Hedging with Automatic Liquidation and Leverage Selection on Bitcoin Futures*

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Abstract

Bitcoin derivatives positions are maintained with a self-selected margin, often too low to avoid liquidation by the exchange, without notice, during periods of excessive volatility. Recently, the size and scale of such liquidations precipitated extreme discontent among traders and numerous lawsuits against exchanges. Clearly, hedgers of bitcoin should account for the possibility of automatic liquidation. That is the mathematical and operational problem that we address, deriving a semi-closed form for an optimal hedging strategy with dual objectives – to minimize both the variance of the hedged portfolio and the probability of liquidation due to insufficient collateral. An empirical analysis based on minute-level data compares the performance of major direct and inverse bitcoin hedging instruments traded on five major exchanges. The products have markedly different speculative trading scores according to new metrics introduced here. Instruments having similar hedging effectiveness can exhibit marked differences in speculative activity. Inverse perpetuals offer greater effectiveness than direct perpetuals, which also exhibit more speculation. We model hedgers with different levels of loss aversion that select their own level of leverage and collateral in the margin account. By following the optimal strategy, the hedger can reduce the liquidation probability to less than 1% and control leverage to a reasonable level, mostly below 5X.

Keywords: Finance; Margin; Operational Risk; Perpetual Swap

JEL Classification: G32; G11

1 Introduction

For decades, the multi-currency revenue streams of international corporations have driven a huge demand for hedging foreign exchange rate risks (Credit-Suisse, 2022).[1] Now, in just the same way, there are large metaverse players and companies emerging whose profits are tied to the market price of bitcoin but expenditures are in local fiat currency. During the last few years these market participants have combined

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[1]In addition to exchange rate risk, firms are exposed to commodity price risk, supply and demand uncertainty, and operational risk, among other risks. This generates strong demand for hedging using financial derivatives (e.g. futures and options); see Caldentey and Haugh (2009), Zhao and Huchzermeier (2017) and Liu and Wang (2019).

to create a noticeable, growing demand for hedging the risks of revenue streams and investments that are linked to the price of bitcoin. These crypto market participants include:

(1) Blockchain miners, who pay operational expenses such as electricity bills in a domestic currency but their income from transaction fees and block rewards is in bitcoin. So unless a miner immediately exchanges each mining reward for fiat, their revenue uncertainty will be dominated by future fluctuations in the price of bitcoin in their domestic currency;

(2) Cryptocurrency derivatives exchanges that list bitcoin futures and options earn transaction fees in bitcoin. Most exchanges also operate as their own clearing house and so keep custody of the total balance of bitcoin-margined accounts of their clients. As daily trading volumes increase so does their bitcoin fee income and the total margin account balance;

(3) Market makers of bitcoin-margined derivatives make profits which depend on bid-ask spreads and these are also denominated in bitcoin. However, like the exchanges their operating costs will also be in some fiat currency;

(4) Brokers and custodians for bitcoin and other coins and tokens, whose profits are determined by the volume of supply and demand and the prices of these crypto assets. Hedging contracts are only widely available for bitcoin and ether. However, the price of most coins and tokens is highly correlated with bitcoin's price so a hedge using bitcoin futures provides a partial hedge for their entire portfolio;

(5) Ferko et al. (2021) identify a growing number of institutional investors holding bitcoin in their asset portfolios. But the price of bitcoin is becoming more highly correlated with equity prices, so instead of regarding bitcoin as a diversification tool for equity risk, these investors may now choose to hedge bitcoin risk. This is supported by the recent wallet analysis conducted by IntotheBlock that reveals a significant shift of bitcoin being held by long-term investors rather than short-term speculators, with around half of all bitcoins being held for one year or more.

The on-chain activities of each type of participant can be analysed using network flows between wallets, such as in Makarov and Schoar (2021). Regarding the first group, these authors show that miners are only responsible for a very small fraction of real on-chain transaction volume because almost all of them have their rewards sent directly to the world's largest crypto exchange, Binance, or another centralised exchange where trades are only recorded off-chain. The trades may be for either speculation or hedging – unfortunately it is not possible to discern which type of trades they make without level 3 order book data – and the spot exchange Coinbase is the only one that provides this. Nevertheless, Binance specifically promotes itself as the number one hedging venue, so one expects that a large portion of trading volume on its futures platform is for hedging purposes.

Regarding the exchanges, and the market makers and broker/custody services that accompany them, Makarov and Schoar (2021) show that transfers to and from cryptocurrency exchange wallets are responsible for 40% of real volume, and at the end of 2020 exchanges held approximately 5.5 million coins, representing about one-third of all bitcoins in active circulation. At the time of writing Binance alone held \$5.5 billion worth of bitcoin in its cold wallet (see report from Cointelegraph). Finally, another

20% of real volume comes from transfers made by individuals or institutions, who together hold on to about 8.5 million coins. Individual investors in the early Bitcoin chain include manufacturers of mining equipment, founders of other blockchains and owners of crypto exchanges. And the 5% of all bitcoins that were mined by its anonymous founder, Satoshi Nakamoto, have not moved wallets for more than decade – so if this individual or group is still active, there is considerable motivation for them to hedge bitcoin's price risk.

The cryptocurrency bitcoin is highly volatile. Since December 2020 its 30-day implied volatility index has ranged between 80% and 150%, according to CryptoCompare. This extremely high price uncertainty imposes tremendous risk to the crypto market participants described above, and a natural need to hedge using bitcoin derivatives.[2] It is common to hedge long (short) spot price risk by buying (selling) futures contracts on the same asset or one that is highly correlated, in a quantity that minimizes the variance of the hedged portfolio. The lower the correlation and greater the mismatch in maturities or underlying, the more the optimal hedge ratio deviates from 1:1. There is a huge literature on hedging with futures examining the use of commodities, currencies, interest rates, stocks and index futures as effective hedges of spot price risk.[3] But at the time of writing there is little in-depth research into the role that bitcoin futures can play to hedge bitcoin spot price risk. To our knowledge only two papers exist, both on the standard minimum-variance problem, which ignores a type of operational risk that is unique to crypto derivatives exchanges.[4]

The hedging problem is actually much richer and more challenging for bitcoin than it is for other assets. First, bitcoin markets are highly segmented. Bitcoin futures were introduced in December 2017 and by May 2021 their monthly trading volume had exceeded $5 trillion notional, with various types of contracts being traded on multiple exchanges. Therefore, a hedging study needs to highlight the important differences between the operations of the main trading venues and characterize the salient features of the possible hedging products. The best choice of hedging instrument, and the exchange on which to trade it, are irrelevant for a traditional hedging study – because there is typically only one futures contract available to match the underlying exposure, being traded on just one exchange. But for bitcoin there are several exchanges offering perpetual and fixed-expiry futures, as well as direct and inverse products – contracts that do not even exist in traditional markets. Our work is the first to consider the alternative benefits of each type of product for hedging purposes, as well as the relative merits of trading such contracts on different exchanges.

There is abundant research on efficient margin mechanisms for traditional futures platforms, including

[2] It is well documented that value-maximizing firms engage in hedging to reduce profit variability; see Chod et al. (2010). A prominent example is jet fuel hedging by the airline industry, and empirical studies confirm the positive correlation between the extent of hedging and the value of an airline firm; see Carter et al. (2006).

[3] See Ederington (1979) and Figlewski (1984) for classical contributions, also Laws and Thompson (2005), Aragó and Salvador (2011), Wang et al. (2015), Mellios et al. (2016), and many others – a useful survey is given in Lien and Tse (2002).

[4] Deng et al. (2020) show that OKEx quarterly inverse contracts are excellent hedging tools for the spot price risk on Bitfinex and OKEx, and are superior to CME standard futures in terms of hedge effectiveness; and Alexander et al. (2020) show that BitMEX perpetuals achieve outstanding hedge effectiveness against the spot price risk on Bitstamp, Coinbase, and Kraken – however, the main focus of Alexander et al. (2020) is price discovery, not hedging.

margin calls and default waterfalls, because the topic is essential for ensuring the stability and integrity of the market (see, for example, Longin (1999) and Alexander et al. (2019)). However, almost all bitcoin futures have no margin calls, the one exception being positions on the Chicago Mercantile Exchange (CME). However, the CME closes every weekend whereas bitcoin spot markets trade continuously, and so should be hedged 24/7, 365 days a year. The exchanges which offer 24/7 hedging have lax know-your-customer (KYC) practices, and operate under self-regulation which allows them to act as their own central clearing counterparty (CCP). These non-KYC exchanges have a matching engine which triggers a liquidation immediately, as soon as the margin account falls below the maintenance level. As a result, it is the traders' own responsibility to monitor their margin accounts and add collateral, if necessary, in order to prevent automatic liquidation without a margin call or any other notice.[5]

As argued in Deep (2002), firms that use leveraged derivatives as hedging instruments should account for the additional risk arising from the need to meet a margin call. This risk is significantly amplified in the case of bitcoin futures, due to the replacement of margin calls by automatic liquidation. Consistent with the definition in Basel II Framework, liquidation can be seen as a type of operational risk, since it is triggered by an inadequate margin set by the hedger (i.e. the failure of an internal decision process) and by an extreme price movement (i.e. the occurrence of an unexpected external event).[6] And when the hedger selects her own maintenance margin and in so doing influences the operational risk of liquidation, the hedging problem alters. Instead of the usual objective to minimize a measure of market risk, the decision becomes one of simultaneously reducing market and operational risk. Our paper is the first deep dive into how the hedger's choice of margin level affects the optimal hedging problem, and the first in the literature to incorporate the operational risk of liquidations into the futures hedging problem.

We have several motivations to incorporate a loss aversion to liquidations into the bitcoin hedging problem. A unique feature of hedging with bitcoin futures is that traders can choose their own level of leverage, and this can be astonishingly high, up to 100X or even higher on some 24/7 non-KYC exchanges such as Binance or Bybit. The initial margin rate is the reciprocal of the leverage level and the maintenance margin rate is around 50% of the initial margin rate.[7] Liquidation is automatically triggered when the margin falls below the maintenance level. When a hedger's position on bitcoin futures is liquidated, it is unlikely that she is financially capable of re-entering the same futures contract, since the initial margin is often double of the maintenance margin and liquidation means she cannot even meet the margin maintenance requirement. Secondly, liquidity or even technical issues may also prevent traders from re-entering the markets in extreme circumstances. For example, on 19 May 2021, the bitcoin

[5]More precisely, the matching engine instantly triggers a liquidation as soon as the bitcoin price falls below a 'liquidation price' which is calculated using a highly complex formula, but is almost exactly equal to the 'fair mark price' (which is simple to calculate) less the collateral in the account. More information about fair mark prices is given in Appendix F.

[6]On a related note, Jarrow (2008) provides an economic and mathematical characterization of operational risk, while Chod et al. (2010) investigate the relationship between operational risk and financial hedging.

[7]For example, Binance BTCUSDT perpetuals set the maintenance margin rate to 0.4% and the initial margin rate to 0.8% for positions with notional value between 0 and 50,000 USDT. Again, the CME margins are in stark contrast to the methods used by the 24/7 non-KYC exchanges. The CME initial margin rates are around 50%, equivalent to a maximum of only 2X leverage.

spot price dropped from \$38,000 to \$30,000 in only 30 minutes, between 12:40 Coordinated Universal Time (UTC) and 13:10 UTC (that is over 20% decline in 30 minutes). The entire Binance futures trading platform froze shortly before 13:30 UTC and it only reopened once the bitcoin price had stabilised around \$37,000.[8] Traders were unable to make any orders for nearly two hours, so if our hedger's position was liquidated before or during the closure that day, she would not have been able to re-enter at the same price and would have lost all the collateral in the margin account. Section 2.4 provides further justification, based on empirical results, for including liquidation loss aversion in the hedging problem.

The above discussion justifies why any assessment of the hedging effectiveness of practical hedging strategies of bitcoin futures must take account of the hedger's ability to select their own leverage. And we analyse the first unique feature of hedging the bitcoin spot price, i.e. the segmentation of bitcoin derivatives markets into different trading platforms and products, by proposing new types of speculation metric which take account of the hedger's own choice of leverage. We explain why these metrics are more appropriate for bitcoin markets than the traditional ratio of trading volume to open interest, and use historical data on liquidations to characterise the speculative trades on different contracts. Then we investigate how the new speculation metrics relate to the hedge effectiveness of each contract, under the optimal strategy derived in this paper.

Our main methodological contribution is to provide a semi-closed solution to a hedging problem that is new to the finance literature, incorporating the hedgers' ability to reduce costs by choosing to lower the margin on the one hand, and their aversion to loss through mitigating the risk of liquidation on the other hand. These are opposing constraints, because the lower the margin constraint the higher the probability of liquidation and the subsequent failure of the hedge. Thus, the hedger seeks an optimal hedging strategy with dual objectives – to minimize both the variance of the hedged portfolio and the liquidation probability. We apply the extreme value theorem to obtain the optimal strategy and show that the margin constraint and liquidation loss aversion both play a key role in determining its characteristics.

Our empirical study uses minute-level data to examine how the threat of liquidation, which depends on the margin level, affects the optimal hedging problem when the hedger is averse to liquidations. This way, we investigate three important topics in risk management – hedge effectiveness, liquidation probability and leverage – under the optimal strategy. Analysing how they depend on the choice of hedging instrument, collateral, loss aversion, leverage and duration of the hedge, our findings include:

(1) The size of the hedge position decreases with both a tighter margin constraint and a greater liquidation loss aversion, justifying the incorporation of these two features in the analysis;

(2) The optimal strategy yields a highly efficient and robust hedge with effectiveness exceeding 90% (sometimes almost 99%) provided the margin level is not too low;

(3) By following the optimal strategy, the hedger is able to reduce the liquidation probability substantially – to less than 1% in most scenarios – and the optimal leverage is only 2X in some scenarios and less than 5X overall;

[8]See this article Binance Froze when Bitcoin Crashed by the Wall Street Journal and Binance's Insurance Fund.

(4) Both inverse and direct products are highly effective hedging instruments, but inverse perpetuals have a slight advantage over their direct counterparts;

(5) In terms of the hedge effectiveness there are no significant differences between the major 24/7 non-KYC exchanges. However, there is a heterogeneous degree of speculative activity, with Deribit having the lowest according to every metric whereas Bybit, OKEx and Binance each exhibit the most speculation, depending on the metric used.

In the following: Section 2 provides the background for this study, describing the trading characteristics of different types of bitcoin futures on various exchanges and introducing new measures of speculation which are more relevant to bitcoin exchanges; Section 3 formulates the optimal hedging problem under both leverage selection (i.e. margin constraint) and loss aversion (to auto-liquidation) and derives the optimal hedging strategy; Section 4 presents empirical results on parameter values, hedge effectiveness and probability of liquidation under the optimal strategy; and Section 5 summarizes and concludes. The Appendix includes proofs of theoretical results, a sensitivity analysis of the optimal strategy to the parameters of our model, and several subsidiary empirical results. Matlab code is available from the authors on reasonable request.

2 Features of Bitcoin Futures Contracts

2.1 Types of Bitcoin Futures

There are two distinctive types of bitcoin futures contracts: (1) standard futures that have a fixed expiry date and a regular schedule for new issues; and (2) 'perpetual futures' or just *perpetuals*, because they have no expiry date. Perpetual futures are an innovative financial product, so far unique to cryptocurrency markets. At the time of writing, standard bitcoin futures are traded on the Chicago Mercantile Exchange (CME), and numerous other 24/7 exchanges. On the largest of these (by trading volume) quarterly standard contracts are traded on BitMEX and Bybit and weekly standard contracts are traded on Huobi, Kraken and OKEx.

Apart from their term, a key difference between the two types of contracts is that bitcoin standard futures are denominated, margined and settled in USD (domestic currency) with bitcoin as the underlying asset (foreign currency), while the majority of bitcoin perpetual futures contracts are margined and settled either in bitcoin (BTC) or tether (USDT). The two dominant types of bitcoin-dollar futures contracts are (1) *direct* perpetuals which are like standard futures but margined and settled in USDT; and (2) *inverse* perpetuals which are margined and settled in BTC. For instance, one CME bitcoin standard futures contract has a notional value of 5 bitcoins (quoted, margined and settled in USD); one Binance BTCUSDT direct perpetuals contract has a notional value of 1 bitcoin (quoted, margined and settled in USDT); and one Binance BTCUSD inverse perpetuals contract has a notional value of 100 USD (quoted in USD but margined and settled in BTC).

2.2 Choice of Futures Contracts in Hedging

Given that there exist various types of bitcoin futures contracts, naturally we ask whether a hedger should use standard futures or perpetuals, and within each type whether she should use USD inverse or USDT direct products, to hedge the spot price risk of bitcoin. In terms of availability, bitcoin spot is traded continuously – the markets do not even close on religious holidays – but the CME closes at weekends and holidays. Therefore, we only consider the 24/7 exchanges. Considering liquidity there is not a huge difference between trading volumes on perpetuals compared with standard futures, on the 24/7 non-KYC derivatives exchanges like BitMEX, OKEx, Binance, Huobi or Bybit. Then, comparing hedging costs for standard futures versus perpetuals, the latter are hardly influenced by the swings between backwardation and contango that can induce a high degree of roll-cost uncertainty into hedging with standard futures contracts. This is because the price of the perpetual futures is kept very close to the underlying price by the funding rate mechanism. Because of this mechanism the basis risk is very small, as the perpetual contract price is continually re-aligned with the spot price. The funding rates are small, typically consisting of a very small base rate, such as one basis point, from the long counterparty to the short counterparty, plus a premium of around three to five basis points which is paid by the long to the short if the perpetual price is greater than the spot, or paid by the short to the long if the spot price is greater than the price of the perpetual.

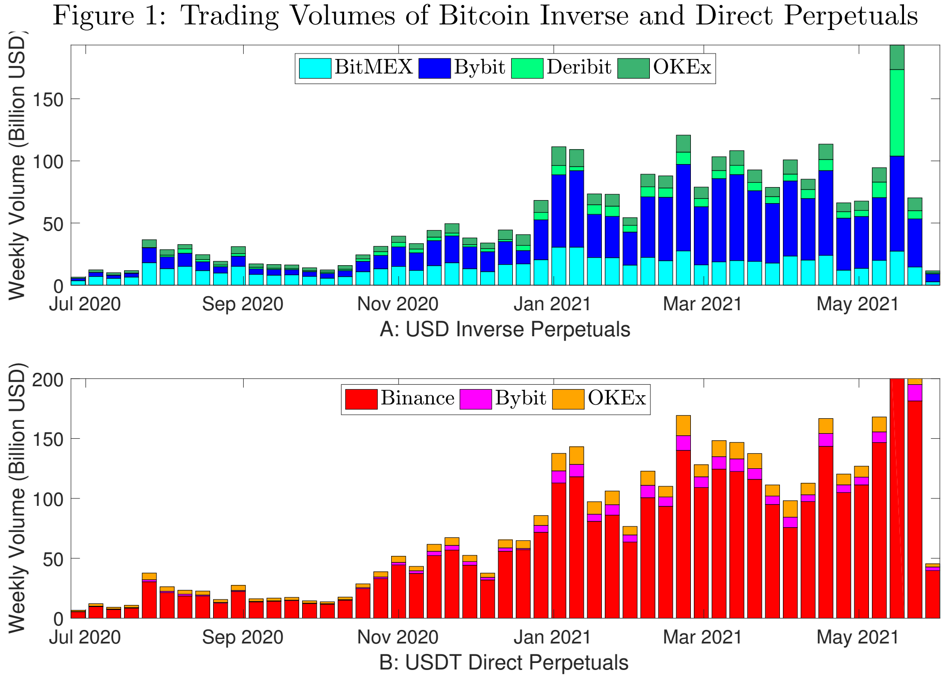
We have both theoretical and practical reasons to assume hedgers favor perpetuals over standard futures. First, hedgers can use such contracts to match their preferred hedge horizon, of any length *exactly*. Second, an immediate benefit of perpetuals having no expiry is that there is no need to consider rolling the hedge, which can be a highly technical issue in hedging.[9] Third, 24/7 exchanges only offer one USD inverse (or USDT direct) perpetual futures contract, but several active fixed-expiry futures contracts. If we otherwise chose fixed-expiry futures, we would need to debate on which term to use or even on how to construct a synthetic contract with a constant maturity using several traded contracts. The open interest and trading volumes of standard bitcoin futures vary considerably over time, so it may be difficult to arrive at a commonly-accepted preferment of one contract over another. Fourth, the perpetual contracts are very much more liquid than any single fixed-expiry futures. For instance, the weekly trading volume on the Deribit perpetual is between three and eight times the total volume on all their standard futures combined. And lastly, a perpetual contract is more cost efficient than a fixed-expiry contract, since no rolling means there is only one entry and one exit in the hedge, moreover the small funding payments often balance out over the hedge horizon as they fluctuate between positive and negative cash flows.

[9] When using standard futures in hedging, one often closes the current contract *one week* prior to its expiry and rolls to the next closest contract. For example, Deng et al. (2020) use OKEx quarterly futures and indeed follow this one-week rule.

2.3 Data Description

At the time of writing, the three exchanges trading the greatest volumes of bitcoin USD inverse perpetuals are BitMEX, Bybit and OKEx, and the top three trading the USDT direct perpetuals are Binance, OKEx and Bybit.[10] Their weekly trading volumes between 1 July 2020 and 31 May 2021 are plotted in Figure 1. The trading volume of the inverse perpetual on Deribit is lower than the others but we include it because around 90% of the bitcoin options market is traded on Deribit where professional option traders use the perpetual for delta hedging. All of these exchanges provide online trading platforms that operate continuously and we refer readers to their websites for full contract specifications. For the bitcoin spot market, we choose Coinbase, which is consistently ranked one of the largest and the most trusted bitcoin exchanges. In earlier versions of this paper we also considered Bitstamp and Gemini, in addition to Coinbase. Those results are not included because the conclusions are the same as those we draw here, from hedging the Coinbase spot price, and instead the Appendix contains other more informative supporting results.

Figure 1: Trading Volumes of Bitcoin Inverse and Direct Perpetuals



Panel A plots the weekly total trading volumes of bitcoin inverse perpetuals on BitMEX, Bybit, Deribit and OKEx. Panel B plots that of direct perpetuals on Binance, Bybit and OKEx. For both panels, data are reported in billion USD units and the sample is from 1 July 2020 to 31 May 2021.

We retrieve data on bitcoin spot and perpetuals from these exchanges at the minute-level (one-min) frequency using the API (application programming interface) provided by CoinAPI. Since the bitcoin

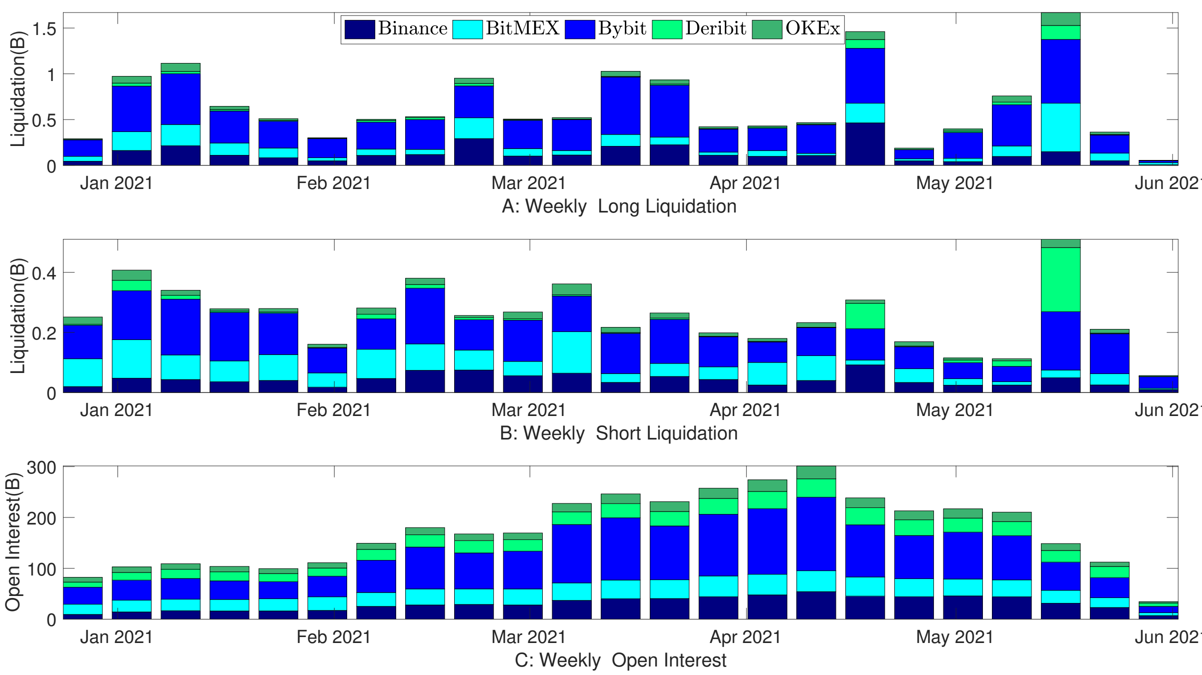
[10] It should be noted that these 24/7 non-KYC derivatives exchanges may engage in wash trading practices that artificially inflate volumes and that new exchanges can quickly rise in volume-based rankings. For instance, Binance's monthly trading volume in November 2020 was up 132% at $405 billion since October, and since then it grew to almost $2.5 trillion in May 2021, representing around 50% of the total volumes on all derivatives exchanges. See CryptoCompare Exchange Review June 2021.

price is highly volatile traders may monitor their perpetuals positions very frequently, which justifies the use of such high-frequency data. In the hedging study all perpetuals except the BitMEX USD inverse product use data from 1 July 2020 to 31 May 2021, containing 482,400 entries. This is because the USDT direct perpetuals data are only available starting from 1 July 2020. Both the Coinbase spot and BitMEX inverse perpetuals price data run from 1 November 2017 to 31 May 2021. This longer data period is useful to obtain more reliable estimates of model parameters in our empirical research.

2.4 Margin and Liquidation Mechanisms

In the previous literature on bitcoin futures hedge, margins are either set to zero or not considered at all – see Baur and Dimpfl (2019), Alexander et al. (2020) and many others. However, we argue that the role of margin requirements is particularly important when trading bitcoin futures, because leverage is allowed to be exceptionally high on the 24/7 non-KYC exchanges, despite the already very high volatility of bitcoin price. Furthermore, as previous described in detail, all these exchanges automatically liquidate a position if the client's collateral in the margin account drops below the required maintenance level. Indeed, according to CryptoQuant, almost \$80bn of positions on centralised exchanges were liquidated during 2021, corresponding to an average of over \$200m per day.

Figure 2: Liquidations of Bitcoin USD Inverse Perpetuals



Panels A and B plot weekly total long and short liquidations in billion USD of bitcoin inverse perpetuals traded on Binance, BitMEX, Bybit, Deribit and OKEx. Panel C plots weekly total open interest in billion USD on the same five exchanges. The relative weights of the total liquidations are 19.1% (Binance), 19.2% (BitMEX), 51.3% (Bybit), 4.6% (Deribit) and 5.8% (OKEx). The data are retrieved from coinalyze.net, spanning from 1 January 2021 to 31 May 2021.

Figure 2 depicts weekly time series of long and short liquidations in USD, and the open interest of bitcoin USD inverse perpetuals from 1 January 2021 to 31 May 2021. The five exchanges Binance, BitMEX, Bybit, Deribit and OKEx attract large trading volumes, with average weekly open interest of 30.68, 29.56, 75.04, 23.42 and 14.64 billion USD, respectively. The Bybit exchange maintains the largest unsettled open perpetuals positions. Upon close inspections of Panels A and B in Figure 2, we find that long liquidations are more significant than short liquidations, and Bybit has the highest volume of liquidations. To be specific, the average daily volume of long liquidations are 130.76, 113.24, 349.70, 22.19 and 37.11 million USD on Binance, BitMEX, Bybit, Deribit and OKEx correspondingly, while the average daily volume of short liquidations are 42.42, 59.88, 116.37, 19.42 and 15.93 million USD accordingly. There is an exceptionally large short liquidation volume on Deribit during the third week of May 2021, when the bitcoin price crashed and many delta hedges were liquidated.

The observations on Figure 2 testify to the utmost importance of considering margin and liquidation mechanisms when trading bitcoin futures. For example, on 19 May 2021 a total of \$8.6 billion of liquidations were reported over all products and exchanges.[11] For the instruments we have selected in this study, only \$339 million liquidations were reported on 19 May, but there is clear evidence that the largest exchange, Binance, which traded around \$100 billion notional on bitcoin futures that day, reported inaccurate data throughout May, which is the last month of our sample.[12] This is the main reason why Bybit accounts for over 50% of the liquidations within our dataset, Binance and BitMEX account for about 20% each, and Deribit and OKEx each report about 5% of all the liquidations in our sample.

We have also conducted a simulation study on the historical liquidation probability using data from Bybit bitcoin direct perpetuals. Table C.1 in Appendix presents detailed results. For positions with the highest leverage (100X), more than 95% of all 1-day positions end up liquidated; that is an enormous amount of risk to traders for such a short horizon. The probability of liquidation becomes very high when a position is held for more than five days. It is lower for long positions than short positions, but hedgers that hold bitcoin require a short position to hedge. Even taking a 5X leverage short position for 15 days results in more than 10% chance that it will be liquidated, on average, and the probability of liquidation will be much grater during volatile periods – which are also when hedging becomes more important. Therefore, our subsequent investigation into hedging with bitcoin futures always takes account of the impact of margin requirements, and only considers hedge horizons of up to five days – because long-term holders of bitcoin would need to re-balance their margin account within that time-frame, to reduce the probability of forced liquidation.

[11] The exchanges report liquidations to Bybt. This is publicly available at the minute-level. See Bybt's liquidations data.

[12] On 18 April 2021 when the bitcoin price fell by 8.7% between 00:00 and 04:00 UTC, long positions worth over \$1 billion were liquidated on Binance. Shortly after, they ceased reporting accurate data. See this detailed article and other press reports.

2.5 Speculation Metrics

Perpetuals have a small notional value, allow very high leverage and are traded on exchanges with few regulations, making them more accessible to speculative investors.[13] Following Garcia et al. (1986), the standard speculative index ($\mathcal{SI}$) is defined as:

$$\mathcal{SI} := \frac{\text{Trading Volume}}{\text{Open Interest}}. \tag{1}$$

The intuition behind this metric is that speculators enter and exit the market within a short period of time when they explore opportunities for profits, in contrast to hedgers who often hold their futures position for a relatively longer period. Therefore, speculations will move trading volumes up but have little impact on open interest; as such, a large $\mathcal{SI}$ indicates a high level of speculation.

If we want to characterize exchanges by speculation, we must consider that their trading platforms offer a wide variety of fee structures to attract different types of professional (larger, more informed) and retail (smaller, less informed) traders. Moreover, acting as CCPs, exchanges employ a vast array of margining mechanisms and liquidation protocols. For these reasons, we argue that the speculative index $\mathcal{SI}$ defined in (1) is not appropriate, because it captures neither leverage nor liquidations, both of which are crucial factors in trading bitcoin futures. It may also be distorted by an activity that is unique to crypto exchanges, i.e. offering negative fees. On Bybit, a market maker earns a rebate (negative fee) for providing liquidity. Rebates increase if she trades small amounts with herself either side of the ticker – a practice known as wash trading. Exchanges encourage this artificial form of liquidity to boost their volume data, and hence rise up the exchange rankings of data providers such as CryptoCompare.

Trading and liquidation volumes and OHLC (open, high, low and close) price data are available every 4 (non-overlapping) hours from data providers such as Bybt. Using these data, Table 1 reports the summary statistics for the products of interest here including trading volumes, open interest and forced (long and short) liquidations. Between 1 January 2021 and 31 May 2021 Bybit (resp. Binance) had the largest trading volumes, open interest and liquidations on USD inverse perpetuals (resp. USDT direct perpetuals). We also report the mean and median of the speculative index $\mathcal{SI}$ defined in (1). Binance attracted the most speculations on both perpetuals. OKEx scored highly for speculation on USDT direct perpetuals, but had a much lower trading volume and open interest. The least speculative exchange is Deribit, which only offers USD inverse perpetuals. This is probably because Deribit is the largest exchange for trading bitcoin options, and the perpetuals here are mainly used for delta hedging. Further results in Appendix examine the dynamic evolution of $\mathcal{SI}$ (see Figure G.1).

We now propose some new measures that we believe are more appropriate to capture speculative trading for the bitcoin futures market. The problem with using $\mathcal{SI}$ for bitcoin futures is that it ignores differences in liquidations. For example, Table 1 seems to indicate that Bybit is much less speculative than

[13] For comparison, one CME standard futures contract has a notional value of 5 bitcoin and the initial margin is about 50%, not to mention the heavy regulations imposed by CME and SEC.

Table 1: Summary Statistics of Trading Volumes, Open Interest and Liquidations of Perpetuals

		USD Inverse Perpetuals					USDT Direct Perpetuals		
		Binance	BitMEX	Bybit	Deribit	OKEx	Binance	Bybit	OKEx
Mean	Volume(B)	1.03	0.49	1.20	0.20	0.25	2.75	0.22	0.29
	OI(B)	0.76	0.73	1.85	0.58	0.36	1.54	0.29	0.14
	Short Liq.(M)	1.05	1.48	2.87	0.48	0.39	7.90	1.50	1.54
	Long Liq.(M)	3.23	2.80	8.64	0.55	0.92	13.92	2.05	1.23
	Total Liq.(M)	4.28	4.28	11.51	1.03	1.31	21.83	3.55	2.77
	$\mathcal{SI}$	1.54	0.71	0.77	0.38	0.76	1.91	0.80	2.11
Median	Volume(B)	0.88	0.40	0.95	0.15	0.20	2.31	0.17	0.23
	OI(B)	0.70	0.75	1.86	0.58	0.34	1.61	0.30	0.14
	Short Liq.(M)	0.59	0.32	1.69	0.01	0.14	3.93	0.72	0.50
	Long Liq.(M)	1.23	0.37	4.08	0.02	0.16	5.08	1.05	0.49
	Total Liq.(M)	2.07	1.33	6.72	0.05	0.52	11.67	2.34	1.45
	$\mathcal{SI}$	1.24	0.57	0.58	0.28	0.60	1.54	0.62	1.66

This table reports the 4-hour mean and median of trading volumes (in billion USD), open interest (O.I. in billion USD), short liquidations (in million USD), long liquidations (in million USD) and total liquidations (in million USD) of USD inverse and USDT direct bitcoin perpetuals across Binance, BitMEX, Bybit, Deribit and OKEx. The time period is from 1 January 2021 to 31 May 2021 with data acquired from Coinalyze.

Binance – yet it has the largest volume of liquidations, on average. Well, this is not surprizing because it also has the largest open interest. Thus, in order to characterise exchanges by their speculative activity, we want a new measure of speculation which accounts for liquidations as a proportion of open interest. This motivates our definition of a *liquidation index* as the ratio of short (or long or total) liquidation volume to open interest. That is, we define three liquidation ($\mathcal{LIQ}$) indices, as:

$$\mathcal{LIQ}_{short} = \frac{\text{Short Liquidation Volume}}{\text{Open Interest}} \quad \text{and} \quad \mathcal{LIQ}_{long} = \frac{\text{Long Liquidation Volume}}{\text{Open Interest}}, \tag{2}$$

so that $\mathcal{LIQ}$ = Total Liquidation Volume/Open Interest is the sum of the two sub-indices (2).

Now suppose that speculators employ similar collateral monitoring processes to hedger, but they select a greater leverage. In this case trades made by speculators are more likely to be liquidated than hedging trades. So we introduce another index, which now takes leverage into account. To this end we estimate the leverage taken at the initiation of a position that is later liquidated.[14] Our estimation assumes a trader takes a long (resp. short) position of one *inverse* perpetual contract at time t_1 using a leverage of λ^i_{long} (resp. λ^i_{short}) when the perpetual price is F_{t_1}. Next, suppose the trader has her position liquidated at a later time t_2 at a perpetual price of F_{t_2}.[15] Set $\lambda^i = \{\lambda^i_{long}, \lambda^i_{short}\}$, and set $\omega^i = 1$ for long and $\omega^i = -1$ for short, and let m_0 be the maintenance margin rate for the contract. Recalling that the initial

[14] Only transaction account data records the exact leverage of each position but these data are not disclosed by exchanges.
[15] So, for the long position we must have $F_{t_1} > F_{t_2}$ and for the short position we have $F_{t_1} < F_{t_2}$.

margin rate is the reciprocal of the leverage, we obtain

$$\frac{1}{\lambda^i}\frac{1}{F_{t_1}} - \frac{m_0}{F_{t_2}} = \omega^i\left(\frac{1}{F_{t_2}} - \frac{1}{F_{t_1}}\right),$$

i.e. 'Initial Margin' - 'Maintenance Margin' = 'Trading Loss'. From the above we derive estimates for λ^i_{long} and λ^i_{short}, as:[16]

$$\lambda^i_{long} = \frac{F_{t_2}}{(1+m_0)F_{t_1} - F_{t_2}} \quad \text{and} \quad \lambda^i_{short} = \frac{F_{t_2}}{F_{t_2} - (1-m_0)F_{t_1}}.$$

We aggregate these leverage estimates to the level of an exchange by weighting them with liquidation volume, yielding a weighted average leverage estimate ($\mathcal{LEV}$) as:

$$\begin{aligned}\mathcal{LEV}_{long} &= \frac{\sum_j \text{Long Liquidation Volume}(j) \times \lambda_{long}(j)}{\text{Total Long Liquidation}},\\ \mathcal{LEV}_{short} &= \frac{\sum_j \text{Short Liquidation Volume}(j) \times \lambda_{short}(j)}{\text{Total Short Liquidation}},\end{aligned} \tag{3}$$

where $\lambda_{long} = \{\lambda^i_{long}, \lambda^d_{long}\}$ and $\lambda_{short} = \{\lambda^i_{short}, \lambda^d_{short}\}$ and the summation is taken over all records on the inverse and direct products on each exchange.

Armed with these average leverage estimates, we can now replace the denominator 'Total Long (Short) Liquidation' in (3) by open interest to propose another type of speculation index, which we call an aggressive index ($\mathcal{AI}$) as:

$$\begin{aligned}\mathcal{AI}_{long} &= \frac{\sum_j \text{Long Liquidation Volume}(j) \times \lambda_{long}(j)}{\text{Open Interest}},\\ \mathcal{AI}_{short} &= \frac{\sum_j \text{Short Liquidation Volume}(j) \times \lambda_{short}(j)}{\text{Open Interest}}.\end{aligned} \tag{4}$$

Compared with the speculative index in (1) which measures a trader's turnover speed and average holding horizon, the proposed aggressive index in (4) is the leverage weighted liquidation to open interest, and measures the aggressiveness of those traders whose positions are liquidated.

Given these new speculative metrics in (2), (3) and (4), we calculate their numerical values for the perpetuals used in this study and report the results in Table 2. As expected, Deribit has the lowest leverage, otherwise leverage is in the region of 14 to 22. We also find that long positions on the inverse perpetuals have less leverage than the short positions. But the most striking result, which is immediately clear, is that the liquidation and aggressiveness indices reveal that direct perpetuals have much more speculative activity than the inverse ones.

[16] In a parallel way, we can estimate the leverages λ^d_{long} and λ^d_{short} for bitcoin *direct* perpetuals, via: $\lambda^d_{long} = \frac{F_{t_1}}{F_{t_1} - (1-m_0)F_{t_2}}$ and $\lambda^d_{short} = \frac{F_{t_1}}{(1+m_0)F_{t_2} - F_{t_1}}$

Table 2: Leverage, Liquidation and Aggressiveness Indices of Bitcoin Perpetuals

	USD Inverse Perpetuals					USDT Direct Perpetuals		
	Binance	BitMEX	Bybit	Deribit	OKEx	Binance	Bybit	OKEx
$\mathcal{LIQ}_{long}$	0.48%	0.45%	0.56%	0.10%	0.27%	0.97%	0.74%	0.90%
$\mathcal{LIQ}_{short}$	0.16%	0.22%	0.21%	0.09%	0.12%	0.55%	0.57%	1.13%
$\mathcal{LIQ}$	0.64%	0.66%	0.77%	0.19%	0.40%	1.52%	1.30%	2.03%
$\mathcal{LEV}_{long}$	17.64	13.81	19.34	12.13	14.33	18.56	20.32	19.86
$\mathcal{LEV}_{short}$	21.40	20.88	21.90	10.35	21.78	19.69	19.62	16.76
$\mathcal{AI}_{long}$	7.52%	5.29%	9.01%	1.15%	3.63%	17.41%	14.92%	17.65%
$\mathcal{AI}_{short}$	2.96%	4.23%	3.40%	0.86%	2.37%	10.65%	10.86%	18.71%
$\mathcal{AI}$	10.48%	9.52%	12.41%	2.01%	6.00%	28.05%	25.78%	36.37%

This tables reports the liquidation indices $\mathcal{LIQ}$ (2), the leverage-volume weighted indices $\mathcal{LEV}$ (3) and the aggressiveness indices $\mathcal{AI}$ (4) of bitcoin perpetuals. We compute all the indices using the same dataset as in Table 1 that ranges from 1 January 2021 to 31 May 2021.

Liquidations: All direct contracts have much higher overall leverage-weighted liquidation volume indices. On Bybit, the overall liquidation index for the direct contract is almost double that of the inverse contract; on Binance the long liquidation index for the direct contract is around twice that of the USD contract; and on OKEx the short liquidation index for the direct contract is even five times that of the inverse contract.

Aggressiveness: The results are even more pronounced when we examine leverage-weighted liquidations as a proportion of open interest – on Bybit, the overall aggressiveness index for the direct contract is about double that of the inverse contract; on Binance, the overall aggressiveness index for the direct contract is nearly three times that of the USD contract; and on OKEx, the overall aggressiveness index for the direct contract is over six times that of the inverse contract.

Why do these metrics reveal more speculative activity in direct (USDT margined) than the inverse (BTC margined) products? They suggest that speculators prefer using USDT than BTC for collateral in their margin account. We note that the astonishing growth in Tether issuance from around \$3 billion in 2019 to almost \$80 billion market cap at the time of writing coincides with the BTC price increase from around \$3,000 in 2019 to almost \$70,000 at its peak. So our results suggest that Tether issuance is driven by speculative trading rather than hedging activities, whereas those holding bitcoin and seeking to hedge its price risk are more likely to prefer a BTC margined product.

We have proposed and applied new measures of speculative activity that are more relevant to our loss-averse hedger who wishes to avoid liquidation of the hedge than standard metrics. Empirical results show that inverse perpetuals are much less speculative than direct perpetuals and that Deribit's inverse perpetual is the lest speculative contract, followed by the OKEx inverse perpetual – whereas its direct perpetual is the most speculative of all. Therefore, if all products were to offer similar hedge effectiveness, we would prefer to hedge on Deribit. But the relative efficiency of the hedges remains an open question, so we now proceed to set up the hedging problem mathematically, solve it, and then implement its solutions in practice for the same seven products that we have analysed here.

3 Hedging with Margin Constraint and Liquidation Loss Aversion

The goal of this section is to study optimal hedging of bitcoin futures under two crucial features – margin constraints and loss aversion in the face of potential liquidation. The motivation of including these in the theoretical hedging problem is our empirical results in Section 2.4, which reveal that the margin and liquidation mechanisms play a vital role in the trading of bitcoin futures.

3.1 P&L on Direct and Inverse Futures

Section 2.2 already discussed the choice of bitcoin futures that are available in crypto futures markets and provided strong arguments why perpetual contracts are more suitable for hedging purposes. However, the theoretical results of this section hold for any futures contract. But in order to apply our results in practice, we also need to formulate the problem for inverse contracts – these being another unique feature of crypto derivative markets. To gain further understanding of direct and inverse futures we briefly review how their trading Profit & Loss (P&L) is calculated. In the following, we always study the P&L of a trading strategy that longs one unit of the corresponding futures contract at time t_1 and later closes the position at time t_2. (1) P&L of standard futures: Consider a standard futures contract with a notional value of 1 bitcoin and the futures price $(F_t)_{t\geq 0}$ is denominated in USD, then the P&L is given by $F_{t_2} - F_{t_1}$ and is settled in USD. (2) P&L of direct perpetuals: Consider a direct perpetuals contract with notional value of 1 bitcoin and the futures price $(F_t)_{t\geq 0}$ is denominated in USDT, *not* fiat currency (e.g. USD), then the P&L is given by $F_{t_2} - F_{t_1}$ and is settled in USDT. (3) P&L of inverse perpetuals: Consider a inverse perpetuals contract with notional value of 1 USD and the futures price $(F_t)_{t\geq 0}$ is now denominated in USD, then the P&L is given by $\frac{1}{F_{t_1}} - \frac{1}{F_{t_2}}$ and is settled in BTC. The P&L of direct perpetuals shares the same expression with the P&L of standard futures, except that the former is margined and settled in USDT while the latter is margined and settled in USD. However, the P&L of inverse perpetuals is completely different from the P&L of standard futures, in both expression and settlement. Under the design of inverse perpetuals, bitcoin (BTC) is the denomination unit (domestic currency) and USD is the underlying asset (foreign currency).

3.2 Optimal Hedging Problem

We now formulate the hedging problem under margin constraint and loss aversion to the possibility of liquidation. We consider a discrete-time economy indexed by $\mathcal{T} := \{n\,\Delta t\}_{n=0,1,2,\ldots}$, with equal time interval Δt, which can be interpreted as the frequency that the hedged position is monitored. We denote the bitcoin spot price by $S = (S_t)_{t\in\mathcal{T}}$, which is denominated in fiat currency USD, and the perpetual futures price by $F = (F_t)_{t \in \mathcal{T}}$ which is denominated in USD for inverse perpetuals or in USDT for direct perpetuals. Define the n-period price differences and (nominal) returns:

$$\Delta_n S_t := S_{t+n\Delta t} - S_t, \quad R^S_{t,n} = \frac{\Delta_n S_t}{S_t}, \quad \text{and} \quad \Delta_n F_t := F_{t+n\Delta t} - F_t, \quad R^F_{t,n} = \frac{\Delta_n F_t}{F_t}, \tag{5}$$

where $t \in \mathcal{T}$ and $n = 1, 2, \cdots$. If $n = 1$ in (5), we simply write ΔS_t and ΔF_t and then – when it is possible without ambiguity – we may also suppress the time subscript t: e.g. $\Delta_n S$ and R^S_n denote the *n-period* price change and return of bitcoin spot. Now the nominal value $\widehat{F}$ of one bitcoin inverse perpetual futures contract, and its difference operators are denoted:[17]

$$\widehat{F}_t := \frac{1}{F_t}, \qquad \Delta_n \widehat{F}_t := \widehat{F}_t - \widehat{F}_{t+n\Delta t}, \qquad \text{and} \qquad R^{\widehat{F}}_{t,n} = \frac{\Delta_n \widehat{F}_t}{\widehat{F}_t}, \tag{6}$$

where $t \in \mathcal{T}$ and $n = 1, 2, \cdots$. The n-period difference $\Delta_n \widehat{F}_t$ is the P&L of a unit long position in bitcoin inverse perpetuals that is opened at t and closed at $t + n\Delta t$. We comment that $\Delta_n \widehat{F}_t$ in (6) is defined as 'opening price $-$ closing price' which is different from the definition of $\Delta_n F_t$ in (5). Consequently, an increase in the inverse futures price leads to a positive P&L for a long position in inverse futures (i.e. $F_{t+n\Delta t} > F_t$ implies $\Delta_n \widehat{F}_t > 0$ in (6)), which is consistent with that of standard or USDT direct futures (i.e. $F_{t+n\Delta t} > F_t$ implies $\Delta_n F_t > 0$ in (5)).

For convenience, and without loss of generality, we consider a representative hedger who holds 1 bitcoin at time $t \in \mathcal{T}$ and who seeks to use a position on a bitcoin USD inverse or direct perpetual contract to hedge the spot price volatility from t to $t + N\Delta t$, where $N \geq 1$ is a positive integer (i.e. $N\Delta t$ is the hedge horizon). The hedger with one bitcoin in possession *shorts* θ units of bitcoin perpetuals, where the hedging position θ is established at time t and carried over for N time periods. In other words, the hedger follows a static hedging strategy to protect against the fluctuation of the bitcoin spot price. Depending on whether direct perpetuals or inverse perpetuals are used, we obtain the P&L of the hedged portfolio in USD for the hedger by

$$\text{Hedger's P\&L} = \Delta_N S_t - \theta\, \Delta_N F_t \quad \text{(direct)} \quad \text{or} \quad \Delta_N S_t - \theta\, \Delta_N \widehat{F}_t \cdot S_{t+N\Delta t} \quad \text{(inverse)}. \tag{7}$$

A few remarks on (7) are in order. When direct perpetuals are used by the hedger, the P&L of the futures position $-\theta \Delta_N F_t$ is settled in USDT *not* USD, so one should multiple it by the USD price of one USDT before adding to the P&L of the spot position $\Delta_N S_t$, which is in USD. However, one USDT is pegged to one USD by design and indeed the historical USDT price is almost equal to one USD; therefore, we implicitly assume the USDT price is one USD and obtain the first result in (7). When USD inverse perpetuals are used by the hedger, the P&L of the futures position $-\theta\, \Delta_N \widehat{F}_t$ is now settled in BTC and

[17] Note that $\widehat{F}$ in (6) is defined in terms of one unit of fiat currency (1 USD), which corresponds to the numerator 1 in the definition. If a bitcoin inverse futures contract has a notional value different from 1 (say 100), we simply multiply $\widehat{F}$ by a factor (100).

is converted into USD by multiplying the spot price $S_{t+N\Delta t}$. The P&L derived in (7) naturally leads to the first optimization objective of hedging: Under the minimum-variance framework, the hedger chooses θ to minimize the variance of the return of the hedged portfolio, given by:

$$\sigma^2_{\Delta h}(\theta) = \mathbb{V}\text{ar}\left(\left(\Delta_N S_t - \theta\,\Delta_N F_t\right)/S_t\right)\ \text{(direct)}\ \ \text{or}\ \ \mathbb{V}\text{ar}\left(\left(\Delta_N S_t - \theta\,\Delta_N \widehat{F}_t \cdot S_{t+N\Delta t}\right)/S_t\right)\ \text{(inverse)}, \quad (8)$$

where $\mathbb{V}\text{ar}(\cdot)$ denotes the variance of a random variable. Minimizing the portfolio variance is a central criterion in portfolio management which dates back to Markowitz's seminal mean-variance portfolio theory. See Capponi et al. (2021) and Dai et al. (2021) for recent applications and developments. Some studies on hedging complex products minimize a risk measure other than the variance, such as the expected shortfall, as in Pflug and Broussev (2009). However, as readily seen from (8), the minimum-variance hedging of inverse perpetuals is already more complex than that of standard futures, and once we have added the margin constraint and liquidation loss aversion, the minimum-variance problem becomes rather complex. We therefore leave the extension of this problem to the minimization under alternative risk measures to future research.

The margin mechanism is key to the integrity and stability of futures markets. In Deng et al. (2020), the authors study hedging of bitcoin futures without imposing any margin constraint, i.e. they implicitly assume the hedger has an *infinite* supply of bitcoin to meet the margin requirement. However, this assumption is contradicted by the empirical findings presented in Figure 2 and in Table C.1. Also, the historical results in Table C.1 indicate very high liquidation probability for positions held for longer than five days without changes in collateral in the margin account. For instance, with 20X leverage the historical probability of liquidation was about 44% for long positions and 62% for short.

To trade bitcoin USD inverse (or USDT direct) perpetuals, the hedger needs to deposit a certain amount of bitcoin (or USDT) to meet the initial margin requirement and maintain the amount at a specified level, both of which are articulated in the specifications of futures contracts. Futures are marked 'period to period' and a positive $\Delta_n \widehat{F}_t$ (or $\Delta_n F_t$) means a marked loss to the hedger from t to $t + n\Delta t$ is $\theta \cdot \Delta_n \widehat{F}_t$ (or $\theta \cdot \Delta_n F_t$). If the loss from trading futures reduces the amount of bitcoin (or USDT) in the hedger's margin account at time $t + n\Delta t$ to or below the maintenance level, a liquidation is triggered.

Now we formalise our discussions on liquidation under the hedging framework. First, we consider the case when the hedger uses direct perpetuals as the hedging instrument. Assume the corresponding maintenance margin rate is m_0, and the hedger is subject to an upper constraint expressed as a percentage $\overline{m}$ of the initial futures price F_t in USDT. Equivalently, the hedger reserves a total of $\overline{m} \cdot F_t$ amount in USDT for meeting the margin requirements of trading θ direct perpetuals, which in turn implies the initial margin rate is $\overline{m}/\theta$ and the (implied) leverage taken by the hedger is $\theta/\overline{m}$.[18] Once the accumulated trading loss in futures reduces the margin account below the maintenance level for the first time at $t+n\Delta t$

[18]Here the initial margin rate $\overline{m}/\theta$ is greater than or equal to the minimum requirement set by the futures exchange. Otherwise, the hedger would not be allowed to open such a leveraged futures position.

(i.e. when loss per contract is greater than $(\overline{m}/\theta) \cdot F_t - m_0 \cdot F_{t+n\Delta t}$), a liquidation event is triggered. Mathematically, we derive the liquidation condition as follows:

$$\underbrace{\Delta_n F_t}_{\text{Trading Loss}} > \underbrace{\frac{\overline{m}}{\theta} \cdot F_t - m_0 \cdot F_{t+n\Delta t}}_{\text{Margin Buffer}} \Leftrightarrow \underbrace{R^F_{t,n} = \frac{\Delta_n F_t}{F_t}}_{\text{Nominal Return}} > \underbrace{\frac{\overline{m}/\theta - m_0}{1+m_0}}_{\text{Adjusted Financial Capacity}} . \tag{9}$$

Second, we study the case of inverse perpetuals. Similarly, we introduce m_0 to denote the maintenance margin rate; differently, the hedger's upper financial constraint is now $\overline{m}$ bitcoins.[19] We obtain the liquidation condition in the case of inverse perpetuals by:

$$\underbrace{\Delta_n \widehat{F}_t}_{\text{Trading Loss}} > \underbrace{\frac{\overline{m}}{\theta} - m_0 \cdot \widehat{F}_{t+n\Delta t}}_{\text{Margin Buffer}} \Leftrightarrow \underbrace{R^{\widehat{F}}_{t,n} = \frac{\Delta_n \widehat{F}_t}{\widehat{F}_t}}_{\text{Nominal Return}} > \underbrace{\frac{\overline{m}/\hat{\theta} - m_0}{1-m_0}}_{\text{Adjusted Financial Capacity}} , \tag{10}$$

where we define $\hat{\theta} = \theta/F_t$ so that (10) takes a similar form as (9). Note that 'Trading Loss' and 'Margin Buffer' are denominated in USDT in (9) and in BTC in (10).

We introduce the following notations for extreme returns:

$$\overline{R}^S_t := \max_{1\le n\le N} R^S_{t,n}, \qquad \overline{R}^F_t := \max_{1\le n\le N} R^F_{t,n}, \qquad \overline{R}^{\widehat{F}}_t := \max_{1\le n\le N} R^{\widehat{F}}_{t,n}, \tag{11}$$

where N is the number of periods of the hedge horizon (see e.g. (8)). As seen from (9) (or (10)), if the extreme return $\overline{R}^F_t$ (or $\overline{R}^{\widehat{F}}_t$) is greater than 'Adjusted Financial Capacity', a liquidation event occurs before the end time $t + N\Delta t$ and the hedger ends up with a naked position. As argued in Section 1, the hedger is loss averse, i.e. a liquidation is seen as an 'undesirable' trading scenario. This motivates us to incorporate the second optimization objective: The hedger wants to minimize the liquidation probability:

$$P(\overline{m}, \theta) := \mathbb{P}\text{rob}\left(\overline{R}^F_t > \frac{\overline{m}/\theta - m_0}{1+m_0}\right) \quad \text{(direct)} \quad \text{or} \quad \mathbb{P}\text{rob}\left(\overline{R}^{\widehat{F}}_t > \frac{\overline{m}/\hat{\theta} - m_0}{1-m_0}\right) \quad \text{(inverse)}. \tag{12}$$

where $\mathbb{P}\text{rob}(\cdot)$ denotes the probability of an event, $\overline{R}^F_t$ and $\overline{R}^{\widehat{F}}_t$ are defined in (11), and $\hat{\theta} = \theta/F_t$. In (12), m_0 is the maintenance margin rate, while $\overline{m}$ captures the hedger's upper financial constraint. In particular, $\overline{m}$ is either the percentage of the initial futures price in the case of direct perpetuals or the amount of reserved bitcoins in the case of inverse perpetuals (recall the hedger holds one bitcoin spot, so the reserved amount is a percentage of the hedger's initial spot position). By definition (12), the liquidation probability $P(\overline{m}, \theta)$ is a decreasing function of the constraint $\overline{m}$ and an increasing function of the position θ. The economic meaning is clear: more financial reserves (larger $\overline{m}$) or less risk taking activities (smaller θ) both reduce the chance of liquidation.

[19] In the first case, the upper constraint is $\overline{m} \cdot F_t$ in USDT, which is approximately equivalent to $\overline{m}$ bitcoins, since $S_t \approx F_t$ and one USDT $\approx$ \$1. We model the hedger's upper constraint in a slightly different way, because the margin account is settled in USDT for direct perpetuals and in BTC for inverse perpetuals.

From the arguments leading to (8) and (12) the hedger has dual objectives to minimize, and a natural way is to aggregate them into one objective. To balance the magnitude of two objectives we multiply the liquidation probability $P(\overline{m}, \theta)$ by a factor σ_S^2, the variance of the N-period return of bitcoin spot, and introduce a parameter γ to aggregate them. This way, we consider the following optimal hedging problem for the hedger:

Problem 3.1. *The hedger, possessing one bitcoin at time t, seeks an optimal static hedging strategy θ^* for N periods that solves the following problem:*

$$\min_{\theta > 0} \quad \left\{ \sigma^2_{\Delta h}(\theta) + \gamma\, \sigma_S^2\, P(\overline{m}, \theta) \right\}, \tag{13}$$

where $\sigma^2_{\Delta h}(\theta)$ is given by (8), $\gamma > 0$ is the aggregation factor, σ_S^2 is the variance of the N-period return of bitcoin spot, $\overline{m} > 0$ represents the margin constraint, and $P(\overline{m}, \theta)$ is given by (12).

One may also interpret γ as a *loss aversion* parameter that captures the extent of the hedger's dislike of the liquidation event defined in (12). As γ increases, the hedger fears liquidation events more and will therefore trade in a more conservative way. The limiting case of $\overline{m} = +\infty$ (or $\gamma = 0$) is studied in Deng et al. (2020) and corresponds to the scenario where liquidation events have no impact on hedging decisions. Note that another extreme case is when $\theta = 0$, in which case liquidation will not occur because the hedger has no position in futures.

The main theoretical result of the paper is a semi-closed solution to Problem 3.1 which is given in the following theorem. The proof is in the Appendix.

Theorem 3.2. *The optimal hedging strategy θ^* to Problem (13) is given by*

$$\theta^* = \omega\, \theta_0, \qquad \text{where } \omega = 1 \text{ (direct) or } F_t \text{ (inverse)}, \tag{14}$$

and θ_0 is a positive root of the following non-linear equation $a(x)x^{-2} + x - b = 0$ with

$$\begin{aligned} a(x) &:= \frac{\gamma\, \nu\, \hat{m}}{2\alpha} \exp\left[-\left(1 + \frac{\tau}{\alpha}\left(\hat{m}x^{-1} - \hat{m}_0 - \beta\right)\right)^{-\frac{1}{\tau}} \right] \left(1 + \frac{\tau}{\alpha}\left(\hat{m}x^{-1} - \hat{m}_0 - \beta\right)\right)^{-\frac{1}{\tau}-1}, \\ b &:= \frac{\sigma_{SF}}{\sigma_F^2}, \quad \nu := \frac{\sigma_S^2}{\sigma_F^2}, \quad \hat{m} := \frac{\overline{m}}{1 + \omega_0 m_0}, \quad \hat{m}_0 := \frac{m_0}{1 + \omega_0 m_0}, \quad \omega_0 := 1 \text{ (direct) or -1 (inverse)}. \end{aligned} \tag{15}$$

In (15), σ_S^2 (resp. σ_F^2) denotes the variance of the N-period return of bitcoin spot (resp. direct futures), and σ_{SF} denotes the covariance between the two random variables. Constants α, β and τ are respectively scale, location and tail index estimation parameters of the right tail of $\overline{R}_t^F$ or $\overline{R}_t^{\widehat{F}}$, both defined in (11), and are based on the extreme value theorem (see (A.2) in Appendix).

To provide more insight to the rather complex expression in (14), let us take a closer look at two extreme cases when $\gamma = 0$ or $\overline{m} = +\infty$. In these two cases, the term $a(x)$ in (15) becomes zero and the optimal strategy $\widetilde{\theta}^*$ $(= \theta^*|_{\gamma=0} = \theta^*|_{\overline{m}=\infty})$ is reduced to

$$\widetilde{\theta}^* = \omega \cdot b = \omega \cdot \rho_{SF} \cdot \frac{\sigma_S}{\sigma_F}, \tag{16}$$

where ρ_{SF} is the correlation coefficient between returns on S and F. Notice that b is exactly the optimal hedging strategy under the classical minimum-variance hedging framework when a *standard* futures contract is used as the hedging instrument. Therefore, even in such simplified cases, the optimal strategy of inverse perpetuals $\widetilde{\theta}^*$ still differs from that of standard futures by a factor F_t (recall from (14) that $\omega = F_t$ in the case of inverse perpetuals). Since the current bitcoin price is in 5-digit USD, missing this factor could lead to catastrophic consequences when using inverse perpetuals to hedge spot risk. Incorporating a margin constraint $\overline{m}$ and loss aversion γ significantly complicates the analysis and introduces a non-linear adjustment to correct b into θ_0 in (14).

4 Empirical Analysis of the Optimal Hedging Strategy

In this section we conduct an empirical analysis to investigate the economic consequences of the optimal hedging strategy θ^*, derived in (14), for the representative hedger. We begin by estimating the parameters of the optimal hedge ratio given in Theorem 3.2. Then we study two important topics related to the hedger's dual objectives: hedge effectiveness in Section 4.3 and liquidation probability in Section 4.4. We close the section with investigations on the implied leverage under θ^* in Section 4.5.

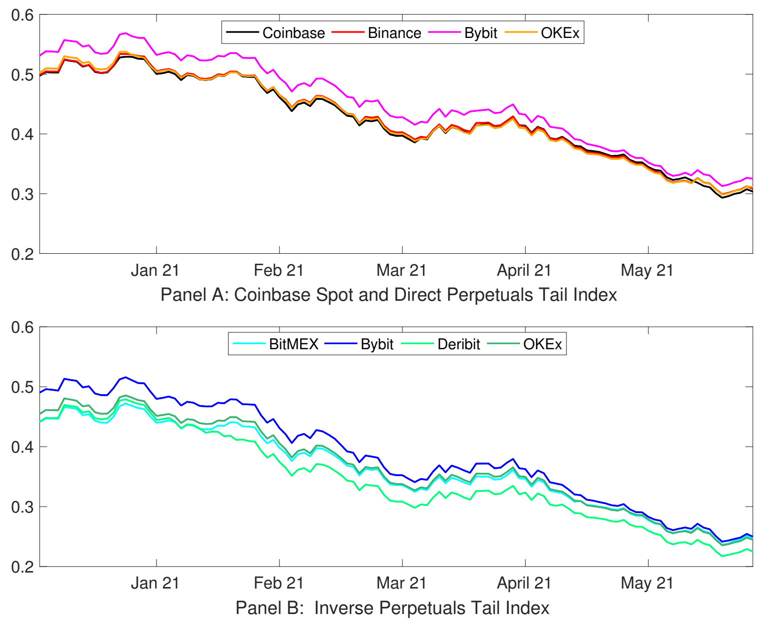
4.1 Parameter Estimation

To calculate the one-period changes of the spot price return R^S, the direct perpetuals price return R^F and the inverse perpetuals return $R^{\widehat{F}}$ (see their definitions in (5) and (6)), we consider the time step at three different values $\Delta t =$ one minute (1min), one hour (1h) and one day (1d). We report the summary statistics of these returns for different bitcoin spot/futures exchanges at difference frequency levels Δt in Appendix E. We observe that the means of all these returns are close to zero and their standard deviations are very large. Moving to higher moments, the skewness of R^S and R^F are small with varying signs at different Δt, while the kurtosis of all these returns are extremely large in positive values. The same section of Appendix E also considers the *extreme* price returns $\overline{R}^S$, $\overline{R}^F$ and $\overline{R}^{\widehat{F}}$ (see their definitions in (11)). As expected, the extreme price movements are enormous, e.g. the maximum value of $\overline{R}^S$ during a 8h time window is 19.5% for Coinbase bitcoin spot price.

To calculate the optimal hedging strategy θ^* in (14), the variances σ_S^2 and σ_F^2 and the covariance σ_{SF} are estimated using a 4-month rolling window of one-minute data. The more difficult task is the estimation of the generalized extreme value (GEV) parameters α, β and τ of the extreme returns $\overline{R}^F$ or $\overline{R}^{\widehat{F}}$ defined in (11), because a reliable estimation requires a long period of data. (Recall $\overline{R}^F$ or $\overline{R}^{\widehat{F}}$ is the maximum return during the hedge horizon $N\Delta t$, which could be as long as five days in our study.)

However, the price data of direct perpetuals on Binance, Bybit and OKEx are only available from 1 July 2020. On the other hand, both Coinbase spot and BitMEX inverse perpetuals price data can be traced back to 1 November 2017. We then ask whether we can use these two much longer price data as proxies to estimate the GEV parameters of other perpetuals only with limited data. Before taking this step, we first estimate τ with one-day horizon for each perpetual contract using its own data from 1 July 2020 to 31 May 2021 and compare the results with those of Coinbase spot and BitMEX inverse perpetuals.[20]

Figure 3: Dynamic Evolution of Estimated Tail Index Parameter τ



These figures illustrate the dynamic evolution of the estimated tail index parameter τ for Coinbase spot, the inverse perpetuals and the direct perpetuals. The sample period is from 1 July 2020 to 31 May 2021 and the hedge period is one day. Panel A shows that the tail indices of Coinbase spot are close to those of Binance, Bybit and OKEx direct perpetuals. Panel B shows that the tail indices of the BitMEX inverse perpetuals are close to those of the other inverse perpetuals.

Figure 3 plots the dynamic evolution of the estimated τ for all available extreme returns data ($\overline{R}^S$, $\overline{R}^F$ and $\overline{R}^{\widehat{F}}$). The upper panel shows that the estimated τ's of Coinbase spot extreme returns $\overline{R}^S$ serves as a good approximation to the direct perpetuals extreme returns $\overline{R}^F$. The approximation is nearly perfect for Binance and OKEx, and very close for Bybit. The lower panel also suggests that it is reasonable to use BitMEX perpetuals data to estimate τ for the other inverse perpetuals on Bybit, Deribit and OKEx.

We further examine the correlations between the estimated τ's and report the results in Table E.5. From there we observe an extremely high positive correlation between the estimated τ's of Coinbase and those of direct perpetuals (with correlation coefficient between 0.95 and 0.98), and between the estimated τ's of BitMEX inverse perpetuals and those of the remaining inverse perpetuals (with correlation coeffi-

[20]We focus on the tail index parameter τ, since it measures the heaviness of the right tail, i.e. the losses to hedgers with short futures positions, and has a major impact on the liquidation probability. The sign of τ determines the family of the GEV distributions; see Appendix A for details. On the other hand, the scale parameter α and the location parameter β represent the dispersion and the average of the extreme value observations, which are less important because we can always change the scale and location by applying a transformation to the data.

cient between 0.96 and 0.98). Therefore, the main difference between tail index estimates is that those for inverse perpetuals are lower than those for direct perpetuals, and it is the direct ones that are closest to the spot tail index. One might suppose that this finding feeds into our hedge effectiveness results, which would be unfortunate because the direct perpetuals have much more speculative trading than the inverse ones. On the other hand, since our speculation metrics include liquidations and leverage, and since our optimal hedge accounts for loss aversion to liquidation, it may be that hedging effectiveness is lower on the direct perpetuals, for these reasons.

The high correlation results in Table E.5 of Appendix E provide additional support to the use of Coinbase and BitMEX data to estimate τ for the direct and inverse perpetuals, respectively, in order to extend the sample size for our hedging study. With that in mind, we use a three-year rolling window (1095 days) when estimating the GEV parameters, which leaves us with enough samples of the extreme returns.[21] Now at each time t we follow the above estimation procedure to obtain all the parameters and then apply (14) to compute the current optimal strategy θ_t^*, which will be followed from t to $t + N\Delta t$. We denote by R_t^* the return of the optimally hedged portfolio over N periods and thus obtain: $R_t^* = (\Delta_N S_t - \theta_t^* \cdot \Delta_N \widehat{F}_t \cdot S_{t+N\Delta t})/S_t$ (inverse) or $R_t^* = (\Delta_N S_t - \theta_t^* \cdot \Delta_N F_t)/S_t$ (direct). Once we have the realized price data at time $t + N\Delta t$, we use the above formulas to compute R_t^* and store its value. Next we roll the fixed three-year window forward by N time periods and repeat the same process to calculate the next portfolio return $R_{t+N\Delta t}^*$, until arriving at the last available time point, i.e. exactly N periods prior to the end of the full sample. This constitutes our realized time series of R^*, the return of the optimally hedged portfolio, used to investigate hedge effectiveness in the next section.

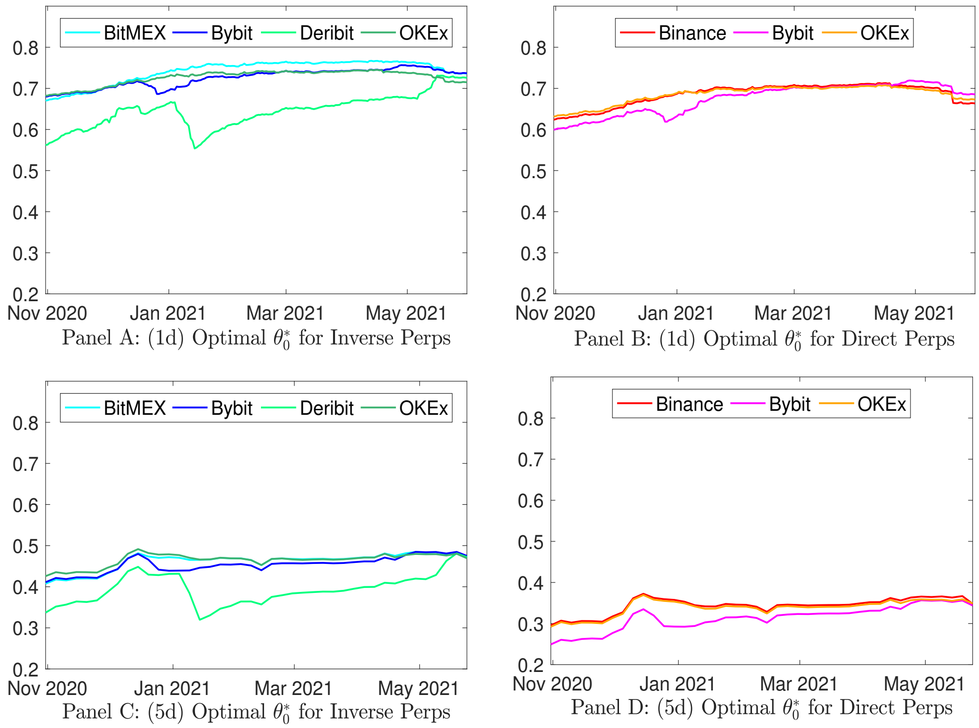
4.2 Optimal Strategy

Before we study the effectiveness of the optimal strategy θ^* we compute θ_0. Recall from (14) that $\theta^* = \omega\theta_0$, where $\omega = 1$ for direct perpetuals and $\omega = F_t$ for inverse perpetuals. Note that θ_0 lies in $[0, 1]$ and makes the positions on inverse and direct products comparable. Figure 4 depicts its value dynamically over time under two possible hedge horizons, one day and five days. We do not report results for hedges held longer than five days, because the liquidation probability becomes so large as to dominate the problem (see Table C.1). That is, we assume the margin account is managed with new collateral within five days. We summarize our findings as follows: (1) The value of θ_0 decreases as the hedge horizon increases. This is because the liquidation risk increases with the hedge horizon and a smaller position in perpetuals helps reduce the liquidation probability, this being one of the dual objectives of the hedger; (2) The values of θ_0 for a fixed hedge horizon are fairly stable over time, and they are marginally lower for the direct perpetuals compared with the inverse perpetuals; (3) The value of θ_0 for the Deribit inverse perpetual is noticeably lower than θ_0 for the other three inverse perpetuals. This is probably because the returns on the Deribit perpetual have much lower correlation with the Coinbase returns than the returns on the

[21] The estimation result is robust to the choice of the rolling window length. We choose a relatively long period of three years (1095 days) to obtain more accurate estimations of α, β and τ.

other perpetuals.

Figure 4: Optimal Hedging Strategy for USD Inverse and Direct Perpetuals



Panel A: (1d) Optimal θ_0^* for Inverse Perps

Panel B: (1d) Optimal θ_0^* for Direct Perps

Panel C: (5d) Optimal θ_0^* for Inverse Perps

Panel D: (5d) Optimal θ_0^* for Direct Perps

This figure plots the scaled optimal hedging strategy θ_0 for inverse perpetuals (Panels A and C) and direct perpetuals (Panels B and D) from 1 July 2020 to 31 May 2021 (see (14) for the definition of θ_0). We set $\Delta t = 1$ minute, the margin constraint $\overline{m} = 20\%$, the maintenance margin rate $m_0 = 1\%$ and a loss aversion of $\gamma = 20$. In Panels A and B the hedge horizon is one day. In Panels C and D the hedge horizon is five days.

4.3 Hedge Effectiveness

To measure the hedge effectiveness of bitcoin perpetual futures in hedging bitcoin spot risk we compare two possible portfolios for the hedger: the first is an *unhedged* portfolio holding one bitcoin, and the second one is an optimally *hedged* portfolio that consists of one bitcoin and a short position of θ^* perpetuals (either USD inverse or USDT direct) contracts, where the optimal strategy θ^* is given by (14). Since the tail index parameter τ is rather stable with respect to the choice of Δt we fix the highest possible monitoring frequency $\Delta t = 1$ minute in the following analysis. Let $R(\theta)$ denote the return of a portfolio with one bitcoin and θ short positions in bitcoin futures. It is clear that $\theta = 0$ corresponds to the no hedging case. Following Laws and Thompson (2005) and many others, we define hedge effectiveness (HE) as the percentage of reduction in portfolio variance:

$$\mathrm{HE}(\theta) = 1 - \frac{\mathbb{V}\mathrm{ar}(R(\theta))}{\mathbb{V}\mathrm{ar}(R(0))}, \qquad \theta > 0, \tag{17}$$

where $\mathbb{V}\mathrm{ar}(\cdot)$ denotes the variance of a random variable. We calculate the hedge effectiveness of the optimal portfolio (i.e. when $\theta = \theta^*$ given by (14)) for different pairs of bitcoin spot and perpetual futures under three margin constraint levels ($\overline{m} = 10\%, 20\%, 50\%$), three loss aversion levels ($\gamma = 10, 20, 40$) and three hedge horizon levels ($N\Delta t =$ 8h, 1d, 5d).[22] We report the detailed results in Table 3 and we summarise the key findings as follows:

First, the hedger's margin constraint, as captured by a percentage parameter $\overline{m}$, has a major impact on the hedge effectiveness of the optimal strategy θ^* in bitcoin futures markets, and hedging is more effective as $\overline{m}$ increases. In all cases, once the hedger has sufficient collateral deposited in the margin account, the optimal hedging strategy achieves superior hedge effectiveness, e.g. when $\overline{m} = 50\%$, HE is above 95% in most cases, even close to 100% in some cases. For a hedger with a more binding constraint ($\overline{m} = 10\%$), HE is still promising if her loss aversion is not too high ($\gamma = 10$ or 20) and the hedge horizon is relatively short ($N\Delta t =$ 8h). This finding offers strong support to the inclusion of margin constraint in the hedging analysis of bitcoin futures. The hedging performance of the optimal strategy may be over-exaggerated if the analysis does not consider the possibility of margin constraint (such as in Deng et al. (2020)). For instance, in the 'worst scenario' (i.e. when $\overline{m} = 10\%$, $\gamma = 40$ and $N\Delta t =$ 5d), HE is only about 20% when using inverse perpetuals to hedge the spot risk on Coinbase. From a different angle, our results suggest that a hedger should allocate 50% of the initial futures contract value into the margin account in order to achieve 99% HE, if she is highly averse to liquidation ($\gamma = 40$) and wants to hedge for a long horizon (five days). We also comment that our numerical analysis can easily be extended to locate a minimum margin constraint $\overline{m}$ needed to achieve a certain level of HE, given the loss aversion γ and the hedge horizon $N\Delta t$.

Secondly, the hedger's loss aversion, as captured by γ, is crucial to the hedge effectiveness when margin constraint is tight (e.g. $\overline{m} = 10\%$). Given $\overline{m} = 10\%$, $N\Delta t =$ 1d and direct perpetuals are used, a hedger with $\gamma = 10$ achieves HE of more than 80%, but less than 40% with $\gamma = 40$, when following the optimal strategy. This finding serves as evidence of the inclusion of loss aversion in the study of optimal hedging problem in (13). However, for a large enough $\overline{m}$ (e.g. $\overline{m} = 50\%$), the impact of γ on HE is insignificant; the reason is because the larger the $\overline{m}$, the smaller the liquidation probability and hence less impact of γ on the hedger's objectives.

Thirdly, generally inverse perpetuals have greater HE than direct perpetuals but the difference is not significant; both perpetuals are effective hedging instruments for Coinbase spot. As an example, with $\overline{m} = 20\%$, $\gamma = 20$ and $N\Delta t =$ 1d, the best inverse perpetuals contract is BitMEX with 93.3% HE, while the best direct perpetuals contract is Binance with 90.1% HE. As for which exchange is the best for hedging, our results show that BitMEX is the best choice for inverse perpetuals and Binance is the best choice for direct perpetuals. However, the superiority of the BitMEX inverse (resp. Binance direct) perpetual relative to other inverse (resp. direct) perpetuals is only marginal. The notable exception is

[22] We do not consider longer hedge horizons, because the volatility of bitcoin futures price is extremely high and in consequence liquidations are substantial, implying that a sufficiently long hedge would not be achieved.

the Deribit inverse perpetual, which is the worst product for hedging Coinbase spot risk.

Finally, as the hedge horizon ($N\Delta t$) increases, the hedge effectiveness drops as expected, since there are naturally more extreme price changes when a longer hedge horizon is chosen. Further the impact of the hedge horizon on HE is more profound when $\overline{m}$ is small (tight margin constraint) or γ is large (high loss aversion). In particular, when $\overline{m} = 10\%$ and $\gamma = 40\%$, HE reduces by more than half (in all cases) when the hedge horizon increases from one day to five days. The rapid drop off in HE as the horizon increases is particularly noticeable in more speculative products, i.e. the Binance and OKEX direct perpetuals. This is because the probability of liquidation is particularly high on these products.

4.4 Liquidation Probability Under the Optimal Strategy

From (12) a liquidation event is regarded as the circumstance that trading losses reduce the collateral in the margin account so that it falls below the maintenance level and the position is automatically liquidated by the exchange. Clearly, as readily seen from Figure 2 and Table C.1, both historical liquidation volumes and simulated liquidation probabilities are very high, imposing enormous risk to bitcoin perpetuals traders. Our approach to alleviating this risk is to take account of the effect that the liquidation probability $P(\overline{m}, \theta)$, defined by (12), has on the hedger's position θ. To be precise, we assume the representative hedger is loss averse and aims to minimize the liquidation probability $P(\overline{m}, \theta)$, as formulated in Problem (13). With this in mind, we naturally hypothesize that following the optimal strategy θ^* should reduce the liquidation probability to a reasonable level. But how much of a reduction is possible? In this sub-section we ask, to what degree does the optimal hedging strategy reduce the liquidation probability?

To investigate this, we obtain the liquidation probability $P(\overline{m}, \theta^*)$ in (18) under the optimal strategy θ^*, which we call the 'optimal liquidation probability' for short. We derive this using the extreme value theorem (A.2) in Appendix as:

$$P(\overline{m}, \theta^*) \simeq 1 - \exp\left[-\left(1 + \frac{\tau}{\alpha}\left(\frac{\hat{m}}{\theta^*} - \hat{m}_0 - \beta\right)\right)^{-1/\tau}\right], \tag{18}$$

where α, β and τ are the GEV parameters of the extreme return $\overline{R}^F$ or $\overline{R}^{\widehat{F}}$ (see their definitions in (11)), and $\hat{m}$ and $\hat{m}_0$ defined by (15) depend on the hedger's margin constraint $\overline{m}$ and the initial margin rate m_0. All the parameters α, β and τ have been estimated (see, e.g. Table E.2 for their summary statistics) and the optimal strategy θ^* has been obtained as well (see Figure 4). Hence, we can calculate the right hand side of (18), and use it as an approximation to the optimal liquidation probability $P(\overline{m}, \theta^*)$.

The middle panel of Table 3 reports the results on the optimal liquidation probability (18) and we now outline the main findings. First, $P(\overline{m}, \theta^*)$ is generally small – less than 1% when $\overline{m} = 50\%$, and less than 3% when $\overline{m} = 20\%$ or 10%. The table provides numerous liquidation probability combinations for different hedge horizons under various margin constraints and loss aversions, and the results range from 0.1% to about 3%. This result strongly supports our hypothesis that the hedger is able to reduce the liquidation probability to a desirable level by following the optimal strategy θ^*, under whatever margin constraint is selected, given her degree of loss aversion. We also emphasize that such a finding is robust, in the sense that it holds regardless of the margin constraint $\overline{m}$, loss aversion γ, hedge horizon $N\Delta t$, and the choice of hedging instrument.

Secondly, the margin constraint $\overline{m}$ has a major impact on the optimal liquidation probability, in particular for hedgers with low loss aversion. When $\gamma = 10$, $N\Delta t = 8$h and Binance inverse perpetuals are used, increasing $\overline{m}$ from 10% to 50% reduces the optimal liquidation probability from 1.5% to 0.1%. Such an impact is less significant for hedgers with high loss aversion, since they already 'overweight' the liquidation in their dual objectives. Indeed, when $\gamma = 40$ and $N\Delta t = 8$h, the optimal liquidation probability is already small at 0.7%, even when the hedger's margin constraint is tight ($\overline{m} = 10\%$). By recalling (15) and (18), we know that the optimal liquidation probability depends directly on the ratio $\overline{m}/\theta^*$. This adds further support to the incorporation of margin constraint into the analysis of optimal futures hedging for a loss averse hedger. However, increasing $\overline{m}$ has a two-fold effect: on the one hand, a greater $\overline{m}$ immediately leads to a decrease in liquidation probability, if the position remains the same; but on the other hand, as $\overline{m}$ increases and the margin constraint alleviates the hedger takes a greater position in perpetuals, to further minimize the variance objective. Therefore, increasing $\overline{m}$ does not necessarily decrease the optimal liquidation probability, especially at longer horizons.

Thirdly, as expected, the optimal liquidation probability decreases as the loss aversion γ increases. Hedgers with higher γ are more loss averse and hence act more conservatively by taking a smaller short position on bitcoin perpetual futures to hedge the spot price risk.[23] For instance, with the Bybit inverse perpetual as a hedge for Coinbase spot risk, for 8 hours and given $\overline{m} = 10\%$, the optimal liquidation probability drops from 1.5% to 0.6% as γ increases from 10 to 40.

Fourthly, in the computations leading to Table 3, we consider four inverse perpetuals (BitMEX, Bybit, Deribit and OKEx) and three direct perpetuals (Binance, Bybit and OKEx). The results on the optimal liquidation probability are very close, even identical, for both types of perpetuals, when the hedge horizon is not long ($N\Delta t = 8$h or 1d). But for a longer horizon ($N\Delta t = 5$d), inverse perpetuals outperform direct perpetuals. Among the four inverse perpetuals, Deribit yields the smallest optimal liquidation probability, exactly the opposite to the hedge effectiveness study, but there is no distinguishable difference between BitMEX, Bybit and OKEx. On the direct perpetuals side, the three exchanges Binance, Bybit and OKEx deliver almost the same performance.

4.5 Optimal Implied Leverage

A distinguishing feature of various bitcoin derivatives is high leverage, e.g. up to 100X for BitMEX inverse perpetual futures. In the last part of empirical analysis, we study the *implied leverage* under the optimal strategy θ^* taken by the representative hedger. Recall that in our setup, for direct perpetuals the

[23]This effect is also seen in the theoretical model, via the sensitivity analysis shown in Figure D.1.

margin constraint $\overline{m}$ is a percentage of the initial contract price F_t in USDT, while for inverse perpetuals the margin constraint $\overline{m}$ is the percentage of bitcoins (with the initial contract value $1/F_t$). For both perpetuals, $\overline{m}$ is applied to the *entire* short position of θ^* futures contracts. Using the derivations of (9) and (10) we obtain the implied leverage under the optimal strategy θ^* as:

$$\text{Optimal Implied Leverage} = \frac{\theta^*}{F_t \cdot \overline{m}} \quad \text{(inverse)} \quad \text{or} \quad \frac{\theta^*}{\overline{m}} \quad \text{(direct)}. \tag{19}$$

We now compute the optimal implied leverage and present the results in the bottom panel of Table 3. The main findings are summarised as follows: First, the implied leverage under the optimal strategy θ^* varies from 1X to around 8X, which is a reasonable level for bitcoin futures, since perpetuals often allow up to 100X, even 125X, leverage in trading. For instance, the maximum leverage of BitMEX inverse perpetuals is 100X, but its optimal implied leverage is less than 5X when $\overline{m} \geq 20\%$ and even less than 2X when $\overline{m} = 50\%$. We thus conclude that the hedger reduces leverage considerably when she follows the optimal strategy θ^*.

Secondly, similar to the analysis of hedge effectiveness and liquidation probability, both the margin constraint $\overline{m}$ and the loss aversion γ play a key role in determining the optimal implied leverage. In particular, the optimal implied leverage is a decreasing function of both $\overline{m}$ and γ, which is self-explanatory. Given a sufficiently large margin constraint (say $\overline{m} = 50\%$), the optimal implied leverage is less than 2X in all cases, and remains stable for different loss aversion γ and various perpetual futures across exchanges. The difference between exchanges is almost negligible except for Deribit, where option delta hedgers predominate, and these take the least leverage overall among all the exchanges considered in the study.

Finally, the optimal implied leverage decreases as the hedge horizon increases. To understand this result, note that both the portfolio variance and the liquidation probability increase with the hedge horizon. As such, all else being equal, when the hedge horizon increases the hedger will reduce her positions – see Figure 4 – and this immediately lowers the optimal implied leverage.

5 Summary and Conclusions

Our research is motivated by the novel clearing methods and margin mechanisms that are unique to cryptocurrency markets. Because of these features, standard hedging objectives should be modified to include the operational risk arising from the threat of automatic liquidation. The solution thus depends on the the hedger's individual level of loss aversion, as well as choice of leverage and the decision about collateral management which determines the horizon over which the hedge is rebalanced. The bitcoin market is also highly fragmented, so that hedgers have a wide choice of possible hedging instruments and exchanges on which to trade them. We find that instruments having similar hedging effectiveness can exhibit marked differences in speculative activity, as measured by new metrics that account for aggressive liquidations on 24/7 non-KYC exchanges acting as their own CCP.

We test our theory by hedging the Coinbase spot bitcoin price with seven possible perpetual products traded on five different exchanges. In addition to choosing the trading venue and the instrument, we model hedgers with different levels of aversion to the loss incurred if their hedge position is liquidated. The hedger also needs to select their own level of leverage and collateral in the margin account. As one may expect, we find that all these parameters affect the optimal strategy and its hedging effectiveness.

Our results show that a 'very wrong' impression of the ability to hedge in this highly-volatile market would be created if the hedger ignores the novel and unique features of bitcoin derivatives. At short (one- or two-day) hedging horizons, all products offer similar effectiveness, except for the Deribit inverse perpetual. This is the least speculative instrument according to all measures, but that does not make it the most effective hedge as one might suppose. In fact, the Deribit perpetual is the *least* effective hedge. The other inverse perpetuals perform better than the direct products as the hedging horizon increases. This too might seem counterfactual, because the correlations between the tail indices of Coinbase extreme returns and the perpetual extreme returns are much higher for direct than inverse products. However, direct perpetuals also yield much higher measures for speculation, especially on Binance and OKEx. Since our hedger is averse to the possibility of her position being liquidated by such exchanges, the indirect perpetuals are preferred, especially when holding the hedge for longer than a couple of days.

As is well known, bitcoin futures are among the most highly leveraged products in any financial market. But this feature is a double-edged sword: on the positive side, it helps attract a large amount of noise traders and speculators, and thus improves the overall market depth and liquidity; but on the negative side, it could easily lead to mass liquidation events, such as occurred on 19 May 2021. Therefore, after examining hedging effectiveness and speculative activity, we also investigate the liquidation probability and the implied leverage under the optimal strategy. By following this, the hedger is able to reduce the liquidation probability to less than 1%, in most scenarios considered, and control the leverage to a reasonable level, mostly below 5X. Moreover, because of the richness of our model, the numerical analysis could easily be extended to locate a minimum margin constraint $\overline{m}$ needed to achieve a certain level of hedging effectiveness, given the hedger's loss aversion γ and hedge horizon $N\Delta t$. This problem is left for further research.

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Table 3: Hedge Effectiveness, Liquidation Probability and Implied Leverage under the Optimal Hedging Strategy

			Inverse Perpetuals												Direct Perpetuals								
			BitMEX			Bybit			Deribit			OKEx			Binance			Bybit			OKEx		
	$\overline{m}$	γ	8h	1d	5d	8h	1d	5d	8h	1d	5d	8h	1d	5d	8h	1d	5d	8h	1d	5d	8h	1d	5d
		10	99.6	99.5	99.2	97.8	97.9	97.7	85.5	90.7	95.4	99.3	99.2	98.5	99.2	99.0	97.1	97.6	97.7	95.9	99.2	99.0	97.1
	50%	20	99.5	99.2	97.6	97.7	97.5	95.8	85.2	90.0	92.6	99.2	98.7	96.7	99.1	98.3	93.0	97.4	97.1	91.2	99.1	98.3	92.9
		40	99.3	98.2	93.2	97.5	96.5	91.2	84.7	88.3	86.8	98.9	97.6	92.3	98.7	96.4	83.1	97.1	95.2	79.7	98.7	96.4	82.7
		10	99.2	97.5	86.9	97.3	95.7	84.7	84.4	87.2	79.3	98.7	96.8	86.3	98.5	95.9	79.7	96.9	94.7	75.7	98.5	95.9	79.2
HE(%)	20%	20	98.3	93.3	72.1	96.2	91.4	70.0	82.8	81.8	64.2	97.6	92.5	72.3	97.2	90.1	58.2	95.6	88.3	52.9	97.1	89.9	57.5
		40	95.4	82.8	53.3	93.1	81.0	51.6	78.9	70.4	46.6	94.5	82.5	54.0	93.7	76.7	34.1	91.7	73.4	29.6	93.6	76.2	33.4
		10	96.3	85.8	52.3	94.1	84.0	50.5	80.1	73.4	45.3	95.4	85.3	53.1	94.9	82.8	41.2	93.0	80.1	36.0	94.7	82.4	40.4
	10%	20	89.9	68.5	33.7	87.6	66.9	32.6	72.8	56.4	29.1	89.2	68.8	34.7	88.0	62.9	20.5	85.4	58.7	17.3	87.8	62.2	20.0
		40	75.8	45.4	20.5	73.6	44.4	19.8	59.0	36.1	17.7	75.8	46.5	21.2	73.7	37.3	9.1	69.8	33.2	7.6	73.2	36.5	8.9
		10	0.1	0.3	0.7	0.1	0.3	0.6	0.1	0.2	0.5	0.1	0.3	0.6	0.1	0.4	1.2	0.1	0.4	1.2	0.1	0.4	1.2
	50%	20	0.1	0.3	0.6	0.1	0.3	0.5	0.1	0.2	0.4	0.1	0.3	0.5	0.1	0.4	0.9	0.1	0.4	0.9	0.1	0.4	0.9
		40	0.1	0.3	0.4	0.1	0.2	0.4	0.1	0.2	0.3	0.1	0.2	0.4	0.1	0.3	0.6	0.1	0.3	0.6	0.1	0.3	0.6
		10	0.6	1.2	2.1	0.6	1.2	2.0	0.5	1.0	1.6	0.5	1.2	2.1	0.6	1.4	2.7	0.6	1.4	2.5	0.6	1.4	2.7
P(%)	20%	20	0.5	1.0	1.1	0.5	0.9	1.1	0.4	0.8	0.8	0.5	0.9	1.1	0.5	1.1	1.3	0.5	1.1	1.1	0.5	1.1	1.2
		40	0.4	0.6	0.5	0.4	0.6	0.5	0.3	0.5	0.3	0.4	0.6	0.5	0.4	0.7	0.4	0.4	0.6	0.3	0.4	0.7	0.4
		10	1.5	2.4	2.0	1.5	2.3	1.9	1.2	1.8	1.4	1.5	2.4	2.1	1.5	2.6	2.0	1.5	2.5	1.6	1.5	2.6	1.9
	10%	20	1.1	1.3	0.7	1.1	1.2	0.6	0.9	0.9	0.5	1.1	1.3	0.7	1.1	1.4	0.5	1.1	1.2	0.4	1.1	1.3	0.5
		40	0.7	0.5	0.2	0.6	0.5	0.2	0.5	0.3	0.1	0.7	0.5	0.2	0.7	0.5	0.1	0.6	0.4	0.1	0.7	0.5	0.1
		10	1.9	1.9	1.8	1.9	1.9	1.8	1.7	1.7	1.6	1.9	1.8	1.7	1.9	1.8	1.6	1.9	1.9	1.7	1.9	1.8	1.7
	50%	20	1.9	1.9	1.7	1.9	1.8	1.6	1.7	1.7	1.5	1.8	1.8	1.6	1.8	1.7	1.5	1.9	1.8	1.4	1.8	1.8	1.5
		40	1.9	1.8	1.5	1.8	1.7	1.4	1.7	1.6	1.3	1.8	1.7	1.4	1.8	1.6	1.2	1.8	1.6	1.1	1.8	1.6	1.1
		10	4.6	4.2	3.1	4.5	4.1	3.1	4.1	3.7	2.7	4.5	4.1	3.1	4.4	4.0	2.7	4.5	4.0	2.6	4.5	4.0	2.7
L.	20%	20	4.4	3.7	2.3	4.3	3.6	2.3	3.9	3.2	2.0	4.2	3.6	2.3	4.2	3.4	1.7	4.2	3.4	1.6	4.2	3.4	1.7
		40	3.9	2.9	1.5	3.8	2.9	1.5	3.4	2.5	1.3	3.8	2.9	1.6	3.8	2.6	0.9	3.8	2.4	0.8	3.8	2.6	0.9
		10	8.1	6.2	3.0	7.9	6.1	3.0	7.1	5.3	2.5	7.9	6.2	3.1	7.8	5.9	2.3	7.8	5.6	2.0	7.8	5.8	2.2
	10%	20	6.8	4.4	1.8	6.7	4.3	1.8	5.9	3.6	1.5	6.7	4.4	1.9	6.6	3.9	1.0	6.4	3.6	0.9	6.6	3.8	1.0
		40	5.0	2.6	1.0	4.9	2.5	1.0	4.2	2.1	0.9	5.1	2.7	1.1	4.9	2.0	0.4	4.6	1.8	0.4	4.9	2.0	0.4

This table reports the hedge effectiveness (17) (HE(%) in the top panel), the liquidation probability (12) (P(%) in the middle panel) and the implied leverage (19) (L. in the bottom panel), all under the optimal hedging strategy (14). We set $\Delta t = 1$ minute and consider three margin constraint levels ($m = 10\%, 20\%, 50\%$), three loss aversion levels ($\gamma = 10, 20, 40$) and three hedge horizon levels ($N\Delta t =$ 8h, 1d, 2d).