FIXED INCOME SECURITIES

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Fixed Income Securities

A bond is an obligation from the borrower to pay:

- Regular coupon payment (e.g. annual, semi-annual, etc.)
- Principal (par value) at a certain time in the future (i.e. maturity)

Example:

- US Treasury Securities: the debt obligation of the US government. We will study these extensively in this course because
  - ✓ Their payments are guaranteed so they can be considered as default free
  - ✓ They are used to finance US cumulative debt so they are highly liquid
  - ✓ They are used as a benchmark for pricing other securities

US Treasury Securities

Treasury securities are categorized based on their coupon payments and maturities:

- Coupon-bearing securities which pay interest every 6 months and the principal at maturity:
  - T-Bonds - Coupon bearing securities with a maturity of more than 10 years
  - T- Notes - Coupon bearing securities with a maturity between 2-10 years

US Treasury Securities

- Discount (zero-coupon) securities which only pay the principal amount at maturity
  - T-bills : Short term zero-coupon securities with a maturity of less than a year

- Callable bonds
  - Bonds that are callable at the discretion of the Treasury within the last five years of their life
  - As interest rates decrease, the Treasury may decide to call these bonds and stop paying high coupons

Example of Treasury Bonds

30 Year bond bearing a 4.0% coupon issued on November 15th 2012 (Maturing on November 15th,2042)

This Bond will pay Coupon payment of 4.0%* 0.5 = $2.00 every six months, starting 5/15/2013 and ending on 11/15/2042. In addition the principal ($100 in this case) will also be paid on 11/15/2042.

The quoted price of this bond on January 23, 2023 was 102-21.

If holder of this bond decide to sell this, for next day settlement, in addition to quoted price of 102-21, the seller is entitled to the accrued interest from 11/15/2022 till 1/24/2023. Therefore, the Invoice price which is the actual payment of buyer to the seller is quoted price + accrued interest.

Invoice price = Quoted price + Accrued interest

Accrued Interest

Why isn't accrued interest included in the quoted price?

Two factors change invoice price:

1- Interest rate change

2- Where the settlement date falls within the coupon cycle

For differentiating between these two factors, the quoted price is only due to the changes in interest rates

Accrued interest =

(Coupon /2)*(Number of days since the last coupon date from settlement)/(number of days in the coupon period)

For above example: Accrued interest = ($4.0/2)*(70/182)=$0.7692

Yield to Maturity (YTM)

- Yield to Maturity is simply the Internal Rate of Return (IRR)

- Instead of discounting future cash flows with its appropriate discount rates, we simply discount ALL cash flows by the one discount rate, which we call Yield to Maturity (y_0).

- $P(y_0) = \sum_{i=1}^{N} C_i(1 + y_i)^{-i} + 1(1 + y_N)^{-N}$

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- Where y_i are exogenous variable and independent of the Bond

Yield to Maturity

- Example

A 10 year bond with semi-annual coupon rate of 5%
and par value of $1000.00 is issued at the price of $970.00.
What is the YTM?

N=20 , C=$25 , F=$1000 , P= $970

Excel function:

RATE(20,25,-970,1000) =2.699% and YTM=5.392%

- Notice that RATE will give YTM in the same unit of coupon payment!

Bond Price from YTM

- Bond price is simply the present value of our cash flows discounted by the YTM

Excel function:

PV (YTM/2 ,N ,coupon, final payment)

In our example:

PV (5.392%/2 , 20, 25,1000) = -970.00

If YTM rises by 1%, the bond price will drop by

7.98% !

Problems with YTM - Why do we use it?

- The PROBLEM is using the same discount rate across all different payments. For example, for 2 bonds with the same coupon rate and only different maturities we derive different YTM.
- However, YTM is an easy way to communicate the price of relatively similar maturity bonds.

Formula when date is between the coupon payments:

$$p = \sum_{t=1}^{n-1} \frac{C_t}{2} \times \left(1 + \frac{y}{2}\right)^{-(t-1+\tau)} + \left(1 + \frac{C_n}{2}\right) \times \left(1 + \frac{y}{2}\right)^{-(n-1+\tau)}$$

τ = (number of Days from settlement until next coupon payment) / (number of days in the coupon period)

Relationship between C, P, YTM and Par Value

- If Coupon = YTM then Price = Par
- If Coupon < YTM then Price < Par
- If Coupon > YTM then Price > Par

- Note that YTM is the return of holding a bond only if you hold a zero coupon bond to the maturity OR if you reinvest all the coupon payments at the current YTM

Risk and Volatility of Debt Securities

- Bond prices and yield moves in opposite directions ,Yield↑(↓) Price↓(↑)

Typical US Treasury bond with semiannual compounding:

$$P = \sum_{j=1}^{N} \frac{C/2}{(1+y/2)^j} + \frac{100}{(1+y/2)^N}$$

- The price risk is the change in price of the bond for changes in the yield:

$$\frac{\partial P}{\partial y} = -\frac{c}{y^2}\left[1 - \frac{1}{(1+\frac{y}{2})^N}\right] + \frac{N(100 - \frac{c}{y})}{2(1+\frac{y}{2})^{N+1}}$$

Price Risk

Usually price risk is divided by 100, which means the change in price for a percentage change in yield.

Price value of a basis point (PVBP)

Change in the bond price per basis point change in the yield, it is also called the dollar value of 0.01. It is simply the price risk divided by 10000.

- **Example:**

A 30 year 5.00% T-bond at yield of 5.00% its price is 100. The price of this bond at yield of 5.01% is 99.8456 ,which makes PVBP = $0.1544 .

Sometimes PVBP is expressed per million dollar of face value:

PVBP(per $1 million par value) = $1544

PVBP Example

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Duration

- Duration is very similar to PVBP and used more widely in industry.

Macaulay Duration:

The average time (in units of period of coupon) taken by the security, on a discounted basis, to pay back the original investment. Thus, it is the discounted time-weighted cash flow of the security divided by the price of security.

$$Duration = \left[PV(CF_1) + 2 \times PV(CF_2) + \ldots + N \times PV(CF_N)\right] / \, Bond\ price$$

Duration

- Example: a four year T-Note , with coupon of 5% and Yield of 6 % :
- The price of this bond is $96.49.

Period	Coupon	PV	i * PV
1	2.5	2.4272	2.4272
2	2.5	2.3565	4.7130
3	2.5	2.2879	6.8636
4	2.5	2.2212	8.8849
5	2.5	2.1565	10.7826
6	2.5	2.0937	12.5623
7	2.5	2.0327	14.2291
8	102.5	80.9144	647.3156

Duration :

Sum (last column)/ Price

=707.77 / 96.49 =

7.33 (semi annual)

= 3.67 year

Duration and Elasticity of Interest Rates

- If we have A= f(B), Elasticity =

(Percentage change of A)/(Percentage change of B)= $(\partial A / A)/(\partial B / B)$

$$\text{Yield Elasticity} = \frac{\text{Percentage change in Price}}{\text{Percentage change in yield (or percentage change in } (1+y))}$$

- Duration is equal to the price elasticity of interest rates, and it is called

Macaulay Duration: $D = -\frac{\partial P}{\partial y} \times \frac{1+y}{p}$

Another measurement of risk is Modified Duration which is the percentage change in price for a change in yield, thus

Modified Duration = $MD = -\frac{\partial p}{\partial y} \times \frac{1}{p}$

Duration and Elasticity of Interest Rates

$$p = \sum_{j=1}^{n} \frac{c_j}{(1+y)^j} + \frac{100}{(1+y)^n}$$

$$-\frac{\partial \mathrm{p}}{\partial \mathrm{y}} = -\frac{\partial \mathrm{p}}{\partial (1+\mathrm{y})} = \sum_{j=1}^{n} \frac{c_j \times j}{(1+y)^{j+1}} + \frac{100 \times n}{(1+y)^{n+1}}$$

$$-\frac{\partial \mathrm{p}}{\partial \mathrm{y}} \times \frac{1+y}{p} = \frac{1}{p} \times \sum_{j=1}^{n} \frac{c_j \times j}{(1+y)^j} + \frac{100 \times n}{(1+y)^n} = \frac{1}{p} \times \sum_{j=1}^{n} j \times PV(CF_j) = D$$

PVBP, D, MD

$$PVBP = \frac{P \times D \times 10^{-4}}{1+y} = p \times MD \times 10^{-4}$$

Example: a zero-coupon 10 year Treasury, yielding 6%, with a notional value of $1,000,000

It is a zero coupon bond, so the duration = maturity = 10 years

At 6% semi-annual yield, the bond price is: $553,676

At 6.01% semi-annual yield, the bond price is: $553,138

PVBP = (553,138 * 10 * 10^-4) / (1+3%) =$537

MD = 537 * 10^4 / 553,138 = 9.71

The actual change in price is $538 but modified duration gives a good approximation (linear change)

Some points for D, MD

- For zero-coupon securities, duration is equal to maturity
- Given two equal maturity bonds bearing different coupon rates, the duration of the lower coupon bearing bond is higher
- When yield decreases, duration increases
- For short to medium term bonds, duration and maturity are very close
- For long term bonds, duration is much shorter than maturity
- Maximum duration is for a zero-coupon bond

Duration of a Portfolio

- Duration of a portfolio is simply the weighted (market value) average of the durations of its bonds

$$P(y) = p(y_1) + p(y_2) + .. + p(y_n)$$

$$\frac{\partial p(y)}{\partial y} = \frac{\partial p(y_1)}{\partial y_1} \times \frac{\partial y_1}{\partial y} + \frac{\partial p(y_2)}{\partial y_2} \times \frac{\partial y_2}{\partial y} + .. + \frac{\partial p(y_n)}{\partial y_n} \times \frac{\partial y_n}{\partial y}$$

$$= \frac{\partial p(y_1)}{\partial y_1} \times \frac{1+y_1}{p_1} \times \frac{p_1}{1+y_1} \times \frac{\partial y_1}{\partial y} \times + .. + \frac{\partial p(y_n)}{\partial y_n} \times \frac{1+y_n}{p_n} \times \frac{p_n}{1+y_n} \times \frac{\partial y_n}{\partial y}$$

$$= D_1 \times \frac{p_1}{1+y_1} \times \frac{\partial y_1}{\partial y} + .. + D_n \times \frac{p_n}{1+y_n} \times \frac{\partial y_n}{\partial y}$$

$$\Rightarrow \frac{\partial p(y)}{\partial y} \times (\frac{1+y}{p}) = D_1 \times \frac{p_1}{p} \times \frac{\partial y_1 / 1+y_1}{\partial y / 1+y} + .. + D_n \times \frac{p_n}{p} \times \frac{\partial y_n / 1+y_n}{\partial y / 1+y}$$

$$if \ \partial y_1 / 1+y_1 = \partial y / 1+y = .. = \partial y_n / 1+y_n \text{ all the yields change with a same percentage, then:}$$

$$\boxed{D_P = w_1 D_1 + W_2 D_2 + .. + W_n D_n}$$

In Sensitivity analysis the duration only matters when there is a parallel shift in the yield curve

Duration Application: Hedging

Bonds	Coupon	Maturity	Price	Duration
A	6%	10	103.8	7.71
B	5%	30	86.16	14.77

In order to have a zero duration portfolio: For each unit of bond A we need to short N(B) of bond B:

N(B)= - P(A)*D(A) / P(B)* D(B)=.629

Now by one bps change in the interest rate, bond A will lose 7.5 cents, and bond B will gain the same amount. Therefore, the overall gain or loss is zero.

Convexity

- For larger changes in yield, we need a better approximation
- Duration measures the “first-order” effects on price due to changes in the yield
- Convexity measures the “second-order” effects on price due to changes in the yield
- Duration tells us how price changes for a change in yield
- Convexity tells us how duration changes for change in yield
- Convexity of the bond is the value weighted average of the individual bond convexity

Approximation of Change in Bond Price - Example

for small Δy, Δy^2 *is very* small, and we can *approximate*
by first order for example $\Delta y = 0.0005,\ \Delta y^2 = 25\times10^{-8}$
for larger changes we can not ignore the second term

Example : T=30 , C= 5% , y=6% , P = 86.16

$$D = -\frac{\partial p}{\partial y}\times\frac{1+y/2}{p} = 14.77$$

$$MD = -\frac{\partial p}{\partial y}\times\frac{1}{p} = \frac{14.77}{1.03} = 14.34$$

$$C_x = \frac{1}{2}\times\frac{1}{p}\times\frac{\partial^2 p}{\partial^2 y} = 157.53$$

Approximation of change in bond price - Example

If yield changes from 6% to 6.01% , price will change to 86.04

Actual percentage change = (86.04-86.16) / 86.16= -13.93 bps

Modified Duration appr.= -14.34* (0.0601-0.06)= -14.34 bps. Off by .41 bps

Convexity appr. = -14.34* (0.0601-0.06)+157.53*(0.0601-0.06)^2= 14.32 bps

When yield changes from 6% to 7%, results are shown in the below table:

	Yield change(bps)	
Method	**1**	**100**
Actual % change	-13.93 bps	1289 bps=12.89%
Modified duration (first order app.)	-14.34 bps	1434 bps =14.34%
Convexity (second order app.)	-14.32 bps	1276 bps =12.76%

A Zero-Coupon Bond: Approximation of Changes in Bond Price

$$p = (1+y)^{-T}, \frac{\partial \mathrm{p}}{\partial \mathrm{y}} = -T \times (1+y)^{-T-1}$$

$$D = -\frac{\partial \mathrm{p}}{\partial \mathrm{y}} \times \frac{1+y}{p} = T \times (1+y)^{-T-1} \times \frac{1+y}{(1+y)^{-T}} = T$$

$$\frac{\partial^2 p}{\partial^2 y} = T \times (T+1) \times (1+y)^{-T-2}$$

$$p(y) = p(y_0) + \frac{\partial \mathrm{p}}{\partial \mathrm{y}}]_{y_0} \times (y - y_0) + \frac{1}{2!}\frac{\partial^2 p}{\partial^2 y}]_{y_0} \times (y - y_0)^2 + \frac{1}{3!}\frac{\partial^3 p}{\partial^3 y}]_{y_0} \times (y - y_0)^3 + \ldots$$

$$\frac{p(y) - p(y_0)}{p(y_0)} = \frac{1}{p(y_0)} \times \frac{\partial \mathrm{p}}{\partial \mathrm{y}}]_{y_0} \times \Delta y + \frac{1}{2}\frac{1}{p(y_0)} \times \frac{\partial^2 p}{\partial^2 y}]_{y_0} \times \Delta y^2 + O(3)$$

$$= -MD \times \Delta y + C_x \times \Delta y^2 + O(3)$$

$$MD = \text{Modified duration}$$

$$C_x = \frac{1}{2}\frac{1}{p(y_0)} \times \frac{\partial^2 p}{\partial^2 y} = \text{Dollarconvexcity}, C_x > 0$$

Summary

- For US Treasury Bond j assume :
  - $P_j^{t_0}(y_j) = \sum_{i=1}^{N} \frac{C_j}{2} \left[\frac{1}{1+y_{T_i}}\right]^{-\frac{T_i - t_0}{365}} + 100 \left[\frac{1}{1+y_{T_N}}\right]^{-\frac{T_N - t_0}{365}} =$
    $\sum_{i=1}^{N} \frac{C_j}{2} \left[\frac{1}{1+y_j}\right]^{-\frac{T_i - t_0}{365}} + 100 \left[\frac{1}{1+y_j}\right]^{-\frac{T_N - t_0}{365}}$
  - $C_j/2$ is the coupon payment and y_j is YTM for Bond j
  - N is the number of coupons in the life of the bond
  - t_0 and T_i are todays date and date the i^{th} coupon
  - $T_i - t_0$ is the number of calendar days between today t_0 and T_i
  - y_{T_i} is exogenous discount rate for T_i
  - y_j is Bond Specific YTM
  - Duration is $D\left(P_j^{t_0}(y_j)\right) = -\frac{(1+y_j)}{P_j^{t_0}} \frac{\partial\left(P_j^{t_0}(y_j)\right)}{\partial y_j}$

Summary

- Portfolio of Bond:
- $P^{t_0}(y) = \sum_{i=1}^{N} C_i \left[\frac{1}{1+y_{T_i}}\right]^{\frac{T_i - t_0}{365}} = \sum_{i=1}^{N} C_i \left[\frac{1}{1+y}\right]^{\frac{T_i - t_0}{365}}$
  - C_i is the i^{th} cash flow and is y YTM for portfolio
  - N is the number of cash flow in the life of the bond
  - t_0 and T_i are todays date and date the i^{th} cash flow
  - $T_i - t_0$ is the number of calendar days between today t_0 and T_i
  - y_{T_i} is exogenous discount rate for T_i
  - $C_x = \frac{1}{2P^{t_0}(y)} \frac{\partial^2 P^{t_0}(y)}{\partial y^2}$
  - $D = -\frac{(1+y)}{P^{t_0}(y)} \frac{\partial P^{t_0}(y)}{\partial y}$

Portfolio Statistics

- Portfolio Hedging
- We have $i = 1, .., n$ bonds maturities T_i and coupon C_i
- These bonds Price, YTM, Duration, Convexity are :
- $P_i, Y_i, D_i, and\ C_{xi}$
- We have m portfolio H_j for $\mathrm{j} = 1, .., m$ where N_{ji} is the number of bond "i" in portfolio "j".
- Let $P_{H_j}, Y_{H_j}, D_{H_j}, C_{xH_j}$ be the price, YTM, Duration and convexity, given by the previous page equations:

Portfolio Statistics

- $P_{H_j} = \sum_{i=1}^{n} N_{ji} P_i$
- $D_{H_j}^{a} = \sum_{i=1}^{n} w_{ij} D_i$
- $C_{xH_j}^{a} = \sum_{i=1}^{n} w_{ij} C_{xi}$
- $w_{ij} = \frac{N_{ji} P_i}{P_{H_j}} = \frac{N_{ji} P_i}{\sum_{i=1}^{n} N_{ji} P_i}$
- Assume we have portfolio H_1 with composition N_{1i}, and $P_{H_1}, D_{H_1} \& C_{xH_1}$, We can modify the risk of this portfolio(i.e. its duration and its convexity) to be $D_{H_1}^{a,new}$ and $C_{xH_1}^{a,new}$ by addition or subtraction of other portfolio :

Portfolio Risk Control

- $D_{H_1}^{a,new} = D_{H_1} w_1 + \ \dots + D_{H_m} w_m$
- $C_{xH_1}^{a,new} = C_{xH_1} w_1 + \ \dots + C_{xH_m} w_m$
- $P_{H_1}^{new} = P_{H_1} + \ \dots + P_{H_m} N_{H_m}$
- Where :
- $w_1 = \frac{P_{H_1}(N_{H_1}=1)}{\sum_{j=1}^{m} N_{H_j} P_{H_j}}$ and $w_i = \frac{P_{H_i} N_{H_i}}{\sum_{j=1}^{m} N_{H_j} P_{H_j}}$ and solve for N_{H_i}
- Note that we may decide to control part or all of the above risk measures.