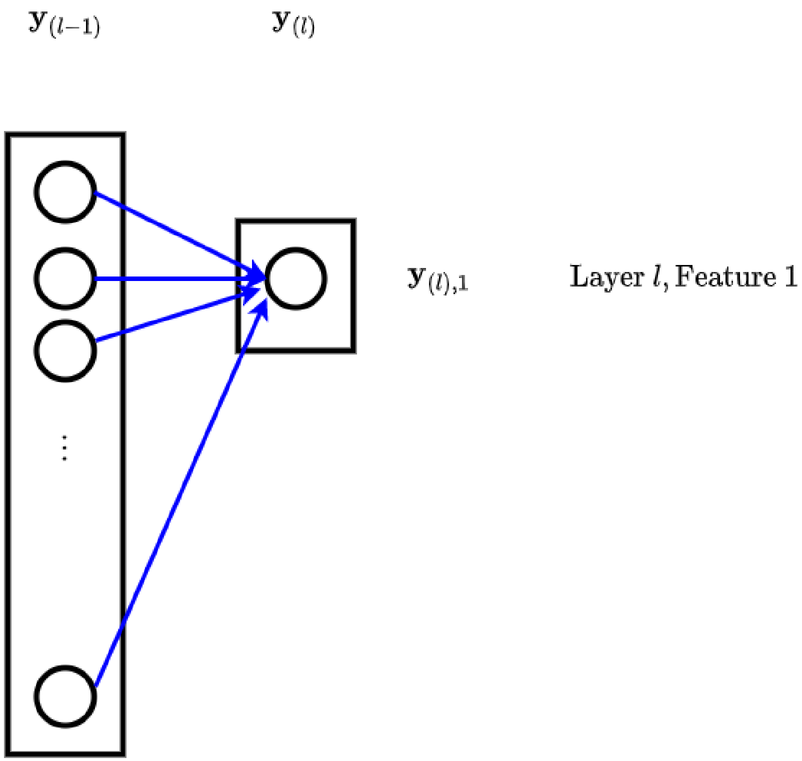
Convolutional Neural Networks

A Fully Connected/Dense Layer with a single unit producing a single feature at layer l computes

$$\mathbf{y}_{(l),1} = a_{(l)}(\mathbf{y}_{(l-1)} \cdot \mathbf{W}_{(l),1})$$

Fully connected, single feature



That is:

- It recognizes one new synthetic feature
- In the entirety ("fully" connected) of $\mathbf{y}_{(l-1)}$
- Using pattern $\mathbf{W}_{(l),1}$ (same size as $\mathbf{y}_{(l-1)}$)
- To reduce $\mathbf{y}_{(l-1)}$ to a single feature.

The pattern being matched spans the entirety of the input

- Might it be useful to recognize a smaller feature that spanned only *part* of the input ?
- What if this smaller feature could occur *anywhere* in the input rather than at a fixed location ?

For example

- A "spike" in a time series
- The eye in a face

A pattern whose length was that of the entire input could recognize the smaller feature only in a *specific* place

This motivates some of the key ideas behind a Convolutional Layer.

- Recognize smaller features within the whole
- Using small patterns
- That are "slid" over the entire input
- Localizing the specific part of the input containing the smaller feature

The spatial dimension

A small pattern (less than full length of input) can match a sub-section of input at any location.

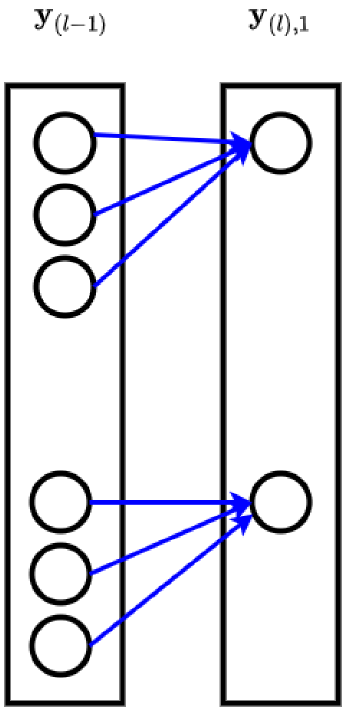
We can imagine centering the pattern on each input element and applying the match.

The output, therefore, will be a vector of length equal to the input length.

Here is the connectivity diagram of a Convolutional Layer producing a **single** feature at layer l

- Using a pattern of length 3
- Eventually we will show how to produce *multiple* features
- Hence the subscript "1" in $\mathbf{y}_{(l),1}$ to denote the first output feature

Convolutional layer, single feature



The output vector $\mathbf{y}_{(l),1}$ is called the first **feature map** as it attempts to match the first feature at each input location.

We refer to the indices of the feature map as the **spatial dimensions**

Thus, the output of layer l is 2 (or higher !) dimensional when layer l is a CNN

- a number of features
    - we have only shown a single feature thus far
- each feature producing a feature map
    - a feature map dimensions are called the spatial dimensions

The output of a CNN layer is a collection of

- $n_{(l)}$ feature maps (one per feature)
- each feature map having the same spatial dimension as its inputs

We can connect multiple CNN layers in sequence

- preserving the spatial dimensions across layers
- but creating more complex features as we get deeper

Technical note: special case

- The input to the first CNN layer l is often a one-dimensional vector of $n_{(l-1)}$ features
- A CNN treats this as a $(n_{(l-1)} \times 1)$ vector
    - 1 feature of a 1D spatial dimensions of shape $((n_{(l-1)},)$

Our convention will be that the **feature dimension** will appear as the **final** dimension of the output of layer l.

- all prior dimensions will be part of the **spatial dimension**

We need to distinguish which dimension is the feature dimension because

- **A convolution finds small patterns in the spatial dimension, not the feature dimension**

To be clear

- the vector of shape $(1 \times d)$ denotes d features at a single spatial location
- the vector of shape $(d \times 1)$ denotes a single features at d spatial locations

Notation

- the feature dimension will be the last index
- $n_{(l)}$ will always denote the *number of features* of a layer l
- $\mathbf{y}_{(l),j',j}$ denotes feature j of layer l at spatial location j'

We say that the above convolutional layer l

- Maps a single feature (defined over a 1D spatial dimension with $d_{(l)} = d_{(l-1)}$ locations) of layer $(l-1)$
- To a single feature, defined over an identical number of spatial locations in layer l

The importance of the spatial dimension

Let's contrast the CNN layer with a Fully Connected layer.

- The Fully Connected layer we depicted matches a pattern over the full *feature* dimension
    - There is no ordering (or spatial relationship) between features
- The CNN layer we depicted matches a pattern over the full *spatial* dimensions

Spatial dimensions are different than feature dimensions

- They have "order" (spatial relationships)

To see this, we show that a FC layer is insensitive to ordering of inputs

- Consider a vector $\mathbf{x}$ of n features (input to the Fully Connected layer)
- Let perm be permutation of the indices of $\mathbf{x}$: $[1 \ldots n]$.

If we permute both $\mathbf{x}$ and weights Θ, the dot product remains unchanged

$$\Theta^T \cdot \mathbf{x} = \Theta[\text{perm}]^T \cdot \mathbf{x}[\text{perm}]]$$

So shuffling inputs to a FC layer does not affect its outputs

- assuming they are shuffled the same way during training and inference

But for certain types of inputs (e.g. images) it is easy to imagine that spatial locality is important.

- Consider a 2D pixel grid depicting a face
- The relative ordering of pixels may be what **defines** a pattern to be recognized
    - The relative location of the pixels within the left eye are important
    - The relative location of the pixels constituting the left eye, right eye, nose and mouth are important

By using a small pattern (and restricting connectivity) we **emphasize the relative locations** of elements

- neighboring elements more important than far away elements.

The "spatial" dimension implies an ordering of elements

- but the ordering does not have to be in space
    - e.g., can be ordered in time

Consider the time series of prices of a single ticker over d days.

Two representations

- $(d \times 1)$: 1 feature ("price") over d spatial ("date") locations
- $(1 \times d)$: 1 ticker with d features $(\text{price } 1, \ldots, \text{price } d)$

The choice of where the singleton dimension appears is sometimes a matter of interpretation.

- but the last index always denotes the feature dimension

Mathematically, the One Dimensional Convolutional Layer (Conv1d) we have shown computes $\mathbf{y}_{(l)}$

$$\mathbf{y}_{(l),1} = \begin{pmatrix} a_{(l)} \left(\, N(\mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, 1) \cdot \mathbf{W}_{(l),1} \, \right) \\ a_{(l)} \left(\, N(\mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, 2) \cdot \mathbf{W}_{(l),1} \, \right) \\ \vdots \\ a_{(l)} \left(\, N(\mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, d_{(l-1)} \cdot \mathbf{W}_{(l),1} \, \right) \end{pmatrix}$$

where $N(\; \mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, j \;)$

- selects a subsequence of $\mathbf{y}_{(l-1),\ldots,1}$ centered at $\mathbf{y}_{(l-1),j,1}$
    - Note the extra spatial dimension in the subscripting; "$\ldots$" denotes the full spatial dimension
    - Centered at the j^{th} element in the spatial dimension of feature 1 of layer $(l-1)$

Note that

- The *same* weight matrix $\mathbf{W}_{(l),1}$ is used for the first feature at *all* locations j
- The size of $\mathbf{W}_{(l),1}$ is the same as the size of the subsequence $N(\; \mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, j)$
    - Since dot product is element-wise multiplication
- The spatial dimension $d_{(l)}$ of $\mathbf{y}_{(l),1}$ is equal to $d_{(l-1)}$

Kernel, Filter

The vector $\mathbf{W}_{(l),1}$ above

- Is a smaller pattern
- That is applied to *each* spatial location j in $\mathbf{y}_{(l-1)}$
- $\mathbf{y}_{(l),j,1}$ recognizes the match/non-match of the smaller first feature at the spatial locations centered at $\mathbf{y}_{(l-1),j,1}$

$\mathbf{W}_{(l),1}$ is called the first convolutional *filter* or *kernel*

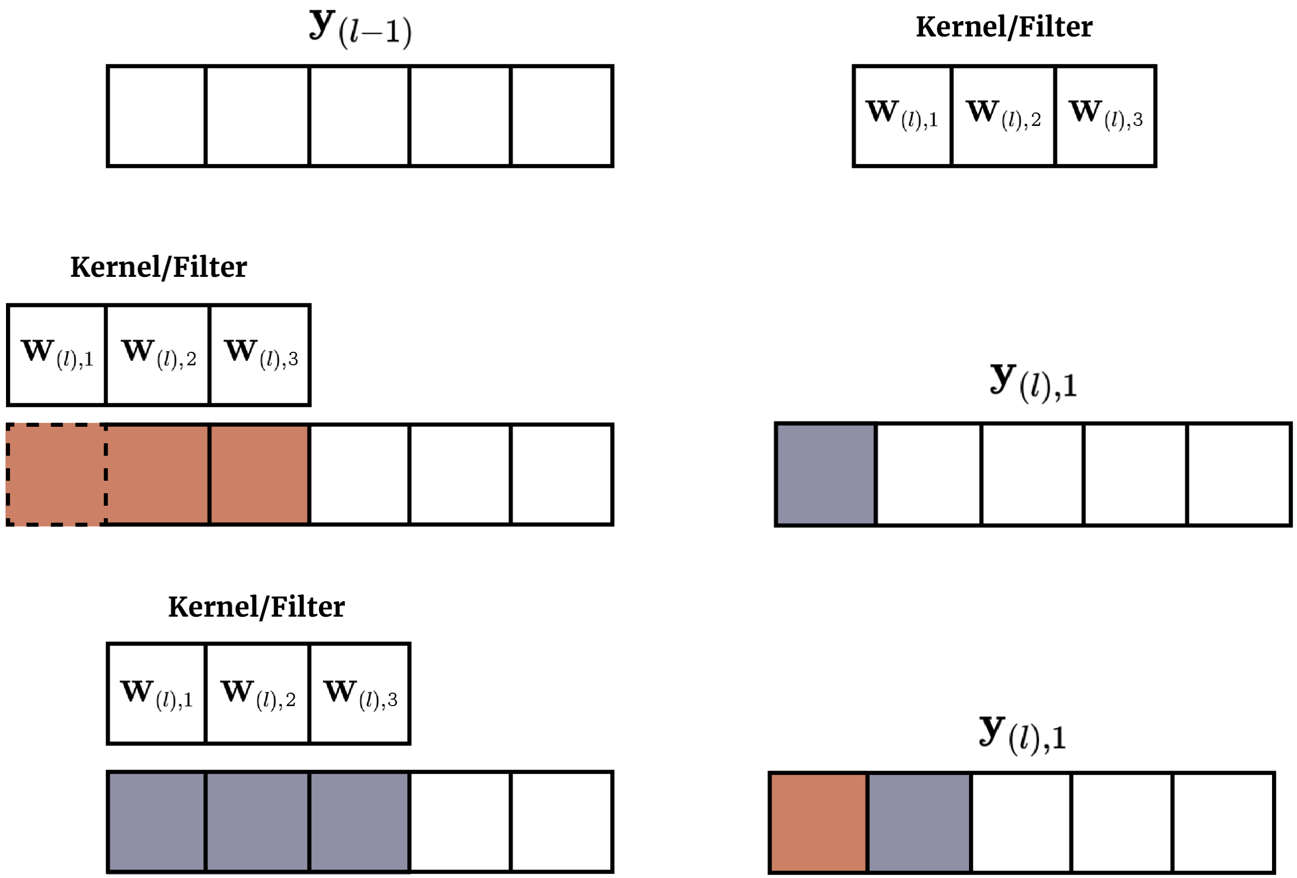
- We will often denote it $\mathbf{k}_{(l),1}$
- But it is just a part of the weights $\mathbf{W}$ of the multi-layer NN.
- We use $f_{(l)}$ to denote the size of the smaller pattern called the *filter size*

A *Convolution* is often depicted as

- A filter/kernel
- That is slid over each location in the input
- Producing a corresponding output for that location

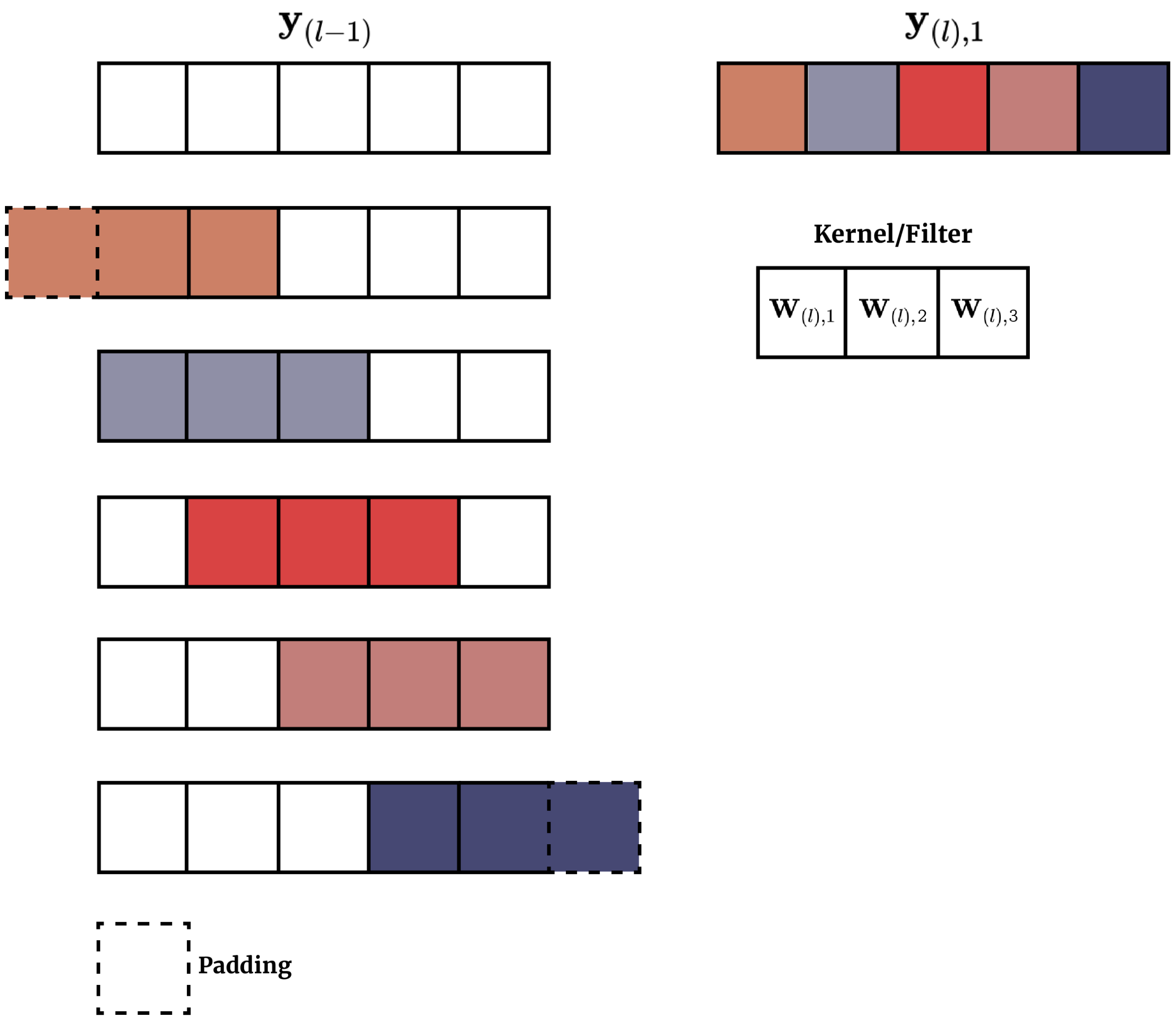
Here's a picture with a kernel of size $f_{(l)} = 3$

Conv 1D, single feature: sliding the filter



After sliding the Kernel over the whole $\mathbf{y}_{(l-1)}$ we get:

Conv 1D, single feature



Element j of output $\mathbf{y}_{(l),\ldots,1}$ (i.e., $\mathbf{y}_{(l),j,1}$)

- Is colored (e.g., $j = 1$ is colored Red)
- Is computed by applying the *same* $\mathbf{W}_{(l),1}$ to
    - The $f_{(l)}$ elements of $\mathbf{y}_{(l-1),1}$, centered at $\mathbf{y}_{(l-1),j,1}$
    - Which have the same color as the output

Note however that, at the "ends" of $\mathbf{y}_{(l-1)}$ the kernel may extend beyond the input vector.

In that case $\mathbf{y}_{(l-1)}$ may be extended with *padding* (elements with 0 value typically)

Activation of a CNN layer

Just like the Fully Connected layer, a CNN layer is usually paired with an activation.

The default activation $a_{(l)}$ in Keras is "linear"

- That is: it returns the dot product input unchanged
- Always know what is the default activation for a layer; better yet: always specify !

Conv2d in action

We have thus far depicted a spatial dimension of length 1.

We can easily expand this into 2 spatial dimensions

- each feature map is 2 dimensional
- each location in the feature map corresponds to a position in a 2D grid

Pre-Deep Learning: manually specified filters have a rich history for image recognition.

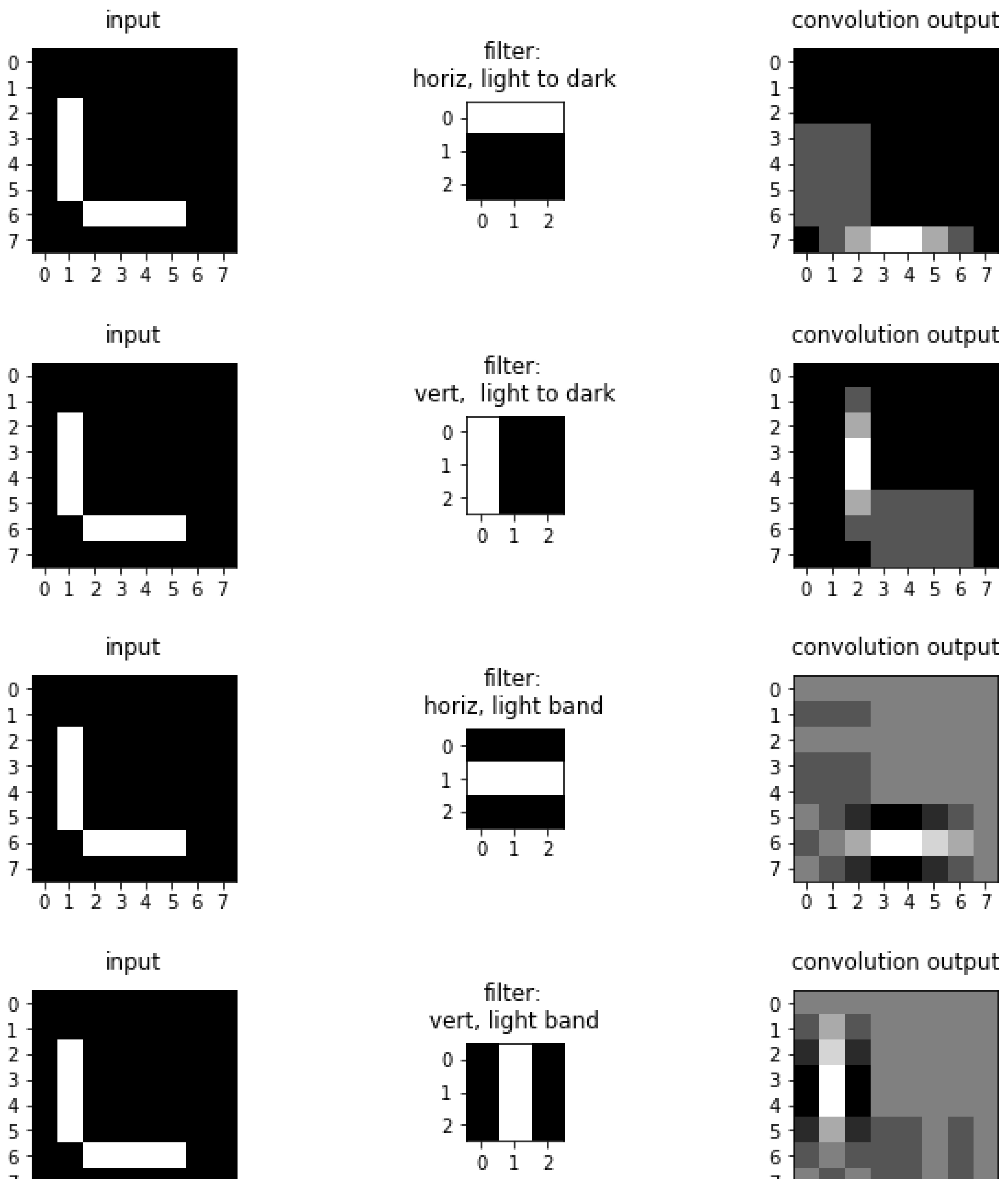
Here is a list of manually constructed kernels (templates) that have proven useful

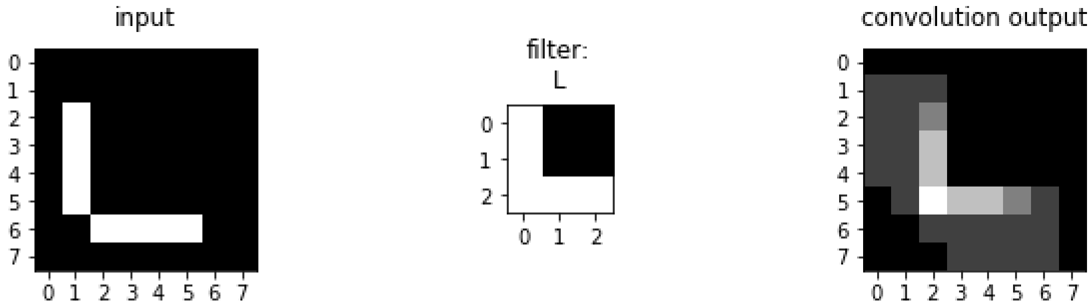
- [list of filter matrices (https://en.wikipedia.org/wiki/Kernel_(image_processing))](https://en.wikipedia.org/wiki/Kernel_(image_processing))

Let's see some in action to get a better intuition.

In [4]:

```
_= cnnh.plot_convs()
```




input
filter:
L
convolution output

- A bright element in the output indicates a high, positive dot product
- A dark element in the output indicates a low (or highly negative) dot product

In our example

- $N = 2$: Two spatial dimensions
- One input feature: $n_{(l-1)} = 1$
- One output feature $n_{(l)} = 1$
- $f_{(l)} = 3$
    - Kernel is $(3 \times 3 \times 1)$.

The template match will be maximized when

- high values in the input correspond to high values in the matching location of the template
- low values in the input correspond to low values in the matching locations of the template

We can have "spatial" dimensions of length even greater than 2

When we want to emphasize the number of features $n_{(l)}$ rather than the number of spatial dimensions, we will use ellipsis (dots)

$$\mathbf{y}_{\ldots,n_{(l)}}$$

where the ellipsis ($\ldots$) is a place-holder for the spatial dimension shape.

In [5]:

```
print("Done")
```

Done