# Valuation for Financial Engineering:

## Part II: Advanced Valuation Benchmarks

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Comments welcomed!

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# Contents

## Preface to Part II

In Part I, we learned many of the foundations of financial asset valuation, and also discovered how to simulate risky cash flows for the purposes of determining expected values. Together with an assumed discount rate, these tools would suffice to value many types of financial assets. However, the valuation would not be very satisfying, since real-world situations do not generally supply ready discount rates. Part II therefore focuses on benchmarking as a way to determine the cost of risk for an asset by linking it to other assets whose values are known or at least knowable.

Chapter 7 introduces risk formally, and how to deal with risk measurement for undercapitalized and overcapitalized assets. For example, if an asset can be safely leveraged with risk-free debt, then it is overcapitalized compared to other assets where the entire value is at-risk. We show that the asset return can be decomposed into time value and risk compensation in these cases, but it is important to note that the risk compensation is not proportional to the asset value, meaning one cannot simply add a risk premium to the discount rate in order to determine an asset's value.

Chapter 7 also converts risk measures into *capital* measures, which can be a very confusing concept for students used to thinking about capital as a cash investment. This leads to a precise relationship between the cost of risk and the cost of capital, and resolves a long-standing problem of how to compare risk and return on long & short positions in assets and all positions in derivatives. This leads to a discussion of generalized Sharpe Ratios to measure investment performance, and the risk relationships between unleveraged and leveraged assets.

In Chapter 8, we value cash flows in the context of benchmark portfolios. In this case, the measure of an asset's risk is its covariance with the benchmark, not its variance *per se*. The model provides a required relationship between expected returns and risk that provides additional insights into portfolio modeling. The model outputs in Chapter 8 provide expected returns an asset return covariance parameters that can be used as inputs to the Markowitz portfolio model, also known as MPT or Modern Portfolio Theory. We therefore conclude with the presentation of the Capital Asset Pricing Model as a method of portfolio benchmarking and also private portfolio valuation.

Chapter 9 recognizes that traded benchmarks include derivative prices as well as asset prices. To that end, valuation relationships are described for futures and forward contracts. In later chapters, futures prices will be used as a way to value cash flows correlated with the underlying asset --- we will not address all the important institutional features or the use of futures contracts by end users. Chapter 10 follows a similar approach in using traded options to value risky cash flows. For example, the cash flows of a real option may be best modeled by combining asset benchmarks, futures benchmarks and option benchmarks.

Part III consolidates the benchmarking techniques of the Part II chapters into a single recommended valuation framework that makes maximal use of traded asset and contract valuations to model cash flows. In Chapter 11, risk uncorrelated to the pricing benchmarks is assumed to be diversifiable or unpriced. In Chapter 12, we explore situations where it makes sense to price idiosyncratic risk, and discuss a methodology for doing so.

# Chapter 7:  Risk, Capital and Valuation

Simple risk measures are easy enough to understand, but few beginning students understand the links between risk and capital.  Also, the connection between risk and value is not always easy, even though we have a sense that increased risk should generally reduce value.  Yet there are some situations risk has no effect on value or even increases value.

We can think of return and risk being two sides of a coin.  They go together, and the students who understand the linkage will master the valuation of risky cash flows.

## WHAT DO WE MEAN BY RISK?

In Part I of this text, we assumed that projected cash flows were either known with certainty, or that we knew what discount rate to apply to expected cash flows.  For example, annuity payment amounts might be known, even though someone's lifetime of annuity payments might have an uncertain termination date.  We also presented default risk in corporate bonds, but then proceeded to model the expected cash flows as if they were deterministic, adding a fixed risk premium to compensate investors for taking bond risk.  The expected cash flows were discounted at a given risk-free rate of interest plus a credit spread to determine their value.

We are interested in how risk is measured, how it affects required returns, how it relates to capital, and how it affects asset valuations.  When cash flows are risky, we would at first expect that the discount rate would be higher than the risk-free rate, reflecting additional compensation (required return) to pay for the risk in cash flows. This is not necessarily the case.  For example, some people might want to take a risk, and be willing to pay to do it!  This happens when taking a risk can help them reduce the risk of their portfolios, such as the purchase of portfolio insurance through put options.  Because the return of the put option is negatively correlated to their portfolio, they actually value it more highly, that is, they require a lower return.

If we are looking at risks through the portfolio lens, we may also ask how diversification affects risk. *Diversification* means holding a broad-based portfolio with many types of assets with the hope that the risk is lower in some sense.  Most people believe that diversification reduces risk, but this is an ambiguous statement.  By adding risks to a portfolio of positively but not perfectly correlated risks, the risk increases with each asset we add.  However, the percentage risk per dollar invested in this portfolio declines. Therefore, diversification reduces risk only under certain conditions, and only if we are speaking of the risk of percentage returns.

**A financial analyst must choose their risk definition, and make sure their risk measures reflect the choice.**

Risky projects play an important role in corporate finance.  Companies may value their projects one way and shareholders may value them another way.  Financial companies and nonfinancial companies approach the problem differently.  Different companies measure risk differently, depending on shareholder preferences. However they measure risk, and translate that into risk charges and project valuations, the tools in this section will provide you the means to value all sorts of risky cash flows in many different situations.

## ASYMMETRIC RISKS

Even in our simplest examples, we saw that risk might be assessed differently between two parties contemplating a transaction. In the first example, where an individual receives a fixed monthly payment for life, their risks are (a) the risk of failure of the insurance company to make payment, (b) the risk of guarantors of the insurance company not making (or delaying) payment in the event of the default of the insurance company, usually the government, (c) the risk of dying sooner than expected, and (d) higher inflation rates lowering the spending power of the payments. There is no single number that will summarize the risk of this contract for the individual. Furthermore, the customer cares mostly about the risk of living without guaranteed income, and the ability of the annuity contract to reduce that risk.

For the insurance company, the risks may have to do with (a) the customer living longer than expected, or medical advances that extend customer lives across the board (b) lower inflation rates and interest rates than projected, and (c) risks of their pricing models being incorrect. Again, no single risk measure will do, but more importantly, the insurance company will look at risk very differently from the customer.

Another way to see this problem is that insurance is an asset for the customer, and a liability for the insurance company. Because the asset has the benefit of reducing his risk of living without income, the customer is willing to pay a premium over expected value for the contract, and will pay a premium to the extent the insurance industry is not fully competitive. For the insurance company, this is all risk, so they must charge a premium for the product over expected value.

## HOW IS RISK MEASURED?

Let's talk about cash flows first. There are many suggested measures for risk in cash flows. Here are several; the numbers in parentheses refer to the results of the following example.

*Symmetric variation*

- Standard deviation (7.12)
- Mean absolute deviation (6.32)

*Downside variation*

- Downside standard deviation (5.41)
- Downside mean absolute deviation (5.27)
- Worst case outcome relative to expected (6.60)

For example, suppose that a risky cash flow is equally likely to be $1, 2, 4, 11, or 20. The mean is 7.6. To begin, we determine the deviation, or difference, of each possible value from its mean. The absolute deviation is simply the absolute value, while the squared deviation is the square. Both are clearly positive, since we would like to obtain a positive measure of dispersion. The *mean absolute deviation* is the average of the absolute deviations, while the *variance* is the mean of the squared deviations. The *standard deviation* is the square root of the variance.

The downside measures compute variability of terms below the mean, which are shaded in the box below. Otherwise, they are analogous to the measures that consider both upside and downside.

| # | Outcome | Deviation | AbsDev | Squared | Sqrt |
|---|---|---|---|---|---|
| 1 | 1 | -6.6 | 6.6 | 43.56 | |
| 2 | 2 | -5.6 | 5.6 | 31.36 | |
| 3 | 4 | -3.6 | 3.6 | 12.96 | |
| 4 | 11 | 3.4 | 3.4 | 11.56 | |
| 5 | 20 | 12.4 | 12.4 | 153.76 | |
| | | | | | |
| Mean | 7.6 | 0 | 6.32 | 50.64 | 7.12 |
| Mean Downside | | | 5.27 | 29.29 | 5.41 |

Why should anyone prefer one risk measure to another? For historical reasons, the standard deviation is the most widely used. The standard deviation of a cash flow is the square root of its expected squared distance from the mean. This may seem slightly contrived to the practical reader, but the mathematical properties of standard deviation make this calculation very tractable for use in applications. It appears in many popular probability distributions, especially the Gaussian or normal distribution, where it plays a prominent role.

A more natural measure of variability might be the mean absolute deviation, i.e. the average absolute value of the deviation from the mean. While it appeals to us in its simplicity, it does present analytical challenges, and for this reason is not widely used in practice.

Downside risk measures were made popular by the asset management industry, as they rightly pointed out that risk to the upside is not such a bad thing! Also, many asset managers use derivative overlays causing return distributions to be asymmetric. Therefore, they benefited from producing downside risk measures for the purpose of making better investment decisions. You can see in our example that a lot of the standard deviation and mean absolute deviation in the cash flows comes from the $20 outlier, a positive outcome. Taking positive results out of the sample therefore leads to smaller estimates of downside risk.

Taking the difference between the expected case and the worst case value seems extreme, yet that is exactly what we do when we quantify stock risk. Since the lowest possible price of a stock is zero, we calculate returns based on the entire value of the stock.

## THE COST OF RISK

Whatever measure of risk you choose, we need to be able to figure out the cost of risk in any given situation. For example, investment banks in the 1990s had simple rules about accepting risky projects. One rule was:

**Accept a project only if**

**the expected profitability exceeds 25% of the risk measure.**

Why would the bank do this? Banks must not take risks beyond their risk capacity lest they increase their chances of bankruptcy and require bailouts from their governments. Of course, in these situations, senior bank managers are usually replaced.

Since total risk capacity is limited, every incremental risk must be judged to make sure that (a) the bank does not take too much risk and (b) that high-risk, low-return projects are not crowding out higher return projects.

Another way of saying this is that risk is a scarce resource for banks, and therefore there must be a cost for taking risk. If there is no cost, then every bank trader or decision-maker will take all risks with a positive return and likely choose a portfolio of risks that is suboptimal for the overall bank.

As another example, consider investment in the stock market. Suppose the risk-free interest rate is r=3%, and the market risk return premium is k=5%. Then the compensation in return for taking market risk equals 5% of the investment per year. The dollar amount of risk is deemed to be the investment amount, and the dollar cost of risk is 5% of the investment.

In some cases, it is reasonable to estimate the cost of risk as a constant percentage of value, and in other cases, it is more reasonable to estimate the cost of risk as a dollar amount. It really does not matter which method we choose, as long as the method reflects assumptions we are willing to accept.

## ECONOMIC CAPITAL

In the last section, we referred to a bank's total risk capacity as the limiting factor in doing business. Banks have plenty of access to cash if they are solvent, but are limited by bank regulations in the amount of risk they can take. Accordingly, the measure of capital for banks is not cash, but risk. If their risk assessments are correct, then their risk capacity is the total amount they could lose, with some small probability. Similar to investing in a stock and losing everything we invested, banks can lose every dollar they place at risk. Hence, risk resembles capital. Some economists will use the phrase *risk capital* to designate the amount of capital put at risk. Regulators prefer the phrase *economic capital*, since it conveys the true economics of the bank's risk-taking decisions. The bank places its economic capital at risk.

Hence, banks are required to hold sufficient economic capital, i.e. shareholder equity based on a formula related to their economic capital --- their risk. When regulators require banks to increase their capital, they mean that more equity capital is required to support the risk-taking of the bank. The higher capital requirements can stem from higher risk-taking by banks, and they can also stem from increased risk aversion of the banking regulators.

How is economic capital measured? While the details are incredibly complex, the principles are simple. We establish a low probability, let's say 1%, and determine what is the worst case loss a bank could experience, based on its own risk models, such that the probability of that loss or greater would be 1% or less over a given time period. This statistic is called the value-at-risk, or VAR. Because it is a dollar-based risk mesasure and not a percentage, it can be computed for all manner of positions, long and short, primary and derivative. The actual process to compute a bank's VAR is extremely complex for most banks, but all banks have to report VAR and similar risk statistics in their assessment of the bank's risk and economic capital. Perhaps the most famous deployment of VAR was made by JPMorgan in their RiskMetrics™ product. Economic capital is determined as a multiple of VAR. The multiple is usually greater than 1 to reflect risk factors not adequately captured by the VAR model.

Coming full circle to the GVE, we may now interpret the cost of risk as the required return on risk capital. The cost of money is still the risk-free rate, but now we see that the real driver of a bank's risk-taking decisions is the expected reward for using its economic capital.

## RISK, THE GVE AND CERTAINTY EQUIVALENTS

Now we are finally prepared to value a single risky cash flow. Let's assume that the letter "C" refers to a risky cash flow in one year, a random variable whose probability distribution is known. The expected value of C is $E[C] = \mu$. For now, assume the risk measure is the cost of the investment, and the cost of risk is k, a constant percentage of the investment.

Writing the GVE, we now realize that we need to be compensated for time value *and for risk* on the left side of the equation, i.e. for "waiting *and worrying*." The right side of the equation is the same for a cash flow in one year: we give up the asset value and collect $\mu$ on average.

$$rV_0 + kV_0 = -V_0 + \mu$$

$$V_0 = \frac{\mu}{1 + r + k}$$

Therefore, we simply increase the discount rate in this case by k in order to find the value of the cash flow. Sometimes, "k" will be called a *risk premium*, and $kV_0$ is the *risk charge*.

What if the risk measure is the standard deviation of cash flow $\sigma$, not proportional to asset value? In this case, the GVE becomes

$$rV_0 + k\sigma = -V_0 + \mu$$

$$V_0 = \frac{\mu - k\sigma}{1 + r}$$

In this formulation, the quantity ($\mu$–$k\sigma$) is called the *certainty-equivalent* of the cash flow, since it can be discounted at the risk-free rate to find the value. In GVE terms, this amounted to moving $k\sigma$ from the left to the right side of the equation. The interpretation is that we discount capital gains plus cash flows minus risk charges at the risk-free rate. In this application, "k" would be called the *cost of risk*, and $k\sigma$ would be called the risk charge.

These two methods yield the same valuation as long as the total risk charge in the first equals the total risk charge in the second. These examples illustrate the two most popular methods for valuing cash flows. The first method is to try to determine what premium to add to the discount rate for valuation purposes. It is used when the risk measure equals the investment value and the cost of risk is a constant proportion of investment value. The second method is to determine the cost of the risk, and deduct it from the mean in order to compute the certainty-equivalent and its risk-free present value. It is used when the risk compensation can be best modeled as a constant. The first method is more popular, but the second method is more applicable to a variety of problems. For example, the risk premium would be difficult to define when considering short positions and derivative positions.

There are of course many models one could use for measuring and pricing risk, but they are all consistent with the GVE.

## REVISITING "RETURN"

In Chapter 1, we pointed out that if an investor buys a share of stock for $20, and the price rises to $21, the percentage return is 5%. However, if the investor sells short, it is impossible to determine the return on his strategy since we do not know how much he might have lost. Therefore we would lack a denominator for the return calculation. We have a similar problem with forwards and futures positions --- returns are not defined. The problems computing return on futures and forward contracts are of course magnified for short call option positions. As in the case of stocks, theoretical losses to short call option positions have no limits.

So far we have seen that percentage return calculations can be frustrating for undercapitalized investments. They can also be frustrating for fully capitalized investments. Consider the investment choice below:

| Investment | Expected return | Worst case |
|---|---:|---:|
| Blue-chip stock at $100/sh | 10% | $80 |
| Startup company at $100/sh | 20% | $0 |

Suppose we wanted to invest $100. At first, it would appear that one could expect to make a higher dollar return on capital on the startup company than the blue-chip firm. However, this conclusion depends on how capital is measured. The conclusion is correct if both investments are fully capitalized, i.e. paid in full with the investor's funds. However, if the worst case blue-chip value were truly $80, the investor could borrow $80 on a risk-free basis without fear of repayment from stock sale proceeds. If the risk-free interest rate were zero, just to keep things simple, this means that the investor could expect a $10 return from a $20 capital investment, rather than a 10% return on a $100 investment. This implies a 50% return on risk-adjusted capital! In short, the investor could buy $500 worth of the blue-chip stock, borrow $400, and expect a $50 return without risking his ability to repay the debt from selling the stock.

In this case, you might say that the actual capital required for one share of the blue-chip stock is $20, while the capital required for one share of the startup company is $100. In general, overcapitalized investments can be divided into a portion that is considered risky and a portion that is considered risk-free. This differs sharply from the traditional approach, which attributes the entire return on investment to its level of cash deployed:

risk.

TRADITIONAL WAY OF THINKING

$100 in S&P

Expected return on S&P = 12%

Expected return on T-bills = 6%

"Worst case" loss (1 yr) = 25%

Expected cash return = $12

In the new way of thinking, $100 of cash capital is replaced with the notion that if only $25 is at risk, that should be the measure of capital. Therefore each $100 earns an investment return that can be divided into two components, a risk-free component associated with the cash requirement and a risk-bearing component based on the actual amount of capital at risk.

NEW WAY OF THINKING

$100 in S&P

$100 risk-free cash investment (6%)

$25 risk capital (24% return)

$6 $6

Expected cash return = $12

The total cash return on an investment is the same as in the traditional method, but the attribution is different. The "cash investment" earns $6 per $100 invested, while the "risk capital" earns $6 per $25 put at risk.

In GVE terms, we could break down the required total return differently. This equation forces the return on risk capital to be 6% of the total 100 investment:

$$Time\ value + Risk\ Compensation = Total\ expected\ return$$

$$0.06(100) + 0.06(100) = 0.12(100)$$

Using the risk capital formulation, if the maximum loss is 25%, then we have

$$0.06(100) + 0.24(25) = 0.12(100)$$

This example shows how overcapitalized investments can also distort the calculation of return and make investment comparisons meaningless. In this case, like the undercapitalized case, a better measure of capital relates to some measure of the maximum loss associated with the investment.

Nonfinancial corporations also have difficulties with the return concept as a measure of performance. A firm that maximizes return may be doing its shareholders a disservice. This occurs because return is normalized to the amount of capital employed. A smaller return on a higher capital base may add significant value to shareholders, and therefore, firms should look to value-added rather than return on capital as a measure of performance. The only time it makes sense to maximize return on capital is when capital is the limited resource. This is sometimes the case in the real world. However, under the perfect markets assumptions of traditional finance theory, unlimited capital is available from shareholders as long as a competitive return is earned. For this reason, real world practices often differ from recommendations of financial theory.

## RISK CAPITAL AND RAROC

When banks make decisions on risky deals, they frequently consider the cost of risk capital as a factor in the decision. *Risk capital* is an estimate of the maximum dollar amount put at risk in an investment. In the previous example, the risk capital was $25, the maximum loss on a blue chip share investment. In practice, risk capital is measured as the maximum likely loss. In other words, it is a statistic, which may be completely different from cash invested.

Using our previous example of the short sale of a stock at $20, if we are comfortable assuming we can lose no more than $10, that is a reasonable measure of capital. Therefore a $1 profit would represent a 10% return on risk capital. In the case of the oil forward contract, if the maximum loss were $5, we could say the $1 profit was a 20% return on risk capital. This ratio, dollar profitability net of interest rate charges divided by maximum estimated loss, has been called RAROC by Bankers Trust. RAROC stands for *Risk-Adjusted Return on Capital*, and Bankers Trust was acquired by Deutsche Bank in 1999.

The RAROC measure is proportional to the more famous Sharpe Ratio, but it qualitatively performs the same function.

## THE SHARPE RATIO

The idea of looking at excess returns per unit risk was not unique to RAROC. Professor William Sharpe of Stanford University promoted the reward to total variability ratio as a primary measure of investment performance. Sharpe attributes the introduction of the measure to A.D. Roy (1952), but Sharpe (1966) developed and applied the concept. The Sharpe Ratio will be used in Chapter 8 to determine the optimally diversified portfolio holdings. You will discover in the subsequent chapters that the Sharpe Ratio is not only the most important measure of investment performance, but it also can be thought of as the cost of risk capital or the return on risk capital.

Starting with the original definition, for fully funded investments, the Sharpe Ratio is the percentage return premium minus the risk-free rate, or the risk premium, divided by a risk measure, in this case the standard deviation of returns.

$$Sharpe\ Ratio = \frac{r_i - r}{\sigma}$$

The ratio can be computed on a prospective or retrospective basis. For investments that are not fully capitalized, it is preferable to work with a dollarized Sharpe Ratio, which can be interpreted in the same fashion:

$$Dollarized\ Sharpe\ Ratio = \frac{Expected\ profitability - Interest\ costs}{Risk\ measure}$$

Taking our GVE from earlier, we started with

$$rV_0 + k\sigma = -V_0 + \mu$$

Solving for k,

$$k = \frac{\mu - (1 + r)V_0}{\sigma}$$

In other words, the Sharpe Ratio can also be interpreted as the cost of risk in the GVE equation if the risk measure is given by $\sigma$. The numerator is the cash flow plus the capital gain minus financing costs, and the denominator is the risk measure, all in cash flow terms rather than percentages.

## LEVERAGE AND RISK

This classical example comes from Modigliani and Miller (1958) although it predates the invention of the Sharpe Ratio. Suppose a firm owns a risky asset worth A now, which it is able to finance up to a level D at the risk-free rate of interest. Its return volatility is given by $\sigma_A$. For simplicity, we may assume this asset is liquidated on one year and the debt will be paid off. The risk-free rate is r.

Without debt, the asset is fully owned by equity investors, and the value is represented by E. With leverage and risk-free debt, some of the asset's value is paid to debtholders and some is paid to equity:

$$r_A A = rD + r_E E$$

Solving for $r_E$, we obtain a slightly different form of the famous Modigliani and Miller Proposition 2, using *A=D+E*:

$$r_E = r_A\left(1 + \frac{D}{E}\right) - r\left(\frac{D}{E}\right)$$

The standard deviation of $r_E$ can be inferred directly from the multiplier of $r_A$ in this equation since r is risk-free:

$$\sigma_E = \left(1 + \frac{D}{E}\right)\sigma_A$$

When we compute the Sharpe Ratio of the levered equity, an interesting result appears:

$$Sharpe\ Ratio\ of\ E = \frac{r_A\left(1 + \frac{D}{E}\right) - r\left(1 + \frac{D}{E}\right)}{\left(1 + \frac{D}{E}\right)\sigma_A} = \frac{r_A - r}{\sigma_A} = Sharpe\ ratio\ of\ A$$

… which seems surprising but should not be. Intuitively, if an asset is financed by risk-free borrowing, there has been no risk transferred to the lender. Hence the owner holds the same dollar risk in the unlevered position and the levered position. However, when the dollar risk is divided by the equity, the percentage cost of risk varies with the amount of leverage. Therefore, expected returns adjust by exactly the amount needed to ensure that the Sharpe Ratio remains the same.

> The Sharpe Ratio of an investment is therefore invariant to the level of risk-free financing, while the expected return and risk of a levered investment both increase with the amount of leverage.

Using the GVE, the effect of risk-free leverage can be shown immediately. If an asset A satisfies the GVE, the cost of risk is k and the risk measure is σ, then

$$rA + k\sigma = r_A A$$

Risk-free borrowings in the amount of D reduce the size of the investment in the asset, but also reduce the cash flows due to the interest expense, while risk level and cost remain the same:

$$r_E E = r(A - D) + k\sigma = r_A A - rD$$

Adding rD to both sides verifies the equivalence.

*Example*

Consider the valuation of a risky cash flow in one year with the parameters below, i.e. mean 100, stdev 20 and so on, as in the table below. Using σ as the risk measure, and a cost of risk of k=0.50, the valuation is $81. The very high 11.11% risk-free interest rate is chosen only to keep the numbers simpler.

Cash flow valuation

| | | | |
|---|---|---|---|
| | Mean | 100 | |
| | StDev | 20 | |
| | Risk-free | 11.11% | rF |
| | k | 0.5 | |
| | Value | 81 | |
| | ExpRet | 23.46% | rU |
| | Vol | 24.69% | σU |
| Leveraged | Borrow | 40 | |
| | Equity val | 41 | |
| | Exp ret | 35.50% | rL |
| | Vol | 48.78% | σL |
| D/E ratio | | 97.56% | |

Valuation statistics for different debt levels (borrow)

| Borrow | D/E | rE | σE | Sharpe |
|---|---|---|---|---|
| 0 | 0.00% | 23.46% | 24.69% | 50.00% |
| 10 | 14.08% | 25.20% | 28.17% | 50.00% |
| 20 | 32.79% | 27.50% | 32.79% | 50.00% |
| 30 | 58.82% | 30.72% | 39.22% | 50.00% |
| 40 | 97.56% | 35.50% | 48.78% | 50.00% |
| 50 | 161.29% | 43.37% | 64.52% | 50.00% |

The cash flow is unleveraged, so we can calculate its mean unleveraged return $r_u$ (100/81-1) and volatility $\sigma_u$ in percentage terms (20/81). Now suppose we buy the asset with risk-free debt of $40. This leaves (81-40=) 41 for equity. The mean return for equity is 100 minus the repayment of the loan 40(1.1111). Dividing by the $41 equity value, the expected return on leveraged equity $r_L$ is 35.50%. The standard deviation $\sigma_L$ is 20/41, since all the risk still belongs to the equity investor. The D/E ratio 97.56% = 40/41.

In the table on the right, different levels of risk-free debt are used in order to demonstrate the earlier claim that the Sharpe Ratio is invariant to risk-free debt level. Of course, we expected this conclusion since the Sharpe Ratio equals k, the cost of standard deviation risk in this valuation model.

## RISK BENCHMARKS: SINGLE FACTOR MODELS

The idea of a *risk benchmark* arises in situations where some of the risk of an investment may be assumed unpriced, and the other part would be perfectly correlated with another risk factor that has already been priced. To be clear in our next few examples, let's refer to $r_m$ as the total expected return of the benchmark asset, r as the risk-free rate, and $r_i$ as the expected return on asset i.

We can decompose the risk of $r_i$ using the usual regression framework, assuming $r_i$ and $r_m$ are jointly normally distributed with correlation $\rho$. The regression equation is

$$r_i = a + b_i r_m + \varepsilon$$

Using a = $E[r_i] - bE[r_m]$, and $b = \rho\sigma_i/\sigma_m$, we can decompose the variance due to the benchmark and due to the residual risk $\varepsilon$, represented as $\sigma_\varepsilon$.

$$Var(r_i) = b_i^2 Var(r_m) + \sigma_\varepsilon^2$$

The two components of the variance are called the *systematic risk* and the *idiosyncratic risk* of the asset if the benchmark is the overall market portfolio. The systematic risk is priced at $(r_m - r_f)$ per dollar value per year, and the idiosyncratic risk is assumed diversifiable, with a price of zero. Therefore, the priced variance of return is $b_i^2 Var(r_m)$, and the priced standard deviation of return is $b\sigma_m$. And, if the standard deviation of the asset return is proportional to the standard deviation of the benchmark return, we may infer that the risk premium is also proportional. The asset's risk premium is therefore $b_i(r_m - r_f)$, and we have the famous equation:

$$E[r_i] = r + b_i(E[r_m] - r)$$

This equation is frequently referred to as the CAPM equation (CAPM=Capital Asset Pricing Model). In the CAPM, M refers to the universe of investable assets. The equation can also be applied to pricing relative to other portfolios, in which case this is called the *market model*. We are referring to this class of models collectively as benchmarked return models. Not surprisingly, the CAPM equation is another expression of the GVE, where the price of risk is *$(E[r_m] - r)$* the risk measure is $b_i$, and any dividends are included in *$E[r_i]$*.

Here's another quick method to derive the CAPM equation. As we have seen previously, the *Sharpe Ratio* is the ratio of two percentages. The numerator is the risk premium, i.e. $r_i$–r, and the denominator is $\sigma_i$. Therefore, the Sharpe ratio is the percentage risk premium per unit risk. However, in a benchmark model, the only risk that is considered for pricing purposes is the benchmark risk, which is $b_i\sigma_m$. If we assert that an asset's Sharpe Ratio relative to its benchmark risk must be the same as the benchmark risk considered on its own, then we have:

$$\frac{E[r_i]-r}{b_i\sigma_m}=\frac{E[r_m]-r}{\sigma_m}$$

The $\sigma_m$ terms in the denominators cancel, and once again we obtain the expected return equation. We will show in a later chapter that this simple trick allows us to derive PDEs for option values quickly.

## PRICING CASH FLOWS IN A SINGLE FACTOR RISK MODEL

We now would like to consolidate the last few sections to value cash flows in a single benchmark model. Let's suppose that the cash flow in one year is C, which is normally distributed with mean $\mu$ and standard deviation $\sigma$ as before. We'd like to use the benchmark model to find the right discount rate, but that requires knowing b, which requires knowing the covariance of the returns $r_i$ and $r_m$.

Fortunately, this can be derived, since we can take advantage of the linearity property of the covariance operator. Let V be the value of the asset that pays C in one year. Then

$$cov(r_i,r_m)=cov\left(\frac{C-V_0}{V_0},r_m\right)=\frac{cov(C,r_m)}{V_0}$$

Substituting into the GVE, and adding $V_0$ to both sides

$$(1+r)V_0+\frac{cov(r_i,r_m)}{\sigma_m^2}(r_m-r)V_0=\mu$$

Making the substitution in the first equation,

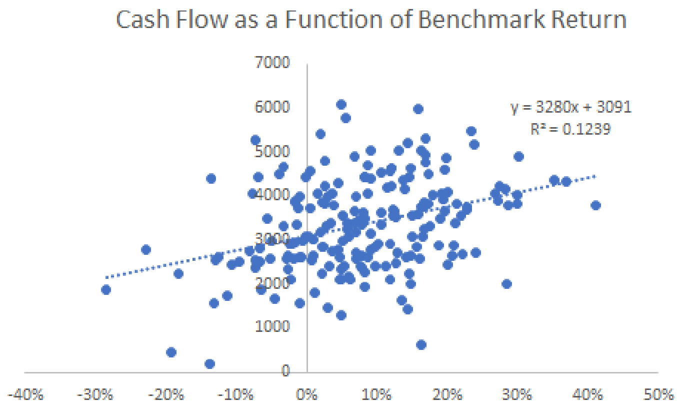
$$V_0=\frac{\mu-\lambda\,cov(C,r_m)}{1+r};\ \ \lambda=\frac{r_m-r}{\sigma_m^2}$$

In this formulation, $\lambda$ is the price per unit of variance risk, and the risk measure is the covariance between the cash flow and the benchmark return. This is a very useful way to compute the certainty equivalent, since we can easily compute the covariance between the cash flows and the market return, even when we don't know the covariance between the asset's return and the market.

This method is also particularly useful when we have cash flows over multiple periods.

## AN EXERCISE USING SIMULATION RESULTS

Suppose you are working for a factory that makes welding equipment. When the economy is robust, they've found that the demand for their product rises as the stock market rises, and their costs also rise, though not as much. You've built a model to simulate different cash flow scenarios correlated to market returns. The results of 200 simulations of first year income is shown in the chart below:




| |
|---|
| *Do you recall these properties of simple linear regression?*<br>• The estimated slope is cov(x,y)/var(x).<br>• The point (average x, average y) is on the regression line.<br>• The previous fact is used to find the estimated intercept, which is average y minus slope times average x. |

What is the current value of the unknown income in one year? Assuming the benchmark model holds, let's use the following parameters

$$E[r_m] = 7.76\%$$
$$\sigma_m = 11.22\%$$
$$r = 3\%$$

A. Value the cash flow using the benchmark model and using risk-free discounting of the expected value.
B. Use the CAPM equation to find the discount rate, and determine the current value of the cash flow.

When is model A used and when is model B used?

*Hints for Part A:*

| | |
|---|---|
| 1. Find the cost of risk $\lambda$ | |
| 2. Find $cov(C,r_m)$ | |
| 3. Find the expected cash flow | |

| | |
|---|---|
| 4. Find the present value | |
| 5. Find the present value using zero cost of risk | |

*Solutions: 3.78,41.29,3345,3097,3248*

*Hints for Part B:*

| | |
|---|---|
| 1. Find cov($r_i$,$r_m$) | |
| 2. Find $\beta_I$ | |
| 3. Find the expected return | |
| 4. Find the present value | |

*Solutions: 0.013335,1.06,8.04%,3097*

The Model A approach is used when cash flows are directly modeled based on their correlation with benchmark factors. Model B is used when relative discount rates for cash flows are based on estimated betas of the assets comprised by the cash flows. The difficulty with Model B is that one needs to know the value of the cash flows in order to determine the return covariance.

## SUMMARY AND CONCLUSION

Risk can be measured in many different ways. For analytic tools, financial analysts usually use variance and standard deviation because of the mathematical tractability, but mean absolute deviation and asymmetric risk measures are often more appropriate for describing risks.

Risk can have an associated risk charge, which may differ by investors. The usual positive risk charges apply when an asset adds risk to someone's portfolio, but a negative charge can apply if risk is reduced. Diversification is a way to reduce risk, but only in percentage terms --- generally, portfolio risk increases as assets are added.

The cost of risk represents one of the most important concepts in finance. It is the critical component of the total return calculation. In financial institutions, it is a mechanism by which risk capital is allocated to different business groups. When risk capacity is scarce, free risk is squandered --- the cost of risk leads to optimal risk-taking. Risk is also a measure of capital. It performs better than cash as a capital measure, and controls for overcapitalized and undercapitalized investments.

This chapter presented a simple formula for valuing normally distributed cash flows when the risk measure is σ. If the risk is proportional to asset value, a risk premium is added to the risk-free rate to discount the cash flow. If risk is not necessarily proportional to asset value, then a risk charge is deducted from the expected cash flow to compute a certainty equivalent, which can be discounted at the risk-free rate.

Percentage returns were problematic in Chapter 1, and they continue to cause trouble. The chapter also shows that when investments are underleveraged, we can get biased views of risk and return. We learned how to incorporate leverage into our risk assessments, and also learned that the Sharpe Ratio is invariant to changes in risk-free leverage levels. As it turns out, the Sharpe Ratio is the cost of risk (k) when σ is the risk measure.

Finally, we touched on the single factor market model, a foreshadowing of what is to come in Chapter 8, linked it back to the GVE and the cost of risk in portfolio-benchmarked valuation problems.

## REFERENCES AND FURTHER READING

Modigliani and Miller

Roy 1952

Sharpe 1966

## SHORT ESSAY QUESTIONS

1. Why do investors tend to prefer downside measures of risk to other measures?
2. Suppose you have one investments A and you would like to make another investment B. Provide an example that shows that the total dollar risk of your portfolio increases whereas the risk per dollar invested decreases. What does this show?
3. How do underleveraged investments distort return and performance measures, and how can this be corrected?
4. How is risk related to capital in a bank?
5. What is meant by the cost of risk? Should it be costly for banks to take risks?
6. What is the Sharpe Ratio and why is it important?
7. How does leverage affect expected return and risk in an investment? Why?
8. What is the impact of risk-free leverage on the Sharpe Ratio of an investment?
9. What is the market model, and how does its pricing follow from the GVE?

## EXCEL EXERCISES

A. Risk Measures

Download the S&P Index data for the last calendar year (ex: 12/31/2016-12/31/2017) as represented by the equity SPX. (We will ignore any dividends for this exercise).

Calculate the following risk measures in dollar and % terms for daily observations:

a. Standard deviation

b. Mean absolute deviation
c. Downside standard deviation
d. Downside mean absolute deviation

B. In the “Leverage and Risk” section, reproduce the Excel spreadsheet, but include volatility levels of 10 and 30 in addition to 20. What can you say about the effect of volatility on the results? Does it make sense?

# Chapter 8: Portfolio-based benchmarks

## INTRODUCTION

This chapter introduces students to portfolio mathematics, and how to price risky cash flows based on their relationship to the risks of a portfolio. It starts with the idea of a normally distributed cash flow, and shows how investors translate their risk aversion into a required rate of return on the cash flow. This required rate of return will generally be greater than the risk-free rate, and the risk premium will depend on the covariance between the asset's return and the return on the market portfolio. The result of this model is an equilibrium expected return and return covariance matrix which can be used as inputs to the Markowitz model, often referred to as MPT or Modern Portfolio Theory. We show the pricing equations to be identical to the Capital Asset Pricing Model. Unlike the CAPM, however, the required returns and covariance matrix will be defined by the cash flows, not assumed to be known in advance. This allows us to discover the relationship between expected return and risk more precisely.

This is the first benchmark valuation we will study, where the benchmark is a reference portfolio of risky cash flows. In later chapters, we will show how to benchmark risky valuations to other comparable assets, including forward contracts and options.

The chapter continues by presenting the original Markowitz portfolio optimization model, and develops a numerical example students can replicate. Finally, since implementation requires the estimation of expected returns, and historical data don't measure expected returns well, we explain the Black-Litterman adjustment and how it is used to make better portfolio recommendations. In the end, the Black-Litterman adjustment finds the implied expected returns of the benchmark portfolio, which is of course where the chapter begins.

## INTRODUCTION TO RISK PRICING

In Part I, we valued assets for which either there was no risk, or we discounted expected cash flows at an assumed rate. What if we knew that various cash flows had risks, and we wanted to figure out what discount rates should be used to value them? One approach, which we discuss in a later chapter, is to try to price the risks by linking it to other risky benchmark assets whose risk prices are known. In this chapter, we assume there are no such benchmarks. We only assume that there are a number of risky assets available in an economy, and we use the economic agents' investment behavior and risk aversion to infer the appropriate discount rates for assets.

This approach was pioneered by Harry Markowitz, a PhD student of Milton Friedman at the University of Chicago, in 1952. It proved to be the backbone for a number of asset valuation methods including the Capital Asset Pricing Model (CAPM) of Sharpe, Lintner, Mossin and Treynor that were developed in the 1960s. In practice, the betas produced by these models are widely used by portfolio managers to manage the market risk in their portfolios.

One limitation of the Markowitz published framework was that he assumed the existence of a number of assets with known expected percentage returns and standard deviations. At that time, it made sense, since capital assets were generally fully funded. As we discussed in Chapter 1, however, there are times when percentage returns cannot be computed, hence we learn how to apply the Markowitz framework to dollar returns and dollar risk measures in this chapter. The key benefit of our approach is that one can use the portfolio framework first to value risky assets, and then determine required returns and percentage standard deviations. One's optimal portfolio holdings are based on the valuations --- the expected returns and covariances are determined by the model once the valuations are known. Asset prices are needed to compute percentage expected returns and standard deviations --- it is very difficult to start our analysis by assuming we know the expected percentage returns and standard deviations, without any reference to asset prices.

## MANY RISKY ASSETS, USING MATRIX NOTATION

In this section, we'll derive the pricing of risky assets in a simple economy. Don't let the matrix notation scare you; while it is needed for the derivation, the ending valuation formula is quite simple. You may refer to the appendix to study the matrix notation and how it relates to traditional algebraic solution methods. If you cannot follow the derivation, it is not critical for applying the valuation model.

Let's begin by assuming we have n+1 assets in a one-period economic model. All but the last one are risky assets, and the assets produce cash flows which in the aggregate are jointly normally distributed with known parameters. That is, asset 1's cash flow is normally distributed with mean $\mu_1$ and standard deviation $\sigma_1$, and so on for all the assets. The correlation between the cash flows i and j is $\rho_{ij}$. Using matrix notation, the mean vector ($\mu$) is an nx1 column vector of the cash flow means, and the covariance matrix ($\Sigma$) is their nxn covariance matrix. The n+1$^{st}$ asset is a risk-free asset that pays a return of r, or allows one to borrow at the rate r in an unrestricted manner. We assume all assets, risky and riskfree, can be bought or sold in unlimited quantity. While this makes it seem like people can choose infinitely risky positions with infinite capital, this will not present any particular challenge, since we are mostly concerned at this point with aggregate holdings. Nevertheless, we assume all investors have the same information and agree on the cash flow risk parameters. They are also assumed to have the same utility functions for wealth.

We now assume there is an investor in this economy who wants to deploy his current wealth ($W_0$). For now, we assume the asset prices (V) are known, but later we will solve for these asset prices. The agent chooses the number of units of each investment (N), pays cash to make the investments, and earns the risk-free rate on the uninvested funds. If he spends more than he has, he needs to borrow at the risk-free rate.

The agent then maximizes his quadratic utility, $U = E[W_1] - ½A\ Var(W_1)$, where A is a risk aversion parameter to be determined later. The quadratic utility function, or "score" suggests that the agent values higher expected wealth but is risk averse. The higher the value of parameter "A", the more risk averse he is, and the greater the penalty he assigns to risk. If A=0, the agent is risk-neutral, and does not factor risk into his valuations.

The first order condition is found by differentiating the utility function with respect to the portfolio holdings vector, and setting that to zero ($\partial U/\partial N=0$). This is a straightforward calculus maximization of a quadratic problem, the only complication being that the variables are expressed in matrix form. The maximization is accomplished by choosing N=N* to clear the market.

The assumed setup and the steps of the computation are shown in the table below. The matrices N' and 1' are transposed versions of N and the 1 vector:

| Description | Notation | Calculation |
|---|---|---|
| The vector describing the holdings of each risky asset, or the number of units of each risky asset | $N$ | |
| The expected dollar value of each asset next year | $\mu$ | |
| The agent's expected wealth in one year, the expected value of the risky asset holdings plus the risk-free return on residual wealth | $E[W_1]$ | $N'\mu + (W_0 - N'V)(1+r)$ |
| The variance of wealth in one year | $Var(W_1)$ | $N'\Sigma N$ |
| The utility of wealth in one year, or a "score" used to compare different choices of N | $U[W_1]$ | $N'\mu + (W_0 - N'V)(1+r) - \frac{1}{2}A\, N'\Sigma N$ |
| The first order condition, or solving for the optimal risky positions | $\partial U/\partial N=0$ | $\mu - V(1+r) - A\Sigma N$ |
| The choice of N that satisfies the first order condition, clearing the capital market | $N^*$ | $\Sigma^{-1}(\mu - V(1+r))/A$ |
| The equilibrium valuation of the risky assets, determined by setting N=1, which clears the market | $V^*$ | $\frac{\mu - A\Sigma 1}{(1+r)}$ |
| A is the expected aggregate risk premium per unit variance, a measure of the reward for taking risk. | $A$ | $\frac{1'(\mu - V(1+r))}{1'\Sigma 1}$ |

Looking at N*, it is immediately apparent that regardless of anyone's wealth, they choose the same relative weights in each of the assets. The size of the position is smaller for more risk averse agents, and the size of the position is larger for less risk averse agents. They choose the same relative weights because this offers the best risk/reward tradeoff of all the possible combinations of the assets.

Because everyone invests in the same portfolio, albeit at a different scale, it makes sense to define this portfolio as the common benchmark, or the *market portfolio*. In the setup of this particular model, the market portfolio is represented by one unit of every asset, hence N*=1, a vector of ones. When we make this substitution in the first order condition, we can solve for the equilibrium prices of the assets V*. The requirement that N*=1 can be thought of as a *market clearing* condition, since the optimal holdings in an economy in equilibrium must reflect the total actual investments.

Finally, since the first order condition is true for all the assets, it must be true for the sum. We can use this fact to derive A from the market portfolio. It turns out that A is the sum of the time 1 risk premia of the assets divided by the variance of the market portfolio --- *the market price of risk*, which we sometimes call $\lambda$. It can also be interpreted as the benefit per unit variance of taking the risk. It is similar to a Sharpe Ratio, but the denominator is the variance instead of the standard deviation.

## NUMERICAL EXAMPLES

*Example 1*

If your matrix algebra is a little rusty, it may be worthwhile to consider a simple case where there is only one risky asset in the economy. Its value in one year is normally distributed with $\mu$=100 and $\sigma$=20. The discount rate is 2%, and the risk aversion coefficient A=0.01.

The derivation can be taken from the table above:

$$E[W_1] = 100N + (W - NV)(1.02)$$

$$Var[W_1] = (20N)^2 = 400N^2$$

$$U[W_1] = 100N + (W - NV)(1.02) - (0.01)200N^2$$

$$\partial U/\partial N = 100 - V(1.02) - (0.01)400N = 0$$

$$N^* = (100 - V(1.02))/(0.01\times 400) = 1$$

$$V^*=(100 - (0.01\times 400))/(1.02) = \$94.12.$$

We set N*=1 to clear the market and then solved for V*. The expected return in percentage terms is 100/94.12 – 1, or 6.25%, the risk premium is 4.25% and the standard deviation is 21.25%. The Sharpe Ratio, or the ratio of the risk premium to total volatility is given by either (4.25/21.25), or A$\sigma$ = 0.20.

The GVE in this case reduces to the following:

$$rV_0 + k\sigma = \mu - V_0$$

$$0.02(94.12) + 0.02(20) = 100 - 94.12$$

When you have more than one risky asset, you do not need to derive this equation every time. All you need to do is value the assets. If you are given the mean cash flows of all the assets and the covariance matrix, the value of asset i in the set of n risky assets is simply:

$$V(i) = \frac{\mu_i - A(\text{sum of row i of the covariance matrix})}{(1+r)}$$

As a check on your math, A is the sum of the risk premiums on the assets divided by the variance of the sum. The individual risk premia are equal to $\mu_i - V(i)(1+r)$, and the variance of the sum is the *grand sum* of the covariance matrix, i.e. the sum of all the terms.

Example 2: Suppose a one-year economy has 5 assets with the following means and covariances between next year's cash flows. If r=2% and A=0.04, what are the valuations, expected percentage return and standard deviation? (Note that the correlations are the same in dollars and percentages.)

Solution:

| Means | Covariance matrix | | | | | Row sum |
|---|---|---|---|---|---|---|
| 100 | 400 | 100 | 50 | 25 | 10 | 585 |
| 150 | 100 | 441 | 50 | 25 | 10 | 626 |
| 50 | 50 | 50 | 225 | 20 | 5 | 350 |
| 75 | 25 | 25 | 20 | 256 | 8 | 334 |
| 125 | 10 | 10 | 5 | 8 | 484 | 517 |
| | | | Grand sum | | | 2412 |

| Valuation | FV Risk charge |
|---|---|
| 75.10 | 23.40 |
| 122.51 | 25.04 |
| 35.29 | 14.00 |
| 60.43 | 13.36 |
| 102.27 | 20.68 |
| Sum | 96.48 |

| Verifying A | |
|---|---|
| - FV Risk charge sum | 96.48 |
| - Grand sum covariance | 2412 |
| - Ratio | 0.04 |

| ExpRet | StDev | Correlation | | | | |
|---|---|---|---|---|---|---|
| 33.16% | 26.63% | 100.00% | 23.81% | 16.67% | 7.81% | 2.27% |
| 22.44% | 17.14% | 23.81% | 100.00% | 15.87% | 7.44% | 2.16% |
| 41.67% | 42.50% | 16.67% | 15.87% | 100.00% | 8.33% | 1.52% |
| 24.11% | 26.48% | 7.81% | 7.44% | 8.33% | 100.00% | 2.27% |
| 22.22% | 21.51% | 2.27% | 2.16% | 1.52% | 2.27% | 100.00% |

## A CLOSER LOOK AT V*: AN APPLICATION OF THE GVE

Since the covariance matrix is Σ, we know that for any two position vectors $N_1$ and $N_2$, we have the following identity:

Covariance between portfolios with holdings $N_1$ and $N_2 = N_1'\Sigma N_2$.

Also, the covariance of anything with itself is the variance. Therefore, since the market N=1, a vector of ones, we know that the variance of the market is $1'\Sigma 1$, and the covariance of any individual asset with the market portfolio is the corresponding element of the vector $\Sigma 1$.

This means the pricing equation is

$$V^* = V_0 = \frac{\mu - A\Sigma 1}{(1+r)} = \frac{\mu - \lambda\, cov(C_i, M_1)}{(1+r)}$$

where M is the market, i.e. the sum of the cash flows.

Rearranging the equation a bit, we can also see the following identity holds:

$$rV_0 + \lambda\, cov(C_i, M_1) = \mu - V_0$$

As you might have suspected, we can now see how to use the GVE in an environment where risk is compensated. The left side of the equation, the required return, now includes two components. The first component reflects the time value of waiting for your money to be returned. The second component reflects the compensation for taking the risk. In this problem, it would first seem that the compensation for risk of asset i would be proportional to the standard deviation of i or its variance. This is the case if there is only one risky asset. However, this is not the case with multiple assets, because investors can diversify some of their risks, and they must evaluate the risk of any given asset in the context of their overall portfolio. On the right side of the equation, we see the exchange of the asset for the expected cash flow.

## THE EXPECTED RETURN RELATIONSHIP

The rearranged pricing equation for assets corresponds to a General Valuation Equation or (GVE), wherein

Required return = Expected return

Time value + Risk compensation = Expected capital gain + Expected cash flow

$$rV + A\Sigma 1 = -V + \mu$$

The capital gain is $-V$ since the asset is surrendered at time 1 in exchange for the cash flow. In this type of model, the risk premium is expressed in dollars, and is not a constant proportion of value. Another interpretation is that $\Sigma 1/1'\Sigma 1$, which could be called the cash beta, is the proportion of risk assigned to each asset from the entire market portfolio. Then the risk measure assigned to each asset is simply its cash beta multiplied by the dollar risk premium of the market.

In contrast, the MPT framework defines the risk to be proportional to asset value:

$$rV + kV = -V + \mu$$

which allows us to discount expected cash flows at a blended rate of r+k, including time and risk compensation. The idea of a blended discount rate applied to mean cash flows including time value and risk compensation only works well when dollar risk is assumed to be proportional to value.

Now that we have determined the equilibrium asset values, we may compute percentage expected returns and standard deviations, perhaps even as inputs to the MPT model. Table 2 demonstrates that the classical CAPM security market line obtains in the variance model, i.e. the Black Ratio for all the assets is identical. The *Black Ratio* is the ratio of expected risk premium to beta, named for Fischer Black. The results are similar for the standard deviation utility model. Results are shown for the traditional security market line in the "returns" column, while the results applied to cash flows are shown in the "dollars" column.

Table 2

Risk Premium Calculations

| Calculation | Dollars | Returns (percentages) |
|---|---|---|
| Total expected return | $\mu - V$ | $(\mu - V) \oslash V$ |
| Risk premium | $\mu - (1 + r)V = A\Sigma 1$ | $[\mu - (1 + r)V] \oslash V = A\Sigma 1 \oslash V$ |
| Covariance with market | $\Sigma 1$ | $\frac{\Sigma 1 \oslash V}{1'V}$ |
| Variance of market | $1'\Sigma 1$ | $\frac{1'\Sigma 1}{(1'V)^2}$ |
| Beta | $\frac{\Sigma 1}{1'\Sigma 1}$ | $\frac{\Sigma 1 \oslash V}{1'\Sigma 1} 1'V$ |
| Risk premium$\oslash$Beta | $A1'\Sigma 1$ | $\frac{A1'\Sigma 1}{1'V}$ |

| | | |
|---|---|---|
| (constant for all securities) | | |

*The symbol "⊘" is being used to represent the Hadamard quotient or the Schur quotient, which is the element-by-element quotient of two vectors or matrices

The equilibrium relationship between expected returns can be written in the usual CAPM manner, since the ratio of risk premium to beta is constant for all assets. Therefore, if we define

$$r_m = \frac{1'(\mu - V)}{1'V} \qquad r_i = \frac{\mu_i - V_i}{V_i} \qquad \beta_i = \frac{\Sigma_i 1}{1'\Sigma 1} \cdot \frac{1'V}{V_i}$$

then the return relationship can be written in the usual way.

$$r_i - r = \beta_i (r_m - r)$$

This is useful because we can now make inferences such as the following:

"Suppose idiosyncratic risk increases across the board. This risk increase causes an overall increase in market risk, this reduces asset values, so expected returns must rise. This also affects asset betas, but the new betas can be computed based on the assumed change in idiosyncratic risks."

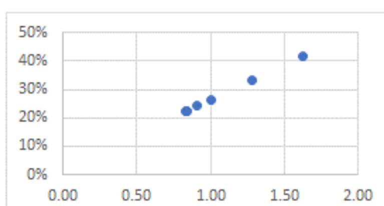
It would have been difficult to formulate this conclusion in the MPT model, or to understand the mechanism by which idiosyncratic risk affects expected returns.

Exercise:

Extend the previous exercise by computing beta for each of the five assets and the market portfolio. Plot the expected return in relationship to beta.

Ans:

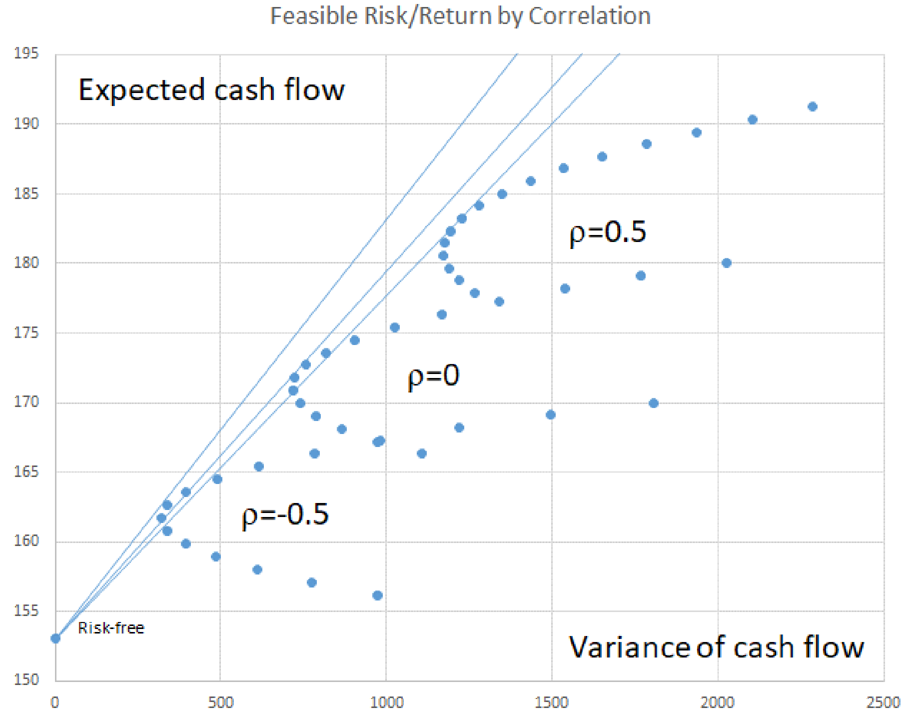
| Beta | ExpRet |
|---|---|
| 1.28 | 33% |
| 0.84 | 22% |
| 1.63 | 42% |
| 0.91 | 24% |
| 0.83 | 22% |
| 1.00 | 26% |




## VISUALIZATION

Let's consider the simple economy described in the exercise just above, with two one-period assets described by their cash flow characteristics.

It is customary for students of portfolio theory to study the effect of diversification upon a portfolio. This is normally done by changing the correlation between two assets, and observing how the feasible investment set changes when correlation changes, *holding everything else constant*. Of course, that assumption is unreasonable --- as correlations among assets increase, there is less of a diversification benefit available, so the total cost of risk increases. As correlations decline, risk premia decline, because diversification improves. Therefore, the set of portfolio investments changes every time the correlation, or any parameter in the model changes.




In this graph, the points represent achievable investment outcomes for a fixed wealth of $150. If the correlation is 0.5, the top parabola represents the achievable outcomes. If one were able to decrease the correlation towards zero, you can see how the risk shrinks due to the effect of diversification. However, expected returns also shrink because the aggregate risk is lower. As we push correlation to -0.5, the results continue. As correlations become negative, we can say that one asset hedges the risk of the other. As this happens, risk premia continue to shrink.

For each of the correlation scenarios, a tangent portfolio is shown to the so called *efficient portfolio frontier*. Since we assumed one could borrow and lend at a single risk-free interest rate, it is possible to combine the risk/reward combination on the curved graph with the risk-free point shown. Since this is possible, no portfolio other than the market portfolio will be considered other than the one that contains the tangency portfolio plus a risk-free position.

The numerical results for the example are shown below using correlation=0.50.

*Calculation of the market portfolio for a two risky asset case*

| Asset | Mu | Sigma | Correl | | Covariance | | x1 Vector |
|---|---|---|---|---|---|---|---|
| 1 | 100 | 20 | 0.5 | | 400 | 300 | 700 |
| 2 | 120 | 30 | | | 300 | 900 | 1200 |

| | | | Inverse | | x1 Vector |
|---|---|---|---|---|---|
| Riskfree | 0.02 | | 0.0033 | -0.0011 | 0.0022 |
| A | 0.02 | | -0.0011 | 0.0015 | 0.0004 |
| W | 150 | | | | |

| | | Total Return | | Risk Premium | | Cov with Market | |
|---|---|---|---|---|---|---|---|
| | Value | Dollars | Percent | Dollars | Percent | Dollars | Returns |
| 1 | 84.31 | 15.69 | 18.60% | 14.00 | 16.60% | 700 | 4.65% |
| 2 | 94.12 | 25.88 | 27.50% | 24.00 | 25.50% | 1200 | 7.15% |
| M | 178.43 | 41.57 | 23.30% | 38.00 | 21.30% | 1900 | 5.97% |

| | Beta | | Risk prem div by beta | |
|---|---|---|---|---|
| | Dollars | Returns | Dollars | Percent |
| 1 | 0.37 | 0.78 | 38.00 | 0.21 |
| 2 | 0.63 | 1.20 | 38.00 | 0.21 |
| M | 1.00 | 1.00 | 38.00 | 0.21 |

| | Variance | | StdDev | | Sharpe Ratio | |
|---|---|---|---|---|---|---|
| | Dollars | Returns | Dollars | Returns | Dollars | Returns |
| 1 | 400 | 5.63% | 20.00 | 23.72% | 0.70 | 0.70 |
| 2 | 900 | 10.16% | 30.00 | 31.88% | 0.80 | 0.80 |
| M | 1900 | 5.97% | 43.59 | 24.43% | 0.87 | 0.87 |

These numerical results show the intermittent calculations in the hopes that the attentive student will endeavor to replicate these results in Excel. The results show the risk and pricing statistics in dollars and percentage terms. To verify the accuracy of the calculations, the figure shows the division of risk premium by beta, which should in all cases be the market risk premium, both in dollar terms and in percentage terms.

The Sharpe Ratio shows the ratio of expected risk premium per unit of standard deviation. It represents another measure of the price of risk. As expected, the Sharpe Ratio is invariant to the dollarized nature of the setup. Also, the Sharpe Ratio of the tangency portfolio is higher than either of the assets individually.

## OTHER COMPARATIVE STATICS

*Comparative statics* generally refers to the change in model results as we change the parameters. The word *statics* reminds us that the model was solved using constant parameters, so it is not always valid to consider changing the parameters to see how the model changes. However, we may ask the question, what if we had a different set of parameters than the one originally given? We could compare the results of the two models, hopefully improving one's understanding of the model implications in the process.

In this risk-based valuation model, we could ask, what happens to asset prices as we change the risk characteristics of the underlying cash flows? Using analytical formulas or the spreadsheet we developed, we can come to the following conclusions for correlations between -1 and 1. *Hint: Use data tables in Excel to perform what-if analyses.*

If we increase the mean cash flow of one designated asset, then

- The value of the designated asset increases
- The expected return of the designated asset falls
- The other asset is unaffected, with the same valuation and expected return
- The percentage weight of the designated asset increases in the market portfolio
- The Sharpe Ratios of the individual assets and the market portfolio are unaffected

If we increase the variance of one designated asset, then

- The values of both assets fall, but the designated asset falls moreso
- The expected percentage return of the designated asset rises along with the other asset and the market, but the designated asset return rises faster
- The weight in the designated asset falls in the market portfolio
- The Sharpe Ratios of both assets and the market portfolio rise, but the designated asset rises fastest, since the expected return rises faster than the risk

If we increase the correlation between the two assets, then

- Valuations fall at a linear rate
- The weight of the first asset (the higher valued asset) falls in the market portfolio
- The expected percent returns for all assets increase in lockstep
- Percent volatilities increase for all assets
- The Sharpe ratios increase for all assets

The main conclusion of this section is that changes in the economy's underlying risk parameters generally affect all the capital markets equilibrium parameters of the assets, i.e. expected returns, standard deviations and correlations. Nevertheless, these are the inputs to the classical Markowitz model in the next section. This is an important reminder that classical portfolio analysis relies critically on the assumptions that returns, standard deviations and correlations are assumed to already be in equilibrium.

## EXPECTED RETURNS AND IDIOSYNCRATIC RISK

Traditional finance teaches that risk can be decomposed into one or more systemic factors, such as market risk, industry risk, and interest rate risk, and so on. The residual risk is considered specific to each asset, and being uncorrelated with systemic factors, the cost of risk is zero. In the model just presented, we demonstrated that this may not be the case. In this model, the idiosyncratic risk affects expected returns because it affects all asset prices, the market expected return and risk, and even the asset betas.

To see this, let's consider a given equilibrium, and increase the systemic risk of one asset (i) by increasing the corresponding diagonal term $\Sigma_{ii}$ in the covariance matrix. We will continue to use the single subscript to refer to the vector, or particular row of the covariance matrix. Since we are not increasing the covariance terms, the covariance risk of the asset with other assets stays the same, so this is a pure increase in firm-specific risk.

The percentage expected return of asset i can be expressed simply as a function of its covariance with the market

$$r_i = \frac{r\mu_i + A\Sigma_i 1}{\mu_i - A\Sigma_i 1}$$

We may now compute the derivative of $r_i$ with respect to $\Sigma_{ii}$. With some simple calculus and algebra, we find

$$\frac{\partial r_i}{\partial \Sigma_{ii}} = \frac{A\mu_i}{V_i^2(1+r)}$$

This derivative is unambiguously positive, demonstrating at least in a comparative static sense, that expected returns rise with idiosyncratic risk.

## THE MARKOWITZ MODEL

This section shows the classical Markowitz results, using notation provided by Jonathan Ingersoll in his famous book, Theory of Financial Decision Making. In this setup, assets are defined by their expected percentage returns, standard deviations and correlations, which may be computed as we derived in the last section. A portfolio is defined by its percentage investment (w) in each of the risky assets, where the sum of the weights adds to one. We assume no individual has the ability to move the market, and compute the individual's optimal portfolio composition.

Again, if you do not have a background in matrix manipulations, the derivation won't be interesting. However, Excel does provide matrix functions which you may use to compute efficient portfolios and tangency portfolios. The formulas for these results are provided in this section.

Notation

w = vector of weights in n risky securities
r = vector of n expected returns
1 = a vector of ones
Σ = covariance matrix of returns
R = given fixed return (a constant)
$r_f$ = Risk-free interest rate (a constant)

The investor's objective is to choose a target return R, and find the lowest risk portfolio composed of risky assets only that has this target return. This is a constrained optimization problem, which can be written as the following

$$\text{Min } \tfrac{1}{2}w'\Sigma w \text{ s.t. } w'1 = 1 \text{ and } w'r = R$$

You can see the mathematics are going to be fairly similar to the previous section. There are a few differences however. First, we are assuming we already know the expected return vector and covariance matrix of returns that is consistent with capital market equilibrium. Second, because we are working with percentage returns, the asset weights must add to one, adding a constraint to the problem. Finally, since we are targeting a fixed return, this adds a second constraint. To maximize a constrained objective, we construct the modified Lagrangean objective, which places a cost λ and γ on the constraints, and then optimizes using ordinary calculus.

The Lagrangean is to maximize L over the choice of w, λ and γ

$$L = \tfrac{1}{2}w'\Sigma w - \lambda(w'1-1) - \gamma(w'r-R)$$

First order conditions (using $w_e$ as the optimum, i.e. an "efficient" portfolio)

$$\Sigma w_e = \lambda 1 + \gamma r$$

$$w_e = \lambda\Sigma^{-1}1 + \gamma\Sigma^{-1}r$$

Therefore, each efficient portfolio is a combination of two portfolios, a property known as "two-fund separation". That is, any portfolio on the efficient frontier can be replicated using any two other portfolios on the frontier. Investors will have no reason to hold more than two portfolios, a market portfolio of risky assets, and the risk-free asset.

Define for convenience

$A = 1'\Sigma^{-1}1$ = the sum of all the elements of the inverse covariance matrix, also called the *grand sum* of the inverse covariance matrix

$$B = r'\Sigma^{-1}1 = 1'\Sigma^{-1}r$$

$$C = r'\Sigma^{-1}r$$

$$D = AC - B^2$$

Premultiplying and applying constraints

$$1 = \lambda A + \gamma B$$

$$R = \lambda B + \gamma C$$

In matrix form

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} \lambda \\ \gamma \end{pmatrix}$$

Or solving for the Lagrange multipliers

$$\begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = \begin{pmatrix} C/D & -B/D \\ -B/D & A/D \end{pmatrix} \begin{pmatrix} 1 \\ R \end{pmatrix}$$

Restating the efficient portfolio set

$$w_e = \lambda\Sigma^{-1}1 + \gamma\Sigma^{-1}r$$

$$w_e = \frac{C-BR}{D}\Sigma^{-1}1 + \frac{AR-B}{D}\Sigma^{-1}r$$

The variance of any efficient portfolio (after some tedious simplification)

$$\sigma^2 = w_e'\Sigma w_e = \frac{AR^2-2BR+C}{D}$$

This parabolic expression is minimized with respect to R when R = B/A. The variance at that point is 1/A.

Note that when $\gamma$=0, this also yields the global minimum variance portfolio, since the R constraint is nonbinding when searching for the global minimum. This means

$$w_g = \Sigma^{-1}1/1'\Sigma^{-1}1 = \Sigma^{-1}1/A$$

$\lambda=1/A$ and $\gamma=0$.

The covariance of this portfolio with any asset or efficient portfolio is 1/A.

*The tangency portfolio equations*

The weights in the tangency portfolio are

$$w_t = \frac{\Sigma^{-1}(r - r_f 1)}{B - A r_f}$$

The mean and variance of the tangency portfolio are

$$r_T = \frac{C - B r_f}{B - A r_f}$$

$$\sigma_T^2 = \frac{C - 2 r_f B + r_f^2 A}{(B - A r_f)^2}$$

Exercise: Using the computed percentage expected returns and covariance matrix of returns in the previous five asset exercise, compute the tangency portfolio using matrix functions in Excel.

Solution: To check your results, multiply your portfolio weights by 395.14. What is the result, and why should it be the result?

## USING HISTORICAL DATA WITH THE MARKOWITZ PORTFOLIO MODEL

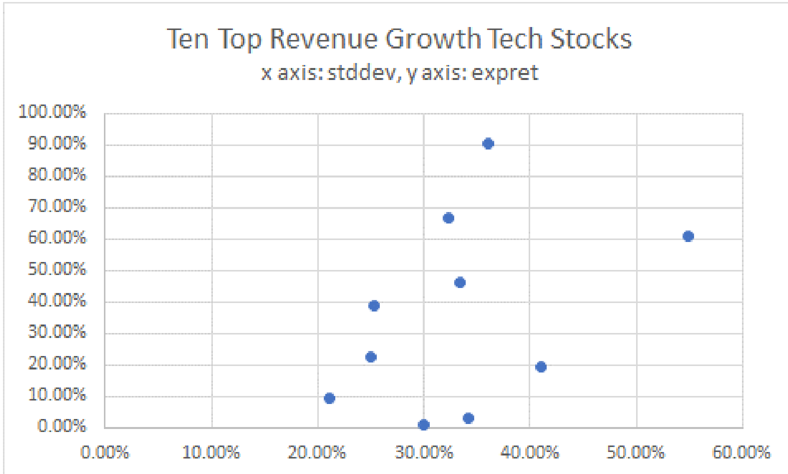
In this section, we will demonstrate how historical data can be used to construct portfolios with optimal risk/return structure, and how the data are sometimes misused. To begin, we build a historical price series associated with a number of stocks. In the following table, we downloaded monthly closing quotes for 10 top revenue producing tech stocks as of 2/17. Since there seems to be a lot of noise in higher frequency data, these analyses are usually done with low frequency data. In the process, we are hoping to pick up some of the correlations between the stock returns.

| Closing Price | MEET | BIDU | MLNX | CRM | NTES | OLED | WEB | SOHU | NTCT | AAPL |
|---|---|---|---|---|---|---|---|---|---|---|
| 29-Feb-16 | 3.11 | 173.42 | 50.81 | 67.75 | 134.61 | 47.78 | 18.15 | 43.47 | 20.67 | 96.69 |
| 31-Mar-16 | 2.84 | 190.88 | 54.33 | 73.83 | 143.58 | 54.1 | 19.82 | 49.54 | 22.97 | 108.99 |
| 29-Apr-16 | 3.42 | 194.3 | 43.25 | 75.8 | 140.7 | 58.31 | 19.99 | 44.93 | 22.26 | 93.74 |
| 31-May-16 | 3.78 | 178.54 | 47.4 | 83.71 | 177.84 | 67.15 | 16.97 | 41.77 | 24.26 | 99.86 |
| 30-Jun-16 | 5.33 | 165.15 | 47.96 | 79.41 | 193.22 | 67.8 | 18.18 | 37.86 | 22.25 | 95.6 |
| 29-Jul-16 | 6.43 | 159.6 | 44.18 | 81.8 | 204.27 | 70.84 | 18.86 | 38.68 | 27.98 | 104.21 |
| 31-Aug-16 | 5.76 | 171.07 | 43.84 | 79.42 | 211.97 | 57.59 | 17.46 | 42.54 | 29.58 | 106.1 |
| 30-Sep-16 | 6.2 | 182.07 | 43.25 | 71.33 | 240.78 | 55.51 | 17.27 | 44.25 | 29.25 | 113.05 |
| 31-Oct-16 | 4.89 | 176.86 | 43.4 | 75.16 | 256.99 | 51.7 | 16.1 | 37.43 | 27.45 | 113.54 |
| 30-Nov-16 | 4.82 | 166.95 | 41.45 | 72 | 224.1 | 54.65 | 15.95 | 34.63 | 31.2 | 110.52 |
| 30-Dec-16 | 4.93 | 164.41 | 40.9 | 68.46 | 215.34 | 56.3 | 21.15 | 33.89 | 31.5 | 115.82 |
| 31-Jan-17 | 4.92 | 175.07 | 47.35 | 79.1 | 253.9 | 66 | 18.95 | 39.67 | 33.3 | 121.35 |
| 22-Feb-17 | 4.94 | 186.01 | 48.8 | 82.08 | 304.07 | 71.25 | 20.4 | 42.16 | 37.75 | 137.11 |

The first step is to convert the prices to returns. If dividends were paid, these should be included in returns. Then we compute the average returns and the covariance matrix of returns.

| Returns | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 31-Mar-16 | -8.68% | 10.07% | 6.93% | 8.97% | 6.66% | 13.23% | 9.20% | 13.96% | 11.13% | 12.72% |
| 29-Apr-16 | 20.42% | 1.79% | -20.39% | 2.67% | -2.01% | 7.78% | 0.86% | -9.31% | -3.09% | -13.99% |
| 31-May-16 | 10.53% | -8.11% | 9.60% | 10.44% | 26.40% | 15.16% | -15.11% | -7.03% | 8.98% | 6.53% |
| 30-Jun-16 | 41.01% | -7.50% | 1.18% | -5.14% | 8.65% | 0.97% | 7.13% | -9.36% | -8.29% | -4.27% |
| 29-Jul-16 | 20.64% | -3.36% | -7.88% | 3.01% | 5.72% | 4.48% | 3.74% | 2.17% | 25.75% | 9.01% |
| 31-Aug-16 | -10.42% | 7.19% | -0.77% | -2.91% | 3.77% | -18.70% | -7.42% | 9.98% | 5.72% | 1.81% |
| 30-Sep-16 | 7.64% | 6.43% | -1.35% | -10.19% | 13.59% | -3.61% | -1.09% | 4.02% | -1.12% | 6.55% |
| 31-Oct-16 | -21.13% | -2.86% | 0.35% | 5.37% | 6.73% | -6.86% | -6.77% | -15.41% | -6.15% | 0.43% |
| 30-Nov-16 | -1.43% | -5.60% | -4.49% | -4.20% | -12.80% | 5.71% | -0.93% | -7.48% | 13.66% | -2.66% |
| 30-Dec-16 | 2.28% | -1.52% | -1.33% | -4.92% | -3.91% | 3.02% | 32.60% | -2.14% | 0.96% | 4.80% |
| 31-Jan-17 | -0.20% | 6.48% | 15.77% | 15.54% | 17.91% | 17.23% | -10.40% | 17.06% | 5.71% | 4.77% |
| 22-Feb-17 | 0.41% | 6.25% | 3.06% | 3.77% | 19.76% | 7.95% | 7.65% | 6.28% | 13.36% | 12.99% |

The average returns are easy to compute, but if we want to work with standardized annual returns, these should be multiplied by 12. Similarly, standard deviations should be multiplied by the square root of 12. If you use the standard deviation function in Excel, use the function *stdevp* instead of *stdev* to ensure they are consistent. These are shown in the figure below, with the axes reversed, i.e. expected return is on the horizontal axis:




To compute the covariance matrix of returns, use the VBA covariance function in your class function library if it is available, or if not, take the following steps. The data begins with n dates (rows) and k columns (stocks), and we will call the matrix D.

- Compute the matrix X = (Identity Matrix(nxn) – Matrix with all elements 1/n) x D
- Then compute $X^TX/n$, which is the historical covariance matrix.
- Multiply every entry by 12 months/year to annualize the covariance matrix

This is a matrix multiplication (*mmult*) and *transpose* function available in Excel. Use *covarp* to check your results between a randomly selected pair of security returns.

The inverse of the annualized covariance matrix of returns is then (using *minverse*)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 32.77 | 121.75 | 60.43 | 25.83 | -33.40 | -21.70 | -4.74 | -81.31 | 27.61 | 13.43 |
| 121.75 | 837.41 | 459.69 | 24.97 | -35.62 | -109.30 | 63.06 | -555.83 | 341.05 | -259.91 |
| 60.43 | 459.69 | 300.26 | -2.19 | 1.81 | -70.36 | 55.54 | -310.80 | 225.80 | -218.61 |
| 25.83 | 24.97 | -2.19 | 77.27 | -43.93 | -29.59 | -9.09 | -20.58 | -23.88 | 68.84 |
| -33.40 | -35.62 | 1.81 | -43.93 | 85.82 | 2.68 | 41.14 | 25.00 | 50.63 | -132.04 |
| -21.70 | -109.30 | -70.36 | -29.59 | 2.68 | 42.51 | -17.65 | 75.28 | -54.59 | 44.86 |
| -4.74 | 63.06 | 55.54 | -9.09 | 41.14 | -17.65 | 39.21 | -42.86 | 66.43 | -107.20 |
| -81.31 | -555.83 | -310.80 | -20.58 | 25.00 | 75.28 | -42.86 | 384.00 | -230.06 | 166.23 |
| 27.61 | 341.05 | 225.80 | -23.88 | 50.63 | -54.59 | 66.43 | -230.06 | 216.26 | -247.29 |
| 13.43 | -259.91 | -218.61 | 68.84 | -132.04 | 44.86 | -107.20 | 166.23 | -247.29 | 395.82 |

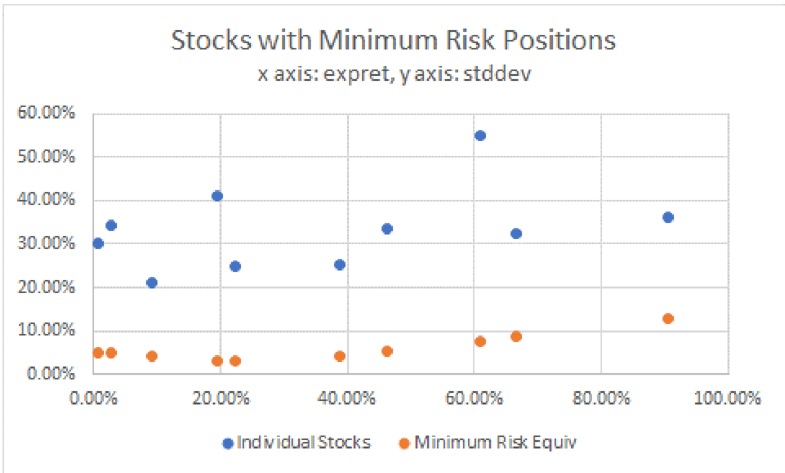
The intermediate calculations are

| | | |
|---|---|---|
| A | | 1010.37 |
| B | | 229.5955 |
| C | | 82.20194 |
| D | | 30340.4 |
| | | |
| Risk-Free Rate | | 0.05 |
| | B-Arf | 179.0769 |

So the weights in the tangency portfolio

| | Tan Wts |
|---|---|
| MEET | 5.92% |
| BIDU | 87.69% |
| MLNX | 51.99% |
| CRM | -7.29% |
| NTES | 21.55% |
| OLED | -12.20% |
| WEB | 19.75% |
| SOHU | -60.06% |
| NTCT | 56.71% |
| AAPL | -64.06% |

We can also compute the minimum risk equivalents for each of the stocks:




Well, this seems too good to be true! Do we really know the optimal tech stock portfolio to hold going forward? Probably not. Take a look at the recommended weights. Do you think a portfolio would go long 88% BIDU and short 60% SOHU? First of all, this does not seem very diversified. Secondly, few would be comfortable with such a concentrated long position or such a large short position.

The problem of course is the use of historical data. A stock that happened to earn 90% in one 12 month period is unlikely to repeat it, and the stock that earned 0% is also unlikely to repeat that. Yet it is these extreme results that drive the computed portfolio allocation.

Some portfolio managers address this problem by re-running the optimization putting on constraints and using Excel solver to find the optimal portfolio. For example, a "long only" fund would constrain all the weights to be positive, which can be accomplished by selecting an option in the Excel solver. This method corrects the problem of negative weights, but still underweights stocks with the lowest average returns. It's a bit like trying to cobble a portfolio together with rules instead of optimization. The resulting constrained portfolio may tend to exclude large numbers of assets based on their historical average returns.

Exercise: Use Excel Solver to compute the optimal portfolio weights using this example for a long only fund. What weights do you determine? *HINT: Be sure to constrain the weights so that they sum to one.*

Answer:

| | |
|---|---|
| MEET | 7.28% |
| BIDU | 0.00% |
| MLNX | 0.00% |
| CRM | 0.00% |
| NTES | 42.04% |
| OLED | 0.00% |
| WEB | 19.87% |
| SOHU | 0.00% |
| NTCT | 30.82% |
| AAPL | 0.00% |

## THE BLACK-LITTERMAN ADJUSTMENTS

Fischer Black and Robert Litterman recognized that the biggest problem with historical data was the historical average returns. If one were willing to accept that the covariance matrix was reasonable, it might make sense to determine what the expected returns *should have* been. To do this, we recognize that we used returns to determine the tangency portfolio composition, so why can't we use the tangency portfolio composition to determine returns?

If we are willing to make the approximating assumption that the market weighted portfolio of the 10 stocks is a reasonably efficient portfolio, we can compute the expected returns that would have generated. Note that a market-weighted portfolio has percentage weights in each stock that are each equal to the ratio of the total value of their traded stock to the total equity value of the portfolio of 10 stocks. This can be calculated by knowing the price per share and the number of shares outstanding of each stock.

Letting B represent the benchmark weights of the portfolio we are studying, the implied expected returns are given by

$$r = \Sigma w_B \left(B - A r_f\right) + r_f 1$$

Provide an expected benchmark return $r_B$ to get a normalizing factor

$$r = \Sigma w_B \left(\frac{r_B - r_f}{w_B^T \Sigma w_B}\right) + r_f 1$$

Now of course, the implied tangency portfolio is simply the capitalization weighted portfolio. The final step is to estimate by how much we expect each stock to underperform or overperform its benchmark expected return, and recompute the new tangency portfolio. This will lead us to an outcome where we overweight the securities we think will outperform, and underweight the securities that we think will underperform.

In general, this approach leads to reasonable results that are consistent with our intuition, and it is a popular correction used by asset managers when they apply Markowitz' portfolio theory to their asset selection processes.

Exercise: Assuming the benchmark portfolio is the equally weighted portfolio of the 10 stocks, and the equally weighted portfolio has an expected return of 15%, what are the new expected returns of the 10 stocks?

Answer:

| | |
|---|---|
| MEET | 11.10% |
| BIDU | 9.41% |
| MLNX | 16.33% |
| CRM | 13.27% |
| NTES | 19.05% |
| OLED | 19.71% |
| WEB | 8.37% |
| SOHU | 20.26% |
| NTCT | 15.69% |
| AAPL | 16.82% |

## REFERENCES

Markowitz

Sharpe Lintner Mossin Treynor

Ingersoll book

## EXERCISES

1. Using the one-asset version of the cash flow model, derive the valuation of the asset, and show the demand curve for the number of units the investor would like to purchase as function of the price of the asset. Also show the supply curve and identify the equilibrium price point.
2. Complete the Appendix worksheet to show the equivalence of the algebraic method and the matrix method for the two asset case.
3. In the two asset case, derive the formulas for the expected percentage return and standard deviation, and differentiate with respect to $\mu_1$, $\sigma_1$ and $\rho$ to show how changes in the parameters affect the asset prices. For each of these three derivatives, explain your result intuitively.

## EXCEL EXERCISES

A. Replicate the two asset case in the text using Excel. Use What-if tables to show how the expected returns, standard deviations and Sharpe Ratios change for different values of the parameters, and produce graphs showing these relationships.

B. Build your own portfolio of stocks! Choose ten stocks using a criterion of your choosing, and collect monthly price data from an Internet source, and determine the optimal portfolio holdings using historical parameters. Show a graph of the expected return and standard deviation of your stocks along with the efficient portfolio frontier. Do your weights look reasonable? What are the optimal weights for a long only fund?

C. Continuing from B, using the Black-Litterman adjustment, consider an equally-weighted portfolio benchmark and a capitalization-weighted benchmark. Determine the expected returns in each case, plotting the individual securities and the efficient frontier. What are the optimal portfolio weights in each case? Show an example of how you could use this computation to design your portfolio.

## APPENDIX: MATRIX NOTATION WORKSHEET

If your matrix algebra needs help, then complete the exercise in this appendix. We will perform the analysis for the two-asset case in the chapter using the usual algebraic representation alongside the matrix notation.

Some preliminaries. If the expected cash flows are $\mu_1$ and $\mu_2$, the standard deviations are $\sigma_1$ and $\sigma_2$, and the correlation is $\rho$, then what are the vectors or matrices that describe the mean cash flows and the covariance matrix? Remember that N is the position vector, which shows how many units of asset 1 and asset 2 we hold. Other notation: r=riskfree rate, A=risk aversion coefficient, $W_0$= initial wealth, V= asset value.

You may like to do this in Excel in parallel using matrix functions, *minverse*, *mmult* and *transpose*.

Mean vector: (column or row?)

Covariance matrix:

Position vector:

Now, how do we compute the variance of the portfolio:

Algebraically:

Using the matrices:

How do we compute the covariance of each asset with the portfolio?

Algebraically:

Using the matrices:

What is the equation for expected wealth at time 1?

Algebraically:

Using the matrices:

What is the equation for expected utility at time 1, using quadratic utility?

Algebraically:

Using the matrices:

Differentiate with respect to N and solve.

Algebraically:

Using the matrices: *HINT: the inverse of* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{(ad-bc)}$

Now set N=1 (market clearing) to solve for the values of the assets.

Algebraically:

Using the matrices:

Now you can see why matrix notation is so popular!  Imagine how difficult this would be to do by hand for more than two assets.

Key takeaways:

1. How do you use matrix notation to compute variance and covariance of a portfolio?
2. What is the role of the covariance matrix inverse in portfolio theory?
3. How to you compute the derivative of a quadratic form matrix, e.g. the derivative of $N'\Sigma N$ with respect to the vector N?
4. How do you solve a matrix equation for N, e.g. $\mu - V(1+r) - A\Sigma N = 0$?

## Chapter 9: Forwards & futures valuation

### INTRODUCTION

Forward contracts are agreements to transact in the future at terms specified today. Forward contracts generally occur between two counterparties, but a standardized forward contract traded on an exchange is a futures contract.

While valuing cash flows, analysts will often determine that a portion of the cash flows correlate with market prices of forward or futures contracts. In this chapter, we show how to use traded futures prices or known forward benchmarks to help value these cash flows. This is not a complete introduction to futures and forwards – there are many great textbooks on this topic. Rather, we focus on the parts of futures and forward pricing that are most relevant to cash flow valuation.

### WAYS TO BUY A HOUSE

Let's imagine that you would like to buy a house, but you have the flexibility to occupy it now or one year in the future. What are the different ways you could buy the house?

First, you could purchase the house outright and start living in it right away. This is called a *spot* transaction. Second, you could purchase the house and rent it out for a year, i.e. a spot contract combined with leasing. Third, you could enter into an agreement to purchase the house in one year's time. This arrangement is called a *forward contract*. In a forward contract, the terms of a transaction are agreed today but the transaction takes place in the future. The *forward price* is the agreed price of the house in the future, which generally differs from the spot price. The subject of this chapter is to understand the pricing of forward contracts, and how these contracts affect the buyers and sellers of the underlying assets. A fourth way to buy the house would be to enter a forward contract, but put money aside upfront in an interest-bearing asset or account. This is known as a *prepaid forward contract*. A prepaid forward contract ensures you will have the cash available to purchase the asset at the contracted future date and price --- but the transaction is still deferred.

The party who will buy in the future is said to be *long* the forward contract or *going long the house*, and the party who has agreed to sell is said to be *short, or going short the house*. We use long/short instead of buy/sell terminology to remind us that the transaction is deferred. We usually use buy/sell terminology for spot transactions.

### PRICING THE FORWARD CONTRACT

To develop a pricing model for forward contracts, it is helpful to consider the factors that might affect the price. Clearly, there must be a relationship between the forward price and the spot price. Ignoring any unusual circumstances, the higher the spot price, the higher the forward price should be. If one house is twice as valuable as another, the forward price should logically be double; just think about the logic of buying two identical houses with two forward contracts. What are the other differences between going long vs buying the house?

If I go long, then I don't have to put up the cash for a year. This means I can keep my other assets earning interest while I take on the forward obligation. Therefore, for me to be indifferent to the forward purchase, the forward price should exceed the spot price by the amount of the earned interest.

The interest savings are a great benefit. The cost, however, is that I would not have the benefit of living in the house or renting it for the year, which would make the forward price lower. I would therefore have to deduct the rent from the forward price. Specifically, I would deduct the future value of the rent at the time I take possession of the house. By the way, any other expenses of operating the home can be deducted from the rent, and any extra benefits, such as tax incentives, could be added to the rent as part of the total return of the property. Therefore, "rental" can include both expenses and non-rent benefits, making the valuation equation below more general.

What about risk? How does the risk of a forward contract relate to the risk of the underlying asset? If I am obligated to buy the house no matter what, and there is no possibility of default on my part, then the obligation to buy in the future *has the same risk as buying now*. **Therefore, I should earn the same risk premium for a forward contract as I would on the underlying asset.**

Expressing this in formulas, suppose $S_0$ is the current value of the house and $S_1$ is the unknown value in one year. The expected appreciation of the house (including the value of the rental) is $\alpha$ in percentage terms, or $\alpha S_0$ in dollar terms. For convenience, define the net rental payment at year end to be a proportion $\delta$ of the house value, or $\delta S_0$ paid in one year. Let r be the risk-free discount rate.

Under these assumptions, the homeowner earns an expected total return of $\alpha S_0$ and a risk premium of $(\alpha\text{-}r)S_0$ for owning the house, which assumes the homeowner earns the rental payment. However, the long forward position forfeits the rental payment. Therefore, for the forward contract:

*Forward contract characteristics*

Time value of funds invested = 0

Compensation for risk = $(\alpha\text{-}r)S_0$

Expected capital gain = $S_0(1+\alpha) - \delta S_0 - F_0$

Cash flow = 0

Note that time value = 0 due to not putting up cash in advance. The expected capital gain includes a capital loss since the long forward position does not receive rental payments.

Plugging these components into the GVE we get

Required return = Expected return

Time value + Risk compensation = Expected capital gain + Expected cash flow

$$0 + (\alpha\text{-}r)S_0 = S_0(1+\alpha) - \delta S_0 - F_0 + 0$$

implying that for the one year forward,

$$F_0 = S_0(1 + r - \delta)$$

This equation comports exactly with our intuition. If the spot price doubles, the forward price doubles. If the interest rate rises, the benefit of not paying in advance is greater, making the forward price higher. The higher the rental income, the lower the forward price. As long as the performance of the forward contract can be guaranteed, its value equals the spot value grossed up by the risk-free interest rate, and scaled down by the future value of the dividends.

Forwards (and futures, which we discuss in the next section) generally use the convention of continuously compounded discount rates. Also, allowing time T to represent the transaction date (or *maturity date*) of the contract, assuming continuous proportional net cash flows δ to the underlying asset accruing to the actual homeowner, we have

$$F_0 = S_0 e^{(r-\delta)T}$$

At any time t between 0 and T, there is a new forward price that could be established, but it is important to understand that this is distinct from the transaction established at time zero. Using similar arguments as above, we can deduce that the new forward price established at each time t is

$$F_t = S_t e^{(r-\delta)(T-t)}$$

One cannot go short after going long and expect the positions to cancel, like buying and selling a share of stock. For forwards, one is obligated to perform both sides of the transaction on the maturity date. We distinguish therefore between the *price* of the forward contract and the *value* of the forward position.

## VALUING THE FORWARD CONTRACT POSITION (NOT THE PRICE)

What if I went short at time t after having gone long at time zero? The short contract would not exactly cancel out the long contract, as if one had sold a share of stock after buying it. Both contracts would have to be held until maturity, at which time the long position would buy the house for $F_0$ and the short position would sell it at $F_t$ (*not* $F_T$). If $F_t$ were higher than $F_0$, the long position would make a profit, and otherwise a loss. Also, these are risk-free contracted values by assumption. Therefore, the *value* of reversing the position V(t) that was implemented at time zero is given by

$$V(t) = (F_t - F_0)e^{-r(T-t)}$$

If the forward contract were executed with a bank or dealer as a *counterparty*, one could likely settle the contract by having the short party pay the long party the present value of the difference between the current forward price and the contracted forward price.

The value of a forward contract position is therefore not the same as the price of a forward contract. At maturity, the long position is worth the difference between the house value at that time and the agreed forward price. The short position is worth the opposite --- hence the sum of the value of the long position and the short position is zero. At inception then, the value of the contract should also be zero, otherwise one of the two parties would not be willing to enter the contract.

## THE LEVERAGE ANALOGY AND THE SHARPE RATIO DERIVATION

Going long is not very different from buying an asset with 100% borrowed funds, except for cash payments prior to maturity. If one borrowed 100% of the value of the house and then purchased the house, the out-of-pocket cost would be zero, and the risk, rent and expected appreciation of the house would belong to the new owner. The only difference is that the leveraged homebuyer earns the rent or lives in the house, whereas the long position forfeits the rent.

This observation is critically important. Forward contracts are usually defined as “derivative contracts”, whose values *derive* from an underlying asset. This definition obscures the economic effect of the forward contract, which in many cases achieves maximum leverage for the long position. While the dangers of excessive leverage are well-known, few understand that forward contracts are simply leverage by a different name.

Analytically, the leverage observation is also helpful in developing an alternative valuation approach. We learned in Chapter 7 that a leveraged asset has the same Sharpe Ratio as the unlevered asset if the debt is risk-free. Let's see what that means for the forward contracts, using the one period discrete model and no dividends:

| Property | Spot position | Forward position |
|---|---|---|
| Expected dollar profit | $\alpha S_0$ | $S_0(1+\alpha) - F_0$ |
| Funding cost in dollars | $rS_0$ | 0 |
| Risk measure | $\sigma S_0$ | $\sigma S_0$ |
| S/$harpe Ratio | $(\alpha - r)/\sigma$ | $[S_0(1+\alpha) - F_0]/[\sigma S_0]$ |

Setting the Sharpe ratio of the spot position equal to the dollarized $harpe Ratio of the forward position yields exactly the same forward pricing equation in the absence of default.

## THE ARBITRAGE ARGUMENT

In the case of a forward contract on a house, expected return equivalence arguments determined what the price of a forward contract should be. In fact, for assets that can be both bought and sold, a stronger statement can be made. In this case, we can show that arbitrage determines the value of a forward contract.

You may have wondered why we chose S as the underlying variable instead of H (for house), and why we chose δ for the rental income instead of ρ (rho). The reason is that there more marketable forward contracts on *S*ecurities than there are on *H*ouses, and securities pay δividends rather than ρentals!

Suppose now we have a forward contract on the basket of securities represented by the S&P 500, a large capitalization-weighted stock index in the U.S. Suppose the index is S, the continuous proportional dividend yield on the S&P 500 is δ, and the risk-free rate is r.

If the forward price were too high, i.e. greater than $S_0 e^{(r-\delta)T}$, then arbitrageurs would engage in a *cash-and-carry* arbitrage. This means they would go short at the forward price, pay *cash* for the stocks in the index, earn dividends, and carry the position until the maturity of the forward contract. At that time, they would have locked in a positive profit. And capital markets do not provide easy money for long --- demand for these arbitrage transactions will force prices to adjust.

Similarly, if the forward price were too low, arbitrageurs would enter a *short sale* arbitrage. In this case, they would go long the forward, borrow the stock, sell the stock, and pay the dividends to earn a risk-free profit. In the process of many arbitrageurs following this strategy, the forward price would have to rise.

In liquid capital markets, the transaction costs of the two arbitrages described are both relatively low, meaning that forward prices cannot deviate much from the theoretical value. When the underlying asset of a forward contract is a financial instrument, we refer to the contract as a *financial forward*. Not all financial instruments trade in liquid markets or can be shorted, but most of the important ones do.

## FOREIGN EXCHANGE FORWARD PRICING: INTEREST RATE PARITY

In the section on Foreign Exchange in Chapter 3, we discussed the pricing of forward contracts for foreign currency. We derived the arbitrage relationship known as Interest Rate Parity. Using this as a way to compute the forward exchange rate, we have for the Chinese Yuan relative to the US dollar at time zero with maturity t

$$F(0,t) = X_{CNY}\frac{(1+r_{CNY})^t}{(1+r_{USD})^t}$$

Foreign exchange fits the requirements of a financial contract, since both buying and selling take place with minimal transaction costs. For direct-quoted currencies like the yuan, the forward price of $1 expressed in yuan is the spot price (X) grossed up at the Chinese risk-free rate.

However, there is one issue. Technically, a dollar bill is not an asset one would invest in, other than having the convenience of cash available for transactions. This is because the dollar bill itself does not earn interest. If it did earn the risk-free rate, it would be an investable asset. Therefore, the dollar bill earns a return that is exactly $r_{USD}$ lower than an invested dollar bill. This is like owning a stock but not getting the dividends, as in a forward contract on a share of stock.

This implies in the equation above that the USD interest rate is like a foregone dividend on investing in dollar bills directly, and therefore can be treated as such for forward pricing. Also, since forward pricing normally uses continuously compounded rates, the discrete compounding formula above can be written as:

$$F(0,t) = X_{CNY}\exp\left[(r_{CNY} - r_{USD})t\right]$$

## THE PRICING AND VALUATION OF NONFINANCIAL FORWARDS

Nonfinancial forwards include commodity forwards, forwards on interest rates, and forwards on non-traded indices. That is, anything that is not an easily traded asset. Examples of these three would be (a) forward contracts on corn (b) forward contracts on the LIBOR 3-month index value and (c) forward contracts on the CPI index.

As we shall see in this section, nonfinancial forward pricing differs substantially from financial forward pricing. This difference occurs because

1. The underlying assets are not investable
2. The underlying assets cannot be easily bought or sold short to perform the required arbitrages associated with financial forwards.

What makes nonfinancial assets noninvestable? Consider the examples above. It would seem corn could be considered an investment. However, corn is primarily stored for future consumption, not investment. Clearly, farmers and grain storage facilities may keep inventories available, but not because of the expected returns. They keep inventories available in order to meet current and unexpected customer requirements and earn a premium by selling corn they have on hand. The value of the ability to capture customer spot business is called a *convenience yield*. While conceptually, it makes a lot of sense, it is difficult for anyone outside of farming or grain storage to earn the convenience yield of corn. And while many acedemics claim that convenience yield is to corn as dividends are to stock, it would be much more difficult to measure the convenience yield of corn than to measure the dividend yield of stock.[1]

Similarly, the LIBOR index is not investable, even though bonds priced on the index are investable. Therefore, one would have to determine the investable equivalent in order to apply the rules for financial futures pricing. The same logic applies to the CPI, a computed index, which is clearly not a tradeable asset.

What makes nonfinancial assets difficult to buy and sell? First, transaction costs may be high. To purchase oil, one would have to buy the oil, pay for transport, pay for storage, pay for insurance, and so on. The cost of an arbitrage requiring the holding of oil could be quite high. On the other side, short-selling oil would be next to impossible, since no market exists for this.

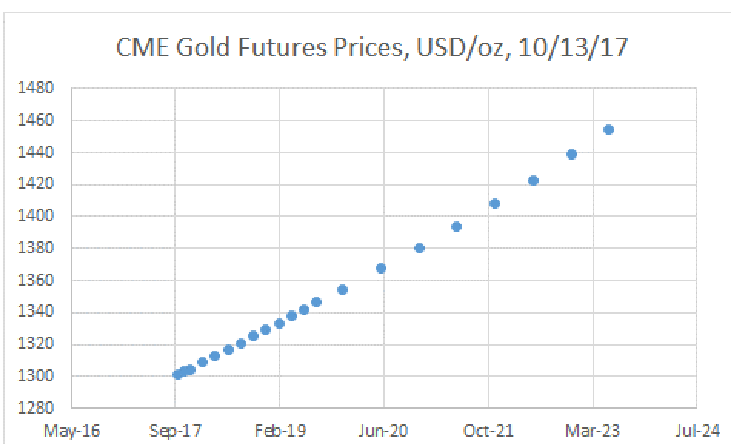
Let's reconsider the case of corn. One can buy corn, but it is not easy or cheap. Corn has to be transported, insured and stored. Cash and carry arbitrage with corn is possible but costly. This puts a maximum value on corn forward prices as long as the price of corn storage is fixed. What about short selling corn? Sorry, not possible. Therefore, there is no arbitrage restriction on how low the corn forward price could go.

*Contango and backwardation*

With fewer restrictions on forward prices, nonfinancial forward curves can take many different shapes. When forward prices are higher for longer maturities, as they are with financial forwards with $r>\delta$, the curve is said to be in *contango*. According to Wikipedia,

> "the term originated in mid-19th century England and is believed to be a corruption of 'continuation', 'continue' or 'contingent'. In the past on the London Stock Exchange, contango was a fee paid by a buyer to a seller when the buyer wished to defer settlement of the trade they had agreed."

The simplest nonfinancial forward may be gold, which almost always trades in contango, as shown in the figure. We are showing futures prices rather than forwards, which have the same function, but some important differences as shown later in the chapter.



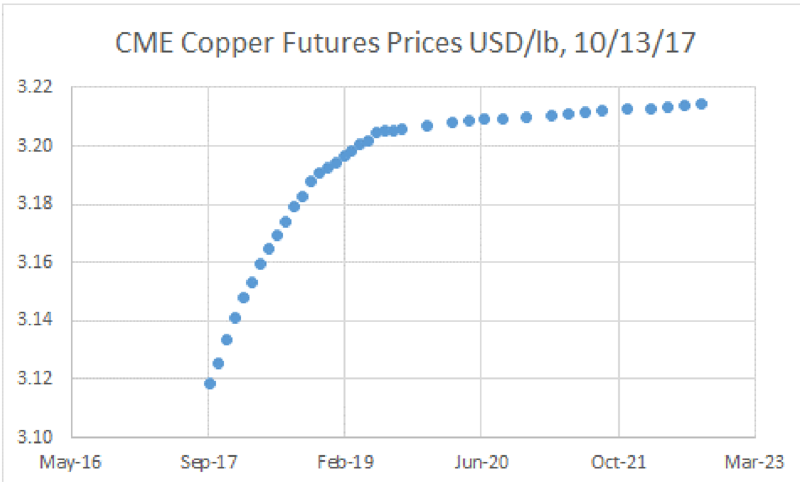

[1] This difficulty arises due to the inability to distinguish between a commodity's convenience yield and the risk premium on its forward contracts.

Why is gold considered nonfinancial? For a financial contract with no dividends, we'd expect F(t) = exp(rt) using continuously compounded rates. This would be due to arbitrage, since it typically represents both the upper and lower bound on forward prices. The upper bound is enforced by cash and carry arbitrage, and the lower bound is enforced by short sale arbitrage. This may present a problem. To short-sell gold, one has to borrow it. The cheapest way to borrow gold is to pay the lease rate, typically 1-3% per year, to the lender of the gold, typically a central bank. So we can say for gold, when δ is the lease rate and S is the spot price:

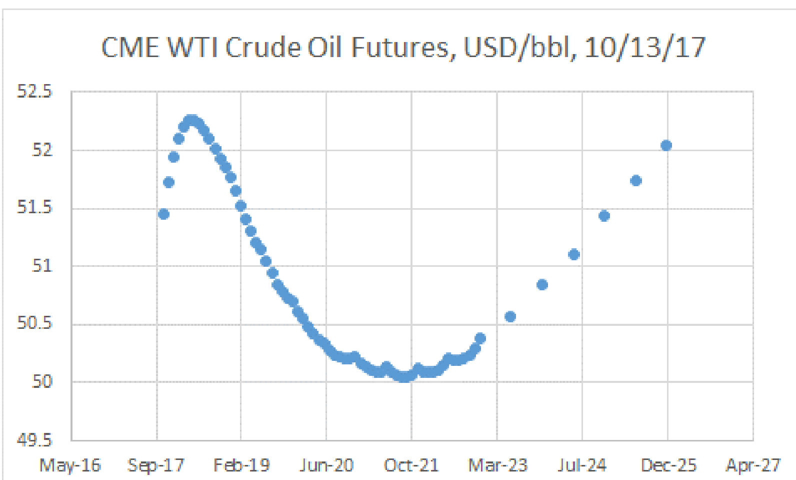
$$Se^{(r-\delta)t} \leq F(t) \leq Se^{rt}$$

Curiously, gold almost always trades at the lower bound. This may be because gold investors impute some convenience yield to holding physical gold that is not available with a forward contract. For example, if one needed to leave one's country in wartime, it might be a good idea to have some physical gold on hand!

Now let's take a look at copper forward prices. On the date shown, the market was in contango, but a contango very different from the convex contango of financial forwards.




Some markets trade in *backwardation* at least part of the time. A backwardated market is one where forward prices are lower than current spot prices. The term originated with John Maynard Keynes, who suggested that hedgers might depress forward prices by selling their goods in the forward markets. As an example, oil frequently trades in backwardation. However, the curve on 10/30/17 was a bit anomalous:



Remarkably, each commodity has its own story to tell. Consider oil for example. Oil storage is limited, because oil needs to be stored in controlled and approved facilities to avoid leakage and toxic damage to the environment. When the demand for oil storage increases, i.e. when forward prices rise relative to spot, the price of storage also rises! It is very difficult to make money in a cash-and-carry arbitrage when the cost of storage rises with the forward price. And of course, oil cannot be shorted.

Electricity is more shocking. It can't be stored, and it can't be shorted. *Circuits* can be shorted, but not electricity! Like oil, there are no arbitrage bounds for electricity forward prices.

Similar logic suggests that arbitrage bounds for forwards on LIBOR or the CPI do not exist. How are forwards priced in these cases, given there are no arbitrage bounds, and the underlying "assets" are not investable in the usual sense? We can't even really speak about the return of physical corn, oil, electricity, LIBOR or CPI.

We can only say in these cases that the forward price for a nonfinancial underlying satisfies the following, unless one of the arbitrage restrictions is active:

$$F_0 = E(S_T) \pm \text{risk premium}$$

The risk premium is determined by the demand for forward contract by buyers and sellers. This argument is due to the economist John Maynard Keynes. Since both $E(S_T)$ and the risk premium are unobservable, it would be difficult to estimate these. Nevertheless, behavior over time may give some clues. For example, long forward positions in oil have been shown to make positive returns on average over time. Extrapolating this into the future, one could infer the market's sense of expected future spot prices.

In the case of a commodity market with constant storage costs, such as copper, which can be stored anywhere, we can say

$$F_0 = \min[E(S_T) \pm \text{risk premium, full carry cost to time T}]$$

Fortunately, the value of a forward contract position is much simpler to determine than its price. Once a long position is offset by a short position, this locks in a gain or loss equivalent to our calculation using financial forwards.

$$V(t) = (F_t - F_0)e^{-r(T-t)}$$

Recall that $F_0$ was the forward price at the inception, and $F_t$ was the forward price for the offsetting transaction, both having the same maturity date T.

## SUMMARY UP TO NOW

In both the discrete and the continuous case, we can summarize with the following facts about forward prices and spot prices in the absence of default risk:

1. The "price" of a forward contract refers to the price that will be paid for the asset on the transaction (maturity) date, not the value of the contract
2. In the absence of default risk, with proportional underlying cash flows, the forward price on an investable asset $F_0 = S_0e^{(r-\delta)T}$ with maturity date T
3. This formula applies whenever the GVE can be used
4. For financial forwards, the formula also represents an arbitrage relationship when the underlying asset can be both bought and sold with low transaction costs
5. The value of any forward contract is zero at inception

6. The forward contract does not earn the time value of money
7. The financial forward contract earns a dollar risk premium identical to that of the underlying asset
8. The forward contract resembles a spot contract with 100% debt financing
9. When the spot and forward are perfectly correlated, the $harpe ratios of an underlying position and the forward contract are identical
10. Nonfinancial futures/forwards may not be subject to the same arbitrage-based pricing limits as financial assets, so the pricing models are different.

## FORWARD PRICES, THE DEFAULT OPTION, AND COUNTERPARTY CREDIT RISK

Until now, we have assumed no risk of default on forward contracts by either the long or the short. In reality, suppose a wheat farmer sold next year's harvest to a baker at a price agreed today. The wheat farmer may fail to deliver due to bankruptcy, crop shortfalls, or by dishonoring the contract and selling the wheat to another party at a higher price. The baker may fail to buy due to bankruptcy, insufficient cash, or refusal to honor the contract if a lower price came along.

Normally, forward contracts include various protections to both the buyer and seller to protect against the risk of counterparty default. These can include collateral posted by either side to ensure performance, guarantees provided by third parties, and *liquidated damages*, for example. *Liquidated damages* refers to a pre-agreed penalty for non-performance that would release the defaulting counterparty from their contractual obligations. Of course, if the defaulting counterparty can't or won't pay liquidated damages, they aren't much protection against default risk.

For this reason, it is necessary for both counterparties in a forward agreement to determine the present value of the expected losses due to default by the other counterparty. Some would call this the value of the *default option*, which is a reasonable characterization. Sometimes defaults are strategic, when the value of exercising the default option exceeds the cost of doing so. In other cases, default is the consequence of a possibly unrelated event that rendered the counterparty unable to meet the terms of its agreement. The increase in the value of a default option for one counterparty decreases the value of the forward contact for the other counterparty.

By this logic, though, one must also consider the value of a counterparty's own default option. If a counterparty thinks the value of its default option exceeds the other's, it might make sense strategically to enter the contract knowing that it was more likely not to execute --- this is like an individual making a bet knowing he could not pay if he lost.

Considering these factors, the value of a forward contract would have to be written as

Forward contract value = Default-free forward contract value

– Value of counterparty's default option + Value of one's own default option

In banking, the value of a counterparty's default option is called the *credit value adjustment* (CVA), and the value of one's own default option is called the *debit value adjustment* (DVA). Indeed, several textbooks have been written on CVA and DVA and they are beyond the scope of this chapter.

## INSTITUTIONAL CHARACTERISTICS OF FORWARDS AND FUTURES

In this chapter so far, we have focused on forward pricing, even though futures contracts perform the same major function as forward contracts. Both are contracts to undertake a transaction in the future at terms that are specified today.

Futures contracts have the following distinct characteristics when compared to forwards:

1. Futures contracts tend to be highly standardized while forward contracts can be highly specialized
2. Futures contracts trade on organized exchanges while forward contracts are negotiated directly between counterparties
3. Futures contracts are subject to initial margin (cash collateral) requirements and maintenance margin requirements that change daily as prices change; forward contracts may or may not have similar requirements.
4. The counterparty in a futures contract is the exchange clearinghouse, while the counterparty in a forward contract is the trading partner.

None of these points is meant to suggest that futures are better than forwards or vice versa. As a general rule, those who use forward contracts have specialized contract requirements. Those who use futures may have any number of different motives for trading, but rarely take or make delivery.

*Standardization*

Standardized contracts are best for those parties whose risk most resembles that of the standard contract. For instance, these are the standard terms of the Chicago Mercantile Exchange wheat contract, as reported on [www.cmegroup.com](http://www.cmegroup.com) :

| | |
|---|---|
| Contract Size | 5,000 bushels (~ 136 Metric Tons) |
| Deliverable Grade | #2 Soft Red Winter at contract price, #1 Soft Red Winter at a 3 cent premium, other deliverable grades listed in Rule 14104. |
| Pricing Unit | Cents per bushel |
| Tick Size (minimum fluctuation) | 1/4 of one cent per bushel ($12.50 per contract) |
| Contract Months/Symbols | March (H), May (K), July (N), September (U) & December (Z) |
| Trading Hours | CME Globex (Electronic Platform)<br>6:00 pm - 7:15 am and 9:30 am - 1:15 pm Central Time, Sunday - Friday<br>Open Outcry (Trading Floor)<br>9:30 am - 1:15 pm Central Time, Monday – Friday |
| Daily Price Limit | $0.60 per bushel expandable to $0.90 and then to $1.35 when the market closes at limit bid or limit offer. There shall be no price limits on the current month contract on or after the second business day preceding the first day of the delivery month. |
| Settlement Procedure | Physical Delivery |
| Last Trade Date | The business day prior to the 15th calendar day of the contract month. |
| Last Delivery Date | Second business day following the last trading day of the delivery month. |

| | |
|---|---|
| Product Ticker Symbols | CME Globex<br>ZW<br>W=Clearing<br>Open Outcry<br>W |
| Exchange Rule | These contracts are listed with, and subject to, the rules and regulations of CBOT. |

This contract is best suited for farmers who have #2 soft red winter wheat at the accepted delivery locations, and for those who agree with the prearranged premium or discount levels for other grades. Other farmers may use the futures market for protection against general drops in the price of wheat, but most will not deliver according to the contract terms. They will generally reverse their position prior to having to deliver.

This is a benefit of standardized contracts. Even if the original buyer wants to take delivery, a new seller can always be found to take on the obligation. Since the farmer offsets his original short contract with an identical long contract, the clearinghouse considers the trade null and void, relieving the farmer of any obligation to deliver. Of course, the farmer is responsible for or benefits from any price change since the time he entered the contract and the time he offset it.

On the other hand, some firms will prefer forward contracts precisely because they want them to be customized to their situation. For instance, a bank may offer a farmer a contract identical to the futures contract, but not require collateral to support it. While this contract would be more expensive to the farmer, in the form of accepting a lower forward price for his wheat, it may be well worth it for him to relieve himself of unknown cash demands associated with a futures contract.

*Public vs private*

Some people argue that futures are better than forwards because the prices are more competitive. This is usually interpreted to mean the bid/offer spread is lower in futures than in forwards, i.e. the offering price is lower and the bidding price is higher. Competitiveness also helps when one wants to exit a contract, because one has a sense that other counterparties will always be available to provide liquidity. With a forward contract, it may be difficult to reverse the contract with a counterparty, or even more difficult to establish an identical offsetting one with another counterparty. Even if one could design a perfectly offsetting contract, the party that thought it had gotten out of the transaction still bears performance risk to each side of the contract.

In spite of these advantages to trading futures, some counterparties prefer forward contracting because of concerns about anonymity, the ability to do large block trades, and the freedom from restrictions imposed by exchanges and their regulators, such as position limits.

*Margining*

Those who trade futures must be prepared to put up collateral to support their position in the event the position moves against them. While the farmer who shorts wheat futures may be hedging, if prices rise, he must come up with the cash today to margin his position, even though the wheat won't be sold until the harvest. It is possible that between the time the farmer enters the contract and he closes it out, the price may double and then return to its previous level. In this case, he would have needed the entire gross value of the crop as collateral to support the position.

In practice, what often happens is that the hedger cannot tolerate such large cash swings, and therefore ends up closing out the position before he otherwise would. This is unfortunate, because it forces the farmer to become unhedged precisely after he has lost a great deal of cash, threatening his solvency even further.

*Counterparty*

In futures contracting, since the counterparty is the exchange clearinghouse, most people don't worry too much about counterparty default risk. Although exchange default is a theoretical possibility, no major has ever defaulted on its obligations, even during the credit crisis of 2008-9. The exchange is able to protect itself through aggressive collateral collection, the capital of its member firms, and in some cases, an extra layer of protection in the form of an insurance 'wrapper' contract.

When one trades over-the-counter with another party, there is always the risk of counterparty default to consider. This is one major factor that differentiates forwards from futures --- you may think you are hedged, but if the counterparty fails to perform, you are not hedged.

*Options*

Both futures and forward contracts may have embedded options. For example suppose a wheat contract gave the long position an option to obtain an additional 10% at the same price. Hopefully, the short position got a small price increase for giving up that option. In the futures markets, most contracts contain delivery options, which allow for different methods to deliver. This is effectively a seller's option, known as the *cheapest to deliver* option. In some contracts, such as the Treasury bond futures contract, this can be an important factor affecting the valuation of the contract.

The next chapter covers options in greater detail.

## POPULAR MISCONCEPTIONS

In this section, we summarize the chapter by listing a number of popular beliefs about futures and forward contracts that are incomplete, misleading or incorrect, along with a corrected interpretation.

| **Incomplete, Misleading or Incorrect Statement** | **Correct Statement** |
|---|---|
| Forward contracts help sellers and buyers reduce risk. | Forward contracts allow sellers and buyers to exchange price risks for counterparty risk and/or margining risk. |
| The expected value of entering a forward contract is zero. (Same as the expectations hypothesis.) | The expected value of entering a forward contract may be positive, zero, or negative, depending on the market and the circumstances. |
| The risk of a forward contract is the same as the risk of the underlying asset. | The dollar risk of a forward contract is about the same as the risk of the underlying |

| | |
|---|---|
| | asset, but on a percentage basis, the forward contract risk may be much greater. |
| The forward price of an asset can never be greater than the price implied by the cost of carrying the physical asset. | The forward price of an asset is not limited from above by the cost of carry in markets where storage is scarce; the price of storage in these cases adjusts. |
| Forward prices are determined by arbitrage. | Forward prices in some markets are completely determined by arbitrage, but only when the cost of carry is the cost of money, and the cost of borrowing is the negative of the cost of money, i.e. interest earned on the cash obtained from selling the borrowed asset. In nonfinancial forward contracts, arbitrage is the exception rather than the rule. |
| Hedgers should always prefer futures to forward contracts since the market is more competitive and there is minimal default risk. | Hedgers may not prefer futures to forwards if they need customized contracts, or if they cannot tolerate daily mark-to-market and the attendant cash requirements. |
| Hedgers should always prefer customized contracts to fit their specific risk profiles. | Depending on the cost of customization, hedgers may prefer imperfect standardized contracts to perfectly tailored ones. |

## PRICING OF FUTURES CONTRACTS AND FUTURES POSITIONS

Absent any complications, the futures *price* is the same as the forward *price* of a contract. Both refer to the transaction price for a delivery in the future negotiated today. However, the value of a futures position differs from the value of the forward position due to marking-to-market. For example, if a contract was initiated at $5, and then went to $6, the long futures position would have earned the $1. The long forward position, however, only earns the present value of the $1. Even if the forward contracts are *booked out* , or offset against each other, the settlement of the contract is based on the present value of the difference between the initially agreed price and the price of the offsetting contract.

This causes futures prices to be slightly more volatile than an otherwise equivalent forward contract. However, it has no influence on the price of the futures unless the volatility correlates with interest rate changes. This effect is significant in bonds, but insignificant for most other contracts. Therefore, we leave further study of this topic to the interested student.

In sum, the value of a forward position currently is the present value of the difference between the original price and the offsetting price, where the present value is taken to the maturity of the contract. The value of the futures position is simply the mark-to-market.

Returning to the main purpose of the chapter, futures provide another benchmark for cash flow valuations that can be used in addition to portfolio-based methods. In Part III of the book, several examples of the application of futures to cash flow valuation will be provided.

## REFERENCES

Futures textbooks

## CHAPTER SUMMARY

1. For liquid markets, the forward price of a financial asset can be determined from the current price, the risk-free interest rate, and the dividend yield (assuming no default risk or additional risk premium).

   In the continuously compounded case,

$$F_0 = S_0 e^{(r-\delta)T}$$

2. A future is an exchange-traded forward. Futures require margin to cover changes in position prices and offer the advantages of greatly lowered credit risk, liquidity, and standardization of terms. However, forwards offer more flexibility.

3. When it is not possible to short an asset, it is not possible to precisely define the relationship between spot and forward prices. Forward prices can go into backwardization (F>S), contango (F<S) or a combination of the two. Factors that go into the shape of the forward curve include consumption, seasonality, production, and political risks.

4. Institutional features of futures and forwards, particularly counterparty risk and margining requirements, may affect the way the traded prices of these contracts can be used to value cash flows.

## EXERCISES

1. The value of the S&P500 index today is 2800, along with the six month T-bill yield of 0.50% and a continuously compounded dividend yield of 1%. Value all of the following, for futures and forwards using an initial 6 month maturity:

| | Forward | Futures |
|---|---|---|
| Price at inception | | |
| New contract price if S&P rises 1% in one month | | |
| Value of position in one month if S&P rises 1% | | |

| | | |
|---|---|---|
| Number of futures needed to hedge a long forward position | N/A | |

2. Suppose the spot price of corn today is \$4 per bushel, and the spot price in one year is normally distributed with mean \$5 and standard deviation \$0.75. Also assume the one year forward price is \$4.50 per bushel. Calculate the expected value of the following, all over the next year, assuming no storage costs or convenience yield, and that the risk-free interest rate is 2%.

| | Dollars | Percent |
|---|---|---|
| Return on spot | | |
| Change in forward | | |
| Risk premium on spot | | |
| Risk premium on forward | | |
| StdDev of Spot | | |
| StdDev of Forward | | |

## EXCEL EXERCISES

A. Using the S&P at 2800, an annualized drift of 10%, an annualized volatility of 15% and a risk-free interest rate of 2%, assuming no dividends, simulate the S&P futures on a daily basis over the next year. Use the formula for a geometric random walk

$$S(t+\Delta t) = S(t)\left[\left(\alpha - \frac{1}{2}\sigma^2\right)\Delta t + z\sigma\sqrt{\Delta t}\right]$$

Where the parameters are adjusted for the correct time step size, and z is a normally distributed independent random number, i.e. z=normsinv(rand( )).

Simulate the maximum cash required for margining a long position and a short position over the course of the year. Use 10,000 simulations, and be sure to report your error margins at the 95% confidence level.

If this was a forward, what would be the 99% worst-case counterparty exposure for a buyer of the long position? How about the seller of the short position?

## Chapter 10: Option Principles

### INTRODUCTION

This chapter introduces options simply, starting with an option to buy real estate. Progressive examples allow the introduction of assumptions necessary to employ risk-neutral pricing concepts, such as continuously defined Sharpe Ratios or assumptions on tradeability. With lognormally distributed prices and risk-neutral pricing, the famous BSM (Black-Scholes-Merton) applies.

While option valuation problems can be solved using differential equations, numerical p.d.e. solutions, integrals, or simulation, this chapter focuses on the use of simulation as a valuation tool, with a particular emphasis on understanding when one values options using the P-measure (actual probabilities) or the Q-measure (risk neutral probabilities).

Upon finishing the chapter, students should be able to value a wide range of traded and nontraded options.

### OPTION REVIEW

- An option is a contract between a buyer and seller, wherein the buyer may elect to enter a transaction in the future but the seller is obligated to comply with the buyer's choice.
- The right to buy an underlying asset is a ***call option***
- The price of the option is the ***premium***
- The agreed price for future purchase of the underlying is the ***strike price*** or ***exercise price***
- The seller or ***writer*** of a call option has an obligation and unlimited liability, since he must deliver the underlying if the option buyer wants it
- The right to sell an underlying asset is a ***put option***
- The last possible exercise date is called the ***option maturity***, and the length of time until that date is the ***term***.
- The **option payout** at maturity is the value assuming correct exercise.
    - The **net option payout** incorporates the value of the option premium
- An option than can be exercised at any time prior to or at maturity is called an ***American option***, and one that can be exercised only at maturity is called a ***European option***.
- The ***intrinsic value*** of an option is the value if it can be exercised today.
    - The extrinsic value, or time value, is the difference between the full premium and the intrinsic value

## MORE WAYS TO BUY A HOUSE

In the last chapter, we compared home purchases using cash, credit, forward contracts and prepaid forward contracts. Using the home value, rental payments and risk-free interest rates, we were able to compute a forward price using the General Valuation Equation. This was possible because, since the risk of the forward contract was the same as that of the underlying asset, the risk premia were identical. That is, we used the cost of risk for the house as the cost of risk for the forward. We now wish to see if and when we can apply this logic to options.

In some real estate markets, it is common for a purchaser to buy an option on a property rather than the property. This can be done outright by paying the owner a fee (the option *premium*) in advance, or by paying some amount every month while renting the property in what is known as a *rent-to-own* arrangement. The option normally specifies an expiration date, and a price at which the prospective homeowner can buy the property up until that expiration date. Unlike the forward contract, which obliges the long position to purchase the house in the future, the option contract gives the prospective property owner the right to choose: he can buy the property or walk away from the transaction for any reason.

The right to buy is also referred to as a *call option*, but in a real estate context, options are usually understood to be calls. Oddly enough, a mortgaged property owner also owns a *put option*, or the right to sell the property to the lender in order to satisfy his debts. You may know already that calls and puts have some similarities, but they have important differences.

The owner of an option has the right to choose. The word "option" comes from "opt", which means "to choose". The seller of the option has the **obligation** to deliver the house if the buyer chooses to exercise. Therefore, a short option position is not an option at all, but an obligation. Only long options positions can truly be called options.

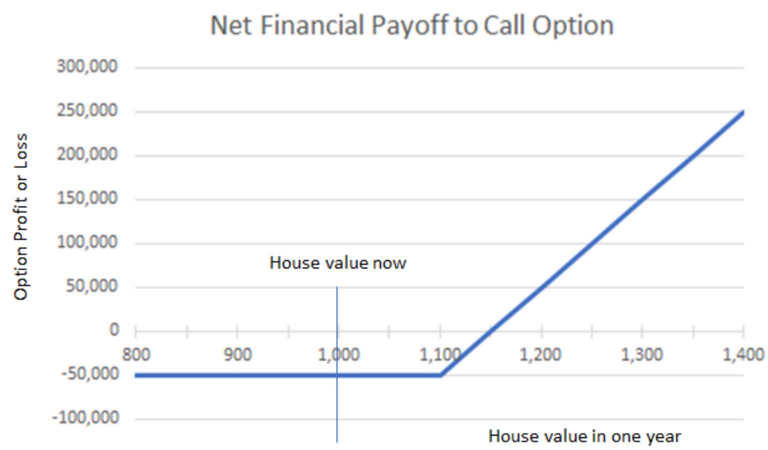
To make this discussion concrete, let's use a numerical example. Suppose I have a house worth $1,000,000 in Brooklyn. You might consider paying me $50,000 for an option to buy my house for $1,100,000 in one year. If you did this, you would actually accomplish several things:

1. I would not be able to sell the house to anyone else, to enable you to exercise your option in one year
2. You would have time to get your finances together
3. You would be able to walk away from the transaction if you got a great new job in a different city, and thereby avoid the high costs of purchasing a house in Brooklyn
4. You would be able to walk away from the transaction if the value of the house were below $1,100,000.

As for me, if I were happy selling the house for $1,100,000, a premium to this year's value, I see your $50,000 payment as additional cash in my pocket in exchange for some obligations I consider acceptable. Meanwhile, I continue to collect rent on the house from my current tenants.

People choose to exercise or not exercise their options for different reasons. Statements 2 and 3 are examples of *autonomous* exercise decisions, which are based on individual-specific variables. If the option is non-transferable, these are an important component to option value, but often very difficult to know. Other people choose to exercise for *financial* reasons, basing their decision to exercise based only on what the house is worth in a year. When options are transferable, they are normally exercised for financial reasons.

The chart below shows the net financial payoff of the call option financial exercise decision:




Mathematically, the call option value at expiration is written as C(S,T) where S is the future house value, X is the exercise price and T is the maturity date:

$$C(S,T) = \text{Max}(S - X, 0)$$

The "S-X" term reminds us that if the option is exercised, the option acted as a way to purchase the underlying property at a fixed price X on a deferred date T, just like a long forward position. The "0" reminds us that the call option buyer achieves a leveraged long position with downside protection. No matter how low S goes, the call option owner can never lose anything other than the initial call premium.

Since the option holder gets the best of both worlds, he must pay a premium for it --- since the seller is getting the worst.

## VALUING THE OPTION

Some readers may have already seen the Black-Scholes-Merton (BSM) model and would be tempted to use this model to value options. Unfortunately, the BSM model assumes that the underlying asset and the call option are tradeable, and moreover, tradeable continuously with no transaction costs. None of these assumptions holds in this example, so we need to take a different approach.

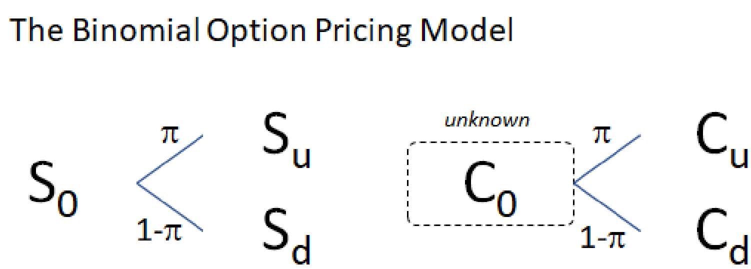
If you are tempted to look at arbitrage possibilities to determine maximum and minimum prices, good luck. The lowest possible value of the option is zero, and the highest possible value is the value of the house minus the present value of the rental income over the year. If the option price exceeded that calculation, it would be cheaper to just buy the house.

Therefore, the only way we can price this is to determine how the risk premium on the call option is related to the risk premium on the property, and price the call option by benchmarking it to the underlying asset. This is how we valued the forward contract, based on its risk relation with the underlying asset.

We may proceed by assuming a probability distribution for the house value in one year's time. The *binomial model* below assumes a Bernoulli or binary distribution for the future home value. The *lognormal model* assumes continuously compounded normally distributed returns.

## THE BINOMIAL MODEL WITHOUT ARBITRAGE

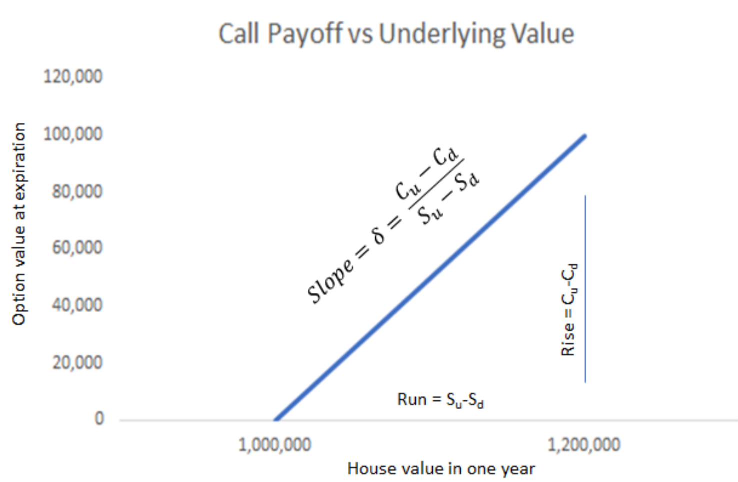
To make this as simple as possible for now, we will assume the house may have only one of two values in the future, a high value of $S_u$=\$1,200,000 with probability $\pi$=0.5, and a low value $S_d$=\$1,000,000 with probability (1-$\pi$). Its current value is $S_0$=\$1,000,000, and the risk-free discount rate is 8%. Under these assumptions, the call option as only two values at maturity, $C_u$=\$100,000 and $C_d$=0.




Ignoring rent, the house has an expected return of 10% under these assumptions. So we might be tempted to value the call option by discounting its expected value at 10%, about \$45,000. As interesting and simple as this approach seems, it is wrong! The expected dollar return cannot be the same since the risk levels of the house and the option are different. The option has lower dollar risk than the house because of the limited loss feature.

*Note to reader: This problem plagued option pricing theorists from Bachelier's work in 1900 into the 1970s. You may compute expected option payouts, but you don't know what discount rate to apply to them. The binomial model (exposited here) and the BSM model were two successful methods used to bypass the direct determination of a discount rate by valuing the assets directly using arbitrage.*

Even though the expected returns are different, we can observe in this simple model that (a) the returns on the house and the call option are perfectly correlated and (b) therefore the dollar risk premium of the option should be proportional to the risk premium for the house. Perfect (linear) correlation is achieved in this case by having only two possible outcomes.




We use the letter delta (δ) to indicate the slope of the regression of the terminal call value on the underlying value. The same notation is used in the BSM model for the derivative of the call option value with respect to the stock. In this example, $\delta$=0.5, indicating that the risk premium for the call option should be half the risk premium for the stock. We may now proceed to use the GVE to value the option.

The risk premium on the stock is given by the difference between its expected value and the value of a risk-free investment of $S_0$ made now:

$$RP[S_1] = E[S_1] - (1+r)S_1$$

Therefore the risk premium on the call is $\delta$ $RP[S_1]$. But the risk premium on the call is also given by

$$RP[C_1] = E[C_1] - (1+r)C_0 = \delta\ RP[S_1]$$

Setting these two quantities equal, it leads us to a pricing model, effectively the GVE result:

$$C_0 = \frac{E[C_1] - \delta(E[S_1] - (1+r)S_0)}{(1+r)}$$

The value of the call equals its expected value minus a risk charge proportional to that of the underlying stock, discounted at the risk-free rate. Continuing with our numerical example from before with r=8% and ignoring rental income, we can now compute all of the following:

Binomial model

| | | | |
|---|---|---|---|
| S | 1,000,000 | X | 1,100,000 |
| Su | 1,200,000 | r | 8.00% |
| Sd | 1,000,000 | | |
| pi | 50.00% | Cu | 100,000 |
| E[S] | 1,100,000 | Cd | 0 |
| | | E[C] | 50,000 |
| | | | |
| RP[S] | 20,000 | RP[Call] | 10,000 |
| delta | 0.5 | Call value | $37,037 |
| ER[S] | 10.00% | ER[Call] | 35.00% |

This leads to two interesting conclusions. First, the call is only worth about $37,000, so we would be overpaying for the option at $50,000. Second, reflecting the implicit leverage of the call option position, the expected return, and therefore the discount rate for expected cash flows, is 35%. Compared to the underlying house's expected return of 10%, this shows 3.5 times leverage. Even though the risk premium for the call is half the risk-premium of the stock, its percentage risk is necessarily much greater.

Exercise: Build the binomial model in Excel. What happens when you change pi? How can you explain your result? How does a change in the exercise price affect the expected return of the option? Why does this happen?

## OPTIONS ON LOGNORMALLY DISTRIBUTED NONTRADABLE ASSETS

In the last section, we assumed the probability distribution of house prices on the option expiration date was limited to two outcomes. In reality, it seems it would be better if one could specify any distribution of future house prices and thereby estimate the option price. However, at this moment, we do not want to assume the house or the option are continuously tradeable. Hence we must find an acceptable valuation approach that does not require tradability to value the option.

*In some treatments, this problem is classified as a* static hedging *problem, i.e. meaning that one can hedge the risk of the option with traded assets, but one can only make one hedging decision rather than changing the hedge ratio constantly.*

Suppose that the return on the house is normally distributed, but continuously compounded, leading to a lognormal distribution of the house's future value. If the expected return is $\alpha$, the risk-free rate is r, and the standard deviation of returns is $\sigma$, the resulting distribution is lognormal with the following properties:

$$\ln(S_T) \text{ is normally distributed}$$

$$E[\ln(S_T)] = \ln(S_0)+(\alpha - \tfrac{1}{2}\sigma^2)T$$

$$\text{Stdev}[\ln(S_T)] = \sigma\sqrt{T}$$

Now suppose we simulate possible values of $S_T$, using the relations just stated, with z being a standard normally distributed random number:

$$S_T = S_0 e^{(\alpha-0.5\sigma^2)T+z\sigma\sqrt{T}}$$

We can write $C_T$ for each simulated value of $S_T$, and thereby estimate $E[C_T]$, $Var[C_T]$ and $cov[S_T,C_T]$ --- that is, using a regression equation to express a linear relationship between $C_T$ and $S_T$, as we did in the binomial case:

$$C_T = (E[C_T] - \beta E[S_T]) + \beta S_T$$

$$\beta = \frac{cov[C_T, S_T]}{var[S_T]}$$

Since the first term in the equation for $C_T$ (the regression intercept) is a constant, it can be discounted at the risk-free rate. Since the second term is equivalent to holding $\beta$ shares of stock at time T, its value today is $\beta S_0$. Hence,

$$C_o = e^{-rT}(E[C_T] - \beta E[S_T]) + \beta S_0$$

To illustrate, let's use parameters roughly consistent with the house example discussed earlier.

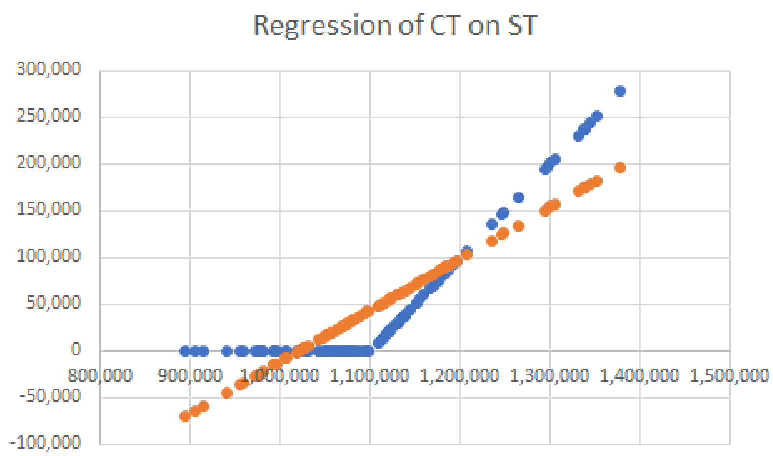
$S_0$=\$1,000,000
$\alpha$ = 10%
r = 8%
$\sigma$ = 10%
T = 1
X = \$1,100,000

Using z=normsinv(rand( )) in Excel to generate values of z, one could simulate values of $S_T$ and compute the corresponding values of $C_T$. Then, running a regression using the SLOPE and INTERCEPT functions, the estimated equation might look something like this:

$$C_T = -563{,}886 + 0.552211\, S_T$$

$$C_0 = -563{,}886 e^{-r1} + 0.552211\, S_0$$

And the resulting valuation might be about $31,678. Of course, every simulation will be different. The scatterplot generated by the simulation and the corresponding regression line are shown in the figure below. Only 100 points are shown although 2500 simulations were run. The benefit of the simulation approach is the scatter of points shown is consistent with the assumed lognormal probability distribution.




*The more advanced reader should note that there is no suggestion of a risk-neutral density up to this point in the presentation. This valuation relies only on the actual distributions of the underlying and option prices.*

This is the first example in this text of benchmarking the value of one nontraded asset to another, i.e. valuing the call option from the known price of the underlying asset. We will use this technique heavily in Chapter 11 to value an even wider range of assets. The logic of benchmarking can be summarized as follows (one period model). The key assumption is that risk uncorrelated to the benchmark is unpriced.

- Given an asset A whose value is unknown, and one whose value is known (the benchmark "M"), simulate cash flows to the asset and to the benchmark.
- Perform a linear regression of the cash flows of A on the cash flows of M, A=intcpt+slope x M
- Discount the intercept at the risk-free rate
- Take the present value of M's cash flows equal to the current market price
- Add these two terms to determine the value of A.

## THE OPTION REGRESSION MODEL IN CLOSED FORM

In the last section, we simulated the underlying probability distribution to obtain the value of the option. Fortunately, if the underlying distribution is normal or lognormal, one can solve the regression model as if one had simulated, by calculating the moments of the joint distribution of $C_T$ and $S_T$.

Recall the equation for pricing the call option

$$C_o = e^{-rT}(E[C_T] - \beta E[S_T]) + \beta S_0$$

$$\beta = \frac{cov[C_T, S_T]}{var[S_T]}$$

The moments of $S_T$ can be taken directly from the properties of the lognormal distribution:

$$E[S_T] = S_0 e^{\alpha T}$$

$$var[S_T] = S_0^2\left(e^{(2\alpha+\sigma^2)T} - 1\right)$$

The distribution of $C_T$ is a truncated lognormal distribution. Therefore

$$E[C_T] = \int_0^\infty Max(S_T - X, 0)f(S_T)dS_T = \int_X^\infty (S_T - X)f(S_T)dS_T$$

$$E[C_T] = S_0 e^{\alpha T} N\left(d_1(\alpha)\right) - XN(d_2(\alpha)) \equiv \Phi(S_0, X, \alpha, \sigma, T)$$

$$d_1(\alpha) = \frac{\ln\left(\frac{S_0}{X}\right) + \left(\alpha + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}; \quad d_2(\alpha) = d_1(\alpha) - \sigma\sqrt{T}$$

$$E[C_T S_T] = \int_X^\infty S_T(S_T - X)f(S_T)dS_T = \Phi(S_0^2, X^2, 2\alpha + \sigma^2, 2\sigma, T) - XE[C_T]$$

$$cov[C_T, S_T] = E[C_T S_T] - E[C_T]E[S_T]$$

Reviewing and rearranging the call pricing formula, we have

$$C_o = e^{-rT}(E[C_T] - \beta E[S_T]) + \beta S_0 = e^{-rT}(E[C_T] - \beta(E[S_T] - S_0 e^{rT}))$$

Which suggests that a risk charge is deducted from the expected call value and the result is discounted at the risk-free rate. The risk charge is proportional to the risk charge for the underlying asset, and the proportion is β.

Using the closed form solution, the intermediate values and the final option value can be calculated in Excel:

| | | |
|---|---|---|
| E(ST) | 1,105,171 | |
| E(ST2) | 1,233,678,059,957 | N( ) |
| | | |
| d1 | 0.096898 | 0.538596 |
| d2 | -0.003102 | 0.498763 |
| E(CT) | 46,602 | |
| | | |
| d1 | 0.196898 | 0.578046 |
| d2 | -0.003102 | 0.498763 |
| E(ST2-X2) | 109,620,458,888 | |
| E(CTST) | 58,357,997,517 | |
| cov(CT,ST) | 6,854,559,793 | |
| var(ST) | 12,275,301,797 | |
| beta | 0.558403 | |
| risk charge | 12,220 | |
| Call | 31738.80758 | |
| rC | 38.41% | |
| leverage | 15.21 | |

Our simulation was off by only \$60; the value of the call is \$31,739 using the closed form solution. Note that the expected continuously compounded call return is 38.41%, representing 15.21 times leverage built into the call option.

A final note, when the growth rate α equals the risk-free rate r, the famous BSM pricing equation obtains for the option:

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2); \quad d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}; \quad d_2 = d_1 - \sigma\sqrt{T}$$

When the underlying asset grows at the risk-free rate, there is not any risk premium, therefore the option will not have a risk premium either. This explains why the expected option payouts can be discounted at the risk-free rate.

## CONTINUOUS RETURNS AND OPTION PRICES

In order to discuss option pricing in continuous time, it is necessary to use a little stochastic calculus. This is because continuous-time mathematics work well for deterministic variables, but don't work well for random variables. Anyway, it is not as hard as it sounds.

We begin by assuming the underlying asset's return is normally distributed over the next instant of time dt. The return is the change in price divided by the price, i.e. dS/S, which we expect to be $\alpha$dt. Let's call the random part dW(t), or simply dW for short, since it is always the same random variable description, and each dW(t) is independent of any other. Now let's set E[dW]=0 and var[dW]=dt. Since the expected value of dW is zero, then var[dW] is the same as E[$dW^2$]. Finally, we want the instantaneous standard deviation to be $\sigma\sqrt{dt}$, so we can write the instantaneous return of the underlying as

$$\frac{dS}{S} = \alpha\ dt + \sigma dW$$

In dollar terms, this is

$$dS = \alpha S\ dt + \sigma S\ dW$$

Written this way, even though the stock price changes, its expected instantaneous return and volatility are constant. Therefore, if the interest rate is constant as well, the continuous version of the Sharpe Ratio, ignoring $\sqrt{dt}$ is

$$\frac{E[dS]/dt - rS}{\sigma S} = \frac{\alpha - r}{\sigma}$$

Now, if we assume that volatility and interest rates are constant over time, we could say the call option value depends only on S and t, i.e. C=C(S,t). Then the instantaneous capital gain on the call option would be given by dC. In ordinary calculus, dC=$C_S$dS+$C_t$dt, where subscripts indicate partial derivatives. **In stochastic calculus, there is one more term.**

If you think of dC as a Taylor series expansion around C, where only terms of order dt appear, that wouldn't be a bad idea. Only in stochastic calculus, one of the second order derivative terms does not disappear. This is called Ito's Lemma, which applied here means

$$dC = C_S\ dS + C_t\ dt + \frac{1}{2} C_{SS}\ dS^2$$

To see this, compute $(dS)^2 = (\alpha S)^2 dt^2 + (\sigma S)^2 dW^2 + 2(\alpha S)(\sigma S)dW\ dt$. After eliminating terms of dt raised to a power greater than 1, the expected value of this quantity is $\sigma^2 S^2 dt$, and its variance is zero. Therefore we have

$$dC = C_S\, dS + C_t\, dt + \frac{1}{2} C_{SS}\, \sigma^2 S^2 dt$$

The expected capital gain on the call option is E[dC], and the instantaneous cash flow is zero. Also, the instantaneous risk can be computed.

$$\frac{E[dC]}{dt} = \alpha S C_s + C_t + \frac{1}{2}\sigma^2 S^2 C_{SS}$$

$$\frac{stdev[dC]}{\sqrt{dt}} = \sigma S C_S$$

Perhaps the most important thing to notice is that the stock volatility is $\sigma S\, dW$, but the option volatility is $\sigma S C_S\, dW$ --- they are locally linear functions of the same normally distributed random variable even though the coefficients change over time, *so they are locally perfectly correlated*. Therefore, it is reasonable to assume that in each moment, the call option would be priced to have the same instantaneous Sharpe Ratio as the stock:

$$\frac{\alpha S - rS}{\sigma S} = \frac{\alpha S C_s + C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} - rC}{\sigma S C_S}$$

Eliminating the denominators and rearranging terms, we see the expected return components related to the stock cancel each other out. The resulting partial differential equation is

$$rC = rSC_S + C_t + \frac{1}{2}\sigma^2 S^2 C_{SS}$$

This equation would be satisfied for any asset whose value is derived from the underlying and time alone. For the call option, we must supply a boundary condition:

$$C(S,T) = Max(S - X, 0)$$

The solution to this famous p.d.e. is of course the BSM formula. Note that we did not explicitly make the assumption of continuous trading or hedging, but rather, that the option should be priced as if it had the same Sharpe Ratio of the stock every moment in time.

## RISK-NEUTRAL PRICING

The implication of risk-neutral pricing is that we can set the expected returns of all assets and derivatives equal to the risk-free rate, so that expected cash flows can be discounted easily. Up to this point, we have seen three different circumstances allowing us to use risk-neutral pricing of derivatives.

1. There is no risk premium on the underlying asset
2. The underlying asset is assumed to be continuously and costlessly traded so that the derivative can be priced using arbitrage arguments relative to the underlying
3. The option and the underlying asset must be priced so that their Sharpe Ratios are the same over (continuous) time.

To use risk-neutral pricing, one sets the growth rate of the underlying asset equal to the risk-free rate, and calculates the derivative payoffs resulting from this change. The expected value of these derivative payments is then discounted at the risk-free rate in order to value the instrument. The valuation can proceed by solving a differential equation, by integration, or by simulation, depending on the situation.

Academic authors refer to the actual price distributions as the "P-measure" and the risk neutral price distributions as the "Q-measure". The examples we gave earlier in the binomial case and the lognormal case were both solved using the P-measure, but there is a Q-measure corresponding to each one that leads to the same solution.

*The Binomial Case*

In the binomial model, the P-measure is described by the actual probability of an upward move in the underlying, i.e. $\pi$. In the Q-measure world, it must be the case that the expected return on the underlying asset must be the risk-free rate. Mathematically, if $\pi^Q$ is the Q-measure probability, this is accomplished by the following:

Choose $\pi^Q$ s.t. $\pi^Q S_u + (1-\pi^Q)S_d = (1+r)S_0$, i.e.

$$\pi^Q = \frac{(1+r)S_0 - S_d}{S_u - S_d}$$

Then

$$C_0 = \frac{\pi^Q C_u + (1-\pi^Q)C_d}{1+r}$$

*The Continuous Case*

In the lognormal or the BSM model, the expected return of the underlying asset is $\alpha$. Because of continuous trading, or because of the assumption that the instantaneous Sharpe Ratios of the underlying and the call must be the same, we learned that the differential equation does not depend directly on $\alpha$, but is the same as if the growth rate were replaced with the risk-free rate.

Hence, if one simulates the underlying asset using the lognormal distribution of the Q-probabilities (risk-neutral probabilities), then the derivative payouts can be computed and discounted at the risk-free rate to determine the derivative price.

## VALUING ALMOST ANY DERIVATIVE USING SIMULATION

In this section, we shall demonstrate univariate and multivariate derivatives which can be valued using the Q-measure together with simulation.

*Univariate problem #1*

If Amazon stock trades at $1500, the annual growth rate is 20%, and the annual standard deviation is 30% with no dividends, what is the value of a bet that pays off $1,000,000 in one year if Amazon exceeds $2000 at that time, if the risk-free rate is 2%?

*Solution*

We'll assume because AMZN is tradeable we can use a BSM approach with risk-neutral pricing. However, this is a binary option, not a regular option. Therefore, we simulate lognormal prices for AMZN in one year's time under the Q-measure, calculate the payoff of the derivative (a million or nothing), and find the expected value. The estimated value is $144,869 with a 95% confidence error of $8,121.

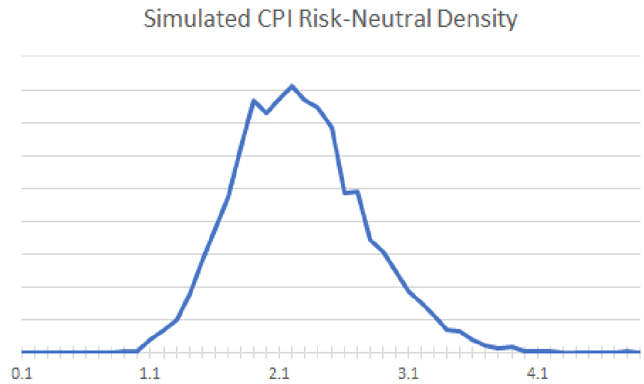
| AMZN RN | BET | PV |
|---|---|---|
| 1,523 | 142,000 | 144,869 |
| 465 | 349,085 | |
| 6.58 | 4,936.81 | |
| 1.65 | 1.65 | |
| 10.83 | 8,121.05 | 8,121 |

*Univariate problem #2*

Suppose the distribution of the CPI inflation rate is one year's time is Gamma, with a mean of 2.0, and standard deviation of 0.5. The forward contract on the CPI inflation rate is priced at 2.2. What is the value of the call option struck at 2.4, if the risk-free interest rate is 0.8%?

*Solution*

The underlying asset (CPI rate) does not trade, but the forward contract has a growth rate of 0 under the Q-measure, and the forward contract trades, hence we can use that as the risk-neutral distribution. If the mean is 2.2 and the standard deviation is 0.5, then the parameters of the gamma distribution are $\alpha$=19.36 and $\beta$=1/8.8=0.113636. Using Excel to generate 5000 random forward price outcomes, using GAMMA.INV(rand( ),$\alpha$,$\beta$) we get the following simulated distribution of CPI rates:




When in doubt of the results check to make sure the simulated underlying values have the desired distribution's moments. Computing the option payouts and discounting continuously at 0.8% for the year, we have an estimated option value of $0.1233 with a maximum 95% confidence error of 0.0060.

| | CPI | Call | PV |
|---|---|---|---|
| Mean | 2.2070 | 0.1243 | 0.1233 |
| StDev | 0.5076 | 0.2596 | |
| N | 5000 | 5000 | |
| StErr | 0.0072 | 0.0037 | |
| #s.d. | 1.6450 | 1.6450 | |
| MaxErr | 0.0118 | 0.0060 | 0.0060 |

*Multivariate problem*

Three stocks are currently valued at $100 each. The option pays out based on the highest of the three stock values in six months time. The payout is Max(max($S_1$,$S_2$,$S_3$)-X,0), where X= 110. The volatility of each stock is 20% per year, and the risk-free rate is 2%. What is the value of the option today, if the correlation between any pair of the three stocks is 0.10?

*Solution*

The correlation matrix of returns together with its Cholesky decomposition is shown below:

| Correlation matrix | | | Cholesky decomposition | | |
|---|---|---|---|---|---|
| 1.000000 | 0.100000 | 0.100000 | 1.000000 | 0.000000 | 0.000000 |
| 0.100000 | 1.000000 | 0.100000 | 0.100000 | 0.994987 | 0.000000 |
| 0.100000 | 0.100000 | 1.000000 | 0.100000 | 0.090453 | 0.990867 |

We generate 5000 sets of 3 correlated random normally distributed numbers, and from this the 3 simulated security prices for each 6-month scenario. Here is a sample of the simulation output

| Sim | Independent | | | Correlated | | | Stock prices | | | Max | Option payout |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -0.26 | 0.21 | 0.49 | -0.26 | 0.18 | 0.48 | 96.41 | 102.62 | 106.98 | 106.98 | 0.00 |
| 2 | -0.94 | 0.01 | -0.20 | -0.94 | -0.08 | -0.29 | 87.51 | 98.84 | 95.95 | 98.84 | 0.00 |
| 3 | -0.13 | 1.35 | -0.49 | -0.13 | 1.33 | -0.37 | 98.21 | 120.77 | 94.88 | 120.77 | 10.77 |
| 4 | -0.91 | -0.19 | 0.81 | -0.91 | -0.28 | 0.69 | 87.93 | 96.10 | 110.25 | 110.25 | 0.25 |
| 5 | 1.96 | -0.57 | 0.75 | 1.96 | -0.37 | 0.89 | 132.01 | 94.94 | 113.38 | 132.01 | 22.01 |

The value of the option is estimated at $6.12 with a 95% confidence maximum error of $0.20.

| | Max | Option | PV |
|---|---|---|---|
| Average | 112.55 | 6.19 | 6.123755 |
| StDev | 12.32 | 8.67 | |
| N | 5000 | 5000 | |
| StdErr | 0.17 | 0.12 | |
| #s.d | 1.65 | 1.65 | |
| MaxErr | 0.29 | 0.20 | 0.199606 |

## SUMMARY

Options play an extremely important role in finance. In many cases, options are explicit, such as traded options or options to buy real estate. In other cases, options are implicit, wherein contracts can make option-like payments, or indeed, any payments that are nonlinear functions of traded or nontraded asset prices.

In this chapter, we discussed a number of ways to value options. The simple binomial method is used mostly as a didactic tool, but it can also be expanded so that a process with a large number of binomial steps can be built to replicate many terminal asset price distributions. The static hedge method suggests that if a nontraded option can be approximated as a linear function of a traded asset's value at maturity, its value can be inferred from the (P-measure) probability distribution of the asset's terminal value and the asset's discount rate.

Numerous famous option pricing problems reduce to (Q-measure) risk neutral probability distribution functions, allowing expected option payouts under the Q-measure to be discounted at the risk-free rate of interest. The Q-measure can be used if

a. The underlying asset has no risk premium
b. One is willing to assume the underlying asset and the call option are priced have the same continuous Sharpe Ratio over time, even if not traded.
c. The underlying and/or the call can be traded continuously and costlessly so as to create an arbitrage pricing

Whether one solves the differential equation under the Q measure, integrates or simulates the asset's payout, the result is often a simpler means to compute an option's price.

After studying this chapter, you should be able to value a wide class of options if they can reasonably meet one of the criteria set above. Also, since options and other nonlinear functions of asset values correlate to cash flows, the option pricing tools covered in this chapter can be used to value a wide range of cash flow simulations.

## EXCEL EXERCISES

A. Build a spreadsheet to value call options under the assumption of lognormally distributed future stock prices, with no trading of the underlying stock or the option. Perform a sensitivity analysis of the option price with respect to the assumed underlying growth rate ($\alpha$), especially noting the relationship when $\alpha$=r.

B. Build a spreadsheet to perform risk-neutral valuations of the following options: European call options & binary options with one set of random numbers, and chooser options (2 stocks) & basket option (2 stocks) with a second set.

# Chapter 11: Advanced Benchmarking

## INTRODUCTION

In this chapter we return to the question of how to value future cash flows. To do so, we assume either that the probability distributions of these cash flows are known, or that some simulation has produced future cash flow scenarios, in which case a probability distribution may be estimated.

If the risk measure and the cost of risk are known, the valuation can be computed easily with the GVE. In Chapter 8, we also showed that the GVE could also be used to value a risky cash flow relative to a portfolio. One important example of a reference portfolio is the "market portfolio" of the CAPM.

In practice, we can do better than the CAPM. This is because future cash flows naturally correlate with priced assets. For example, a company's income may rise and fall according to the value of the S&P 500 or risk-free interest rates. These are called procyclical and countercyclical cash flows. A company's income may depend on commodity prices, if they are producers or consumers of commodities. An international airline's income depends on foreign exchange rates, since rates affect both demand and domestic translation of foreign ticket sales. A company's income may also depend on the options they are able to exercise in different market environments. In this chapter, we will value a simple refinery with the real option to decide to produce or not given market prices of energy products.

Because you have studied futures and options in the previous two chapters, you are now well-equipped to use traded asset prices (securities) and contract prices (futures and options) to value cash flows more accurately. In some cases, this leads to a much higher quality valuation. The improvement inaccuracy depends on how well the traded assets replicate the cash flows.

The theoretical foundation of this methodology comes from Brennan (19xx).

## STATIC REPLICATION AND VALUATION

We will use static portfolios of traded asset and contract prices to replicate a cash flow as well as possible. The value of a portfolio is a linear combination of the component values, therefore we can use simple linear regression to determine the best replicating portfolio. The residual uncorrelated risk may be assumed unpriced in some cases, or may be priced using the methods of Chapter 12. For now, we assume residual risk is unpriced.

Philosophically, instead of trying to find a discount rate for a cash flow, we are trying to replicate it from traded asset and contract prices. This process is called *benchmarking*. Benchmarking resembles a discrete-time option-pricing valuation framework, where no rebalancing is possible. See the references on option pricing with static hedges for further study.

If benchmarking seems new or difficult to you, consider that many equity analysts use P/E (price to earnings) ratios to value company A's shares to a comparable company B's shares:

$$\text{Price}_A = \text{Earnings}_A \times \text{Price}_B/\text{Earnings}_B$$

$$\text{Price}_A = \text{Price}_B \times (\text{Earnings}_A/\text{Earnings}_B)$$

In the language of this chapter, company A's share is being replicated in some sense by a position in B's shares, and the quality of the replication depends on the choice of a comparable portfolio. For example, one might also use the industry P/E ratio as a benchmark, or the market P/E ratio.

Most analysts use P/E as a starting point and may make adjustments for leverage, growth or risk. All of these translations use a simple adjustment to the P/E in order to attempt to improve its predictive power, but the multiples are used in the same way as the basic P/E multiple. In this chapter, we will do a multi-asset equivalent of the simple P/E ratio. This will likely dominate the replication ability of standard P/E analysis, in exchange for some analytic complexity.

## VALUING A SINGLE CASH FLOW (REVIEW OF ONE-PERIOD MODEL)

By way of a brief review, let's repeat two simple valuation ideas. The standalone valuation $V_0$ of an unknown cash flow C paid at time 1 is determined according to the GVE by its appropriate risk measure $\sigma$ and its cost of risk k:

$$rV_0 + k\sigma = (0 - V_0) + E[C]$$

$$V_0 = \frac{E[C] - k\sigma}{(1+r)}$$

The numerator is the certainty equivalent, which is of course discounted at the risk-free rate.

In the CAPM, the risk measure is taken to be the covariance of the cash flow with terminal asset values in one period. If we assume the cash flow does not affect the reference portfolio, then the valuation as shown in Chapter 8 would be

$$V_0 = \frac{E[C] - A\,\Sigma_c' 1}{(1+r)}$$

where $\Sigma_C$ is the column vector comprised of the dollar covariances of the cash flow with each of the assets in the reference portfolio. Additionally, we assumed jointly normal cash flow distributions, and that the risk uncorrelated with the reference portfolio was unpriced in order to derive this result. The parameter A is a measure of the price of variance risk for the reference portfolio.

Recalling the valuation of the portfolio in equilibrium in vector notation,

$$V^* = \frac{\mu - A\,\Sigma\,1}{(1+r)}$$

Recall that $\mu$ is the column vector of the expected cash flows of the N assets in one period, $\Sigma$ is the dollar covariance matrix, and "1" is a column vector of scalars equal to 1.

We may now proceed to replicate the cash flow as well as possible using a portfolio of **all** the securities in the reference portfolio, instead of constraining ourselves to work with the portfolio as a single unit.

## THE CAPM IN REPLICATION FORMAT

Let's represent any portfolio holdings of the assets in the reference portfolio by a column vector $\beta$. A project generates a cash flow C which may correlate with assets in the reference portfolio. We wish to choose the replicating portfolio that minimizes the variance of the final net cash flow. Letting M represent the vector of random cash flows in the reference portfolio, we can write

$$Var(C - \beta' M) = \sigma_C^2 - 2\beta'\Sigma_C + \beta'\Sigma\beta$$

and minimize over β to get the usual regression β when covariances are known rather than estimated:

$$\beta = \Sigma^{-1}\Sigma_C$$

We may now solve for the valuation equation of the cash flow in terms of all the assets in the reference portfolio:

*The CAPM Cash Flow Replication Equation*

$$\boxed{V_0 = \frac{E[C] - \beta'\mu}{(1+r)} + \beta' V^* \;\; with \;\; \beta = \Sigma^{-1}\Sigma_C}$$

In other words, the valuation may also be written as a linear combination of a constant and the values of all the assets in the reference portfolio.

## RESIDUAL RISK AND IDIOSYNCRATIC RISK

We now face a definitional challenge. In the CAPM, the term *residual risk* refers to the risk of as security after hedging the market risk, essentially borrowing the word *residual* from the regression framework. *Idiosyncratic risk* is often used synonymously, since any risk is independent of the market portfolio has no price. The mistake arises when we conclude that the idiosyncratic risk of one security does not correlate to anything else, a more popular interpretation of the word *idiosyncratic*. In fact, the residual risk of one security will in general correlate to the risk of other securities in the market portfolio individually even though it has no correlation to the market portfolio.

To resolve this nomenclature problem, we propose the use of another term, the *unique risk* of a security, which is the residual risk implied by the CAPM Cash Flow Replication formula rather than the one that only considers the correlation to the market as a whole. The unique risk is purely idiosyncratic, and uncorrelated to any other risk in the market portfolio. The unique risk variance can be computed from the definition above, substituting the replicating portfolio's beta:

$$Var(C - \beta' M) = \sigma_C^2 - 2\beta'\Sigma_C + \beta'\Sigma\beta = \sigma_C^2 - \Sigma_C'\beta$$

As one would expect, the positivity constraint on the variance places a lower bound on the value of $\sigma_C^2$

$$\sigma_C^2 \geq \Sigma_C'\beta$$

These distinctions are particularly important when we value an asset relative to a portfolio other than the market portfolio.

## WORKED NUMERICAL EXAMPLE

Let's re-open an example from Chapter 8. In this case, the cash flow for each of five assets in the reference portfolio has a given mean vector and covariance matrix, as shown below. We seek to value a cash flow represented in the 6th row of the table. The cash flow is to be valued relative to the reference portfolio. The expected cash flow is 10, the covariance with the first reference portfolio cash flow is 2, and so on for assets 2-5. The standard deviation of the cash flow is $\sigma_C = 2$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| r | | 0.02 | | A | | 0.04 | | |

| | Means | Covariance | | | | | RowSum | Valuation |
|---|---|---|---|---|---|---|---|---|
| | 100 | 400 | 100 | 50 | 25 | 10 | 585 | 75.10 |
| | 150 | 100 | 441 | 50 | 25 | 10 | 626 | 122.51 |
| | 50 | 50 | 50 | 225 | 20 | 5 | 350 | 35.29 |
| | 75 | 25 | 25 | 20 | 256 | 8 | 334 | 60.43 |
| | 125 | 10 | 10 | 5 | 8 | 484 | 517 | 102.27 |
| CF | 10 | 2 | 1 | 5 | 0 | -3 | 5 | 9.61 |

The valuation of the cash flow using the method of Chapter 8 is to find the certainty equivalent cash flow, 10 – 0.04(5), and discounting at the 2% risk-free rate to arrive at $9.61. Now let's value the cash flow using the Replication Equation.

We multiply the inverse covariance matrix times the cash flow covariance vector with the reference assets to obtain the dollar betas. The sumproduct of the beta vector is taken with the mean vector and the pricing vector V*. The calculation is shown below:

| Inverse Covariance | | | | | $Betas |
|---|---|---|---|---|---|
| 0.00271 | -0.00055 | -0.00046 | -0.00017 | -0.00004 | 0.0027 |
| -0.00055 | 0.00245 | -0.00041 | -0.00015 | -0.00003 | -0.0006 |
| -0.00046 | -0.00041 | 0.00466 | -0.00028 | -0.00003 | 0.0221 |
| -0.00017 | -0.00015 | -0.00028 | 0.00396 | -0.00006 | -0.0017 |
| -0.00004 | -0.00003 | -0.00003 | -0.00006 | 0.00207 | -0.0064 |

| | |
|---|---|
| β'μ | 0.3457 |
| constant term | 9.4650 |
| β'V* | 0.1428 |
| Replication value | 9.6078 |

The valuations are identical of course. The first valuation may be preferred for simplicity and ease of calculation. The benchmark valuation will be preferred when one wants to know the sensitivity of the valuation to the values of a large number of securities. For example, a tech company may believe its value is tracked sufficiently well by the S&P500, in which case the CAPM valuation would be useful. If the tech company stock is better tracked by other tech companies or tech indices, it may be better to build a benchmarking model using other tech stocks to understand those pricing relationships.

The lower bound of $\sigma_C^2$ for the covariances is 0.1343. Therefore, the unique risk variance is the difference between $\sigma_C^2 = 4$ and its lower bound, 3.8657.

## A ONE-PERIOD ILLUSTRATION

Suppose a company determines its revenues are procyclical. It has developed a model to suggest that its expected cash flow over the next year will be a function of the value of the S&P index (M), grossed up for any received dividends over the year. If the expectation is given by $E[C_1] = b\ E[M_1]$, then the value of the cash flow today is simply $V_0 = b\ M_0$. We don't need to worry about discounting, since M is already priced by the market, and we don't need to deduct for dividends.

Now suppose the expected cash flow formula is $E[C_1] = a + b\ E[M_1]$, with "a" being constant. Because "a" is constant it can be discounted at the risk-free rate. Therefore the valuation is $V_0 = a/(1+r) + b\ M_0$ today. The cash flow is being replicated by a position in the S&P 500 and in a zero-coupon bond maturing in one year. If the price of that bond today is $B_0$, then we can write the valuation as $V_0 = a\ B_0 + b\ M_0$.

Suppose we don't know the actual S&P value because of asynchronous trading of the underlying components. If we want to use S&P futures to value the cash flow, we can find the present value equivalent by discounting at the risk-free rate. Hence if $F_{0,1}$ is the futures price of the S&P index today for settlement in one year, then

$$V_0 = a\ B_0 + b\ F_{0,1}\ B_0$$

In other words, futures prices can be converted to asset price equivalents by discounting at the risk-free rate.

The same logic can be used to value European options on the cum-dividend S&P index. After all, an option payment at maturity is simply a cash flow determined as a function of the S&P index value at maturity. In Chapter 10, we showed how one could determine the value of a European option using a regression model --- this was your first example of a benchmarking exercise.

Finally, suppose we have a nonlinear relationship, e.g. $E[C_1] = b\ E[M_1] + c\ E[M_1^2]$. What is the value of the cash flow today? We could replicate the cash flow with stocks or futures, but we would not capture the nonlinearity. To do so, we should include options in our replication strategy. Options do not need to be discounted since we know their values today. The numerical example in the next section illustrates this last valuation.

## NUMERICAL WORKED EXAMPLE – ONE PERIOD CASH FLOW

PROBLEM:

You are working as a financial engineer in May 2019 at an oil company. The CEO has negotiated an investment contract with a wildcatter company. The investment requires a $25 million nonrefundable deposit. It is automatically triggered in 6 months if the oil price equals or exceeds Q=$70. If the Price (P) equals or exceeds Q, then the investment is estimated to be worth ($100 + $(P-Q)^2$) millions. The nonlinear effect is due to the ability to undertake more aggressive extraction strategies as the price rises.

Use market data as of May 20, 2019, including these assumptions:

- Oil futures (6 months) $62.65 per barrel
- Oil price volatility 25%
- Risk-free rate 2% p.a.

What is the value of the contingent investment on May 20, 2019?

SOLUTION:

Assuming the oil futures price follows a geometric random walk, we can use the drift value of zero to simulate 10,000 outcomes in 6 months.

For each of these outcomes, simulate the payoff.

You can use the basic simulation method to take the average of the payoffs and discount to the present. However, in this case we want to see the benefit of benchmarking to the known futures price. We therefore regress the payoffs on the price, and obtain the following statistics (your results will vary of course, due to the simulation):

| | | | | | | |
|---|---|---|---|---|---|---|
| intercept | -512.429 | | | b' | -512.43 | 9.00 |
| slope | 9.003219 | | | Xh' | 0.99 | 62.03 |
| steyx | 100.0449 | | | | | |
| | | | | | | |
| X'X | 10000 | 625678.122 | | | | |
| | 625678.1 | 40407066.8 | | | | |
| | | | | | | |
| INV | 0.003208 | -4.9667E-05 | | | | |
| | -5E-05 | 7.93805E-07 | | Est value | | 51.11 |
| | | | | StErr (mean response) | | 0.99 |
| INV x Xh | | 9.49636E-05 | | | | |
| | | 6.45919E-08 | | | | |
| | | | | | | |
| Xh'(inv)Xh | | 9.80251E-05 | | | | |

The value of this investment is currently estimated at $51.11 million minus the $25 million upfront cost. Because this is a benchmark valuation, we could change the futures price to determine the sensitivity of the value to oil prices. Since the regression is linear, every $1 increase in the oil price (measured as the futures price discounted to the present) will bring the estimated value higher by $9.00 mm. However, the standard error of the mean response will increase as the futures price moves away from its mean value.

## SUMMARY OF THE BENCHMARKING ALGORITHM

Suppose we are using N simulated outcomes for market-priced variables and cash flows in the future. Market-priced variables can include securities, futures, and options, for example. To make things simple, let's use a zero-coupon bond worth $1 at time t, a dividend-protected stock (S), and a futures contract price (F). For this purpose, the option behaves like stock since it is a fully paid-up contract.

We may write the value of the cash flow at time t as follows, performing a linear regression on the outcomes provided by the simulation:

$$V_t = b_0 + b_1 S_t + b_2 F_t + \varepsilon_t$$

Then if we assume the unique risk is unpriced, the expected value at time 0 is given by

$$V_0 = b_0 e^{-rt} + b_1 S_0 + b_2 F_0 e^{-rt}$$

If we are willing to assume approximate joint normality in the regression variables, we can also compute the error in the value estimate, which is the square root of the mean response variance (MRV). Taking $s_e$ as the residual standard error of the regression, and X as the N×3 matrix of the simulated market prices, the first column of which is all ones, we have

$$b' = (b_0 \quad b_1 \quad b_2)$$

$$X_h' = (e^{-rt} \quad S_0 \quad F_0 e^{-rt})$$

$$MRV = s_e^2 X_h' (X'X)^{-1} X_h$$

The MRV can be used to estimate the error in the simulated mean of the replicating portfolio. The final solution for the value with an error of z standard deviations from the mean estimate is therefore

$$V_0 = X_h' b \pm z s_e \sqrt{X_h' (X'X)^{-1} X_h}$$

*Reminder on risk neutrality in simulations*

Note that if the benchmarking procedure is used to value a derivative instrument, and the underlying variables have been simulated using the risk-neutral growth rates, the value of $V_0$ will converge to the value of the derivative in the risk-neutral framework.

The following table summarizes the growth rate to use when simulating risk-neutral future values:

| | Actual growth rate | Risk-neutral growth rate |
|---|---|---|
| Stock | $\alpha$ | r |
| Stock with dividends | $\alpha - \delta$ | $r - \delta$ |
| Futures | $\alpha - r$ | 0 |

## THE MULTIPERIOD CAPM MODEL

In practice, many analysts will estimate a discount rate for cash flows as if they were in a one period setting, and then assume the result does not change in the future. One such assumption is the “constant opportunity set,” which requires that the moments of the asset return distributions stay constant over time. In addition, because the risk is usually measured as the standard deviation of returns, the dollar risk of the asset is always proportional to its value. This is a critical limitation of the CAPM and related models --- and the single period CAPM should not be used to extrapolate to multiple periods without this assumption.

We can see this by writing the GVE in this manner, where volatility is always proportional to asset value by a constant u:

$$rV_0 + kuV_0 = E[V_1] - V_0 + E[C_1]$$

The denominator absorbs the risk charge by adding a risk premium to the risk free rate. This can only happen in the case where dollar volatility is proportional to dollar value.

$$V_0 = \frac{E[V_1 + C_1]}{(1 + (r + ku))}$$

which leads to the conclusion that expected values should be adjusted at a constant risk-adjusted rate.

In fact, there is no other theoretical justification for applying a constant discount rate to multiperiod cash flows to determine value.

Bogue and Roll (19xx) demonstrated that the preferred way to extrapolate the CAPM to multiple periods would be to apply the CAPM sequentially moving backward in time from the last cash flows to the first. This dynamic programming approach resembles the methodology used in pricing derivatives. Rubinstein (19xx) ….. other support for multiperiod modelling?

In summary, the constant opportunity set assumption fails to capture changes in risk profiles and correlations between cash flows over time. The cash flows may be correlated with assets or contracts whose values are known, in which case that knowledge would enter the valuation. Finally, the cases of private valuations and idiosyncratic risk charges, including intertemporal correlations, is addressed in Chapter 12.

## REFERENCES

Bogue and Roll

Brennan

Rubinstein

Static hedging

## EXERCISES

1. Replicate the 5-asset reference portfolio example from this chapter in Excel. Recall that we assumed the cash flow was outside the reference portfolio. For this exercise, revalue the reference portfolio including

the new cash flow. An alternative valuation benchmark is the change in value of the reference portfolio caused by adding the new asset.

Calculate the new asset valuations. Compared to the "outside" case, how can the resulting valuations of all the assets be reconciled?

2. Expand the oil company example in the text. Now you have been asked whether options should be used in the replication model. Call options on May 20, 2019 for the 6 month maturity for strikes 69, 70 and 71 were $2.07, 1.80 and 1.57.

   Determine the new valuation based on a replicating portfolio including these options, and compute the standard error of the estimate (mean response). What is your valuation equation, and how does the estimate compare to one without the option benchmarks?

3. Using your results in #2, how would you advise the company to hedge its risks, if it wanted to do so? Prepare probability distributions of the outcomes of the hedged and unhedged investments to justify your answer.

4. The CEO has just informed you that the trigger price may be negotiable downward. Using your full model, what is the value of being able to reduce the trigger to 69.5 instead of 70? *Hint: Make sure you use the same random numbers for the 70 trigger and the 69.5 trigger.*