FIXED INCOME SECURITIES

FRE : 6411

Sassan Alizadeh, PhD
Tandon School of Engineering of NYU

2023

Constant Maturity Treasury (CMT)

- Constant maturity is the theoretical value of a U.S. Treasury that is based on recent values of auctioned U.S. Treasuries. The value is obtained by the U.S. Treasury on a daily basis through interpolation of the Treasury yield curve which, in turn, is based on closing bid-yields of actively-traded Treasury securities. It is calculated using the daily yield curve of U.S. Treasury securities.
- Constant maturity yields are often used by lenders to determine mortgage rates. The one-year constant maturity Treasury index is one of the most widely used, and is mainly used as a reference point for adjustable-rate mortgages (ARMs) whose rates are adjusted annually.

Yield Curve Modelling

- Assume we have $i = 1, \dots, m$ US government T-bonds and T-note each paying semi-annual coupon $C_j/2$
- For Bond j assume :
  - $P_j^{t_0}(y_j) = \sum_{i=1}^{N} \frac{C_j}{2} \left[\frac{1}{1+y_{T_i}}\right]^{\frac{T_i - t_0}{365}} + 100 \left[\frac{1}{1+y_{T_N}}\right]^{\frac{T_N - t_0}{365}}$
  - $C_j/2$ is the coupon payment and y_j is YTM for Bond j
  - N is the number of coupons in the life of the bond
  - t_0 and T_i are todays date and date the i^{th} coupon
  - $T_i - t_0$ is the number of calendar days between today t_0 and T_i
  - y_{T_i} is unobservable rate for T_i
- Consider T 3 ⅛ 11/15/28 on 1/29/2019 close at 103-16 3/4
  - For this bond $P_j^{t_0}(y_j) = 103.5234375, C_j = 3.125$, $T_1 =$ 5/15/2019 ,…, $T_N =$ 11/15/2028, $N = 20$ and $y_j = 2.712$

Yield Curve Modelling

Cash Security	Price	Conventional Yield
T 3 ⅛ 11/15/28	103-16 3/4	2.712
Todays Date	1/29/2019	
Coupon Date	# days to Coupon date	Fraction of year
5/15/2019	106	0.290411
11/15/2019	290	0.794521
5/15/2020	472	1.293151
11/15/2020	656	1.797260
5/15/2021	837	2.293151
11/15/2021	1021	2.797260
5/15/2022	1202	3.293151
11/15/2022	1386	3.797260
5/15/2023	1567	4.293151
11/15/2023	1751	4.797260
5/15/2024	1933	5.295890
11/15/2024	2117	5.800000
5/15/2025	2298	6.295890
11/15/2025	2482	6.800000
5/15/2026	2663	7.295890
11/15/2026	2847	7.800000
5/15/2027	3028	8.295890
11/15/2027	3212	8.800000
5/15/2028	3394	9.298630
11/15/2028	3578	9.802740

Yield Curve Modelling

- Our Objective is, given certain criteria (simplicity, predictability, etc.) to construct a continuous zero coupon yield curve $y(t)$, and approximate $y_{T_i} \approx y(T_i)$, such that approximated bond prices and actual market observed prices to be "as close to each other as possible".
  - $P_j^{t_0}(y_j) \approx \sum_{i=1}^{N} \frac{C_j}{2} \left[\frac{1}{1+y(T_i)}\right]^{-\frac{T_i - t_0}{365}} + 100 \left[\frac{1}{1+y(T_i)}\right]^{-\frac{T_N - t_0}{365}}$
- Three method:
  - Non-parametric Modelling. (curve fitting, interpolation, etc.)
  - Parametric Modelling. (Nelson-Siegel, AR(1),etc.)
    - $y(t) = y_\infty + (y_0 - y_\infty)\alpha^{\frac{t-t_0}{365}}$ then $y(T_i) = y_\infty + (y_0 - y_\infty)\alpha^{\frac{T_i - t_0}{365}}$ for $\alpha < 1$
  - Arbitrage free base modelling. (Affine Term Structure Models)

Continuous Time MM

- $B(T, s_{T-1}) = B(0)\ \prod_{i=0}^{T-1} R(i, s_i)$
- $R(i, s_i) = \ (1 + r_i(s_i)\Delta t)\ where\ \ \ r_i(s_i)$ is annualized Rate
- Now divide time $(0, T)$ into "n“ equal interval and $\Delta t = \frac{T}{n}$
- $B(T, s_{T-1}) = B(0)\ \ \prod_{i=0}^{n-1}\left[1 + \ r_i(s_i)\frac{T}{n}\right]$
- For Continuous Compounding let $\Delta t \to 0$ or $\ n \to \infty$
- $B(T, s_{T-1}) = B(0)e^{\int_0^T r_s ds} = e^{\int_0^T r_s ds}$
- $\frac{1}{B(T, s_{T-1})} = \frac{1}{e^{\int_0^T r_s ds}} = e^{\int_0^T -r_s ds}$
- $P(0, T) = E_0^*(\frac{P(T,T)=1}{B(T,S_{T-1})}) = E_0^*\left[e^{\int_0^T -r_s ds}\right]$

Continuous Forward Rate

- $G(t, T_1, T_2) = \frac{P(t,T_1)}{P(t,T_2)} = [1 + f(t, T_1, T_2)(\Delta T_{12})]$, $\Delta T_{12} = T_2 - T_1$
- If Continuous Compounding then :
- $f(t, T_1, T_2) = \frac{1}{\Delta T_{12}} [\ln P(t, T_1) - \ln P(t, T_2)]$
- $f(t, T) = \lim_{\Delta T_{12} \to 0} \frac{1}{\Delta T_{12}} [\ln P(t, T_1) - \ln P(t, T_2)] = -\frac{\partial \ln P(t,T)}{\partial T}$
- $p(t, T) = e^{-\int_0^T f(t,s)ds}$
- $p(t, T) = e^{-\int_0^T f(t,s)ds} = E_0^*\left[e^{-\int_0^T r_s ds}\right]$
- Where $f(t, t) = r_t$

Continuous Yield Curve & Instantaneous Forward Rate

- Recall zero coupon bond price $p(t,T)$ & its continuous compounded YTM, $y_t(T)$:
  - $p(t,T) = e^{-Ty_t(T)}$ & $y_t(T) = -\frac{\ln[p(t,T)]}{T}$
- Recall the annualized Forward Rate $f_t(T, T+\delta)$:
  - $1 + \delta f_t(T,T+\delta) = \frac{p(t,T)}{p(t,T+\delta)}$ & $f_t(T,T+\delta) = \frac{p(t,T)-p(t,T+\delta)}{\delta p(t,T+\delta)}$
- Define Instantaneous Forward Rate (IFR) as :
  - $f_t(T) = \lim_{\delta \to 0} \frac{p(t,T)-p(t,T+\delta)}{\delta p(t,T+\delta)} = -\frac{d\ln[p(t,T)]}{dT} = \frac{dTy_t(T)}{dT} = y_t(T) + T\frac{dy_t(T)}{dT}$
- Construct continuous yield curve and Instantaneous Forward Rate . (Note for IFR to be continuous $\frac{df_t(T)}{dT}$ & $\frac{d^2y_t(T)}{dT^2}$ must exist or yield curve $y_t(T)$ must be CONTINOUS and SMOOTH)

Non-Parametric Model

- Partition the yield curve in to distinct 'n' intervals such as 0, 1, 2, 5, 7, 10, 30, years and its corresponding zero coupon yields $y_i, for \ i = 1, \dots, n$ (constant maturity yield). (note y_i are usually unobservable and need to be estimated)
- Assume that yield curve is a non-parametric function (spline interpolation) of these constant maturity yield, y_i:

$$y(t) = y(t;\, y_1, y_2, \dots . y_n)$$

- Approximate the discount rate $y_{T_i} \approx y(T_i; y_1, y_2, \dots . y_n)$ and bond price:
  - $P_j^{t_0}(y_j) \approx \sum_{i=1}^{N} \frac{C_j}{2} \left[\frac{1}{1+y(T_i; y_1, \dots, y_n)}\right]^{-\frac{T_i - t_0}{365}} + 100 \left[\frac{1}{1+y(T_i; y_1, \dots, y_n)}\right]^{-\frac{T_N - t_0}{365}}$

Spline Interpolation $S(x)$

- Assume we have a set of discrete observation $\{x_i, y_i\}\ i = 1, \dots, n$ from a continuous function $F(x)$. Define y_i as "knot" & WLOG assume $x_1 < \cdots < x_n$. (knots is our constant maturity yields)
- We would like to approximate $F(x)$ by function $S(x)$, such that:
  - $S(x)$ is an $n - 1$ piece polynomial $S_i(x)$ of q^{th} degree.
  - $S(x)$ = $S_i(x)$ for $x_i \leq x \leq x_{i+1}$
  - $S(x_i) = y_i$ knot condition
  - $\frac{d^j S_i(x)}{dx^j}|_{x_{i+1}} = \frac{d^j S_{i+1}(x)}{dx^j}|_{x_{i+1}}\ for\ j = 0, \dots, q-1$
- Then $S(x)$ is q^{th} order spline
- $y_{i+1} = S_i(x_{i+1}) = S_{i+1}(x_{i+1}) = \sum_{j=0}^{q} a_{ji}\ x_{i+1}^{j} = \sum_{j=0}^{q} a_{ji+1}\, x_{i+1}^{j}$
- Function and its first $q - 1$ derivatives are continuous

First & Second order Spline

- First Order Spline or Linear Interpolation:
  - $S_i(x) = a_{0i} + a_{1i}x$ for $x_i \le x \le x_{i+1}$
  - $S_i(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$ for $x_i \le x \le x_{i+1}$
  - $a_{0i} = y_i - \frac{y_{i+1} - y_i}{x_{i+1} - x_i} x_i$ & $a_{1i} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$
  - Note that the function is continuous $S_i(x_{i+1}) = S_{i+1}(x_{i+1}) = y_{i+1}$
  - But the first derivative is not continuous (piecewise constant)
- Second Order Spline or Quadratic Interpolation :
  - $S_i(x) = a_{0i} + a_{1i}x + a_{2i}x^2$ for $x_i \le x \le x_{i+1}$
  - $a_{0i} + a_{1i}x_{i+1} + a_{2i}x_{i+1}^2 = a_{0i+1} + a_{1i+1}x_{i+1} + a_{2i+1}x_{i+1}^2 = y_{i+1}$
  - $a_{1i} + 2a_{2i}x_{i+1} = a_{1i+1} + 2a_{2i+1}x_{i+1}$
  - The function and its first derivative are continuous
  - But the second derivative is not continuous (piecewise constant)

Second order Spline

- Note for Quadratic spline we need to find $n-1$ second order equations, S_i where each equation has 3 unknown a_{oi}, a_{1i}, a_{2i} for a total of $3n-3$ unknown.
  - Knot condition : We have $n-1$ equations from :
    $$S_i(x_i) = y_i \; for \; i = 1,..,n-1$$
  - Continuity of function : We have $n-1$ equations from :
    $$S_i(x_{i+1}) = y_{i+1} \;\; i = 1,..,n-1$$
  - Continuity of first derivative : We have $n-2$ equations from:
    $$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}) \; i = 1,..,n-2$$
  - For total of $3n-4$ equations , add one more constrain such as $S_1'(x_1) = 0 \; (i.e. \; a_{11} + a_{21}x_1 = 0)$; for total of $3n-3$ equations
  - $S_i(x) = y_i + \delta_i(x - x_i) + \frac{\delta_{i+1}-\delta_i}{2(x_{i+1}-x_i)} (x - x_i)^2$ where

    $$\delta_{i+1} = -\delta_i + 2\left(\frac{y_{i+1}-y_i}{x_{i+1}-x_i}\right) \; and \; S_1'(x_1) = \delta_1 = 0$$

Cubic or Third order Spline

- Cubic Spline is a n-1, 3rd order piecewise polynomial such that the function meet the knot condition and the function, its first and second derivative are continuous:
  - $S(x) = S_i(x)$ for $x_i \leq x \leq x_{i+1}$ & $i = 1, \dots, n-1$
  - $S_i(x) = a_{0i} + a_{1i}x + a_{2i}x^2 + a_{3i}x^3$
  - $S_i(x_i) = a_{0i} + a_{1i}x_i + a_{2i}x_i^2 + a_{3i}\, x_i^3 = y_i$
  - $S_i(x_{i+1}) = S_{i+1}(x_{i+1}) = a_{0i} + a_{1i}\, x_{i+1} + a_{2i}x_{i+1}^2 + a_{3i}\, x_{i+1}^3 =$
    $a_{0i+1} + a_{1i+1}x_{i+1} + a_{2i+1}x_{i+1}^2 + a_{3i+1}\, x_{i+1}^3 = y_{i+1}$
  - $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}) = a_{1i} + 2a_{2i}x_{i+1} + 3a_{3i}x_{i+1}^2 =$
    $a_{1i} + 2a_{2i+1}x_{i+1} + 3a_{3i+1}x_{i+1}^2$
  - $S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}) = 2a_{2i} + 6a_{3i}x_{i+1} = 2a_{2i+1} + 6a_{3i+1}x_{i+1}$

Cubic Spline

- For Cubic spline we need to find $n-1$, third order linear equations, S_i where each equation has 4 unknown $a_{oi}, a_{1i}, a_{2i}, a_{3i}$ for a total of $4n-4$ unknown.
  - 1 .Knot condition : We have $n-1$ equations from :
    $$S_i(x_i) = y_i \; for \; i = 1, .., n-1$$
  - 2 .Continuity of function : We have $n-1$ equations from :
    $$S_i(x_{i+1}) = y_{i+1} \;\; i = 1, .., n-1$$
  - 3 .Continuity of 1st derivative : We have $n-2$ equations from:
    $$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}) \; i = 1, .., n-2$$
  - 4 .Continuity of 2nd derivative : We have $n-2$ equations from:
    $$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}) \; i = 1, .., n-2$$
  - 5 .For total of $4n-6$ equations , add two more constrain such as $S_1''(x_1) = S_{n-1}''(x_n) = 0$ for total of $4n-4$ equations (natural spline)

Cubic Spline

- Since $S_{i+1}(x)$ is cubic then S''_{i+1} is linear and from continuity condition :
  - $S''_i(x_{i+1}) = S''_{i+1}(x_{i+1}) = 2a_{2i} + 6a_{3i}x_{i+1} = 2a_{2i+1} + 6a_{3i+1}x_{i+1} = M_{i+1}$
  - $S''_i(x) = \frac{(x_{i+1}-x)M_i+(x-x_i)M_{i+1}}{h_i}$ where $h_i = x_{i+1} - x_i$ for $x_i \leq x \leq x_{i+1}$
- Now Integrate $S''_i(x)$ twice and using the knot condition $S_i(x_i) = y_i$ to solve for binary condition we obtain :
  - $S_i(x) = \iint S''_i(x) \ s.t. \ S_i(x_i) = y_i \ \& \ S_i(x_{i+1}) = y_{i+1}$
  - $S_i(x) = M_i\frac{(x_{i+1}-x)^3}{6h_i} + M_{i+1}\frac{(x-x_i)^3}{6h_i} + \left[y_i - \frac{M_i h_i^2}{6}\right]\left(\frac{x_{i+1}-x}{h_i}\right) + \left[y_{i+1} - \frac{M_{i+1}h_i^2}{6}\right]\left(\frac{x-x_i}{h_i}\right)$

Cubic Spline

- Note that if we can solve for M_i then we are done.
- We have already used three (conditions 1, 2, & 4) of the 5 conditions. We can use conditions 3 (continuity of first derivative) and condition 4(natural spline) to solve for M_i
- Differentiate S_i :
  - $S_i'(x) = -\frac{M_i(x_{i+1}-x)^2}{2h_i} + \frac{M_{i+1}(x-x_i)^2}{2h_i} + \frac{(y_{i+1}-y_i)}{h_i} - \frac{h_i(M_{i+1}-M_i)}{6}$
- Now Note from condition 3 :
  - $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$
  - $\frac{M_{i+1}(x_{i+1}-x_i)^2}{2h_i} + \frac{(y_{i+1}-y_i)}{h_i} - \frac{h_i(M_{i+1}-M_i)}{6} = -\frac{M_{i+1}(x_{i+2}-x_{i+1})^2}{2h_{i+1}} + \frac{(y_{i+2}-y_{i+1})}{h_{i+1}} - \frac{h_{i+1}(M_{i+2}-M_{i+1})}{6}$

Cubic Spline

- Collecting terms we get :
  - $M_i \frac{h_i}{h_{i+1}+h_i} + 2M_{i+1} + M_{i+2} \frac{h_{i+1}}{h_{i+1}+h_i} = \frac{6}{h_{i+1}+h_i} \left[\frac{y_{i+2}-y_{i+1}}{h_{i+1}} - \frac{y_{i+1}-y_i}{h_i} \right]$
- Let :
  - $\theta_i = \frac{h_i}{h_{i+1}+h_i}$ & $\lambda_i = 1 - \frac{h_i}{h_{i+1}+h_i}$ & $\zeta_i = \frac{6}{h_{i+1}+h_i} \left[\frac{y_{i+2}-y_{i+1}}{h_{i+1}} - \frac{y_{i+1}-y_i}{h_i} \right]$
- Then :
  - $M_i\theta_i + 2M_{i+1} + M_{i+2}\lambda_i = \zeta_i \;\; for \; i = 1, \dots, n-2$
- Note that we have ‘n-2’ equations but ‘n’ unknowns. Add two more equations using the natural spline condition :
  - $M_1 = M_n = S_1^{"}(x_1) = S_{n-1}^{"}(x_n) = 0$
- Now we have ‘n’ equations & ‘n’ unknowns solve for M_i

Cubic Spline & Minimum Curvature

- Minimum Curvature Property : (Cubic Spline is the smoothest function:
- Let $F(x)$ be a smooth (twice differentiable) function such that $F(x_i) = y_i$ (knot condition) then we want to show then we can show :
  - $\int_{x_1}^{x_n} (S''(x))^2 dx \le \int_{x_1}^{x_n} (F''(x))^2 \, dx \;\; or \;\; \int_{x_1}^{x_n} (F''(x))^2 \, dx - \int_{x_1}^{x_n} (S''(x))^2 dx \ge 0$
- Proof :
  - $\int_{x_1}^{x_n} \left(F''(x) - S''(x)\right)^2 \, dx \ge 0 \; or \; \int_{x_1}^{x_n} (F''(x))^2 \, dx$ - $\int_{x_1}^{x_n} (S''(x))^2 \, dx -$
$$2 \int_{x_1}^{x_n} \left(F''(x) - S''(x)\right) S''(x) \, dx$$
- We need to show $2 \int_{x_1}^{x_n} \left(F''(x) - S''(x)\right) S''(x) \, dx = 0$

Cubic Spline & Minimum Curvature

- We need to show $2\int_{x_1}^{x_n}\left(F^{"}(x) - S^{"}(x)\right) S^{"}(x)\, dx = 0$
- Show $\sum_{i=1}^{n}\int_{x_i}^{x_{i+1}}\left(F^{"}(x) - S_i^{"}(x)\right) S_i^{"}(x)\, dx \ = 0$
- Integrate by part :
  - $\int_{x_i}^{x_{i+1}} u_i dv_i = u_i v_i|_{x_i}^{x_{i+1}} - \ \int_{x_i}^{x_{i+1}} v_i du_i$
- Now define :
  - $u_i \ = S_i^{"}(x)$
  - $dv_i = \left(F^{"}(x) - S_i^{"}(x)\right) dx$

Cubic Spline & Minimum Curvature

- Integrate last equation by part, First Part:
  - $\sum_{i=1}^{n}\big(F'(x_{i+1}) - S_i'(x_{i+1})\big)S_i''(x_{i+1}) - \big(F'(x_i) - S_i'(x_i)\big)S_i''(x_i)$
- Now from Condition 3 and 4 :
  - $S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}) = M_{i+1}$ & $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}) = D_{i+1}$
  - $\sum_{i=1}^{n}(F'(x_{i+1}) - D_{i+1})M_{i+1} - (F'(x_i) - D_i)M_i =$ $(F'(x_n) - D_n)(M_n) - (F'(x_0) - D_0)(M_0) = 0\,;\ (M_n = M_0 = 0)$
- Second Part :
  - $2\int_{x_1}^{x_n}\big(F'(x) - S'(x)\big)S'''(x)\,dx = 0$ since $S'''(x)$ is constant and $F(x_i) = S(x_i) = y_i$(knot Condition)
- Therefore:
  - $\int_{x_1}^{x_n}\big(F''(x) - S''(x)\big)^2\,dx = \int_{x_1}^{x_n}\big(F''(x)\big)^2\,dx$ - $\int_{x_1}^{x_n}\big(S''(x)\big)^2\,dx \geq 0$

Home Work

- On 1/29/2019 we have the following zero coupon YTM was published by the FED

	1mon	2mon	3mon	6mon								
Maturity in year	.0833	.1667	0.25	0.5	1 yr	2 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
01/29/19	2.39	2.41	2.42	2.51	2.60	2.56	2.54	2.55	2.61	2.72	2.90	3.04

- Using First Order Spline find the interpolated price of the T 3 ⅛ 11/15/2028 on 1/29/2019
- Note you need to Interpolate the zero coupon yields of dates the coupons are being paid . For example for first, second and last payments we need to interpolate $y_{.290411}$, $y_{0.794521}$ & $y_{9.802740}$

Home Work

- Use Second & Third Order Spline to do these interpolations and compare your result with the actual bond closing price
- T 3 ⅛ 11/15/28 on 1/29/2019 close at 103-16 3/4