FIXED INCOME SECURITIES

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Affine Term Structure Model

- From these equations we get YTM using price P(0,T):
- $Y(0,T) = P(0,T)^{\frac{-1}{T}} = E_0^*\left(\frac{1}{B(T,S_{T-1})}\right)^{\frac{-1}{T}}$
- $Y(0,T) = -\frac{\ln(P(0,T))}{T} = -\frac{\ln(E_0^*\left[exp(-\int_0^T r_t dt)\right]}{T}$
- Consider find eYTM using E(0,T) instead of P(0,T)
- $eY(0,T) = E(0,T)^{\frac{-1}{T}} = E_0\left(\frac{1}{B(T,S_{T-1})}\right)^{\frac{-1}{T}}$
- $eY(0,T) = -\frac{\ln(E(0,T))}{T} = -\frac{\ln(E_0\left[exp(-\int_0^T r_t dt)\right]}{T}$

Affine Term Structure Model

- For a generic asset with payoff $X(T, s_T)$:
- $P(0, X) = \; E_0^*\left(\frac{X(T,s_T)}{B(T,s_{T-1})}\right) = E_0^*\left(X(T, s_T)e^{\int_0^T -r_t\, dt}\right)$
- when $X(T, s_T) = 1$ ie zero coupon bond we get yield curve which is :
- $Y(0, T) = -\frac{\ln(P(0,T))}{T} = -\frac{\ln(E_0^*\left[exp(-\int_0^T r_t dt)\right]}{T}$
- Assume we have observation of yield at $T = 1, 5, 7, 10, 30$ years
- $Y_1 = -\frac{\ln\left(E_0^*\left[\exp(-\int_0^{T=1} r_t dt)\right]\right)}{T=1}, \dots, Y_{30} = -\frac{\ln\left(E_0^*\left[\exp(-\int_0^{T=30} r_t dt)\right]\right)}{T=30}$
- How to get 6 years yield Y_6?

Affine Term Structure Model

- Let us assume we fit a first order spline and approximate Y_6 as $Y_6 = -\frac{\ln\left(E_0^*\left[\exp(-\int_0^{T=6} r_t dt)\right]\right)}{T=6}$ $by\ \hat{Y}_6 = Y_5 + \frac{Y_7 - Y_5}{(7-5)}(6-5) =$
- $-\frac{\ln\left(E_0^*\left[\exp(-\int_0^{T=5} r_t dt)\right]\right)}{T=5} + -\frac{\ln\left(E_0^*\left[\exp(-\int_0^{T=7} r_t dt)\right]\right)}{T=7} \neq -2 * \frac{\ln\left(E_0^*\left[\exp(-\int_0^{T=6} r_t dt)\right]\right)}{T=6}$
- Modeling the yield curve by itself is equivalent to modeling a very unique function of the underlying exogenous short term rate (i.e. log of the above expectation)
- Modeling the yield curve without understanding the stochastic process of short term is like working backward . It would be very hard to infer the underlying model and to insure no arbitrage opportunity

Affine Term Structure Model

- To avoid this problem first Model the dynamic short term rate and from this model using the above equation find the price of zero coupon bond and then yield curve and the dynamic of the yield curve.
- One popular model is to assume short term rate can be written as an affine model of factors X_t where X_t themselves are affine also stochastic processes.
- $r_t = \alpha + \beta X_t$
- Where in continuous time X_t dynamic is :
- $dX_t = \mu(X_t)\mathrm{dt} + \Sigma\sqrt{S(X_t)}dW_t$
- Where we can write:

Affine Term Structure Model

- We assume X_t is (Nx1) vector of OBSERVED factor, Σ is (NxN) matrix, and dW_t is (Nx1) standard Brownian Motion.

- For Affine model we need additional assumption that mean vector $\mu(X_t)$ and covariance matrix $(\Sigma\sqrt{S(X_t)})(\Sigma\sqrt{S(X_t)})^T$ are linear function of X_t or $S(X_t)$ is a diagonal matrix

- $\mu(X_t) = \Theta(M - X_t)$ and $S_t^{ii} = S_{i0} + S_{i1}X_t$

Affine Term Structure Model

- Why Affine Model?
- It is easy to deal with linear models and they are general enough to encompass most of the observed behavior of the data
- Duffie and Kan showed under these assumption then log price of zero coupon bond is also affine model of Factors X_t
- $P(X_t, T) = \exp(A(T) + B(T)' X_t)$
- Or the YTM are linear function of X_t:
- $y(X_t . T) = -\frac{A(T)+B(T)' X_t}{T}$

Vasicek Model

- Discrete Vasicek model assume short rate is an AR(1) Gaussian Process :
- $(r_{t+1} - r_t) = \theta(\mu - r_t) + \sigma\epsilon_{t+1}\ where\ \epsilon_{t+1} \sim N(0,1)$
- Short term rate is mean reverting normally distributed random variable with constant volatility and possibility of negative short rate.
- $dr = \theta(\mu - r_t)dt + \sigma\, dW_t$
- Comparing to affine Model :
- $X_t = r_t\,, M = \mu\ , \Sigma = \sigma\,, \Theta = \theta\ and\ S(X_t) = 1\ (S_{10} = 1\ and S_{11} = 0)$

Vasicek Model

- Then :
- $P(X_t, T) = \exp(A(T) + B(T)' X_t)$

- Where :
- $A(T) = -\left[\left(\mu - \frac{\sigma^2}{2\theta^2}\right)(B(T) + T) + \frac{\sigma^2}{4\theta} B(T)^2\right]$
- $B(T) = -\frac{1 - \exp(-\theta T)}{\theta}$

Cox–Ingersoll–Ross (CIR) Model

- Discrete CIR Model assumes short rate process is an AR1 process with non-constant square root volatility:
- $(r_{t+1} - r_t) = \theta\,(\mu - r_t) + \sigma\sqrt{r_t}\varepsilon_{t+1}$ where $\varepsilon_{t+1} \sim N(0,1)$
- Here the short rate is r_{t+1} only condition on r_t is normally distributed
- Continuous time version which is called Square Root Process:
- $dr_t = \theta\,(\mu - r_t)dt + \sigma\sqrt{r_t}\,dW_t$ for $2\theta\mu > \sigma^2$
- If $2\theta\mu > \sigma^2$ guarnante $r > 0$ and r has a non-central χ^2
- Comparing to affine model we have :
- $X_t = r_t, M = \mu\ and\ \Sigma = \sigma\ and\ S(X_t) = X_t$: ($S_{10} = 0$ and $S_{11} = 1$)

Cox–Ingersoll–Ross (CIR) Model

- Since we have an affine model we know :
- $p(r_t, T) = exp(A(T) + B(T)\, r_t)$ where :
- $B(T) = -\frac{2(exp(hT)-1)}{g}$ and $A(T) = \left[\left(\frac{2\theta\mu}{\sigma^2}\right)\left(\frac{(h+\theta)T}{2} + ln\frac{2h}{g}\right)\right]$ where
- $h = \sqrt{\theta^2 + 2\sigma^2}$
- $g = 2h + (h+\theta)(exp(hT) - 1)$

CIR vs. Vasicek

- Issues with Both Models
  - Both Are one Factor model, Implying Bond Prices are 100% correlated across all maturities.
  - Very hard and close to impossible to match the real time market yield data
- CIR advantage to Vasicek :
  - General vs. Partial Equilibrium model
  - Non-negative short term rate
  - Non-constant volatility for short rate(fat tail distribution)
  - Most shape of yield cure is achievable
- Vasicek advantage vs. CIR:
  - Possibility of negative short term rate (JOB, SCB, ECB)
  - Conditional an Unconditional Normal distribution of the short rate
  - Easy to Manipulate and simulate and extending to Multifactor.

Nelson Siegle Model (NS)

- Non-Dymanic NS :
- $y(T) = x_1 + x_2\left(\frac{1-e^{-\theta T}}{\theta T}\right) + x_3\left(\frac{1-e^{-\theta T}}{\theta T} - e^{-\theta T}\right)$
- Where x_1 , x_2 and x_3 are the three factor (Level, Slope and Curvature), and log bond price or yield curve is affine model of these three factor
- $p(X_t, T) = exp(A(T) + B(T)' X_t)$
- $A(T) = 0 \; and \; B(T) = -\begin{bmatrix} 1 \\ \left(\frac{1-e^{-\theta T}}{\theta T}\right) \\ \left(\frac{1-e^{-\theta T}}{\theta T} - e^{-\theta T}\right) \end{bmatrix}$

Nelson Siegle Model (NS)

- What will happen if we choose some stochastic dynamic for our three factor NS model (DNS) (i.e. the three factors follow some stochastic process)
- Sine $A(T) = 0$, it is impossible to preclude arbitrage by DNS (Filipovic)
- If we relax $A(T) = 0$, What type of model would be consistent with both NS and Arbitrage Free Dynamic Nelson Siegle (AFDNS). (i.e. We get the yield curve to have the same representation as in NS and as we eliminate the dynamic behavior we get NS)

Nelson Siegle Model (NS)

- Christensen, Diebold and Rudebusch : Dynamic of Factor :
- $$\begin{bmatrix} dx_{1t} \\ dx_{2t} \\ dx_{3t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \theta & -\theta \\ 0 & 0 & \theta \end{bmatrix} \begin{bmatrix} -x_{1t} \\ -x_{2t} \\ -x_{3t} \end{bmatrix} + \Sigma \begin{bmatrix} dw_{1t} \\ dw_{2t} \\ dw_{3t} \end{bmatrix}$$
- Which implies that X_t is affine model (comparing to affine model)
- $$X_t = \begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \end{bmatrix}, \theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \theta & -\theta \\ 0 & 0 & \theta \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, S = I_3, M = 0$$
- Now assuming short rate is affine in X_t, $r_t = \alpha + \beta^t X_t$
- $p(X_t, T) = exp(A(T) + B(T)' X_t)$

Nelson Siegle Model (NS)

- Then we can write
- $p(X_t, T) = exp(A(T) + B(T)' \, X_t)$
- Where
- $A(T) = A(T, \Sigma) and \; B(T) = -\begin{bmatrix} 1 \\ \left(\frac{1-e^{-\theta T}}{\theta T}\right) \\ \left(\frac{1-e^{-\theta T}}{\theta T} - e^{-\theta T}\right) \end{bmatrix}$
- We can show $A(T, \Sigma)$ is only function of maturity and Σ and not factor X_t and when $\Sigma = 0$ then we get NS model (non-dynamic)