FIXED INCOME SECURITIES

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Continuous Time MM

- $B(T, s_{T-1}) = B(0) \prod_{i=0}^{T-1} R(i, s_i)$
- $R(i, s_i) = (1 + r_i(s_i)\Delta t)\ where \quad r_i(s_i)$ is annualized Rate
- Now divide time $(0, T)$ into "n" equal interval and $\Delta t = \frac{T}{n}$
- $B(T, s_{T-1}) = B(0) \prod_{i=0}^{n-1} \left[1 + r_i(s_i)\frac{T}{n}\right]$
- For Continuous Compounding let $\Delta t \to 0$ or $n \to \infty$
- $B(T, s_{T-1}) = B(0)e^{\int_0^T r_s ds} = e^{\int_0^T r_s ds}$
- $\frac{1}{B(T, s_{T-1})} = \frac{1}{e^{\int_0^T r_s ds}} = e^{\int_0^T -r_s ds}$
- $P(0, T) = E_0^*\left(\frac{P(T,T)=1}{B(T, S_{T-1})}\right) = E_0^*\left[e^{-\int_0^T r_s ds}\right]$

Continuous Forward Rate

- $G(t,T_1,T_2) = \frac{P(t,T_1)}{P(t,T_2)} = [1 + f(t,T_1,T_2)(\Delta T_{12})]$, $\Delta T_{12} = T_2 - T_1$
- If Continuous Compounding then :
- $f(t,T_1,T_2) = \frac{1}{\Delta T_{12}} [\ln P(t,T_1) - \ln P(t,T_2)]$
- $f(t,T\ ) = \lim_{\Delta T_{12} \to 0} \frac{1}{\Delta T_{12}} [\ln P(t,T_1) - \ln P(t,T_2)] = -\frac{\partial \ln P(t,T)}{\partial T}$
- $p(t,T) = e^{-\int_0^T f(t,s)ds}$
- $p(t,T) = e^{-\int_0^T f(t,s)ds} = E_0^*\left[e^{-\int_0^T r_s ds}\right]$
- Where $f(t,t\ ) = r_t$

Local Expectation Hypothesis

- $P(0,T) = E_0^*\left(\frac{P(T,T)=1}{B(T,S_{T-1})}\right) = E_0^*\left[\exp\left(-\int_0^T r_t dt\right)\right]$
- $E(0,T) = E_0\left(\frac{P(T,T)=1}{B(T,S_{T-1})}\right) = E_0\left[\exp\left(-\int_0^T r_t dt\right)\right]$
- Here E(0,T) is the time 0 expected value of $1 at T
- Here P(0,T) is the time 0 value of $1 at T
- Note for P(0,T) we use the pseudo-probability or the risk adjusted probability where as for E(0,T) we use the real probability

Local Expectation Hypothesis

- From the above equations we get YTM using price P(0,T):
- $Y(0,T) = P(0,T)^{\frac{-1}{T}} = E_0^*\left(\frac{1}{B(T,S_{T-1})}\right)^{\frac{-1}{T}}$
- $Y(0,T) = -\frac{\ln(P(0,T))}{T} = -\frac{\ln(E_0^*\left[exp(-\int_0^T r_t dt)\right]}{T}$
- Consider find eYTM using E(0,T) instead of P(0,T)
- $eY(0,T) = E(0,T)^{\frac{-1}{T}} = E_0\left(\frac{1}{B(T,S_{T-1})}\right)^{\frac{-1}{T}}$
- $eY(0,T) = -\frac{\ln(E(0,T))}{T} = -\frac{\ln(E_0\left[exp(-\int_0^T r_t dt)\right]}{T}$

Local Expectation Hypothesis

- $P(0,T)=E_0^*\left(\frac{P(T,T)}{B(T,S_{T-1})}\right) = E_0^*\left[\exp(-\int_0^T r_t dt)\right]$
- $E(0,T)= E_0\left(\frac{P(T,T)}{B(T,S_{T-1})}\right) = E_0\left[\exp(-\int_0^T r_t dt)\right]$
- $Y(0,T)= -\frac{\ln\left(P(0,T)\right)}{T} = -\frac{\ln(E_0^*\left[exp(-\int_0^T r_t dt)\right]}{T}=\frac{1}{T}\int_0^T f(t,s)\, ds$
- $eY(0,T)= -\frac{\ln\left(E(0,T)\right)}{T} = -\frac{\ln(E_0\left[exp(-\int_0^T r_t dt)\right]}{T}$

Local Expectation Hypothesis

- LEH : $y(0,T) = ey(0,T)$

- That is the expectation under the pseudo-probability and actual probability is the same .
- Actual probability is the same as risk-adjusted probabilities. (risk neutrality)
- We can show under LEH :

$$E_t\left(\frac{P(t+1,T,s_{t+1})}{P(t,T,s_t)}\right) = E_t^*\left(\frac{P(t+1,T,s_{t+1})}{P(t,T,s_t)}\right) = r(t,s_t)$$

- The expected actual return is the same as expected risk adjusted return which is risk free rate.

Expectation Hypothesis

- Expectation Hypothesis:

$$EH: \quad y(0,T) = \frac{E((R_0-1)+\cdots+(R(T-1,s_{T-2})-1)}{T} = \frac{1}{T}E\left(\int_0^T r_t\,dt\right)$$

$$Y(0,T) = -\frac{\ln(P(0,T))}{T} = \frac{1}{T}\int_0^T f(t,s)\,ds = \frac{1}{T}E\left(\int_0^T r_s\,ds\right)$$

$$\int_0^T E[f(t,s)-r_s]\,ds = 0$$

- Note the difference between the two hypothesis

Or YTM is the expected average of the short rates

Expectation Hypothesis

- $LEH: \quad y(0,T) = -\frac{\ln(E_0\left[exp(-\int_0^T r_t dt)\right]}{T}$

$$EH: \quad y(0,T) = \frac{1}{T} E_0\left(\int_0^T r_t \, dt\right)$$

By Jensen's Ineqaulity we know :

$$-\frac{\ln(E\left[exp(-\int_0^T r_t dt)\right]}{T} <= -\frac{\mathrm{E}(ln\left[\exp(-\int_0^T r_t dt)\right]}{T} = \frac{\mathrm{E}(\int_0^T r_t dt)}{T}$$

So if LEH hold then EH usually will not hold