A Guarded Syntactic Model of Gradual Dependent Types

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Overview

Two Problems

Two Problems - One Solution

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- Implementing Gradual Dependent Types

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- Denotational Semantics + Metatheory

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- Approximate Normalization + Translation to Static Dependent Types

Implementing Gradual Dependent Types

Don’t Reinvent the Wheel

Long-term goal: gradual types in a full-scale dependent language

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    - E.g. Idris: experimental normalization by compilation to ChezScheme

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- Dependent languages restricted in how non-termination is used
  - Ensures consistency and decidablity of type checking

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- Ensures type-checking terminates

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- More on this later

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- Type computations only use approximate part

The Other Side: Denotational Semantics

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What kind of reasoning principles hold for gradual dependent types?

What kind of guarantees can we give the programmer?

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Need Approx. Normalization for decidable type-checking

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    - Even more complex when add approximation

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If syntactic model is in consistent calculus, then can prove these things in the target theory itself (unlike $GCIC^{\mathcal{G}}$)

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Then can prove things about the language using the model

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Consistent: model in Topos of Trees

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- Expression semantics: if $t : T$ then $t : (El_{approx}\ \mathcal{T}[\![\mathbf{T}]\!]) \times (L\ (El_{exact}\ \mathcal{T}[\![\mathbf{T}]\!]))$

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- Expression semantics: if $t : T$ then $t : (El_{approx}\, \mathcal{T}[\![\mathbf{T}]\!]) \times (L\ (El_{exact}\, \mathcal{T}[\![\mathbf{T}]\!]))$
- Functions : $El_{exact}\, \mathcal{T}[\![(\mathbf{x} : \mathbf{T_1}) \to \mathbf{T_2}]\!] = El_{exact}\, \mathcal{T}[\![\mathbf{T_1}]\!] \to L\ (El_{exact}\, \mathcal{T}[\![\mathbf{T_2}]\!])$

A Model in Guarded Type Theory

Universe à la Tarski

- Data-type of "codes" $\mathbb{C}_\ell : \mathit{Type}$
- "Elements-of" interpretations
  - $El_{approx} : \mathbb{C}_\ell \to \mathit{Type}$
  - $El_{exact} : \mathbb{C}_\ell \to \mathit{Type}$

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  - Allows for non-termination

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- Then write $cast : (c_1\ c_2 : \mathbb{C}_\ell) \to El_{exact}\ c_1 \to (El_{exact}\ c_2)$
- Have $f : Unk \to L\ Unk$ is cast to $\lambda(x : \triangleright Unk) \to f >>= (\theta\ x)$

Mapping GTT to Idris

Straightforward mapping of GTT to partial Idris

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- So neither does Idris typing

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- GGs as corollary