Let us suppose that the system of stochastic equations

$$(\dagger)\quad \left\{ \begin{aligned} S_t &= \quad + \int_0^t r S_u \, du + \int_0^t \Sigma_u \, dB_u \\ \Sigma_t^2 &= \frac{1}{t} \int_0^t \left(\frac{dS_u}{S_u} \right)^2 \end{aligned} \right\}$$

has a solution.

Then the product rule applied to

$$(*)\quad t\Sigma_t^2 = \int_0^t \frac{(dS_u)^2}{S_u^2} = \int_0^t \left(r\,du + \Sigma_u \, dB_u \right)^2 = \int_0^t \Sigma_u^2 \, du$$

gives

$$d(t\Sigma_t^2) = t\,d\Sigma_t^2 + \Sigma_t^2 \, dt.$$

But from (*) we have: $d(t\Sigma_t^2) = \Sigma_t^2 \, dt$,

thus

$$d(\Sigma_t^2) = 0, \text{ i.e., } t \mapsto \Sigma_t^2 \text{ is constant.}$$

- Let's call this constant σ^2; then ($\star$) becomes

$$(**) \qquad S_t = 1 + r\int_0^t S_u\,du + \sigma B_t$$

which gives $(dS_t)^2 = \sigma^2 \cdot dt$ as well as

- $$\sigma^2 t = t\Sigma_t^2 = \int_0^t \frac{\sigma^2\,du}{S_u^2},$$

i.e,
$$t = \int_0^t \frac{du}{S_u^2}, \quad \forall\ t \geq 0.$$

But this forces $S_t \equiv 1$, in blatant contradiction of (**).

- CONCLUSION: The system ($\star$) has no solution.