

A Graded Syntactic Model of Gradual Dependent Types

Translation to Support Implementation and Metatheory

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TODO write abstract

1 INTRODUCTION

Gradual dependent give principled static and dynamic semantics to programs where part of a type or term is missing. By allowing imprecision, gradual dependent types allow for smooth migration of code from non-dependently typed languages, or even untyped languages, to full dependent types, allowing the programmer run and test their code even when the full type details haven't been figured out. This migration is easiest in languages fulfilling the *gradual guarantees* of Siek et al. [2015], which state replacing part of a program with `?` creates no new static or dynamic type errors. The gradual guarantees ensure that, when an error is encountered, the problem is never too few types, but that two types in the program are fundamentally incompatible.

However, the benefits of gradual dependent types have not been realized, since existing developments have enabled the gradual guarantees at the expense of other desirable properties. Eremondi et al. [2019] presented GDTL, a dependent calculus supporting the gradual guarantees, but relied on a termination argument that does not scale to inductive types. Lennon-Bertrand et al. [2022] presented GCIC, with two extensions of the Calculus of Inductive Constructions (CIC) that satisfy the gradual guarantees, but one has undecidable type checking and the other rejects some well-typed static CIC programs.

Lennon-Bertrand et al. [2022] show that, to a degree, these sacrifices are unavoidable, to a degree: no dependently typed language can satisfy all of strong normalization, conservative extension of CIC, and *graduality*, a strengthening of the gradual guarantees.

Another obstacle to the adoption of gradual dependent types is that gradual dependent types have not yet been meaningfully implemented. Constructing a compiler for a dependently typed language is a massive engineering effort, and involves writing a type checker, a convertability check for terms, and unification engine for inference, in addition to the code generation and optimization. Writing a compiler for a gradual dependently typed language involves all of this work, plus extra handling to ensure safety in the presence of type imprecision.

We address both these shortcomings in GrInd, a Gradual language with Inductive types. GrInd sacrifices strong normalization and graduality, but keeps the properties that we actually want: decidable type checking, (weak) canonicity, the gradual guarantees, and conservatively extending CIC. Because type checking is decidable, GrInd can be translated into the core calculi of existing dependently typed compilers. Moreover, we show that GrInd does not violate static reasoning principles: propositionally-equal CIC terms embedded in GrInd are observationally-equivalent, and casts only change the error-behavior of terms, not the concrete results produced, a notion which we call *weak graduality*.

Our main contribution is a translation from GrInd to a static type theory:

- For implementation, the translation means that existing normalizers and code generators can be used “off-the-shelf” to compile GrInd programs;
- For metatheory, the translation serves as a syntactic model in the style of Boulier et al. [2017], which we use to prove the gradual guarantees and other metatheoretic properties;
- To enable decidable type checking, we adapt approximate normalization from Eremondi et al. [2019] to a cast calculus, using the syntactic model to prove termination;

- To model run-time semantics, we translate to *guarded type theory* [?], whose non-positive recursive types allow the non-termination of gradual types to be exactly represented in a consistent target language;
- Our translation and the theorems about it have been mechanized in Guarded Cubical Agda [?]

2 TWO PROBLEMS, ONE SOLUTION

Our work attacks two main problems that have a common solution. First, we want to allow GrInd to be implemented by translating them to static dependent types without needing to add features to the static target language, so that existing technology can be used when compiling them. The challenge here is accommodating the non-termination of gradual typing, since dependently typed core languages typically forbid or restrict non-terminating function definitions. Second, we want to prove properties about GrInd, namely that approximate normalization terminates (for decidable type checking) and that the gradual guarantees are satisfied. Unlike the approach of [Eremondi et al. \[2019\]](#), the syntactic model approach scales to handle inductive types, as well as logical-relation style proofs [\[Bernardy et al. 2012\]](#).

In this section, we explain these two problems, the challenges in solving them, and a birds-eye view of our approach to solving them.

Throughout, we refer to terms from several languages. For clarity, we write terms from CIC, the static starting language, in **red sans-serif font**, terms from GrInd, the gradual cast calculus, in **blue, bold serif font**, and terms from the guarded type theory, the target of translation, in *green, italic serif font*.

2.1 Translation to Support Implementation

2.1.1 Don't Reinvent the Wheel. Even without gradual types, implementing a dependently typed language is a formidable task. Because types may depend on terms, checking that two types are equal involves determining if those two types *convertible*, i.e., if they are equal after some number of reductions. Such a check must be implemented carefully, usually using Normalization by Evaluation (NbE) [?], to ensure that only well-typed terms are normalized so the check terminates. Support for type inference and implicit arguments to functions involves a writing higher-order unification algorithm. Most dependently typed languages have some form of proof-search, tactics, or reflection, to automate the generation of trivial but tedious proofs.

Even with the above features implemented, implementing them efficiently is even more difficult. Conversion checking is a costly operation, and while it can be implemented by fully normalizing the compared terms, this likely performs more computation than is necessary to see whether the terms are definitely convertible or not. Recent experiments in Idris have shown that on-the-fly compilation of terms can provide efficient normalization [?]. For code generation, since the return type of a function can depend on the value of its argument, the generation of machine code more closely resembles a dynamic language, but from an optimization standpoint, programs contain a wealth of type information that can be used to optimize said dynamic code. Moreover, dependently typed programs may contain indices or function arguments that are needed only because of type dependency, but are unused during run time [?]. Additional optimizations are needed to identify and remove unused terms from the generated code. Gradual dependent types should support the latest and greatest techniques for efficiency in both the compiler and compiled code.

2.1.2 An Implementation Strategy. To avoid re-implementing the above tools for gradual dependent types, our implementation strategy is to first translate gradual dependent programs to a static dependently typed language. Then, existing implementations of conversion checking, unification,

optimization and code-generation can be freely used on the result of translation. A gradual dependently typed language can simply be a new front-end to an existing dependently typed language, such as Agda or Idris.

The challenge with this approach is that, to reuse existing infrastructure, we must be able to translate gradual dependent types to static dependent types *without changing the static core language*. Gradual dependently typed programs do not always terminate, and may raise run-time type errors. However, the termination of static tools relies on the termination of static programs. Escape hatches exist, like Agda's `{-# TERMINATING #-}` pragma, but using such features is morally wrong, as these features are intended for cases where programs do terminate, but their termination is difficult or impossible to prove within the type theory itself. Instead, we must work within more conservative features for taming non-termination.

2.1.3 Idris and Agda's Models of Non-Termination. In Idris, all definitions are marked as either partial or total. Those that are marked total can only refer to others that are marked total. Recursive total functions are checked to ensure that recursive calls are only made to smaller arguments, and total inductive datatypes must be strictly positive (i.e. self-reference may only occur) to the right of function arrows in fields). By contrast, partial functions allow for general recursion and non-positive datatypes, and

The key to keeping type-checking decidable is to never normalize partial terms. In Idris, when a type depends on a partial function, the definition of that function is kept abstract.¹ This convention allows for trivial equalities to be satisfied, like a term being equal to itself. Two terminating terms that are marked partial are not considered definitionally equal even if they have the same evaluation, because normalization is required to see that they are equal. Agda has a similar model: definitions may be marked as `{-# NON_TERMINATING #-}`, which allows for general recursion but prevents such definitions from reducing during type checking.

The issue with these models is that they are difficult to reason about in the abstract. Running the type-checker immediately reveals whether a particular definition type checks with partial annotations. However, we wish to prove that our gradual to static translation produces well typed code for *all* outputs, but doing so requires reasoning about the internal details of Idris or Agda's normalization algorithm.

To simplify the proof of our translation's type preservation, we instead turn to *guarded type theory*. We explain more details of guarded type theory in §2.3.2, but for now it suffices to say that guarded type theory allows for a restricted form of general recursion and non-positive datatypes, by placing such self-reference behind a separate modality, and establishing fixed-point equalities as propositional equalities, rather than definitional equalities. Guarded type theory provides a convenient formalism for reasoning about when definitions are normalized, and allows us to mechanize our translation in Guarded Cubical Agda ?. Then, the primitives of guarded type theory can be easily implemented using the less restrictive partial annotations of Agda and Idris, allowing us to prove that the resulting code is always well typed.

2.1.4 Translating Approximate Normalization. While guarded type theory allows us to represent non-terminating terms at the type level, it provides no clues as to actually devise a gradual type system for which checking is decidable. However, previous work on gradual dependent types does provide clues. Lennon-Bertrand et al. [2022] were able to provide a syntactic model of GCIC without

¹In Idris 1, this behaviour was implemented in the compiler. Idris 2 has yet to implement this behavior, and happily runs the type checker forever when comparing non-terminating terms. However, safety can be recovered by defining partial functions in their own module, and exposing them with `export` rather than `public export`, which reveals their types while keeping their definitions abstract.

Language	Decidable type checking	Gradual Guarantees	Conservative Over CIC	Weak Graduality	Weak Canonicity	Inductives	Graduality	Strong Normalization
GDTL	✓	✗*	✗	?	✓	✗	✗	✗
GCIC ^G	✗	✓	✓	✓	✓	✓	✓	✗
GCIC ^N	✓	✗	✓	✗	✓	✓	✗	✓
GCIC [†]	✓	✓	✗	✓	✓	✓	✓	✓
GrInd	✓	✓	✓	✓	✓	✓	✗	✗

Table 1. Gradual Dependent Languages: Feature Comparison

decidable type-checking by extending CIC with a non-positive datatypes, then translating the unknown type $?$ to a non-positive inductive type $Unk := (Unk \rightarrow Unk) + (Unk \times Unk) + (List\ Unk) \dots$ Eremondi et al. [2019] were able to obtain terminating type level computations in GDTL with *approximate normalization*: terms occurring within types were given a separate semantics that produced $?$ whenever there was not enough type information to ensure termination.

We combine these two strategies to obtain a translation with decidable type checking. Like GDTL, GrInd has separate semantics for compile-time normalization and run-time execution. However, GDTL had totally separate approximate and exact semantics, where approximation was based on hereditary substitution, no formal notion related the two semantics. Our approximate and exact semantics are identical except for one part: casts between $?$ and $? \rightarrow ?$. Instead of relying on hereditary substitution, we use the GCIC approach and translate it to CIC: the approximations mean that only strictly-positive datatypes are used, and hence termination is guaranteed.

2.2 Metatheory

2.2.1 Extinguishing the Fire Triangle. When developing GCIC, Lennon-Bertrand et al. [2022] establish a “fire-triangle” theorem stating that no language could conservatively extend CIC while having strong normalization and graduality. Here, graduality is the property defined by New and Ahmed [2018], where casting to a more precise type and back produces a term that is as precise as the original, and casting to a less precise type and back produces an observationally-equivalent term. The name is chosen by analogy with parametricity: just as parametricity is a logical relation strengthening type safety, graduality is a logical relation strengthening the gradual guarantees. GCIC is then presented with three variants: one without decidable type checking, one without the gradual guarantees, and one that rejects some well-typed CIC programs.

In this paper, we take the view that for programming languages specifically, strong normalization and graduality are means to an end, rather than ends unto themselves. Specifically, we want strong normalization to prove decidable type checking and weak canonicity i.e. that all normal-forms of a term are either canonical, $?$, or an error \mathcal{U} . Strong normalization is also required for weak logical consistency: GrInd does not have this, but we discuss how to recover a form of it in §2.2.2. We want graduality to prove the gradual guarantees, as well as to establish that $?$ is never needlessly produced as a result. (Such a property is desirable because a language that produces $?$ as a result of all casts is useless, but still technically satisfies the gradual guarantees.)

So while GrInd cannot simultaneously have all three fire-triangle properties, it can have all the properties we actually want: decidable type checking, the gradual guarantees, and conservatively extending CIC. Section 2.2.1 compares the features of GDTL, the three variants of GCIC, and GrInd: while GrInd sacrifices strong normalization and graduality, we are able to recover all the other desired properties despite this.

However, proving these properties is not trivial. GDTL proved decidability of type checking using hereditary substitution, which does not scale well to inductive types. The gradual guarantees were

proved using a simulation argument, but the proof was complex and delicate. While a syntactic model was used, the $GCIC^G$ variant that embeds CIC and has the gradual guarantees required non-positive datatypes, so the model was by a translation to an inconsistent logic, and the target type theory could not be used to prove theorems about $GCIC^G$. For the other variants, strong normalization was shown in a syntactic model by sacrificing either embedding CIC or the gradual guarantees.

Our implementation strategy works equally well for proving that type checking is decidable: if we can translate approximate normalization into the total fragment of a type theory, then it must be terminating. Then, by translating our exact semantics into guarded type theory, we are able to prove the gradual guarantee in the type theory itself. Unlike CIC with non-positive types, guarded type theory is logically consistent, so the theorems proved about the translations of gradual programs can be believed.

2.2.2 Static Reasoning in Gradual Code. In addition to the gradual guarantees, there are two properties about GrInd that our translation allows us to prove:

- **Preservation of Propositional Equality:** this is the property that, for any two static CIC terms $s, t : T$, if we have a proof that $s =_T t$, then when we embed s and t into GrInd, the resulting s and t are observationally equivalent. This guarantees that static reasoning principles still hold for fully static code that is used in gradual programs. For example, any optimizations that are valid for static code are still valid if that code is imported by a gradual library.

The preservation of equality is a weaker version of the gradual full-abstraction property described by Jacobs et al. [2021], which says that all observationally-equivalent static programs are observationally-equivalent when embedded in the gradual language. Full abstraction is difficult to prove for GrInd, since the conventional strategy requires the ability to translate any GrInd context into an equivalent CIC one, which is impossible since CIC lacks non-termination as an effect. Proving that propositionally-equal static terms are gradually-equivalent provides a good first step, however, to ensuring that static is preserved gradually.

JE ▶ *TODO: check if this prevents the bad example from DeVrise paper* ◀ JE ▶ *TODO: can we prove FA with funExt, by turning $(x : T) \text{ into } \lambda(C : T \rightarrow B). Cx$ and using fun-ext?* ◀ JE ▶ *TODO: can we prove FA from CIC \rightarrow GuardedTT FA?* ◀

- **Weak graduality:**

Approximate normalization also let us recover a form of weak logical consistency: any closed term of type **False** must have an approximate normal form that is either the unknown term **?** or a run-time type error **U**. This is useful for proofs that are not used in computation, but serve as validation that some desired property holds. Because some GrInd programs are non-terminating, it is possible to write closed terms of statically-empty types that do not evaluate to **?** or **U**. However, inspecting the approximate-normal forms of such proofs can inform the programmer about places where type-imprecision caused approximation, directing them to where type imprecision forced approximation. If a term of a fully-static type has a fully-static approximate normal form, then the programmer can be confident in the truth of its Curry-Howard interpretation, even if the proof itself uses imprecise types or terms.

2.3 Modelling Gradual Dependent Types

2.3.1 Modelling Approximate Normalization.

2.3.2 Guarded Type Theory.

2.3.3 Relating Approximate and Exact Normalization.

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