Strictly Monotone Brouwer Trees for Well Founded Recursion Over Multiple Values

Anonymous Author(s)

Abstract

Ordinals can be used to prove the termination of dependently typed programs. Brouwer trees are a particular ordinal notation that make it very easy to assign sizes to higher order data structures. They extend unary natural numbers with a limit constructor, so a function's size can be the least upper bound of the sizes of values from its image. These can then be used to define well founded recursion: any recursive calls are allowed so long as they are on values whose sizes are strictly smaller than the current size.

Unfortunately, Brouwer trees are not algebraically well behaved. They can be characterized equationally as a join-semilattice, where the join takes the maximum of two trees. However, this join does not interact well with the successor constructor, so it does not interact properly with the strict ordering used in well founded recursion.

We present Strictly Monotone Brouwer trees (SMB-trees), a refinement of Brouwer trees that are algebraically well behaved. SMB-trees are built using functions with the same signatures as Brouwer tree constructors, and they satisfy all Brouwer tree inequalities. However, their join operator distributes over the successor, making them suited for well founded recursion or equational reasoning.

This paper teaches how, using dependent pairs and careful definitions, an ill behaved definition can be turned into a well behaved one. Our approach is axiomatically lightweight: it does not rely on Axiom K, univalence, quotient types, or Higher Inductive Types. We implement a recursively-defined maximum operator for Brouwer trees that matches on successors and handles them specifically. Then, we define SMB-trees as the subset of Brouwer trees for which the recursive maximum computes a least upper bound. Finally, we show that every Brouwer tree can be transformed into a corresponding SMB-tree by joining it with itself an infinite number of times. All definitions and theorems are implemented in Agda.

Keywords: dependent types, Brouwer trees, well founded recursion

1 Introduction

1.1 Recursion and Dependent Types

Dependently typed languages, such as Agda [?], Coq [Bertot and Castéran 2004], Idris [?] and Lean [?], bridge the gap between theorem proving and programming.

Functions defined in dependently typed languages are typically required to be *total*: they must provably halt in all inputs. Since the halting problem is undecidable, recursively-defined functions must be written in such a way that the type checker can mechanically deduce termination. Some functions only make recursive calls to structurally-smaller arguments, so their termination is apparent to the compiler. However, some functions cannot be easily expressed using structural recursion. For such functions, the programmer must instead use *well founded recursion*, showing that there is some ordering, with no infinitely-descending chains, for which each recursive call is strictly smaller according to this ordering. For example, the typical quicksort algorithm is not structurally recursive, but can use well founded recursion on the length of the lists being sorted.

1.2 Ordinals

While numeric orderings work for first-order data, they are ill suited to recursion over higher-order data structures, where some fields contain functions.

There are many formulations of ordinals in dependent type theory, each with their own advantages and disadvantages.

1.3 Contributions

This work defines *strictly monotone Brouwer Trees*, henceforth SMB-trees, a new presentation of ordinals that hit a sort of sweet-spot for defining functions by well founded recursion. Specifically, SMB-trees:

- are strictly ordered by a well founded relation;
- have a maximum operator which computes a least-upper bound;
- are *strictly-monotone* with respect to the maximum: if $a < b$ and $c < d$, then $\mathsf{max}\ a\ c < \mathsf{max}\ b\ d$;
- can compute the limits of arbitrary sequences;
- are light in axiomatic requirements: they are defined without using axiom K, univalence, quotient types, or higher inductive types.

1.4 Uses for SMB-trees

1.4.1 Well Founded Recursion.

Having a maximum operator for ordinals is particularly useful when traversing over multiple higher order data structures in parallel, where neither argument takes priority over the other. In such a case, a lexicographic ordering cannot be used.

As an example, consider a unification algorithm over some encoding of types, and suppose that α-renaming or some other restriction prevents structural recursion from being used. To solve a unification problem $\Sigma(x : A). B = \Sigma(x : C). D$ we must recursively solve $A = C$ and $\forall x. B[x] = D[x]$. However, the type of x in the latter equation depends on the solution to the first equation, which is bounded by the size of the maximum of the sizes of both A and C. So for each recursive call to be on a smaller size, the size of $a = c$ and $b = d$ must both be strictly smaller than $(a, b) = (c, d)$. In a lexicographic ordering where the size of the left-hand size dominates, we know that a is strictly smaller than (a, b), but we have no guarantees that TODO. Conversely, if we order unification problems by the size of the maximum of their two sides.

This style of well founded induction was used to prove termination in a syntactic model of gradual dependent types [?]. There, Brouwer trees were used to establish termination of recursive procedures for combining the type information in two imprecise types. The decreasing metric was the maximum size of the codes for the types being combined. Brouwer trees' arbitrary limits were used to assign sizes to dependent function and product types, and the strict monotonicity of the maximum operator was essential for proving that recursive calls were on strictly smaller arguments.

1.4.2 Syntactic Models and Sized Types.

An alternate way view of our contribution is as a tool for modelling sized types [?]. The implementation of sized types in Agda has been shown to be unsound [?], due to the interaction between propositional equality and the top size ∞ satisfying $\infty < \infty$. [Chan 2022] defines a dependently typed language with sized types that does not have a top size, proving it consistent using a syntactic model based on Brouwer trees.

SMB-trees provide the capability to extend existing syntactic models to sized types with a maximum operator. This brings the capability of consistent sized types closder to feature parity with Agda, which has a maximum operator for its sizes [?], while still maintaining logical consistency.

1.4.3 Algebraic Reasoning.

Another advantage of SMB-trees is that they allow Brouwer trees to be interpreted using algebraic tools. SMB-trees can be described as In algebraic terminology, SMB-trees satisfy the following algebraic laws, up to the equivalence relation defined by $s \approx t := s \leq t \leq s$

- Join-semlattice: the binary max is associative, commutative, and idempotent
- Bounded: there is a least tree Z such that max t $Z \approx t$
- Inflationary endomorphism: there is a successor operator ↑ such that max (↑t) $t \approx$ ↑t and ↑(max s t) = max(↑s) (↑t)

Bezem and Coquand [2022] describe a polynomial time algorithm for solving equations in such an algebra, and describe its usefulness for solving constraints involving universe levels in dependent type checking. While equations involving limits of infinite sequences are undecidable, the inflationary laws could be used to automatically discharge some equations involving sizes. This algebraic presentation is particularly amenable to solving equations using free extensions of algebras [Allais et al. 2023; Corbyn 2021].

1.5 Implementation

We have implemented SMB-trees in Agda 2.6.4. Our library specifically avoids Agda-specific features such as cubcal type theory or Axiom K, so we expect that the library can be easily ported to other proof assistants.

This paper is written as a literate Agda document, and the definitions given in the paper are valid Agda code. Several definitions are presented with their body omitted due to space restrictions. The full implementation can be found in the supplementary materials section of this submission.

2 Brouwer Trees: An Introduction

Brouwer trees are a simple but elegant tool for proving termination of higher-order procedures. Traditionally, they are defined as follows:

```
data SmallTree : Set where
  Z : SmallTree
  ↑ : SmallTree → SmallTree
  Lim : (ℕ → SmallTree) → SmallTree
```

Under this definition, a Brouwer tree is either zero, the successor of another Brouwer tree, or the limit of a countable sequence of Brouwer trees. However, these are quite weak, in that they can only take the limit of countable sequences. To represent the limits of uncountable sequences, we can paramterize our definition over some Universe à la Tarski:

```
module RawTree {ℓ}
  (ℂ : Set ℓ)
  (El : ℂ → Set ℓ)
  (Cℕ : ℂ) (CℕIso : Iso (El Cℕ) ℕ ) where
```

Our module is paramterized over a universe level, a type $\mathbb{C}$ of *codes*, and an "elements-of" interpretation function *El*, which computes the type represented by each code. We require that there be a code whose interpretation is isomorphic to the natural numbers, as this is essential to our construction in ??. Increasingly larger trees can be obtained by setting $\mathbb{C}$:= Set ℓ and *El* := *id* for increasing ℓ. However, by defining an inductive-recursive universe, one can still capture limits over some non-countable types, since Tree is in Set whenever $\mathbb{C}$ is.

We then generalize limits to any function whose domain is the interpretation of some code.

```
data Tree : Set ℓ where
  Z : Tree
  ↑ : Tree → Tree
  Lim : ∀ (c : ℂ ) → (f : El c → Tree) → Tree
```

The small limit constructor can be recovered from the natural-number code

```
ℕLim : (ℕ → Tree) → Tree
ℕLim f = Lim Cℕ (λ cn → f (Iso.fun CℕIso cn))
```

Brouwer trees are a the quintessential example of a higher-order inductive type.[1]: Each tree is built using smaller trees or functions producing smaller trees, which is essentially a way of storing a possibly infinite number of smaller trees.

2.1 Ordering Trees

Our ultimate goal is to have a well-founded ordering[2], so we define a relation to order Brouwer trees.

```
data _≤_ : Tree → Tree → Set ℓ where
  ≤-Z : ∀ {t} → Z ≤ t
  ≤-sucMono : ∀ {t₁ t₂}
    → t₁ ≤ t₂
    → ↑ t₁ ≤ ↑ t₂
  ≤-cocone : ∀ {t} {c : ℂ} (f : El c → Tree) (k : El c)
    → t ≤ f k
    → t ≤ Lim c f
  ≤-limiting : ∀ {t} {c : ℂ}
    → (f : El c → Tree)
    → (∀ k → f k ≤ t)
    → Lim c f ≤ t
```

This relation is reflexive:

```
≤-refl : ∀ t → t ≤ t
≤-refl Z = ≤-Z
≤-refl (↑ t) = ≤-sucMono (≤-refl t)
≤-refl (Lim c f)
  = ≤-limiting f (λ k → ≤-cocone f k (≤-refl (f k)))
```

Crucially, it is also transitive, making the relation a pre-order. We modify our the order relation from that of Kraus et al. [2023] so that transitivity can be proven constructively, rather than adding it as a constructor for the relation. This allows us to prove well-foundedness of the relation without needing quotient types or other advanced features.

```
≤-trans : ∀ {t₁ t₂ t3} → t₁ ≤ t₂ → t₂ ≤ t3 → t₁ ≤ t3
≤-trans ≤-Z p23 = ≤-Z
≤-trans (≤-sucMono p12) (≤-sucMono p23)
  = ≤-sucMono (≤-trans p12 p23)
≤-trans p12 (≤-cocone f k p23)
  = ≤-cocone f k (≤-trans p12 p23)
≤-trans (≤-limiting f x) p23
  = ≤-limiting f (λ k → ≤-trans (x k) p23)
≤-trans (≤-cocone f k p12) (≤-limiting .f x)
  = ≤-trans p12 (x k)
```

We create an infix version of transitivity for more readable construction of proofs:

```
_≤⨟_ : ∀ {t₁ t₂ t3} → t₁ ≤ t₂ → t₂ ≤ t3 → t₁ ≤ t3
lt1 ⨟≤ lt2 = ≤-trans lt1 lt2
```

2.1.1 Strict Ordering.

We can define a strictly-less-than relation in terms of our less-than relation and the successor constructor:

```
_<_ : Tree → Tree → Set ℓ
t₁ < t₂ = ↑ t₁ ≤ t₂
```

That is, a t_1 is strictly smaller than t_2 if the tree one-size larger than t_1 is as small as t_2. This relation has the properties one expects of a strictly-less-than relation: it is a transitive sub-relation of the less-than relation, every tree is strictly less than its successor, and no tree is strictly smaller than zero. JE ►TODO more?◄

```
≤↑t : ∀ t → t ≤ ↑ t
≤↑t Z = ≤-Z
≤↑t (↑ t) = ≤-sucMono (≤↑t t)
≤↑t (Lim c f)
  = ≤-limiting f λ k →
    (≤↑t (f k))
    ⨟≤ (≤-sucMono (≤-cocone f k (≤-refl (f k))))
```

```
<-in-≤ : ∀ {x y} → x < y → x ≤ y
<-in-≤ pf = ≤-trans (≤↑t _) pf
```

```
<∘≤-in-< : ∀ {x y z} → x < y → y ≤ z → x < z
<∘≤-in-< x<y y≤z = ≤-trans x<y y≤z
```

```
≤∘<-in-< : ∀ {x y z} → x ≤ y → y < z → x < z
≤∘<-in-< {x} {y} {z} x≤y y<z = ≤-trans (≤-sucMono x≤y) y<z
```

```
¬<Z : ∀ t → ¬(t < Z)
¬<Z t ()
```

2.2 Well Founded Induction

Recall the definition of a constructive well founded relation:

```
data Acc {A : Set a} (_<_ : A → A → Set ℓ) (x : A) : Set (a ⊔ ℓ) where
  acc : (rs : ∀ y → y < x → Acc _<_ y) → Acc _<_ x
```

```
WellFounded : (A → A → Set ℓ) → Set _
WellFounded _<_ = ∀ x → Acc _<_ x
```

That is, an element of a type is accessible for a relation if all strictly smaller elements of it are also accessible. A relation is well founded if all values are accessible with respect to that relation. This can then be used to define induction with arbitrary recursive calls on smaller values:

[1]Not to be confused with Higher Inductive Types (HITs) from Homotopy Type Theory [Univalent Foundations Program 2013]

[2]Technically, this is a well-founded quasi-ordering because there are pairs of trees which are related by both $\leq$ and $\geq$, but which are not propositionally equal.

```
wfRec : (P : A → Set ℓ)
  → (∀ x → ((y : A) → y < x → P y) → P x)
  → ∀ x → P x
```

Following the construction of Kraus et al. [2023], we can show that the strict ordering on Brouwer trees is well founded. First, we prove a helper lemma: if a value is accessible, then all (not necessarily strictly) smaller terms are are also accessible.

```
smaller-accessible : (x : Tree)
  → Acc _<_ x → ∀ y → y ≤ x → Acc _<_ y
smaller-accessible x (acc r) y x<y
  = acc (λ y' y'<y → r y' (<∘≤-in-< y'<y x<y))
```

Then we use structural reduction to show that all terms are accesible. The key observations are that zero is trivially accessible, since no trees are strictly smaller than it, and that the only way to derive ↑t ≤ (Lim c f) is with ≤-cocone, yielding a concrete index k for which ↑ t ≤ f k, on which we can recur.

```
ordWF : WellFounded _<_
ordWF Z = acc λ _ ()
ordWF (↑ x)
  = acc (λ { y (≤-sucMono y≤x)
    → smaller-accessible x (ordWF x) y y≤x})
ordWF (Lim c f) = acc helper
  where
    helper : (y : Tree) → (y < Lim c f)
      → Acc _<_ y
    helper y (≤-cocone .f k y<fk)
      = smaller-accessible (f k)
        (ordWF (f k)) y (<-in-≤ y<fk)
```

3 First Attempts at a Join

In this section, we present two faulty implmentations of a join operator for trees. The first uses limits to define the join, but does not satisfy strict monotonicity. The second is defined inductively. Its satisfies strict monotonicity, but fails to be the least of all upper bounds, and requires us to assume that limits are only taken over non-empty types. In ??, we define SMB-trees a refinement of Brouwer trees that combines the benefits of both versions of the maximum.

3.1 Limit-based Maximum

Since the limit constructor finds the least upper bound of the image of a function, it should be possible to define the maximum of two trees as a special case of general limits. Indeed, we can compute the maximum of t_1 and t_2 as the limit of the function that produces t_1 when given 0 and t_2 otherwise.

```
limMax : Tree → Tree → Tree
limMax t₁ t₂ = ℕLim λ n → if0 n t₁ t₂
```

This version of the maximum has several of the properties we want from a maximum function: it is monotone, idempotent, commutative, and is a true least-upper-bound of its inputs.

```
limMax≤L : ∀ {t₁ t₂} → t₁ ≤ limMax t₁ t₂
limMax≤L {t₁} {t₂}
  = ≤-cocone _ (Iso.inv CℕIso 0)
    (subst
      (λ x → t₁ ≤ if0 x t₁ t₂)
      (sym (Iso.rightInv CℕIso 0))
      (≤-refl t₁))


limMax≤R : ∀ {t₁ t₂} → t₂ ≤ limMax t₁ t₂
limMax≤R {t₁} {t₂}
  = ≤-cocone _ (Iso.inv CℕIso 1)
    (subst
      (λ x → t₂ ≤ if0 x t₁ t₂)
      (sym (Iso.rightInv CℕIso 1))
      (≤-refl t₂))


limMaxIdem : ∀ {t} → limMax t t ≤ t
limMaxIdem {t} = ≤-limiting _ helper
  where
    helper : ∀ k → if0 (Iso.fun CℕIso k) t t ≤ t
    helper k with Iso.fun CℕIso k
    ... | zero = ≤-refl t
    ... | suc n = ≤-refl t

limMaxMono : ∀ {t₁ t₂ t₁' t₂'}
    → t₁ ≤ t₁' → t₂ ≤ t₂'
    → limMax t₁ t₂ ≤ limMax t₁' t₂'
limMaxMono {t₁} {t₂} {t₁'} {t₂'} lt1 lt2 = extLim _ _ helper
  where
    helper : ∀ k →
      if0 (Iso.fun CℕIso k) t₁ t₂
        ≤ if0 (Iso.fun CℕIso k) t₁' t₂'
    helper k with Iso.fun CℕIso k
    ... | zero = lt1
    ... | suc n = lt2

limMaxLUB : ∀ {t₁ t₂ t} → t₁ ≤ t → t₂ ≤ t → limMax t₁ t₂ ≤ t
limMaxLUB lt1 lt2 = limMaxMono lt1 lt2 ⨾≤ limMaxIdem

limMaxCommut : ∀ {t₁ t₂} → limMax t₁ t₂ ≤ limMax t₂ t₁
limMaxCommut = limMaxLUB limMax≤R limMax≤L
```

3.1.1 Limitation: Strict Monotonicity.

The one crucial property that this formulation lacks is that it is not strictly monotone: we cannot deduce max t_1 t_1 < max t_1' t_2' from $t_1 < t_1'$ and $t_2 < t_2'$. This is because the only way to construct a proof that ↑t ≤ Lim c f is using the ≤-cocone constructor. So we would need to prove that ↑(max t_1 t_2) ≤ t_1' or that ↑(max t_1 t_2) ≤ t_2', which cannot be deduced from the premises alone. What we want is to have ↑max (t_1) t_2 ≤ max(↑t_1) (↑t_2), so that strict monotonicity is a direct consequence of ordinary monotonicity of the maximum. This is not possible when defining the constructor as a limit.

3.2 Recursive Maximum

In our next attempt at defining a maximum operator, we obtain strict monotonicity by making indMax (↑t_1) (↑t_2) = ↑(indMax t_1 t_2) hold definitionally. Then, provided indMax is monotone, it will also be strictly monotone.

To do this, we compute the maximum of two trees recursively, pattern matching on the operands. We use a *view* [?] datatype to identify the cases we are matching on: we are matching on two arguments, which each have three possible constructors, but several cases overlap. Using a view type lets us avoid enumerating all nine possibilities when defining the maximum and proving its properties.

To begin, we parameterize our definition over a function yielding some element for any code's type.

```
module IndMax {ℓ}
  (ℂ : Set ℓ)
  (El : ℂ ⟶ Set ℓ)
  (Cℕ : ℂ) (CℕIso : Iso (El Cℕ) ℕ )
  (default : (c : ℂ) ⟶ El c) where
```

We then define our view type:

```
private
  data IndMaxView : Tree ⟶ Tree ⟶ Set ℓ where
    IndMaxZ-L : ∀ {t} ⟶ IndMaxView Z t
    IndMaxZ-R : ∀ {t} ⟶ IndMaxView t Z
    IndMaxLim-L : ∀ {t} {c : ℂ} {f : El c ⟶ Tree}
      ⟶ IndMaxView (Lim c f) t
    IndMaxLim-R : ∀ {t} {c : ℂ} {f : El c ⟶ Tree}
      ⟶ (∀ {c' : ℂ} {f' : El c' ⟶ Tree} ⟶ ¬ (t ≡ Lim c' f'))
      ⟶ IndMaxView t (Lim c f)
    IndMaxLim-Suc : ∀ {t1 t2 } ⟶ IndMaxView (↑ t1) (↑ t2)
opaque

  indMaxView : ∀ t1 t2 ⟶ IndMaxView t1 t2
```

Our view type has five cases. The first two handle when either input is zero, and the second two handle when either input is a limit. The final case is when both inputs are successors. *indMaxView* computes the view for any pair of trees.

The maximum is then defined by pattern matching on the view for its arguments:

```
indMax : Tree ⟶ Tree ⟶ Tree
indMax' : ∀ {t1 t2} ⟶ IndMaxView t1 t2 ⟶ Tree

indMax t1 t2 = indMax' (indMaxView t1 t2)
indMax' {.Z} {t2} IndMaxZ-L = t2
indMax' {t1} {.Z} IndMaxZ-R = t1
indMax' {(Lim c f)} {t2} IndMaxLim-L
  = Lim c λ x ⟶ indMax (f x) t2
indMax' {t1} {(Lim c f)} (IndMaxLim-R _)
  = Lim c (λ x ⟶ indMax t1 (f x))
indMax' {(↑ t1)} {(↑ t2)} IndMaxLim-Suc = ↑ (indMax t1 t2)
```

The maximum of zero and t is always t, and the maximum of t and the limit of f is the limit of the function computing the maximum between t and f x. Finally, the maximum of two successors is the successor of the two maxima, giving the definitional equality we need for strict monotonicity.

This definition only works when limits of all codes are inhabited. The ≤-limiting constructor means that Lim c f ≤ Z whenever *El c* is uninhabited. So indMax ↑Z Lim c f will not actually be an upper bound for ↑Z if c has no inhabitants. In **??** we show how to circumvent this restriction.

Under the assumption that all code are inhabited, we obtain several of our desired properties for a maximum: it is an upper bound, it is monotone and strictly monotonicity, and it is associative and commutative.

```
opaque
  unfolding indMax indMax'

  indMax-≤L : ∀ {t1 t2} ⟶ t1 ≤ indMax t1 t2
  indMax-≤L {t1} {t2} with indMaxView t1 t2
  ... | IndMaxZ-L = ≤-Z
  ... | IndMaxZ-R = ≤-refl _
  ... | IndMaxLim-L {f = f} = extLim f (λ x ⟶ indMax (f x) t2) (λ k ⟶ in
  ... | IndMaxLim-R {f = f} _ = underLim λ k ⟶ indMax-≤L {t2 = f k}
  ... | IndMaxLim-Suc = ≤-sucMono indMax-≤L

  indMax-≤R : ∀ {t1 t2} ⟶ t2 ≤ indMax t1 t2
  indMax-≤R {t1} {t2} with indMaxView t1 t2
  ... | IndMaxZ-R = ≤-Z
  ... | IndMaxZ-L = ≤-refl _
  ... | IndMaxLim-R {f = f} _ = extLim f (λ x ⟶ indMax t1 (f x)) (λ k ⟶
  ... | IndMaxLim-L {f = f} = underLim λ k ⟶ indMax-≤R
  ... | IndMaxLim-Suc {t1} {t2} = ≤-sucMono (indMax-≤R {t1 = t1} {t2 = t2})


  indMax-monoR : ∀ {t1 t2 t2'} ⟶ t2 ≤ t2' ⟶ indMax t1 t2 ≤ indMax t1 t2'
  indMax-monoR' : ∀ {t1 t2 t2'} ⟶ t2 < t2' ⟶ indMax t1 t2 < indMax (↑ t1)

  indMax-monoR {t1} {t2} {t2'} lt with indMaxView t1 t2 in eq1 | indMaxVi
  ... | IndMaxZ-L | v2 = ≤-trans lt (≤-reflEq (cong indMax' eq2))
  ... | IndMaxZ-R | v2 = ≤-trans indMax-≤L (≤-reflEq (cong indMax' eq2)
```

```
  ... | IndMaxLim-L {f = f1} | IndMaxLim-L = extLim _ _ λ k → indMax-monoR {t1 = f1 k} lt
  indMax-monoR {t1} {(Lim _ f')} {.(Lim _ f)} (≤-cocone f k lt) | IndMaxLim-R neq | IndMaxLim-R neq'
    = ≤-limiting (λ x → indMax t1 (f' x)) (λ y → ≤-cocone (λ x → indMax t1 (f x)) k (indMax-monoR {t1 = t1} {t2 = f' y} (≤-trans (≤-cocon
  indMax-monoR {t1} {.(Lim _ _)} {t2'} (≤-limiting f x1) | IndMaxLim-R x | v2 =
    ≤-trans (≤-limiting (λ x2 → indMax t1 (f x2)) λ k → indMax-monoR {t1 = t1} (x1 k)) (≤-reflEq (cong indMax' eq2))
  indMax-monoR {(↑ t1)} {.(↑ _)} {.(↑ _)} (≤-sucMono lt) | IndMaxLim-Suc | IndMaxLim-Suc = ≤-sucMono (indMax-monoR {t1 = t1} lt)
  indMax-monoR {(↑ t1)} {(↑ t2)} {(Lim _ f)} (≤-cocone f k lt) | IndMaxLim-Suc | IndMaxLim-R x
    = ≤-trans (indMax-monoR' {t1 = t1} {t2 = t2} {t2' = f k} lt) (≤-cocone (λ x1 → indMax (↑ t1) (f x1)) k (≤-refl _)) --indMax-monoR' {!lt!}

  indMax-monoR' {t1} {t2} {t2'} (≤-sucMono lt) = ≤-sucMono ( (indMax-monoR {t1 = t1} lt))
  indMax-monoR' {t1} {t2} {.(Lim _ f)} (≤-cocone f k lt)
    = ≤-cocone _ k (indMax-monoR' {t1 = t1} lt)

  indMax-monoL : ∀ {t1 t1' t2} → t1 ≤ t1' → indMax t1 t2 ≤ indMax t1' t2
  indMax-monoL' : ∀ {t1 t1' t2} → t1 < t1' → indMax t1 t2 < indMax t1' (↑ t2)
  indMax-monoL {t1} {t1'} {t2} lt with indMaxView t1 t2 in eq1 | indMaxView t1' t2 in eq2
  ... | IndMaxZ-L | v2 = ≤-trans (indMax-≤R {t1 = t1'}) (≤-reflEq (cong indMax' eq2))
  ... | IndMaxZ-R | v2 = ≤-trans lt (≤-trans (indMax-≤L {t1 = t1'}) (≤-reflEq (cong indMax' eq2)))
  indMax-monoL {.(Lim _ _)} {.(Lim _ f)} {t2} (≤-cocone f k lt) | IndMaxLim-L | IndMaxLim-L
    = ≤-cocone (λ x → indMax (f x) t2) k (indMax-monoL lt)
  indMax-monoL {.(Lim _ _)} {t1'} {t2} (≤-limiting f lt) | IndMaxLim-L | v2
    = ≤-limiting (λ x1 → indMax (f x1) t2) λ k → ≤-trans (indMax-monoL (lt k)) (≤-reflEq (cong indMax' eq2))
  indMax-monoL {.Z} {.Z} {.(Lim _ _)} ≤-Z | IndMaxLim-R neq | IndMaxZ-L = ≤-refl _
  indMax-monoL {.(Lim _ f)} {.Z} {.(Lim _ _)} (≤-limiting f x) | IndMaxLim-R neq | IndMaxZ-L
    with () ← neq refl
  indMax-monoL {t1} {.(Lim _ _)} {.(Lim _ _)} (≤-cocone _ k lt) | IndMaxLim-R {f = f} neq | IndMaxLim-L {f = f'}
    = ≤-limiting (λ x → indMax t1 (f x)) (λ y → ≤-cocone (λ x → indMax (f' x) (Lim _ _)) k
    (≤-trans (indMax-monoL lt) (indMax-monoR {t1 = f' k} (≤-cocone f y (≤-refl _)))))
  ... | IndMaxLim-R neq | IndMaxLim-R {f = f} neq' = extLim (λ x → indMax t1 (f x)) (λ x → indMax t1' (f x)) (λ k → indMax-monoL lt)
  indMax-monoL {.(↑ _)} {.(↑ _)} {.(↑ _)} (≤-sucMono lt) | IndMaxLim-Suc | IndMaxLim-Suc
    = ≤-sucMono (indMax-monoL lt)
  indMax-monoL {.(↑ _)} {.(Lim _ f)} {.(↑ _)} (≤-cocone f k lt) | IndMaxLim-Suc | IndMaxLim-L
    = ≤-cocone (λ x → indMax (f x) (↑ _)) k (indMax-monoL' lt)

  indMax-monoL' {t1} {t1'} {t2} lt with indMaxView t1 t2 in eq1 | indMaxView t1' t2 in eq2
  indMax-monoL' {t1} {.(↑ _)} {t2} (≤-sucMono lt) | v1 | v2 = ≤-sucMono (≤-trans (≤-reflEq (cong indMax' (sym eq1))) (indMax-monoL lt))
  indMax-monoL' {t1} {.(Lim _ f)} {t2} (≤-cocone f k lt) | v1 | v2
    = ≤-cocone _ k (≤-trans (≤-sucMono (≤-reflEq (cong indMax' (sym eq1)))) (indMax-monoL' lt))


indMax-mono : ∀ {t1 t2 t1' t2'} → t1 ≤ t1' → t2 ≤ t2' → indMax t1 t2 ≤ indMax t1' t2'
indMax-mono {t1' = t1'} lt1 lt2 = ≤-trans (indMax-monoL lt1) (indMax-monoR {t1 = t1'} lt2)

indMax-strictMono : ∀ {t1 t2 t1' t2'} → t1 < t1' → t2 < t2' → indMax t1 t2 < indMax t1' t2'
indMax-strictMono lt1 lt2 = indMax-mono lt1 lt2

indMax-sucMono : ∀ {t1 t2 t1' t2'} → indMax t1 t2 ≤ indMax t1' t2' → indMax t1 t2 < indMax (↑ t1') (↑ t2')
indMax-sucMono lt = ≤-sucMono lt

indMax-Z : ∀ t → indMax t Z ≤ t
indMax-Z Z = ≤-Z
indMax-Z (↑ t) = ≤-refl (indMax (↑ t) Z)
indMax-Z (Lim c f) = extLim (λ x → indMax (f x) Z) f (λ k → indMax-

indMax-↑ : ∀ {t1 t2} → indMax (↑ t1) (↑ t2) ≡ ↑ (indMax t1 t2)
indMax-↑ = refl

indMax-≤Z : ∀ t → indMax t Z ≤ t
indMax-≤Z Z = ≤-refl _
indMax-≤Z (↑ t) = ≤-refl _
indMax-≤Z (Lim c f) = extLim _ _ (λ k → indMax-≤Z (f k))

indMax-limR : ∀ {c : ℂ} (f : El c → Tree) t → indMax t (Lim c f) ≤ Lim c
indMax-limR f Z = ≤-refl _
indMax-limR f (↑ t) = extLim _ _ λ k → ≤-refl _
indMax-limR f (Lim c f1) = ≤-limiting _ λ k → ≤-trans (indMax-limR f (

indMax-commut : ∀ t1 t2 → indMax t1 t2 ≤ indMax t2 t1
indMax-commut t1 t2 with indMaxView t1 t2
... | IndMaxZ-L = indMax-≤L
... | IndMaxZ-R = ≤-refl _
... | IndMaxLim-R {f = f} x = extLim _ _ (λ k → indMax-commut t1 (f k)
... | IndMaxLim-Suc {t1 = t1} {t2 = t2} = ≤-sucMono (indMax-commut t1 t2
... | IndMaxLim-L {c = c} {f = f} with indMaxView t2 t1
... | IndMaxZ-L = extLim _ _ λ k → indMax-Z (f k)
... | IndMaxLim-R x = extLim _ _ (λ k → indMax-commut (f k) t2)
... | IndMaxLim-L {c = c2} {f = f2} =
  ≤-trans (extLim _ _ λ k → indMax-limR f2 (f k))
  (≤-trans (≤-limiting _ (λ k → ≤-limiting _ λ k2 → ≤-cocone _ k2 (≤-coc
  (≤-trans (≤-refl (Lim c2 λ k2 → Lim c λ k → indMax (f k) (f2 k2)))
  (extLim _ _ (λ k2 → ≤-limiting _ λ k1 → ≤-trans (indMax-commut (f

indMax-assocL : ∀ t1 t2 t3 → indMax t1 (indMax t2 t3) ≤ indMax (indMa
indMax-assocL t1 t2 t3 with indMaxView t2 t3 in eq23
... | IndMaxZ-L = indMax-monoL {t1 = t1} {t1' = indMax t1 Z} {t2 = t3} indM
... | IndMaxZ-R = indMax-≤L
... | m with indMaxView t1 t2
... | IndMaxZ-L rewrite sym eq23 = ≤-refl _
... | IndMaxZ-R rewrite sym eq23 = ≤-refl _
... | IndMaxLim-R {f = f} x rewrite sym eq23 = ≤-trans (indMax-limR (λ x
indMax-assocL .(↑ _) .(↑ _) .Z | IndMaxZ-R | IndMaxLim-Suc = ≤-refl _
indMax-assocL t1 t2 .(Lim _ _) | IndMaxLim-R {f = f} x | IndMaxLim-Suc
indMax-assocL (↑ t1) (↑ t2) (↑ t3) | IndMaxLim-Suc | IndMaxLim-Suc = ≤-
... | IndMaxLim-L {f = f} rewrite sym eq23 = extLim _ _ λ k → indMax-a

indMax-assocR : ∀ t1 t2 t3 → indMax (indMax t1 t2) t3 ≤ indMax t1 (indM
indMax-assocR t1 t2 t3 = ≤-trans (indMax-commut (indMax t1 t2) t3) (≤-t
  (≤-trans (indMax-assocL t3 t2 t1)) (≤-trans (indMax-commut (indMax t

indMax-swap4 : ∀ {t1 t1' t2 t2'} → indMax (indMax t1 t1') (indMax t2 t2') ≤ i
indMax-swap4 {t1}{t1'}{t2 }{t2'} =
  indMax-assocL (indMax t1 t1') t2 t2'
  ⨾≤ indMax-monoL {t1 = indMax (indMax t1 t1') t2} {t2 = t2'}
```

```
  (indMax-assocR t1 t1' t2 ⨾≤ indMax-monoR {t1 = t1} (indMax-commut t1' t2) ⨾≤ indMax-assocL t1 t2 t1')
  ⨾≤ indMax-assocR (indMax t1 t2) t1' t2'

indMax-swap6 : ∀ {t1 t2 t3 t1' t2' t3'} → indMax (indMax t1 t1') (indMax (indMax t2 t2') (indMax t3 t3')) ≤ indMax (indMax t1 (indMax t2 t3))
indMax-swap6 {t1} {t2} {t3} {t1'} {t2'} {t3'} =
  indMax-monoR {t1 = indMax t1 t1'} (indMax-swap4 {t1 = t2} {t1' = t2'} {t2 = t3} {t2' = t3'})
  ⨾≤ indMax-swap4 {t1 = t1} {t1' = t1'}

indMax-lim2L :
  ∀
  {c1 : ℂ}
  (f1 : El c1 → Tree)
  {c2 : ℂ}
  (f2 : El c2 → Tree)
  → Lim c1 (λ k1 → Lim c2 (λ k2 → indMax (f1 k1) (f2 k2))) ≤ indMax (Lim c1 f1) (Lim c2 f2)
indMax-lim2L f1 f2 = ≤-limiting _ (λ k1 → ≤-limiting _ λ k2 → indMax-mono (≤-cocone f1 k1 (≤-refl _)) (≤-cocone f2 k2 (≤-refl _)))

indMax-lim2R :
  ∀
  {c1 : ℂ}
  (f1 : El c1 → Tree)
  {c2 : ℂ}
  (f2 : El c2 → Tree)
  → indMax (Lim c1 f1) (Lim    c2 f2)
    ≤ Lim c1 (λ k1 → Lim    c2 (λ k2 → indMax (f1 k1) (f2 k2)))
indMax-lim2R f1 f2 = extLim _ _ (λ k1 → indMax-limR _ (f1 k1))
```

3.2.1 Limitation: Idempotence.

The problem with an inductive definition of the maximum is that we cannot prove that it is idempotent. Since indMax is associative and commutative, proving idempotence is equivalent to proving that it computes a true least-upper-bound.

The difficulty lies in showing that indMax (Lim c f) (Lim c f) $\leq$ (Lim c f). By our definition, indMax (Lim c f) (Lim c f) reduces to

$$(\text{Lim } c\ \lambda x \to (\text{Lim } c\ \lambda y \to \text{indMax } (f\ x)\ (f\ y))) \leq \text{Lim } c\ f$$

We cannot use ≤-cocone to prove this, since the left hand side is not necessarily equal to $f\ k$ for any $k : El\ c$. So the only possibility is to use ≤-limiting. Applying it twice, along with a use of commutatativity of indMax, we are left with the following goal:

$$(\forall x \to (\forall y \to \text{indMax } (f\ x)\ (f\ y))) \leq \text{Lim } c\ f$$

There is no a priori way to prove this goal without already having a proof that indMax is a least upper bound. But proving that was the whole point of proving idempotence! An inductive hypothesis would give that indMax $(f\ x)\ (f\ x) \leq f\ x \leq Lim\ c\ f$, but it does not apply when the arguments to indMax are not equal. Because we are working with constructive ordinals, we have no trichotomy property [?], and hence no guarantee that indMax $(f\ x)\ (f\ y)$ will be one of $f\ x$ and $f\ y$.

We now have two competing definitions for the maximum: the limit version, which is not strictly monotone, and the inductive version, which is not actually a least upper bound. In the next section, we describe a large class of trees for which indMax is idempotent, and hence does compute a true upper bound. We then use that in ?? to create a version of ordinals whose join has the best properties of both limMax and indMax. JE ►TODO recall the algebraic definition of semilattice◄

4 Trees with a Striclty-Monotone Idempotent Join

4.1 Well-Behaved Trees

```
opaque
  unfolding indMax indMax'
  --Attempt to have an idempotent version of indMax

  nindMax : Tree → ℕ → Tree
  nindMax t ℕ.zero = Z
  nindMax t (ℕ.suc n) = indMax (nindMax t n) t

  nindMax-mono : ∀ {t1 t2 } n → t1 ≤ t2 → nindMax t1 n ≤ nindMax t2 n
  nindMax-mono ℕ.zero lt = ≤-Z
  nindMax-mono {t1 = t1} {t2} (ℕ.suc n) lt = indMax-mono {t1 = nindMax

  --

  indMax∞ : Tree → Tree
  indMax∞ t = ℕLim (λ n → nindMax t n)

  indMax-∞lt1 : ∀ t → indMax (indMax∞ t) t ≤ indMax∞ t
  indMax-∞lt1 t = ≤-limiting _ λ k → helper (Iso.fun CℕIso k)
    where
    helper : ∀ n → indMax (nindMax t n) t ≤ indMax∞ t
    helper n = ≤-cocone _ (Iso.inv CℕIso (ℕ.suc n)) (subst (λ sn → nindM

  indMax-∞ltn : ∀ n t → indMax (indMax∞ t) (nindMax t n) ≤ indMax∞
  indMax-∞ltn ℕ.zero t = indMax-≤Z (indMax∞ t)
  indMax-∞ltn (ℕ.suc n) t =
    ≤-trans (indMax-monoR {t1 = indMax∞ t} (indMax-commut (nindMa
    (≤-trans (indMax-assocL (indMax∞ t) t (nindMax t n))
    (≤-trans (indMax-monoL {t1 = indMax (indMax∞ t) t} {t2 = nindMax

  indMax∞-idem : ∀ t → indMax (indMax∞ t) (indMax∞ t) ≤ indMax∞
  indMax∞-idem t = ≤-limiting _ λ k → ≤-trans (indMax-commut (nind

  indMax∞-self : ∀ t → t ≤ indMax∞ t
  indMax∞-self t = ≤-cocone _ (Iso.inv CℕIso 1) (subst (λ x → t ≤ nindM

  indMax∞-idem∞ : ∀ t → indMax t t ≤ indMax∞ t
  indMax∞-idem∞ t = ≤-trans (indMax-mono (indMax∞-self t) (indMax

  indMax∞-mono : ∀ {t1 t2} → t1 ≤ t2 → (indMax∞ t1) ≤ (indMax∞ t2)
  indMax∞-mono lt = extLim _ _ λ k → nindMax-mono (Iso.fun CℕIso
```

```
nindMax-≤ : ∀ {t} n → indMax t t ≤ t → nindMax t n ≤ t
nindMax-≤ ℕ.zero lt = ≤-Z
nindMax-≤ {t = t} (ℕ.suc n) lt = ≤-trans (indMax-monoL {t₁ = nindMax t n} {t₂ = t} (nindMax-≤ n lt)) lt

indMax∞-≤ : ∀ {t} → indMax t t ≤ t → indMax∞ t ≤ t
indMax∞-≤ lt = ≤-limiting _ λ k → nindMax-≤ (Iso.fun CℕIso k) lt

-- Convenient helper for turning < with indMax□ into < without
indMax<-∞ : ∀ {t₁ t₂ t} → indMax (indMax∞ (t₁)) (indMax∞ t₂) < t → indMax t₁ t₂ < t
indMax<-∞ lt = ≤∘<-in-< (indMax-mono (indMax∞-self _) (indMax∞-self _)) lt

indMax-<Ls : ∀ {t₁ t₂ t₁′ t₂′} → indMax t₁ t₂ < indMax (↑ (indMax t₁ t₁′)) (↑ (indMax t₂ t₂′))
indMax-<Ls {t₁} {t₂} {t₁′} {t₂′} = indMax-sucMono {t₁ = t₁} {t₂ = t₂} {t₁′ = indMax t₁ t₁′} {t₂′ = indMax t₂ t₂′}
  (indMax-mono {t₁ = t₁} {t₂ = t₂} (indMax-≤L) (indMax-≤L))

indMax∞-<Ls : ∀ {t₁ t₂ t₁′ t₂′} → indMax t₁ t₂ < indMax (↑ (indMax (indMax∞ t₁) t₁′)) (↑ (indMax (indMax∞ t₂) t₂′))
indMax∞-<Ls {t₁} {t₂} {t₁′} {t₂′} = <∘≤-in-< (indMax-<Ls {t₁} {t₂} {t₁′} {t₂′})
  (indMax-mono {t₁ = ↑ (indMax t₁ t₁′)} {t₂ = ↑ (indMax t₂ t₂′)}
  (≤-sucMono (indMax-monoL (indMax∞-self t₁)))
  (≤-sucMono (indMax-monoL (indMax∞-self t₂))))

indMax∞-lub : ∀ {t₁ t₂ t} → t₁ ≤ indMax∞ t → t₂ ≤ indMax∞ t → indMax t₁ t₂ ≤ (indMax∞ t)
indMax∞-lub {t₁ = t₁} {t₂ = t₂} lt1 lt2 = indMax-mono {t₁ = t₁} {t₂ = t₂} lt1 lt2 ⨾≤ indMax∞-idem _

indMax∞-absorbL : ∀ {t₁ t₂ t} → t₂ ≤ t₁ → t₁ ≤ indMax∞ t → indMax t₁ t₂ ≤ indMax∞ t
indMax∞-absorbL lt12 lt1 = indMax∞-lub lt1 (lt12 ⨾≤ lt1)

indMax∞-distL : ∀ {t₁ t₂} → indMax (indMax∞ t₁) (indMax∞ t₂) ≤ indMax∞ (indMax t₁ t₂)
indMax∞-distL {t₁} {t₂} =
  indMax∞-lub {t₁ = indMax∞ t₁} {t₂ = indMax∞ t₂} (indMax∞-mono indMax-≤L) (indMax∞-mono (indMax-≤R {t₁ = t₁}))

indMax∞-distR : ∀ {t₁ t₂} → indMax∞ (indMax t₁ t₂) ≤ indMax (indMax∞ t₁) (indMax∞ t₂)
indMax∞-distR {t₁} {t₂} = ≤-limiting _ λ k → helper {n = Iso.fun CℕIso k}
  where
  helper : ∀ {t₁ t₂ n} → nindMax (indMax t₁ t₂) n ≤ indMax (indMax∞ t₁) (indMax∞ t₂)
  helper {t₁} {t₂} {ℕ.zero} = ≤-Z
  helper {t₁} {t₂} {ℕ.suc n} =
    indMax-monoL {t₁ = nindMax (indMax t₁ t₂) n} (helper {t₁ = t₁} {t₂} {n})
    ⨾≤ indMax-swap4 {indMax∞ t₁} {indMax∞ t₂} {t₁} {t₂}
    ⨾≤ indMax-mono {t₁ = indMax (indMax∞ t₁) t₁} {t₂ = indMax (indMax∞ t₂) t₂} {t₁′ = indMax∞ t₁} {t₂′ = indMax∞ t₂}
    (indMax∞-lub {t₁ = indMax∞ t₁} (≤-refl _) (indMax∞-self _))
    (indMax∞-lub {t₁ = indMax∞ t₂} (≤-refl _) (indMax∞-self _))

indMax∞-cocone : ∀ {c : ℂ} (f : El c → Tree) k →
  f k ≤ indMax∞ (Lim c f)
indMax∞-cocone f k = indMax∞-self _ ⨾≤ indMax∞-mono (≤-cocone _ k (≤-refl _))
```

5 A Strictly-Monotone, Idempotent Join

```
module Idem {ℓ}
  (ℂ : Set ℓ)
  (El : ℂ → Set ℓ)
  (Cℕ : ℂ) (CℕIso : Iso (El Cℕ) ℕ ) where

module Raw where
  open import RawTree ℂ (λ c → Maybe (El c)) Cℕ (maybeNatIso CℕIs

record Tree : Set ℓ where
  constructor MkTree
  field
    sTree : Raw.Tree
    sIdem : (indMax sTree sTree) Raw.≤ sTree
open Tree

record _≤_ (t₁ t₂ : Tree) : Set ℓ where
  constructor mk≤
  inductive
  field
    get≤ : (sTree t₁) Raw.≤ (sTree t₂)
open _≤_

opaque
  unfolding indMax
  Z : Tree
  Z = MkTree Raw.Z Raw.≤-Z

  ↑ : Tree → Tree
  ↑ (MkTree o pf) = MkTree (Raw.↑ o) (subst (λ x → x Raw.≤ Raw.↑ o) (s

  ≤↑ : ∀ t → t ≤ ↑ t
  ≤↑ t = mk≤ (Raw.≤↑t _)

_<_ : Tree → Tree → Set ℓ
_<_ t₁ t₂ = (↑ t₁) ≤ t₂

opaque
  unfolding indMax Z ↑ indMaxView
  max : Tree → Tree → Tree
  max t₁ t₂ = MkTree (indMax (sTree t₁) (sTree t₂)) (indMax-swap4 Raw.≤
  Lim : ∀     (c : ℂ) → (f : El c → Tree) → Tree
  Lim c f =
    MkTree
    (indMax∞ (Raw.Lim c (maybe′ (λ x → sTree (f x)) Raw.Z)))
    (indMax∞-idem _)
  MkTree (( indMax□)(Lim c (□ x → sTree (f x))) ( indMax□-id

  ≤-Z : ∀ {t} → Z ≤ t
```

```
≤-Z = mk≤ Raw.≤-Z

≤-sucMono : ∀ {t1 t2} → t1 ≤ t2 → ↑ t1 ≤ ↑ t2
≤-sucMono (mk≤ lt) = mk≤ (Raw.≤-sucMono lt)

infixr 10 _≤⨾_
_≤⨾_ : ∀ {t1 t2 t3} → t1 ≤ t2 → t2 ≤ t3 → t1 ≤ t3
_≤⨾_ (mk≤ lt1) (mk≤ lt2) = mk≤ (Raw.≤-trans lt1 lt2)

≤-refl : ∀ {t} → t ≤ t
≤-refl =    mk≤ (Raw.≤-refl _)

≤-limUpperBound : ∀ {c : ℂ} → {f : El c → Tree}
  → ∀ k → f k ≤ Lim c f
≤-limUpperBound {c = c} {f = f} k = mk≤ (Raw.≤-cocone _ (just k) (Raw.≤-reflL _) Raw.≤⨾ indMax∞-self (Raw.Lim c _))

≤-limLeast : ∀ {c : ℂ} → {f : El c → Tree}
  → {t : Tree}
  → (∀ k → f k ≤ t) → Lim c f ≤ t
≤-limLeast {f = f} {t = MkTree o idem} lt
  = mk≤ (
     indMax∞-mono (Raw.≤-limiting _ (maybe (λ k → get≤ (lt k)) Raw.≤-Z))
     Raw.≤ ⨾ (indMax∞-≤ idem) )

≤-extLim : ∀ {c : ℂ} → {f1 f2 : El c → Tree}
  → (∀ k → f1 k ≤ f2 k)
  → Lim c f1 ≤ Lim c f2
≤-extLim lt = ≤-limLeast (λ k → lt k ⨾≤ ≤-limUpperBound k)

≤-extExists : ∀ {c1 c2 : ℂ} → {f1 : El c1 → Tree} {f2 : El c2 → Tree}
  → (∀ k1 → Σ[ k2 ∈ El c2 ] f1 k1 ≤ f2 k2)
  → Lim c1 f1 ≤ Lim c2 f2
≤-extExists {f1 = f1} {f2} lt = ≤-limLeast (λ k1 → proj₂ (lt k1) ⨾≤ ≤-limUpperBound (proj₁ (lt k1)))

--□-limLeast (□ k1 → proj□ (lt k1) □ ⨾ □-limUpperBound (proj□ (lt k1)))

¬Z<↑ : ∀   t → ¬ ((↑ t) ≤ Z)
¬Z<↑ t pf = Raw.¬<Z (sTree t) (get≤ pf)

max-≤L : ∀ {t1 t2} → t1 ≤ max t1 t2
max-≤L = mk≤ indMax-≤L

max-≤R : ∀ {t1 t2} → t2 ≤ max t1 t2
max-≤R = mk≤ indMax-≤R

max-mono : ∀ {t1 t1' t2 t2'} → t1 ≤ t1' → t2 ≤ t2' →
  max t1 t2 ≤ max t1' t2'
max-mono lt1 lt2 = mk≤ (indMax-mono (get≤ lt1) (get≤ lt2))

max-monoR : ∀ {t1 t2 t2'} → t2 ≤ t2' → max t1 t2 ≤ max t1 t2'
max-monoR {t1} {t2} {t2'} lt = max-mono {t1 = t1} {t1' = t1} {t2 = t2} {t2' = t2'} (≤-refl {t1}) lt

max-monoL : ∀ {t1 t1' t2} → t1 ≤ t1' → max t1 t2 ≤ max t1' t2
max-monoL {t1} {t1'} {t2} lt = max-mono {t1} {t1'} {t2} {t2} lt (≤-refl {t2})

max-idem : ∀ {t} → max t t ≤ t
max-idem {t = MkTree o pf} = mk≤ pf

max-idem≤ : ∀ {t} → t ≤ max t t
max-idem≤ {t = MkTree o pf} = max-≤L

max-LUB : ∀ {t1 t2 t} → t1 ≤ t → t2 ≤ t → max t1 t2 ≤ t
max-LUB lt1 lt2 = max-mono lt1 lt2 ⨾≤ max-idem

max-commut : ∀ t1 t2 → max t1 t2 ≤ max t2 t1
max-commut t1 t2 = mk≤ (indMax-commut (sTree t1) (sTree t2))

max-assocL : ∀ t1 t2 t3 → max t1 (max t2 t3) ≤ max (max t1 t2) t3
max-assocL t1 t2 t3 = mk≤ (indMax-assocL _ _ _)

max-assocR : ∀ t1 t2 t3 → max (max t1 t2) t3 ≤ max t1 (max t2 t3)
max-assocR t1 t2 t3 = mk≤ (indMax-assocR _ _ _)

max-swap4 : ∀ {t1 t1' t2 t2'} → max (max t1 t1') (max t2 t2') ≤ max (max t1
max-swap4 = mk≤ indMax-swap4

max-strictMono : ∀ {t1 t1' t2 t2'} → t1 < t1' → t2 < t2' → max t1 t2 < max t
max-strictMono lt1 lt2 = mk≤ (indMax-strictMono (get≤ lt1) (get≤ lt2))

max-sucMono : ∀ {t1 t2 t1' t2'} → max t1 t2 ≤ max t1' t2' → max t1 t2 < ma
max-sucMono lt = mk≤ (indMax-sucMono (get≤ lt))

ℕLim : (ℕ → Tree) → Tree
ℕLim f = Lim Cℕ (λ cn → f (Iso.fun CℕIso cn))

max' : Tree → Tree → Tree
max' t1 t2 = ℕLim (λ n → if0 n t1 t2)

max'-≤L : ∀ {t1 t2} → t1 ≤ max' t1 t2
max'-≤L {t1} {t2}
  = subst (λ x → t1 ≤ if0 x t1 t2) (sym (Iso.rightInv CℕIso 0)) ≤-refl ⨾≤
    ≤-limUpperBound (Iso.inv CℕIso 0)

max'-≤R : ∀ {t1 t2} → t2 ≤ max' t1 t2
max'-≤R {t1} {t2}
  = subst (λ x → t2 ≤ if0 x t1 t2) (sym (Iso.rightInv CℕIso 1)) ≤-refl ⨾≤
    ≤-limUpperBound (Iso.inv CℕIso 1)

max'-Idem : ∀ {t} → max' t t ≤ t
max'-Idem {t} = ≤-limLeast helper
  where
  helper : ∀ k → if0 (Iso.fun CℕIso k) t t ≤ t
  helper k with Iso.fun CℕIso k
  ... | zero = ≤-refl
  ... | suc n = ≤-refl

max'-Mono : ∀ {t1 t2 t1' t2'}
  → t1 ≤ t1' → t2 ≤ t2'
  → max' t1 t2 ≤ max' t1' t2'
max'-Mono {t1} {t2} {t1'} {t2'} lt1 lt2 = ≤-extLim helper
  where
  helper : ∀ k → if0 (Iso.fun CℕIso k) t1 t2 ≤ if0 (Iso.fun CℕIso k) t1' t2'
  helper k with Iso.fun CℕIso k
  ... | zero = lt1
  ... | suc n = lt2
```

```
max'-LUB : ∀ {t₁ t₂ t} → t₁ ≤ t → t₂ ≤ t → max' t₁ t₂ ≤ t
max'-LUB lt1 lt2 = max'-Mono lt1 lt2 ⨾≤ max'-Idem


max≤max' : ∀ {t₁ t₂} → max t₁ t₂ ≤ max' t₁ t₂
max≤max' = max-LUB max'-≤L max'-≤R

max'≤max : ∀ {t₁ t₂} → max' t₁ t₂ ≤ max t₁ t₂
max'≤max = max'-LUB max-≤L max-≤R

limSwap : ∀ {c₁ c₂ } {f : El c₁ → El c₂ → Tree} → (Lim c₁ λ x → Lim c₂ λ y → f x y) ≤ Lim c₂ λ y → Lim c₁ λ x → f x y
limSwap = ≤-limLeast (λ x → ≤-limLeast λ y → ≤-limUpperBound x ⨾≤ ≤-limUpperBound y )

max-swapL : ∀ {c} {f g : El c → Tree} → Lim c (λ k → max (f k) (g k)) ≤ max (Lim c f) (Lim c g)
max-swapL {c} {f} {g} = ≤-extLim (λ k → max≤max') ⨾≤ limSwap ⨾≤ ≤-extLim helper ⨾≤ max'≤max
  where
    helper : (k : El Cℕ) →
      Lim c (λ x → if0 (Iso.fun CℕIso k) (f x) (g x)) ≤
      if0 (Iso.fun CℕIso k) (Lim c f) (Lim c g)
    helper kn with Iso.fun CℕIso kn
    ... | zero = ≤-refl
    ... | suc n = ≤-refl

max-swapR : ∀ {c} {f g : El c → Tree} → max (Lim c f) (Lim c g) ≤ Lim c (λ k → max (f k) (g k))
max-swapR {c} {f} {g} = max≤max' ⨾≤ ≤-extLim helper ⨾≤ limSwap ⨾≤ ≤-extLim (λ k → max'≤max)
  where
    helper : (k : El Cℕ) →
      if0 (Iso.fun CℕIso k) (Lim c f) (Lim c g) ≤
      Lim c (λ z → if0 (Iso.fun CℕIso k) (f z) (g z))
    helper kn with Iso.fun CℕIso kn
    ... | zero = ≤-refl
    ... | suc n = ≤-refl
```

References

Guillaume Allais, Edwin Brady, Nathan Corbyn, Ohad Kammar, and Jeremy Yallop. 2023. Frex: dependently-typed algebraic simplification. arXiv:2306.15375 [cs.PL]

Yves Bertot and Pierre Castéran. 2004. Interactive Theorem Proving and Program Development. Springer-Verlag.

Marc Bezem and Thierry Coquand. 2022. Loop-checking and the uniform word problem for join-semilattices with an inflationary endomorphism. Theoretical Computer Science 913 (2022), 1–7. https://doi.org/10.1016/j.tcs.2022.01.017

Jonathan H.W. Chan. 2022. Sized dependent types via extensional type theory. Master's thesis. University of British Columbia. https://doi.org/10.14288/1.0416401

Nathan Corbyn. 2021. Proof Synthesis with Free Extensions in Intensional Type Theory. Technical Report. University of Cambridge. MEng Dissertation.

Nicolai Kraus, Fredrik Nordvall Forsberg, and Chuangjie Xu. 2023. Type-theoretic approaches to ordinals. Theoretical Computer Science 957 (2023), 113843. https://doi.org/10.1016/j.tcs.2023.113843

The Univalent Foundations Program. 2013. Homotopy Type Theory: Univalent Foundations of Mathematics. https://homotopytypetheory.org/book, Institute for Advanced Study.