Bilateral and Multilateral Exchanges for Peer-Assisted Content Distribution

Christina Aperjis, Ramesh Johari, *Member, IEEE*, and Michael J. Freedman, *Member, IEEE*

***Abstract*—Users of the BitTorrent file-sharing protocol and its variants are incentivized to contribute their upload capacity in a bilateral manner: Downloading is possible in return for uploading to the same user. An alternative is to use multilateral exchange to match user demand for content to available supply at other users in the system. We provide a formal comparison of peer-to-peer system designs based on bilateral exchange with those that enable multilateral exchange via a price-based market mechanism to match supply and demand. First, we compare the two types of exchange in terms of the equilibria that arise. A multilateral equilibrium allocation is Pareto-efficient, while we demonstrate that bilateral equilibrium allocations are not Pareto-efficient in general. We show that Pareto efficiency represents the "gap" between bilateral and multilateral equilibria: A bilateral equilibrium allocation corresponds to a multilateral equilibrium allocation if and only if it is Pareto-efficient. Our proof exploits the fact that Pareto efficiency implies reversibility of an appropriately constructed Markov chain. Second, we compare the two types of exchange through the expected percentage of users that can trade in a large system, assuming a fixed file popularity distribution. Our theoretical results as well as analysis of a BitTorrent dataset provide quantitative insight into regimes where bilateral exchange may perform quite well even though it does not always give rise to Pareto-efficient equilibrium allocations.**

***Index Terms*—Asymptotic analysis, market equillibria, Markov processes, peer-to-peer systems, random graphs.**

I. Introduction

EARLY peer-to-peer systems did not provide any incentives for participation, leading to extensive *free riding*: Many peers were using the resources of other peers without contributing their own [2], [18]. The peer-to-peer community responded with mechanisms to prevent free riding by incentivizing sharing on a *bilateral exchange* basis, as used by BitTorrent [11] and its variants [32], [39], [35].

According to the BitTorrent protocol, each user splits his available upload rate among users from which he gets the highest download rates. As a result, an increase in the upload rate to one user may increase the download rate from that particular user. However, it does not increase the download rate from other users. Thus, two users are incentivized to exchange only if each has content the other wants. This results in a significant drawback of bilateral exchange: It breaks down between users that do not have reciprocally desired files.

The difficulties of bilateral exchange (or barter) in an economy have been long known, the most important being the improbability of coincidence between persons wanting and possessing. As discussed in [21], "there may be many people wanting, and many possessing those things wanted; but to allow of an act of barter, there must be a double coincidence, which will rarely happen." In modern economies, the aforementioned difficulty is eliminated by the use of *money*. Money can enable multilateral exchange by serving as a medium of exchange and a common measure of value. Even though modern societies take the use of money for granted, the same is not the case in peer-to-peer systems.

Peer-to-peer systems could potentially also use *market-based multilateral exchange* to match user demand for content to available supply at other users in the system. This can be done by using virtual currency and assigning a budget to each user that decreases when downloading and increases when uploading. Monetary incentives with a virtual currency have been previously proposed to encourage contribution in peer-to-peer systems [17], [38], [36], [6], [5]. However, such designs are usually more complex than bilateral protocols and are not widespread.

The two system designs present a significant tradeoff: *Bilateral exchange without money is simple, while multilateral exchange allows more users to trade*. In this paper, we provide a formal comparison of two peer-to-peer system designs: bilateral barter systems such as BitTorrent and a market-based exchange of content enabled by a price mechanism to match supply and demand. Our main goal is to identify precisely what benefits a currency-based system might offer and whether these benefits are sufficient to actually warrant all the complexity of implementation presented by such systems.

We start in Section II with a fundamental abstraction of content exchange in systems like BitTorrent: *exchange ratios*. The exchange ratio from one user to another gives the download rate received per unit upload rate. Exchange ratios are a useful formal tool because they allow us to define and study the equilibria of bilateral exchange. In the model of bilateral exchange we consider, each user optimizes with respect to exchange ratios. In Section III, we define *bilateral equilibrium* as a rate vector and a vector of exchange ratios, where all users have simultaneously optimized given exchange ratios. We also define *multilateral equilibrium*, where users optimize with respect to prices; our definition of multilateral equilibrium is the same as competitive equilibrium in economics [26]. In a multilateral

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C. Aperjis is with HP Labs, Palo Alto, CA 94304 USA (e-mail: christina.aperjis@hp.com).

R. Johari is with Stanford University, Stanford, CA 94305 USA.

M. J. Freedman is with the Department of Computer Science, Princeton University, Princeton, NJ 08540 USA.

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equilibrium, the presence of money enables a potentially wider set of exchanges than is possible in bilateral equilibrium.

Our main results are the following.

1) *Characterizing efficient equilibria*. We compare bilateral and multilateral peer-to-peer systems through the allocations that arise at equilibria. A multilateral equilibrium allocation is always Pareto-efficient, while bilateral equilibria may be inefficient. Our main result is that a bilateral equilibrium allocation is Pareto-efficient if and only if it is a multilateral equilibrium allocation—in other words, efficient bilateral equilibria must effectively yield "supporting prices" as in a multilateral equilibrium. This result provides formal justification of the efficiency benefits of multilateral equilibria. The proof exploits an interesting connection between equilibria and Markov chains: An important step of the proof is to show that Pareto efficiency of a bilateral equilibrium rate allocation implies reversibility of an appropriately defined Markov chain, and that this chain has an invariant distribution that corresponds to a price vector of a multilateral equilibrium.
2) *Quantifying the efficiency gap between bilateral and multilateral exchange*. From a practical standpoint, the preceding insight is somewhat unsatisfying because it does not *quantify* the benefits of multilateral exchange. Although all efficient equilibria are multilateral equilibria, if the potential loss of efficiency in bilateral equilibrium is small, then it may be an acceptable tradeoff in return for a significantly simpler system design. To address this issue, we perform a quantitative comparison of bilateral and multilateral exchange by quantifying how rarely a double coincidence of wants occurs under different assumptions on the popularity of files in the system.
   a) *Theoretical analysis*. We first perform an asymptotic analysis assuming that file popularity follows a power law. We find that asymptotically all users are able to trade bilaterally when the file popularity is very concentrated (i.e., when the popularity distribution has a relatively light tail). On the other hand, multilateral exchange may perform significantly better than bilateral exchange when the file popularity is not concentrated (i.e., when the distribution has a heavy tail). Importantly, we also find that increasing the number of files that each user shares or wants by a small amount can significantly improve the performance of bilateral exchange.
   b) *Empirical validation*. We complement our theoretical analysis by studying file popularity from a large BitTorrent dataset [33]. We find that on this dataset, bilateral exchange may in general exhibit significant inefficiency relative to multilateral exchange. However, consistent with our theoretical observation, the gap between bilateral and multilateral exchange can be narrowed significantly if each user shares or is interested in a sufficiently large number of files. For example, for systems of the size in the dataset, over 96% of users can trade bilaterally if each user shares at least 10 files. The last result is informative: It suggests that taking small steps to increase the number of bilateral matches possible can actually significantly eliminate almost all the advantage of multilateral exchange.

The remainder of the paper is organized as follows. Section IV characterizes efficient equilibria. Section V provides both a theoretical and an empirical quantification of the efficiency gap between bilateral and multilateral exchange. Section VI discusses the related literature, and Section VII concludes the paper.

II. Exchange Ratios in Bilateral Protocols

Many peer-to-peer protocols enable exchange on a *bilateral* basis between users: A user i uploads to a user j if and only if user j uploads to user i in return. Of course, such an exchange is only possible if each user has something the other wants; this is known as "double coincidence of wants" in economics. The foremost examples of such a protocol are BitTorrent and its variants. While such protocols are traditionally studied solely through the rates that users obtain, this section provides an interpretation of these protocols through *exchange ratios*. As exchange ratios can be interpreted in terms of prices, these ratios allow us to compare bilateral barter-based peer-to-peer systems with multilateral price-based peer-to-peer systems.

Let r_{ij} denote the rate sent from user i to user j at a given point in time in a bilateral peer-to-peer protocol. We define the *exchange ratio* between user i and user j as the ratio $\gamma_{ij} = r_{ji}/r_{ij}$; this is the download rate received by i from j per unit of rate uploaded to j. By definition, $\gamma_{ij} = 1/\gamma_{ji}$. Clearly, a rational user i would prefer to download from users with which he has higher exchange ratios.

The exchange ratio has a natural interpretation in terms of prices. In particular, assume that users charge each other for content in a common monetary unit, but that all transactions are *settlement-free*, i.e., no money ever changes hands. In this case, if user i charged user j a price p_{ij} per unit rate, the exchange of content between users i and j must satisfy

$$p_{ij}r_{ij} = p_{ji}r_{ji}.$$

We refer to p_{ij} as the *bilateral price* from i to j. Note that the preceding condition thus shows the exchange ratio is equivalent to the ratio of bilateral prices: $\gamma_{ij} = p_{ij}/p_{ji}$ (as long as the prices and rates are nonzero).

What is the exchange ratio for BitTorrent? A user splits his upload capacity equally among those users in his active set from which he gets the highest download rates. Let α be the size of the active set. Suppose all rates r_{kj} that user j receives from users $k \neq i$ are fixed, and let R_j^α be the αth highest rate that j receives. Let B_j be the upload capacity of user j. Then, r_{ji} depends on r_{ij}. In particular

$$r_{ji} = \begin{cases} B_j/\alpha, & \text{if } r_{ij} > R_j^\alpha \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for BitTorrent, the exchange ratio is $\gamma_{ij} = B_j/(\alpha \cdot r_{ij})$ if user i is in the active set, and zero otherwise. Note that the exchange ratios $\gamma_{i_1,j}$ and $\gamma_{i_2,j}$ may be different for two users i_1,i_2 in j's active set.

The exchange ratio γ_{ij} *decreases* with r_{ij} as long as user i is in user j's active set (in which case r_{ji} is constant). Hence, a strategic user i would prefer to choose r_{ij} as small as possible while remaining in j's active set. This behavior is exactly the approach taken by the BitTyrant [32] variation on BitTorrent. In fact, if all users follow this policy, then $r_{ij} = R_j^\alpha$ for all users i in j's active set. Note that in this case, $\gamma_{ij} = B_j/(\alpha \cdot R_i^\alpha)$. Thus, user j has the same exchange ratio to all users i with which he bilaterally exchanges content.

The preceding discussion highlights the fact that the rates in a bilateral peer-to-peer system can be interpreted via exchange ratios. Thus far, we have assumed that *transfer rates* are given, and exchange ratios are computed from these rates. In Section III, we turn this relationship around: We explicitly consider an abstraction of bilateral peer-to-peer systems where users react to given exchange ratios, and we compare the resulting outcomes to price-based multilateral exchange.

III. Bilateral and Multilateral Equilibria

In this section, we define bilateral equilibrium (BE) and multilateral equilibrium (ME), i.e., the market equilibria corresponding to bilateral and multilateral exchange. In the formal model we consider, a set of users U shares a set of files F. User i has a subset of the files $S_i \subseteq F$ and is interested in downloading files in $T_i \subseteq F - S_i$. Throughout, we use r_{ijf} to denote the rate at which user i uploads file f to user j. We then let $x_{if} = \sum_j r_{jif}$ be the rate at which user i downloads file f. We denote the vector of download rates for user i by $\boldsymbol{x}_i = (x_{if}, f \in T_i)$. Finally, let $y_i = \sum_{j,f} r_{ijf}$ be the total upload rate of user i. We measure the desirability of a rate vector to user i by a *utility function* $v_i(\boldsymbol{x}_i, y_i)$, according to the following assumption. *This assumption remains in force throughout the paper.*

Assumption 1: The preference relation of a user on the set of feasible rate vectors is represented by a continuous strictly concave utility function $v_i(\boldsymbol{x}_i, y_i)$, which is strictly increasing in each download rate x_{if} for all $f \in T_i$ and strictly decreasing in the upload rate y_i. We further assume that $v_i(\boldsymbol{x}_i, y_i)$ has finite derivative everywhere with respect to all x_{if}, $f \in T_i$.

Note that utility functions in our model depend on instantaneous transfer rates rather than the number of bytes exchanged. This is consistent with the "snapshot" view that our model of peer-to-peer file sharing adopts: Informally, it is as if we are analyzing efficiency of the system at a fixed moment in time. This is also why our model keeps fixed both the set of files available for upload and the set of files desired for download at a given user; these sets remain constant on the timescale we are considering. An important open direction related to this work concerns the analysis of *dynamic* efficiency, where these sets might change over time.

Each user is assumed to have a constraint on the available upload rate. Let B_i denote this upper bound for user i. We assume that users do not face any constraint on their download rate; this is consistent with most end-users' asymmetric connections today, where upload capacity is far exceeded by download capacity. Furthermore, for the purposes of this paper, we also assume that there are no constraints in the middle of the network, though our prior work suggests a natural approach for including such constraints [5].

Let

$$\mathcal{X} = \left\{ \boldsymbol{r} : \boldsymbol{r} \geq 0; r_{kjf} = 0 \text{ if } f \notin S_k; \sum_{j,f} r_{ijf} \leq B_i \forall i \in U \right\}$$

be the set of feasible rate vectors. In particular, this ensures that: 1) all rates are nonnegative; 2) users only upload files they possess; and 3) each user does not violate his upload capacity constraint.

In Sections III-A and III-B, we look at two different models of equilibrium, corresponding to exchange in bilateral and multilateral systems, respectively. In bilateral exchange, we assume that users maximize their utility given the exchange ratios they see to other users. Thus, in a bilateral equilibrium, the exchange ratios must be chosen to exactly balance the rates each user considers optimal—this is informally the condition that "supply equals demand." In multilateral exchange, on the other hand, we assume that users can earn currency by uploading to others, and they can spend that currency in downloading from any users they wish. In this model, users maximize their utility given the current prices in the system, and the prices must be set to exactly balance the rates each user considers optimal. Thus, in both types of exchange, "supply equals demand" at an equilibrium or, equivalently, the market clears.

A. Bilateral Equilibrium

We start by considering users' behavior in bilateral schemes, given a vector of exchange ratios $(\gamma_{ij}, i, j \in U)$. User i solves the Bilateral Peer Optimization problem given in Fig. 1.[1]

In this optimization problem, in addition to the definition of upload and download rates, each user i faces one constraint for *each potential peer* j with which i might exchange content: the restriction that $\sum_f r_{jif} = \gamma_{ij} \sum_f r_{ijf}$ ensures that the rate at which i downloads from j is exactly the exchange ratio times the rate at which i uploads to j.

We can now define bilateral equilibrium.

Definition 1: The rate allocation $\boldsymbol{r}^*$ and the exchange ratios $\boldsymbol{\gamma}^* = (\gamma_{ij}^*, i, j \in U)$, with $\gamma_{ij}^* \cdot \gamma_{ji}^* = 1$ for all i, j, constitute a BE if, for each user i, $\boldsymbol{r}^*$ solves the Bilateral Peer Optimization problem given exchange ratios $\boldsymbol{\gamma}^*$.

Definition 1 requires that: 1) all users have optimized with respect to the exchange ratios; and 2) the market clears. Even though the market clearing condition is not explicitly stated, it is implicitly required since the same vector $\boldsymbol{r}^*$ is an optimal solution of the Bilateral Peer Optimization problems of all users. The following proposition shows that a BE exists.

Proposition 1: A BE exists.

B. Multilateral Equilibrium

By contrast, in a *multilateral price-based exchange*, the system maintains one price per user, and users optimize with

[1]Note that we allow users to bilaterally exchange content over multiple files. This is partly supported by swarming systems like BitTorrent through bundles [28]. BitTorrent also has a mixed bundling mechanism, which could be adapted to allow trade across files in a bundle (but currently is not flexible).

Bilateral Peer Optimization:

maximize $v_i(\boldsymbol{x}_i, y_i)$

subject to $x_{if} = \sum_j r_{jif}$ for all f

$y_i = \sum_{j,f} r_{ijf}$

$\boldsymbol{r} \in \mathcal{X}$

$\sum_f r_{jif} = \gamma_{ij} \sum_f r_{ijf}$ for all j

Multilateral Peer Optimization:

maximize $v_i(\boldsymbol{x}_i, y_i)$

subject to $x_{if} = \sum_j r_{jif}$ for all $f \in T_i$

$y_i = \sum_{j,f} r_{ijf}$

$\boldsymbol{r} \in \mathcal{X}$

$\sum_{j,f} p_j r_{jif} = p_i \sum_{j,f} r_{ijf}$

Fig. 1. Optimization problems for price-based exchange.

respect to these prices.[2] We denote the price of user i by p_i. Fig. 1 also gives the Multilateral Peer Optimization problem. Note that the first three constraints (giving download and upload rates and ensuring that the rate allocation is feasible) are identical to the Bilateral Peer Optimization; only the last constraint is different. While the bilateral exchange requires user i to download only from those users to whom he uploads, no such constraint is imposed on multilateral exchanges: User i accrues capital for uploading, and he can spend this capital however he wishes for downloading.

We next give the definition of multilateral equilibrium, which corresponds to the concept of competitive equilibrium in economics [26].

Definition 2: The rate allocation $\boldsymbol{r}^*$ and the user prices $\boldsymbol{p}^* = (p_i^*, i \in U)$ with $p_i^* > 0$ for all $i \in U$ constitute an ME if, for each user i, $\boldsymbol{r}^*$ solves the Multilateral User Optimization problem given prices $\boldsymbol{p}^*$.

Similar to Definition 1, Definition 2 requires that: 1) all users have optimized with respect to prices; and 2) the market clears. Again, even though the market clearing condition is not explicitly stated, it is implicitly required since the same vector $\boldsymbol{r}$ is used in the optimization problems of all users. The following proposition shows that an ME exists.

Proposition 2: An ME exists.

Our model is closely related to exchange economies [26]. In an *exchange economy*, there is a finite number of agents and a finite number of commodities. Each agent is endowed with a bundle of commodities and has a preference relation on the set of commodity vectors. Given a price vector, each agent finds a vector of commodities to exchange that maximizes his utility. In particular, if $\boldsymbol{p}$ is the vector of prices and agent i has endowment $\boldsymbol{w}_i$, he sells it at the market and obtains wealth $\boldsymbol{p}\cdot\boldsymbol{w}_i$. Then, the agent buys goods for his consumption at the same price (he may buy back some of the goods he sold).

A straightforward reformulation reveals that our model shares much in common with a standard exchange economy. It is *as if* agent i has B_i units of his own "good," priced at p_i. He can trade this for goods from other users on the open market at prices $\boldsymbol{p}$. With this interpretation, $B_i - y_i$ is the amount of his own good that he chooses to keep. However, notice that this is not a standard exchange economy, as the upload rate is not a true commodity. Rather, the commodities are the rates of specific files that are uploaded. Since B_i imposes a *joint* constraint on the upload rates of these files, our model is a generalization of the standard exchange economy.

[2]It can be shown that in our setting, this is equivalent to having either one price per file or one price per user per file [5]. As explained in that paper, the choice of one price per user affords certain advantages for system design, so we adopt it as our approach in this paper.

C. Convergence to Equilibrium

The focus of Sections III-A and III-B is equilibria, i.e., a static setting. However, since it is hard to know a system's equilibrium prices or allocation in advance, we need to consider how out-of-equilibrium prices are updated (price discovery). We briefly discuss this issue here.

If demand is equal to supply at a given price vector, then this price vector constitutes an equilibrium. The standard approach of updating prices out of equilibrium is to increase prices when demand exceeds supply, and decrease prices when supply exceeds demand. In a standard exchange economy, this process converges to equilibrium under the condition of *gross substitutes* [26], [8], which requires that if the price of one commodity increases, then the demand for all other commodities increases. This approach and its variants have been used in many settings to establish convergence to equilibrium. (Dynamics that do not depend on mismatches between demand and supply have also been considered for convergence to ME, e.g., [39].)

In the setting of a peer-to-peer system, demand and supply consist of desired download and upload rates, respectively. Suppose first that a central server tracks total demand and supply of each file at current prices. If prices are updated based on total demand and supply as described in the previous paragraph, it can be shown that the resulting process converges to equilibrium.[3]

On the other hand, we can consider less centralized versions of this price updating process. For instance, in the case of multilateral exchange with $|S_i| = 1$, suppose the total excess demand (i.e., the amount by which demand exceeds supply) is allocated equally among users that have the file and are currently charging the minimum price. Peers then raise or lower their prices depending on whether excess demand is positive or negative, respectively. A user that does not currently have the minimum price for the file it is uploading sees no demand and, as a result, has a negative excess demand, which forces its price to decrease. Similar ideas can be applied in the case of bilateral exchange.[4]

In prior work [5], we considered implementation of a price discovery mechanism for ME in a decentralized setting appropriate for a peer-to-peer system. Our approach was inspired by the theoretical price dynamics above. Given prices of users, a user requests download rates from other users in the network.

[3]Suppose that the aggregate excess demand for each file satisfies the gross substitutes property. Then, with the definition of ME considered in this paper, convergence is guaranteed if each user is uploading one file. In the BE setting considered in this paper, convergence is guaranteed if each user is uploading and downloading one file. (The formal statements are provided in [7].) We conjecture that similar convergence results hold even if users are uploading and/or downloading multiple files.

[4]To construct a similar procedure for BE, note that we cannot directly use exchange ratios in a decentralized setting because then it is not clear whether i or j should update the exchange ratio γ_{ij}. Instead, we use bilateral prices (cf. Section II) so that user i updates a price p_{ij} and user j updates a price p_{ji}, with $\gamma_{ij} = p_{ij}/p_{ji}$.

A user serves requests sequentially without preemption and updates its price according to the mismatch between requests received and available capacity. Simulations suggest that such price dynamics converge for a variety of topologies in the case of multilateral exchange [5]. The same approach can be applied to BE by considering exchange ratios instead.

IV. Efficiency of Equilibria

This section rigorously analyzes the efficiency properties of bilateral and multilateral exchange. We assume users explicitly react to exchange ratios or prices, and we compare the schemes through their resulting equilibria. In order to proceed, we first formally define Pareto efficiency.

Definition 3: Given rate allocations $\boldsymbol{r}, \boldsymbol{r}' \in \mathcal{X}$, let $\boldsymbol{x}, \boldsymbol{x}'$ be the corresponding download rates, and let $\boldsymbol{y}, \boldsymbol{y}'$ be the corresponding upload rates. Then, $\boldsymbol{r}$ *Pareto-dominates* $\boldsymbol{r}'$ if $v_i(\boldsymbol{x}_i, y_i) \geq v_i(\boldsymbol{x}'_i, y'_i)$ for all i, with strict inequality for at least one i.

A rate allocation $\boldsymbol{r} \in \mathcal{X}$ is *Pareto-efficient* if it is not Pareto-dominated by any other rate allocation $\boldsymbol{r}' \in \mathcal{X}$.

Thus, a rate allocation is Pareto-efficient if there is no way to increase the utility of some user without decreasing the utility of some other user. An ME allocation is always Pareto-efficient; this is the content of the first fundamental theorem of welfare economics [26]. For completeness, we include the result here.

Proposition 3: If the rate allocation $\boldsymbol{r}^*$ and the user prices $(p_i^*, i \in U)$ with $p_i^* > 0$ for all $i \in U$ constitute an ME, then the allocation $\boldsymbol{r}^*$ is Pareto-efficient.

BE may not be Pareto-efficient. Inefficiencies may arise because trade does not occur at a BE, while users do trade at an ME of the same system. Moreover, even when all users trade at a BE, the allocation may not be Pareto-efficient, as the following example shows.

Example 1: Consider a system with n users and n files, for $n > 2$. Each user i has file f_i and wants files f_{i+1} and f_{i-1}. The utility of user i is $v_i(x_{i,f_{i-1}}, x_{i,f_{i+1}}, y_i) = x_{i,f_{i-1}} + 4x_{i,f_{i+1}} + \ln(2 - y_i)$, i.e., user i wants the files of both user $i+1$ and user $i-1$, but derives a higher utility from the file of user $i+1$.

We first consider a symmetric BE with exchange ratios $\gamma^*_{i,i+1} = 2$ and $\gamma^*_{i,i-1} = 1/2$. The equilibrium rates are $r^*_{i-1,i} = 1$ and $r^*_{i+1,i} = 1/2$, and the download rates are $x^*_{i,f_{i-1}} = 1$ and $x^*_{i,f_{i+1}} = 1/2$. The utility of each user i is $3 - \ln(2) \approx 2.3$. On the other hand, prices $p_i^* = 1$ for all i, and rates $r^*_{i+1,i} = 1.75$, $r^*_{i-1,i} = 0$ constitute an ME. The utility of each user is $7 - \ln(4) \approx 5.61$, i.e., significantly larger than the utility of a user at the BE. This demonstrates that the BE allocation is not Pareto-efficient.

The previous example shows that BE may not be Pareto-efficient. By changing the utility function of a user in this example, we next provide an example of a BE rate allocation that is Pareto-efficient.

Example 2: Consider a system with n users and n files, for $n > 2$. Each user i has file f_i and wants files f_{i+1} and f_{i-1}. The utility of user i is $v_i(x_{i,f_{i-1}}, x_{i,f_{i+1}}, y_i) = x_{i,f_{i-1}} + x_{i,f_{i+1}} + \ln(2 - y_i)$.

We consider a symmetric BE with exchange ratios $\gamma^*_{i,i+1} = 1$ and $\gamma^*_{i,i-1} = 1$. The equilibrium rates are $r^*_{i-1,i} = 1/2$ and $r^*_{i+1,i} = 1/2$. The BE rate allocation is Pareto-efficient. In particular, it corresponds to an ME: Prices $p_i^* = 1$ for all i, and rates $r^*_{i+1,i} = 1/2$, $r^*_{i-1,i} = 1/2$ constitute an ME.

Thus, BE may be inefficient, while ME always has Pareto-efficient allocations (Proposition 3). In Example 2, the BE rate allocation is Pareto-efficient and corresponds to an ME. Our main result is that *a BE allocation is Pareto-efficient if and only if it is an ME allocation*. In particular, if a BE allocation is Pareto-efficient, then there exist "supporting prices," i.e., prices such that the BE rate allocation is optimal for the Multilateral Peer Optimization problem of each user. Informally, Pareto efficiency represents the "gap" between BE and ME.

Proposition 4: Assume that for every user i and any fixed $\boldsymbol{x_i}$, $v_i(\boldsymbol{x_i}, y_i) \to -\infty$ as $y_i \to B_i$. Let $(\boldsymbol{r}^*, \boldsymbol{\gamma}^*)$ be a BE. The rate allocation $\boldsymbol{r}^*$ is Pareto-efficient if and only if there exists a price vector $\boldsymbol{p}$ such that $\boldsymbol{r}^*$ and $\boldsymbol{p}$ constitute an ME.

Proposition 4 assumes that $v_i(\boldsymbol{x_i}, y_i) \to -\infty$ as $y_i \to B_i$ for every user i and every fixed $\boldsymbol{x_i}$. This assumption ensures that the total upload rate of a user is strictly smaller than his upload capacity at the BE. This is a reasonable assumption for a peer-to-peer setting since we do not expect users to use all their upload capacity. We note that if the total upload rate of a user is equal to his upload capacity, then there may exist Pareto-efficient BE that do not correspond to ME simply because users have already "maxed out" their available upload capacity.

We provide an overview of the proof of Proposition 4, which demonstrates an interesting connection between equilibria and Markov chains; the details of the proof are provided in the Appendix. From a BE rate allocation $\boldsymbol{r}^*$, we construct a transition rate matrix $\boldsymbol{Q}$ for a continuous-time Markov chain, such that $Q_{ij} = \sum_f r^*_{ijf}$ if $i \neq j$, and $Q_{ii} = -\sum_{j,f} r^*_{ijf}$. We first observe that $\boldsymbol{\pi Q} = 0$ implies that the multilateral budget constraint is satisfied with price vector $\boldsymbol{\pi}$. Therefore, for any invariant distribution $\boldsymbol{\pi}$, $\boldsymbol{r}^*$ is feasible for the Multilateral Peer Optimization problem of every user when prices are equal to $\boldsymbol{\pi}$. We then show that if $\boldsymbol{r}^*$ is also Pareto-efficient, there exists an invariant distribution of $\boldsymbol{Q}$, say $\boldsymbol{p}$, such that $\boldsymbol{r}^*$ is an optimal solution of the Multilateral Peer Optimization problem of each user when the prices are equal to $\boldsymbol{p}$. We conclude that $\boldsymbol{r}^*$ and $\boldsymbol{p}$ constitute an ME.

A key step of the proof is to show that Pareto efficiency of $\boldsymbol{r}^*$ implies *reversibility* of $\boldsymbol{Q}$. This is proven by contradiction: If $\boldsymbol{Q}$ is not reversible, then we can find a cycle of users that can change their rates only along successive pairs of users on the cycle and, in doing so, make all their utilities strictly higher.

Now let $\boldsymbol{\pi}$ be an invariant distribution of $\boldsymbol{Q}$ with all entries positive. If the matrix $\boldsymbol{Q}$ is reversible, then $\gamma^*_{ij} = \pi_i/\pi_j$ for all pairs of users i and j that trade at the BE. We conclude that if $\boldsymbol{r}^*$ is Pareto-efficient, then $\boldsymbol{r}^*$ solves the Multilateral Peer Optimization problem for each user given prices $\boldsymbol{\pi}$ *if the user is restricted to trade with peers he trades with at the BE*. Much of the complexity in the proof is to show that this result holds even if user i is not restricted to trade only with those users it transacts with at the BE.

The matrix corresponding to the BE allocation of Example 1 is not reversible, which implies that the BE allocation is not Pareto-efficient. On the other hand, the matrix corresponding to the BE allocation of Example 2 is reversible, and the BE allocation is Pareto-efficient and corresponds to an ME allocation.

In closing, we note that if the rate matrix corresponding to an ME is reversible, then the ME allocation is a BE allocation and can be realized through bilateral trade. In particular, from an ME rate allocation $\boldsymbol{r}^*$, we construct a transition rate matrix $\boldsymbol{Q}$ for a continuous-time Markov chain as described. If $\boldsymbol{Q}$ is reversible, then there exists an invariant distribution π with all entries positive such that $\pi_i r_{ij}^* = \pi_j r_{ji}^*$ for all i, j. Setting $\gamma_{ij} = \pi_i/\pi_j$ for all i, j, we conclude that $(\boldsymbol{r}^*, \boldsymbol{\gamma})$ is a BE.

V. Ability to Trade

Propositions 1 and 2 show that both BE and ME exist under general conditions (in particular if Assumption 1 holds). However, not all users are guaranteed to trade at these equilibria. For instance, if user i does not have reciprocally desired files with any other user (that is, for any user j, either $T_i \cap S_j = \emptyset$ or $T_j \cap S_i = \emptyset$), at a BE γ_{ij}^* is sufficiently low for all j that have files desired by i so that i does not trade. In this section, *we compare bilateral and multilateral exchange through the corresponding percentages of users that can trade*. Though distinct from Pareto efficiency, this metric provides *quantitative* insight into the comparison of the two types of exchange. In particular, we expect that systems that perform well will also generally encourage high levels of participation. We characterize regimes where bilateral exchange performs very well with respect to this metric, and for which, as a result, it may not be worth the effort to use multilateral exchange.

In Section V-A, we introduce the framework we use to study the percentage of users that can trade bilaterally and multilaterally. Our analysis is based on a random model, where we assume file popularity follows a power law. In Section V-B, we carry out an asymptotic theoretical analysis as the number of users and files grows large. In Section V-C, we complement our theoretical analysis by studying file popularity from a large BitTorrent dataset. Here, we find that the ability to trade bilaterally improves significantly if each user shares or is interested in a sufficiently large number of files.

A. Framework

1) Definitions: We start by formally defining the quantities we compare. For a given peer-to-peer system, we define the *system profile* to consist of the specification of which files each user desires and possesses, i.e., $\mathcal{P} = \{T_i, S_i, i \in U\}$.

We say that user i *can trade bilaterally under* $\mathcal{P}$ if there exists some user j such that $T_i \cap S_j \neq \emptyset$ and $S_i \cap T_j \neq \emptyset$, that is, i and j have reciprocally desired files. Given a system profile $\mathcal{P}$, let $\rho_{\rm BE}(\mathcal{P})$ be the percentage of users that cannot trade bilaterally.

Similarly, we say that user i *can trade multilaterally under* $\mathcal{P}$ if there exist users $k_1, k_2, \ldots, k_n$ such that $T_i \cap S_{k_1} \neq \emptyset$; $T_{k_j} \cap S_{k_{j+1}} \neq \emptyset$ for $j = 1, \ldots, n$ and $T_{k_n} \cap S_i \neq \emptyset$. In other words, user i is able to trade multilaterally if and only if there exists a cycle of users starting (and ending) at i such that each user possesses a file that is desired by the next user in the cycle. Clearly, if user i can trade bilaterally under $\mathcal{P}$, then he can also trade multilaterally under $\mathcal{P}$. Let $\rho_{\rm ME}(\mathcal{P})$ be the percentage of users that cannot trade multilaterally.

Note that whether or not a user *actually* trades in equilibrium depends on the specific utility functions chosen. For example, if the marginal utility for downloading is sufficiently low, the optimal decision for a user may be to not download at all, even if he can. However, the definition above ensures that if a user *cannot* trade bilaterally (resp., multilaterally), then this user *never* trades in a BE (resp., ME), regardless of the utility functions.

2) Random Model: We assume that the system profile $\mathcal{P}$ is chosen according to some distribution that depends on the popularity of different files, and that the sets S_i and T_i are chosen independently for each user i. We denote by q_i the *popularity* of the ith file and assume that the *probability* that the ith file is desired or possessed by a user is proportional to q_i. We assume that each q_i does not depend on the number of files in the system. On the other hand, the probability that the ith file is chosen clearly depends on the number of files K in the system since it is $q_i / \sum_{j=1}^{K} q_j$.

We are interested in comparing the expected proportions of users that cannot trade bilaterally and multilaterally—that is, the expected values of $\rho_{\rm BE}(\mathcal{P})$ and $\rho_{\rm ME}(\mathcal{P})$—for given file popularities.

B. Asymptotic Analysis

This section theoretically compares the two types of exchange through the expected percentages of users that cannot trade. We focus on large systems and consider the asymptotic regime where the number of files and users in the system becomes large.

We assume the files that users possess and desire are drawn independently from a Zipf file-popularity distribution that is identical for each user. Our motivation to study this distribution comes from the fact that Zipf's law has been observed in many settings and has been suggested as a good model for file popularity (e.g., [1], [4], and [10]).[5] Zipf's law states that the popularity of the rth largest occurrence is proportional to a power of its inverse rank. We adjust this definition to our setting.

Definition 4: File popularity has a *Zipf distribution* with exponent s if the rth most popular file has popularity $q_r = r^{-s}$.

Note that $s = 0$ corresponds to the uniform distribution. On the other hand, the distribution becomes more concentrated as s increases.

Recall that we are interested in the expected percentage of users that cannot trade. This is a function of the number of users N, the number of files K, and the Zipf exponent s. Let $\bar{\rho}_{\rm BE}(K, N, s)$ and $\bar{\rho}_{\rm ME}(K, N, s)$ be the expected percentages of users that cannot trade bilaterally and multilaterally, respectively. In particular, $\bar{\rho}_{\rm BE}(K, N, s)$ (resp., $\bar{\rho}_{\rm ME}(K, N, s)$) is the expected value of $\rho_{\rm BE}(\mathcal{P})$ (resp., $\rho_{\rm ME}(\mathcal{P})$) over system profiles.

We consider a sequence of peer-to-peer systems indexed by N. The Nth system has N users and $K(N)$ files, where $K(N)$ is a nondecreasing function of N. The function $K(N)$ represents how the number of files scales with the number of users. For simplicity, we suppress the dependence of K on N. We study an asymptotic regime where $N \to \infty$.

Since the number of users that cannot trade bilaterally is always greater than or equal to the number of users that cannot trade multilaterally, we have $\bar{\rho}_{\rm BE}(K, N, s) \geq \bar{\rho}_{\rm ME}(K, N, s)$. The following propositions imply that in a large system,

[5]Gummadi *et al.* find that peer-to-peer file popularity follows a flattened Zipf-link distribution [16]. However, the Zipf distribution is still the closest approximation for which analytical work is possible.

$\bar{\rho}_{\mathrm{BE}}(K,N,s) - \bar{\rho}_{\mathrm{ME}}(K,N,s)$ may be significant when $s<1$, but is always negligible when $s>1$.

Proposition 5: If $s>1$, then $\bar{\rho}_{\mathrm{BE}}(K,N,s) \to 0$ as $N \to \infty$ for any nondecreasing K.

Since $\bar{\rho}_{\mathrm{ME}}(K,N,s) \le \bar{\rho}_{\mathrm{BE}}(K,N,s)$, we conclude that if files are chosen according to a Zipf distribution with $s>1$, then both $\bar{\rho}_{\mathrm{BE}}(K,N,s) \to 0$ and $\bar{\rho}_{\mathrm{ME}}(K,N,s) \to 0$ as $N \to \infty$. Thus, when $s>1$, bilateral exchange performs very well asymptotically for any scaling of K and N. This result holds regardless of the number of files that users possess or desire. For details, see the proof of Proposition 5 in the Appendix. We note that the result of Proposition 5 can be generalized to all popularity distributions for which $\sum_{i=1}^{\infty} q_i < \infty$; the Zipf distribution with $s>1$ is of course a special case.

This is an interesting result: Even though bilateral exchange significantly restricts trade compared to multilateral exchange, almost all users can trade in expectation under both types of exchange when the system is large and file popularity follows a Zipf distribution with exponent $s>1$. The intuition behind this result is that when s is large, the popularity distribution is more concentrated, i.e., the most popular files are chosen with relatively high probability. As a result, for any user i, both T_i and S_i probably consist of one of the most popular files, and it is more likely that there exists a user j such that i and j have reciprocally desired files.

When $s<1$, the asymptotic behavior can be quite different, as the following proposition shows.

Proposition 6: Assume $0 \le s < 1$, and $|S_i| = |T_i| = 1$ for all $i \in U$. As $N \to \infty$, we have the following.

i) If $K/\sqrt{N} \to \infty$, then $\liminf_{N\to\infty} \bar{\rho}_{\mathrm{BE}}(K,N,s) \ge (1-s)^2$.

ii) If $K/\sqrt{N} \to 0$, then $\bar{\rho}_{\mathrm{BE}}(K,N,s) \to 0$.

iii) If $K/N \to \infty$, then $\liminf_{N\to\infty} \bar{\rho}_{\mathrm{ME}}(K,N,s) \ge 1-s$.

iv) If $K \log K/N \to 0$, then $\bar{\rho}_{\mathrm{ME}}(K,N,s) \to 0$.

The case where N scales slower than K^2 but faster than $K \log K$ is of particular interest. In this case, according to Proposition 6, $\bar{\rho}_{\mathrm{BE}}(K,N,s) \ge (1-s)^2$ and $\bar{\rho}_{\mathrm{ME}}(K,N,s) \to 0$ as $N \to \infty$. That is, when the system is large, almost all users *can* trade multilaterally, but a constant proportion of users *cannot* trade bilaterally. Thus, for this case, multilateral exchange performs significantly better than bilateral exchange in terms of the ability of users to trade. We note that if $s=0$, in this regime $\bar{\rho}_{\mathrm{BE}}(K,N,s) \to 1$; that is, almost all users can trade multilaterally, but cannot trade bilaterally.

By contrast, when $0 \le s < 1$ and N scales faster than K^2, both bilateral and multilateral exchange perform well. On the other hand, if N scales slower than K, then neither type of exchange performs well. We note that Proposition 6 has a small gap since it does not say how well multilateral exchange performs when N scales faster than K yet slower than $K \log K$.

Fig. 2 shows the percentages of users (from simulations) that cannot trade bilaterally when popularities follow a Zipf distribution for various values of the exponent s, assuming that users desire and possess one file. (We assume the system consists of 7323 files, as this is the number of files in the dataset considered in Section V-C.) We observe that bilateral exchange does not perform well when the exponent s is small, which agrees with our theoretical results. As the exponent increases, the performance of bilateral exchange improves. When the exponent is greater than one, bilateral exchange performs reasonably well.

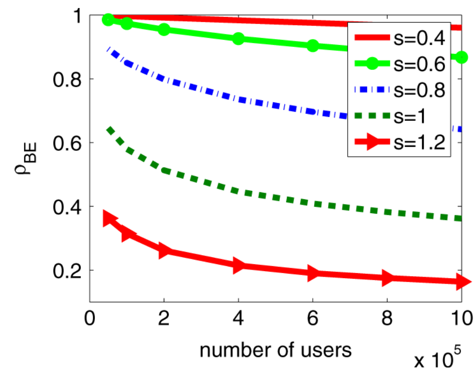

Fig. 2. Percentages of users (from simulations) that cannot trade bilaterally (ρ_{BE}) when file popularities follow a Zipf distribution with exponent $s \in \{0.4, 0.6, 0.8, 1, 1.2\}$. Users desire and possess one file, i.e., $|T_i| = |S_i| = 1$ for all users i. There are 7323 files in the system, and the number of users is shown on the horizontal axis.

In [7], we show that if $s=0$, then the conclusions of Proposition 6 hold even if users possess and desire multiple files. In fact, the proof of Proposition 7 in [7] shows that when each user possesses σ files and desires τ files, there exist constants c_1 and c_2 such that

$$\begin{aligned} c_1 \exp\left(-\frac{\sigma^2\tau^2 N}{K^2}\right) &\\ \le \bar{\rho}_{BE}(K,N,0) &\\ \le c_2 \exp\left(-\frac{(\sigma^2\tau^2/2 - \sigma^2\tau - \sigma\tau^2)N}{K^2} + o\left(\frac{1}{K^2}\right)\right) & \end{aligned}$$

for all sufficiently large K and N. Thus, to first order, the exponent scales like $-\sigma^2\tau^2 N/K^2$. This suggests that small increases in the number of files that agents are willing to trade can lead to significant improvements in system performance. Indeed, we observe precisely this phenomenon in the Section V-C analysis.

C. Data Analysis

This section quantitatively compares bilateral and multilateral exchange using data on BitTorrent peer-to-peer file-sharing collected by Piatek *et al.* [33]. We find that a significant percentage of users cannot trade bilaterally when each user is sharing one file. However, the percentage becomes negligible as users share more files. We conclude by discussing this finding's implications for the design of peer-to-peer content exchange systems.

The dataset consists of 1 364 734 downloads, 679 523 users, and 7323 files.[6] We use the number of downloads of each file in the dataset to estimate the popularities of different files. We thus abstract from the details of the specific BitTorrent trace and only use the information on the preferences of the users for different files in order to compare bilateral and multilateral exchange through simulations.

The estimated popularities are shown in Fig. 3. As before, we assume that the probability that a given file is selected is

[6]We equate files to torrents, and neglect bundles, as a first-order approximation.

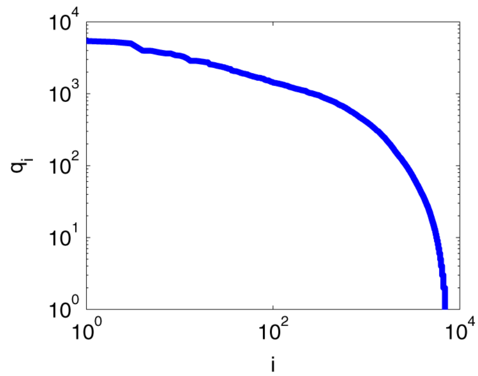

Fig. 3. Popularity of each unique file (q_i for all files i)—that is, the number of times file i was downloaded—shown in decreasing popularity on a log–log scale.

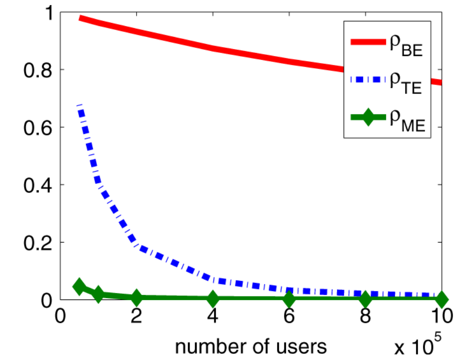

Fig. 4. Percentages of users (from simulations) that cannot trade bilaterally, trilaterally, and multilaterally when users desire and possess one file, i.e., $|T_i| = |S_i| = 1$ for all i. The horizontal axis shows the number of users in the system.

proportional to its popularity. We then use these probabilities to generate system profiles and compute the percentages of users that cannot trade bilaterally and multilaterally. We assume that there are 7323 files with the given distribution and vary the number of users in the system.[7]

The algorithm we use to compute ρ_{BE} is exact. For every user i, we check whether there is some user j such that i and j have reciprocally desired files. On the other hand, computing the exact value of ρ_{ME} for a large system appears computationally intractable. Therefore, we use an approximation: We recursively remove users that either possess files not desired by others or desire files not possessed by others since such users cannot trade multilaterally. Simulations for small numbers of users suggest that this algorithm provides a very good approximation for ρ_{ME}.[8]

Bilateral and Multilateral Trade: We first assume that each user possesses and desires exactly one file, i.e., $|T_i| = |S_i| = 1$ for every $i \in U$. Fig. 4 shows the percentages of users that cannot trade bilaterally and multilaterally from simulations for various numbers of users in the system. We observe that a significant majority of users cannot trade bilaterally, while nearly all users can trade multilaterally. Finally, as the number of users increases, the percentages of users that can trade increase for both bilateral and multilateral exchange.

Trading Trilaterally: Fig. 4 also shows the percentage ρ_{TE} of users that cannot trade in *triangles*, i.e., triples (i, j, k), where i uploads to j, j uploads to k, and k uploads to i.[9] We observe that a very large percentage of users is able to trade in triangles when there are at least 600 000 users in the system.[10]

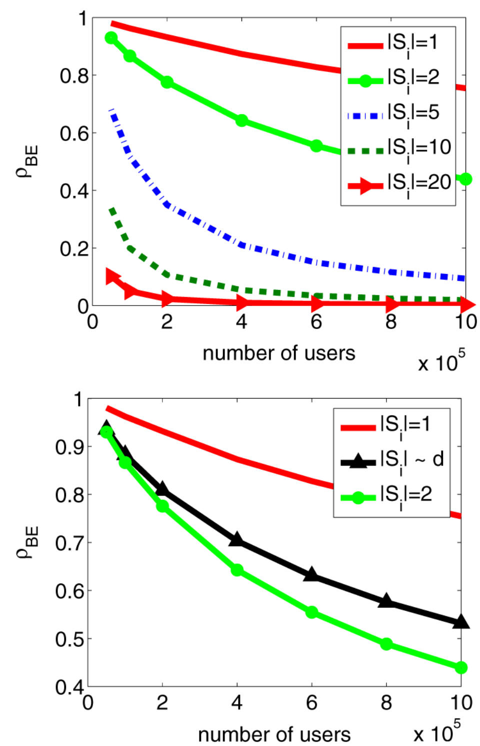

Fig. 5. Percentages of users (from simulations) that cannot trade bilaterally when each user desires one file ($|T_i| = 1$) and possesses multiple files. The legend shows $|S_i|$ for each line. The horizontal axis shows the number of users in the system. In the top graph, all users possess the same number of files ($|S_i| \in \{1, 2, 5, 10, 20\}$). In the bottom graph, we consider the case where different users possess different numbers of files, where this number is drawn from the dataset distribution d (denoted by $|S_i| \sim d$).

Uploading (or Downloading) Multiple Files: We next assume each user desires one file and possesses multiple files. As the number of files that each user has increases, the number of possible trades increases, and as a result the percentage of users that can trade bilaterally increases. In Fig. 5 (top graph), we show the percentages of users that cannot trade bilaterally when each user desires one file ($|T_i| = 1$) and possesses multiple files ($|S_i| \in \{2, 5, 10, 20\}$). In these experiments, all users possess the same number of files, i.e., $|S_i| = |S_j|$ for all $i, j \in U$ (the case of $|S_i| = 1$ is already shown in Fig. 4). Note that, by symmetry, we get exactly the same graph if users possess one file ($|S_i| = 1$) and desire multiple files ($|T_i| \in \{2, 5, 10, 20\}$).

[7] We have also considered the case that the number of users N is fixed and the number of files K varies; as expected, performance degrades as K increases.

[8] For instance, suppose there are 1000 users and 200 files in the system whose popularities are equal to the popularities of the 200 most popular files of the dataset. In 100 simulations, 972 users can trade multilaterally on average, while our heuristic finds that 976 users can trade multilaterally on average (99.6% accuracy).

[9] We estimate ρ_{TE} by sampling at least a few thousand users and using an exact algorithm to compute the proportion of the sampled users that cannot trade in triangles.

[10] Analytically, it can be shown that if we allow trade in triangles, then performance under uniform file popularity (i.e., $s = 0$) is good as long as $N^2/(K^3 \log K) \to \infty$ as $N \to \infty$ (see [7, Appendix, Prop. 8]). This is a significant improvement on the corresponding result for bilateral equilibrium.

From these simulations, we observe a significant decrease in the percentage of users that cannot trade when $|S_i|$ increases from 1 to 20 (resp., when $|S_i| = 1$ and $|T_i|$ increases from 1 to 20). We can illustrate this by considering the minimum required number of users in the system so that at most 10% are not able to trade: 1 000 000 users are required when each user has five files, but only 50 000 users are needed when each user possesses 20 files.

Distribution of $|S_i|$ *Across Users:* Our simulations up to now have assumed that all users in the system possess the same number of files, i.e., $|S_i| = |S_j|$ for all i, j. We next assume that the number of files that users possess varies across different users, inferring this distribution for $|S_i|$ from the dataset.[11] We are interested in whether the percentage of users that can trade bilaterally increases as the variance of the distribution of $|S_i|$ increases (assuming that the mean remains the same). At first, it may seem plausible that users with very large $|S_i|$ would be able to accommodate a lot of trades, and as a result, ρ_{BE} should increase as the $|S_i|$'s become more dispersed. However, this is not the case, as we discuss next.

The dataset shows that most users, in fact, possess only a few files. Only 32% of users possess more than a single file, and only 2% of users possess more than 10 files. There are, however, a few users that have more than 400 files. Since the mean value of $|S_i|$ in the dataset is 2.0084, we are interested in whether ρ_{BE} increases compared to the case that $|S_i| = 2$ for all i. This comparison is shown in Fig. 5 (bottom graph).

We observe that when $|S_i|$ is drawn from the dataset distribution d, the percentages of users that cannot trade bilaterally are between the cases of $|S_i| = 1$ and $|S_i| = 2$, even though the expected value of $|S_i| \sim d$ is slightly greater than 2. This occurs because, even though some users have a large number of files and are thus more likely to be able to trade bilaterally when the distribution of $|S_i|$ is more dispersed, the percentage of users that only have one file also increases. Moreover, since each user desires one file, the probability that a user that has one file is matched with a user with multiple files does not significantly increase.

VI. Related Work

In this paper, we have provided a formal comparison of peer-to-peer system designs, and have studied the advantages and disadvantages of bilateral and multilateral exchange. Menasché *et al.* investigate direct and indirect reciprocity in peer-to-peer systems [27], which correspond to bilateral and multilateral exchange in our model. They upper bound the efficiency loss of direct reciprocity assuming that users are willing to download files they do not desire for bartering purposes. They also consider dynamic scenarios in simulations. On the other hand, we compare bilateral and multilateral exchange through equilibrium outcomes and through the expected percentage of users that can trade, and we do not assume that users download files they do not desire. DeFigueiredo *et al.* also consider direct and indirect reciprocity [12], but do not focus on comparing the two.

[11]We assume that the number of files that a user possesses is equal to the number of files he downloads in the dataset; note that this may be a bit optimistic, as it ignores the possibility of deletions over time from a user's set of shared files.

The "gap" between bilateral and multilateral exchange in terms of both efficiency and complexity has motivated the study of incentive mechanisms that lie between the two types of exchange in terms of both metrics. Through trace-driven analysis and measurements of a deployment on PlanetLab, Piatek *et al.* find that allowing trades to pass through one intermediary improves performance and incentives relative to BitTorrent [33]. Liu *et al.* study a similar mechanism assuming that peers belong to an underlying social network [25]. Finally, the performance implications of bundling have been considered [28].

Our work is also related to the study of equilibria in economies where not all trades are allowed. Kakade *et al.* introduce a graph-theoretic generalization of classical Arrow–Debreu economics, in which an undirected graph specifies which consumers or economies are permitted to engage in direct trade [22]. However, the inefficiencies of bilateral exchange do not arise in their model. The monetary economics literature has long studied how money reduces the double coincidence problem. The implementation of a competitive equilibrium is a central theme in this literature. The superiority of monetary exchange has been studied [37], and dynamics of bilateral trading processes have been considered [29], [13]. The transactions role of money is surveyed in [40].

Finally, as discussed in the Introduction, we note that a number of studies consider incentives in peer-to-peer systems (e.g., [3], [9], [11], [14], [15], [24], [33], [38], [40]). Our paper contributes to this broad line of literature.

VII. Conclusion

This paper provides a formal comparison of two peer-to-peer system designs: bilateral barter systems such as BitTorrent, and a market-based exchange of content enabled by a price mechanism to match supply and demand. Our results demonstrate that even though bilateral equilibria are not Pareto-efficient in general, bilateral exchange may perform very well in terms of the expected percentage of users that can trade for certain file probability distributions. Moreover, our data analysis shows a significant increase in the percentage of users that can trade bilaterally when each user shares or desires multiple files. Informally, the fact that BitTorrent breaks files into chunks greatly increases the number of matches possible, leading to performance gains similar to those described in Section VI. More generally, our insight suggests that bilateral incentives in BitTorrent could be made even stronger if the protocol considered exchanges across *different* files, rather than restricting exchange in a single swarm to chunks of the same file.[12]

We conclude by noting that our paper has considered a static "snapshot" view of a file-sharing system. This is complemented by our earlier work in [5], where we considered a system design for multilateral exchange in a dynamic setting. In a dynamic system, the number of *simultaneous* matches possible may be quite small, even if *all* content possessed by the users is available for upload. In such systems, money plays another important role: It can act as a *store of value* (e.g., see [21]) over time, allowing a user to upload *now* and earn the ability to download

[12]One can think of this as a more flexible version of mixed bundling, adapted to allow trade across files in a bundle.

later. Our design in [5] leverages this advantage by designing a system where pricing mechanisms serve only as algorithmic devices to ensure efficient exchange; to keep the system simple, we never expose prices directly to the end-user.[13] However, in our system design, currency functions as a store of value and allows dynamic trades over time. Quantifying this advantage (in the sense of Section V) remains an important direction for future work.

APPENDIX

Note: Throughout the Appendix, we write $\boldsymbol{x} \gg 0$ if all components of $\boldsymbol{x}$ are positive.

Proof of Proposition 4: Define $r^*_{ij} \equiv \sum_f r^*_{ijf}$, the total rate that user i sends to user j. We define the matrix $\boldsymbol{Q}$ such that $Q_{ij} = r^*_{ij}$ if $i \neq j$, and $Q_{ii} = -\sum_j r^*_{ij}$. By construction, $\boldsymbol{Q}$ is a transition rate matrix of a continuous-time Markov chain with no transient subclasses since $r^*_{ij} > 0$ implies that $r^*_{ji} > 0$ (by the definition of BE). In what follows, we consider the communicating classes of $\boldsymbol{Q}$: If $r^*_{ij} > 0$, then users i and j are in the same communicating class. For the purposes of this proof, let $\mathcal{N}_i(\boldsymbol{r}^*)$ be the set of users with which i trades under $\boldsymbol{r}^*$, i.e., $\mathcal{N}_i(\boldsymbol{r}^*) = \{j \in U : r^*_{ji} > 0\}$. Note that $\mathcal{N}_i(\boldsymbol{r}^*)$ is a subset of the communicating class containing i. Finally, a transition rate matrix $\boldsymbol{Q}$ is called *reversible* if for every cycle of distinct nodes $i_1, i_2, \ldots, i_n, i_1$, there holds: $Q_{i_1 i_2} Q_{i_2 i_3} \ldots Q_{i_{n-1} i_n} Q_{i_n i_1} = Q_{i_1 i_n} Q_{i_n i_{n-1}} \ldots Q_{i_3 i_2} Q_{i_2 i_1}$; i.e., the product of transition rates is the same in each direction around the cycle. If a matrix $\boldsymbol{Q}$ is reversible, and $\boldsymbol{\pi}$ is a strictly positive invariant distribution, i.e., $\boldsymbol{\pi} \gg 0$ and $\boldsymbol{\pi Q} = 0$, then the *detailed balance conditions* hold: For all i, j, $\pi_i Q_{ij} = \pi_j Q_{ji}$. See [23] for details.

Let $\boldsymbol{\pi}$ be an invariant distribution of $\boldsymbol{Q}$, i.e., $\boldsymbol{\pi Q} = 0$. We observe that $\boldsymbol{\pi Q} = \boldsymbol{0}$ implies that the budget constraint in the Multilateral Peer Optimization problem is satisfied with prices $\boldsymbol{\pi}$. Therefore, for any invariant distribution $\boldsymbol{\pi}$, $\boldsymbol{r}^*$ is feasible for the Multilateral Peer Optimization problem of every user when prices are equal to $\boldsymbol{\pi}$. We show that if $\boldsymbol{r}^*$ is Pareto-efficient, then for some invariant distribution $\boldsymbol{p}$ of $\boldsymbol{Q}$, $\boldsymbol{r}^*$ and $\boldsymbol{p}$ constitute an ME. In particular, we show that for each user i, $\boldsymbol{r}^*$ solves the Multilateral Peer Optimization problem under $\boldsymbol{p}$.

This is done in three steps. First, we show that if $\boldsymbol{r}^*$ is Pareto-efficient, then $\boldsymbol{Q}$ corresponds to a reversible Markov chain. This implies that if $\boldsymbol{\pi}$ is an invariant distribution of $\boldsymbol{Q}$ with all components strictly positive, then $\gamma^*_{ij} = \pi_i/\pi_j$ whenever $r^*_{ij} > 0$, and as a result $\boldsymbol{r}^*$ solves the Multilateral Peer Optimization problem of user i given prices $\boldsymbol{\pi}$ *if user i is restricted to trade with users in* $\mathcal{N}_i(\boldsymbol{r}^*)$ (Step 1). We then show that if user i is restricted to trade with users in the same communicating class under prices $\boldsymbol{\pi}$, then $\boldsymbol{r}^*$ is an optimal solution of the Multilateral Peer Optimization problem (Step 2). Step 2 completes the proof if $\boldsymbol{Q}$ consists of one communicating class. Finally, we show that if there are multiple communicating classes, there exists an invariant distribution $\boldsymbol{p}$ (derived as a convex combination of the invariant distributions corresponding to the communicating classes) such that $\boldsymbol{r}^*$ is an optimal solution of the Multilateral Peer Optimization problem of each user (Step 3). We show each of these steps by demonstrating that if the desired conclusion of the step does not hold, then there exists a rate vector $\boldsymbol{r}$ that Pareto-improves $\boldsymbol{r}^*$—violating the assumption that $\boldsymbol{r}^*$ is Pareto-efficient.

[13]See recent work in [34] for a similar approach in peer-to-peer distributed backup.

Before beginning the proof, we derive a simple condition that allows us to test whether a new rate vector improves user i's utility. Suppose $\boldsymbol{r}^*$ solves the Bilateral Peer Optimization problem of user i under $\boldsymbol{\gamma}^*$. Let $(x^*_{if}, f \in T_i)$ and y^*_i be the corresponding download and upload rates for user i. Consider a rate allocation $\boldsymbol{r}$ where $(x_{if}, f \in T_i)$ and y_i are the corresponding download and upload rates for user i. Fix a file $f \in T_i$, and assume that $x_{ig} = x^*_{ig}$ for all files $g \neq f$. If $x_{if} - x^*_{if}$ and $y_i - y^*_i$ are sufficiently small, we can use Taylor's approximation to conclude that user i is strictly better off under $\boldsymbol{r}$ if

$$(x_{if} - x^*_{if}) \frac{\partial v_i(\boldsymbol{x}^*_i, y^*_i)}{\partial x_{if}} + (y_i - y^*_i) \frac{\partial v_i(\boldsymbol{x}^*_i, y^*_i)}{\partial y_i} > 0. \quad (1)$$

Let $\gamma^*_{ig} = \max\{\gamma^*_{ij} : g \in S_j\}$. If user i is getting file g from user j at the BE, it must be that $\gamma^*_{ij} = \gamma^*_{ig}$. Substituting x_{if} in the other constraints of the Bilateral Peer Optimization problem of user i (given in Fig. 1), we conclude that $y_i = \sum_{g \in T_i} x_{ig}/\gamma^*_{ig}$. Thus, user i wishes to maximize $v_i(\boldsymbol{x}_i, \sum_{g \in T_i} x_{ig}/\gamma^*_{ig})$. The first-order optimality conditions (which are necessary and sufficient since the objective is concave) and the fact that $\gamma^*_{ij} = \gamma^*_{ig}$ $r^*_{jig} > 0$ yield that

$$\frac{\partial v_i(x^*_{ig}, g \in T_i)}{\partial x_{if}} + \frac{1}{\gamma^*_{ij}} \frac{\partial v_i(\boldsymbol{x}^*_i, y^*_i)}{\partial y_i} = 0 \quad (2)$$

whenever $r^*_{jif} > 0$. (Here, we use the fact that $v_i(\boldsymbol{x}_i, y_i) \to -\infty$ as $y_i \to B_i$ to ignore the constraint $y_i \leq B_i$.) Combining (2) with (1), we see that user i strictly prefers $\boldsymbol{r}$ to $\boldsymbol{r}^*$ if

$$\frac{x_{if} - x^*_{if}}{y_i - y^*_i} > \gamma^*_{ij} \quad (3)$$

assuming that $x_{if} - x^*_{if}$ and $y_i - y^*_i$ are sufficiently small. Thus, if $r^*_{jif} > 0$, and we increase the download rate of file f to user i as well as the total upload rate of user i such that the previous condition holds, then user i is strictly better off.

Step 1. *If $\boldsymbol{r}^*$ is Pareto-efficient, then $\boldsymbol{Q}$ is reversible. Furthermore, if $\boldsymbol{\pi} \gg 0$ is an invariant distribution of $\boldsymbol{Q}$, then $\boldsymbol{r}^*$ solves the Multilateral Optimization Problem of user i given prices $\boldsymbol{\pi}$ if user i is restricted to trade only with the users in $\mathcal{N}_i(\boldsymbol{r}^*)$.* Let $\boldsymbol{\pi}$ be a strictly positive invariant distribution of $\boldsymbol{Q}$, i.e., $\boldsymbol{\pi} \gg 0$ and $\boldsymbol{\pi} \cdot \boldsymbol{Q} = \boldsymbol{0}$. If $\boldsymbol{Q}$ is reversible, then the detailed balance equations hold for every $i, j \in U$, i.e., $\pi_i r^*_{ij} = \pi_j r^*_{ji}$. We note that if $r^*_{ij} = 0$, then it must be that $r^*_{ji} = 0$, so the detailed balance equation trivially holds for i and j. On the other hand, the budget constraint of the Bilateral Peer Optimization problem of user i implies that $\gamma^*_{ki} r^*_{ki} = r^*_{ik}$. We conclude that $\boldsymbol{Q}$ is reversible if and only if $\gamma^*_{ij} = \pi_i/\pi_j$ whenever $r^*_{ij} > 0$.

We show that Pareto efficiency of $\boldsymbol{r}^*$ implies reversibility of $\boldsymbol{Q}$. Assume that $\boldsymbol{Q}$ is not reversible. Then, $\pi_j/\pi_i > \gamma^*_{ji}$ for some i, j with $r^*_{ij} > 0$. Since π is an invariant distribution, $\boldsymbol{\pi Q} = \boldsymbol{0}$, and thus $\sum_k (\pi_k/\pi_i) r^*_{ki} = \sum_k r^*_{ik}$. On the other hand, the budget constraint of the Bilateral Peer Optimization problem of user i implies that $\gamma^*_{ki} r^*_{ki} = r^*_{ik}$. Summing over k and substituting, we conclude that

$$\sum_k \gamma^*_{ki} r^*_{ki} = \sum_k \frac{p_k}{p_i} r^*_{ki}.$$

If $\pi_j/\pi_i > \gamma^*_{ji}$ for some i, j with $r^*_{ij} > 0$, the previous equation implies that there exists some user k such that $\pi_{j+1}/\pi_k > \gamma^*_{j+1,k}$ and $r^*_{j+1,k} > 0$. Without loss of generality, we relabel i to be $j+1$, and k to be $j+2$. Then, $\pi_{j+1}/\pi_{j+2} > \gamma^*_{j+1,j+2}$. Applying this reasoning recursively, we can find a sequence of users $1, 2, \ldots, K, K+1$ such that $1 \equiv K+1$ and $\pi_k/\pi_{k+1} > \gamma^*_{k,k+1}$ for all k.

We show how the utility of each user in $D = \{1, 2, \ldots, K\}$ can increase while the rate allocation to users outside D remains the same. In particular, we increase $r^*_{k,k-1}$ and y^*_k by a_k for all $k \in D$. We note that users' upload capacity constraints do not bind (i.e., remain inactive) at the BE, a consequence of the assumption that $v_i(\boldsymbol{x}_i, y_i) \to \infty$ as $y_i \to B_i$. Therefore, it is feasible to slightly increase the upload rates of all users. Applying (3), user k is better off if

$$\frac{a_{k+1}}{a_k} > \gamma^*_{k,k+1}.$$

Since $\pi_k/\pi_{k+1} > \gamma^*_{k,k+1}$, it follows that $\prod_k \gamma^*_{k,k+1} < 1$. Then, it is possible to make all users in the set D better off by, e.g., choosing δ and ε small enough, and setting $a_1 = \delta$; $a_{k+1} = \gamma^*_{k,k+1} \cdot a_k + \varepsilon$, for all $k \in S$.

We conclude that if $\boldsymbol{r}^*$ is the rate allocation of a BE and is Pareto-efficient, then $\boldsymbol{Q}$ is reversible. Furthermore, if $\boldsymbol{\pi} \gg 0$ is an invariant distribution of $\boldsymbol{Q}$, then $\gamma^*_{ij} = \pi_i/\pi_j$ whenever $r^*_{ij} > 0$. This means that $\boldsymbol{r}^*$ solves the Multilateral Peer Optimization problem of user i given prices $\boldsymbol{\pi}$ if he is restricted to trade with users in $\mathcal{N}_i(\boldsymbol{r}^*)$. The remainder of the proof shows that there exists an invariant distribution $\boldsymbol{p}$ such that $\boldsymbol{r}^*$ is optimal for the Multilateral Peer Optimization problem under $\boldsymbol{p}$. Due to space limitations it is provided in [7]. ■

Proof of Proposition 5: We first show the result when $|T_i| = |S_i| = 1$ for all users. Let

$$\theta(K, s) \equiv \sum_{i=1}^{K} \sum_{j=1, j\neq i}^{K} (ij)^{-s}. \tag{4}$$

The probability that $S_k = \{i\}$ and $T_k = \{j\}$ is equal to

$$\frac{(ij)^{-s}}{\theta(K, s)} \equiv p_{ij}.$$

Peer k cannot trade bilaterally if there exists no user k' such that $S_{k'} = T_k$ and $T_{k'} = S_k$. The event $S_{k'} = T_k$ and $T_{k'} = S_k$ occurs with probability $(1 - p_{ji})^{N-1}$ (since there are $N-1$ users to choose from). Since (S_k, T_k) is chosen independently for each user k, the expected percentage of users that cannot trade bilaterally when there are K files and N users satisfies

$$\bar{\rho}_{\text{BE}}(K, N, s) = \sum_{i=1}^{K} \sum_{j=1, j\neq i}^{K} \frac{(ij)^{-s}}{\theta(K, s)} \left(1 - \frac{(ij)^{-s}}{\theta(K, s)}\right)^{N-1}. \tag{5}$$

We first assume that $K \not\to \infty$ as $N \to \infty$. Then, $\bar{\rho}_{\text{BE}}(K, N, s)$ is the sum of a finite number of terms, each of which approaches 0 as $N \to \infty$. Thus, $\bar{\rho}_{\text{BE}}(K, N, s) \to 0$ as $N \to \infty$.

Now, assume that $K \to \infty$ as $N \to \infty$, and let

$$\theta(s) \equiv \sum_{i \neq j : i, j \in \{1, 2, \ldots\}} (ij)^{-s}.$$

We observe that $\theta(K, s) \uparrow \theta(s)$ as $K \to \infty$. Since $s > 1$, $\theta(s)$ is finite. We have

$$\bar{\rho}_{\text{BE}}(K, N, s) \leq \frac{1}{\theta(K, s)} \sum_{i=1}^{K} \sum_{j=1, j\neq i}^{K} (ij)^{-s} \left(1 - \frac{(ij)^{-s}}{\theta(s)}\right)^{N-1}.$$

Let

$$B_N \equiv \sum_{i=1}^{K} \sum_{j=1, j\neq i}^{K} (ij)^{-s} \left(1 - \frac{(ij)^{-s}}{\theta(s)}\right)^{N-1}.$$

Since $\theta(s)$ is finite and $\theta(K, s) < \theta(s)$, it suffices to show that $B_N \to 0$ as $N \to \infty$. We observe that for any fixed $\bar{K}$

$$B_N < \sum_{i \neq j : i \cdot j \leq \bar{K}} \left(1 - \frac{(ij)^{-s}}{\theta(s)}\right)^{N-1} + \sum_{i \neq j : i \cdot j > \bar{K}} (ij)^{-s}.$$

The first term approaches zero as $N \to \infty$; the second term does not depend on N, and approaches zero as $\bar{K} \to \infty$. Thus, first taking the limit as $N \to \infty$, then taking the limit as $\bar{K} \to \infty$, the result follows.

We have shown the result for the case that $|S_i| = |T_i| = 1$ for all users i. When users desire or possess more files, the chance of bilateral exchange increases, so again $\bar{\rho}_{\text{BE}}(K, N, s) \to 0$ as $N \to \infty$. ■

The following result is used in the proof of Proposition 6.

Lemma 1: If $y \in [0, 1)$ and $N > 0$, then

$$(1 - y)^N \leq \frac{1}{1 + N \cdot y}.$$

Proof of Lemma 1: Let

$$f(y) \equiv (1 - y)^{-N} - (1 + Ny).$$

It suffices to show that $f(y) \geq 0$ for $y \in [0, 1]$. We observe that $f(0) = 0$ and

$$f'(y) = N((1 - y)^{-N-1} - 1) \geq 0$$

for $y \in [0, 1)$. This completes the proof. ■

Proof of Proposition 6: We follow the same notation as in the proof of Proposition 5. In particular, we define $\theta(K, s)$ as in (4) and conclude that $\bar{\rho}_{\text{BE}}$ is given by (5).

We observe that

$$\begin{aligned}\theta(K,s) &\le \left(\sum_{i=1}^{K} i^{-s}\right)^2 = \left(1+\sum_{i=2}^{K} i^{-s}\right)^2 \\ &\le \left(1+\int_{i=1}^{K} i^{-s}\right)^2 \le \frac{K^{2(1-s)}}{(1-s)^2} \\ \theta(K,s) &\ge \left(\sum_{i=1}^{K} i^{-s}\right)\cdot\left(\sum_{i=2}^{K} i^{-s}\right) \\ &\ge \left(\int_{1}^{K} x^{-s}dx\right)\cdot\left(\int_{2}^{K} x^{-s}dx\right) \\ &= \frac{K^{2(1-s)}}{(1-s)^2}\left(1-\frac{1+2^{1-s}}{K^{2(1-s)}}\right).\end{aligned}$$

Thus

$$\frac{K^{2(1-s)}}{(1-s)^2}\left(1-\frac{1+2^{1-s}}{K^{1-s}}\right) \le \theta(K,s) \le \frac{K^{2(1-s)}}{(1-s)^2}. \quad (6)$$

We first show i). Let $A_K(\delta) \equiv \{\frac{\lceil\delta K\rceil}{K}, \frac{\lceil\delta K\rceil+1}{K}, \ldots, 1\}$. We have

$$\begin{aligned}&\bar{\rho}_{\mathrm{BE}}(K,N,s) \\ &\ge \sum_{i=1}^{K}\sum_{j=2}^{K} \frac{(ij)^{-s}}{\theta(K,s)}\left(1-\frac{(ij)^{-s}}{\theta(K,s)}\right)^{N-1} \\ &\ge \frac{(1-s)^2}{K^2}\sum_{u,v\in A_K(\delta)} (uv)^{-s}\left(1-\frac{(uv)^{-s}K^{-2s}}{\theta(K,s)}\right)^{N-1} \\ &\ge \frac{(1-s)^2}{K^2}\left(1-\frac{(\delta K)^{-2s}}{\theta(K,s)}\right)^{N-1}\sum_{u,v\in A_K(\delta)} (uv)^{-s}. \quad (7)\end{aligned}$$

In the previous inequalities, we set $u=i/K$, $v=j/K$ and use the upper bound in (6).

Define

$$\gamma(\delta,K,s) \equiv 1-\frac{(\delta K)^{-2s}}{\theta(K,s)}.$$

Using the lower bound in (6) and the fact that $K/\sqrt{N} \to \infty$ as $N \to \infty$, we have $\gamma(\delta,K,s)^{N-1} \to 1$ as $N \to \infty$. Also observe that since the cardinality of $A_K(\delta)$ is at least $(1-\delta)K$, we have

$$\frac{1}{K^2}\sum_{u,v\in A_K(\delta)} (uv)^{-s} \ge (1-\delta)^2.$$

It follows from (7) that

$$\liminf_{N\to\infty} \bar{\rho}_{\mathrm{BE}}(K,N,s) \ge (1-s)^2(1-\delta)^2.$$

Since $\delta > 0$ was arbitrary, the result follows.

We next show ii). By Lemma 1

$$\begin{aligned}&\bar{\rho}_{\mathrm{BE}}(K,N,s) \\ &\le \sum_{i=1}^{K}\sum_{j=1}^{K} \frac{(ij)^{-s}}{\theta(K,s)}\left(1-\frac{(ij)^{-s}}{\theta(K,s)}\right)^{N-1} \\ &\le \sum_{i=1}^{K}\sum_{j=1}^{K} \frac{(ij)^{-s}}{\theta(K,s)}\frac{1}{1+(ij)^{-s}(N-1)/\theta(K,s)} \\ &= \sum_{i=1}^{K}\sum_{j=1}^{K} \frac{1}{(ij)^{s}\theta(K,s)+(N-1)} \\ &\le \frac{K^2}{N} \to 0 \text{ as } N \to \infty.\end{aligned}$$

We now show iii). We observe that a user cannot trade multilaterally if he has a file that no user wants. Thus

$$\bar{\rho}_{\mathrm{ME}}(K,N,s) \ge \sum_{i=1}^{K} \frac{i^{-s}}{\sum_{j=1}^{K} j^{-s}}\left(1-\frac{(ij)^{-s}}{\sum_{j=1}^{K} j^{-s}}\right)^{N-1}.$$

A similar argument to the argument used in i) shows that

$$\liminf_{N\to\infty} \bar{\rho}_{\mathrm{ME}}(K,N,s) \ge 1-s.$$

Finally, we show iv). Our proof exploits a connection to classical Erdös–Rényi random graphs. Throughout, we assume without loss of generality that $N \le K(K-1)$. Let $G(K,N)$ denote a graph drawn uniformly at random from the $\binom{K(K-1)}{N}$ possible directed graphs on K nodes with N edges. We let $H(K,N,\boldsymbol{p})$ denote a random multigraph on K nodes with N edges, where each edge is independently placed from f to g with probability p_{fg}. Typically, $\boldsymbol{p}$ will be a probability distribution. However, in the subsequent analysis it is convenient if we allow the possibility $q(\boldsymbol{p}) = \sum_{f,g} p_{fg} < 1$. If $q(\boldsymbol{p}) < 1$, then we assume that each edge is not placed at all with probability $1-q(\boldsymbol{p})$.

Now consider a random system with N users and K files, where each user desires and has one file. For each user i, draw an edge from f to g if user i wants file f and has file g. This is exactly the random multigraph $H(K,N,\boldsymbol{p})$ described in the preceding paragraph, with

$$p_{fg} = \frac{(fg)^{-s}}{\theta(K,s)} \quad (8)$$

where $\theta(K,s)$ is defined as in (4).

Furthermore, observe that if $H(K,N,\boldsymbol{p})$ is strongly connected, then all users can trade multilaterally. This follows because if a user i has f and wants g, then there is an edge from f to g in $H(K,N,\boldsymbol{p})$. If $H(K,N,\boldsymbol{p})$ is strongly connected, then there must exist a path from g to f as well. This path identifies a collection of users that, together with user i, form a cycle. Thus, user i can trade multilaterally. It suffices to show that with probability approaching 1 as $N \to \infty$, $H(K,N,\boldsymbol{p})$ is strongly connected. (Convergence in probability implies convergence in expectation, as ρ_{ME} is bounded.)

It is known that if $N/(K \log K) \rightarrow \infty$, then $\mathbb{P}(G(K,N) \text{ is connected}) \rightarrow 1$ as $N \rightarrow \infty$ [31], where we use "connected" to mean strongly connected.[14] In Lemma 2, we use this threshold to establish the same threshold for a special class of $H(K,N,\boldsymbol{p})$ multigraphs, where $p_{fg} = \alpha/(K(K-1))$ for all f, g, with $0 < \alpha \leq 1$.

To complete the proof, fix s such that $0 \leq s < 1$, and observe that from (6) we have for fixed f and g

$$\frac{(fg)^{-s}}{\theta(K,s)} \geq \frac{(1-s)^2 (f/K)^{-s} (g/K)^{-s}}{K^2} \geq \frac{(1-s)^2}{K(K-1)}.$$

Let $p_{fg} = (fg)^{-s}/\theta(K,s)$, and let $r_{fg} = (1-s)^2/(K(K-1))$. It follows that

$$\mathbb{P}(H(K,N,\boldsymbol{p}) \text{ is connected}) \geq \mathbb{P}(H(K,N,\boldsymbol{r}) \text{ is connected}).$$

Since the right-hand side approaches 1 as $N \rightarrow \infty$ by Lemma 2, we conclude that the left-hand side approaches 1 as well. Thus, the probability that all users can trade multilaterally approaches 1. ■

The following lemma uses the same definitions as the preceding proof.

Lemma 2: Suppose $N/(K \log K) \rightarrow \infty$ as $N \rightarrow \infty$, and $p_{fg} = \alpha/(K(K-1))$ for all f, g, where $0 < \alpha \leq 1$. Then $\mathbb{P}(H(K,N,\boldsymbol{p}) \text{ is connected}) \rightarrow 1$ as $N \rightarrow \infty$.

Proof: The random multigraph $H(K,N,\boldsymbol{p})$ differs in two ways from the random graph $G(K,N)$. First, we may sample the same edge twice (this is why $H(K,N,\boldsymbol{p})$ is a multigraph). Second, with probability $1-\alpha$, a given edge may not be placed at all. Informally, neither of these effects change the order scaling of the number of edges needed to ensure connectivity. We now formally justify this intuition.

Where clear from context, to compress notation, we let H and G denote $H(K,N,\boldsymbol{p})$ and $G(K,N)$, respectively. Let $\Gamma(H)$ denote the simple graph obtained from H by replacing any multiedges by a single edge. Observe that conditional on $\Gamma(H)$ having N' edges, $\Gamma(H)$ has the same distribution as $G(K,N')$. Thus it suffices to show that almost surely, the number of edges N' in $\Gamma(H)$ satisfies $N'/(K \log K) \rightarrow \infty$.

Since $N/(K \log K) \rightarrow \infty$, we can choose a sequence $M(N)$ such that $N/M \rightarrow \infty$ and $M/(K \log K) \rightarrow \infty$. (For example, for each k, choose N_k such that $N/(K(N) \log K(N)) \geq k^2$ for all $N \geq N_k$. For N such that $N_k \leq N < N_{k+1}$, let $M(N) = N/k$.) Since $M/N \rightarrow 0$ and $N \leq K(K-1)$, it follows that $M/(K(K-1)) \rightarrow 0$ as $K \rightarrow \infty$. It suffices to show that $\mathbb{P}(\Gamma(H) \text{ contains at least } M \text{ edges}) \rightarrow 1$ as $N \rightarrow \infty$.

Consider the following procedure for sampling with replacement from $K(K-1)$ items. Each time we draw an item, we record its number with probability α; with probability $1-\alpha$, we discard the observation. If we have recorded n distinct items, the time until we record the next distinct item is geometric with parameter $\alpha - \alpha n/(K(K-1))$. Let $T(K)$ denote the time until we observe M distinct items. Then

$$\mathbb{E}[T(K)] = 1 + \frac{1}{\alpha - \alpha/(K(K-1))} + \cdots + \frac{1}{\alpha - \alpha(M-1)/(K(K-1))} \leq \frac{M}{\alpha - \alpha M/(K(K-1))}.$$

Since $M/(K(K-1)) \rightarrow 0$ as $K \rightarrow \infty$, we conclude

$$\lim_{N \rightarrow \infty} \frac{\mathbb{E}[T(K)]}{M} \leq \frac{1}{\alpha}.$$

Using Markov's inequality

$$\mathbb{P}(T(K) > N) \leq \frac{\mathbb{E}[T(K)]}{N} = \frac{\mathbb{E}[T(K)]}{M} \cdot \frac{M}{N} \rightarrow 0$$

as $N \rightarrow \infty$. In other words, if we sample N items with replacement from a bin of $K(K-1)$ items as described above, then we obtain M distinct items with probability approaching 1 as $N \rightarrow \infty$. It follows that $\mathbb{P}(\Gamma(H) \text{ contains at least } M \text{ edges}) \rightarrow 1$ as $N \rightarrow \infty$, as required. ■

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Christina Aperjis received the B.S. degree in electrical and computer engineering from the National Technical University of Athens, Athens, Greece, in 2003, the M.S. degree in computer science from Columbia University, New York, NY, in 2004, and the Ph.D. degree in operations research from Stanford University, Stanford, CA, in 2009.

She is currently a Researcher with the Social Computing Group, HP Labs, Palo Alto, CA.

Ramesh Johari (M'05) received the A.B. degree in mathematics from Harvard University, Cambridge, MA, in 1998, the Certificate of Advanced Study in mathematics from the University of Cambridge, Cambridge, U.K., in 1999, and the Ph.D. in electrical engineering and computer science from the Massachusetts Institute of Technology (MIT), Cambridge, in 2004.

He is currently an Assistant Professor of management science and engineering and, by courtesy, computer science and electrical engineering with Stanford University, Stanford, CA.

Michael J. Freedman (S'99–M'09) received the B.S. and M.Eng. degrees in electrical engineering and computer science from the Massachusetts Institute of Technology (MIT), Cambridge, in 2001 and 2002, respectively, and the Ph.D. degree in computer science from the Courant Institute, New York University, New York, in 2007, during which he spent two years at Stanford University, Stanford, CA.

He is currently an Assistant Professor of computer science with Princeton University, Princeton, NJ.