Fixed Income

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Abstract

Moving fixed cash flows in time.

A *zero coupon bond* with maturity u pays unit notional at time u and has price, or *discount*, $D(u)$. A *fixed income* instrument is specified by cash flows (c_j) at times (u_j). This is just a portfolio of zero coupon bonds and its *present value* is $p = \sum_j c_j D(u_j)$.

The *(continuously compounded) yield* of a bond, $y = y(p)$, given a price is determined by $p = \sum_j c_j e^{-yu_j}$. All yields require an adjective, and there are many such adjectives. In the bond world the most common is *semiannual yield* determined by $p = \sum_j c_j (1 + y/2)^{-2u_j}$. Yield is a proxy for price, similar to Black-Scholes/Merton implied volatility. The adjective indicates the formula used to convert the proxy value to a price.

Zero coupon prices determine the *(continuously compounded) forward rate*, $f(u)$, defined by $D(u) = \exp(-\int_0^u f(s)\,ds)$.

Exercise. *Show* $f(u) = (d/du)(-\log D(u))$.

The *spot rate*, $r(u)$, is defined by $D(u) = e^{-ur(u)}$. It is the continuously compounded yield of a zero coupon bond.

Exercise. *Show* $r(u) = (1/u)\int_0^u f(t)\,dt$.

Exercise. *Show* $f(u) = r(u) + ur'(u)$.

The definitions and exercises show how to convert either discount, forward, or spot to any of the other two. It is preferable to work with forward rates since since integration smooths out functions while differentiation can exaggerate small, local variations.

Discount Curve

It is not the case zero coupon bonds of all maturities are traded. The *discount curve* $D(t)$ is used to interpolate a discount for all maturities. An instrument with cash flows (c_k) at times (u_k) and initial price p *fits* the curve if $p = \sum_k c_k D(u_k)$. Typically a collection of such instruments and prices are given and we wish to find a discount curve that fits them. This is a highly under-determined problem and there is a vast literature on various methods of constructing a solution.

Bootstrap

Given a collection of fixed income instruments ordered by increasing maturity and corresponding prices we can *bootstrap* a discount curve using a piece-wise constant forward curve that fits each price. The first forward, f_0, is the constant where

$$D(t) = e^{-f_0 t}, 0 \leq t \leq t_0$$

fits the first instrument. The second forward, f_1, is determined using the discount

$$D(t) = D(t_0)e^{-f_1(t-t_0)}, t_0 < t \leq t_1,$$

where t_1 is the maturity of the second instrument. Repeating this for each maturity determines a unique piece-wise constant forward curve that reprices every instrument in the collection.

A piece-wise constant forward curve f is determined by times (t_j) and forwards (f_j), $j < n$, where $f(t) = f_j$ for $t_{j-1} < t \leq t_j$. Note $f(t_j) = f_j$. The curve is undefined for $t > t_{n-1}$. We assume $f(0) = f_0$ and $f(t)$ is undefined for $t < 0$.

Given a forward curve and an instrument with maturity t_n and cash flows c_j at u_j we must find f_n such that

$$p = \sum_{u_j \leq t_{n-1}} c_j D(u_j) + \sum_{u_j > t_{n-1}} c_j D(t_{n-1}) e^{-f_n(u_j - t_{n-1})}$$

where p is the instrument price. This can be solved using one-dimensional root finding to produce the next point (t_n, f_n) of the piece-wise constant forward curve.

If an instrument has only one cash flow past t_{n-1} then this equation has a closed-form solution.

Exercise. *If c_k at u_k is the only cash flow greater then t_{n-1} show*

$$f_n = -\frac{\log((p - p_n)/c_k D(t_{n-1}))}{u_k - t_{n-1}}$$

where $p_n = \sum_{u_j \leq t_{n-1}} c_j D(u_j)$.

There are also closed-form solutions for f_n when adding an instrument with exactly two cash flows (u_0, c_0) and (u_1, c_1) and price 0. This is the case for forward rate agreements. We assume $u_0 < u_1$ and $u_1 > t_{n-1}$.

Exercise. *if $u_0 \geq t_{n-1}$ show show*

$$f_n = \frac{\log(-c_1 D(t_{n-1})/c_0 D(t_{n-1}))}{u_1 - u_0}$$

Clearly c_0 and c_1 must have opposite signs for this to exist.

Exercise. *Find a closed-form solution for f_n when $u_0 < t_{n-1}$.*

It is important that no two instruments have nearly equal maturity since the forward between those dates may require a large adjustment to fit the price.

The vast literature on interpolating discount curves should be ignored. Splining curves introduces mathematical artifacts into the discount. A cubic Hermite tension spline may produce a forward curve that is pleasing to the eye, but makes it difficult to explain to a trader why their rho bucketing is off. It is better to add synthetic instruments at intermediate maturities with prices determined by an interpolation method that traders can understand.

Repurchase Agreement

A *repurchase agreement* is a contract to lend 1 unit of notional and receive that plus interest a short period later, typically one day or over the weekend: $X_0 = 1$, $C_{\Delta t} = 1 + r\Delta t$ where r is the *repo rate* over $[0, \Delta t]$. More generally, the repo rate at t is determined by $X_t = 1$, $C_{t+\Delta t} = 1 + r_t\Delta t$. The *short term forward rate*, f_t, is defined by $1 + r_t\Delta t = e^{f_t\Delta t}$.

If D_t is a deflator then $D_t = E_t[e^{f_t\Delta t}D_{t+\Delta t}]$. The *canonical deflator* is determined by $D_{t+\Delta t} = \exp(-f_t\Delta t)$. Note the deflator to $t + \Delta t$ is known at t. In continuous time $D_t = \exp(-\int_0^t f(s)\,ds)$.

In the fixed income world the short term forward/repo rates determine the value of all instruments.

Zero Coupon Bond

Let $D(u)$ denote a zero coupon bond paying unit notional maturing at u, so $C_u^{D(u)} = 1$ is its only cash flow. We write $D_t(u)$ for the price $X_t^{D(u)}$ of the zero coupon bond at time t. An arbitrage free model requires the price at time t to satisfy $D_t(u)D_t = E_t[D_u]$ so $D_t(u) = E_t[D_u]/D_t$. The forward at time t, f_t, is defined by $D_t(u) = E_t[\exp(-\int_t^u f_s\,ds)]$.

A *fixed income* instrument is specified by times (u_j) and cash flows (c_j). This is just a portfolio of zero coupon bonds so its *present value* is $p = \sum_j c_j D(u_j)$ where $D(u)$ is the discount to time u. The present value at time t is $p_t = \sum_{u_j > t} c_j D_t(u_j)$.

Risky Bonds

A *risky zero coupon bond* with *recovery* R and *default time* T has a single cash flow $C_u = 1$ if default occurs after maturity or $C_T = R$ if $T \leq u$. It is customary to assume R is constant. As with American options, we must expand the sample space to $\Omega \times (0, \infty]$ where $(\omega, t) \in \Omega \times (0, \infty]$ indicates default occurred at time t. The partition of $(0\infty]$ representing information available at time t for the default time is $\{(t, \infty]\} \cup \{\{s\} : s \leq t\}$. If default has not occurred prior to t we only know $T > t$. If default occurred prior to time t we know exactly when it happened.

We write $D_t^{R,T}(u)$ for the price $X_t^{D^{R,T}(u)}$ of the risky zero coupon bond at time t. The dynamics of a risky zero are determined by

$$D_t^{R,T}(u) D_t = E_t[R 1(T \leq u) D_T + 1(T > u) D_u].$$

The *credit spread* $s_t = s_t^{R,T}(u)$ defined by $D_t^{R,T}(u) = D_t(u) e^{-u s_t}$ incorporates both recovery and default.

If rates are zero then $D_t = 1$ for all t and this simplifies to $D_0^{R,T}(u) = RP(T \leq u) + P(T > u)$ when $t = 0$. If T is exponentially distributed with hazard rate λ then $P(T > t) = e^{-\lambda t}$ and

$$D_0^{R,T}(u) = R + (1 - R) e^{-\lambda u}.$$

When $\lambda = 0$ the right hand side is 1. When $R = 0$ the credit spread equals the hazard rate. If λu is small then the approximation $e^x \approx 1 + x$ for small x gives the rule of thumb $s = \lambda(1 - R)$ where $s = s_0 = s_0^{R,T}(u)$ is the credit spread.

For general t we have

$$D_t^{R,T}(u) = RP(T \leq u | T > t) 1(t < T \leq u) + P(T > u | T > t) 1(T > u).$$

Unlike in the credit default swap market, mathematical finance literature likes to assume recovery is delayed until maturity. It is also popular to make the unrealistic assumption that default time is independent of the deflator. Under these assumptions we have

$$D_t^{R,T}(u) = D_t(u) \big(RP(T \leq u | T > t) 1(t < T \leq u) + P(T > u | T > t) 1(T > u) \big).$$

In principal, R could be random and joint distributions involving the default time and deflators could be specified.

Cash Deposit

A *cash deposit* is specified by a *settlement convention*, *period*, *coupon*, *day count basis*, and *notional*. On a *valuation date* d_v the parties doing the transaction agree on a coupon c and notional N. The settlement convention is the number of days (T+n) after valuation that the buyer transfers the notional N to the seller. This is the *effective date* d_e of the cash deposit. The period determines the *termination date* d_t and is usually specified as a number of weeks or months after the effective date. At termination the seller transfers $N(1 + c\delta)$ to the buyer where δ is the *day count fraction* determined by the daycount basis from the effective date to the termination date.

There are a number of daycount bases used to convert a period of time to a *day count fraction* that is approximately equal to the time in years of the period. The *Actual/365* convention uses the number of days in the period divided by 365. A cash deposit with effective date d_e and termination date d_t has day count fraction equal to the number of days from d_e to d_t divided by 365.

Roll Convention

Transactions can only occur on business days so dates must be *adjusted* using a *roll convention* based on weekends and a *holiday calendar*. The *following* convention advances a date to the next business day. The *previous* convention rolls a date back to a business day. *Modified following* uses following unless the date falls into the next month, in which case the previous is used.

Forward Rate Agreement

A *forward rate agreement* is specified by an *effective date*, *period*, *forward*, *day count basis*, and *notional*. On a *valuation date* d_v the parties doing the transaction agree on a coupon f and notional N. The buyer transfers the notional N to the seller on the effective date d_e. The period determines the *termination date* d_t and is usually specified as a number of months after the effective date. At termination the seller transfers $N(1 + f\delta)$ to the buyer where δ is the *day count fraction* determined by the daycount basis from the effective date to the termination date.

A forward rate agreement can be thought of as a forward starting cash deposit but it is better to think of a cash deposit as a forward rate agreement with an effective date determined by the settlement convention.