Valuation for Financial Engineering:

Part I: Foundations

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Comments welcomed!

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Contents

Preface for students

Some of you will savor the preface in order to forestall the inevitable labor of the seemingly unending chapters that follow. Others will skip the preface thinking that they should save time and avoid any material that won't appear on the midterm. I intend to surprise and please the former. In fact, I the book reveals many unexpected surprises with only modest departures from and extensions of rigorous traditional finance theory.

But let's not get ahead of ourselves. *Financial engineering* is a young field, lacking well-defined boundaries and consistent self-definition. Some authors define financial engineering as anything mathematical or computational in finance. Others restrict financial engineering to the mathematics and analysis of derivative securities and structured products. I prefer to adapt the traditional application of the term *engineering* to the field of finance, such as the one below from Wikipedia. The clarifications for finance are my own.

- **Engineering** – /ˌenjəˈniriNG/ - Engineering is the use of scientific principles to design and build machines, structures, and other things, including bridges, roads, vehicles, and buildings
- **Financial Decisions** – Valuation, trading, structuring (product design), hedging, risk management, investment, financing, lending, risk-based performance assessment.
- **Financial Engineering** – The branch of finance concerned with the design, building and use of financial decision-making tools and processes, financial computing processes and financial products.

What do financial engineers design and build? Here is a partial list:

- ✓ Financial decision frameworks, such as valuation, lending and trading
- ✓ Financial decision support, including data collection and analysis
- ✓ Design of securities, contracts and strategies used to raise capital and fund companies
- ✓ Other Innovative financial product and strategy design
- ✓ Risk measurement and management strategies

To test the definition, does it correctly discriminate between activities that should be considered financial mathematics and those that should be considered financial engineering? Most financial economists would agree that financial engineering should include the study of financial derivative pricing and risk models. Even further, algorithmic trading models or any number of big or small data implementations that support derivatives pricing and trading activity should be included.

Second, financial risk models would reasonably be expected to fall within the scope of financial engineering. This can be attributed to the incredible efforts taken to implement accurate risk measurement and reporting when financial theory provides no easy answers. I recall one discussion in 2019 with an eminent financial economist who asked what the big deal was with risk management. *"Isn't it just sigma?"* A financial engineer would address the practical complications, such as variable volatilities, nonlinear relationships and unstable correlations within shaky volatility models --- maybe even using implied volatility --- unlikely to be content with a mere standard deviation calculation.

Third, while there is not as much literature in this area, many finance folks would probably agree that financial product design or security structuring falls solidly within the boundaries of financial engineering. After all, this is just another type of consumer or industrial product design, highly customized to particular clients and market environments. Financial products are the means by which we express trading opinions, design portfolios to meet investment objectives, and raise money in the most efficient manner.

Fourth, and perhaps most controversial, I submit that corporate finance can own significant real estate in the field of financial engineering. Though we still benefit from the celebrated theoretical work of Modigliani, Miller, Sharpe, Lintner, Mossin and Treynor (and their many worthy successors unnamed here), financial theory has focused on highly idealized or stylized assumptions that were likely never meant to apply to all firms in practice at all times. Financial engineering applies financial theory to the vicissitudes of practice.

I realized this in my 25-year consulting practice with corporations. No company relies on the CAPM (Capital Asset Pricing Model) to determine discount rates for its expected cash flows; if they start with the CAPM, a hefty risk premium is usually added. Companies do value real options, but often lack the necessary engineering tools to apply financial theory rigorously to practice. They often end up mis-applying standardized option pricing models to unsuitable problems. Corporations make capital structure decisions, including security design and decisions about leverage levels, but using considerations far afield of those considered important in theoretical *classical* finance. Risk management decisions, which should be recast as *contingent capital structure and cash management decisions*, are roundly ignored or minimized in corporate finance textbooks --- which is ***shocking*** when you consider that so many risk management debacles have destroyed previously strong corporations. Brought down by contingent liabilities! Unreported by FAS 5![1]

Which brings me to the next point. With so much applied work to be done in corporate finance, who will actually do the work? Financial theorists would have to develop new theories for all types of corporations in many different circumstances. More likely, financial engineers well-versed in the theory should apply the tools of their trade to solve each problem that comes up. They should pay particular attention to their employers' circumstances, objectives, constraints, and contemporaneous market conditions. Hopefully, you are reading this book in order to become one of those people.

So much for the *engineering* term. The single most important tool for any financial professional is *valuation*. The CAPM doesn't work for the kinds of complex cash flows we have today. We can't even define percentage returns or discount rates for large classes of securities, contract and positions. Valuations need to be both public and private, since our differences in valuation opinions drive trading. Capital constraints affect private valuations dramatically. Benchmarking is critical, but tools like P/E ratios are far too primitive. Valuation models developed in the 1960s and taught to MBAs ever since need vigorous review and revision, even though textbooks continue to focus on these models.

This book therefore aspires to compile, develop and explain a body of theoretical and practical valuation knowledge learned in a career in applied finance --- corporate, sovereign and fund consulting. With applied, bespoke valuation tools, corporations can:

- Value projects and their embedded options; make investment decisions
- Decide when to buy and sell assets
    - e.g. corporate assets differ from projects when they have limited liability
- Determine value-maximizing capital structure & risk management strategy
- Design securities
- Write optimal contracts

Indeed, nearly every financial decision requires a valuation. Therefore, *Valuation for Financial Engineers* meets a well-defined need for financial problem-solvers to apply valuation theory rigorously in the face of specific corporate constraints and market circumstances.

[1] "[FAS 5] establishes standards of financial accounting and reporting for loss contingencies. It requires accrual by a charge to income (and disclosure) for an estimated loss from a loss contingency if two conditions are met: (a) information available prior to issuance of the financial statements indicates that it is probable that an asset had been impaired or a liability had been incurred at the date of the financial statements, and (b) the amount of loss can be reasonably estimated." - https://www.fasb.org/summary/stsum5.shtml Many derivative positions meet neither condition necessary for reporting under FAS 5.

It won't be that easy however. The book focuses on discrete-time modeling since our real world is not continuous. Closed-form solutions and easy formulas are rare. Nearly all risk-related and valuation problems in the real world require simulation to get a flexible solution. We will challenge long-held beliefs of financial theory, for example, the role of idiosyncratic risk in asset pricing. We provide cases, like real life, which should be expected to display a degree of complexity.

Finally, *Valuation for Financial Engineers* is not a glorified MBA textbook, though one trained as an MBA might see an occasional familiar equation. It is a toolkit for the modern financial engineer --- that applies advanced finance, mathematics, statistics, and computational algorithms to practical problems in corporate and investment finance. It is particularly useful for applications where joint cash flow probabilities are known or can be simulated, and not useful for continuous time or finite state valuations.

Supplemental preface for faculty (Part I)

Most faculty don't read others' textbooks, but will perhaps suffer a preface. This preface summarizes the unusual and different approach we take to standard valuation topics. It does not follow the formula of a valuation text or a corporate finance text. Generally, we approach the topic of valuation using a comprehensive simulation and benchmarking framework, where the benchmarks are based on traded assets or contract prices. For example, if one were to value an oil refinery, one would build a stochastic pro-forma, simulate cash flows, benchmark the cash flows to traded futures and option prices in energy, discount the cash flow components separately to reflect their different risks or benchmarks, and determine a final valuation. The value of an oil refinery should depend closely on energy futures prices, and options as well --- since a refinery in its essence comprises a collection of options to transform energy products from one form to another. In other words, the valuation of an oil refinery should take a similar approach to a complex derivative instrument, without the benefit of continuous-time financial mathematics.

Part I addresses valuation when discount rates are known. This assumption is surprisingly unrestrictive --- even real options can be modeled by estimating the effect of nonlinear risks on expected cash flows. Each chapter contains some surprises that I have not been able to find in other textbooks, but that I found extremely useful in practice. All the models herein are based on financial theory, but have not gotten much play outside academic papers. For example, the simple form valuations follow Lintner, Bogue and Roll, Brennan and Rubinstein's papers on cash flow valuation --- methodologies displaced by the detailed calibrations of the CAPM approximation undertaken by many valuation texts.

Here are some surprises you may expect: (yes I know this ruins the surprise, but in case you don't read the book…)

Chapter 1	• Characterizing valuation as a modeling process --- "always wrong, sometimes useful" • Introducing portfolio-based valuation as an important tool, especially in private valuations • Dealing with undefined percentage returns, e.g short positions of all types and long forward/futures or other leveraged positions • Introducing a single equation, the GVE, that unites all the valuation formulae in classical finance: Required return = Expected return, special cases resulting in the CAPM, Modigliani & Miller proposition 2, forward and option pricing formulae.
Chapter 2	• New and easier ways to develop annuity formulas using financial intuition rather than mathematical brute force; includes exponentially growing annuities • Focus on using Excel, as hand-held financial calculators have little use in modern finance

	• New types of annuities, including linear growth and mean reversion, foreshadowing the introduction of ABM, GBM and Mean Reversion stochastic processes in Chapter 6
Chapter 3	• A grab-bag of complications to finding basic present values of cash flows: payment frequency, inflation and taxes • Teaches students how to handle numerous cash flow types and risks • The introduction of the zero-coupon bond pricing curve and FX forward pricing • Pricing event risk including default, in the style of Poisson processes • Floating rate bonds, including the valuation of individual FRN payments and cash flows that correlate to interest rates
Chapter 4	• A comprehensive introduction to bonds traded in the U.S. and their informational content • A closed-form solution for defaultable bonds, using the BIS framework (PD and LGD)
Chapter 5	• A comprehensive introduction to simulation techniques in finance, using Excel or Python • Variance reduction: Moment matching and antithetical variables • The control variate simulation technique • Multivariate methods including the Cholesky decomposition and Gaussian copula transformations • Simulation and optimization with common random variables
Chapter 6	• Thorough discussion of introductory stochastic processes: Arithmetic & Geometric Random Walks and Mean Reversion in discrete time • Knowing which process to apply to any given financial variable of interest • Statistical properties, simulation and parameterization techniques

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I especially thank my NYU colleagues Dan Gode and Brian Lessing, and who bravely used my notes to teach the masters level valuation class in the Spring of 2020. I am continuing to revise the material based on their explicit comments and implicit needs.

Of course, I am sure there are still many errors, typos and unclear sentences, but with the help of future volunteers I hope to minimize those.

Chapter 1: Valuation Methods for Financial Engineers

INTRODUCTION

Problems in valuation extend far beyond the scope of this text. Most financial engineers will not be called upon to value paintings or used cars, so we may consider it safe to exclude these interesting valuations from consideration. However, financial engineers would clearly be expected to value financial instruments, like stocks and bonds, most of which would have no value were it not for the contractual or otherwise expected cash flows generated by those instruments. We may call these financial instruments *capital assets*. In this text, we define our coverage of assets to include both capital assets and *physical assets* whose values derive from the cash flows they produce, rather than the intangible benefits they provide to any individual or institution. Indeed, the valuation of intangible benefits is a far more difficult problem we cannot presently solve.

If we limited our valuation scope to capital assets and physical assets that generate cash, it would not suffice. We must also include contracts in our valuation models, such as derivatives, even though they may require little or no initial capital investment. Why? Because contracts may become significant assets or liabilities over time. If you have a contract to allow you the exclusive right to sell hot dogs at Yankee Stadium, the value of that contract may increase significantly over time. Hopefully you could eventually sell the rights of the contract and realize a gain on the trade. We may simply refer to this category of potential assets as *contracts*. Consequently, we may not ignore the valuation of liabilities, since one man's asset may be another's liability.

Finally, the valuation of assets in many cases depends on how the assets are managed, so any comprehensive discussion of asset valuation should include corporate assets and liabilities. These assets and liabilities are actively managed in order to maximize shareholder value. The values of these assets are determined in part by the choices made at corporate headquarters. These choices include the capital budgeting decision (when and how to invest), the capital structure decision (how to finance the company), and the risk management decision (how to select and manage the company's risk profile). Each of these decisions depends on how the company values the alternative available asset and liability management strategies: ultimately they are all valuation problems. Ultimately, every corporate decision of consequence can be thought of as a valuation problem.

Mathematically, the text will focus on assets or contracts that produce cash flows with known probability distributions, or at least cash flows that can be stochastically defined or simulated.   These probability distributions may be altered within defined bounds at the asset manager's discretion, in which case their values can be maximized. At any given time, we assume the risk-free term structure is known. Our goal is to value any cash flow that can be defined stochastically. We determine the risk adjustments to discount rates based on market benchmarks, or deduct risk charges first in order to discount cash flows at the risk-free rate.

Part I of the text focuses on valuation problems when discount rates are known. In this section, we assume the discount rates already include any risk adjustments, and we need only apply those discount rates to the expected cash flows in order to determine value. Part II offers several methods for valuing cash flows when the risk adjustments are unknown ahead of time. This means that the cost of risk, or the *risk charges*, are based on utility theory, or based on careful benchmarking to traded assets.

After completing the text, financial engineers will have learned a single consistent framework within which capital assets, contracts, and corporate assets & liabilities can be valued. In each section, the relevant theories of valuation are presented first, followed by detailed chapters on applications. The theoretically inclined student may wish to focus on the valuation theory chapters, but the practical financial engineer will find significant value in the detailed applications. There are also many opportunities in the text to develop or sharpen the student's Excel/VBA or Python programming skills.

ASSET TYPES

Assets may be tangible, such as a building or a ship. They may be intangible, such as a corporate investment opportunity, a celebrity's brand or even an education. Somewhere in between tangible and intangible assets lie capital assets, which are claims to cash flows. An example of a capital asset is a share of stock, which provides its owner with a percentage share of the appreciation and cash flows of an enterprise. The appreciation in price is called the *capital gain*, and corporate payouts to shareholders are called *dividends*. Bondholders receive interest payments or *coupons* over time in exchange for an initial cash investment, plus capital gains and scheduled *principal* payments. Owners of real estate receive capital gains and rental payments over time.

Can anything be a capital asset? Even owners of dairy cows receive cash from the sale of milk, and make payments for livestock feed, labor and facilities to produce the milk. One sh*udders* to think of the fatal consequences (to the cow) of a termination date, but this produces cash as well. Yikes. Even physical assets can often be thought of as capital assets if they can be used to produce cash flows. Coming full circle, buildings and ships generally have value because they produce cash flows for their owners, in the form of net rental fees and future cash proceeds from the sales of the assets. Even in the case of a private yacht used for personal cruises, the owner benefits from the use of the yacht while avoiding the payment of rental fees to others. However, in this case, there are no doubt intangible benefits of a private yacht in addition to the measured cash savings, making it difficult to fully value the yacht in a market context.

Assets can be affected by their circumstances, particularly the legal environment. For example, in the U.S., most corporations enjoy government-granted *limited liability* which protects them against claims that would exceed the value of its assets. For example, a drug company could go out of business to avoid paying claims to consumers who were suing for damages caused by the drugs. A firm with limited liability protection may abandon its assets at any time, even if there are no lenders involved. However, the decision is irrevocable.

In *bankruptcy*, firms have the right to default on their debts by surrendering the assets of the firm to the lenders, or *debtholders*. Again, the decision is irrevocable. Tax incentives, such as the tax deductibility of interest paid on debt issued by corporations, may increase cash flow to a firm's assets and therefore increase the value of the company. Therefore, the existence of limited liability, bankruptcy protection and tax subsidies all add value to a corporation's unenhanced assets. Shareholders in a limited liability company never have to pay a penny in damages to litigants, creditors or the government on behalf of the company in which they invested. They can never lose more than they invest. Therefore, the company may enjoy limited liability, and the individual shareholders may do likewise.

Contracts can also be thought of as contingent assets. Contracts have value because they commit two or more parties to actions that hopefully create value for all parties. Even if contracts have no value at inception, they may end up destroying value for one party over time, in which case the contract value to that party would be negative. For example, if a farmer commits to sell wheat in the future for $4 per bushel, but the market price in the meantime increases to $5, he faces an opportunity loss of $1 per bushel, and the value of the contract is negative to him.[2]

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[2] This type of contract is known as a forward contract, since the transaction occurs on a forward (future) date.

In all of these cases, it will be necessary to value assets or contracts from time to time. In the case of futures contracts, these revaluations occur continually. In the case of real estate, valuations occur mostly contemporaneously with financing, sales and purchases. This text has to do with valuing financial assets and contracts, regardless of the asset type or the reasons for valuation.

VALUATION MODELS

The *price* of an asset is the amount for which it is bought or sold. In a negotiation, there may be a price *offered* or *asked* by the seller, and another price *bid* by the buyer. The *transaction price* reflects the final agreed price of the asset at the time of sale. A market order to buy on an exchange will be executed at the ask price, and a market order to sell will be executed at the bid. **However, the *value* of an asset might differ significantly from its price.** Also, the *public value* of the asset likely differs from one party's *private valuation*. Indeed, one could argue that a public asset should only be bought by an individual if their private valuation meets or exceeds the public valuation.

In sum, the value of an asset is not a simple calculation. It is the result of a model, based on assumptions. When one prepares a valuation, assumptions are made, and therefore the results may always be questioned. The following quote from George Box and Norman Draper (1987) sums it up nicely:

> "Remember that all models are wrong; the practical question is how wrong do they have to be to not be useful."

Modelers should always humbly keep this aphorism in mind when they prepare valuation analyses. For example, present value is not the same as value; it is but one model for valuation. There are many possible models of course, and any one of them might be considered suitable for a particular purpose at a point in time. Six of these models are listed in the table below:

Some useful valuation models

#	Name	Description
1	Liquidation Value	How much the highest available bidder will pay for an asset if sold today
2	Replacement Cost	How much it would cost to replace or rebuild the asset
3	Comparable Value	The value of comparable assets, with valuation adjustments for imperfect comparables
4	Benchmark Valuation	Similar to comparables, but allows a wide range of assets and contracts to be used to replicate the cash flows of the asset as well as possible, using securities and assets whose prices are known.

5	Discounted Cash Flow	The present value of the net appreciation and cash flows that an asset creates incrementally; or, the amount of money one would have to set aside at the prescribed discount rate to produce a given pattern of future cash flows
6	Portfolio Valuation	The increase in the value of a portfolio as a result of the addition of the asset.

Liquidation Value is a popular and generally reasonable valuation model. However, it does not always apply. For example, we sometimes refer to the value of a *going concern*, i.e. an asset in the context of an operating business. In *liquidation*, the forced sale of an asset may necessarily reduce its ability to produce income. Method 1 may be interpreted in some situations as a liquidation value rather than a going concern value. In other cases, the definition of available bidders can have a significant impact on asset value. In general, the more restricted the bidder pool, the lower the inferred asset price will be.

Replacement Cost is also practical in many cases. A designer home may have cost $2 million to build the first time, but it would cost much less to duplicate. The owner may have "overpaid" for the asset because he or she placed a personal value on innovation, beauty or customization that went with the house. Not surprisingly, these owners often do not recoup the cost of designer homes when they are resold --- unless they can find buyers who value the unique characteristics of the home, and who do not have the ability to easily replicate it.

Comparable Value applies when assets can be directly compared. Two apartments in the same building, with the same features and similar views should be worth about the same amount. Therefore, if one apartment sells for $500,000, it can be used as a benchmark to estimate the value of a second apartment. But no valuation benchmark is perfect. One apartment may be in better physical condition than another, for example. Also, as time passes, the valuation benchmark becomes less relevant, since last year's sale price of one apartment may not reflect current price changes in the local real estate market.

Replacement Cost and Comparable Value are the most widely used valuation methods in accounting. Accounting valuation rules differ from financial valuation rules principally because accounting requires a degree of objectivity and verifiability --- accounting reports are for managerial and public consumption. Financial valuation is expected to be subjective and private, with no need for consistency or verification. Accounting rules can help prevent owners of assets from abusing subjective aspects of their measurements to overstate or understate asset value. However, accounting procedures can but also lead to values which would be unrealistic from a financial valuation standpoint.

Benchmark Valuation is used when the cash flows of an asset can be approximated by other assets or contracts whose valuations are known. For example, when modeling Amazon stock, you might expect that some of the company's performance is related to the retail sector, while some is related to high-tech comparables. Benchmark valuation chooses the portfolio of retail and tech stocks that best matches Amazon's projected earnings in order to determine a valuation.

Discounted Cash Flow valuation is used when we have a good sense of the expected cash flows that an asset will produce, and an appropriate discount rate reflecting the risk of those cash flows. In this chapter, we will take the discount rate as given, but in Part II, we will discuss models that can be used to determine appropriate discount rates or risk adjustments.

Portfolio Valuation is used when new assets interact with existing assets in a portfolio. For example, an oil drilling company with 50 wells already in operation will value a new oil well differently than a new entrant into the industry. The established company will likely be able to enjoy operating efficiencies and other *synergies* with the new acquisition, allowing them to pay a higher price for the well than less-established firms. To value the new well, the company would determine the value of the 51 wells together, and subtract the value of the 50 wells in order to determine a marginal private valuation for the new well. This methodology would allow the company to capture the value of synergies of the new well with their existing portfolio, and thereby come to a more accurate private valuation.

In traditional financial asset pricing models, portfolio valuation is unnecessary, since all parties value an asset in the same way. In practice, private valuations can differ markedly from public valuations ---- both much higher and much lower. In this text, private valuations are considered as important as public valuations. This is because many investments are either unavailable or too specialized to be of interest to the general public, especially in the early stages. Venture Capital firms, for example, often invest heavily in a small number of startup firms that lack access to the public markets. Accordingly, they value the portfolio not as well-diversified investors, but as investors with serious constraints on their capital and managerial resources.

Considering the goals of the text, we will focus mainly on *Benchmark Valuation*, *Discounted Cash Flow* and *Portfolio Valuation* methods. An asset for our purposes is anything that creates cash flows and capital appreciation or depreciation. The total return on asset is defined as the incremental net benefit produced by the asset over a period of time. The cash flows can be positive or negative, depending on whether we will be receiving or making payments. The change in asset value, called the *capital gain*, records the appreciation or depreciation of the asset over time.

RATE DEFINITIONS

Students will encounter many different *rates* in finance, and may be confused by many terms that sound similar at first. For example, we have already mentioned discount rates, which determine the annual discount applied to future cash flows to find their valuation today. Today's valuation is called the *present value*, and it is one way to compare the values of cash flows that occur at different times. We may also compare cash flows received in different time periods by computing a *future value*, which represents the value of the cash flows at some fixed future date. The time value of money reflects the percentage rate of change in the value of a riskless cash flow over time.

The rate of return reflects the expected growth rate of an asset or portfolio over time. The actual future rate of return may be unknown, but we usually assume the expected return is known or computable in traditional finance. The expected return depends on the price, and may be expressed as a rate in dollars or as a percentage. After time passes, we see the realized rate of return, no longer unknown. The *capital gain rate* reflects the change in the price of an asset, but is only one component of the total return, which may include cash payments as well.

In some analyses, we will use *growth rates* to describe the change in cash flows over time. The growth rate of cash flows should be distinguished from the rate of return on the asset composed of the cash flows. We use *default rates* to reflect the likelihood that the payer of the cash flows may fail due to bankruptcy or other reason. Normally, all rates are quoted on annual basis unless stated otherwise.

In the next sections, we will discover economic laws governing the determination of expected return rates, a critical determinant of asset prices.

PRICES, RETURNS, EXPECTATIONS AND REALIZATIONS

Suppose you purchased a share of Walmart for $70 one year ago, expecting it would pay a $1.00 dividend and appreciate in value by $6.00. Then, it actually paid $0.50 in dividends and appreciated by $10 in value. Then we could say any and all of the following:

- The expected one-year total dollar return was $7.00 (or 10%) one year ago
- The expected one-year capital gain was $6.00 one year ago
- The expected dividend was $1.00 one year ago
- The realized one-year total dollar return was $10.50
- The realized one-year capital gain was $10.00
- The realized one-year dividend was $0.50

These quantities are often expressed as percentages, when divided by $70.

For public valuation purposes, of course, if the share trades now at $80, that is the value. But suppose you are making a private investment decision. You might buy Walmart if you thought the public market valuation was too low, or sell Walmart (possibly by *selling short*[3]), if you thought the public valuation was too high. Your private valuation today might differ from the market because you have different expectations of future dividends and capital gains. How would you then value Walmart stock? *Hint: Do not skip this footnote!*

One answer is to project dividends and expected capital gains into the future. Once you did that, you would determine the opportunity cost of your cash, and how much you should be compensated for the risk. You would *discount* future cash flows and expected appreciation at the rate of return you require for putting up the cash and taking the risk. In the Walmart example earlier, in other words, you bought the share a year ago thinking that a 10% total return was sufficient to justify your investment of cash and your assumption of risk, i.e. at a minimum:

$$\text{Required dollar return} \leq \text{Expected total dollar return (\$7)}$$

But how would you determine the required return? If the investment funds are to be redirected from a profitable activity, such as risk-free lending, one would expect to be compensated just for having to divert cash. Then, to the extent the investment was risky, an additional return or premium would have to be earned. Therefore, the required dollar return comprises the time value of money and a risk charge. Putting this all together any buyer of a security would require:

$$\text{Return on cash} + \text{Compensation for risk} \leq \text{Expected capital gain} + \text{Expected cash flow}$$

and any seller would require the inequality to be reversed. In a competitive capital market, if everyone agrees on the assumptions, the two sides of the equation would have to converge, forcing the value of the asset or contract to satisfy the equation. This equality can be called the General Valuation Equation (GVE). It is an extremely useful tool which can be used to value many different kinds of assets. In this text, we focus on three broad categories:

- Cash flows
- Derivative instruments and contracts with embedded derivatives

[3] *Selling short* means that one borrows the asset and sells it, with an agreement that the borrower will repurchase the asset in the future and return it to its owner. Investors purchase stocks generally believing their value will increase; they sell short when they think the value will fall.

- Corporate assets and liabilities

Indeed, there is no single concept that is more important for financial decision-making than valuation, and therefore no equation more important than the General Valuation Equation.

RETURNS IN DETAIL

In a phrase, the word *returns* refers to the *cost of money*, or in a risk-free context, the *time value of money*. When we borrow, we pay the cost of borrowing. When we lend, we earn income, or a return from our lending. Usually, when people speak of returns in the future, they assume annual returns (e.g. 6% simple annual interest rate), even if they have to divide by 12 to get the simple monthly interest rate (e.g. 0.5%). When we speak of returns in the past, like "I made a 30% return on my house," we are likely to mean the return over the time period we owned the house.

As you can see, returns may be retrospective or prospective. Retrospectively, returns are a measure of performance, and tell us something about how successfully our money has been invested. Prospectively, we don't always know what return we are going to get, so we often describe returns as random variables. The *expected return* is the expected value of the unknown return. The *required return* refers to the minimum acceptable expected return on an investment.

What is your required return on capital? If I borrow money from you at 3% per year, assuming my default risk is low, would you be happy with that? The answer may differ for each person. For someone earning 1% interest (i.e. a 1% return) on a savings account at a bank, with no better alternative, a chance to earn 3% might look very attractive. For someone with low risk investments earning 8% on average, it might look unattractive to lend at 3%. Therefore, when we establish our required returns on investment, it is important to know our own *opportunity cost*, i.e. what we are giving up on our current investments to invest in a new project.

The term *discount rate* is used specifically to denote the rate of return used to discount future expected cash flows. It is also the return on investment if we purchase the cash flows at that price, the cash flows are realized, and the asset is held until the last cash payment is made. The term *interest rate* is usually used to describe the return on low risk or riskless investments, usually expressed as an annual return, and likely to differ for different maturities. For risk-free cash flows, you may see the terms *discount rate* and *interest rate* being used interchangeably.

WHEN RETURNS ARE UNDEFINED

Sometimes, the definition of returns can be difficult. In the Walmart example of the last section, we discussed dollar returns, which is the amount of increase in value one receives in an asset over a period of time plus any cash flows. We were able to divide these dollar returns by the price of Walmart stock at the beginning of the period to determine the percentage returns, since there was no possibility that additional cash would be required to fund the stock position. Said differently, because of limited liability, we can divide dollar returns by the share price to determine percentage returns because the share price represents the maximum investment we would have to make in one share of the stock. The share price also measures the maximum loss on investment. Much of traditional financial theory is based on percentage returns, but surprisingly, percentage returns often cannot be calculated, because the maximum loss is unknown.

For example, consider a short position in Walmart stock. Suppose you sell short at $70, the share drops to $60, and you repurchase the share to earn $10. What was your percentage return? You might be tempted to say $10/70 \approx 14\%$, but you would be incorrect. Why? Because if the share had gone to \$1070, you would have lost \$1000 per share! Your loss could even have been higher than that. Since your maximum loss is undefined, your percentage return cannot be computed. We do not have any idea (yet) how to measure the amount of capital required by the short position.

Similarly, suppose you enter a contract to buy a barrel of oil in one year's time for \$50. The amount \$50 is called the *forward price* since it specifies the price of a future transaction and not a current price. The value of the contract is zero at inception. If the forward price of oil rises to \$51, you could cancel your contract and take a \$1 profit. You might be tempted to say you made a return of 1/50, or 2%. Again, you would be wrong, since it cost you nothing to enter the contract, and you could have lost \$50. Your percentage return cannot be computed for the simple reason that division by zero is undefined. If you think you should use \$50 as your maximum loss, that is arguable, but oil is much less likely to go to zero than a share of stock, so \$50 would be very conservative.[4] Of course, if you had agreed to *sell* oil in one year's time, again, we would not be able to compute the maximum loss.

For these reasons, we use percentage returns in the case of assets that are purchased and paid for in cash, and have limited liability protection. This means percentage returns are limited to long stock, bond and option positions. For short positions and forward contracts, we must use dollarized returns. In this text, we generally use dollar returns.

Finally, a *very small* note on return vocabulary. A *basis point* is equivalent to 0.01%. Hence, 0.25% would be called either 25 basis points, 25 bp when written, or when spoken on a trading floor, 25 *"bips"*. In some situations, a *point* is used to refer 1% of a transaction's value. For example, a real estate mortgage broker might collect a transaction fee of one *point* on a \$250,000 mortgage loan, or \$2500 upfront.

THE MOST BEAUTIFUL EQUATION IN FINANCE

The most powerful valuation concept in finance can be captured in a single equation, which we mentioned briefly earlier in the chapter. This equation values cash flows, securities, derivatives, and corporate assets. In words, the equation states that the price of an asset must be set so that the required return on the asset will equal its expected return. Since the expected return is a function of the asset price, we can then often solve the equation to determine the asset value.

The General Valuation Equation (GVE)

Required return = Expected return

The required return is what the asset buyer "requires" or must earn for giving up the time value of money, plus some compensation for risk as appropriate. In this chapter and the next, there is no risk, so the compensation for risk is necessarily zero. In later chapters, the risk compensation will be the cost per unit of risk multiplied by the risk measure. With no risk, the required return is the risk-free rate of interest. The required return can also be thought of as the opportunity cost of funds for similar assets.

The expected return of the asset is related to how the asset is managed and what the asset produces. Normally, assets produce some combination of capital gains, i.e. appreciation or depreciation, and cash flows. In addition, we normally have a *boundary condition* which specifies the value of the asset at a point in time, usually at maturity. For example, if a stream of payments ends at time n, the value of the cash flows is 0 at

[4] There is one commodity that can trade at a negative price, which is electricity. This means that the seller would pay to deliver electricity, which they would only do if the cost of not delivering the electricity were higher.

time n, after the last payment has been made. If a business earns $1,000,000 a year for 20 years, and then can be sold for scrap value at that time for $500,000, then the boundary condition requires that the asset be worth $500,000 in 20 years.

As another example, suppose we have a bond that makes annual payments of $50 starting next year for 10 years. On the tenth year, we also receive $1000 in principal and the bond is retired. In this case, the capital gain of the bond is the change in value from one year to the next. The cash flow is the coupon being paid. The boundary condition is that the bond is worth $1000 after 10 years. Using these three facts, we can derive the value of the bond.

How can we derive the value from the capital gains, cash flows and boundary conditions? The GVE tells us how the returns are related, and the boundary condition ensures we have defined the asset value at a single point in time correctly. Together with a fixed valuation point and a law governing returns, we can find the current value of the asset. To see how this works, we begin with the *discrete time* case. This term implies that cash flows and valuations will occur on fixed dates, as opposed to the *continuous time* models, where cash flows and valuations change continuously in time. In practice, most corporate valuations are based on discrete time analysis, and most derivative valuations are based on continuous time analysis. Some applications require both, such as the valuation of a derivative contract at a number of set future dates. Examples of these applications will be covered in future chapters.

Let's take the simplest possible discrete time case, one for which you already know the answer. Begin with a single risk-free cash flow of "C" paid in one year. Its value today will be denoted as V_0, and the risk-free interest rate "r" is a simple annual risk-free interest rate. How can we determine V_0?

The GVE must hold over the next year

$$\textit{Required return} = \textit{Time value of money} + \textit{Compensation for risk}$$

$$\textit{Required return} = rV_0 + 0$$

The compensation for time value can also be thought of as the interest an investor would have earned on the funds needed to purchase the asset, had he kept those funds in an account earning the risk-free rate of interest --- a risk-free opportunity cost of funds, i.e. rV_0. There is no compensation for risk, since the cash flow is assumed risk-free in this example.

$$\textit{Expected return} = \textit{Expected capital gain} + \textit{Expected cash flow}$$

$$\textit{Expected return} = (0 - V_0) + C$$

The expected capital gain is $0-V_0$ since the asset holder gives up their asset for 0 at time 1, i.e. receives nothing more after the cash flow C is paid. After the cash flow payment, the asset is worthless. Alternatively, one may think of exchanging the asset V_0 for the final payment of C, in which case the capital gain would be $C-V_0$, and the additional "cash flow" would be 0. Both characterizations lead to the same result.

Now we may solve the GVE:

$$\textit{Required return} = \textit{Expected return} \qquad rV_0 = -V_0 + C \qquad V_0 = C/(1+r)$$

The value of a single risk-free payment of C in one year is C/(1+r), as expected. While this result is obvious and well-known, the purpose of the derivation is to introduce the GVE and how to use it. This knowledge will aid us in valuing a large class of simple and complex assets, including futures and options. The same equation that values a cash flow in one year can value the most complex derivative securities.

In later chapters, we will show that the GVE can be used to value all of the following:

- **Annuities**: Sequences of fixed or growing future payments
- **Perpetuities**: Infinite sequence of fixed or growing future payments
- **Stocks and bonds**
- Structured debt products
- Risky cash flows
- Futures and forward contracts
- Financial options and real options
- Corporate assets and liabilities

In addition, we will show that many of the famous equations in finance governing returns and prices are special cases of the GVE:

- Benchmarking models used for equity
- The Modigliani-Miller Proposition II
- The Capital Asset Pricing Model (CAPM)
- The option pricing differential equation of Black-Scholes-Merton (BSM)

This makes the GVE the most beautiful equation in finance!

REFERENCES AND FURTHER READING

Box, G. E. P., & Draper, N. R. (1987). **Empirical model-building and response surfaces**, Wiley series in probability and mathematical statistics. Empirical model-building and response surfaces. Oxford, England: John Wiley & Sons.

INTERVIEW QUESTIONS

Answer the following questions using no more than 2-3 sentences each. Be concise, and prioritize your responses to the more complex questions.

1. What is the difference between valuation and a valuation model?
2. What valuation models are primarily used by accountants and why?
3. What valuation models are primarily used by financial engineers and why?
4. How does a discount rate differ from a rate of return?
5. In what situations are percentage returns undefined?
6. What is the relationship between expected returns and required returns in a competitive capital market? Why does this relationship hold?

Chapter 2: Annuities and Perpetuities, Basic and Complex

Cash flows come in many forms. Sometimes it is convenient to be able to value common cash flow patterns quickly, using a formula instead of a spreadsheet or code.   This chapter therefore focuses on several different variants of annuities, with the objective of providing quick present value formulas. In all cases, as in all the Part I chapters, we assume the discount rate is known.

Annuities are defined as a sequence of equally spaced fixed payments, such as a fixed loan payment of $1000 per month for 24 months. *Perpetuities* are perpetual annuities, but these are rare outside of a few examples. By default, the first payment of an annuity is made in one period. A *deferred annuity* starts at a future date.

Annuities are often contracted to grow or shrink at a fixed percentage growth rate, in which case we must consider how the growth rate affects valuation. For example, an employment contract may provide for a 5% annual pay raise for an employee. In some instances, annuities may increase by a fixed dollar amount each period. In other cases, particularly when annuity payments are linked to commodity prices or interest rates, we may assume that the underlying index decays exponentially to a higher or lower long run mean level.

These annuities and perpetuities are all valued in this chapter, using the GVE to get the most efficient derivation. You may have previously learned other methods to value these instruments based on mathematical relationships. In this chapter we focus on the more intuitive GVE to obtain the values. This also sets the stage for more advanced valuations in the text using the GVE.

NOTATION

In order to proceed with the valuation of many different types of annuities, it is convenient to define notation that is consistently applied throughout the chapter. In valuing cashflows occurring at discrete intervals, or in discrete time, we will generally use time 0 as the present, and the indexes or subscripts "i" or "t" to represent floating point in time, and "s" to represent a fixed point in time. **Unless otherwise specified, the time step between cash flows is one period, the first cash flow occurs in one period, and the discount rate is the simple periodic rate.** We will address non-annual cash flows in Chapter 3.

Definition of terms

n,t = date of final payment, also in some cases, the number of payments

r = risk-free rate of interest, simple annual, or interpreted as a given discount rate

C = amount of a single cash flow, or a constant cash flow depending on context

C_i = cash flow on a particular date i

g = percentage growth rate in cash flow over a period, e.g. $C_2 = C_1(1+g)$

b = fixed dollar increase in a cash payment per period, e.g. $C_2 = C_1 + b$

μ = long-run mean (asymptotic) cash payment

κ = speed of reversion of cash flow to long-run mean per period; the expected percentage movement toward the long-run mean in one period

V_i = valuation of a financial instrument as of date i, so V_0 = value today

$V_1 - V_0$ = the capital gain on an asset over one period, after cash flows have occurred

Functions you will create later in the chapter in Excel or Python

PVA(C,n,r,V_n,s) = the present value of a flat annuity making n equally spaced payments of C, discounted at the rate r. V_n is any value paid in addition to the last payment, and s is the starting date of the first payment, taking time 0 as the present.

PVGA(C_1,n,r,g,V_n,s) = the present value of n cash flows discounted at rate r, where C_1 is the starting payment, g is the one period percentage growth rate in cash flows, V_n is the ending value (i.e. additional payment) after n cash flows have been paid, and "s" is the time period of the first cash flow

PVLGP(a,b,r,s) = the present value of a linearly growing perpetuity, where the increase in payment from one period to the next is a constant "b". The payment formula is given by a+bt at time t, and the first payment occurs at time s. The constant discount rate is r.

PVLGA(a,b,n,r,s) = the present value of a linearly growing annuity, where n is the number of payments, and the other parameters are defined as in PVLGP.

PVMRA(C_1,κ,μ,n,r,s) = the present value of a mean-reverting annuity with n periodic payments starting at time s, a long-run mean cash flow of μ, and periodic mean-reverting speed κ. The constant discount rate is r, and the first payment is C_1.

A SINGLE PAYMENT AT ANY FUTURE DATE

In Chapter 1, we valued a single risk-free cash flow in one year. We now begin with a two year time period, and use mathematical induction to derive the general case. To value a single payment in two years, it is helpful to work *backwards*. Using this method, we first value the asset at time 1, and then value the asset at time 0 knowing the value at time 1. Following this logic, at time 1, the GVE holds:

$$rV_1 = (0 - V_1) + C_2$$

and moving back to time zero the GVE must also hold, substituting the now known value of V_1:

$$rV_0 = (V_1 - V_0) + 0$$

Note how the capital gain is treated differently at times 0 and 1. At time 1, we receive the general capital gain on the asset $(V_1\text{-}V_0)$ and no cash flow. At time 2, we are giving up the asset and receiving a cash flow. Solving the first equation, we compute $V_1 = C/(1+r)$. Substituting this into the second equation, we arrive at the valuation of the cash flow in the second period

$$V_0 = C_2/(1+r)^2$$

Using this backward recursion technique, it follows that the value of a risk-free payment of C at any time i is given by

$$V_0 = C_2/(1+r)^i$$

Furthermore, the present value of a sequence of cash flows would be valued as the sum of the present values of the individual cash flows. The general formula for valuing a possibly infinite sequence of cash flows is then

$$V_0 = \sum_{i=1}^{\infty} \frac{C_i}{(1+r)^i}$$

As a practical matter, this is the most widely used discounting equation in finance. In practice, cash flows are usually variable and complex, and are not valued on handheld calculators, but by machines using spreadsheets and code. The valuation shortcuts proposed in this chapter only apply to special cases.

Even though this equation is the most general we will see in this chapter, it exhibits two important properties. From it, we can easily show that if all the cash flows are multiplied by the same constant, the resulting value equals the constant times the PV of the original sum. This is called *homogeneity of degree 1*. We will use this property together with additivity to derive many of the results in this chapter.

Exercise: You are expecting a legal settlement of $2,000,000 to be paid in 3 years. If your discount rate is 5%, what is the value of the claim today? If someone offers you $1.5 million for your settlement, paid today, should you take it?

Answer: The PV is $1.73 mm. If your true discount rate is 5%, you should reject the offer. If the cash allows you to do something else which is worth more than $0.23 mm to you, such as buying a house for your family, or paying for cancer treatments, perhaps you should take the offer. Remember, present value is only one model for value.

VALUING A PERPETUITY, OR A RECURRING PERPETUAL CASH FLOW

What if an asset makes a payment of C every year in perpetuity, starting one year from today? There are many ways to value this asset; we will demonstrate the traditional method and the GVE method. Perpetuities include debt obligations with no specified repayment date. Preferred stock, issued by corporations, promises to pay dividends perpetually at a fixed rate, meaning they can be valued using the perpetuity formula.

The traditional method notices that because of value-additivity of risk-free cash flows, we already have an equation

$$V_0 = \sum_{i=1}^{\infty} \frac{C}{(1+r)^i}$$

where the constant C has replaced the variable C_i. To solve this, let $d = 1/(1+r)$ for convenience. Then we have

$$V_0 = Cd \times [1 + d + d^2 + d^3 + \ldots]$$

The term in brackets is an infinite geometric series, which converges as long as d<1, i.e. r>0. Therefore

$$[1 + d + d^2 + d^3 + \ldots] = 1/(1-d)$$

$$V_0 = Cd/(1-d) = C/r$$

Now let's solve using the GVE. An asset that pays C in perpetuity today must be worth the same amount next year, after the first payment is made --- the future cash flow diagram stays the same. Therefore $V_1=V_0$ and the capital gain on V is always 0. Then, using the GVE, we get

$$rV_0 = 0 + C$$

$$V_0 = C/r$$

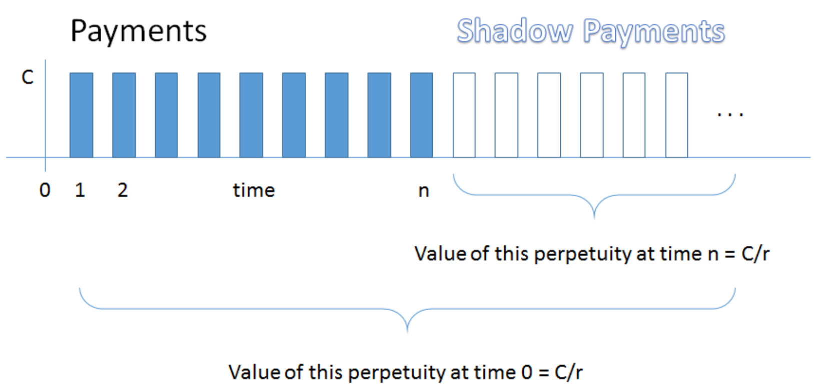
Isn't the GVE method simpler and more intuitive?

Exercise: If you expect a real estate investment to produce $100,000 per year net return forever, and your discount rate for real estate is 4% per year, what is the private value of your investment? If the market value is $3,000,000, should you sell the property?

Answer: The present value of the perpetuity is $2,500,000. You might sell the property for $3,000,000 if you could can earn the same 4% return, i.e. $120,000 per year, in another investment with similar risk.

VALUING AN ANNUITY, A FINITE NUMBER OF PERIODIC PAYMENTS

To value an annuity, we can use the traditional approach, express the annuity's value as the sum of its payments, and use the known formula for the sum of a finite geometric series. Perhaps the most popular approach is to observe that an annuity, a sequence of n payments of C, starting in one year for n years, is really the difference between two perpetuities, as shown in the figure below.



The value of all the payments, including the shadow payments, would have been C/r. However, the shadow payments have a present value of

$$\text{PV of shadow payments (the deferred perpetuity)} = \frac{C}{r}\left[\frac{1}{(1+r)^n}\right]$$

Therefore the value of a regular annuity is the difference between them

$$V_0 = \frac{C}{r}\left(1-(1+r)^{-n}\right) \quad \text{for } r\neq 0$$

$$V_0 = nC \text{ for } r=0$$

Note that the deferred perpetuity is discounted from time n, not n+1. This is due to the convention of these pricing models that the first payment is at the end of the first period (in this case n+1) and the valuation is at time 0 (in this case n).

Now let's try the GVE. The value of the annuity can be computed at time 0 and at time 1 for the purposes of determining the capital gain V_1 - V_0, which is simply the negative of the present value of the last payment.

$$V_1 = \sum_{i=1}^{n-1} \frac{C}{(1+r)^i}$$

$$V_0 = \sum_{i=1}^{n} \frac{C}{(1+r)^i}$$

$$V_1 - V_0 = \frac{-C}{(1+r)^n}$$

The required return is the risk-free rate, and the cash flow is C, therefore according to the GVE:

$$rV_0 = \frac{-C}{(1+r)^n} + C$$

$$V_0 = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right) \quad \text{for } r \neq 0$$

Exercise: Your client has been in an automobile accident and was awarded $50,000 per year for 25 years as a settlement, starting in one year. If his discount rate is 8%, what is the present value of the settlement? Assuming the insurance company is willing to double the annual payment as long as the present value stays the same, for how many years can he receive $100,000 per year?

Answer: The present value is about $534K. If the payment is doubled, it would last for only 7.24 years. Hint: If you had trouble with the second part, solve the annuity equation for n. If you get stuck, consider taking natural logs.

VALUING A SIMPLIFIED BOND

In its simplest form, a bond normally offers investors a combination of interest payments or coupons (C), together with a final repayment of principal, face value (F) or return of investment. For example, if a corporation were to issue a bond for $1000 with annual payments of 5% (C=50) for 10 years, it would return the principal in 10 years. As such, the valuation of a bond is quite straightforward, since it is the sum of an annuity together with a final payment.

$$V_0 = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{F}{(1+r)^n}$$

Exercise: What is the value of the bond described above if the discount rate is 4, 5, or 6%? ($1081.11, $1000.00, $926.40) What do we learn from this calculation?

The exercise shows us three facts:

1. Par bonds (for which the face value and the present value are the same) have a discount rate equal to the coupon rate
2. Higher discount rates are associated with lower bond prices
3. Bond prices are convex in the discount rate: They decrease in value at a decreasing rate as interest rates rise

Zero-coupon bonds (ZCBs)

A special case of a bond that pays no coupons is known as a *zero-coupon bond* or ZCB. Two examples of these are U.S. government-issued T-Bills and STRIPS, which are discussed in Chapter 3. These securities are useful to us if we want to value a stream of cash flows with different discount rates for different maturities. Of course, if the discount rate (r(t)) is known at time t, the value of a ZCB is its discounted face value (F), or $V_0=F/(1+r(t))^t$.

VALUING A GROWING PERPETUITY

We may have a situation where a payment recurs perpetually, but grows or declines by a fixed percentage every period. One example of this might be a perpetual cash flow adjusted for expected inflation. How would one value this?

Let's try the brute force method first. Allow g to represent the percentage growth rate from one period to the next. We can write the present value of the growing perpetuity directly by adding up the present values of all the payments, noting that $C_i = C_0(1+g)^i$:

$$V_0 = \sum_{i=1}^{\infty} \frac{C_i}{(1+r)^i} = C_0 \sum_{i=1}^{\infty} \frac{(1+g)^i}{(1+r)^i}$$

This is a convergent geometric sum as long as r>g. Making this assumption, and recognizing that the sum starts at one rather than zero, the summation is:

$$\sum_{i=1}^{\infty} \frac{(1+g)^i}{(1+r)^i} = \frac{1}{1-\frac{1+g}{1+r}} - 1 = \frac{1+g}{r-g}$$

Making the further substitution that $C_1 = C_0(1+g)$, we have $V_0 = C_1/(r-g)$.

Now let's try the GVE derivation. Suppose the first cash payment is C_1, and future payments grow at a percentage rate of g, so $C_2=(1+g)C_1$ and so on. Even if we don't know the value V_0 at time zero, we can see that at time 1, the cash flows have all been multiplied by (1+g), hence it must be the case that $V_1 = V_0(1+g)$.

From the GVE we know

$$rV_0 = (V_1-V_0) + C_1$$

therefore

$$rV_0 = V_0(1+g) - V_0 + C_1$$

$$\text{or } V_0 = C_1/(r-g)$$

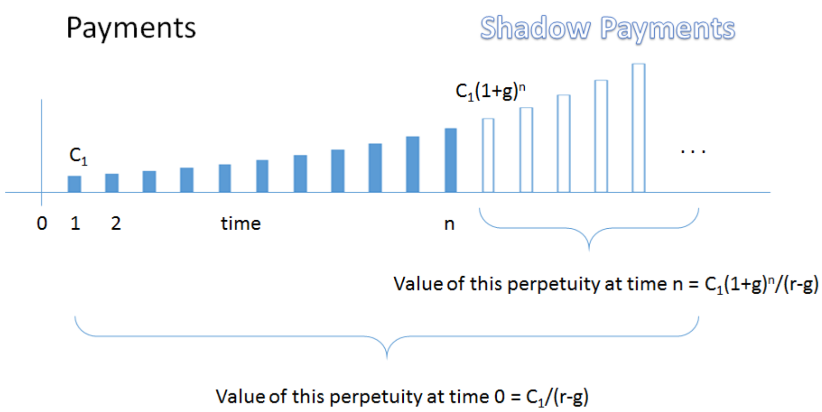
This equation only holds if $g<r$. If $g \geq r$ then the value is infinite.

It may be convenient at times to define $C_0=C_1/(1+g)$, and $r^*=(r-g)/(1+g)$. With these substitutions, the value of the growing perpetuity is expressed as if it were a constant perpetuity of C_0 starting one period from now and discounted at a rate of r^*

$$V_0 = C_0/r^*$$

VALUING A GROWING ANNUITY

Now suppose an annuity makes an initial payment of C, but the payment grows at a rate of g per year for n payments in total. Then using the same observation we made for an annuity, we can draw a chart for a growing annuity:



The value of the payments is the total value of payments and shadow payments, minus the present value of the shadow payments:

$$V_0 = \frac{C_1}{(r-g)}\left[1 - \frac{(1+g)^n}{(1+r)^n}\right] \equiv PVGA(C_1, n, r, g, 0, 1)$$

To simplify the computation, make the same substitution we made for a growing perpetuity

$$r^* = \frac{r-g}{1+g} \qquad\qquad C_0 = \frac{C_1}{(1+g)}$$

and then value the asset using the regular annuity formula:

$$V_0 = \frac{C_0}{r^*}\left(1 - \frac{1}{(1+r^*)^n}\right) \equiv PVGA(C_0, n, r^*, 0, 0, 1)$$

This is particularly useful when using financial calculators or spreadsheets to compute the value of a growing annuity, or any time the function is not provided.

It is interesting to note that the formulas for a growing annuity are valid *even if* g>r. If g=r, then growth exactly cancels out discounting, so the value of the annuity is $V_0 = nC_0$, the same as an annuity when r=0.

Repeating the previous exercise, but adding an inflation adjustment:

Exercise: Your client has been in an automobile accident and was awarded $50,000 next year **growing at 2% per year** for a total of 25 annual payments as a settlement. The 2% growth rate is meant to cover expected increases in lost wages. If his discount rate is 8%, what is the present value of the settlement? Assuming the insurance company is willing to double the monthly payment for the same present value, for how many years can he receive $100,000 per year? Answers: $634K, 8.37 years.

SUMMARY OF DISCRETE TIME VALUATIONS SO FAR

This table summarizes the discrete-time cash flow types we have valued so far, using a risk-free payment or first payment of C, an annual growth rate of g where applicable, and a constant risk-free annual interest rate of r.

Type of Cash Flow	**Present Value**	**Restrictions**
Single payment of C at time n	$V_0 = C/(1+r)^n$	
Any payments C_i at different times i	$V_0 = \Sigma_1^\infty C_i/(1+r)^i$	
Perpetuity of C per period forever	$V_0 = C/r$	r>0
Annuity payment of C for n periods (first payment in one period)	$V_0 = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)$ *for* $r \neq 0$ $V_0 = nC$ *if* $r=0$	
Growing perpetuity	$V_0 = C/(r-g)$	r-g>0
Growing annuity	$V_0 = \frac{C_1}{(r-g)}\left[1 - \frac{(1+g)^n}{(1+r)^n}\right]$ $V_0 = nC_1$ *if* $r-g=0$	
Growing annuity equivalent (using the regular annuity formula)	$V_0 = \frac{C_0}{r^*}\left[1 - \frac{1}{(1+r^*)^n}\right]$ *for* $r^* \neq 0$ $r^* = (r-g)/(1+g)$ $C_0 = C_1/(1+g)$ $V_0 = nC_0$ *if* $r^*=0$	

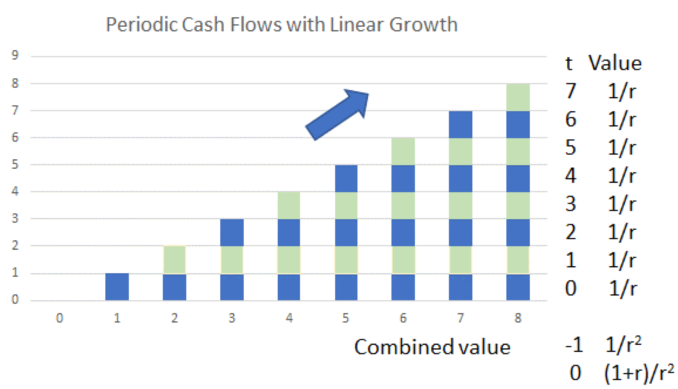
ADVANCED ANNUITIES: LINEAR GROWTH AND MEAN REVERSION

In practice, one encounters annuities of various types that do not fit the constant or exponential growth model discussed in the last section. The two most common of these alternative models are *linear growth* and *mean reversion*. These annuities were chosen because they are the discrete equivalent of geometric random walks, arithmetic random walks, and Ornstein-Uhlenbeck mean-reverting stochastic processes, which are explained in detail in Chapter 6.

LINEAR GROWTH

In a linear growth payment pattern, the equally spaced payments increase by a constant each period. For example, a broker who introduces a client to a bank might receive a step-down compensation scheme that defers part of his commission. The broker's payment might be \$100 in year 0, \$80 in year 1, \$60 in year 2, and so on until zero is reached. The broker would of course want to know the present value of his payments.

To see how to value linear-growth annuities, let's begin with a perpetuity, as we did when we first studied flat annuities. Suppose a payment stream begins with \$1 in one year, \$2 in two years, \$3 in three years and so on indefinitely, as in the figure below:



To value this particular annuity, it is useful to "cut" the cash flows horizontally, as shown with the alternate color shading. Then, we can see that the linear growth perpetuity can be valued as the sum of an infinite number of perpetuities. The first perpetuity, the bottom block in each column shaded darker, pays \$1 per year forever, and is worth $1/r$ at time 0 --- this is recorded in the right column under "value". The second perpetuity, the second-from-the-bottom line shaded lighter, pays \$1 per year forever starting at time 2, so it is worth $1/r$ **at time 1**. The present value of the perpetuity of perpetuities, i.e. all the cash flows, is therefore

$$V_{-1} = (1/r)/r = 1/r^2$$

at time -1. The present value at time 0 is $(1+r)$ times higher, or $V_0 = (1+r)/r^2$.

Alternatively, we can do this the easy way! After the first payment, we can see that the remaining payments are all exactly one dollar higher than at time 0, hence the value of the capital gain $V_1 - V_0 = 1/r$. Substituting into the GVE, we have

$$rV_0 = (V_1 - V_0) + C_1 = \frac{1}{r} + 1$$

implying $V_0 = (1+r)/r^2$. ☺

Abstracting to the more general case, suppose we want to value a perpetuity that pays a+bt at unit intervals starting at time s. This can be decomposed into a perpetuity worth (a+bs)/r at time s-1, and a perpetuity of perpetuities worth b/r each starting at time s, worth b/r^2 at time s-1. Hence the present value of the linear growth perpetuity with payments starting at s is given by

$$PVLGP(a,b,r,s) = \frac{\left(\frac{a+bs}{r} + \frac{b}{r^2}\right)}{(1+r)^{s-1}}$$

The annuity of n payments can then be valued as the difference between two perpetuities, with at least one of them deferred. For example, if the annuity payment starts at time 1 and ends at n, we have

$$PVLGA(a,b,n,r,1) = PVLGP(a,b,r,1) - PVLGP(a,b,r,n+1)$$

Please note, as with exponentially growing annuities, it may be helpful to define a=C_0=C_1-b, so that the payment at time 1 equals the desired C_1.

Exercise: A new B2B (business to business) marketing program expects to bring in one major client per year for the next 15 years. If each client is worth $50,000 per year to the company in perpetuity, and the corporate discount rate is 12%, what is the value of the marketing program? Answer: About $3.2 mm

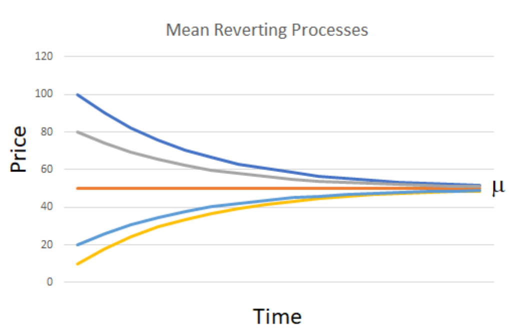
MEAN REVERSION

Mean-reversion occurs frequently in the modeling of interest rates, foreign exchange rates and commodity prices. For example, when oil prices are high, demand falls while supply increases, forcing future prices downward. When oil prices are low, demand for oil increases and supply decreases, causing prices to rise in the future.

If oil follows a simple mean reverting process, then the expected spot price (S) moves in the direction of the long-run mean price mu (μ, pronounced "mew") at a speed of kappa (κ), as described by the following equation:

$$S_{t+1} = S_t + \kappa(\mu - S_t)$$

For different possible starting values of S, we might see either an upward sloping or downward sloping exponential decay towards the long run mean, as in the figure below.



Solving the above equation recursively, we can see that S_{t+j} can be expressed as a function of S_t and the parameters:

$$S_{t+j} = \mu + (S_t - \mu)(1-\kappa)^j$$

Note that the equation works equally well for upward and downward sloping curves, since S_t - μ changes sign. Also, if a cash flow is linked to this formula, it can be expressed as the sum of a fixed cash flow and an exponentially growing cash flow. The growth rate is -κ, and the starting value is S_t - μ.

Therefore we have the present value of a mean-reverting cash flow starting in one period is given by

$$PVRMA(C_1, \kappa, \mu, n, r, 1) = PVA(\mu, n, r, 0{,}1) + PVGA(C_1 - \mu, n, r, -\kappa, 0{,}1)$$

Exercise: Farmer John believes that corn prices, currently \$3.50 per bushel (bu) will rise to a long run mean of \$5/bu over time, at a rate of 10% per year. The first harvest is one year away, and Farmer John expects to operate the farm for 20 years. If the discount rate is 5%, and he produces 1,000,000 bu of corn per year, what is the present value of his revenues? Check your answer by computing all the cash flows in Excel. Answer: \$53.7 mm

PRESENT VALUE IN PRACTICE: USING MICROSOFT EXCEL

Some professionals use financial calculators on their jobs to provide instant answers to their clients' financial questions. These professionals include real estate agents, mortgage brokers, bond salespeople, leasing specialists, and financial advisors, as examples. In most cases, bankers and corporate finance professionals do all their analysis in Excel, because of its flexibility, transparency and ease of sharing files among peers and clients. For this reason, it is important for financial engineers to learn Excel --- it provides a way to analyze financial data and share it with colleagues in an open format that every finance professional understands. Furthermore, complex calculations available in VBA, C++ or Python can be incorporated into Excel when additional firepower is needed. Advanced proficiency in Excel should be listed among the computing skills on your resume!

There are many basic Excel tutorials online available free of charge, so we will focus only on the financial functions here. This is a list of functions directly relevant to the materials covered in this chapter. Arguments in square brackets are optional. Parenthetical values refer to the equivalent notation used in this chapter.

Excel Function	Arguments	Purpose and Use
=PV(rate,nper,pmt,[fv],[type])	rate=periodic interest rate (r) nper=number of periods (n) pmt=payment (C) fv=future value of annuity after all payments are made type=1 if first payment is immediate, 0 if in one year	Finds the present value of a flat annuity (PVA)
=NPER(rate,pmt,pv,[fv],[type])	pv=present value	Finds the value of n that makes the present value of the flat annuity equal to pv
=RATE(nper,pmt,pv,[fv], [type],[guess])	guess=initial value guess to speed up calculation	Finds the periodic interest rate that makes the present value of the payments equal to pv.
=EFFECT(nominal_rate,npery)	nominal_rate=simple annual interest rate (r) npery=number of compounding periods per year	Finds the effective annual return for cash flows compounded more than once per year
=NPV(rate,value1,[value2],…)	value1=first cash flow in one period (C_1)	Finds the present value of a range of equally spaced cash flows at the given rate.
=IRR(values,[guess])	values=set of cash flows	Finds the rate that makes a sequence of cash flows have pv=0. *May not be unique if there are more that one sign changes in the cash flow.
=XNPV(rate,values,dates)	dates=Excel-format dates corresponding to the cash flows	Finds the present value of a set of cash payments made on specific dates.
=XIRR(values,dates,[guess])		Finds the discount rate that makes the present value of a set of cash flows on given dates equal to zero.

REFERENCES AND FURTHER READING

PRESENT VALUE EXERCISES

Note: For many of these problems, it is helpful to think in terms of periods of time rather than years. For example, if a problem has a simple annual interest rate of 6% but payments are made monthly, compute the simple monthly rate of 0.5%, and count months instead of years. These problems are meant to be solved by hand, but may be solved or checked in Excel as well, depending on your instructor's preferences.

1. A government bond pays $100 in coupons per year starting in one year for 8 years, and $1000 principal along with the last coupon. If the discount rate is 4% per year, what is the value of the bond today?
2. A homeowner wishes to borrow $500,000, but must repay interest and principal with monthly flat payments over 30 years. If the simple annual interest rate is 6%, i.e. the monthly rate is 0.5%, what is the monthly payment? What is the remaining balance (amount owed) after five years of payments?
3. Suppose a bank wanted to offer the borrower in #2 an option to have the payments grow at 1% a month for the full 30 year period. What would the initial payment be in this case, in order to have the same present value?
4. A retiree saved $250,000 and wants to withdraw $5000 per month for expenses, starting next month. How many months (n) will the money last if the annual discount rate is 5%? What is the general formula to solve for n as a function of r? Hint: You may need to take a natural log.
5. How does your answer to #4 change if the retiree increase the withdrawal by 0.5% per month?
6. Annuity Plan A has an initial annual payment that starts at $10,000 and grows at 5% per year. Plan B has a flat payment of $12,500 for the 10 years.
    a. Which 10 year annuity is better, if the annual r=8%?
    b. At what discount rate would the annuity owner be indifferent between the two plans?
7. An alternating annuity pays $1 in one year, –$2 in two years, $4 in three years, –$8 in four years, which each subsequent payment doubling and changing sign for 20 years. If the risk-free discount rate is 5%, what is the value of this alternating annuity? (*Hint:* Check your result with Excel.)
8. A company plans to pay a dividend of $0.50 per share next year rising by $0.25 per year until it reaches $2.50, at which point it will remain flat forever. If the discount rate is 9% per year, what is the present value of the dividends, i.e. the value of a share of stock?
9. The price of oil is currently $50 per barrel (bbl), but the long-run mean is believed to be $75, and the annual speed of mean reversion is 15%. An oil drilling company wishes to value a prospective new property. The financial details are below:
    a. Production levels will start at 500,000 bbl for the first 10 years, and will drop to 250,000 bbl for the next 10 years, at which time the oil will be exhausted.
    b. The fixed costs of producing the oil will be $5,000,000 per year.
    c. The variable costs of producing oil will be $10 per barrel
    d. The discount rate is 9% per year, and the first cash flow is in one year.

    What is the value of the oil well today?

EXCEL EXERCISES AND YOUR PERSONAL VBA OR PYTHON FUNCTION LIBRARY

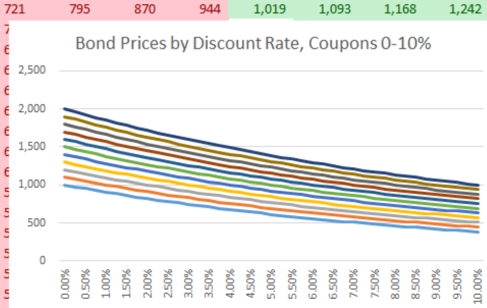
A. Consider a bond that makes annual payments. For example, a 6% coupon bond with face value $1000 pays $60 per year in payments. Create a "bond table" that would provide a reference to a bond trader. The trader chooses a bond term in one cell of the spreadsheet. The table shows coupon rates (0%-10% in 1% increments) across the columns and discount rates (0-10% in 25 bp increments) across the rows, and in each cell of the report, the bond price is quoted. Using your table, answer the following questions.

a. Graph the bond prices as a function of discount rates. How would you describe the relationship between discount rates and bond prices?
b. On your table, use conditional formatting to identify bonds that trade at a premium or discount to par value of $1000.

Your solution should look like this:

Face value	1000
Bond term (years)	10

		Coupon rates										
		0.00%	1.00%	2.00%	3.00%	4.00%	5.00%	6.00%	7.00%	8.00%	9.00%	10.00%
Discount rates	0.00%	1,000	1,100	1,200	1,300	1,400	1,500	1,600	1,700	1,800	1,900	2,000
	0.25%	975	1,074	1,173	1,271	1,370	1,469	1,567	1,666	1,764	1,863	1,962
	0.50%	951	1,049	1,146	1,243	1,341	1,438	1,535	1,632	1,730	1,827	1,924
	0.75%	928	1,024	1,120	1,216	1,312	1,408	1,504	1,600	1,696	1,792	1,888
	1.00%	905	1,000	1,095	1,189	1,284	1,379	1,474	1,568	1,663	1,758	1,852
	1.25%	883	977	1,070	1,164	1,257	1,350	1,444	1,537	1,631	1,724	1,818
	1.50%	862	954	1,046	1,138	1,231	1,323	1,415	1,507	1,599	1,692	1,784
	1.75%	841	932	1,023	1,114	1,205	1,296	1,387	1,478	1,569	1,660	1,751
	2.00%	820	910	1,000	1,090	1,180	1,269	1,359	1,449	1,539	1,629	1,719
	2.25%	801	889	978	1,066	1,155	1,244	1,332	1,421	1,510	1,598	1,687
	2.50%	781	869	956	1,044	1,131	1,219	1,306	1,394	1,481	1,569	1,656
	2.75%	762	849	935	1,022	1,108	1,194	1,281	1,367	1,454	1,540	1,626
	3.00%	744	829	915	1,000	1,085	1,171	1,256	1,341	1,427	1,512	1,597
	3.25%	726	810	895	979	1,063	1,147	1,232	1,316	1,400	1,484	1,569
	3.50%	709	792	875	958	1,042	1,125	1,208	1,291	1,374	1,457	1,541
	3.75%	692	774	856	938	1,021	1,103	1,185	1,267	1,349	1,431	1,513
	4.00%	676	757	838	919	1,000	1,081	1,162	1,243	1,324	1,406	1,487
	4.25%	660	740	820	900	980	1,060	1,140	1,220	1,300	1,381	1,461
	4.50%	644	723	802	881	960	1,040	1,119	1,198	1,277	1,356	1,435
	4.75%	629	707	785	863	941	1,020	1,098	1,176	1,254	1,332	1,410
	5.00%	614	691	768	846	923	1,000	1,077	1,154	1,232	1,309	1,386
	5.25%	599	676	752	828	905	981	1,057	1,134	1,210	1,286	1,362
	5.50%	585	661	736	812	887	962	1,038	1,113	1,188	1,264	1,339
	5.75%	572	646	721	795	870	944	1,019	1,093	1,168	1,242	1,317
	6.00%	558	632									1,294
	6.25%	545	618									1,273
	6.50%	533	605									1,252
	6.75%	520	591									1,231
	7.00%	508	579									1,211
	7.25%	497	566									1,191
	7.50%	485	554									1,172
	7.75%	474	542									1,153
	8.00%	463	530									1,134
	8.25%	453	519									1,116
	8.50%	442	508									1,098
	8.75%	432	497									1,081
	9.00%	422	487									1,064
	9.25%	413	476									1,048
	9.50%	404	466									1,031
	9.75%	394	457	519	581	643	705	767	829	891	953	1,016
	10.00%	386	447	508	570	631	693	754	816	877	939	1,000



Excel skills learned: PV function, absolute and relative addressing, formatting and conditional formatting, graphing.

B. Create and test the following VBA functions to run in Excel. Make sure to trap any errors. Also, test your models to ensure they deliver the correct results.

a. PVA
b. PVGA
c. PVLGP
d. PVLGA
e. PVMRA

This will form a basis for your own personal VBA library.

There are many online resources available to show you how to build VBA functions. It is not difficult! You may search in Google for "create custom vba excel functions", for example, the Microsoft resource link is:

https://support.office.com/en-us/article/Create-custom-functions-in-Excel-2F06C10B-3622-40D6-A1B2-B6748AE8231F

CHALLENGE EXERCISES

1. Derive the valuation formula for linearly growing annuities using the GVE.

2. Using the GVE, find the value of a growing perpetuity that pays t^2 on every payment date t starting one period from now. HINT: $(t+1)^2 = ?$ Value the perpetuity for an arbitrary starting time s.

WORKSHEET EXERCISES

Fill in the following table using the GVE for discrete time valuations. Assume all payments are annual, and the first payment is in one period.

Name		Capital Gain		Cash Flow	Boundary Condition	Valuation
Single payment in one period	$rV_0=$	$0-V_0$	+	C	$V_1=0$	$V_0=C/(1+r)$
Constant perpetuity	$rV_0=$		+			
Constant annuity	$rV_0=$		+			
Growing perpetuity	$rV_0=$		+			
Growing annuity	$rV_0=$		+			
Linear growing perpetuity	$rV_0=$		+			
Linear growing annuity	$rV_0=$		+			
Mean reverting perpetuity	$rV_0=$		+			
Mean reverting annuity	$rV_0=$		+			

MINI-PROJECT: CONSUMER LOANS, POINTS AND THE ANNUAL PERCENTAGE RATE (APR)

An important term used in consumer loans is the APR, or *annual percentage rate*. Be careful with this one, because definitions can vary! APR is used to integrate financing costs into the determination of the all-in cost of a loan. For example, suppose Loan A offers you a 5% simple annual rate with monthly payments for 5 years, but Loan B offers you a 4.5% rate for the same loan, as long as you pay 1.5% in fees to the lender, or 1.5 *points* either deducted from the loan proceeds at inception or financed. APR is designed to help you choose between these two loans. In fact, the U.S. Truth in Lending Act (TILA) requires lenders to supply an APR to their prospective retail customers.

To see which loan is "better", let's assume the loans above are both 5 year loans for $100,000. Let's use Excel to figure out the APR:

Term	5	years
Months	60	
Loan amount	100,000	
Fees (deducted)	1,500	

	Rate	Payment	Actual borrowed	Monthly rate	APR	EAR
LOAN A	5.00%	$1,887.12	100,000	0.42%	5.00%	5.12%
LOAN B	4.50%	$1,864.30	98,500	0.43%	5.12%	5.24%

You should verify these calculations using your Excel spreadsheet or financial calculator. HINTS: Use the PMT and RATE functions.

The payment on the loans is computed using the standard flat annuity formula, since the loans must be fully paid off at the end of five years, and the payment must be constant. The borrowed amount for Loan B is $98,500, reflecting the origination costs paid to the lender. The monthly rate is the rate that makes the present value of the computed payments equal to the loan proceeds. The bottom line: Loan B is more expensive than Loan A when the financing costs are included, by 12 bppa (basis points per annum).

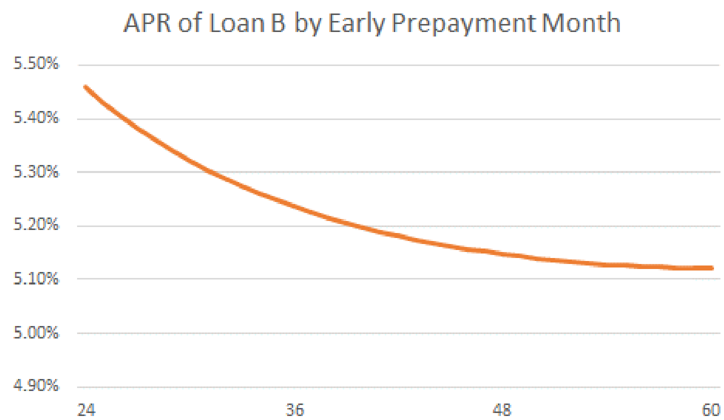
Note that the APR is not an effective annual rate. If you wanted to compare these loans to loans with payments of different frequencies, it would be necessary to convert all of them to effective annual rates to determine the least expensive loan. For the purposes of this problem, it is sufficient to compare APRs, and we do not need to compute an effective annual rate. Furthermore, nearly all consumer loans have monthly payments, so APR works well in practice. Or does it?

> Question: Quicken Loans offered a 30-year fixed rate loan with monthly payments at a contract interest rate of 3.875% per year. The APR was 4.136%. What was the effective annual interest cost in percentage terms? Assuming the points were financed, i.e. included in the calculation of the loan amount, what was the implied fee as a percentage of the original loan amount? (4.215%, 3.10% or 3.1 points)

Questioning the usefulness of APR

Clearly, the APR is a useful tool for comparing similar consumer loans. However, the APR calculation implicitly assumes that loans will be held to maturity. In reality, while many unsecured consumer loans may be paid off at maturity, nearly all mortgage loans are paid off prior to maturity. In finance, we call this loan *prepayment*. The possibility of loan prepayment means that the standard APR calculation could cause borrowers to underestimate their true financing costs.

In particular, with fewer compounding periods to pay off the origination fees, the APR calculation not only understates the origination costs, it is actually the lowest possible annualized loan cost! To see this, reconsider our last example but now compute the APR for different time periods. The results are shown graphically here:



For example, if you pay off your loan in the 24th month, the effective APR is 5.46%, not 5.12%. Of course, people generally don't know in advance when they will pay off their loans. Nevertheless, borrowers who have some idea that they will prepay their loans early should beware of loans with higher origination fees.

Problem:

Develop an Excel spreadsheet to help a borrower estimate their actual all-in APR based on their own payoff assumptions. For example, the user would put in a number for each month indicating the percentage of the loan balance that would be expected to be paid off in that month. Of course, these numbers should add to one.

Your spreadsheet should be intuitive enough for an average student to be able to use it to make a loan decision between Loans A and B. Provide an example where A is preferable, and where B is preferable.

Chapter 3: Complications and resolutions

In Chapter 2, we valued some cash flow patterns commonly seen in practice, but we made several simplifying assumptions. First, we assumed that interest was compounded once per period, that the first cash flow occurred in one period, and that the cash flows were fixed in dollars. Cash flows don't generally follow such simple patterns. Second, in practice, cash flows are affected by inflation and also by taxes; hence we must be able to adjust our valuations to reflect these phenomena. Third, when we are receiving cash payments, we are always concerned with events that may cause our cash flows to cease, for example, a default by the payer, the death of the receiver, or a contract provision triggering early termination. Fourth and finally, we would like to be able to value cash flows that correlate with future unknown interest rates or inflation rates.

Perhaps the most important valuation adjustment we will make in this text will be for risk, but this is left for Part II, where we will explicitly address risks in cash flows. For now, Chapter 3 may help you value cash flows in practice, even some risky cash flows, and apply the Chapter 2 formulas more broadly. To help you understand all the rate adjustments in this chapter, we offer you the following lexicon as a reference:

RATE	DEFINITION
Simple annual rate	The discount rate used to discount annual cash flows, e.g. compounded once a year
Periodic rate	The simple annual rate divided by the number of periods a year
Effective annual rate	The rate of interest earned or paid for a year considering the effect of periodic compounding and earning interest on interest
Nominal rate	The discount rate used to value actual dollars paid (ignoring inflation)
Real rate	The discount rate used to value inflation-adjusted cash flows, or the rate of growth in spending power of savings
Spot rate	The annualized rate of return on a bond bought today and held to maturity.
Forward rate	The year-by-year implied future returns embedded in a bond price.

CHANGING THE FREQUENCY OF PAYMENTS

But it is not particularly easy for one to climb up out of the working class—especially if he is handicapped by the possession of ideals and illusions. I lived on a ranch in California, and I was hard put to find the ladder whereby to climb. I early inquired the rate of interest on invested money, and worried my child's brain into an understanding of the virtues and excellencies of that remarkable invention of man, compound interest.

- Jack London, 1906

In Chapter 2, we assumed all payments were annual, and were discounted at the simple annual interest rate. Suppose we have payments every month, like consumer loans? Or every quarter (3 months), like some corporate bonds? Or every half-year, like some corporate bonds and all U.S. government bonds? How are valuations affected? In practice, we usually simply divide the quoted simple annual rate by the number of periods per year. Therefore, if r is the quoted simple annual rate, a monthly time step would imply a simple monthly interest rate of r/12. A 6% interest rate on a loan with monthly payments therefore has a monthly interest rate of 0.5% or 50 bp per month.

At first, this concept may puzzle some students. If we invested $1,000,000 and earned 0.5% per month for 12 months, the future value would be $1{,}000{,}000 \times (1.005)^{12} =$ 1,061,678, which really amounts to 6.17% over the year, well over 6% per year thanks to the effect of compounding interest. That is because interest is being paid on the interest. This seems like the more honest calculation, since it considers the return using actual compounding --- multiplying and dividing an interest rate by a constant seems intuitively incorrect, even though it is both convenient and a widely used convention.

It is important to realize that in almost every capital market, there are different legacy quoting conventions, such as the simple annual rate, but usually some adaptation needs to be made in order to compare effective rates of return between investments. To help investors or borrowers distinguish between the quoted rate and the actual earned return, we often call the correctly compounded return the *effective annual yield* or *effective annual rate*.

Quick quiz 1: If the simple annual rate is 8%, and interest is compounded quarterly, what is the effective annual rate? (8.24%)

Quick quiz 2: If the effective annual rate is 12.55%, and the payment frequency is quarterly, what was the simple annual rate? (12%)

To summarize in an equation, if r is the simple annual interest rate, y is the effective annual rate, and n is the number of compounding periods per year:

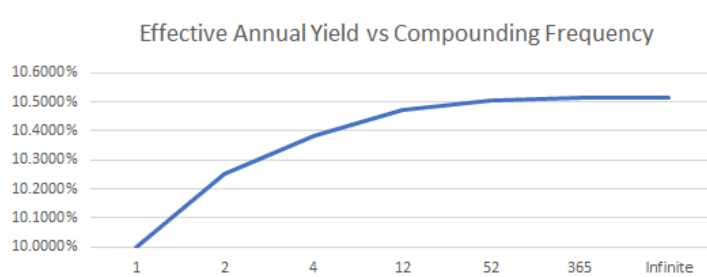
$$1 + y = (1 + r/n)^{n}$$

The effective annual rate is rarely computed in practice when simple rates can be compared among similar instruments. However, it is a necessary tool in careful return analysis, particularly when comparing the returns of assets with different payment frequencies.

Continuous compounding

As we saw in the last section, increasing the compounding frequency to 2, 4 or 12 times a year increases the effective annual return. What would happen if we increased to 52, 365, or even an infinite number of periods per year?

Frequency	EAY
1	10.0000%
2	10.2500%
4	10.3813%
12	10.4713%
52	10.5065%
365	10.5156%
Infinite	10.5171%



This exhibit shows the effective annual yield for a 10% simple annual rate at different compounding frequencies. The maximum rate that can ever be attained is 10.5171%. As the number of compounding periods tends to infinity, the discrete model is said to become *continuous*. We will refer to the 10% figure as the continuously discounted rate, and continue to refer to 10.5171% as the effective annual yield.

In general, we may continue to interpret r as the simple annual interest rate, but $e^r - 1$ is the effective annual rate when the number of compounding periods is infinite. This can be derived by taking the limiting value of a $1 investment in one year, and compounding n times a year. As n goes to infinity, the future value of $1 is given by[5]:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

This means that the present value of any payment C_t at time t, discounted continuously at the simple annual rate of r is given by

$$PV = C_t e^{-rt}$$

In many applications, you may see all discount rates converted to their continuously compounded equivalents. Like the EAY, this has the benefit of eliminating differences in yield calculations caused by cash flows being discounted at different periodic rates.

Fortunately, there is an easy conversion between effective annual yield and the continuously compounded interest rate:

$$r_{cont} = \ln(1 + EAY)$$

Question: If an investment offers 3% simple return per half-year, what is the equivalent continuously compounded discount rate? (5.91%=$\ln(1.03^2)$)

In practice, continuously compounded rates are rare in corporate finance, but are used almost exclusively in valuations involving derivatives. For this reason, the financial engineering student must be very comfortable with continuous discounting.

Payment frequency and discounting frequency

When you encounter problems with different payment frequency and discounting frequency, it is best to begin by computing the discount rate corresponding to the payment frequency. For example, the interest rate on consumer loans is quoted on an annual basis, but payments were made on a monthly basis. In this case, we divide the simple annual rate by 12 in order to get the monthly rate, understanding that this is the standard rate convention --- ignoring the temptation to “de-compound” the return.

Now suppose we have a biennial payment of $10,000 starting in one year, repeating every second year for 5 payments. If the simple annual discount rate is 8%, what is the present value? We’d like to use the annuity formula to find the present value, but there are two problems. First, the quoted interest rate is annual, while

[5] This can be verified using L’Hôpital’s rule in computing the limit of the natural log.

payments occur every two years. Second, the first payment is in one year, which is in one-half a period into the future.

To solve this problem, we first compute the effective two-year discount rate, which is 1.08^2-1 or 16.64%. Then we find the PV of the annuity, or $32,260.01. But this is the present value *a year ago,* one period (of two years) before the first payment in one year. Hence we must multiply by 1.08 to find the present value today, or $34,840.41. HINT: If you are having trouble seeing this, draw a cash flow diagram.

In sum, this section discussed how to compute discount rates for different compounding periods and starting dates. Because consumer loans make monthly payments, corporate loans make quarterly or semiannual payments and government loans make semiannual payments, financial analysts frequently need to be able to convert yields from one compounding frequency to another.

The usual convention is to take a simple annual rate and divide by the number of compounding periods, ignoring the effects of compound interest. Then, an effective annual yield may be determined considering the effect of compounding. Finally, continuously compounded rates are sometimes used as a discounting standard that allows analysts to compare yields at different compounding frequencies on an equivalent basis.

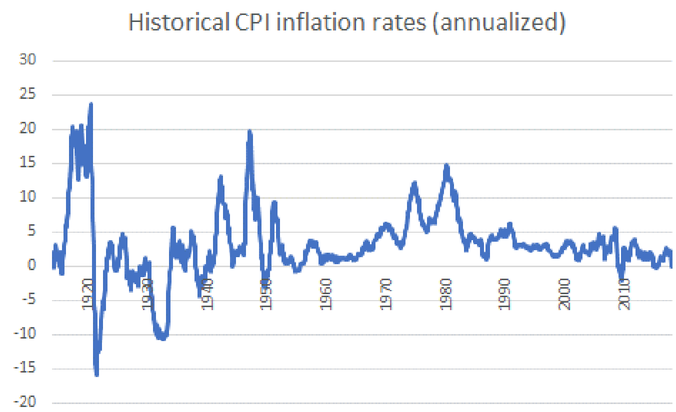
INFLATION

"I was reading in the paper today that Congress wants to replace the dollar bill with a coin. They've already done it. It's called a nickel."
— Jay Leno

"*The arithmetic makes it plain that inflation is a far more devastating tax than anything that has been enacted by our legislature. The inflation tax has a fantastic ability to simply consume capital. It makes no difference to a widow with her saving in a 5 percent passbook account whether she pays 100 percent income tax on her interest income during a period of zero inflation, or pays no income taxes during years of 5 percent inflation. Either way, she is 'taxed' in a manner that leaves her no real income whatsoever. Any money she spends comes right out of capital. She would find outrageous a 120 percent income tax, but doesn't seem to notice that 5 percent inflation is the economic equivalent."*
— Warren Buffett

Until now, we have ignored inflation. In reality, the spending power of a currency may decline over time, so that the value of $1 today differs from the value in 10 years. We can call this a decline in purchasing power, where the rate of decline is measured by the *inflation rate*. For example, if a can of beans cost $1 ten years ago, and it costs $1.50 today to buy the same can of beans, then the inflation rate, as measured by beans would be $(1.50/1.00)^{1/10}-1 = 4.14\%$ per year. It is also possible that a currency may *deflate*, indicating a negative rate of inflation. In this rare case, the spending power of a currency increases.

Last decade in the U.S. (the 2010s), the rate of inflation was about 2% per year. The chart below shows a historical perspective. In the 1970s and early 1980s, the U.S. experienced its highest modern inflation rates, both high in level and in volatility. Since then rates have fallen and become more stable. The inflation rate is usually measured by the CPI or *consumer price index*, which is based on what is considered a typical consumer consumption basket rather than simply a can of beans!



The CPI includes consumer costs in the following categories: Food & beverages, housing, apparel, transportation, medical care, recreation, education, communication, and other goods & services. The CPI includes taxes such as sales taxes that are associated with consumer expenses, but excludes other taxes. It excludes all investment related items.

Of course, the CPI will not in general be able to match anyone's actual increase in expenses, since some will necessarily spend relatively more on medical care vs education and vice versa. Nevertheless, it is considered close enough for most consumers. In fact, many consumer payments, such as cost-of-living adjustments to wages (so-called *COLA* or inflation-protection clauses) are usually based on the CPI. Also, pension benefits may be tied to the CPI. If your wages are *indexed* to the CPI, this means your actual cash income will rise in proportion with the CPI. If you make $50,000 in 2017, your base wage does not increase, you have a COLA clause in your wage contract, and the CPI ends up being 2%, you will make $51,000 in 2018.

As an index choice, the CPI does not work in all situations. There are custom regional indices available for example, which you might use to negotiate a labor contract specific to Syracuse or Seattle. There are also indexes related to producer prices (PPI), since companies purchase baskets of commodities that differ significantly from consumer baskets. Finally, some analysts prefer to use the GDP (gross domestic product) deflator, which necessarily considers all goods and services produced in an economy as the "basket", and measures the rate of price increase of that basket. The GDP deflator better reflects the ability of USD to purchase a pro-rated basket of total U.S. production of goods and services, as compared to the previous year.

Can inflation be a good thing?

We've been considering the effect of inflation on wage-earners, who would be hurt by inflation if they receive fixed wage payments while inflation is present. However, if the same wage-earner was making fixed payments on a mortgage, he would actually be pleased to see his purchasing power increasing on a relative basis rather than decreasing. This happens because the worker's net fixed payment is lower as a result of the debt payment --- the only part of his income that is hurt by inflation is his after-debt income.

Nominal and real cash flows and rates

Whatever inflation index you use going forward, we will need to differentiate between *nominal* cash flows and *real* cash flows. Real cash flows have been adjusted for inflation so that the purchasing power of a real dollar in one year is the same as another year. For example, a few paragraphs ago, your nominal income went from $50,000 to $51,000 with 2% inflation, but your real income in terms of constant spending power stayed the same at $50,000 in 2017 dollars.

The valuations we have done so far have all been based on nominal cash flows, indeed in practice, almost all valuations are like this, since cash payments are normally expressed without inflation adjustment. Income and expenses are estimated in nominal dollars, and are discounted at nominal interest rates. When we use "r" in this text, and it represents the nominal rate of interest.

Real cash flows are those that have been adjusted for spending power. With inflation present, the spending value of a nominal dollar will depend on the inflation rate. Real dollars need to be discounted at a different rate of interest. We will use the Greek letter rho "ρ" to denote the real rate of interest. Also, "i" is the inflation rate.

The famous **Fisher Equation** links one period inflation to one period spending power --- interest raises the currency's spending power in the future, while inflation reduces it:

$1 for 1 yr → (1+r)/(1+i) = (1+ρ) or $r = \rho + i + \rho i \approx \rho + i$ for small ρ and i

$$\text{or } \rho = (r - i)/(1+i)$$

With continuously discounted inflation and interest rates, we have exactly

$$e^r = e^\rho e^i \text{ or } r = \rho + i$$

For example, if the interest rate is 5%, but the inflation rate is 2%, a risk-free investment increases in spending power by about 3% per year. One nominal dollar now has the same real value in today's dollars, unless we define it otherwise using a historical benchmark, such as "1960 dollars". After a year, a nominal dollar is worth $1.05, but its spending power is about $1.03 in today's dollars.

To value real annuities, therefore, we must adjust the growth rate to reflect a currency's decline in spending power. However, we have already solved this problem analytically. Let the cash flow growth rate $g = -i$ to convert nominal cash flows to real cash flows. Then apply a real discount rate to the real cash flows.

As an example, let's consider a COLA wage annuity that pays an inflation-linked cash flow, with C_0 being the base income, $C_1 = C(1+i)$ being the first nominal payment in one year, $C_2 = C(1+i)^2$, and so on.

We can easily value this as a growing annuity with g=i

$$V_0 = \frac{C(1+i)}{(r-i)}\left[1 - \frac{(1+i)^n}{(1+r)^n}\right]$$

But if we let $C^*=C_0$, and $r^*=(r-i)/(1+i)$, we have the regular annuity formula

$$V_0 = \frac{C_0}{r^*}\left[1 - \frac{1}{(1+r^*)^n}\right]$$

Which is exactly the simplification we learned in Chapter 2 for valuing growing annuities. Of course r* is actually ρ in this case. If ρ=0, we can simply add up the real cash flows to determine the value

$$V_0 = nC_0 \qquad \text{if } r^* = 0$$

This relationship is particularly important to understand when inflation rates are unknown. In this case, it would be very difficult to forecast nominal dollars and know what rate to use to discount them. We could however assume that the real payment was the same in every year, and that the cash flows were a constant annuity, to be discounted at the rate of ρ.

Exercise: Paul earned \$100,000 in the past year. He just signed a 3-year employment contract with a COLA clause pay his base salary adjusted upward according to the CPI, currently 2%. If the nominal rate of interest is 4.5%, what is the present value of the contract revenues? (\$286K)

Inflation CPI calculations

To learn the rate of inflation, one may consult government statistics, as reported below by the Federal Reserve Bank of the U.S. The CPI index is quoted as a *level*, not a percentage. The inflation *rate* is measured as a rate of change in the overall price level. For example, from March 2018 to March 2019, the inflation index grew from 249.554 to 254.202. This implies a one-year inflation rate of 1.86%.

Click and drag in the plot area to zoom in. Hover over chart to view data.
Source: U.S. Bureau of Labor Statistics.

Data extracted on: Apr 30, 2019 (10:42:21 AM)

CPI-All Urban Consumers (Current Series)

Series Title	: All items in U.S. city average, all urban consumers, not seasonally adjusted
Series ID	: CUUR0000SA0
Seasonality	: Not Seasonally Adjusted
Survey Name	: CPI-All Urban Consumers (Current Series)
Measure Data Type	: All items
Area	: U.S. city average
Item	: All items

Latest Observation:
March 2019
254.202

Jan 2017 - Mar 2019
Minimum Value: January 2017
242.839

Maximum Value: March 2019
254.202

Data Availability:
1913 - 2019

.XLSX .CSV

Year	Period	Label	Value
2018	M03	2018 Mar	249.554
2018	M04	2018 Apr	250.546
2018	M05	2018 May	251.588
2018	M06	2018 Jun	251.989
2018	M07	2018 Jul	252.006
2018	M08	2018 Aug	252.146
2018	M09	2018 Sep	252.439
2018	M10	2018 Oct	252.885
2018	M11	2018 Nov	252.038
2018	M12	2018 Dec	251.233
2019	M01	2019 Jan	251.712
2019	M02	2019 Feb	252.776
2019	M03	2019 Mar	254.202

If you are fortunate enough to have a cost-of-living-adjustment (COLA) clause in your wage contract, this means that your future wage will be determined by the formula:

$$\textit{Future wage} = \textit{Starting wage} \times \textit{Future CPI index level} / \textit{Starting CPI level}$$

Of course, actual inflation is unknown ahead of time, but COLA clauses protect people from unforeseen increases in the inflation rate as defined by the CPI.

In sum, the most important things to know about inflation are:

- ✓ The CPI is the most popular inflation index used in the U.S., but it is not suited to all purposes. It would be impossible to create a single index to measure everyone's change in purchasing power. The PPI and the GDP deflator are the most popular alternatives.
- ✓ *Nominal* cash flows are actual dollar cash flows, while *real* cash flows are adjusted for inflation.
- ✓ Discount nominal or real cash flows at their respective interest rates r and ρ, never mix them.
- ✓ The growing annuity tools can be used to value real and nominal cash flows.
- ✓ $\rho = (r-i)/(1+i)$, nearly equal to $r - i$ for small i. This is exact with continuous discounting: $\rho = (r-i)$
- ✓ In today's dollars, the value of nominal cash flows discounted at nominal rates equals the value of real cash flows discounted at real rates.

The last point may be the most important one. It may seem difficult to value a stream cash flows linked to an unpredictable future CPI index level. However, if the cash flows are inflation-adjusted, we can discount the future payments expressed in today's dollars at the real rate of interest.

TAXES

One year while I was at his Princeton home preparing his return, Mrs. Einstein asked me to stay for lunch. During the course of the meal, the professor turned to me and with his inimitable chuckle said: "The hardest thing in the world to understand is income taxes." I replied: "There is one thing more difficult, and that is your theory of relativity." "Oh, no," he replied, "that is easy." To which Mrs. Einstein commented, "Yes, for you."

LEO MATTERSDORF New York City

In real life, taxes are complicated. Nevertheless, taxes can play a pivotal role in valuations. Governments use tax rules to encourage people to invest in government-favored assets, increasing the value of some assets relative to others. Individuals seek ways to classify or offset income to reduce their taxes, and the tax rules are complicated. A precise understanding of the taxes in any activity will be important for you to determine accurate valuations. Your private valuations will depend on your tax situation. However, it will be impossible to cover all the complications of taxes here. Simulation methods covered later in the text can deal fairly easily with complex tax schemes.

Let's begin with some terminology. *Nominal tax rates* refer to the stated tax rate for a given situation. *Effective tax rates* refer to the rate actually paid, or the rate of tax implied by the price of an asset. Taxes may be levied on capital gains and cash flows, usually both, albeit at different marginal tax rates. The percentage tax paid on a marginal dollar of return is called the *marginal tax rate*. Total taxes divided by investment is called the *average tax rate*. Taxes may be assessed on capital gains, but often, tax is only due in the tax period the capital gain is *realized*, which occurs by definition when the asset is sold.

While it is impossible to list all the complications surrounding real life taxes, here is a list of important things to know:

Realistic assumptions:

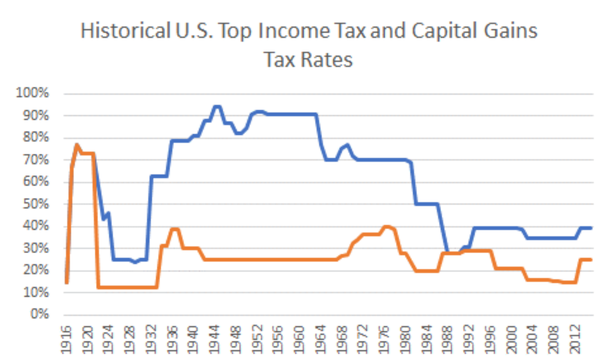
- Capital gains are generally taxed only when they are realized, on the sale of an asset
    - Capital losses are deducted from taxes, but must also be realized before they are taxed
    - There are generally limits to allowable deductions
    - Losses may be allowed to be *carried forward*, i.e. recognized in future periods
- Capital gains tax rates differ from ordinary income (cash) tax rates, and both change over time
- Tax rates are progressive, meaning that people with higher incomes have higher marginal tax rates
- Tax rates differ for different individuals and corporation types
- Tax rates in some countries go up and down with the overall market over time

For the purposes of this chapter, we will make the simplifying assumptions as follows:

- Capital gains and losses are taxed at the time they occur
- Tax rates are constant, and do not change based on income or ownership
- Loss deductions are unlimited
- Tax rates do not change in response to time, economic conditions or anything else.

For the moment, we will allow capital gains tax rate (τ_{CG}) to differ from the ordinary income tax rate (τ_{OI}). A rule of thumb in finance states that you may discount pretax cash flows at pretax interest rates, or after tax (AT) cash flows at after tax interest rates. Sometimes, these calculations should yield the same present value, and in many cases, this equivalence defines the effective after-tax discount rate. When capital gains (CG) and ordinary income (OI) are taxed at the same rate τ, the after tax interest rate $r_{AT} = r(1-\tau)$, if r is the pretax interest rate.

Historically the two tax rates have generally differed in the U.S. since 1922, with the top income tax rate exceeding the capital gains rate:



Simple Example

Consider a par bond with annual coupons, which costs \$1000 and pays annual coupons at a rate of 5% (\$50 per year) until maturity. At maturity \$1000 is repaid along with the final coupon. If the tax rate τ=40%, then the bondholder only keeps \$30 from the coupon payment, which is (1-τ) or 60% of the \$50, since \$20 is paid in taxes. There is no capital gain to tax.

The pretax present value is \$1000 if the interest rate is also 5%. The after-tax cash flows resemble a pretax bond paying \$30 per year and \$1000 at maturity, so the realized rate of return, r_{AT} = 3%. Indeed, in this simple case the equation holds --- 3% = (1-0.40)×5%.

The effect on the GVE of taxes is simple in theory. If τ is the tax rate, and capital gains and cash flows are taxed at the same rate, and our assumptions from above hold, then after-tax required and expected returns are $(1-\tau)$ times the pretax returns in dollars:

Ignoring risk the GVE states,

$$\text{Pretax required return} \times (1-\tau) = \text{Pretax capital gain} \times (1-\tau) + \text{Pretax cash flows} \times (1-\tau)$$

and we can see that the equation is not affected by the taxes under our assumptions, if there is one tax rate. One simply must express all the values in pretax terms or post-tax terms. Note that the appropriate change must be made to any boundary condition as well.

Fortunately, there is a way to integrate these into the GVE. Ignoring risk,

$$\text{Required return (after tax)} = (1-\tau_{CG})\text{ Capital gains} + (1-\tau_{OI})\text{ Cash flow}$$

This equation assumes that all capital gains are taxed at the same constant rate, all cash income (or losses) are taxed at a potentially different contract rate. Implicitly, it also assumes any losses are fully deductible from taxes paid --- negative taxes would reduce an investor's tax burden or even lead to a refund. If it were not possible to deduct capital losses, then "$(1-\tau_{CG})$ Capital gains" would have be replaced with "$(1-\tau_{CG})$ Max(Capital gains,0)".

We can see now what happens when the two tax rates are the same and losses are fully deductible. If $\tau_{CG} = \tau_{OI} = \tau$, we can divide the GVE by $(1-\tau)$ to get the pretax GVE. In this case, $r_{AT} = r(1-\tau)$. Here is a numerical example for a 10 year annuity of \$50 per year, and r=5%, and both tax rates equal to 40%:

	ACTUALS			TAX			
Time	CF	V	CG	CF	CG	Total	AT
0		386.09					
1	50.00	355.39	(30.70)	20.00	(12.28)	7.72	42.28
2	50.00	323.16	(32.23)	20.00	(12.89)	7.11	42.89
3	50.00	289.32	(33.84)	20.00	(13.54)	6.46	43.54
4	50.00	253.78	(35.53)	20.00	(14.21)	5.79	44.21
5	50.00	216.47	(37.31)	20.00	(14.92)	5.08	44.92
6	50.00	177.30	(39.18)	20.00	(15.67)	4.33	45.67
7	50.00	136.16	(41.14)	20.00	(16.45)	3.55	46.45
8	50.00	92.97	(43.19)	20.00	(17.28)	2.72	47.28
9	50.00	47.62	(45.35)	20.00	(18.14)	1.86	48.14
10	50.00	0.00	(47.62)	20.00	(19.05)	0.95	49.05

In each time period, the annuity value is determined as the PV of the remaining payments. The capital gain (CG) is the change in value, which is negative for its life. The tax is 40% of the payment minus a credit for the capital loss. The after tax cash flow (AT) is the pretax cash flow (payment) minus taxes. In this case, as expected, the present value of the after tax cash flows at r_{AT}=3% is \$386.09, exactly as in the pretax case.

What if $\tau_{CG} \neq \tau_{OI}$? To value an asset in this situation, the problem is that we don't know the value of r_{AT} from the outset. To solve the after-tax problem, we therefore take the following steps:

- ✓ Compute the value of the pretax instrument V_0 at discount rate r and set this equal to the after-tax value
- ✓ Find the discount rate r_{AT} that makes the present value of the after tax cash flows equal to this value.
- ✓ The effective tax rate τ_e satisfies $r_{AT}=(1-\tau_e)r$.

Example

A risk-free security sale is expected to generate $20 in dividends, $30 in capital gains, and a $1000 return of principal in one year. The security's value today is $1000, implying a 5% risk-free discount rate. The tax rate for ordinary income is 40%, and the capital gains rate is 15%. What is the effective tax rate and the after-tax rate of return on the security?

Solution

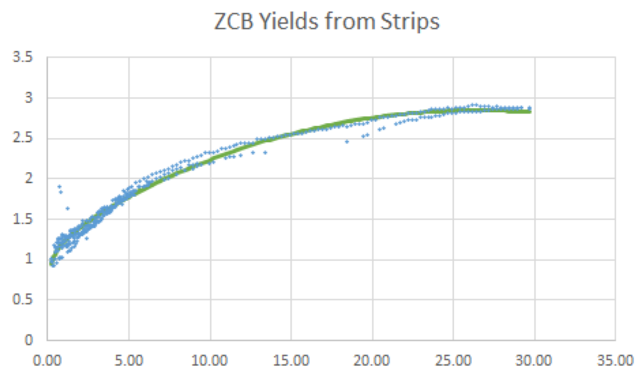
The pretax valuation is $1000, as is the post-tax value. The after-tax cash flows are $37.50 (= 0.6(20) + 0.85(30)) in one year. This suggests r_{AT} = 3.75% so $(1-\tau_e) = \frac{3.75}{5} = 0.75$, and the effective overall tax rate is 25%.

VARIABLE INTEREST RATES

In previous chapters, we assumed discount rates were constant over time. Of course, it is obvious from business news reports that interest rates are changing all the time. Even at a point in time, longer term discount rates will differ from shorter-term rates closer to the present. The graph of yield as a function of time or term is known as the *term structure* of interest rates. How can we determine the present value of cash flows when rates differ by time?

For expositional convenience, let's work with one-year intervals and focus on zero-coupon bonds. Using a randomly chosen historical date, one can see the term structure of zero-coupon bonds by examining prices of U.S. STRIPS, which are explained in detail in Chapter 4. For now, let's examine the rates and see what information they contain.

This chart shows the spot prices of ZCBs as a function of different maturities measured in years, together with a smooth curve used to estimate ZCB yield for any spot maturity. The prices are called spot prices because the bonds are purchased and paid for on a spot basis. The STRIPS are risk-free and pay $100 at maturity, so the present value is given by $V = 100/(1+r_S)^T$, where r_S is the annualized spot rate and T is the number of years to maturity. The values are listed in the spot column in the table. Therefore, one could invest for 1 year and earn 1.22%, or 2 years and earn 1.40% per year.



Year Start	End	Spot	Forw	Simple Avg
0	1	1.22	1.22	1.22
1	2	1.40	1.58	1.40
2	3	1.54	1.83	1.54
3	4	1.67	2.04	1.67
4	5	1.78	2.24	1.78
5	6	1.89	2.41	1.89
6	7	1.98	2.57	1.98
7	8	2.07	2.71	2.08
8	9	2.16	2.84	2.16
9	10	2.24	2.96	2.24

One might also ask, what return is earned from year 1 to year 2? Can we determine the implied return from the given data? You might say, if I make 1.22% for one year (0-1) and 1.40% on average for 2 years (0-2), it is as if I am earning 1.58% in the second year, so that the average rate would be 1.40%. If you said this, you would be approximately correct. If you were discounting continuously, you would be exactly correct! The exact discrete formula would suggest comparing two strategies, specifically a two-year investment, or a one-year investment being guaranteed a given *forward rate* in year 1 for 1 year. These strategies should yield the same result. Therefore,

$$(1+r_{s2})^2 = (1+r_{s1})(1+f_{12}) \qquad discrete\ formula$$

$$e^{2r_{s2}} = e^{r_{s1}}e^{f_{12}} \rightarrow 2r_{s2} = r_{s1} + f_{12} \qquad continuous\ formula$$

The term f is called a forward rate because it implicitly allows us to compute the return as if we had a contract to purchase the 1 year bond in 1 year with the proceeds of a one year bond invested now. The forward rate curve, we can see, rises faster than the spot rate. Since the spot rate is approximately the average of the forward rates, this makes sense. For the average spot rate to rise over time, the forward price must be increasing faster.

What determines the shape of the term structure? Forward rates are determined by two factors. First, the market has implicit expectations of future interest rates. Expectations of rising rates would be sufficient to produce an upward-sloping yield curve. However, even when the market expects rates to fall, we often see upward-sloping yield curves. Why is this the case?

If all interest rates were the same, bond investors would prefer to lend short-term rather than long-term. However, governments and corporations would prefer to borrow long-term than borrow short-term. Because of this natural imbalance in preferences, long-term rates generally exceed short-term rates to induce lenders (investors) to buy long term bonds. Of course, lower short-term rates give an incentive for corporations to borrow shorter-term. In sum, there is a *term premium* in equilibrium, i.e. an additional component of the interest rate necessary to clear the risk-free capital markets. We usually write it in the form

$$Forward\ rate = Expected\ future\ spot\ rate + Term\ premium$$

This equation is implied by the *liquidity premium hypothesis*, or can be seen as the likely implication of the *preferred habitat theory*. In the *expectations hypothesis* of the term structure, the term premium is zero. However, few economists would support the expectations hypothesis any longer.

Suppose we have a sequence of known cash flows of $100 each in years 1-10, continuing the example above. The present value of the cash flows is quite simple in the case, since we can discount each cash flow separately at the corresponding spot rate of interest. The result is $900.77, which is shown in the Excel calculation below, along with a computation of the *Internal Rate of Return* (IRR),a single average compounded yield corresponding to the pricing of the cash flows.

Date	Spot	PV(100)	CF
0			-900.77
1	1.22%	98.79	100
2	1.40%	97.26	100
3	1.54%	95.51	100
4	1.67%	93.60	100
5	1.78%	91.55	100
6	1.89%	89.40	100
7	1.98%	87.16	100
8	2.07%	84.85	100
9	2.16%	82.51	100
10	2.24%	80.14	100
	Total	900.77	
	IRR		1.95%

The more interesting application associated with variable interest rates has to do with cash flows that change as a function of the interest rates. In particular, a *floating rate bond* is one whose coupons vary based on a pre-specified interest rate index. This is the case with *floating rate notes* or *FRN*s. How should we value floating rate bonds?

First, we should know that the interest rate on a floating rate bond is normally determined at the beginning of the payment period, not the end. This convention of declaring the interest rate at the beginning of the effective period is known as quoting interest *in arrears*. For example, a floating rate bond with annual payments would be structured so that the first payment equals the one-year spot rate now at time 0 (r_{01}) multiplied by the principal balance (F). Using the GVE, we can write it this way:

$$r_{01}V_0 = (F - V_0) + r_{01}F$$

This implies that the floating rate bond is priced at par, or $V_0 = F$. If we had a two-year floating rate bond, the price at time 1 and 0 would be determined by

$$r_{01}V_1 = (F - V_1) + r_{01}F \;\; or \;\; V_1 = F$$

$$r_{12}V_0 = (V_1 - V_0) + r_{12}F \;\; or \;\; V_0 = F$$

implying that after each floating coupon is paid, the value of the bond resets to the principal value, *regardless of the value of the new spot rate*. This is important especially since r_{12} is not known at time 0. The value of floating rate bonds constructed in this matter is straightforward --- they always stay at par value if they are risk-free, at the moment the last payment is made. Interest may accrue, i.e. become owed but not included in the bond price, but the bond price resets to par after the floating interest rate is paid.

What if we needed to value a single cash flow that was a function of the future spot interest rate, and not part of a floating rate bond? In this case, the unknown r_1 would not cancel as neatly as with the FRN. This would occur for example if you were trying to value a company whose earnings fluctuated with interest rates. In this case, we would need to find a number that represented the market value of the variable payment. Fortunately, we already know the forward rate serves that role, and that information is embedded in the spot yields. Therefore, if the actual cash flow equals $r_t F$ at time t+1, the present value is given by

$$PV = \frac{f_{t-1,t} F}{(1 + r_{st})^t}$$

The forward rate f in this formula represents the forward rate one year prior for a payment on the same payment date. Continuing the previous example, suppose we want to value each FRN bond payment separately. We substitute the forward price for the unknown future spot price and discount it at the risk-free spot rate. Here are the details:

FRN Payments - Individual Present Values

F=100	Spot	Forw Equiv	PV using Spot
1	1.22%	1.22	1.21
2	1.40%	1.58	1.54
3	1.54%	1.83	1.75
4	1.67%	2.04	1.91
5	1.78%	2.24	2.05
6	1.89%	2.41	2.16
7	1.98%	2.57	2.24
8	2.07%	2.71	2.30
9	2.16%	2.84	2.34
10	2.24%	102.96	82.51
		Total	100.00

Of course, the total value is *F = 100*, as expected. This demonstration shows us how to value any risk-sensitive cash flow at any time to a risk-free forward equivalent, and determine its value by discounting at the risk-free spot rate.

Example

A company has determined that its cash flows for the next three years are sensitive to interest rates. The estimated cash flow each year is estimated to be \$100 MM times the one-year spot rate in the previous year. What is the present value of the first 3 cash flows using the same term structure as the one shown in this section?

Answer: \$4.5 mm.

FOREIGN EXCHANGE

The U.S. Federal Reserve Bank reports spot exchange rates for the major currencies on every trading day. "Spot rates" refer to the exchange rate on a given day, while "Forward rates" refer to exchange rates for future transactions whose terms are agreed today. In particular, the exchange rate to be applied in the future is determined today.

Most currencies are quoted in terms of the number of units of foreign currency received for each $1 USD. This reporting standard is called a *direct quote*. There are a few exceptions, as noted in the following table (from 12/22/2017):

COUNTRY	CURRENCY	CODE	12/22/2017
*AUSTRALIA	DOLLAR	AUD	0.7713
BRAZIL	REAL	BRL	3.3209
CANADA	DOLLAR	CAD	1.2748
CHINA, P.R.	YUAN	CNY	6.575
DENMARK	KRONE	DKK	6.2866
*EMU MEMBERS	EURO	EUR	1.1839
HONG KONG	DOLLAR	HKD	7.8137
INDIA	RUPEE	INR	64.02
JAPAN	YEN	JPY	113.3
MALAYSIA	RINGGIT	MYR	4.077
MEXICO	PESO	MXP	19.5905
*NEW ZEALAND	DOLLAR	NZD	0.7015
NORWAY	KRONE	NOK	8.3387
SINGAPORE	DOLLAR	SGD	1.3441
SOUTH AFRICA	RAND	ZAR	12.605
SOUTH KOREA	WON	KRW	1076.87
SRI LANKA	RUPEE	LKR	152.85
SWEDEN	KRONA	SEK	8.358
SWITZERLAND	FRANC	CHF	0.9898
TAIWAN	DOLLAR	TWD	29.89
THAILAND	BAHT	THB	32.73
*UNITED KINGDOM	POUND	GBP	1.3379
VENEZUELA	BOLIVAR	VEF	9.975

*indirect quotes: USD per currency unit

Near the end of 2017, the spot exchange rate for Chinese Yuan (CNY) was 6.50 Yuan per $1 U.S. dollar (USD).

We can see that the spot CNY exchange rate grew from about 6 to about 7 in three years 2014-2017, implying an annual growth rate of about 5%. A positive growth rate indicates that the CNY was weakening relative to the dollar, since more units of the Chinese currency were required to purchase 1 USD as time passed. We might like to know how we should expect to see the exchange rate move in the future.

There are many theories relating to the determination of an exchange rate between two currencies, but the main drivers are relative interest rates, inflation, current account (trade) deficits, public debt levels, political stability and economic performance[6]. The best available indicator of expected future exchange rates is the *forward* exchange rate, ignoring any risk premium. A forward contract may specify for example that $1 could be exchanged for CNY 7 in 5 years time. This would indicate expected future weakening of the currency.

There is a strict relationship between spot rates, forward rates and interest rates governing relative prices known as IRP or **Interest Rate Parity**. The table below demonstrates IRP by showing two ways to invest 1 USD to produce CNY in the future. It can either be invested in the U.S. and converted to CNY using a forward contract, or it can be converted now and invested in a Chinese bond.

Interest rate parity illustration for direct quotation currencies

Time	Now	In t years	In CNY
USD	$1	$(1+r_{USD})^t$	$F(t)(1+r_{USD})^t$
CNY	¥1 (X_{CNY})	$X_{CNY}(1+r_{CNY})^t$	$X_{CNY}(1+r_{CNY})^t$

The final column entries are guaranteed to be equal by the threat of arbitrage; any discrepancy can be made profitable by borrowing in one currency to invest in the other, until the discrepancy disappears. This implies that F(t) forward prices at time t are completely determined by the spot exchange rate (X) and the ZCB interest rates of the two currencies.

The interest rate parity equation is a restatement of the equality of the last two table entries:

$$\frac{F(t)}{X_{CNY}} = \frac{(1+r_{CNY})^t}{(1+r_{USD})^t} \equiv (1+\delta)^t$$

If the Chinese interest rate exceeds the US interest rate, then the USD will become more expensive to purchase over time at the rate of δ, the CNY currency depreciation rate.

Looking at the situation through the GVE framework, say that we buy $1 worth of CNY today at the prevailing exchange rate and then invest that money to earn the CNY interest rate for t years.

Expected Return + Risk Compensation = Capital Gains + Cashflows

$$((1+r_{USD})^t - 1) + 0 = \left(\frac{X_{CNY}}{F(t)} - \$1\right) + \frac{1}{F(t)}((1+r_{CNY})^t - 1)X_{CNY}$$

[6] There are also instances where the spot exchange rate is fixed or pegged. One famous example is Argentina's pegging of the peso to the USD from 1991 to 2002.

This easily rearranges to the same result given by Interest Rate Parity.

The purpose of this section is to allow you to model foreign-currency denominated cash flows. Rather than trying to estimate the expected future exchange rate, one can convert future foreign payments into domestic payments at the forward rate and then discount them at the risk-free rate. Alternatively, foreign risk-free payments can be discounted at the foreign interest rate and converted to domestic currency using the spot exchange rate.

EVENT RISKS

Cash flows or capital gains may be subject to shocks at random times. If that is the case, we need to know how to value them. In this section, we demonstrate the application of the GVE to event risks when the probability of the occurrence of the event is constant in each period.

A life insurance policy

Consider the example of a single premium life insurance policy:

- V_0 = Value of the insurance policy now.
    - The policy is purchased for V_0, and there are no recurring payments.
- C = Payoff at the time of death. The policy has no residual value after the payment is made
- π = Probability of death each year, assumed constant.

For now, let's assume there is no risk premium. Then according to the GVE we have

$$\text{Required return} = rV_0$$

$$\text{Expected return} = \text{expected capital gain} + \text{expected cash flow} = -\pi V_0 + \pi C$$

The capital gain is $-\pi V_0$ because in any given year the policy is active, the possibility of terminating the policy is π and the capital loss is V_0. Similarly, the probability in any year of getting the cash flow is π so the expected cash flow is πC.

Setting these returns equal and solving for V_0

$$V_0 = \frac{\pi C}{r + \pi}$$

For example, suppose an insurance policy pays C=$100,000 at the time of death, and for a 30-year old, the conditional likelihood of death is 1% per year, at least for a while. The risk-free rate is assumed constant ay 5%. Then the value of the insurance policy is $16,667 regardless of when it is bought, provided the probability of death remains constant. The resulting formula may be interpreted as a kind of perpetuity: every year, the expected payment is πC. We cannot discount at r since the policy will not last forever, but increasing the discount rate to $r+\pi$ automatically adjusts for the expected time to collect on the policy.

Now we may turn our attention to the risk premium. Under the assumptions made above, the insurance company would not make a return on the policy and could not recover any of the underwriting or management costs. If we assume the insurance company must earn a fraction k of the policy's value every year (let's say 0.5%), then the buyer must pay that total risk premium as follows:

$$\text{Required return} = rV_0 - kV_0 = (r - k)V_0$$

We could also add kV0 to the right-hand side of the GVE, reflecting the fact that the insurance company earns part of the capital gain and cash flow of the investment. Whether we adjust the left or the right side of the GVE, to solve for V_0, we adjust the discount rate downward to 4.5%, and the cost of the policy rises to 100,000(0.01)/(0.045+0.01) = $18,182.

The increase in cost should not surprise anyone. In practice, sales, origination, administrative and risk and regulatory charges can be a significant part of the overall cost of any insurance policy.

Auto insurance

Consider the example of a single premium auto insurance policy:

- V_0 = Value of the insurance policy now.
    - The policy is purchased for V_0, and there are no recurring payments.
- C = Payoff in case of an accident. The policy remains in force after the accident and covers the policyholder for future accidents, until death.
- δ = Probability of death in any year, assumed constant.
- π = Probability of death each year, assumed constant.
- k = Premium paid to the insurance company, as a percentage of the policy value

We assume for simplicity that the two events, death and accidents, do not interact with each other. This would be the case, for example, if death always occurred at the end of the period. Then according to the GVE we have

$$\text{Required return} = rV_0 - kV_0$$

$$\text{Expected return} = \text{expected capital gain} + \text{expected cash flow} = -\delta V_0 + \pi C$$

The capital gain is $-\delta V_0$ because in any given year the policy is active, the possibility of terminating the policy is π and the capital loss is V_0. Similarly, the probability in any year of getting the cash flow is π so the expected cash flow is πC. The single premium value is therefore

$$V_0 = \frac{\pi C}{(r-k)+\delta}$$

If the automobile policy that will pay C=$15,000 for each year you are in an accident, the probability of an accident every year is π=10%, the risk-free rate is 5%, the death probability is 1% per year and the risk premium s 0.5%, then the value of the policy is $27,273.

Notice in this example, we can separate the probability of the cash event (in the numerator) from the probability of the termination event (in the denominator). The adjustment to the discount rate appears in the denominator due to the assumption of proportional fees & costs.

Modelling default

Default risk is one of the most important topics in finance. We can model default as an event that occurs with some probability π per year, so it is an event risk. This type of default model is called a *reduced form* model, since we assume we know the likelihood of default, but not the actual triggers or structural forces causing default. Normally, when a company defaults on its debt, creditors are able to recover some value, even if they are not paid in full. We can say that the *loss rate* λ for investors reflects the fraction of value lost in the event of default. Some analysts prefer to work with the *recovery rate*, i.e. $1-\lambda$.

Suppose we have a one-year ZCB promising to pay F at maturity. Can we use the GVE to value the bond? The promised capital gain is $F - V_0$, but the expected loss is $\pi\lambda F$:

$$rV_0 = (F - V_0) - \pi\lambda F$$

The value can be written in two ways. In the first equation below, we deduct expected losses from the promised payment and discount at the risk-free rate. In the second equation, we find we can reach the same approximate result, which is fairly accurate as long as $\pi\lambda$ is small.

$$V_0 = \frac{F(1 - \pi\lambda)}{(1 + r)} \approx \frac{F}{1 + r + \pi\lambda}$$

Analysts frequently refer to the increase in the risk-free rate for credit risk as the *credit spread*. In practice, the term $\pi\lambda$ is referred to as an *expected loss rate*, and the market normally prices bonds using a higher discount rate including a risk premium --- reflecting the market's view of the additional risk of the bonds. In sum, the yield on a defaultable bond can be approximated for any bond, not just a ZCB as

$$r_{def} = r + \pi\lambda + r.p.$$

Where $\pi\lambda$ represents expected percentage losses, and r.p. represents a percentage risk premium required by the market for owning that particular bond.

Example:

A one year corporate ZCB (commercial paper) trades at $95 per 100 par value. The risk-free discount rate is 3%. If the assumed loss rate is 60% in default, what is the implied probability of default in the next year, assuming no additional risk premium for holding the bond?

Ans: Using the approximate formula, we get 3.77%.

SUMMARY

This chapter looks like a seemingly unconnected list of useful cash flow tricks, but it is more than that. Our objective by the end of Part I is to be able to simulate cash flows of any type. The purpose of this chapter was to introduce you to the many alternatives. For example, your cash flows might be subject to inflation adjustments. If not, their purchasing power will decrease in the future. The Fisher equation tells us how to move from inflation-adjusted to unadjusted cash flows, and how to choose the real rate of interest to express returns as the rate of increase in purchasing power.

Taxes affect everyone's cash flows, and hence must be addressed. Because investors have different tax rates, they value cash flows differently, but the price they pay for assets incorporates an effective embedded tax rate.

Finally, income and expense items for corporations are often denominated in foreign currency. Future foreign-denominated cash flows can be converted at the forward rate and discounted at the risk-free domestic interest rate to find the present value. Events that affect our cash flows, such as insurance events and default events can have a dramatic impact on present value calculations. Hence we must know how to address them. Fortunately for bonds with default risk, the yield can be expressed as a simple sum of the risk-free rate, compensation for expected losses and compensation for risk. We will sharpen this valuation in the next chapter.

REFERENCES AND FURTHER READING

EXERCISES FOR THE CALCULATOR

1. Which has a higher effective annual yield, 10% interest compounded quarterly or 9.8% compounded continuously?

2. *(Harder than it looks!)* Consumer home loans (mortgages) usually have monthly payments. Some companies are advertising the possibility of making 26 biweekly payments instead of 12 monthly payments. If the simple annual rate for a normal mortgage is 4.5%, and we have a $500,000 loan for 15 years, what are the payments on the regular mortgage and on the biweekly mortgage? What is the sum of all the payments in both cases? Which do you think is better? *Note: The biweekly mortgage payment is computed as half of the regular monthly payment, meaning that the mortgagor would make 13 monthly payments a year.*

3. Rank the following mortgage offers in terms of which is best for the consumer, using a 30 year loan of $225,000:

   a. A 3.5% mortgage with 3 points in closing costs
   b. A 3.75% mortgage with 1.5 points in closing costs
   c. A 4.00% mortgage with no closing costs

4. If the nominal interest rate is 5% and the inflation rate is 2% (simple annual rates), then what is the real interest rate? What is the answer if the two rates are continuously compounded rates?

5. Your employer gives you a choice of two compensation plans. In Plan A, your base wage will be increased now by 10% and stay flat for the next 10 years. In Plan B, your current wage will be indexed to inflation. The nominal interest rate is 5%, and the expected inflation rate is 2%. What is the present value of each plan? What other factors might cause you to choose Plan A over Plan B?

6. Your pension plan offers you one of two options:

   a. A fixed annuity of $5000 per month
   b. An inflation-indexed annuity starting at a $4500 base level today, but with the first payment in one month increasing at the rate of inflation.

   If the interest rate is 5% and the inflation rate is 2%, what is the value of each plan, assuming you are going to live 20 years longer?

   If you have a 0.25% chance of dying per month, what are the new plan values? HINT: Divide all annual rates by 12 to determine monthly rates.

7. A stock is expected to pay 2% of its value in dividends per year, and grow at a net rate (after dividends) of 7% per year. If the capital gains tax rate is 15% and the ordinary income tax rate is 40%, then what is the expected after-tax return on the stock? Note: Dividends are taxed as ordinary income.

8. A bank's energy loan portfolio is estimated to have an average LGD of 70% given that oil prices have halved in the last few years. If the default probabilities are 8% per year, the principal is $100,000,000, and the scheduled interest payments are fixed at 6% per year for 12 years, what is the value of the energy loan portfolio today? How does your answer change if oil prices fall further? Assume the risk-free rate is 4%.

9. A senior unsecured corporate bond has a yield of 6.8% per year, even though the risk-free rate is 5%. According to bank regulations, "senior claims on corporates, sovereigns and banks not secured by recognized collateral attract a 45% LGD." If a bank owns the corporate bond, assuming no extra risk compensation is implied in the yield, what would be the implied PD, or annual estimated probability of default on the bond?

10. If you bought a single payment life insurance policy for $150,000 today which paid $2,000,000 on death, the risk-free interest rate is 4%, and there is no risk premium, what is the implied annual probability of death?

EXERCISES TO BE DONE IN EXCEL

A. Modify your VBA library functions to include different compounding periods. Also, create a library function PVDB to value defaultable bonds.

B. Revise the Excel tax computation in the chapter to value a retirement annuity instead of a bond. The annuity pays $10,000 per year for 20 years, and the cost of the annuity is determined by discounting the payments at the contract rate of 3% per year.

    1. What is the valuation and effective after tax rate for the annuity if the capital gains rate is 15%, the ordinary income rate is 40%, and the market risk-free discount rate is 4%? Assume the annuity is purchased from pretax funds. What is the value from the annuity provider's point of view?
    2. What is the answer to part 1 if the annuity is purchased from after tax funds?

    <u>NOTE:</u> Annuity taxation differs from bonds. This quote is from Kiplinger in the context of retirement annuities: "If you buy the annuity with pretax money, then the entire balance will be taxable. If you use after-tax funds, however, then you'll be taxed only on the earnings. If you cash out a deferred annuity in a lump sum, then you'll have to pay income taxes on all of the earnings higher than your original investment. Jul 10, 2009".

CHALLENGE PROBLEMS

A. Suppose you have two defaultable bonds in a portfolio with different π and λ values, but otherwise identical. The "first to default" bond protection product will cover all losses in the first bond that defaults, but not the second. What is the present value of this protection? Use a simulation to value the protection, and show how the value fluctuates with changes in the relative values of π and λ. HINT: Use fixed random numbers so that the results can be easily compared. Also, use a WHAT-IF table and a 3-dimensional graph to display your results.

Chapter 4: Bonds

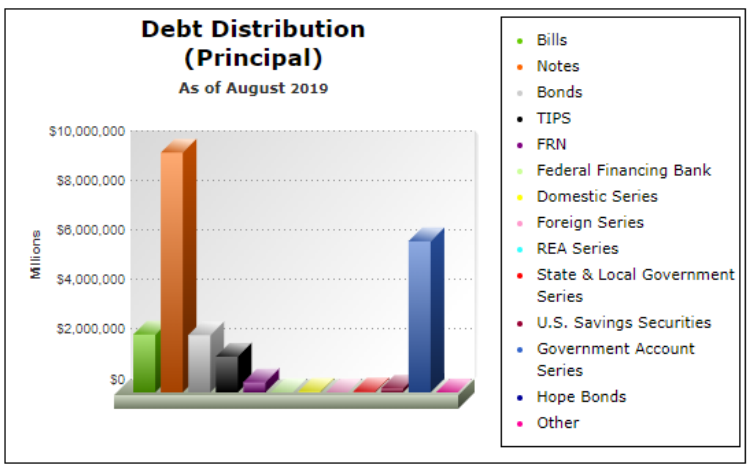
From one perspective, any financial engineer expecting to work in a capital markets function must learn the operational details of bonds and bond valuation. For example, once an analyst knows the yield to maturity, the coupon accrual conventions, the quoting conventions, applicable tax treatment, liquidity, inflation rate, and everything else that affects the value of the bond, he or she may proceed to value the bond with some confidence.

For many bonds, however, the better indicator of the bond's value is its market price, not the result of a theoretical model. Hence, bond prices can be used to infer the parameters that might be unknown or difficult to estimate. In Chapter 3, for example, we learned how to discover bond forward rates, and therefore currency forward prices from the observed ZCB term structure. In this chapter, we will extend this logic to explore the determination of implied real rates of interest, inflation rates, tax rates, default rates and estimates of loss given default for corporate bonds.

The chapter begins with U.S. government bonds and extends through corporate bonds.

U.S. GOVERNMENT BONDS

As of August 2019, the principal amount of government bonds outstanding is shown below, in this chart from TreasuryDirect.com:



Treasury bills account for $2.3 trillion, Notes $9.7 trillion, Bonds $2.3 trillion, TIPS $1.4 trillion and FRNs of $0.4 trillion. Treasury bills are ZCBs with less than a year to maturity. Notes and bonds are fixed-rate bonds, the only difference being that notes have no more than 5 years to maturity at issuance, while bonds extend to 30 years. TIPS are securities designed to protect long-term investors against risks of unknown inflation rates, and FRNs are 2-year floating rate notes with variable interest rates. These five bond types are discussed in this chapter. Hope bonds, not discussed in this chapter, represent a discontinued, non-marketable security that was issued to fund the "HOPE for Homeowners" program authorized by the Housing and Economic Recovery Act of 2008. It is not considered part of the U.S. government debt.

Interest income for all U.S. government bonds is taxable at the Federal level, but exempt from taxes at the state and local levels. Principal increases in the case of TIPS have the same tax treatment as interest income.

TREASURY BILLS

With less than one year to maturity, and no coupons, T-Bills are an ideal ZCB benchmark for valuing short-term cash flows. They are issued by the U.S. government and therefore are considered risk free. The face value at maturity is $10,000 per T-Bill.

Here is a WSJ quote page

TREASURY BILLS

GO TO: Notes and Bonds

Friday, May 26, 2017

Treasury bill bid and ask data are representative over-the-counter quotations as of 3pm Eastern time quoted as a discount to face value. Treasury bill yields are to maturity and based on the asked quote.

Maturity	Bid	Asked	Chg	Asked yield
6/1/2017	0.650	0.640	-0.013	0.649
6/8/2017	0.660	0.650	unch.	0.659
6/15/2017	0.680	0.670	0.028	0.680
6/22/2017	0.710	0.700	0.025	0.710
6/29/2017	0.728	0.718	0.028	0.728
7/6/2017	0.733	0.723	0.013	0.733
7/13/2017	0.783	0.773	0.020	0.784
7/20/2017	0.823	0.813	0.020	0.825
7/27/2017	0.810	0.800	0.040	0.812
8/3/2017	0.878	0.868	0.023	0.881
8/10/2017	0.893	0.883	0.007	0.896
8/17/2017	0.905	0.895	0.013	0.909
8/24/2017	0.920	0.910	0.010	0.925
8/31/2017	0.915	0.905	0.028	0.920
9/7/2017	0.933	0.923	0.018	0.938
9/14/2017	0.930	0.920	0.008	0.935
9/21/2017	0.938	0.928	0.010	0.943
9/28/2017	0.945	0.935	0.015	0.951
10/5/2017	0.973	0.963	0.010	0.979
10/12/2017	0.983	0.973	0.013	0.990
10/19/2017	0.995	0.985	0.015	1.003
10/26/2017	0.998	0.988	0.013	1.005
11/2/2017	1.013	1.003	0.013	1.021
11/9/2017	1.023	1.013	0.013	1.031

The only slight challenge with using T-Bills as a pricing benchmark is the legacy quotation system for trading and reporting T-Bill prices. This can easily be resolved once we know the system:

$$TBill\ Price = 10{,}000\left(1 - Quoted\ yield\ \times \frac{Days}{360}\right)$$

From the price, we can compute the continuously compounded yield using:

$$r = \frac{-\ln\left(\frac{Price}{10{,}000}\right)}{\left(\frac{Days}{365}\right)}$$

For which the effective annual yield (EAY) would be $e^r - 1$.

For example, the 10/19/17 T-Bill maturity (146 days) has an asking quote of 0.985%. This means the price is $10{,}000\left(1-0.00985\left(\frac{146}{360}\right)\right)=\$9{,}960.05$. The continuously compounded yield is 1.00068%, and the effective annual rate is 1.0057%.

Please note: though there are other ways to compute the effective annual yield, this is slightly simpler.

THE IDEALIZED U.S. GOVERNMENT COUPON BOND

U.S. government notes and bonds pay interest semiannually. Their face value is $1000, and the semiannual coupon is given by 1000c/2, where c is the annual coupon rate, so a 6% coupon bond pays $30 every six months for n years. On the maturity date, the bond pays the last coupon and the $1000 principal. For the purposes of the next few sections, we will value U.S. bonds under the following assumptions:

1. The valuation date is just after a coupon is paid
2. The coupons are evenly spaced at 6 month intervals
3. The annual yield is based on the APR of the semiannual yield
4. Bonds are priced in decimals

In reality, (1) the bond's price depends on the day it is bought in between its coupon paying dates[7], (2) the coupons are about 6 months apart, but the number of days between coupon payments varies by a few days, (3) there are many ways to compute yield, this is just the simplest, and (4) bonds may be priced in 32nds, not always decimal. For these reasons, we focus on valuing the idealized bond under these assumptions, recognizing that in practice, these adjustments would have to be made.

Under the idealized assumptions, even though each cash flow in the bond might be discounted at a different rate of interest, the *yield to maturity, YTM* or "yield" for short, is the constant discount rate that makes the present value of the coupons and principal equal to the bond value.

$$P=\sum_{i=1}^{2n}\left(\frac{\frac{1000c}{2}}{\left(1+\frac{y}{2}\right)^{i}}\right)+\frac{1000}{\left(1+\frac{y}{2}\right)^{2n}}$$

The yield to maturity is an implied rate of return for the bond, assuming the yield is constant, and it is held to maturity. Note that by convention, YTM is calculated on a semiannual basis, and converted to annual YTM by multiplying by two. This is also called the Bond Equivalent Yield (BEY).

On most financial calculators or using Excel, the idealized bond value can be computed easily with the annuity functions. The general steps are:

- Clear financial memories if applicable
- Set the payment equal to 1000c/2
- Set the future value (value in addition to the last payment) equal to 1000

[7] There is an adjustment for accrued interest (AI). For example, if we are looking to buy a bond half-way between payment dates the price paid (value) would be equal to the quoted price plus the accrued interest – which in this case would be ½ of the normal coupon payment.

- Set the discount rate equal to the yield divided by 2 (some calculators use whole number yields, and others use decimal)
- Set n equal to the number of coupon payments (not the number of years)
- Touch PV to compute present value

For example, a 6% coupon 10 year bond with a 5% yield would be priced at $1077.95. The coupon annuity is worth $467.67, and the final payment is worth $610.27 today[8].

Note that for *par bonds*, the value equals 1000 exactly. This happens when the coupon equals the bond yield. If the coupon rate is higher than the bond yield, we call this a *premium bond*, and it trades at a premium to par. Similarly a *discount bond* trades below par since the coupon rate is less than the yield.

Here is a sample treasury quotation summary, with prices quoted in decimals. The yield is reported as twice the semiannual yield.

Treasury Yields

NAME	COUPON	PRICE	YIELD	1 MONTH	1 YEAR	TIME (EDT)
GB3:GOV **3 Month**	0.00	0.91	0.92%	+13	+61	5/26/2017
GB6:GOV **6 Month**	0.00	1.05	1.07%	+10	+59	5/26/2017
GB12:GOV **12 Month**	0.00	1.12	1.15%	+9	+47	5/26/2017
GT2:GOV **2 Year**	1.25	99.91	1.29%	+3	+39	5/26/2017
GT5:GOV **5 Year**	1.75	99.81	1.79%	-2	+40	5/26/2017
GT10:GOV **10 Year**	2.38	101.14	2.25%	-3	+40	5/26/2017
GT30:GOV **30 Year**	3.00	101.75	2.91%	-4	+27	5/26/2017

STRIPS

U.S. government bonds, while representing one of the largest security issuances in the world, are not ideal for determining discount rates to apply to single cash flows. This is because bonds consist of many different payments at different dates; the yield is a complex average of the yields of the individual coupons and principal payments.

Fortunately for those who build ZCB yield curves, the U.S. decided in 1985 to "strip" the coupons and principal components of bonds and offer them as separate securities. Each bond is therefore converted into a portfolio of ZCBs. These are priced and traded separately in the STRIPS market, an acronym for *Separate Trading of Registered Interest and Principal of Securities*. STRIPS are sold by dealers, and individuals cannot purchase STRIPS directly from the U.S. government.

[8] Many business calculators have exact government bond valuations, which can determine accrued interest and yields based on actual day counts. There are many sources that cover this topic, and it is not our primary interest here.

Because STRIPS were introduced relatively recently, they do not suffer from the same type of arcane price quotation as T-Bills. Investors see the price of each strip per $100 principal in the quotation, which can be converted to an effective yield using a 365 day calendar. In the formula below, *days* represents the number of days to maturity.

$$EAY = \left(\frac{100}{Price}\right)^{365/days} - 1 \qquad\qquad r = \ln(1 + EAY)$$

> Exercise: For the STRIPS quotes on the following page, find the STRIP for 8/15/23, Using the asking price, determine the EAY and the continuously compounded yield.
>
> Answer: Days=2161, 1.89%, 1.88%

As you can see, the computed yield matches the one reported by the WSJ. This means we have an ideal dataset from which to build a ZCB yield curve.

Home | Markets

U.S. Treasury Strips

Wednesday, September 13, 2017

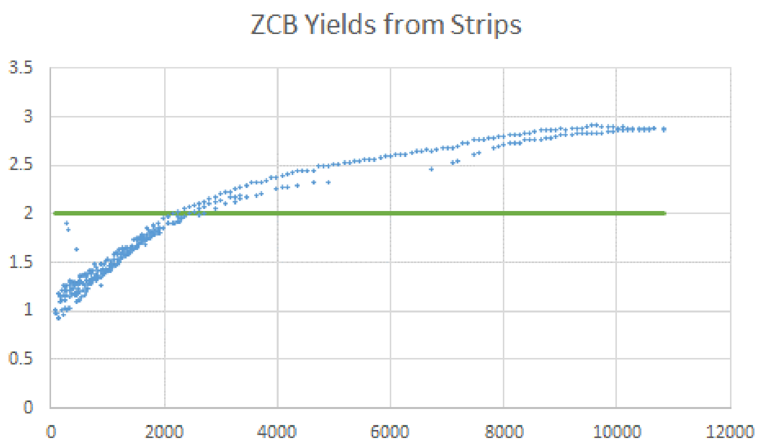
U.S. zero-coupon STRIPS allow investors to hold the interest and principal components of eligible Treasury notes and bonds as separate securities. STRIPS offer no interest payment; investors receive payment only at maturity. Quotes are as of 3 p.m. Eastern time based on transactions of $1 million or more. Yields calculated on the ask quote.

Maturity	Bid	Asked	Chg	Asked yield
Treasury Bond, Stripped Principal				
2018 May 15	99.242	99.249	0.001	1.13
2018 Nov 15	98.662	98.673	-0.017	1.15
2019 Feb 15	98.331	98.345	-0.021	1.18
2019 Aug 15	97.500	97.518	-0.034	1.31
2020 Feb 15	96.806	96.829	-0.046	1.34
2020 Feb 29	96.676	96.700	-0.035	1.37
2020 May 15	96.363	96.389	-0.041	1.38
2020 Aug 15	95.785	95.813	-0.048	1.47
2021 Feb 15	94.739	94.771	-0.063	1.58
2021 May 15	94.366	94.400	-0.064	1.58
2021 Aug 15	93.860	93.897	-0.080	1.61
2021 Nov 15	93.314	93.352	-0.081	1.66
2022 Aug 15	91.563	91.607	-0.119	1.79
2022 Nov 15	91.048	91.095	-0.121	1.81
2023 Feb 15	90.499	90.548	-0.114	1.84
2023 Aug 15	89.429	89.482	-0.115	1.89
2024 Nov 15	86.757	86.819	-0.104	1.98
2025 Feb 15	86.180	86.243	-0.158	2.01
2025 Aug 15	84.957	85.023	-0.193	2.06
2026 Feb 15	83.736	83.805	-0.206	2.11
2026 Aug 15	82.713	82.786	-0.188	2.13
2026 Nov 15	82.089	82.163	-0.175	2.15
2027 Feb 15	81.521	81.597	-0.183	2.17
2027 Aug 15	80.481	80.560	-0.188	2.19
2027 Nov 15	79.865	79.945	-0.198	2.21
2028 Aug 15	78.117	78.201	-0.250	2.27
2028 Nov 15	77.591	77.677	-0.221	2.28
2029 Feb 15	77.066	77.153	-0.233	2.28
2029 Aug 15	76.024	76.114	-0.215	2.30
2030 May 15	74.580	74.674	-0.250	2.32
2031 Feb 15	73.279	73.376	-0.279	2.32
2036 Feb 15	63.492	63.608	-0.215	2.47
2037 Feb 15	61.165	61.282	-0.230	2.54

The scatterplot is shown in the chart below. It may seem strange that STRIPS from one bond may differ in price from other strips. This can happen for several reasons:

1. Quotations are asynchronous, meaning they are recorded at different times. One bond might have last traded at 12:00 p.m. and another might have last traded at 3:00 p.m.
2. There are slight taxation differences for principal and interest strips that can cause pricing differences
3. The market is simply not active enough or deep enough in certain maturities to give perfectly reliable prices.

As of 9/14/17	42992	243				
	Bid	Ask	Chg	Ask yield	Date text	Datevalue
2018 May 15	99.225	99.231	-0.017	1.16	May 15, 2018	43235
2018 Nov 15	98.648	98.659	-0.014	1.16	Nov 15, 2018	43419
2019 Feb 15	98.313	98.327	-0.018	1.2	Feb 15, 2019	43511
2019 Aug 15	97.478	97.496	-0.022	1.33	Aug 15, 2019	43692
2020 Feb 15	96.783	96.806	-0.023	1.35	Feb 15, 2020	43876
2020 Feb 29	96.645	96.668	-0.031	1.38	Feb 29, 2020	43890
2020 May 15	96.33	96.355	-0.033	1.4	May 15, 2020	43966
2020 Aug 15						44058
2021 Feb 15						44242
2021 May 15						44331
2021 Aug 15						44423
2021 Nov 15						44515
2022 Aug 15						44788
2022 Nov 15						44880
2023 Feb 15						44972
2023 Aug 15						45153
2024 Nov 15						45611
2025 Feb 15						45703
2025 Aug 15						45884
2026 Feb 15						46068
2026 Aug 15						46249
2026 Nov 15	82.085	82.159	-0.004	2.18	Nov 15, 2026	46341
2027 Feb 15	81.513	81.589	-0.008	2.17	Feb 15, 2027	46433
2027 Aug 15	80.488	80.567	0.007	2.19	Aug 15, 2027	46614
2027 Nov 15	79.865	79.945	unch.	2.21	Nov 15, 2027	46706



To arrive at a single ZCB yield curve, we take a regression approach to find a curve of best fit and not try to fit all the points exactly. We are not performing any statistical inference, so there is no assumption regarding the residual distribution. We can specify a general nonlinear function of t that seems to fit y(t) very well, for example:

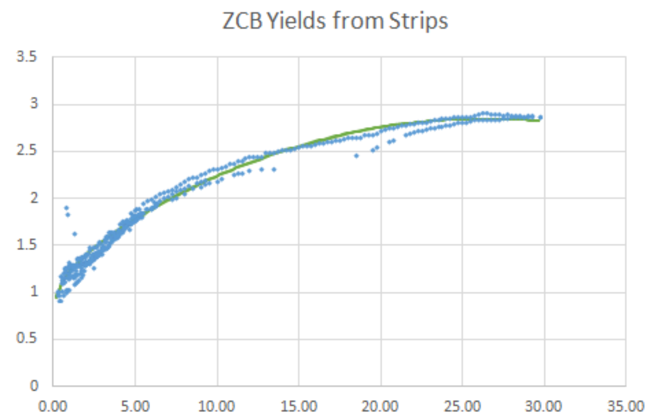
$$y(t) = a_1 + a_2 t + a_3 t^2 + a_4 e^{a_5 t} + a_6 ln(t - a_7) + \varepsilon$$

Our objective will be to choose the parameters to minimize the sum of squared deviations between model and actual yields. We can perform this nonlinear regression analysis in Excel using the SOLVER function. The add-in must be loaded before it is called; it is accessible under FILE OPTIONS. You may also want to load the DATA ANALYSIS tool pack as well. Once loaded, the functionality can be accessed using the DATA ribbon.

Once SOLVER is called, the user specifies that they would like to minimize the sum of squared differences between the reported yields and the computed yields, by changing the parameters in the yield calculation. The parameter values should NOT be constrained to be non-negative, as this may compromise the fit. When the solver is completed running, the selection KEEP SOLVER SOLUTION shows the optimized parameters.

This figure shows the optimized parameters along with the fitted curve and the data on 9/14/17:

Parameters	
a_1	0.90167
a_2	0.108333
a_3	-0.00213
a_4	0.214606
a_5	-2E-05
a_6	0.10953
a_7	-0.00061
Sum of squared deviations	
	2.97056



At this point, the yield for any maturity needed to discount a cash flow could be easily computed from the estimated smooth yield curve function.

BOOTSTRAPPING WITHOUT STRIPS

STRIPS data was widely reported in the financial press prior to 2019, but has become more difficult to obtain. Also, financial engineers working in banks will want to use all available bond pricing data to get the most accurate ZCB curve possible. Fortunately, the method described in the last section works well even with coupon bonds.

With coupon-paying bonds, the procedure is as follows:

1. Find the exact coupon dates (t) and payment amounts C_t for each payment date
2. Use the same function above to model the ZCB discount rate function y(t):
$$y(t) = a_1 + a_2 t + a_3 t^2 + a_4 e^{a_5 t} + a_6 ln(t - a_7) + \varepsilon$$
3. The model bond price is the present value of $\Sigma C_t e^{-y(t)t}$
4. The squared deviation is the square of the difference between the observed price and the market price.
5. Make sure the market prices include accrued interest (dirty prices)
6. Choose the “a” parameters to minimize the sum of squared deviations for all available risk-free bonds

This procedure provides a ZCB discount rate curve using continuous compounding, and can therefore be used to value any risk-free at cash flows at known times.

TIPS AND INFLATION

“TIPS” stands for *Treasury Inflation-Protected Securities*. These securities were first issued by the U.S. government in 1997 partly in response to demand for long-term inflation-protected assets by pension funds and long-term investors.

From Treasurydirect.gov:

> "Treasury Inflation-Protected Securities, or TIPS, provide protection against inflation. The principal of a TIPS increases with inflation and decreases with deflation, as measured by the Consumer Price Index. When a TIPS matures, you are paid the adjusted principal or original principal, whichever is greater.
>
> "TIPS pay interest twice a year, at a fixed rate. The rate is applied to the adjusted principal; so, like the principal, interest payments rise with inflation and fall with deflation."

Investors who purchase TIPS can expect their coupons and principal to appreciate at the same rate as the CPI index. However, the tax code inserts a slight wedge in the inflation hedge:

> "Interest payments from Treasury Inflation-Protected Securities (TIPS), and increases in the principal of TIPS, are subject to federal tax, but exempt from state and local income taxes."

Whereas the return of principal in a regular bond is untaxed, the return of inflation-adjusted principal above the face value is taxed. Therefore, this bond is not a complete inflation hedge. However, all is not lost --- the principal also has an embedded option since it pays the higher of the original principal amount or the inflation adjusted principal. This protects against the seemingly remote risk of deflation.

Treasury Inflation Protected Securities (TIPS)

NAME	COUPON	PRICE	YIELD	1 MONTH	1 YEAR	TIME (EDT)
GTII5:GOV **5 Year**	0.13	100.43	0.04%	+5	+17	5/26/2017
GTII10:GOV **10 Year**	0.38	99.57	0.42%	+6	+18	5/26/2017
GTII20:GOV **20 Year**	3.38	140.37	0.55%	+5	+3	5/26/2017
GTII30:GOV **30 Year**	0.88	98.05	0.95%	+5	+8	5/26/2017

Details aside, the TIPS product is best for those looking to get protection from long-term inflation increases. Also, we can use the Bloomberg quotes in the exhibit to estimate the market's expected inflation rate.

To first approximation, the TIPS yield is like a real yield. From the treasury bond exhibit earlier, we can take the nominal yield. Taking the expected inflation rate equal to the difference, we see that the 5 year expected inflation rate is being priced at 175 bp. Similarly, the 10 and 30 year expected inflation rates are being priced at 183 and 196 basis points.

Therefore, if we were valuing inflation-linked cash flows, such as a cost-of-living-adjusted wage contract, this would suggest to us that we could deduct between 1.75% and 1.96% from the discount rate and proceed to compute the present value at the reduced real rate of interest.

FLOATING RATE BONDS

Floating rate bonds have not traditionally been issued by the U.S. government. However, the U.S. Treasury began auctioning Floating Rate Notes (FRNs) in January 2014, and has continued on a monthly basis since then. FRNs are issued for a term of two years and pay coupons quarterly. Unlike a fixed rate bond, the FRNs pay a variable rate of interest determined in arrears, implying that the first coupon payment is always known, but later payments are unknown.

The FRN coupon rate is the sum of the 13-week Treasury-bill auction rate and a spread determined by the auction result on the issue date. The Treasury bill is auctioned every week, and the index rate is re-set every week. The spread is constant over the life of the FRN.

Because the interest payment includes a floating component and a fixed component, it is not a pure floating instrument as described in Chapter 3. The instrument should be valued as a combination of a floating rate note and a fixed rate note. For example, the 8/30/19 FRN auction settled with a high discount margin of 0.238% and a spread of 0.22%. Note that these are quarterly rates, not annual. Also, nearly half of the interest payment is fixed for the life of the bond.

Here are some other rates from 2019 auctions from the TreasuryDirect website:

Bills | CMBs | Notes | Bonds | TIPS | FRNs

Security Term	CUSIP	Reopening	Issue Date	Maturity Date	High Discount Margin	Spread
2-Year	9128287G9	Yes	08/30/2019	07/31/2021	0.238%	0.220%
2-Year	9128287G9	No	07/31/2019	07/31/2021	0.220%	0.220%
2-Year	9128286Q8	Yes	06/28/2019	04/30/2021	0.210%	0.139%
2-Year	9128286Q8	Yes	05/31/2019	04/30/2021	0.140%	0.139%
2-Year	9128286Q8	No	04/30/2019	04/30/2021	0.139%	0.139%
2-Year	9128285Y2	Yes	03/29/2019	01/31/2021	0.180%	0.115%
2-Year	9128285Y2	Yes	02/22/2019	01/31/2021	0.149%	0.115%
2-Year	9128285Y2	No	01/31/2019	01/31/2021	0.115%	0.115%
2-Year	9128285H9	Yes	12/28/2018	10/31/2020	0.150%	0.045%
2-Year	9128285H9	Yes	11/30/2018	10/31/2020	0.050%	0.045%
2-Year	9128285H9	No	10/31/2018	10/31/2020	0.045%	0.045%
2-Year	912828Y53	Yes	09/28/2018	07/31/2020	0.050%	0.043%
2-Year	912828Y53	Yes	08/31/2018	07/31/2020	0.047%	0.043%
2-Year	912828Y53	No	07/31/2018	07/31/2020	0.043%	0.043%
2-Year	9128284K3	Yes	06/29/2018	04/30/2020	0.042%	0.033%
2-Year	9128284K3	Yes	05/25/2018	04/30/2020	0.028%	0.033%
2-Year	9128284K3	No	04/30/2018	04/30/2020	0.033%	0.033%
2-Year	9128283T5	Yes	04/02/2018	01/31/2020	0.049%	0.000%
2-Year	9128283T5	Yes	02/23/2018	01/31/2020	0.016%	0.000%
2-Year	9128283T5	No	01/31/2018	01/31/2020	0.000%	0.000%

FRNs also trade outside the U.S. in the Eurodollar market, using a structure similar to the U.S. Treasury FRNs. In the past, issuers used LIBOR, or London Interbank Offer Rate for US dollars as a reliable benchmark of the cost of short term dollars. LIBOR also had the benefit of longer maturities being available, particularly 1-month, 3-month and 1-year LIBOR. In fact, the 3-month LIBOR became so popular it was chosen as the benchmark for Eurodollar futures by the Chicago Mercantile Exchange.

These quotes are from bankrate.com:

LIBOR, other interest rate indexes — Updated 5/24/2017

	This week	Month ago	Year ago
Bond Buyer's 20 bond index	3.73	3.71	3.26
FNMA 30 yr Mtg Com del 60 days	3.52	3.61	3.21
1 Month LIBOR Rate	1.02	0.99	0.45
3 Month LIBOR Rate	1.19	1.17	0.66
6 Month LIBOR Rate	1.41	1.42	0.96
Call Money	2.75	2.75	2.25
1 Year LIBOR Rate	1.72	1.77	1.30

Regrettably, in 2012, "multiple criminal settlements by Barclays Bank revealed significant fraud and collusion by member banks connected to the rate submissions, leading to the scandal."[9] . The LIBOR index is now managed by ICE, the Intercontinental Exchange, and is scheduled to be discontinued in 2021 or 2022. A new standard, the *Secured Overnight Funding Rate* (SOFR) has been recommended as an alternative benchmark. See https://www.financialresearch.gov/data/libor-alternatives/ for details.

One of the key feature of floating rate bonds is that, at every coupon date, the value of the bond equals its par value. In most cases, the next coupon is determined at that time, so the value of a floating rate bond can change between coupon dates due to the next payment being fixed.

MUNICIPAL BONDS

Municipal bonds (munis) are issued by non-federal government entities, such as states and cities. These bonds are not guaranteed by the full faith and credit of the U.S. government, so default is a possibility. However, in many cases, these bonds are rated AAA due to the underlying collateral, the ability of the municipality to collect taxes, or insurance provided by a third party insurance company.

We will focus on AAA munis so that we can focus more on implied tax rates as opposed to default risk. Bloomberg creates an index of AAA nontaxable municipal yields based on their proprietary data and pricing models. The chart below shows the yields reported on 5/28/17:

[9] See the Wikipedia entry "LIBOR scandal" for more details

Municipal Bonds

NAME	YIELD	1 DAY	1 MONTH	1 YEAR	TIME (EDT)
BVMB1Y:IND **Muni Bonds 1 Year Yield**	0.77%	0	-7	+17	5/26/2017
BVMB2Y:IND **Muni Bonds 2 Year Yield**	0.91%	0	-10	+18	5/26/2017
BVMB5Y:IND **Muni Bonds 5 Year Yield**	1.28%	-1	-17	+21	5/26/2017
BVMB10Y:IND **Muni Bonds 10 Year Yield**	1.95%	-1	-21	+33	5/26/2017
BVMB30Y:IND **Muni Bonds 30 Year Yield**	2.82%	-2	-18	+34	5/26/2017

At first, it appears strange that the yields on municipal bonds seem to be *lower* than the yields on U.S. government bonds --- we might have expected that the yields should be higher to reflect some element of default risk and losses given default. The reason for this is taxes. From Investopedia.com:

> "While interest income is usually tax-exempt for municipal bonds, capital gains are usually not tax-exempt. Capital gains realized from selling a bond are subject to federal and state taxes. ... These investors are subject to taxation on the interest income from some munis."

Assuming bonds are held to maturity, then, municipal coupons will be untaxed whereas U.S. government bond coupons are taxed at the federal level. The spread between the bonds would then be derived from the following equation

$$\text{Municipal yield} = (1-\tau)\text{ Treasury yield}$$

$$+ \text{ Expected loss rate} + \text{Additional risk premium}$$

Note that municipal bonds, other things equal, will attract investors in the highest tax brackets. Since the highest marginal federal tax rate is 33% currently, we would expect the embedded implied tax rate to be around 33%, and the data show that we are close.

	Yields		Implied Tax for Assumed Loss Rates	
Years	Treasuries	Muni	Loss=0bp	Loss=5bp
1	1.15	0.77	33.04%	37.39%
2	1.29	0.91	29.46%	33.33%
5	1.79	1.28	28.49%	31.28%
10	2.25	1.95	13.33%	15.56%
30	2.91	2.02	30.58%	32.30%

Allowing a small loss rate, which in the certainty case would be simply the product of the default rate with the loss given default, we get a little closer. For example, a 5 basis point loss charge will raise the implied tax rate, as shown in the table.

CORPORATE BONDS

We would like to learn about a company's implied credit quality from the market value of the bonds it has issued. However, data for corporate bonds are hard to find, given the relative lack of trading in that market and the unwillingness of dealers to provide transparency. Nevertheless, one can usually find prices where the reporting is slightly delayed. These sample data are from FINRA, the regulatory body, as of 1/18/18.

Most Active Investment Grade Bonds

Issuer Name	Symbol	Coupon	Maturity	Moody's®/S&P	High	Low	Last	Change	Yield%
GE CAP INTL FDG CO MEDIUM TERM NTS BOOK	GE4373444	2.342%	11/15/2020	A2/AA-	99.24900	98.91400	99.00900	-0.296000	2.708436
CITIGROUP INC	C4114236	2.550%	04/08/2019	Baa1/BBB+	100.45400	100.08000	100.15000	-0.009000	2.422841
TRANSCANADA PIPELINES LTD	TRP4566319	2.125%	11/15/2019	A3/	99.60000	99.48800	99.48800	-0.138000	2.413118
JPMORGAN CHASE & CO	JPM4260166	2.750%	06/23/2020	A3/A-	100.66400	100.23840	100.50500	-0.130000	2.526826
WELLS FARGO BK N A SAN FRANCISCO CALIF M	WFC4432931	2.150%	12/06/2019	Aa2/AA-	100.04700	99.46300	99.48200	-0.374000	2.432825
PNC BK N A PITTSBURGH PA MEDIUM TERM SR	PNC4341038	1.950%	03/04/2019	A2/A	99.70700	99.64300	99.67500	-0.020000	2.243243
GE CAP INTL FDG CO MEDIUM TERM NTS BOOK	GE4373445	4.418%	11/15/2035	A2/AA-	106.11400	105.10300	105.29900	-1.266000	3.999004
GENERAL ELEC CO	GE4105158	4.500%	03/11/2044	A2/AA-	107.78600	106.69200	107.31300	-0.813000	4.043973
JPMORGAN CHASE & CO	JPM4203225	2.250%	01/23/2020	A3/A-	100.35900	99.54400	99.62900	-0.107000	2.440085
BOARD TRUSTEES LELAND STANFORD JR UNIV	STUY4236061	3.460%	05/01/2047	/AAA	100.49200	100.26400	100.49200	0.751000	3.433025

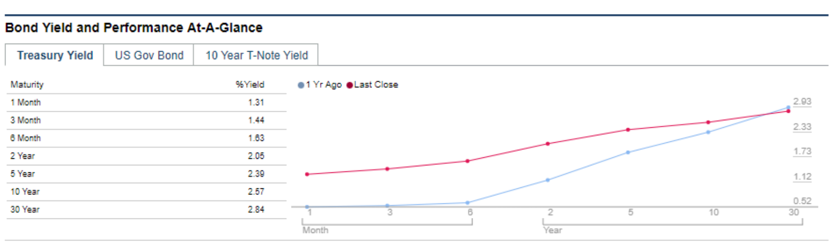
Investment grade bonds are those higher in credit quality than other bonds, with a minimum rating of Baa/BBB on Moody's/S&P bond rating services. Bonds with lower ratings are called *high yield* bonds by marketers and *junk bonds* by investors.

Ratings are chosen by the ratings agencies to give their best possible estimate of the likelihood of default of the bond. They do not attempt to measure loss given default. Here are three recent agency studies of historical default rates for bonds of different ratings. The A.M. Best ratings for guaranteed insurance companies are based on impairment, rather than default, since theoretically, no default is possible on insurance company obligations if the payments are guaranteed by the U.S. government.

Historical default rates from recent ratings agency studies

Moody's	Nov 2006 Special Comment p 14		
	20-year	1-year	1-month
Aaa	0.64%	0.03%	0.0027%
Aa	0.65%	0.03%	0.0027%
A	1.55%	0.08%	0.0065%
Baa	4.27%	0.22%	0.0182%
Ba	13.84%	0.74%	0.0620%
B	25.20%	1.44%	0.1209%
Caa-C	41.23%	2.62%	0.2212%
S&P	2015 Annual Global Corporate Default Study p 10		
		1-year	1-month
AAA		0.01%	0.0008%
AA		0.02%	0.0017%
A		0.06%	0.0050%
BBB		0.19%	0.0158%
BB		0.73%	0.0610%
B		3.77%	0.3197%
CCC/C		26.36%	2.5176%
AM Best	Best's Impairment Rate & Rating Transitions Study 1977-2015		
	15 year	1-year	1-month
A++	0.15%	0.01%	0.0008%
A+	1.62%	0.11%	0.0091%
A	2.86%	0.19%	0.0161%
A-	3.44%	0.23%	0.0194%
B++	5.24%	0.36%	0.0299%
B+	6.30%	0.43%	0.0361%
B	10.06%	0.70%	0.0589%
B-	12.53%	0.89%	0.0743%
C++/C+	13.80%	0.99%	0.0825%
C/C-	17.03%	1.24%	0.1037%
D	22.85%	1.71%	0.1440%

Recall that the corporate bond spread, the difference between the yield on the corporate bond and that of a U.S. government bond with a similar maturity, approximately equals the expected loss ($\pi\lambda$) plus a risk premium. Let's look at the General Electric Bond, yielding 4.04% for roughly a 27 year bond. Using the 30 year bond as a proxy (table below), the spread is about (4.04-2.84=) 1.24%.


Bond Yield and Performance At-A-Glance

Treasury Yield | US Gov Bond | 10 Year T-Note Yield

Maturity	%Yield
1 Month	1.31
3 Month	1.44
6 Month	1.63
2 Year	2.05
5 Year	2.39
10 Year	2.57
30 Year	2.84

The GE Bond is rated AA- by S&P indicating a historically based expected default rate of 0.02%. Even with λ=100%, this implies a risk premium of 1.22%.

Clearly, there are many assumptions one could make about default probabilities, losses given default and the risk premium, and it is impossible to get estimates of all three from a single spread observation. Nevertheless, skilled corporate bond analysts use all the means at their disposal to estimate these parameters as carefully as possible.

The bottom line: Corporate bond prices hold information about default risk and risk premia that can be useful not only for bond buyers, but for any analyst needing to understand more about the companies they are studying.

VALUING A DEFAULTABLE BOND – EXACT FORMULA

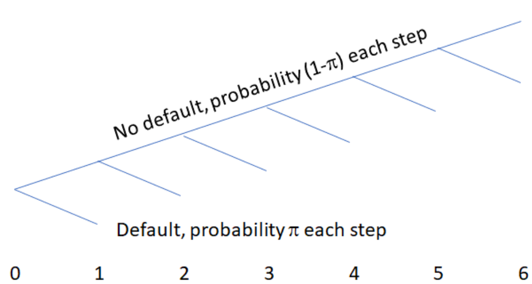
Default is one of the most important valuation topics in finance. Technically, the word *default* in legal terms refers to any breach of contract. We generally use the word default in a bond or loan context specifically to mean "default on payments." Indeed, the valuation of government bonds, corporate bonds and loans, and personal loans all depend on assumptions about the risk of payment failure, and the eventual recovery of assets by lenders in the event of default through the bankruptcy process.

In reality, loan default is a complex legal process, and it is difficult to predict exactly what and the cash flows of a defaulted bond will be, and when they will be paid. For this reason, we will have to make some simplifying assumptions. Using terminology of the Bank of International Settlements (BIS), the standard-setter in bank risk management, we refer to the constant annual probability of default in one year as the probability of default PD or π. We also use the term loss given default, LGD or λ, to represent what percentage of a bond's book value is lost in the event of default. The product of the two, $\pi\lambda$, can be understood as the expected loss on a bond in any given year.

Measuring loss can be tricky. It may seem logical to estimate LGD by considering the percentage loss in the *market value* of the bond on default. However, with loans, there is no reported market value, and LGD would be impossible to calculate. **For this reason, we measure LGD relative to the *scheduled* payments of the bond discounted at the scheduled contract rate, i.e. the book value**. We then ask what fraction of this value will be lost in the event of default. Default-adjusted cash flows are then discounted at the risk-free rate. This standardization is extremely useful, particularly when collections from creditors often take place long after the cash payments are missed.

Bond and loan rating agencies have conducted comprehensive studies to determine historical PD and LGD for many different kinds of loans and bonds. These include Moody's, Standard & Poors, Fitch and A.M. Best, for example. Letter ratings generally correspond to estimated PDs, and LGDs are determined according to a number of factors. These factors include country, industry, debt priority in bankruptcy, loan security and other factors. For example, Moody's has calculated that historically, the average LGD for all U.S. loans in their database was 60%.

To illustrate the valuation of a defaultable bond, we begin with a risk-free valuation. To make things simple at first, we assume that the default probability is constant every year. Further, the bond only survives to the next year if there is no default, as shown in the tree diagram below:



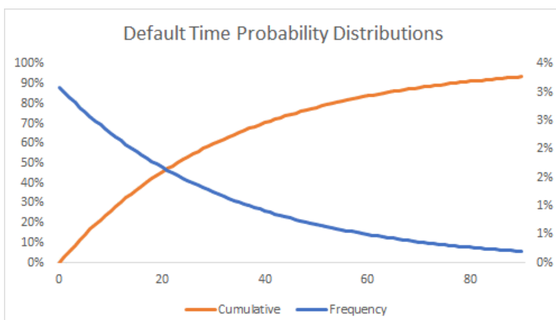
Now suppose we have a bond with scheduled annual interest payments of \$55 and a principal repayment of \$1000 in 5 years. If the simple annual risk-free rate is 4%, then the value would be \$1,066.78 in the absence of default. Now, let's assume the annual default probability is constant at π=3%, and the LGD is λ=60%. What does this mean?

There are four steps to this process:

1. Compute the probability distribution associated with the default dates
2. For each default date, determine the cash flows conditional on default
3. For each default date, find the present value of the default-adjusted cash flows at the risk-free rate
4. Value the cash flows by computing the expected value over all the possible default dates

Step 1. Let's begin with the default dates. Using a Bernoulli process, we can determine the probability distribution of the default date. Clearly, the probability of default in the first year is π. In the second year, we must have a non-default followed by a default, implying a probability of $\pi(1-\pi)$. Proceeding by induction, the probability of default in any given year i preceded by (i-1) nondefault years, is $f(i)=\pi(1-\pi)^{i-1}$. You should verify that the sum of these probabilities is one.

The cumulative probability of default in year i can also be easily computed using sums of geometric sequences: $F(i)=1-(1-\pi)^{i}$. The probability distribution generated by this stochastic process is known as the *geometric distribution,* or in continuous models, the *exponential distribution*. Here's what it looks like for π=3%:



The expected default time is $1/\pi$, or in this case 33.33 years. The standard deviation is $\frac{\sqrt{1-\pi}}{\pi}$ = 32.83 years.

Step 2. We now specify the cash flows contingent on each default date. If default occurs on or before the one year mark, we get 40% (1-λ) of the scheduled cash flows (\$55) and principal (\$1000). If the bond defaults in the second year, we get the first full payment, and then the payments drop for the rest of the period. This continues through year 5. If the default date exceeds 5, the full scheduled payments are received.

These calculations are reflected in the boxed portion of the figure below:

	Default						
Prob	Date	Cash flows as function of default date					PV
		1	2	3	4	5	
3.00%	1	22	22	22	22	422	426.71
2.91%	2	55	22	22	22	422	458.44
2.82%	3	55	55	22	22	422	488.95
2.74%	4	55	55	55	22	422	518.29
2.66%	5	55	55	55	55	422	546.50
85.87%	over	55	55	55	55	1055	1,066.78
	ECF	54.01	53.05	52.12	51.21	965.58	984.73
	Growth		-1.78%	-1.76%	-1.73%		
	StDev						202.92

Steps 3 and 4. Now we compute the present values for each possible default date by discounting the conditional cash flows at the risk-free rate of 4%. HINT: Use the NPV function in Excel to do this.

We can also compute expected cash flows and valuations by taking the probability-weighted sum of the contingent cash flows or valuations. In Excel, this can be done using the SUMPRODUCT function.

The final value of $984.73 can be reached either by discounting the expected cash flows at 4%, or finding the probability-weighted average of the default contingent valuations.

This may seem like a relatively difficult valuation compared to others we have covered to date. Let's see if there is a quicker way.

AN ANALYTIC SOLUTION FOR THE DEFAULTABLE BOND VALUE

To value this bond using conventional tools, it is helpful to realize that the LGD assumption allows us to assume that part of the bond is risk-free (the nondefaultable part) and part is risky due to the fact we are guaranteed to receive a fraction (1-λ) of the cash flows. Subtracting the minimum cash flows from the cash flows in the last figure, we obtain the following:

Prob	Date	Cash flows minus the nondefaultable part				
		1	2	3	4	5
3.00%	1	0	0	0	0	0
2.91%	2	33	0	0	0	0
2.82%	3	33	33	0	0	0
2.74%	4	33	33	33	0	0
2.66%	5	33	33	33	33	0
85.87%	over	33	33	33	33	633
	ECF	32.01	31.05	30.12	29.21	543.58
	%Δ		-3.00%	-3.00%	-3.00%	

Notice that once we find the expected cash flows, they have a predictable pattern, since the growth rate is negative (-3% per year), or -π. This means we can value the defaultable piece as a growing annuity! Therefore, the overall valuation of a defaultable bond is simply the sum of a flat annuity and a growing annuity.

$$\text{Non-defaultable part} = (1-\lambda)\ PVA(C,n,r,F,1) = \$426.71$$

$$\text{Defaultable part} = \lambda\ PVA(C,n,r^*,F,1) = \$558.02 \text{ with } r^* = \frac{r+\pi}{1-\pi} = 7.22\%$$

Total valuation = \$984.73

The non-defaultable part is the recovered cash flow $(1-\lambda)C$ and $(1-\lambda)F$, and the defaultable cash flows are λC and λF, which decline over time in expected value due to the risk of default.

Approximate yield comparison from Chapter 3

Notice that the yield of a bond with these scheduled payments and priced at \$984.73 is 5.86%. If you subtract the risk-free interest rate of 4%, the *spread* is 1.86%. Some would call this a credit spread, but it would be better to refer to this as an *expected loss* spread. Since the expected loss $\pi\lambda$ is 1.80%, this provides a short-cut approximation to valuing the defaultable bond. The approximation $\frac{\pi\lambda}{1-\pi}$ is even closer, at 1.86%. However, it is still an approximation, which becomes less accurate as π increases.

Bottom line: "*Wall Street rule of thumb: Expected loss spread is approximately $\pi\lambda$. Total credit spread includes an additional risk premium.*"

Example

Suppose we have a corporate bond that makes 10 annual payments of \$50, returning \$1000 in principal at the end of 10 years. The risk-free rate is 5%, but the price of this bond is \$828.40, not \$1000. What can we say about the implied default probability, assuming that the loss given default λ=60%, and there is no additional risk premium?

Solution

The rate that makes the present value of these payments equal to \$828.40 is 7.5%. Therefore the "credit spread" is 2.5%. Using the rule of thumb, and assuming no additional risk premium, this is the same as the expected loss premium ($\pi\lambda$), and λ=60%. Therefore π=4.17%, implying an average of about 4.17% expected default rate per year. The exact valuation and yield can be computed using the exact formulae provided above.

EXERCISES

1. A 5 year idealized U.S. government bond with a 1.75% coupon is valued at 99.81.
    a. What is its YTM (i.e. twice the semiannual yield)?
    b. If the YTM increases by 1%, what is the new price?
2. A Treasury bill with 135 days to maturity has an asking yield of 0.95%.
    a. What is the price of the T-Bill?
    b. What is the effective annual return?
3. Suppose the workers at Gideon Industries negotiated a contact that paid out \$10,000,000 per year for the next five years.

   a. If the current risk-free rate is 5%, what would be the present value of a COLA (cost of living adjustment) clause in their contract for the five year period? Use the TIPS data provided in the chapter.
   b. What is the fixed wage increase that would make the union indifferent between the COLA and a fixed wage structure?

4. A five-year AAA rated municipal bond yields 1.5% when a five-year government bond yield 2.0%. You have two investor clients, one in the 20% tax bracket and one in a 45% tax bracket. What would you advise them about buying this particular municipal bond compared to the U.S. government bond?

5. Download current TIPS yields and yields for U.S. government bonds. What can you infer about the
   a. Expected inflation rate
   b. Real rate of interest

6. A corporate junk bond is rated CCC by S&P and yields 14% when the yield of a comparable government bond is 2%.
   a. What can you infer about loss given default?
   b. If the additional risk premium is 5%, what can you infer about LGD?

EXERCISES TO BE DONE IN EXCEL

A. Download today's prices for U.S. Treasury strips, TBills and TBonds from the Wall St Journal or other source, and build a smooth ZCB yield curve from the data. Use spot rates in increments of 1 month to build a forward rate curve for future one month loans.

Chapter 5: Simulation

INTRODUCTION

As we transition from known cash flows to unknown cash flows in Part II, simulation increases in importance. As long as *laws of motion* can be prescribed for financial variables of interest, we can simulate possible movements in those variables for the purpose of determining expected cash flows, risk and ultimately, valuation. For example, we may want to assume that a security's future returns are normally distributed, or that corn prices follow a mean-reverting process on average, but with volatility.

Simulation provides one way to value assets, and in many cases, other methods can be used. These methods may include any of the following

- Closed-form solutions
- Binomial and trinomial trees
- Numerical integration
- Numerical solution of partial differential equations

All of these methods can be faster and more accurate than simulation, but unfortunately, each of these methods requires restrictions on the model complexity. Simulation is often the slowest and the least accurate method, but it is by far the most flexible, particularly when there are many stochastic variables involved – e.g. interest rates, exchange rates, commodity prices, event risks, or anything else that one might use to simulate corporate income. However, with advances in computing software and hardware, especially cloud computing, simulation can become much more accurate at a fraction of its previous cost, making it a viable contender as a valuation method.

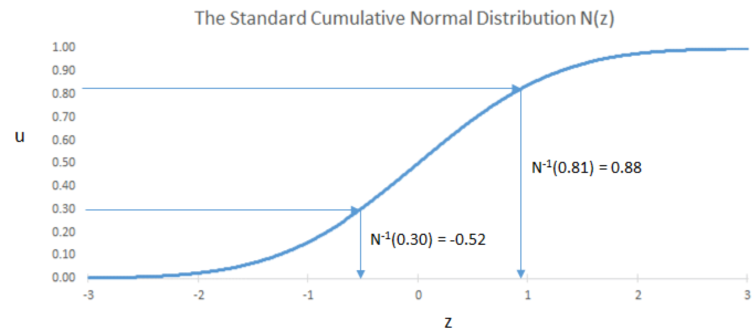
Simulation requires that we generate a large number of random outcomes of our model, and estimate the result of the model using the average result of our outcomes. We first generate random numbers, then that we compute the average outcome, and find some measure of the reliability of the result. This is the core of simulation, the rest is detail.

RANDOM NUMBER GENERATION

Of course, we cannot generate numbers that are truly random. When we say we are "generating random numbers", we really mean that we want to generate a series of numbers which have the key *properties* of random numbers. For example, the generated numbers should follow a prescribed probability distribution. Also, the numbers should be statistically independent of each other, meaning that I shouldn't be able to determine the elements of the sequence using some predictive model.

Numbers that meet these properties are called *pseudorandom numbers*, but we still call them random numbers for short. Entire books have been written about the creation of pseudorandom numbers, see for example von Neumann (1951) for an early example. The importance of these numbers extends deeply into the fields of computer science, cryptography and computer security. In finance, we use pseudorandom numbers to generate price processes, default states, and the occurrence of all sorts of unpredictable events. For many applications, we require uniform and/or normally distributed random numbers, all independent of each other.

Suppose for example that we wish to generate a large number of normally distributed variables (z) with the standardized mean of 0 and standard deviation of 1. We know the probability density function or PDF, n(z), and also the cumulative density function or CDF, N(z) for values of z. Although z can take any value on the real line, the most common values range from -3 to 3, i.e. within 3 standard deviations of the mean. The graph below shows the CDF of N(z):



Notice that the range of N(z) is (0,1). To generate random values of z, therefore, it is easiest to generate pseudorandom numbers (u) that are uniformly distributed on the interval (0,1), and map them to the corresponding values of z. For example, if a uniform random number generator provided 0.81 as a value, we could map it to z by computing the inverse of the CDF at 0.81, obtaining the z value of 0.88. Similarly, 0.30 would map to -0.52.

There are many techniques to generate values of u. The only real challenge with this method is that the normal CDF is not analytic. However, this can easily be addressed in Excel or Python. In Excel, uniformly distributed random numbers can be generated with the function *u=rand()*. The inverse standard normal CDF in Excel is given by $N^{-1}(u)$ *= normsinv(u)*. Therefore, the single line *=normsinv(rand())* will generate a standard normally distributed random number. When the formula is copied, each cell provides a new random number, allowing us to generate as many random numbers as we like. A similar function is available in Python: numpy.random.normal(mean, stdev, size).

Important: This technique works for all CDFs. If you would like to generate random number of any standard distribution, you can do so as long as you have some way to compute the inverse CDF and some way to generate uniform (0,1) random numbers. In some cases, this leads to a simple numerical result, such as the exponential distribution, which we will see later in this chapter.

Exercise

Using the uniform random numbers 0.36 and 0.98, generate two random numbers from a Chi-squared distribution with 1 degree of freedom using Excel functions.

Solution

Using the inverse Chi-squared CDF function, this is computed as =chisq.inv(u,1), resulting in 0.22 and 5.41.

THE SIMULATION CORE LOGIC

The underlying core logic of simulation is simple. These steps are common to all simulations. Let's say we are going to simulate a large number of values of z, and we want to know the expected value of f(z), a function of z.

We also want to define our maximum likely error based on a 2 standard deviation range (though we might like to use 3 or more standard deviations for critical applications)

Core logic of simulation

1. Generate a large number N of random numbers z
2. Compute outcome values f(z)
3. Compute the average value $\bar{X}$of the f(z) results --- this is your estimate of E[f(z)]
4. Compute the sample standard deviation of the values of f(z). Let's call this "s"
5. Compute the *standard error*, which is the standard deviation divided by $\sqrt{N}$, i.e. $s/\sqrt{N}$
6. Compute the maximum likely error, which is 2 times the standard error = $2s/\sqrt{N}$
7. If the maximum likely error is too large, increase the size of N in proportion to the square of the desired increase in precision.
8. The solution is then given by $E[f(z)] = \bar{X} \pm 2s/\sqrt{N}$

The only tricky step in the process is to remember to compute the standard error by dividing by the square root of N. We use the words standard *error* to remind ourselves that it is the standard *deviation* of the measure of the average.

Note that the variance of a sum of N independent identically distributed random variables is N times the individual variance. Also, the variance of (sum/N) is the variance of the sum divided by N^2. Accordingly, the variance of the average is the variance of the underlying data divided by N, and the standard deviation is the square root of that. Furthermore, as N increases, the distribution of $\bar{X}$ asymptotically approaches a normal distribution has a mean equal to the true value of the population mean, and a *standard error* approaching $\sigma/\sqrt{N}$. This is known as the *Central Limit Theorem*.

A detailed example

Let's suppose we want to compute the expected value of max(z,c), where z is the standard normally distributed random variable and c is a constant. In this case, we actually know the formula for this, so we can check the result when we are finished:

$$E[\max(z, c)] = n(c) + c\,N(c)$$

For example, when c = 0.5, the result is 0.6978.

To begin, let's generate 1000 values of z in Excel by copying the formula "*=normsinv(rand())*" into 1000 cells in a column. Then for each cell, we compute the value of max(z,c) with a given value of c of 0.50. Finally, we compute the mean, standard deviation, standard error and maximum likely error. Here is an illustration of how your spreadsheet might look (only showing the first 10 simulations):

First 10 simulations shown of 1000

c		0.5
	z	max(z,c)
1	-0.45	0.50
2	-1.84	0.50
3	-0.31	0.50
4	1.54	1.54
5	0.02	0.50
6	-0.66	0.50
7	0.30	0.50
8	-0.26	0.50
9	-0.58	0.50
10	0.50	0.50

Statistics for 1000 simulations

	z	max(z,c)
Average	-0.0373	0.6874
StDev	1.0006	0.3911
n	1000	1000
StErr	0.0316	0.0124
#StErr	2	2
MaxErr	0.0633	0.0247
Min		0.5000
Max		2.9665
True value		*0.6978*

Our estimate of the value of E[max(z,0.5)] is 0.6874 $\pm$ 0.0247. The true computed value of 0.6978 falls in the lower end of that range. If we need to reduce the size of the maximum likely error by a factor of 30, we can multiply the number of simulations by 900 or 30^2 for a total of 900,000 simulations.

You may have noticed that the randomized z values have a mean of -0.0373 and a standard deviation of 1.0006, not the prescribed values of 0 and 1. Of course, if we are simulating any set of random numbers, we will get values of the mean and standard deviations changing with every simulation, i.e. every recalculation of the spreadsheet. We will see in the next section how we can improve on the simulated values of z to get more accurate simulation results.

Exercise

Before proceeding to the next section, replicate this example in Excel, but adjust the random numbers so that the mean is zero and the standard deviation is 1. Does the adjusted set of random numbers seem to provide a more accurate result?

VARIANCE REDUCTION TECHNIQUES

Much has been written about methods to increase simulation accuracy or computation speed, i.e. the reduction in the sample variance of f(z) in the previous example. In this section, we will discuss two of them, known as *moment matching* and *antithetic sampling*. Please refer to Botev and Ritter (2017) for a comprehensive treatment of these and other techniques. *Control variate* techniques can also be used to reduce variance if the variables correlate with the simulation results.

One simple and effective way to reduce sampling variance is to force the first two moments, i.e. the mean and standard deviation of the random sample of z values to be equal to their theoretical values, i.e. 0 and 1. This can be accomplished by replacing each simulated value of z with its standardized equivalent, that is,

$$z \leftarrow \frac{z - \bar{z}}{s_z}$$

where we subtract the sample mean from each value of z, and divide by the sample standard deviation. This technique, *moment matching*, eliminates two sources of uncertainty from the random sample and can have a positive impact on simulation performance.

Antithetic sampling means that each time we choose a uniformly distributed random number, we include the value of (1-u) as a separate random number. Notice that for the normal distribution, this has the effect of forcing the mean of the sample random numbers to be zero, but does not affect the variance, since z and –z are included in every simulation. However, the antithetic sample creates negative correlations between observations in the dataset, allowing for some variance reduction. Antithetic sampling used to be considered beneficial because the computation time for generating an antithetic number was less than the time of creating a new pseudorandom number. Nowadays, this benefit is not as highly valued due to rapid advances in computing speed and parallel processing.

To test the impact of moment matching and antithetic sampling, we will now perform 1000 iterations of our simulation of 1000 values, for a total of 1,000,000 values. To judge the efficiency of the variance reduction, we will compute the sample standard deviation of the results of every 1000 sims and then compile the results in a two stage process. In Excel, therefore, we will use our original setup to simulate 1000 outcomes, and then use a *what-if table* to generate 1000 results of the 1000 individual simulations. The results are as follows:

								Basic	Moment Matching	Antithetic
Mean	-0.0233	0.0000	0.0000	0.6911	0.6986	0.6926		0.6974	0.6977	0.6970
StDev	0.9992	1.0000	0.9806	0.3956	0.4039	0.3955		0.0127	0.0035	0.0116
StErr	0.0316	0.0316	0.0310	0.0125	0.0128	0.0125		0.0004	0.0001	0.0004
	Single simulaton of 1000 outcomes						*What-if table of 1000 simulations*			
	z	moment	antith	f(z)	moment	antith	means			
1	0.93	0.96	0.93	0.93	0.96	0.93	1	0.69	0.70	0.68
2	-2.45	-2.43	-2.45	0.50	0.50	0.50	2	0.70	0.70	0.68
3	-1.16	-1.14	-1.16	0.50	0.50	0.50	3	0.68	0.70	0.67
4	0.53	0.55	0.53	0.53	0.55	0.53	4	0.69	0.70	0.68
5	1.35	1.38	1.35	1.35	1.38	1.35	5	0.68	0.69	0.69
6	-0.46	-0.44	-0.46	0.50	0.50	0.50	6	0.72	0.70	0.69
7	-0.98	-0.96	-0.98	0.50	0.50	0.50	7	0.69	0.70	0.70
8	-1.04	-1.01	-1.04	0.50	0.50	0.50	8	0.69	0.70	0.67
9	0.40	0.42	0.40	0.50	0.50	0.50	9	0.69	0.70	0.68
10	1.07	1.09	1.07	1.07	1.09	1.07	10	0.71	0.70	0.69

The first three columns contain the raw z-values, the z values with corrected mean (0) and standard deviation (1), and the antithetic values; entries 501-1000 are the negative values of 1-500. The next 3 columns show the values of f(z) for the three methods. The final table on the right tabulates 1000 simulations of the results of the single simulation done in the spreadsheet. For each column, the statistics are computed on the top.

Using the basic technique, or the z column, the standard deviations of the 1000-sim values was 0.0127, which as expected, were lower by a factor of about sqrt(1000) from the original f(z) simulation result of 0.3956. Remarkably, the moment matching technique reduced the standard error by a factor of more than 3.5 times. The antithetic technique did not have much success with this particular problem, though in other problems it may be much more successful. In all cases, the large number of final simulations leads to good results for all three techniques.

Perhaps needless to say, different variance reduction techniques work better with different simulations, but the results can be significant.

CONTROL VARIATE TECHNIQUE

In some situations, we may be able to condition the simulated variable (let's say "Y") on values of an explanatory vector of other observed variables "X", one of which may be a constant. That is, with a constant, all the first column elements of the X matrix equal "1". Furthermore, let's assume the relationship between Y and X is linear for the time being, so Y = bX, and b is a coefficient vector derived from a linear regression.

In this situation, if "b" were known with certainty, we would expect that the variance of Y would drop to the variance of the residuals. But in most practical cases, b is estimated, so we must also consider the variance in those estimates. For any given vector X, let's call it X_h, then the *variance of the mean response* conditional on X_h is given by:

$$s^2(X_h'b) = s^2 X_h'(X'X)^{-1} X_h$$

where "s" is the residual standard deviation from the regression. he further X_h deviates from its mean, the greater the mean response variance. In some cases, we know the theoretical value of the expected value of X, in which case, we can set this vector equal to X_h to minimize the mean response variance.

Example

For the sake of demonstrating the calculation, let's assume that the X variable is a vector of the numbers 1-10. The Y values are secretly equal to X^2, but we are pretending we don't know this and are using X as a control variate to predict the theoretical mean of E[Y|X]. The calculations in Excel are shown below:

Y	1	X	E[Y|X]	MR std
1	1	1	-11	4.77
4	1	2	0	4.05
9	1	3	11	3.41
16	1	4	22	2.90
25	1	5	33	2.61
36	1	6	44	2.61
49	1	7	55	2.90
64	1	8	66	3.41
81	1	9	77	4.05
100	1	10	88	4.77

Regression statistics	
a	-22
b	11
s	8.12
sy	10.81
X'X	
10	55
55	385
Inv(X'X)	
0.4667	-0.0667
-0.0667	0.0121

The calculation demonstrates that the mean response (MR) standard deviations are indeed lowest near the mean value of X (5.5), and increase as one moves away from the mean. More importantly for our purposes, for values near the mean, the mean response standard deviation is much smaller than the standard error of the Y values without regressing on X (10.81). In this case, the control variate technique is likely to reduce error standard deviations by more than 50%.

We will use regression in Part II in our effort to value assets by benchmarking them to others. In these cases, we will compute the error variance from the mean response variance rather than the unconditional variation of the simulated asset values.

SIMULATION AND THE DEFAULTABLE BOND VALUE

This section provides a second simulation example, using the defaultable bond valuation from Chapter 4. In this case, we will actually use simulation to do a valuation, and we can check the results with the pricing from Chapter 4. Also, we will work the geometric distribution to simulate the bond's default date and therefore its value conditional on the default date.

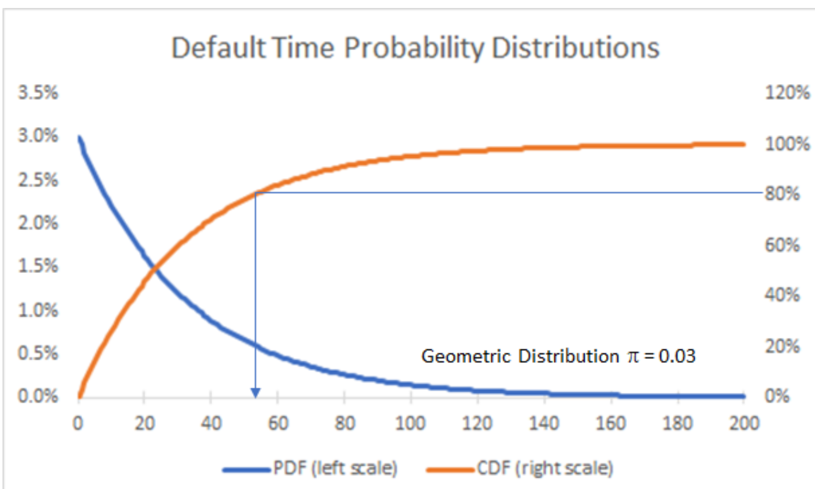
To simulate the bond's value, we take the following steps, adapted for this problem:

1. Determine the bond's value conditional on defaulting at time t for each t.
2. Simulate default dates, and assign a bond value to each simulation using step 1.
3. Calculate the average of the values as an estimate of the bond value
4. Calculate the maximum error of the simulation

Step 1. We already know the conditional bond value based on the default dates, from our earlier spreadsheet analysis:

Date	PV
1	426.71
2	458.44
3	488.95
4	518.29
5	546.50
over	1,066.78

Step 2. To simulate dates, examine the following chart. The geometrically distributed default dates can be converted to a uniformly distributed random number on [0,1] when the cumulative density is computed.



For example, the value of 0.8 maps to a default time of 53 periods. We can generate geometrically distributed random numbers from uniformly distributed random numbers. Letting $u=1-(1-\pi)^t$, we can solve for t as a function of u:

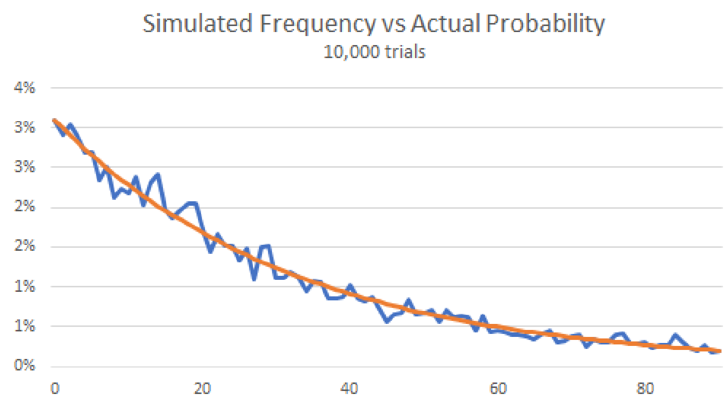
$$t = \frac{\ln(1-u)}{\ln(1-\pi)}$$

For our purposes, we want to generate discrete random times, so we have to round up. In Excel, we can use the INT function or the ROUNDUP function:

$$t = INT\left(\frac{\ln(1-u)}{\ln(1-\pi)}\right) + 1$$

$$t = ROUNDUP\left(\frac{\ln(1-u)}{\ln(1-\pi)}, 0\right)$$

Step 3. We generated 10,000 equally likely possible default dates using this formula. The empirical frequency distribution is shown alongside the theoretical distribution in the following chart:



Here's a small sample of 25 of the 10,000 sims, with the matching conditional bond values on the right:

Simulations				
95	88	10	29	15
23	4	28	14	29
6	47	19	2	145
16	109	69	39	1
5	37	12	128	45

Valuations				
1,067	1,067	1,067	1,067	1,067
1,067	518	1,067	1,067	1,067
1,067	1,067	1,067	458	1,067
1,067	1,067	1,067	1,067	427
546	1,067	1,067	1,067	1,067

The values should be 1,067 about 85% of the time. The valuations were matched using the VLOOKUP function, and the cells with defaults before 5 years were shaded using conditional formatting. Excel note: Be sure to replace the word "over" with the number 6 in the lookup table to guarantee the right VLOOKUP results.

Step 4. We already know the computed bond value is $984.73, and the standard deviation is $202.92. Make sure you know how to compute the standard deviation on your own. When we consider the results of 10,000 simulations, we will all get different but similar results. In one case, the results were:

Average	987.77
StDev	200.04
Count	10,000
StErr	2.00

The standard deviation is calculated using the STDEV function. AVERAGE and COUNT are also Excel functions.

Since this is a simulation, our estimate of \$987.77 has a natural error range. To compute the maximum likely error range, recall that the distribution of a sample average of values from any distribution approaches a normal distribution. The simulated standard deviation was quite close to the theoretical value of 202.92. The sample standard error was 2.03, i.e. $202.92/\sqrt{10{,}000}$. Conservatively then, we could go 3 standard deviations around our estimate to get a range where the true value should lie:

$$\text{Estimate range} = 987.77 \pm 3(2.03) = \{981.68, 993.86\}$$

Of course, the value 984.73 falls comfortably within that range. To increase our accuracy by any factor A, the sample size must be multiplied by A^2.

Example: Suppose a simulation of a defaultable bond value yielded an estimate of \$1040 for the bond. The standard deviation of the simulations was \$35, and 1,000 simulations performed. What is the 3 standard deviation confidence estimate of the true bond value? {1036.68, 1043.32}

MULTIVARIATE SIMULATIONS

In most problems of practical interest, we will want to simulate many different random variables simultaneously. If the variables were statistically independent, we could simply simulate them separately, but of course, there is bound to be some kind of interaction required between the variables. Most frequently, we expect random variables to have specified correlations. Since the (Pearson) correlation is the measure of the strength of a linear relationship applied to jointly normally distributed random variables, it makes sense to begin there.

Suppose we wish to simulate two standardized normal random numbers z_1 and z_2 with a given correlation ρ. One method is to begin with two independent random numbers z_1 and z_3. We can keep z_1 and then compute z_2 as a linear combination of z_1 and z_3 that meets these requirements:

- The mean of z_2 must be 0 (this is guaranteed)
- The variance of z_2 must be 1
- The correlation between z_1 and z_2 must be ρ

These requirements are uniquely met by the equation

$$z_2 = \rho z_1 + z_3\sqrt{1-\rho^2}$$

To implement this, we could do so one line at a time in the data. However, it may be more convenient to express these equations in matrix form, especially when we have more than two variables:

$$\begin{pmatrix} z_1 & z_2 \end{pmatrix} = \begin{pmatrix} z_1 & z_3 \end{pmatrix} \begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix}$$

We can think of $\begin{pmatrix} z_1 & z_3 \end{pmatrix}$ as the uncorrelated data, and $\begin{pmatrix} z_1 & z_2 \end{pmatrix}$ as the correlated result. The 2x2 matrix is known as the transpose of the (lower diagonal) Cholesky decomposition of the correlation matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. In fact, if you matrix multiply the Cholesky decomposition by its transpose, you will obtain the correlation matrix as the result.

Note that for standardized z values, the correlation matrix is the same as the covariance matrix. When nonstandard mean and variance are required, the covariance matrix may be used instead of the correlation matrix to account for different variance levels for the k variables, and the means are simply added to the result. The transposed Cholesky decomposition of the covariance matrix above is $\begin{pmatrix} \sigma_1 & \rho\sigma_2 \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}$.

When simulating more than two correlated normal random numbers, the Cholesky decomposition is invaluable. In this case, let's suppose we have k random variables with correlation matrix R, and we want to simulate N sets of random variables. We first produce an Nxk matrix of uncorrelated random z values. We then post-multiply the data by the transpose of the Cholesky decomposition of R. The resulting matrix of random numbers has the desired correlation matrix.

The appendix covers the construction of a Cholesky decomposition, moment matching to "purify" the sample data, and the Cholesky transformation to create data with the specified covariance matrix.

COPULAE AND THE CORRELATION OF NON-NORMAL RANDOM VARIABLES

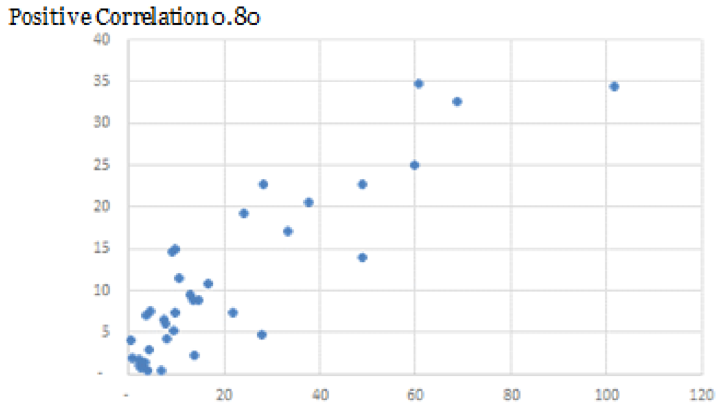
In the last section, we showed how to convert simulated normal independent random numbers into correlated random numbers. Technically, this approach works only normally distributed variables, so the question naturally arises, "how do we deal with non-normal variables?"

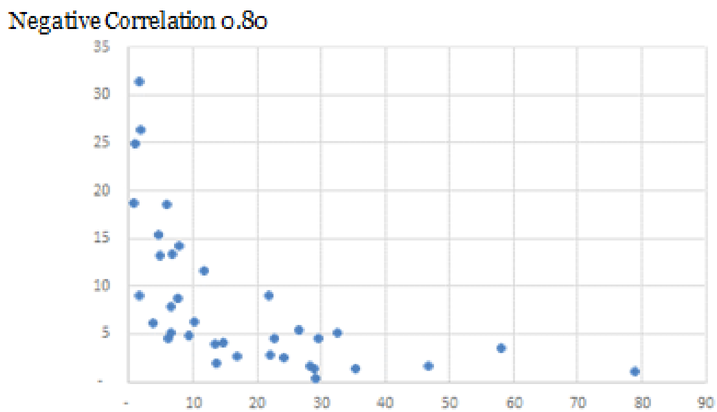
To illustrate our approach, consider the geometrically distributed default dates for the bonds discussed earlier in this chapter. Suppose we have two bonds. Bond A has a default rate of 5% per year, and Bond B has a default rate of 10% per year. We would like to introduce systemic risk to the analysis of our bond portfolio, so we will assume an 80% correlation in the default dates. What does that mean from a computational perspective?

We can generate normally distributed random numbers with a correlation of 80% easily using the Cholesky method described in the last section. To convert these to geometrically distributed default dates, we may first convert them to uniform numbers by computing the normal CDF, and then convert them to geometric distributions using the inverse CDF of the geometric distribution. This is an example of a Gaussian copula, the calculations for which are shown below.

The first two columns are random z values with a correlation of 0.80. The second two columns are the CDFs or uniform equivalents of those values. The third set of two columns are the transformed default date simulations using the inverse function of the geometric CDF.

Correlated Normal		Uniform		Simulated Defaults	
	80%			5%	10%
0.19	(0.23)	0.58	0.41	17	5
(1.29)	(2.00)	0.10	0.02	2	0
0.70	0.94	0.76	0.83	28	17
0.53	0.32	0.70	0.63	24	9
1.18	0.75	0.88	0.77	42	14
0.29	0.58	0.62	0.72	19	12
2.59	2.78	1.00	1.00	104	56
1.03	0.64	0.85	0.74	37	13
1.12	1.59	0.87	0.94	40	27
0.89	(0.19)	0.81	0.42	33	5
(1.37)	(0.73)	0.09	0.23	2	3
(1.55)	(1.10)	0.06	0.14	1	1
0.11	0.71	0.55	0.76	15	14
0.15	(0.51)	0.56	0.30	16	3
(0.65)	(0.85)	0.26	0.20	6	2
(2.37)	(2.50)	0.01	0.01	0	0
(1.02)	(0.60)	0.15	0.27	3	3
(1.74)	(0.60)	0.04	0.28	1	3
(1.01)	(0.06)	0.16	0.48	3	6
(0.86)	(0.37)	0.20	0.36	4	4
0.77	0.39	0.78	0.65	29	10
0.71	0.34	0.76	0.63	28	10
(0.71)	(1.13)	0.24	0.13	5	1
(0.44)	(0.47)	0.33	0.32	8	4
2.34	1.62	0.99	0.95	90	28
0.39	0.22	0.65	0.59	21	8
0.41	0.12	0.66	0.55	21	8
0.67	0.37	0.75	0.64	27	10
(0.56)	1.24	0.29	0.89	7	21
0.20	0.14	0.58	0.55	17	8
1.23	0.87	0.89	0.81	43	16
(0.62)	(1.23)	0.27	0.11	6	1
0.06	(0.37)	0.52	0.36	14	4
0.49	(0.10)	0.69	0.46	23	6
2.29	0.75	0.99	0.77	88	14
1.47	1.24	0.93	0.89	52	21





The graphs illustrate the impact of correlation in defaults on the timing of joint default. As expected, when defaults are positively correlated, we see many more outcomes where both defaults occur within 10 years of the starting date, as compared to the simulations with the negative correlation. The copula can be used to capture the essence of correlation while remaining true to the geometric distribution model of bond default dates.

SIMULATION AND OPTIMIZATION

In some cases, we may be required to optimize the expected value of some function of one or more random variables. If the function is complex or the number of random variables is large, it may be necessary to find the optimal value(s) of the choice variable(s) using simulation.

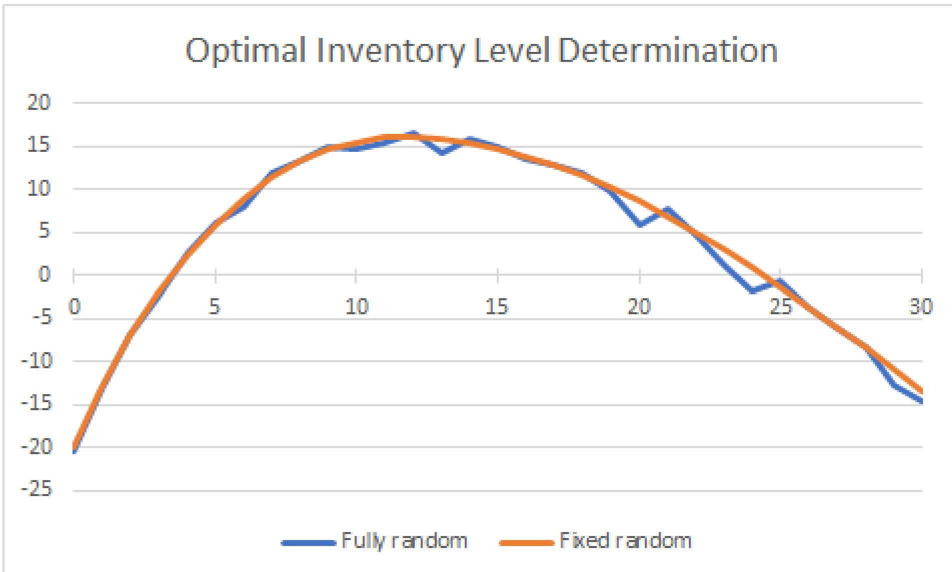
For example, consider a simplified inventory management problem. Assuming a one-period problem, we purchase or stock "s" units of a product, but we do not know the demand "d". The demand level is given by a geometric distribution with parameter 0.10, so expected demand is 10 units. The actual number of units sold is the minimum of s and d. For each unit we sell, we obtain a gross profit of $5.00. Let's suppose there is a cost of overstocking of $3.00 per unit, and a cost of understocking of $7.00 per unit. Overstocking leads to return or storage costs, while understocking leads to disappointed customers. Overstocked quantity is given by max(s-d,0) and understocked quantity is given by max(d-s,0).

Therefore net profit is given by the equation

$$Net\ profit = 5\min(s,d) - 3\max(s-d,0) - 7\max\,(d-s,0)$$

Our objective is to choose the stock level to maximize expected net profit. We could go with s = E(d) = 10, but since the cost of overstocking is lower than the cost of understocking, it would be better to err on the side of having s > 10. But how much greater? With the geometric distribution, the probability of higher demand levels is decreasing. Also we have the problem of optimizing a discrete choice, so calculus is unlikely to provide an easy answer. Therefore we turn to simulation.

The obvious approach seems to estimate expected profits for several different values of s, and then choose the one the gives the highest value. However, because of the natural errors in the simulation process, the result might appear like the choppy curve in the diagram below. While it does not cause much harm in this case, this simulation error could cause serious problems in other applications. Therefore, we need a better method.



Fortunately, another variance reduction method can be employed. If we fix the random numbers for our experiment, and then calculate expected profit for each given level of s, we obtain the smooth line is the chart, which is easily optimized at s = 12.

This is known as the method of *common random numbers*, an example of which appears in Hamrick (2016). In general, the technique of common random numbers requires that the same set of random numbers be used in comparisons when an inference will be drawn about the *change* in the simulation result. One weakness of this approach is that it uses fewer random numbers than the fully random method, so any analysis you do of this type should be checked with multiple sets of fixed random numbers, each of which will provide a specific optimal solution. For example, if we had a “bad” set of random numbers in our optimization above, we may have arrived at the incorrect optimal value, having biased all the profitability estimates in the same way.

SUMMARY

From this chapter, you should have learned the following.

1. Many pseudorandom number generation techniques exist to produce statistically independent uniformly distributed random numbers.
2. Uniformly distributed random numbers (u) can be converted to any other distribution by computing the inverse CDF of the target distribution for each value of u.

3. Simulation is used to estimate the expected value of an arbitrary function of an arbitrary number of random variables.
4. Simulation's core logic requires the computation of an average outcome, the standard error of the average outcome, and the maximum likely error based on a number of standard deviations needed to produce the desired confidence interval. $E[f(z)] = \bar{X} \pm 2s/\sqrt{N}$
5. Simulation results can be improved with variance reduction techniques. The most notable and easiest to implement are the moment matching technique and the antithetic variable technique.
6. When available, control variate techniques can be used when predictor variables X explain some of the variance of the prediction Y. In this case, the confidence intervals conditioned on X are given by a Mean Response variance.
7. Multivariate normal simulations are achieved by simulating independent normal variables and post-multiplying the dataset by the transpose of the Cholesky decomposition of the desired covariance matrix to obtain simulated data with the correct covariance matrix.
8. For non-normal variables, the copula can be used as a linkage between two random variables. Correlated normal variables lead to correlated uniform CDF variables, which can be converted to any distribution using #2 above.
9. In optimization problems involving simulation, it is normally best to use fixed (or common) random numbers as a variance reduction technique.
10. (Appendix) The Cholesky decomposition can be computed in Excel or called as a function in Python.

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EXERCISES TO BE COMPLETED IN EXCEL OR PYTHON

A. Calculate 1000 values of the random variable z, a standard normally distributed random number. Find the expected value range of z^4 within two standard deviations. Does your range include the correct value? (Hint: What is the kurtosis of the normal distribution?)

B. Repeat A, but calibrating the normally distributed numbers so that the mean is zero and the standard deviation is 1. How does your result compare to A?

C. For a given correlation matrix of three standard normally distributed random variables, estimate the expected maximum value of the set of three variables. (Hint: Requires computation of a Cholesky decomposition)

D. Design a spreadsheet to produce 100 outcomes of two dice. The dice are independent, and the probabilities of the outcomes {1, 2, … 6} are equally likely.

E. Repeat D for a pair of dice with a "correlation" of 0.50 between the two outcomes. Show the simulated frequencies of the uncorrelated dice pairs from part D with the correlated dice pairs in part E.

F. For the inventory management problem described in the optimization section, show how the optimal stock level depends on gross profit, overstock costs and understock costs.

Appendix: Computing the Cholesky decomposition of a covariance matrix

This appendix shows a worked example of how to compute a Cholesky decomposition for a 5x5 matrix, and how to create a random dataset with the properties of a given covariance matrix. In Python, use numpy.linalg.cholesky(A) or scipy.linalg.cholesky(A). The scipy function returns the upper triangular decomposition by default, and the numpy function returns the lower triangular decomposition.

Here is an algorithm you can implement in Excel, or another programming language from Higham (2008):

The Cholesky factorization can be computed by a form of Gaussian elimination that takes advantage of the symmetry and definiteness. Equating (i, j) elements in the equation $A = R^T R$ gives

$$\begin{aligned} j = i: \quad a_{ii} &= \sum_{k=1}^{i} r_{ki}^2, \\ j > i: \quad a_{ij} &= \sum_{k=1}^{i} r_{ki} r_{kj}. \end{aligned}$$

These equations can be solved to yield R a column at a time, according to the following algorithm:

for $j = 1\colon n$
    for $i = 1\colon j-1$
        $r_{ij} = (a_{ij} - \sum_{k=1}^{i-1} r_{ki} r_{kj}) / r_{ii}$
    end
    $r_{jj} = (a_{jj} - \sum_{k=1}^{j-1} r_{kj}^2)^{1/2}$
end

Here is a numerical example:

Covariance matrix

100	80	70	60	50
80	144	80	70	60
70	80	196	80	70
60	70	80	256	80
50	60	70	80	324

Step 1: First column

10	- Square root of the first element
8	- Each element of this column is divided by 10
7	
6	
5	

Step 2: Second column

10	0	0	0	0
8	8.94 - Sum of squares of this row must be 144			
7	2.68 = (80-8x7)/8.94			
6	2.46 = (70-8x6)/8.94			
5	2.24 = (60-8x5)/8.94			

Step 3: Third column

10	0	0	0	0
8	8.94	0	0	0
7	2.68	11.82 - Sum of squares of this row must be 196		
6	2.46	2.66 = (80-7x6-2.68x2.46)/11.82		
5	2.24	2.45 =(70-7x5-2.68x2.24)/11.82		

Step 4: Fourth column

10	0	0	0
8	8.94	0	0
7	2.68	11.82	0
6	2.46	2.66	14.38 - Sumsq = 256
5	2.24	2.45	2.64 =(80-6x5-2.46x2.24-2.66x2.45)/14.38

Step 5: Last row sum of squares must be 324

10	0	0	0	0
8	8.94	0	0	0
7	2.68	11.82	0	0
6	2.46	2.66	14.38	0
5	2.24	2.45	2.64	16.76

Step 6: Multiply by transpose to recover original matrix

100	80	70	60	50
80	144	80	70	60
70	80	196	80	70
60	70	80	256	80
50	60	70	80	324

Now we wish to simulate data with the desired covariance matrix. Begin by generating a 20x5 dataset of “independent” z values. Of course, the correlations will not be zero because this is a sample. Nevertheless, for illustration purposes, we’d like to know how to “purify” the data so that the correlation matrix is the identity matrix.

This is accomplished by post-multiplying the data matrix by the inverse of the Cholesky decomposition of the sample covariance matrix, essentially undoing what a normal moment matching exercise would do. The results are shown below.

Random "independent" data

	A	B	C	D	E
1	1.21	-1.72	0.85	-1.52	0.89
2	0.09	1.89	-0.22	-0.71	0.39
3	0.43	0.97	-0.42	0.41	0.55
4	-0.18	-0.51	1.84	-1.02	-0.39
5	-0.25	0.49	-0.41	-0.49	-1.23
6	-0.05	0.03	0.56	-0.08	-0.19
7	0.11	0.24	0.61	0.62	-0.35
8	0.32	-0.01	0.19	-1.93	-1.99
9	1.45	0.34	-0.55	0.44	-0.15
10	-0.20	0.13	1.52	0.34	2.13
11	1.05	1.08	-0.35	-0.32	2.17
12	-0.17	-0.06	-1.35	1.83	-0.36
13	0.51	0.37	-0.73	1.98	0.23
14	-0.51	0.82	1.34	2.98	1.49
15	1.17	1.92	-0.24	-0.22	-0.51
16	0.76	0.38	0.97	1.51	0.22
17	-0.64	0.92	0.35	-0.28	1.46
18	-0.57	0.00	-0.44	-1.35	-0.10
19	-1.69	0.39	-0.45	-1.17	1.53
20	-1.60	0.37	1.27	-1.37	0.09
Mean	0.06	0.40	0.22	-0.02	0.29
Covariance	0.68	0.00	-0.13	0.17	-0.12
	0.00	0.59	-0.20	0.21	0.09
	-0.13	-0.20	0.71	-0.08	0.17
	0.17	0.21	-0.08	1.62	0.29
	-0.12	0.09	0.17	0.29	1.09
Cholesky	0.83	0.00	0.00	0.00	0.00
	0.00	0.77	0.00	0.00	0.00
	-0.16	-0.26	0.79	0.00	0.00
	0.21	0.27	0.02	1.23	0.00
	-0.14	0.12	0.23	0.23	0.97

Moment matching
- De-mean & post-multiply by inv of Cholesky transpose

	A	B	C	D	E
1	1.39	-2.76	0.19	-0.87	1.31
2	0.04	1.93	0.07	-0.99	0.08
3	0.45	0.73	-0.48	0.12	0.32
4	-0.29	-1.18	1.61	-0.54	-0.85
5	-0.37	0.11	-0.83	-0.33	-1.36
6	-0.14	-0.49	0.25	0.08	-0.53
7	0.06	-0.21	0.45	0.54	-0.86
8	0.31	-0.53	-0.14	-1.49	-1.85
9	1.68	-0.09	-0.66	0.12	-0.08
10	-0.32	-0.35	1.47	0.40	1.45
11	1.19	0.87	-0.19	-0.64	2.18
12	-0.28	-0.60	-2.23	1.72	-0.52
13	0.55	-0.05	-1.10	1.57	-0.09
14	-0.69	0.55	1.46	2.42	0.15
15	1.34	1.97	0.33	-0.82	-0.76
16	0.85	-0.04	1.11	1.09	-0.47
17	-0.85	0.67	0.22	-0.21	0.99
18	-0.76	-0.51	-1.16	-0.82	0.02
19	-2.12	-0.01	-1.28	-0.55	1.40
20	-2.01	-0.03	0.92	-0.77	-0.53
Mean	0.00	0.00	0.00	0.00	0.00
Covariance	1.00	0.00	0.00	0.00	0.00
	0.00	1.00	0.00	0.00	0.00
	0.00	0.00	1.00	0.00	0.00
	0.00	0.00	0.00	1.00	0.00
	0.00	0.00	0.00	0.00	1.00

In this exercise, we computed the sample covariance matrix for the simulated “independent” data. After de-meaning the data, we multiplied by the inverse of the transpose of the Cholesky decomposition to obtain the dataset on the right. These data have zero mean and a correlation matrix equal to the identity matrix.

Now, to create a dataset with the prescribed covariance matrix, we take the purified data and post-multiply by the transpose of the Cholesky decomposition, as follows:

Moment matching to target correlation

- De-mean & post-multiply by inv of Cholesky transpose

A	B	C	D	E
1.39	-2.76	0.19	-0.87	1.31
0.04	1.93	0.07	-0.99	0.08
0.45	0.73	-0.48	0.12	0.32
-0.29	-1.18	1.61	-0.54	-0.85
-0.37	0.11	-0.83	-0.33	-1.36
-0.14	-0.49	0.25	0.08	-0.53
0.06	-0.21	0.45	0.54	-0.86
0.31	-0.53	-0.14	-1.49	-1.85
1.68	-0.09	-0.66	0.12	-0.08
-0.32	-0.35	1.47	0.40	1.45
1.19	0.87	-0.19	-0.64	2.18
-0.28	-0.60	-2.23	1.72	-0.52
0.55	-0.05	-1.10	1.57	-0.09
-0.69	0.55	1.46	2.42	0.15
1.34	1.97	0.33	-0.82	-0.76
0.85	-0.04	1.11	1.09	-0.47
-0.85	0.67	0.22	-0.21	0.99
-0.76	-0.51	-1.16	-0.82	0.02
-2.12	-0.01	-1.28	-0.55	1.40
-2.01	-0.03	0.92	-0.77	-0.53
0.00	0.00	0.00	0.00	0.00
1.00	0.00	0.00	0.00	0.00
0.00	1.00	0.00	0.00	0.00
0.00	0.00	1.00	0.00	0.00
0.00	0.00	0.00	1.00	0.00
0.00	0.00	0.00	0.00	1.00

Moment matching to original target covariance

A	B	C	D	E
13.92	-13.58	4.60	-10.43	20.98
0.37	17.55	6.31	-9.06	3.36
4.45	10.13	-0.59	4.92	8.33
-2.92	-12.88	13.84	-8.17	-15.81
-3.75	-1.99	-12.14	-8.93	-27.35
-1.43	-5.51	0.62	-0.29	-9.95
0.62	-1.36	5.17	8.84	-12.01
3.10	-2.28	-0.94	-21.31	-34.88
16.76	12.64	3.75	9.78	5.52
-3.24	-5.76	14.16	6.77	26.50
11.90	17.31	8.40	-0.38	42.36
-2.79	-7.55	-29.93	15.72	-12.34
5.46	3.95	-9.35	22.76	2.56
-6.94	-0.62	13.87	35.81	10.20
13.41	28.31	18.53	1.92	-2.96
8.45	6.43	18.95	23.59	1.87
-8.52	-0.80	-1.61	-5.96	13.76
-7.63	-10.68	-20.41	-20.78	-9.65
-21.18	-16.99	-29.98	-24.03	8.20
-20.06	-16.30	-3.27	-20.77	-18.69
0.00	0.00	0.00	0.00	0.00
100.00	80.00	70.00	60.00	50.00
80.00	144.00	80.00	70.00	60.00
70.00	80.00	196.00	80.00	70.00
60.00	70.00	80.00	256.00	80.00
50.00	60.00	70.00	80.00	324.00

Cholesky transpose of original target cov

10.00	8.00	7.00	6.00	5.00
0.00	8.94	2.68	2.46	2.24
0.00	0.00	11.82	2.66	2.45
0.00	0.00	0.00	14.38	2.64
0.00	0.00	0.00	0.00	16.76

Chapter 6: Stochastic Processes in Finance

INTRODUCTION

In Chapter 5, we discussed the simulation of one or more random variables at a point in time. In this chapter, we will discover how simulation can be applied to generate future financial asset prices, interest rates, commodity prices and cash flows. We will also learn how to estimate the parameters of these processes using historical data, and how to calibrate these processes to assumed future probability distributions. By the way, "stochastic" is a synonym of "random."

Many financial valuations derive from assumed stochastic processes for cash flows, so the ultimate objective is to have all the needed tools at our disposal to generate cash flows that can be valued using one or more of the techniques we will learn in Part II of this text.

THE RANDOM WALK

The random walk is one of the simplest stochastic processes, yet it forms the basis for many of the more complex ones. In fact, the French mathematician Louis Bachelier used this process in 1900 to describe stock price movements on the Paris Bourse, or stock exchange. Now clearly, there are many factors causing stock prices to change, but the most important factor comprises the unknown desires of investors looking to buy or sell in the future. Traders attempt to forecast these demands and take advantage of their information to buy or sell now and thereby profit. If they collectively do a good job, it should be impossible for us to predict the short-term future price movement, and we should not expect to earn more or less than the normal compensation for owning stocks in the long term.

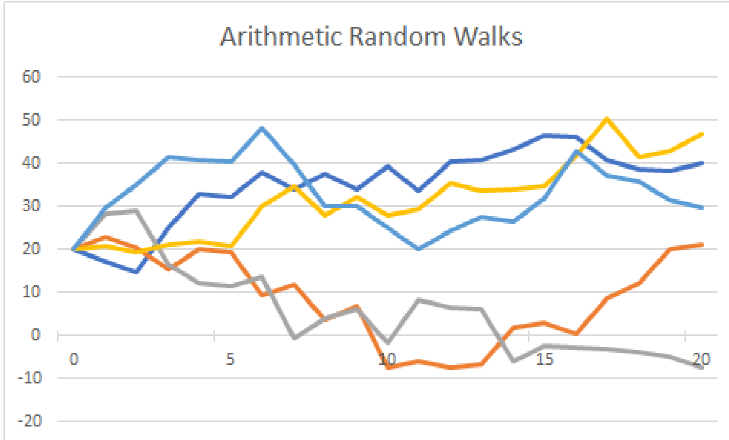
In other words, even though we don't believe the price changes are random, we may believe that based on the information available to us, there is no better model to explain their behavior. This observation has come to be known as the Efficient Market Hypothesis --- the idea that we cannot forecast prices better than the market if our available information is the same or less. However, price forecasting is not our only goal. We also want to capture potential future volatility and correlation between prices.

Mathematically, we could write down this relationship as a word-equation and then formalize it. For simplicity and convenience, as before, we will use the standardized normal distribution to generate the random variables "z". Also, time t represents any specified time, but can be thought of as the present, implying that the price at time t+1 is unknown to us.

Future change in stock price = Expected compensation for stock price risk + Random shock

$$S_{t+1} - S_t = \mu + \sigma z_{t+1}$$

This is a stochastic process known as an Arithmétic Random Walk. The accent indicates the stressed syllable. If the time step is dt instead of 1, this is also called Arithmétic Brownian Motion or ABM. The non-stochastic version of this model appeared in Chapter 2: a linearly growing perpetuity. The term μ is called the drift of the process, and σ is called the volatility. The stochastic (random) term is also sometimes referred to as the *innovation* in the process. The random variable z is indexed based on the time it is known, so z_{t+1} is unknown until time t+1. The drift of the process reflects the expected growth in S over time, and the volatility relates to how much we may deviate over time from that straight-line growth in a single period. Here is a sample of five Arithmetic Random Walk outcomes, with a starting value of 20, a drift of 0, and volatility of 5.



The graph shows a simulation of 20 periods, all starting from the same point. As you can see, the average growth rate appears to be about zero, and the standard deviation of the outcomes seems to increase over time. Can we quantify these?

Using repeated applications of the one-step stochastic process, $S_{t+1} - S_t = \mu + \sigma z_{t+1}$, we can extrapolate to a j period forecast made at time t. In particular,

$$S_{t+j} = S_t + j\mu + \sigma \sum_{i=1}^{j} z_{t+i}$$

From this equation we can determine the properties of S_{t+j}. Except for the constants, S_{t+j} is the sum of k normally distributed independent values of z. The sum of normally distributed random variables is also normal. We can compute the mean and variance conditional on S_t easily due to the independence of the z values:

Moments of arithmetic random walk

$$E[S_{t+j}|S_t] = S_t + j\mu$$

$$Var[S_{t+j}|S_t] = j\sigma^2$$

The mean of the process is expected to grow linearly with time. However, the standard deviation of the process outcomes increases indefinitely with the square root of time --- the stochastic process values become more and more uncertain as we forecast longer into the future.

The covariance terms are also straightforward in this context, since each value of z correlates perfectly with itself, and does not correlate to any other value of z. Therefore,

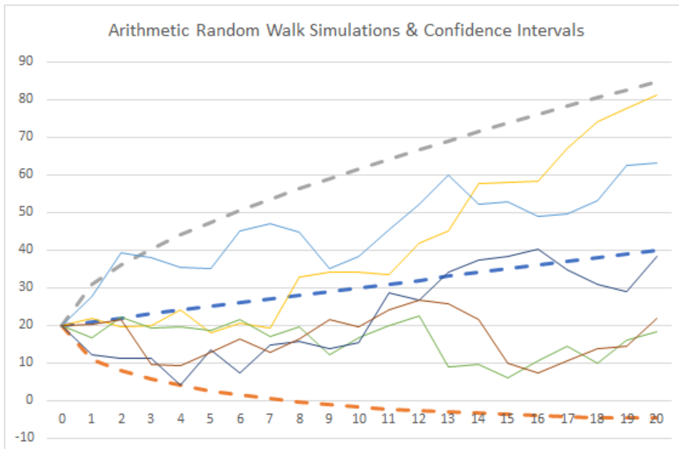
$$Cov[S_{t+j}, S_{t+m}|S_t] = \sigma^2 \min(j, m)$$

In other words, we can think of S_t as a path, or if it is more convenient, we may also think of S_t as a vector of jointly normally distributed variables. That is, we could generate random outcomes for the vector by taking a dataset of independent standard normally distributed numbers and multiplying by the transpose of the Cholesky decomposition of this covariance matrix. This is particularly convenient in this case because the Cholesky decomposition of the matrix defined by C_{jm} = min(j,m) is simply a lower diagonal matrix of ones.

Exercise

Using S_0=20, μ=1, σ=5 and N=20, simulate and graph five Arithmetic Random Walk outcomes, and provide the mean and two standard deviation confidence intervals around the simulated processes in Excel.

Solution



FINANCIAL APPLICATIONS OF THE ARITHMETIC RANDOM WALK

Although Bachelier used this model for stock price fluctuations in the short term, a quick inspection of the simulations shown in the figure above reveals that the process may become negative over longer time periods. Clearly, for stocks with limited liability, prices can never become negative, so the choice of this process to model asset prices may be questionable.

However, there are financial variables that can become negative, for which the arithmetic random walk can be used as a stochastic process model. For example, the cash flows of a company may be modeled as an Arithmetic Random Walk, since cash flow can become negative. If cash flows become negative enough, then the company may decide to go bankrupt or go out of business, but nevertheless, the cash flow model could be reasonably modeled as an Arithmetic Random Walk.

Another commonly cited possible application for the ABM is a basis, or the difference between two prices. For example, suppose S represents the difference in oil prices between West Texas (WTI) and in the North Sea (Brent). This difference can become negative, but in the long term, we would not expect the standard deviation of a forecast to increase indefinitely --- there should be some fundamental relationship between the two that draws prices back together. Hence we would like to be able to model some variables in a way that recognizes their long-term joint relationship. For this purpose, we turn to the mean-reverting process, also known as the Ornstein-Uhlenbeck process.

MEAN REVERSION AND THE ORNSTEIN-UHLENBECK PROCESS

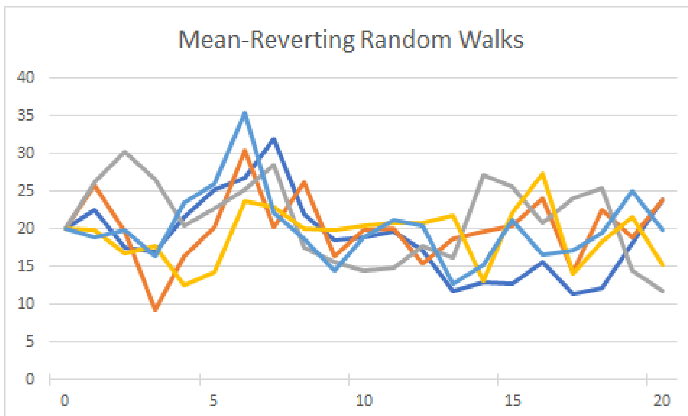
We have already seen the non-stochastic version of the OU process in Chapter 2: the mean-reverting perpetuity. Restating the deterministic process, $S_{t+1} = S_t + \kappa(\mu - S_t)$. The term μ represents the long-run mean of the process, and $0 \leq \kappa \leq 1$ represents the speed of mean reversion. When κ=0, there is no mean-reversion, and when κ=1, there is 100% mean reversion in one period. When $\mu > S_t$, the process drifts positively, and when $\mu < S_t$ the process drifts negatively --- in both cases, in the direction of the long-run mean.

The stochastic version of mean-reversion is credited to Ornstein-Uhlenbeck, adding an innovation identical to the Arithmetic Random Walk:

$$S_{t+1} = S_t + \kappa(\mu - S_t) + \sigma z_{t+1}$$

Because the innovation is normally distributed, there is a possibility of a negative value of S when μ>0. However, mean-reversion will tend to pull the process back from negative territory so it won't stay negative indefinitely. Some modelers will avoid this "problem" by replacing σ with σS_t^{γ} with γ>0 which reduces the likelihood of a negative outcome by forcing the volatility lower as the value of S falls.[10] Of course, this may not be a problem at all for some variables, such as interest rates, that can become slightly negative for short periods of time.

The figure below shows some sample mean-reverting processes, with μ=20, κ=0.50 and σ=5.



Unlike the Arithmetic Random Walk, it appears that the variance of the outcomes does not increase in a linear fashion, but rather, seems to reach a maximum level. This can be verified by constructing the k-step forecast of S in an OU mean-reverting process:

$$S_{t+j} = \mu + (S_t - \mu)(1-\kappa)^j + \sigma \sum_{i=0}^{j-1} (1-\kappa)^i \, z_{t+j-i}$$

Note that the sum runs backwards, giving greater weight to later innovations. Earlier innovations are dampened by the mean reversion; the higher the mean reversion, the more dampened the historical innovations. As before, we can use this expression, involving a sum of normally distributed random variables, to determine the parameters of the normally distributed S_{t+j}.

Moments of the mean-reverting OU process

$$E[S_{t+j}|S_t] = \mu + (S_t - \mu)(1-\kappa)^j$$

$$Var[S_{t+j}|S_t] = \sigma^2 \left(\frac{1-(1-\kappa)^{2j}}{\kappa(2-\kappa)} \right)$$

[10] Note that in continuous time models, this prevents S from becoming negative, but in discrete models, there is still a possibility.

$$Cov[S_{t+j}, S_{t+m}|S_t] = \sigma^2 \left(\frac{1-(1-\kappa)^{2j}}{\kappa(2-\kappa)}\right)(1-\kappa)^{m-j}\ for\ m \geq j$$

The expected value follows the expected trajectory of exponential decay to the long-run mean. The variance increases at a decreasing rate, and reaches a maximum value at $\sigma^2/(\kappa(2\text{-}\kappa))$. As $\kappa \rightarrow 0$, the covariance approaches σ^2 *min(j,m)*, the value for the Arithmetic Random Walk.

The Cholesky decomposition of the covariance matrix of the OU process (with σ=1) has ones on the diagonal, and entries below the diagonal shrinking at a constant rate of κ. For example, if we have a 5-period forecast of a mean-reverting process with σ=1 and κ=0.20, we obtain the following covariance matrix and Cholesky decomposition:

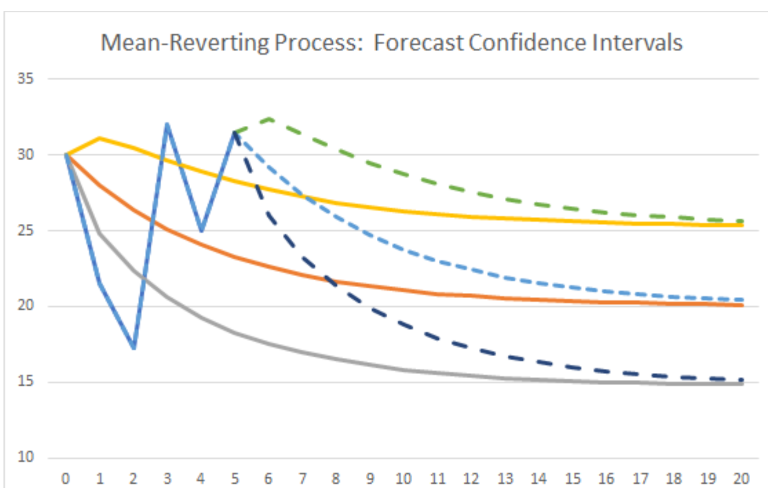
Covariance matrix (σ=1, κ=20%)

1	0.8	0.64	0.512	0.4096
0.8	1.64	1.312	1.0496	0.83968
0.64	1.312	2.0496	1.63968	1.311744
0.512	1.0496	1.63968	2.311744	1.849395
0.4096	0.83968	1.311744	1.849395	2.479516
			sum	30.20826

Cholesky decomposition (σ=1, κ=20%)

1	0	0	0	0
0.8	1	0	0	0
0.64	0.8	1	0	0
0.512	0.64	0.8	1	0
0.4096	0.512	0.64	0.8	1

Recall the distinguishing characteristic of the mean-reverting process is that the long-run variance approaches a maximum level of $\sigma^2/(\kappa(2\text{-}\kappa))$, regardless of the trajectory of the value of S. Imagine graphically that we can put a 2-standard deviation bound around future values of the mean-reverting process, and examine our future forecasts from the standpoint of the present. In the following figure, we have S_0=30, σ=5, κ=0.2 and μ=20. The jagged line represents one simulation of the process for the next 5 periods, exaggerated for graphical effect. Notice that wherever the jagged line ends up, the long-term mean and confidence interval for future values of S are unaffected.



This makes the mean-reverting process a natural choice for modelling the WTI-Brent oil basis, or interest rates. However, mean reversion would not be suitable for modeling the price of a share of stock or a stock index. Why? Mean-reversion implies predictability in the mean. If everyone believed share prices would fall in the future, they would end up falling now, as people sell their shares. The same could be said if prices were expected to rise by more than their equilibrium expected rates of return. For this reason, we need a different process, which is discussed in the next section.

Exercise

Simulate the annual price of corn for the next five years, using a current price of \$5, a long-run mean of \$4, annual mean reversion of 20% and volatility of \$0.75 per year. Find the average of the next five years' prices within a 2 standard deviation range of \$0.10. Can you find the exact values without simulation using the information only in this chapter and a hand calculator?

Solution

The simulation itself is left to the reader, using the methods presented in Chapter 5. The analytic mean of the five years is \$4.54. The variance of the sum of five years' prices is the sum of all the covariance elements above (conveniently having the same κ=20% ☺), or 30.21, multiplied by the variance of 0.75^2, or 16.99. The variance of the average is therefore $16.99/5^2$, and its square root, the standard deviation of the 5-year average, is 0.82.

THE GEOMETRIC RANDOM WALK

Harry Markowitz of Modern Portfolio Theory fame assumed the expected percentage return and standard deviation of percentage returns were constant, and used this as a basis for determining efficient and optimal portfolios. We will modify this specification to assume that *continuously compounded* returns are normally distributed with mean α and standard deviation σ per period. This assumption allows us to work more easily with the natural logarithms of the prices, and also to ensure that security prices will not become negative.

The process we are developing is called the Geometric Random Walk. This is also known as *Geometric Brownian Motion (GBM)* when the time step is infinitesimal. The specific equation used to generate the stochastic process in a single step from t to t+j is the following:

$$S_{t+j} = S_t \, exp\left[\left(\alpha - \frac{\sigma^2}{2}\right)j + z\sigma\sqrt{j}\right] \qquad \text{"}the\ GBM\ generator\text{"}$$

The equation exhibits exponential growth, but there is a puzzling adjustment to α. Why are we using a lower drift? The reason is a convexity adjustment due to the nonlinearity of the exponential function. In particular, the expected value of $\exp\,(z\sigma\sqrt{j})$ is $\exp\,(\frac{1}{2}\sigma^2 j)$ when z is a standard normally distributed random variable. The drift adjustment offsets the convexity adjustment so that the expected value relationship below is satisfied:

$$E\left[S_{t+j}\middle|S_t\right] = S_t e^{\alpha j}$$

… exactly as we wanted, with constant expected exponential growth of α.

Therefore we can express the moments of *ln*(S_{t+j}) to be consistent with this outcome:

Log-moments of the Geometric Random Walk

$$E\left[ln\left(S_{t+j}\right)\middle|S_t\right] = ln(S_t) + \left(\alpha - \frac{\sigma^2}{2}\right)j$$

$$Var\left[ln\left(S_{t+j}\right)\middle|S_t\right] = \sigma^2 j$$

$$Cov\left[ln\left(S_{t+j}\right), ln(S_{t+m})\middle|S_t\right] = \sigma^2\, min(j,m)$$

A simulated price process can therefore be computed from the GBM generator, or from these three equations --- generating the Arithmetic Random Walk first, and then computing the exponential to get the Geometric values.

The GBM process is the most popular choice for modeling security values, and in fact forms the basis for the Black-Scholes-Merton option pricing model (1973) and for most risk models used in equity markets. However, it does have its limitations. Security returns tend to be negatively skewed and leptokurtic, meaning that sudden drops in price are more likely than sudden rises, and the probability of extreme returns is higher than would be implied by a normal distribution; the distribution has *fat tails*. More advanced security models address these limitations of the simple version of GBM when applied to security returns.

Exercise

Assume the market index is currently at 1300. If the expected annual growth rate is 10%, and the annual volatility is 18%, using monthly simulated index values. What is the probability that the index will rise over 2000 any time in the next 24 months?

Solution

Proceed using simulation, taking care to make the time step 1/12 of a year, and making each successive value of the index depend on the previous one. Tabulate the number of simulations where the maximum index level exceeds 2000. The correct answer is about 20%.

STOCHASTIC PROCESS SELECTION AND PARAMETERIZATION

There are obviously many different types of stochastic processes, and we have only covered three of the most basic models. Nevertheless, it is worthwhile to summarize the models we have discussed, explain how to parameterize them, and examine some of the most frequent adjustments to these processes.

First, of these three models, the best model for security prices is the GBM model. Even if it assumes constant expected returns and volatility, it ensures that prices can never become negative. Also, a constant proportional dividend rate can be subtracted from the drift to obtain ex-dividend simulations. However, it does not allow security values to reach zero in the event of bankruptcy or termination, and it does not allow negatively skewed or leptokurtic returns in the simple version.

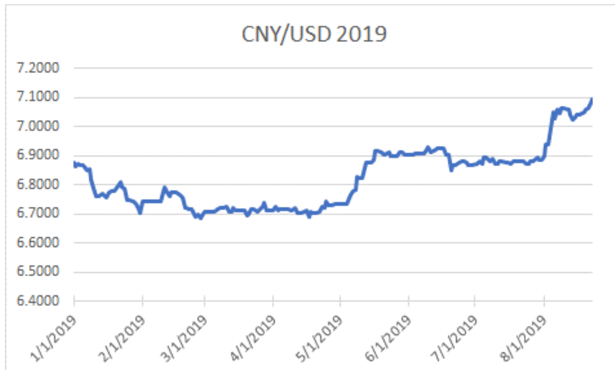
Second, the best of these three models for commodity prices and interest rates is the mean-reverting process. Importantly, commodities are not assets in the financial sense. High commodity prices stimulate production and reduce demand, causing everyone to expect lower prices in the future. If you tried to go short commodities in expectation of this, you would be disappointed to find no market for short sales of commodities! Low commodity prices have the opposite effect, causing price expectations to rise for the future. Interest rates resemble commodities in this respect since high rates encourage lending and discourage borrowing, causing future rates to fall. Finally, in many financial models, volatility itself is assumed to change over time, and an MR process can reasonably describe these movements over time.

Finally, most financial variables may be modeled short-term as ABM processes, but this generally fails long-term due to the possibility of negative values and nondecreasing forecasted volatility. The ABM process is most important as a foundation for mean-reverting and GBM processes.

Suppose we have a doubt which process to use? For some markets like foreign exchange, it is not clear if mean reversion exists. But if it does exist, we'd like to be able to estimate the parameters of the process. How can we test if mean-reversion exists, and how do we parameterize it?

Parameterizing CNY drift and volatility

The following chart shows the Chinese Yuan (CNY) exchange rate in units per dollar for about eight months in 2019.



Even though the currency may be considered partially targeted or controlled, market forces do play a role in the determination of the exchange rate. Suppose we had an income stream expected in yuan over the next few years. We might want to assess the expected appreciation or depreciation of the CNY over that time, and the volatility of the exchange rate. Not being able to see the future, we turn to the past: available data in 2019 on a daily basis, from 1/1/19 to 8/23/19.

Here is a sample from the dataset:

Annualized mean (Average x 252)		3.92%
Annualized StdDev (Stdev x Root 252)		3.72%
Adjusted mean (adding 0.5 x variance)		3.99%
	CNY	Δln(CNY)
1/1/2019	6.8785	
1/2/2019	6.8621	-0.0024
1/3/2019	6.8720	0.0014
1/4/2019	6.8694	-0.0004
1/5/2019	6.8694	0.0000
1/7/2019	6.8510	-0.0027
1/8/2019	6.8532	0.0003
1/9/2019	6.8180	-0.0051
1/10/2019	6.7885	-0.0043
1/11/2019	6.7627	-0.0038
1/12/2019	6.7627	0.0000
1/14/2019	6.7681	0.0008
1/15/2019	6.7610	-0.0010

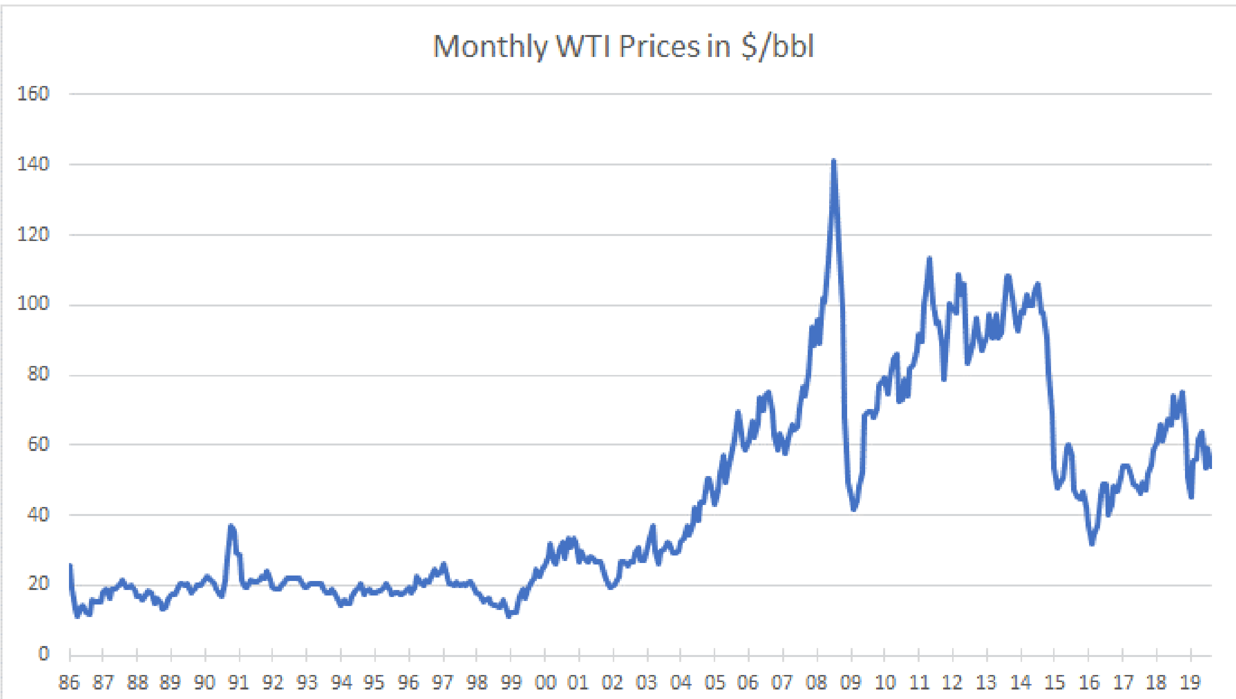
Assuming this followed a GBM process in the past, and that the first 8 months of 2019 could accurately reflect the future, both heroic assumptions indeed, we would determine the change in log prices historically. Using the convention that there are 252 trading days a year, the average log change would be multiplied by 252, and the standard deviation of the log change would be multiplied by $\sqrt{252}$.

The results are shown in the table. Based on this short history, the yuan is depreciating relative to the dollar at a continuous rate of 3.99% with a volatility of 3.72% per year. If we would like to simulate the future using these historical parameters, we may do so as long as we are clear about our assumptions.

The historical price series does not seem to show any evidence of mean-reversion, and the statistics bear that out. How can we check for mean reversion and determine the parameters?

Parameterizing a mean-reverting process

Let's consider oil prices. Using Monthly WTI prices available on MacroTrends, we obtained these data:



The x-axis denotes the beginning of each year from Jan 1986 to Aug 2019. The prices are quoted in dollars per barrel, and are not inflation-adjusted. The prices appear at times to follow a GBM process, and at other times an MR process.

To test for the presence of mean reversion, we calculate price change as the y variable, and the previous price as the x variable, and perform a simple regression. Time series statisticians may shudder at the simplicity of the test, and you should probably test for a unit root, but that is beyond the scope of this course. At least, doing a simple regression will give us an indicator. The equation is

$$(S_{t+1} - S_t) = \kappa\mu - \kappa S_t$$

So the constant is $\kappa\mu$, and the slope is $-\kappa$. The Excel regression output is copied below:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.085955
R Square	0.007388
Adjusted R Square	0.004913
Standard Error	4.931793
Observations	403

ANOVA

	df	SS	MS	F	Signif F
Regression	1	72.59619	72.59619	2.984724	0.084824
Residual	401	9753.355	24.32258		
Total	402	9825.951			

	Coeff	StdErr	t Stat	P-value
Intercept	0.7065	0.442604	1.596234	0.111224
Slope	-0.01441	0.00834	-1.72764	0.084824

The t-statistic doesn't appear to be overwhelmingly in favor of rejecting the null hypothesis, but could be considered marginal. Most interestingly, we have estimated a mean reversion speed of 1.44% per month (the slope coefficient) and a long-run mean oil price of $49 (0.7065/0.01441). Perhaps more testing is needed; mean reversion may have existed over some periods in the data and not others. The point is, if we were looking to simulate oil prices in the future, it would not be a terrible idea to use a mean-reverting process.

If we examine only the last 5 years of monthly data, a slightly different story emerges:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.404129
R Square	0.16332
Adjusted R Square	0.148895
Standard Error	5.335624
Observations	60

ANOVA

	df	SS	MS	F	Signif F
Regression	1	322.3145	322.3145	11.32164	0.001363
Residual	58	1651.195	28.46889		
Total	59	1973.51			

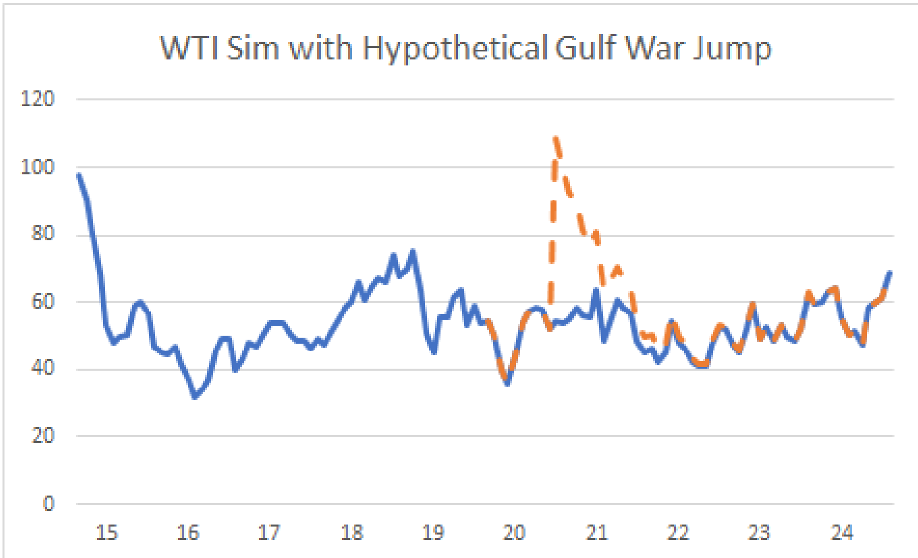
	Coeff	StdErr	t Stat	P-value
Intercept	8.810743	2.918485	3.018944	0.003766
Slope	-0.17098	0.050814	-3.36476	0.001363

In this period, we are seeing 17% per month mean reversion, a high degree of statistical significance, and a long-run mean value of $51 per barrel.

JUMP PROCESSES

Sometimes, prices move quickly and far. People may say "we had a 10 standard deviation move" in price on one day, but that is fairly unlikely. Most likely, they are using GBM or a MR process to compute volatility, and failed to account for the occasional jumps that occur in real-world prices outside of these model parameters. To address this concern in financial modeling, we can overlay a jump on top of any price process. The jump may represent the unlikely event that a company becomes a takeover target suddenly, or that electricity prices in the Midwest spike due to a power plant or transmission failure.

As an example, let's consider an oil shock, such as a war in the gulf. If we assume there is a 1% chance in any given month of a sudden 100% rise in the price of oil, then the future price of oil might look like this, using the 60 month parameters from the last example.



In a mean-reverting process, the impact of jumps dissipates over time, as shown in the figure. In the case of ABM or GBM processes, however, the shift is permanent.

CONCLUSIONS

In sum, there are a large number of complex and sophisticated stochastic processes used to model financial indicators such as prices and rates. This chapter explained the most basic processes understanding that in practice, you will invest significant time determining what process to use in order to model the variables that are most important to you at the time. For example, suppose you were simulating future stock prices. Your model might include any or all of the following, or even more:

- Stochastic drift or volatility (changes in expected return and risk)
- Event risks
- Changes in bid-offer spreads
- Changes in economic variables affecting the company's performance
- Changes in investor sentiments in favor or against the company's industry

… and so on

With the tools you have learned in this chapter, you will be able to design and simulate any stochastic process needed, generate cash flows from a contract, security, or activity, and value those cash flows.

With the tools we have covered in Part I of this text (through Chapter 6), you can find expected cash flows for almost any practical situation you may encounter and discount them to find the present value when discount rates are known. Chapter 7 begins Part II, which explores the determination of risk charges and discount rates used to value cash flows --- since they are never really known to you in practice.

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Markowitz

Ornstein, Uhlenbeck

EXERCISES TO BE DONE IN EXCEL

1. Use the same series of 20 independent normally distributed numbers z to generate all three of the following (annual) prices. Take initial values to be 100; volatility is 18% for GBM, and 18% of the price level in the other cases. Show your results in a graph.
   a. A GBM process with drift 10%.
   b. An ABM process with growth rate equal to a constant 10.
   c. A MR process with speed of MR of 0.35 and a long run mean of 200.

2. For the mean-reverting process in #1, what is the mean and standard deviation of the price level in periods 10 and 20? What is their covariance?

3. How would you adjust from annual means and standard deviations to monthly means and standard deviations for all three of these processes? (*Hint:* Go from monthly to annual first.)

4. How does your answer to #1 change if the processes are correlated? Implement this in your spreadsheet.