differs in every future year?
6. How are bonds priced (theoretically) in the presence of stochastic interest rates?
7. Explain spot rates, forward rates and expected future spot rates?
8. What is the general shape of the term structure? Why does it take that shape?
9. What might a downward sloping term structure suggest?
10. What is a forward bond?

Review Questions

11. What is an FRN and how is it priced?
12. What is the present value of a payment that is proportional to a future interest rate?
13. What is mean by a *factor model* of the term structure?
14. What are the differences between bonds and loans?
15. What is a ZCB and why is it important?
16. Why do we see negative interest rates?
17. What happened to the composition of U.S. debt due to COVID?
18. Why is the yield on a T-Bill not really the yield?
19. What is a STRIP? What does it offer that differentiates it from T-Bills?
20. What is yield curve bootstrapping?

Review Questions

21. Why are T-bonds preferred for bootstrapping?
22. How does accrued interest relate to bond pricing?
23. What is meant by the terms real and nominal?
24. How is the CPI computed?
25. What is a COLA, and how is present value computed in the presence of a COLA?
26. How can one determine implied inflation?

Challenge problem

- Suppose X is a binomial random variable with two equally likely outcomes.
  - The two outcomes are: $X = a + b$ and $X = a - b$
  - Then $E[X] = a$ and $SD[X] = |b|$

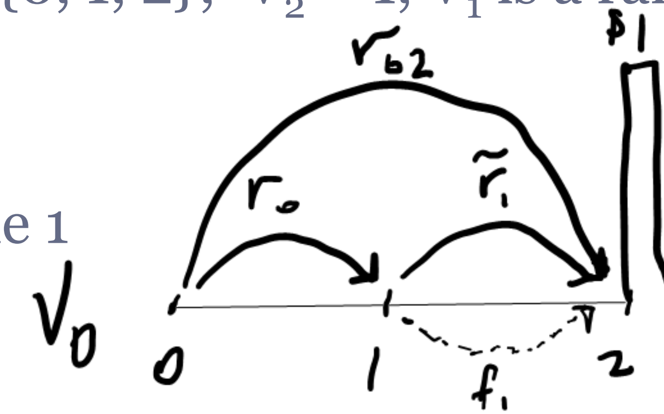
- Partial fractions: $\frac{1}{a+b} + \frac{1}{a-b} = \frac{2a}{a^2-b^2}$ $\qquad \frac{1}{a+b} - \frac{1}{a-b} = \frac{2b}{a^2-b^2}$

- Long division: $\frac{a^2-b^2}{a-kb} = a + kb - \frac{b^2(1-k^2)}{a-kb}$

Fun formulas!

Binomial model of a two-period ZCB

- Definitions
  - r_0 = spot rate for one period at time 0, known
  - r_1 = spot rate for one period at time 1, a random variable
  - r_{02} = spot rate for two-year debt
  - μ_r = expected change in spot rate from time 0 to 1
  - measure of risk
    σ_r = standard deviation of change in spot rate from time 0 to 1
  - V_t = Value of the ZCB at time $t \in \{0, 1, 2\}$; $V_2 = 1$, V_1 is a random variable
  - k = cost of interest rate risk
  - f_1 = one-year forward rate at time 1
    - constant rate equivalent to the random variable rate
    - risk-neutral rate
    - market value of the random variable



Short-term interest rate evolution

Binomial tree

$$r_o \quad \xrightarrow{1/2} \quad r_1 = r_0 + \mu_r + \sigma_r$$

$$r_o \quad \xrightarrow{1/2} \quad r_1 = r_0 + \mu_r - \sigma_r$$

Hints: Let a = $(1 + r_o + \mu_r)$ and b = σ_r; use discrete discounting

Determining $V_1(r_1)$

one for each value of r

$$V_o$$

1/2

$$V_1 = \frac{1}{(1 + r_0 + \mu_r) + \sigma_r}$$

=1/(a+b)

1/2

$$V_1 = \frac{1}{(1 + r_0 + \mu_r) - \sigma_r}$$

=1/(a-b)

Hint: Let a = $(1 + r_o + \mu_r)$ and b = σ_r

V_o with the GVE

- GVE $\quad V_0 = \dfrac{E(V_1) - k\, SD(V_1)}{1+r_0}$

also

$$V_0 = \frac{1}{(1+r_{02})^2}$$

- $E[V_1] = \frac{1}{2}\left(\frac{1}{a+b} + \frac{1}{a-b}\right) = \frac{a}{a^2-b^2}$
- $SD[V_1] = \frac{1}{2}\left(\frac{1}{a+b} - \frac{1}{a-b}\right) = \frac{b}{a^2-b^2}$
- $V_0 = \dfrac{\frac{a}{a^2-b^2} - \frac{kb}{a^2-b^2}}{1+r_0}$

$$V_0(1+r_0) = \frac{a-kb}{a^2-b^2}$$

$$V_0(1+r_0) = \frac{1}{a+kb + h.o.t.} = \frac{1}{1+f_1}$$

Solved for value of the bond, r02, f1

Finding the spot and forward rate

- $V_0 = \frac{1}{(1+r_{02})^2} = \frac{1}{(1+r_0)(1+f_1)}$

- $(1 + f_1) = \frac{1}{(1+r_0)V_0} =$

- *Interpretations*

Interpretations

- In this model, $f_1 \approx r_0 + \mu_r + k\sigma_r = E[r_1] + k\sigma_r$

- Forward prices rise approx. linearly with increasing spot rates, expected future spot rates, and the cost of interest rate risk
- Set some parameters equal to zero to isolate the effects of other parameters
- The pure convexity effect is found by setting k=0.
- The negative impact of convexity on the forward rate is negligible, on the order of σ_r^2