

Functional Programming 1: Recursion and Immutable Data

CS 350

Dr. Joseph Eremondi

Last updated: July 3, 2024

- Topic: Functional Programming in Racket and plait

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- Required Reading:

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 - Plait videos, HtDP videos

Programming in CS 350

All coding for this class uses:

- The Racket Programming Language

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- The `plait` library for Racket

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- The Dr. Racket editor

Racket

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- Lisp-style language

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 - (((((((((Parentheses))))))))

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- Language for making languages

What is Dr. Racket?

- IDE for Racket

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- IDE for Racket
 - Syntax highlighting

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 - see <https://docs.racket-lang.org/guide/other-editors.html>

Plait

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- Adds syntax to Racket
 - Declaring and pattern matching on data types
 - Type annotations for functions
- Minimal
 - Has what you need to write programming languages
 - Not much else
 - You can do a lot with very little

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- Algebraic Data Types

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 - Racket writes `(f x)`, not `f(x)`
- `x` is not the same as `(x)`
 - `x` gets the value of the variable `x`
 - `(x)` is calling a function named `x` with zero arguments

Numbers

(+ 2 7)

Numbers

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Numbers

```
(+ 2 7)  
(- 10 0.5)
```

9

Numbers

```
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Numbers

```
(+ 2 7)  
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(* 1/3 2/3)
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```
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9.5  
2/9
```

Numbers

```
(+ 2 7)  
(- 10 0.5)  
(* 1/3 2/3)  
(/ 1 100000000000000.0)
```

```
9  
9.5  
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(+ 2 7)  
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```
9  
9.5  
2/9  
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```


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(+ 2 7)
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(max 10 20)
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Numbers

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(+ 2 7)
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(max 10 20)
(modulo 10 3)
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1
```


Booleans

```
(= (+ 2 3) 5)
```


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```
(= (+ 2 3) 5)  
(> (/ 0 1) 1)
```

#t

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```

#t

#f

Booleans

```
(= (+ 2 3) 5)  
> (/ 0 1) 1  
(zero? (- (+ 1 2) (+ 3 0)))
```

```
#t
```

```
#f
```

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```
(= (+ 2 3) 5)  
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(zero? (- (+ 1 2) (+ 3 0)))
```

```
#t  
#f  
#t
```

Booleans

```
(= (+ 2 3) 5)  
(> (/ 0 1) 1)  
(zero? (- (+ 1 2) (+ 3 0)))  
(and (< 1 2) (> 1 0))
```

```
#t  
#f  
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(> (/ 0 1) 1)  
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#f  
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Booleans

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(> (/ 0 1) 1)  
(zero? (- (+ 1 2) (+ 3 0)))  
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(or (zero? 1) (even? 3))
```

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```
#t  
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```
(if (< 2 3) "hello" "goodbye")  
(+ 3  
  (if (= 2 (+ 1 1))  
    3  
    40))
```

Conditionals

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```
(if (< 2 3) "hello" "goodbye")  
(+ 3  
  (if (= 2 (+ 1 1))  
    3  
    4))
```

```
"hello"
```

```
6
```

- Calling a function replaces variable with concrete argument

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```
(define (addOne [x : Number]) : Number  
  (+ x 1))  
(addOne 10)
```

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```
11
```


Functions

```
(define (isRemainder [x : Number]
                  [y : Number]
                  [remainder : Number])
  : Boolean
  (= remainder (modulo x y)))
(isRemainder 10 3 1)
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  : Boolean
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```

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#f
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- General form:

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Functions (ctd.)

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```
(define (functionName  
    [argName : argType]  
    ...  
    [argNameN : argTypeN]) : returnType  
  functionBody)
```

- Later in the course we'll see another way of defining functions

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 - (symbol=? 'a 'b)
 - Compares pointers, so very fast

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  (let ([xy (+ x y)])
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 - We'll see this later when we learn about lambdas

Functional Thinking and Recursion

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 - Fast, memory efficient

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 - Some optimizations easier

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 - Sometimes faster because you need fewer safety checks in your code
- Farther from what the CPU is actually doing
- Some algorithms are more concise with mutation
 - But lots aren't

How to design functional programs

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Further reference:

<http://htdp.org>, Matthew Flatt's Notes (URCourses)

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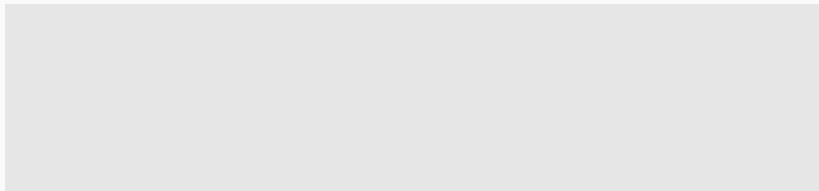
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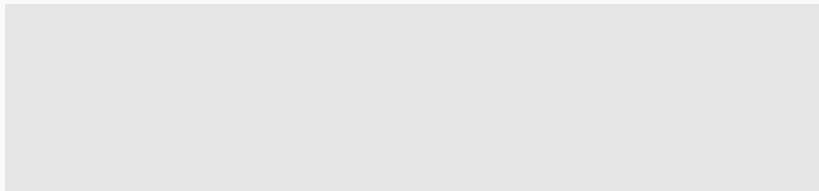
```
(define (factorial [n : Number]) : Number  
  (error 'factorial "TODO"))
```

Factorial - Examples

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Factorial - Examples

```
(test (factorial 0) 1 )
```

Factorial - Examples

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(test (factorial 0) 1 )  
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(test (factorial 0) 1 )  
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(test (factorial 2) 2 )
```

Factorial - Examples

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(test (factorial 0) 1 )  
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(test (factorial 1) 1 )  
(test (factorial 2) 2 )  
(test (factorial 3) 6 )  
(test (factorial 4) 24 )
```

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Factorial - Examples

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(test (factorial 0) 1 )  
(test (factorial 1) 1 )  
(test (factorial 2) 2 )  
(test (factorial 3) 6 )  
(test (factorial 4) 24 )  
(test (factorial 5) 120 )
```

- Notice the pattern?

- A natural number is either

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 - Zero

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```
(define (factorial [n : Number]) : Number
  (if (zero? n)
    (error 'zero "TODO")
    (let ([n-1 (- n 1)])
      (error 'suc "TODO")))
  ))
```

Factorial - Recursion

- Divide problem into base case and recursive cases

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```
(define (factorial [n : Number]) : Number
  (if (zero? n)
      (error 'zero "TODO")
      (let*
          ([n-1 (- n 1)]
           [fn-1 (factorial n-1)])
        (error 'suc "TODO")))
  ))
```

Factorial - Filling holes

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```
(define (factorial [n : Number]) : Number
  (if (zero? n)
    1
    (let*
      ([n-1 (- n 1)]
       [fn-1 (factorial n-1)])
      (* n fn-1))
  ))
```

Run Tests

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```
(test (factorial 0) 1 )  
(test (factorial 5) 120 )
```

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```
good (factorial 0) at line 11  
  expected: 1  
  given: 1
```

```
good (factorial 5) at line 12  
  expected: 120  
  given: 120
```

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 - Aside: I research languages where you *can* express this with types

Another Example: Exponentiation

- Live coding in Dr. Racket

Unbounded Data: Lists

Functional Linked Lists

- Every linked list is one of:

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- `Cons` does **not** change its input
 - Creates a new list whose tail is the old list

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 - `(cons 1 (cons 2 (cons 3 (cons 4 '()))))`
- Lots more helper functions, see the documentation

Template for Lists

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```
(define (list-template
  [xs : (Listof Number)])
  (if (empty? xs)
      (error 'nil "TODO")
      (let ([h (first xs)]
            [t (rest xs)])
        [tRet (list-template t)])
      (error 'cons "TODO")))
))
```

Example: Sum

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```
(define (sum [xs : (Listof Number)])  
  : Number  
  (if (empty? xs)  
      0  
      (let* ([h (first xs)]  
              [t (rest xs)]  
              [tRet (sum t)])  
        (+ h tRet))))  
(sum '())  
(sum '(1 2 3))  
(sum '(100 20 3))
```


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(define (sum [xs : (Listof Number)])  
  : Number  
  (if (empty? xs)  
      0  
      (let* ([h (first xs)]  
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              [tRet (sum t)])  
        (+ h tRet))))  
(sum '())  
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```

0

6

123

Pattern Matching: Motivation

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 - e.g. `first '()` will raise an error

Pattern Matching:

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```
(define (increment [xs : (Listof Number)])  
  : (Listof Number)  
  (type-case (Listof Number) xs  
    [empty  
      empty]  
    [(cons h t)  
      (cons (+ h 1) (increment t))]))  
(increment '(2 3 4))
```

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```
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(increment '(2 3 4))
```

```
'(3 4 5)
```

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 - Input is list whose elements are some type 'a
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 - e.g. `first '(1 2 3)` is a Number, but `first '(#t #f #t)` is a Boolean
- Later, this will be very useful for writing generic list operations

Example: List Concatenation

- We can combine two lists into a single list

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 - This function is built into Plait as `append`

```
(define (concat [xs : (Listof 'elem)]  
           [ys : (Listof 'elem)])  
  : (Listof 'elem)  
  (type-case (Listof 'elem) xs  
    [empty  
     ys]  
    [(cons h t)  
     (cons h (concat t ys))]))
```

Example: List Concatenation (ctd.)

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Example

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Example

```
(concat '(1 2 3) '(4 5 6))
```

Example: List Concatenation (ctd.)

Example

```
(concat '(1 2 3) '(4 5 6))
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Example: List Concatenation (ctd.)

Example

```
(concat '(1 2 3) '(4 5 6))  
(concat '("3" "5") '("0"))
```

```
'(1 2 3 4 5 6)
```

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Example

```
(concat '(1 2 3) '(4 5 6))  
(concat '("3" "5") '("0"))
```

```
'(1 2 3 4 5 6)  
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```


Example: List Concatenation (ctd.)

Example

```
(concat '(1 2 3) '(4 5 6))  
(concat '("3" "5") '("0"))  
(concat '() '(#t))
```

```
'(1 2 3 4 5 6)  
'("3" "5" "0")
```

Example: List Concatenation (ctd.)

Example

```
(concat '(1 2 3) '(4 5 6))  
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```
'(1 2 3 4 5 6)  
'("3" "5" "0")  
'(#t)
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Example

```
(concat '(1 2 3) '(4 5 6))  
(concat '("3" "5") '("0"))  
(concat '() '(#t))  
(concat '(#f) '())
```

Results

```
'(1 2 3 4 5 6)  
'("3" "5" "0")  
'(#t)
```

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```
(concat '(1 2 3) '(4 5 6))  
(concat '("3" "5") '("0"))  
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```

Results

```
'(1 2 3 4 5 6)  
'("3" "5" "0")  
'(#t)  
'(#f)
```

More Examples

- Demo: Dr. Racket (as time permits)

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