Ho-Lee

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The Ho-Lee model assumes the stochastic short rate process is $f_t = r + \sigma B_t$ where r and σ are constant and B_t is standard Brownian motion. It allows negative interest rates and has only two parameters, r and σ, so it is not possible to fit the model to market data. If we make $r(t)$ a function of t we can fit the discount curve $D(t)$. If we make $\sigma(t)$ a function of t we can fit caplets/floorlets. If we use multi-dimensional Brownian motion and make $\sigma(t)$ vector-valued then we can also fit market swaption prices.

The stochastic discount is

$$D_t = \exp(-\int_0^t f_s\,ds) = \exp(-\int_0^t r + \sigma B_s\,ds) = e^{-rt}\exp(-\sigma\int_0^t B_s\,ds).$$

The discount to time t is $D(t) = E[D_t]$.

Exercise. *Show* $\operatorname{Var}(\int_0^t B_s\,ds) = t^3/3$.

Hint: $\operatorname{Var}(\sum_i X_i) = \sum_{i,j}\operatorname{Cov}(X_i, X_j)$.

▼ Solution

We have

$$
\begin{aligned}
\mathrm{Var}\left(\int_0^t B_s\,ds\right) &= \int_0^t \int_0^t \mathrm{Cov}(B_u, B_v)\,du\,dv \\
&= \int_0^t \int_0^t \min(u, v)\,du\,dv \\
&= \int_0^t \int_0^v u\,du\,dv + \int_0^t \int_v^t v\,du\,dv \\
&= \int_0^t v^2/2\,dv + \int_0^t v(t-v)\,dv \\
&= \int_0^t v^2/2 + v(t-v)\,dv \\
&= \int_0^t vt - v^2/2\,dv \\
&= t^3/2 - t^3/6 \\
&= t^3/3
\end{aligned}
$$

This shows $D(t) = E[D_t] = \exp(-rt + \sigma^2 t^3/6)$ since $E[\exp(N)] = \exp(E[N] + \mathrm{Var}(N)/2)$ if N is normal. Recall the *forward curve* $f(t)$ is defined by $D(t) = \exp(-\int_0^t f(s)\,ds)$.

Exercise. *Show $f(t) = r - \sigma^2 t^2/2$ for the Ho-Lee model.*

▼ Solution

We have $\int_0^t f(s)\,ds = rt - \sigma^2 t^3/6$ so $f(t) = r - \sigma^2 t^2/2$.

The futures quote is $\phi(t) = E[f_t]$ for the contact expiring at t. For the Ho-Lee model $\phi(t) = r$ for all t. The previous exercise shows $\phi(t) - f(t) = \sigma^2 t^2/2$.

Convexity is defined to be the difference between the futures quote and forward rate. Let F be the (random) rate at futures expiration so $\phi = E[F]$ is the futures quote. The (par) forward f is defined by $0 = E[(f - F)D]$ where D is the stochastic discount to expiration. We have $fE[D] = E[FD] = E[F]E[D] + \mathrm{Cov}(F, D)$ so

$\phi - f = -\operatorname{Cov}(F, D)/E[D]$. This is positive since the forward and discount are negatively correlated. For most models it is approximately proportional to the the square of time to expiration.

Dynamics

The price at time t of a zero coupon bond maturing at u is $D_t(u) = E_t[D_u/D_t]$. The *stochastic forward curve* $f_t(u)$ is defined by $D_t(u) = \exp(-\int_t^u f_t(s)\,ds)$. For the Ho-Lee model $D_t(u) = E_t[\exp(-\int_t^u r + \sigma B_s\,ds)]$.

Exercise. *Show* $\int_t^u B_s\,ds = \int_t^u (u-s)\,dB_s + (u-t)B_t$.

Hint Use $d(tB_t) = t\,dB_t + B_t\,dt$ and $0 = -uB_t + uB_t$.

▼ Solution

We have

$$
\begin{aligned}
\int_t^u B_s\,ds &= -\int_t^u s\,dB_s + uB_u - tB_t \\
&= -\int_t^u s\,dB_s + uB_u - uB_t + uB_t - tB_t \\
&= -\int_t^u s\,dB_s + u\int_t^u dB_s + uB_t - tB_t \\
&= \int_t^u (u-s)\,dB_s + (u-t)B_t
\end{aligned}
$$

Exercise. *Show* $E_t[\exp(\int_t^u \Sigma(s)\,dB_s)] = \exp(\frac{1}{2}\int_t^u \Sigma(s)^2\,ds)$.

Hint: Use $X_t = \exp(\int_0^t \Sigma(s)\,dB_s - \frac{1}{2}\int_0^t \Sigma(s)^2\,ds)$ is a martingale. Note the right-hand side is not random.

▼ Solution

$1 = E_t[X_u/X_t] = E_t[\exp(\int_t^u \Sigma(s)\,dB_s - \frac{1}{2}\int_t^u \Sigma(s)^2\,ds)]$

Note also $E_t[\exp(-\int_t^u \Sigma(s)\, dB_s)] = \exp(\frac{1}{2}\int_t^u \Sigma(s)^2\, ds)$ by replacing Σ with $-\Sigma$, or B_t by $-B_t$. We use this below.

We now show $D_t(u) = \exp(-r(u-t) + \sigma^2(u-t)^3/6 - \sigma(u-t)B_t)$.

$$
\begin{aligned}
D_t(u) &= E_t[D_u/D_t] \\
&= E_t[\exp(-\int_t^u r + \sigma B_s\, ds)] \\
&= E_t[\exp(-r(u-t) - \sigma\int_t^u B_s\, ds)] \\
&= E_t[\exp(-r(u-t) - \sigma\int_t^u (u-s)\, dB_s - \sigma(u-t)B_t)] \\
&= \exp(-r(u-t) + \frac{1}{2}\sigma^2\int_t^u (u-s)^2\, ds - \sigma(u-t)B_t) \\
&= \exp(-r(u-t) + \sigma^2(u-t)^3/6 - \sigma(u-t)B_t)
\end{aligned}
$$

using $\int_t^u (u-s)^2\, ds = (u-t)^3/3$. Since $\int_t^u f_t(s), ds = r(u-t) - \sigma^2(u-t)^3/6 + \sigma(u-t)B_t$ we have $f_t(u) = r - \sigma^2(u-t)^2/2 + \sigma B_t$.

The Ho-Lee model has a closed form solution for the dynamics of zero-coupon bond prices.

Options

A *caplet* with strike k and expiration t pays $\max\{f_t - k, 0\} = (f_t - k)^+$ and a *floorlet* pays $(k - f_t)^+$ at t. The risk-neutral value of a floorlet is $p = E[(k - f_t)^+ D_t]$.

Exercise *Show* $p = E[(k + \sigma^2 t^2/2 - f_t)^+]D(t)$.

Hint. Recall $E[f(M)e^N] = E[f(M + \mathrm{Cov}(M,N))]E[e^N]$ if M and N are jointly normal.

▼ Solution

We have $\mathrm{Cov}(f_t, \log D_t) = \mathrm{Cov}(\sigma B_t, -\int_0^t \sigma B_s\, ds) = -\sigma^2\int_0^t s\, ds = -\sigma^2 t^2/2$.

Note the floorlet value can be calculated using the Bachelier model. If $F = f + sZ$ where Z is standard normal then $E[(k - F)^+] = (k - F)\Phi(z) + s\phi(z)$ where $z = (k - f)/s$, Φ is the standard normal cumulative distribution, and $\phi = \Phi'$ is the standard normal density function.

Exercise. *Find a closed form solution for the floorlet value* $E[(k - f_t)^+ D_t]$.