FIXED INCOME SECURITIES

FRE : 6411

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Risk Neutral Pricing Example

- $p_{io} \;=\; \beta\, E_0\left[\frac{U'(c_1)}{U'(c_0)} P_{i1}\right] = \frac{1}{r_f} E_0\left[\beta r_f \frac{U'(c_1)}{U'(c_0)} P_{i1}\right] = \frac{1}{r_f} E_0[Z' P_{i1}]$
- In Price Format :
- $p_{is_t} \;=\; \beta\, E_{s_t}\left[\frac{U'(c_{s_{t+1}})}{U'(c_{s_t})} P_{is_{t+1}}\right] = \frac{1}{r_f(s_t)} E_{s_t}\left[Z'_{s_{t+1}} P_{is_{t+1}}\right] = \frac{1}{r_f(s_t)} \tilde{E}_{s_t}\left[P_{is_{t+1}}\right]$
- Or in Rate of Return Format :
- $r_f(s_t) = E_{s_t}\left[Z'_{s_{t+1}} \frac{P_{is_{t+1}}}{p_{is_t}}\right] = E_{s_t}\left[Z'_{s_{t+1}} R_{is_{t+1}}\right] = \tilde{E}_{s_t}\left[R_{is_{t+1}}\right]$

Risk Neutral Pricing Example

0	1	2	3	4
				1
			0.99184	
			1.035	1
		0.926356		
		1.04		1
			0.956938	
			1.045	1
	0.880426			
	1.045			1
			0.956938	
			1.045	1
		0.905322		
		1.05		1
			0.947867	
0.822983			1.055	1
1.05				
				1
			0.953938	
			1.045	1
		0.908778		
		1.05		1
			0.947867	
			1.055	1
	0.847838			
	1.055			1
			0.947867	
			1.055	1
		0.888337		
		1.06		1
			0.938967	
			1.065	1

Risk Neutral Pricing Example

- Consider the above 4 period Economy. Where we have two traded assets in each state s_t , a risk free asset $r_f(s_t)$ and 4 period zero coupon bond $p(t, 4, s_t)$. Assume the actual probability of moving from state s_t to state $s_t u$, is 0.5
- $\pi_{s_t}(s_t u) = \pi_{s_t}(s_t d) = \frac{1}{2}$
- We want to see if we can find the risk-adjusted or risk-neutral probabilities and the random variable $Z'_{s_{t+1}}$. ($\tilde{\pi}_{s_t}(s_t u), \tilde{\pi}_{s_t}(s_t d)$, $z_{s_{tu}}$ *and* $z_{s_{td}}$)
- Let us consider if we are in period 2 and state $s_2 = ud$. In this state risk free rate $r_f(s_2 = ud)$ = 1.05 and price of 4 period zero coupon $p(2,4, ud\ ) = 0.905322$

Risk Neutral Pricing Example

- Next period we either move to $s_3 = udu$ or $s_3 = udd$, where the prices of 4 period zeros coupon bonds are $p(3{,}4, udu\ ) =$ 0.956398, and $p(3{,}4, udd\ ) =$ 0.947867
- To find $\tilde{\pi}_{s_t}(s_t u), \tilde{\pi}_{s_t}(s_t d)$ we can use either the price equation or return equation:
  - $0.905322 = \frac{1}{1.05}[\tilde{\pi}_{ud}(udu) * 0.956938 + \tilde{\pi}_{ud}(udd) * 0.947867]$
  - $1 = \tilde{\pi}_{ud}(udu) + \tilde{\pi}_{ud}(udd)$
- Return Equation
  - $1.05 = [\tilde{\pi}_{ud}(udu) * 1.05701 + \tilde{\pi}_{ud}(udd) * 1.04699]$
  - $1 = \tilde{\pi}_{ud}(udu) + \tilde{\pi}_{ud}(udd)$
- $\tilde{\pi}_{ud}(udu) = 0.3\ \&\ \tilde{\pi}_{ud}(udd) = 0.7$
- $z_{udu} = \frac{\tilde{\pi}_{ud}(udu)}{\pi_{ud}(udu)} = \frac{0.3}{0.5}\ \&\ z_{udd} = \frac{\tilde{\pi}_{ud}(udd)}{\pi_{ud}(udd)} = \frac{0.7}{0.5}$

Risk Neutral Pricing Example

- To summarize , define $B(t+1, s_t)$ as the value of MM account.
  - $B(t+1, s_t) = B(t, s_{t-1})\, r_f(s_t)$
- Define :
  - $\frac{P(t,s_t)}{B(t,s_{t-1})} = A(t, s_t)$
- Then :
  - $\frac{P(t,s_t)}{B(t,s_{t-1})} = \tilde{E}_{t,s_t}\left[\frac{P(t+1,s_{t+1})}{B(t+1,s_t)}\right] = A(t, s_t) = \tilde{E}_{t,s_t}[A(t+1, s_{t+1})]$
- Or $A(t, s_t)$ is a Martingale :
  - $A(0) = \tilde{E}_0[A(1, s_1)] = \tilde{E}_0[A(2, s_2)] = \ldots = \tilde{E}_0[A(t, s_t)]$
  - $A(t, s_t) = \tilde{E}_{t,s_t}[A(t+1, s_{t+1})] = \tilde{E}_{t,s_t}[A(t+n, s_{t+n})]$