

Strictly Monotone Brouwer Trees for Well Founded Recursion Over Multiple Values

Anonymous Author(s)

Abstract

Ordinals can be used to prove the termination of dependently typed programs. Brouwer trees are a particular ordinal notation that make it very easy to assign sizes to higher order data structures. They extend unary natural numbers with a limit constructor, so a function's size can be the least upper bound of the sizes of values from its image. These can then be used to define well founded recursion: any recursive calls are allowed so long as they are on values whose sizes are strictly smaller than the current size.

Unfortunately, Brouwer trees are not algebraically well behaved. They can be characterized equationally as a join-semilattice, where the join takes the maximum of two trees. However, this join does not interact well with the successor constructor, so it does not interact properly with the strict ordering used in well founded recursion.

We present Strictly Monotone Brouwer trees (SMB-trees), a refinement of Brouwer trees that are algebraically well behaved. SMB-trees are built using functions with the same signatures as Brouwer tree constructors, and they satisfy all Brouwer tree inequalities. However, their join operator distributes over the successor, making them suited for well founded recursion or equational reasoning.

This paper teaches how, using dependent pairs and careful definitions, an ill behaved definition can be turned into a well behaved one. Our approach is axiomatically lightweight: it does not rely on Axiom K, univalence, quotient types, or Higher Inductive Types. We implement a recursively-defined maximum operator for Brouwer trees that matches on successors and handles them specifically. Then, we define SMB-trees as the subset of Brouwer trees for which the recursive maximum computes a least upper bound. Finally, we show that every Brouwer tree can be transformed into a corresponding SMB-tree by joining it with itself an infinite number of times. All definitions and theorems are implemented in Agda.

Keywords: dependent types, Brouwer trees, well founded recursion

1 Introduction

1.1 Recursion and Dependent Types

Dependently typed languages, such as Agda [Norell 2009], Coq [Bertot and Castéran 2004], Idris [Brady 2021], and Lean [de Moura et al. 2015], bridge the gap between theorem proving and programming.

Functions defined in dependently typed languages are typically required to be *total*: they must provably halt in all inputs. Since the halting problem is undecidable, recursively-defined functions must be written in such a way that the type checker can mechanically deduce termination. Some functions only make recursive calls to structurally-smaller arguments, so their termination is apparent to the compiler. However, some functions cannot be easily expressed using structural recursion. For such functions, the programmer must instead use *well founded recursion*, showing that there is some ordering, with no infinitely-descending chains, for which each recursive call is strictly smaller according to this ordering. For example, the typical quicksort algorithm is not structurally recursive, but can use well founded recursion on the length of the lists being sorted.

1.2 Ordinals

While numeric orderings work for first-order data, they are ill suited to recursion over higher-order data structures, where some fields contain functions.

There are many formulations of ordinals in dependent type theory, each with their own advantages and disadvantages.

1.3 Contributions

This work defines *strictly monotone Brouwer Trees*, henceforth SMB-trees, a new presentation of ordinals that hit a sort of sweet-spot for defining functions by well founded recursion. Specifically, SMB-trees:

- are strictly ordered by a well founded relation;
- have a maximum operator which computes a least-upper bound;
- are *strictly-monotone* with respect to the maximum: if $a < b$ and $c < d$, then $\max a c < \max b d$;
- can compute the limits of arbitrary sequences;
- are light in axiomatic requirements: they are defined without using axiom K, univalence, quotient types, or higher inductive types.

1.4 Uses for SMB-trees

1.4.1 Well Founded Recursion. Having a maximum operator for ordinals is particularly useful when traversing over multiple higher order data structures in parallel, where neither argument takes priority over the other. In such a case, a lexicographic ordering cannot be used.

As an example, consider a unification algorithm over some encoding of types, and suppose that α -renaming or some

other restriction prevents structural recursion from being used. To solve a unification problem $\Sigma(x : A). B = \Sigma(x : C). D$ we must recursively solve $A = C$ and $\forall x. B[x] = D[x]$. However, the type of x in the latter equation depends on the solution to the first equation, which is bounded by the size of the maximum of the sizes of both A and C . So for each recursive call to be on a smaller size, the size of $a = c$ and $b = d$ must both be strictly smaller than $(a, b) = (c, d)$. In a lexicographic ordering where the size of the left-hand side dominates, we know that a is strictly smaller than (a, b) , but we have no guarantees that $\boxed{\text{JE}} \triangleright \text{TODO} \triangleleft$. Conversely, if we order unification problems by the size of the maximum of their two sides.

This style of well founded induction was used to prove termination in a syntactic model of gradual dependent types [Ermondi 2023]. There, Brouwer trees were used to establish termination of recursive procedures for combining the type information in two imprecise types. The decreasing metric was the maximum size of the codes for the types being combined. Brouwer trees' arbitrary limits were used to assign sizes to dependent function and product types, and the strict monotonicity of the maximum operator was essential for proving that recursive calls were on strictly smaller arguments.

1.4.2 Syntactic Models and Sized Types. An alternate way view of our contribution is as a tool for modelling sized types [Hughes et al. 1996]. The implementation of sized types in Agda has been shown to be unsound [Developers 2017], due to the interaction between propositional equality and the top size ∞ satisfying $\infty < \infty$. Chan [2022] defines a dependently typed language with sized types that does not have a top size, proving it consistent using a syntactic model based on Brouwer trees.

SMB-trees provide the capability to extend existing syntactic models to sized types with a maximum operator. This brings the capability of consistent sized types closer to feature parity with Agda, which has a maximum operator for its sizes, while still maintaining logical consistency.

1.4.3 Algebraic Reasoning. Another advantage of SMB-trees is that they allow Brouwer trees to be interpreted using algebraic tools. SMB-trees can be described as In algebraic terminology, SMB-trees satisfy the following algebraic laws, up to the equivalence relation defined by $s \approx t := s \leq t \leq s$

- Join-semilattice: the binary `max` is associative, commutative, and idempotent
- Bounded: there is a least tree Z such that `max t $Z \approx t$`
- Inflationary endomorphism: there is a successor operator `↑` such that `max (↑ t) $t \approx \uparrow t$` and `↑(max s t) = max(↑ s) (↑ t)`

Bezem and Coquand [2022] describe a polynomial time algorithm for solving equations in such an algebra, and describe its usefulness for solving constraints involving universe levels in dependent type checking. While equations involving limits of infinite sequences are undecidable, the inflationary laws could be used to automatically discharge some equations involving sizes. This algebraic presentation is particularly amenable to solving equations using free extensions of algebras [Allais et al. 2023; Corbyn 2021].

1.5 Implementation

We have implemented SMB-trees in Agda 2.6.4. Our library specifically avoids Agda-specific features such as cubical type theory or Axiom K, so we expect that the library can be easily ported to other proof assistants.

This paper is written as a literate Agda document, and the definitions given in the paper are valid Agda code. Several definitions are presented with their body omitted due to space restrictions. The full implementation can be found in the supplementary materials section of this submission.

2 Brouwer Trees: An Introduction

Brouwer trees are a simple but elegant tool for proving termination of higher-order procedures. Traditionally, they are defined as follows:

```
data SmallTree : Set where
  Z : SmallTree
  ↑ : SmallTree → SmallTree
  Lim : (ℕ → SmallTree) → SmallTree
```

Under this definition, a Brouwer tree is either zero, the successor of another Brouwer tree, or the limit of a countable sequence of Brouwer trees. However, these are quite weak, in that they can only take the limit of countable sequences. To represent the limits of uncountable sequences, we can parameterize our definition over some Universe à la Tarski:

```
module Brouwer {ℓ}
  (C : Set ℓ)
  (El : C → Set ℓ)
  (CN : C) (CNIso : Iso (El CN) ℕ) where
```

Our module is parameterized over a universe level, a type \mathbb{C} of codes, and an “elements-of” interpretation function El , which computes the type represented by each code. We require that there be a code whose interpretation is isomorphic to the natural numbers, as this is essential to our construction in Section 4.1. Increasingly larger trees can be obtained by setting $\mathbb{C} := \text{Set } \ell$ and $El := id$ for increasing ℓ . However, by defining an inductive-recursive universe, one can still capture limits over some non-countable types, since `Tree` is in `Set` whenever \mathbb{C} is.

We then generalize limits to any function whose domain is the interpretation of some code.

```
data Tree : Set ℓ where
  Z : Tree
  ↑ : Tree → Tree
  Lim : ∀ (c : C) → (f : El c → Tree) → Tree
```

The small limit constructor can be recovered from the natural-number code

```
NLim : (N → Tree) → Tree
NLim f = Lim CN (λ cn → f (Iso.fun CNIso cn))
```

Brouwer trees are a the quintessential example of a higher-order inductive type.¹ Each tree is built using smaller trees or functions producing smaller trees, which is essentially a way of storing a possibly infinite number of smaller trees.

2.1 Ordering Trees

Our ultimate goal is to have a well-founded ordering², so we define a relation to order Brouwer trees.

```
data _≤_ : Tree → Tree → Set ℓ where
  ≤-Z : ∀ {t} → Z ≤ t
  ≤-sucMono : ∀ {t1 t2}
    → t1 ≤ t2
    → ↑ t1 ≤ ↑ t2
  ≤-cocone : ∀ {t} {c : C} (f : El c → Tree) (k : El c)
    → t ≤ f k
    → t ≤ Lim c f
  ≤-limiting : ∀ {t} {c : C}
    → (f : El c → Tree)
    → (∀ k → f k ≤ t)
    → Lim c f ≤ t
```

This relation is reflexive:

```
≤-refl : ∀ t → t ≤ t
≤-refl Z = ≤-Z
≤-refl (↑ t) = ≤-sucMono (≤-refl t)
≤-refl (Lim c f)
  = ≤-limiting f (λ k → ≤-cocone f k (≤-refl (f k)))
```

Crucially, it is also transitive, making the relation a pre-order. We modify our the order relation from that of Kraus et al. [2023] so that transitivity can be proven constructively, rather than adding it as a constructor for the relation. This allows us to prove well-foundedness of the relation without needing quotient types or other advanced features.

¹Not to be confused with Higher Inductive Types (HITs) from Homotopy Type Theory [Univalent Foundations Program 2013]

²Technically, this is a well-founded quasi-ordering because there are pairs of trees which are related by both \leq and \geq , but which are not propositionally equal.

```
≤-trans : ∀ {t1 t2 t3} → t1 ≤ t2 → t2 ≤ t3 → t1 ≤ t3
≤-trans ≤-Z p23 = ≤-Z
≤-trans (≤-sucMono p12) (≤-sucMono p23)
  = ≤-sucMono (≤-trans p12 p23)
≤-trans p12 (≤-cocone f k p23)
  = ≤-cocone f k (≤-trans p12 p23)
≤-trans (≤-limiting f x) p23
  = ≤-limiting f (λ k → ≤-trans (x k) p23)
≤-trans (≤-cocone f k p12) (≤-limiting .f x)
  = ≤-trans p12 (x k)
```

We create an infix version of transitivity for more readable construction of proofs:

```
_≤%_ : ∀ {t1 t2 t3} → t1 ≤ t2 → t2 ≤ t3 → t1 ≤ t3
lt1 ≤% lt2 = ≤-trans lt1 lt2
```

2.1.1 Strict Ordering. We can define a strictly-less-than relation in terms of our less-than relation and the successor constructor:

```
_<_ : Tree → Tree → Set ℓ
t1 < t2 = ↑ t1 ≤ t2
```

That is, a t_1 is strictly smaller than t_2 if the tree one-size larger than t_1 is as small as t_2 . This relation has the properties one expects of a strictly-less-than relation: it is a transitive sub-relation of the less-than relation, every tree is strictly less than its successor, and no tree is strictly smaller than zero. JE ►TODO more?◀

```
≤↑t : ∀ t → t ≤ ↑ t
≤↑t Z = ≤-Z
≤↑t (↑ t) = ≤-sucMono (≤↑t t)
≤↑t (Lim c f)
  = ≤-limiting f λ k →
    (≤↑t (f k))
  ≤% (≤-sucMono (≤-cocone f k (≤-refl (f k))))
```

```
<-in-≤ : ∀ {x y} → x < y → x ≤ y
<-in-≤ pf = (≤↑t _) ≤% pf
<≤-in-≤ : ∀ {x y z} → x < y → y ≤ z → x < z
<≤-in-≤ x<y y≤z = x<y ≤% y≤z
≤≤-in-≤ : ∀ {x y z} → x ≤ y → y < z → x < z
≤≤-in-≤ {x} {y} {z} x≤y y<z = (≤-sucMono x≤y) ≤% y<z
¬<Z : ∀ t → ¬(t < Z)
¬<Z t ()
```

2.2 Well Founded Induction

Recall the definition of a constructive well founded relation:

```
data Acc {A : Set a} (ℓ : A → A → Set ℓ) (x : A) : Set (a ⊔ ℓ) where
  acc : (rs : ∀ y → y < x → Acc _≤_ y) → Acc _≤_ x
```

WellFounded : $(A \rightarrow A \rightarrow \text{Set } \ell) \rightarrow \text{Set } _$
WellFounded $_ \leq _ = \forall x \rightarrow \text{Acc } _ \leq x$

That is, an element of a type is accessible for a relation if all strictly smaller elements of it are also accessible. A relation is well founded if all values are accessible with respect to that relation. This can then be used to define induction with arbitrary recursive calls on smaller values:

wfRec : $(P : A \rightarrow \text{Set } \ell)$
 $\rightarrow (\forall x \rightarrow ((y : A) \rightarrow y < x \rightarrow P y) \rightarrow P x)$
 $\rightarrow \forall x \rightarrow P x$

Following the construction of Kraus et al. [2023], we can show that the strict ordering on Brouwer trees is well founded. First, we prove a helper lemma: if a value is accessible, then all (not necessarily strictly) smaller terms are also accessible.

smaller-accessible : $(x : \text{Tree})$
 $\rightarrow \text{Acc } _ \leq x \rightarrow \forall y \rightarrow y \leq x \rightarrow \text{Acc } _ \leq y$
smaller-accessible $x (\text{acc } r) y x < y$
 $= \text{acc } (\lambda y' y' < y \rightarrow r y' (< \leq \text{-in-} < y' < y x < y))$

Then we use structural reduction to show that all terms are accessible. The key observations are that zero is trivially accessible, since no trees are strictly smaller than it, and that the only way to derive $\uparrow t \leq (\text{Lim } c f)$ is with $\leq\text{-cocone}$, yielding a concrete index k for which $\uparrow t \leq f k$, on which we can recur.

ordWF : **WellFounded** $_ \leq _$
ordWF $Z = \text{acc } \lambda _ ()$
ordWF $(\uparrow x)$
 $= \text{acc } (\lambda \{ y (\leq\text{-sucMono } y \leq x) \}$
 $\rightarrow \text{smaller-accessible } x (\text{ordWF } x) y y \leq x \})$
ordWF $(\text{Lim } c f) = \text{acc helper}$
where
helper : $(y : \text{Tree}) \rightarrow (y < \text{Lim } c f)$
 $\rightarrow \text{Acc } _ \leq y$
helper $y (\leq\text{-cocone } f k y < f k)$
 $= \text{smaller-accessible } (f k)$
 $(\text{ordWF } (f k)) y (< \text{-in-} \leq y < f k)$

3 First Attempts at a Join

In this section, we present two faulty implementations of a join operator for trees. The first uses limits to define the join, but does not satisfy strict monotonicity. The second is defined inductively. It satisfies strict monotonicity, but fails to be the least of all upper bounds, and requires us to assume that limits are only taken over non-empty types. In Section 4, we define SMB-trees a refinement of Brouwer trees that combines the benefits of both versions of the maximum.

3.1 Limit-based Maximum

Since the limit constructor finds the least upper bound of the image of a function, it should be possible to define the maximum of two trees as a special case of general limits. Indeed, we can compute the maximum of t_1 and t_2 as the limit of the function that produces t_1 when given 0 and t_2 otherwise.

limMax : $\text{Tree} \rightarrow \text{Tree} \rightarrow \text{Tree}$
limMax $t_1 t_2 = \text{INLim } \lambda n \rightarrow \text{if0 } n t_1 t_2$

This version of the maximum has several of the properties we want from a maximum function: it is monotone, idempotent, commutative, and is a true least-upper-bound of its inputs.

limMax $\leq L : \forall \{t_1 t_2\} \rightarrow t_1 \leq \text{limMax } t_1 t_2$
limMax $\leq L \{t_1\} \{t_2\}$
 $= \leq\text{-cocone } _ (\text{Iso.inv } \text{CNIso } 0)$
 $(\text{subst } (\lambda x \rightarrow t_1 \leq \text{if0 } x t_1 t_2))$
 $(\text{sym } (\text{Iso.rightInv } \text{CNIso } 0))$
 $(\leq\text{-refl } t_1))$

limMax $\leq R : \forall \{t_1 t_2\} \rightarrow t_2 \leq \text{limMax } t_1 t_2$
-- Symmetric

limMaxIdem : $\forall \{t\} \rightarrow \text{limMax } t t \leq t$
limMaxIdem $\{t\} = \leq\text{-limiting } _ \text{helper}$
where
helper : $\forall k \rightarrow \text{if0 } (\text{Iso.fun } \text{CNIso } k) t t \leq t$
helper $k \text{ with Iso.fun CNIso } k$
 $\dots \mid \text{zero} = \leq\text{-refl } t$
 $\dots \mid \text{suc } n = \leq\text{-refl } t$

JE ► **TODO update description** ◄ From these properties, we can compute several other useful properties: monotonicity, commutativity, and that it is in fact the least of all upper bounds.

limMaxMono : $\forall \{t_1 t_2 t'_1 t'_2\}$
 $\rightarrow t_1 \leq t'_1 \rightarrow t_2 \leq t'_2$
 $\rightarrow \text{limMax } t_1 t_2 \leq \text{limMax } t'_1 t'_2$

limMaxCommut : $\forall \{t_1 t_2\} \rightarrow \text{limMax } t_1 t_2 \leq \text{limMax } t_2 t_1$

limMaxLUB : $\forall \{t_1 t_2 t\} \rightarrow t_1 \leq t \rightarrow t_2 \leq t \rightarrow \text{limMax } t_1 t_2 \leq t$

It is not surprising that this version of the maximum is a least upper bound: by definition **Lim** computes the least upper bound of a function's image, and **limMax** is simply **Lim** applied to a function whose image has (at most) two elements.

3.1.1 Limitation: Strict Monotonicity. The one crucial property that this formulation lacks is that it is not strictly monotone: we cannot deduce $\text{max } t_1 t_1 < \text{max } t'_1 t'_2$ from

$t_1 < t'_1$ and $t_2 < t'_2$. This is because the only way to construct a proof that $\uparrow t \leq \text{Lim } c f$ is using the $\leq\text{-cocone}$ constructor. So we would need to prove that $\uparrow(\max t_1 t_2) \leq t'_1$ or that $\uparrow(\max t_1 t_2) \leq t'_2$, which cannot be deduced from the premises alone. What we want is to have $\uparrow \max (t_1) t_2 \leq \max(\uparrow t_1) (\uparrow t_2)$, so that strict monotonicity is a direct consequence of ordinary monotonicity of the maximum. This is not possible when defining the constructor as a limit.

3.2 Recursive Maximum

In our next attempt at defining a maximum operator, we obtain strict monotonicity by making $\text{indMax } (\uparrow t_1) (\uparrow t_2) = \uparrow(\text{indMax } t_1 t_2)$ hold definitionally. Then, provided indMax is monotone, it will also be strictly monotone.

To do this, we compute the maximum of two trees recursively, pattern matching on the operands. We use a *view* [MCBRIDE and MCKINNA 2004] datatype to identify the cases we are matching on: we are matching on two arguments, which each have three possible constructors, but several cases overlap. Using a view type lets us avoid enumerating all nine possibilities when defining the maximum and proving its properties.

To begin, we parameterize our definition over a function yielding some element for any code's type.

```
module IndMax {ℓ}
  (C : Set ℓ)
  (El : C → Set ℓ)
  (CN : C) (CNIso : Iso (El CN) ℕ)
  (default : (c : C) → El c) where

  We then define our view type:

private
  data IndMaxView : Tree → Tree → Set ℓ where
    IndMaxZ-L : ∀ {t} → IndMaxView Z t
    IndMaxZ-R : ∀ {t} → IndMaxView t Z
    IndMaxLim-L : ∀ {t} {c : C} {f : El c → Tree}
      → IndMaxView (Lim c f) t
    IndMaxLim-R : ∀ {t} {c : C} {f : El c → Tree}
      → (∀ {c' : C} {f' : El c' → Tree} → ¬(t = Lim c' f'))
      → IndMaxView t (Lim c f)
    IndMaxLim-Suc : ∀ {t1 t2} → IndMaxView (↑ t1) (↑ t2)
```

opaque

```
indMaxView : ∀ t1 t2 → IndMaxView t1 t2
```

Our view type has five cases. The first two handle when either input is zero, and the second two handle when either input is a limit. The final case is when both inputs are successors. indMaxView computes the view for any pair of trees.

The maximum is then defined by pattern matching on the view for its arguments:

```
indMax : Tree → Tree → Tree
indMax' : ∀ {t1 t2} → IndMaxView t1 t2 → Tree
```

```
indMax t1 t2 = indMax' (indMaxView t1 t2)
indMax' {Z} {t2} IndMaxZ-L = t2
indMax' {t1} {Z} IndMaxZ-R = t1
indMax' {(Lim c f)} {t2} IndMaxLim-L
  = Lim c λ x → indMax (f x) t2
indMax' {t1} {(Lim c f)} (IndMaxLim-R _)
  = Lim c (λ x → indMax t1 (f x))
indMax' {(↑ t1)} {(↑ t2)} IndMaxLim-Suc = ↑ (indMax t1 t2)
```

The maximum of zero and t is always t , and the maximum of t and the limit of f is the limit of the function computing the maximum between t and $f x$. Finally, the maximum of two successors is the successor of the two maxima, giving the definitional equality we need for strict monotonicity.

This definition only works when limits of all codes are inhabited. The $\leq\text{-limiting}$ constructor means that $\text{Lim } c f \leq Z$ whenever $El c$ is uninhabited. So $\text{indMax } \uparrow Z \text{ Lim } c f$ will not actually be an upper bound for $\uparrow Z$ if c has no inhabitants. In Section 4.2 we show how to circumvent this restriction.

Under the assumption that all code are inhabited, we obtain several of our desired properties for a maximum: it is an upper bound, it is monotone and strictly monotonicity, and it is associative and commutative.

opaque

```
unfolding indMax indMax'
```

```
indMax-≤L : ∀ {t1 t2} → t1 ≤ indMax t1 t2
indMax-≤L {t1} {t2} with indMaxView t1 t2
... | IndMaxZ-L = ≤-Z
... | IndMaxZ-R = ≤-refl _
... | IndMaxLim-L {f = f}
  = extLim f (λ x → indMax (f x) t2) (λ k → indMax-≤L)
... | IndMaxLim-R {f = f} _
  = underLim λ k → indMax-≤L {t2 = f k}
... | IndMaxLim-Suc
  = ≤-sucMono indMax-≤L
```

```
indMax-≤R : ∀ {t1 t2} → t2 ≤ indMax t1 t2
```

```
-- Symmetric
```

```
indMax-monoL : ∀ {t1 t'1 t2}
  → t1 ≤ t'1 → indMax t1 t2 ≤ indMax t'1 t2
```

```
indMax-monoR : ∀ {t1 t2 t'2}
  → t2 ≤ t'2 → indMax t1 t2 ≤ indMax t1 t'2
```

```
indMax-mono : ∀ {t1 t2 t'1 t'2}
  → t1 ≤ t'1 → t2 ≤ t'2 → indMax t1 t2 ≤ indMax t'1 t'2
```

```
indMax-strictMono : ∀ {t1 t2 t'1 t'2}
  → t1 < t'1 → t2 < t'2 → indMax t1 t2 < indMax t'1 t'2
```

```
indMax-strictMono lt1 lt2 = indMax-mono lt1 lt2
```

$\text{indMax-assocL} : \forall t_1 t_2 t_3$
 $\rightarrow \text{indMax } t_1 (\text{indMax } t_2 t_3) \leq \text{indMax } (\text{indMax } t_1 t_2) t_3$
 $\text{indMax-assocR} : \forall t_1 t_2 t_3$
 $\rightarrow \text{indMax } (\text{indMax } t_1 t_2) t_3 \leq \text{indMax } t_1 (\text{indMax } t_2 t_3)$
 $\text{indMax-commut} : \forall t_1 t_2$
 $\rightarrow \text{indMax } t_1 t_2 \leq \text{indMax } t_2 t_1$

3.2.1 Limitation: Idempotence. The problem with an inductive definition of the maximum is that we cannot prove that it is idempotent. Since indMax is associative and commutative, proving idempotence is equivalent to proving that it computes a true least-upper-bound.

The difficulty lies in showing that $\text{indMax } (\text{Lim } c f) (\text{Lim } c f) \leq (\text{Lim } c f)$. By our definition, $\text{indMax } (\text{Lim } c f) (\text{Lim } c f)$ reduces to

$(\text{Lim } c \lambda x \rightarrow (\text{Lim } c \lambda y \rightarrow \text{indMax } (f x) (f y))) \leq \text{Lim } c f$

We cannot use $\leq\text{-cocone}$ to prove this, since the left hand side is not necessarily equal to $f k$ for any $k : El c$. So the only possibility is to use $\leq\text{-limiting}$. Applying it twice, along with a use of commutativity of indMax , we are left with the following goal:

$(\forall x \rightarrow (\forall y \rightarrow \text{indMax } (f x) (f y))) \leq \text{Lim } c f$

There is no a priori way to prove this goal without already having a proof that indMax is a least upper bound. But proving that was the whole point of proving idempotence! An inductive hypothesis would give that $\text{indMax } (f x) (f x) \leq f x \leq \text{Lim } c f$, but it does not apply when the arguments to indMax are not equal. Because we are working with constructive ordinals, we have no trichotomy property [Kraus et al. 2023], and hence no guarantee that $\text{indMax } (f x) (f y)$ will be one of $f x$ and $f y$.

We now have two competing definitions for the maximum: the limit version, which is not strictly monotone, and the inductive version, which is not actually a least upper bound. In the next section, we describe a large class of trees for which indMax is idempotent, and hence does compute a true upper bound. We then use that in Section 4.2 to create a version of ordinals whose join has the best properties of both limMax and indMax . JE ▶ TODO recall the algebraic definition of semilattice ◀

4 Trees with a Strictly-Monotone Idempotent Join

4.1 Well-Behaved Trees

Our first step in defining an ordinal notation with a well behaved maximum is to identify a class of Brouwer trees which are well behaved with respect to the inductive maximum. As we saw in

The answer, it turns out, is more limits: if we indMax a term with itself an infinite number of times, the result will be idempotent with respect to indMax . First, we define a

function to indMax a term with itself n times or a given number n :

$\text{nindMax} : \text{Tree} \rightarrow \mathbb{N} \rightarrow \text{Tree}$
 $\text{nindMax } t \ \mathbb{N}.\text{zero} = Z$
 $\text{nindMax } t (\mathbb{N}.\text{suc } n) = \text{indMax } (\text{nindMax } t n) t$

To compute a tree equivalent to the infinite chain of applications $\text{indMax } t (\text{indMax } t (\text{indMax } t \dots))$, we take the limit of n applications over all n :

$\text{indMax}^\infty : \text{Tree} \rightarrow \text{Tree}$
 $\text{indMax}^\infty t = \mathbb{N}\text{Lim } (\lambda n \rightarrow \text{nindMax } t n)$

This operator has useful basic properties: it is monotone, and it computes an upper bound on its argument.

$\text{indMax}^\infty\text{-self} : \forall t \rightarrow t \leq \text{indMax}^\infty t$

$\text{indMax}^\infty\text{-mono} : \forall \{t_1 t_2\}$

$\rightarrow t_1 \leq t_2$

$\rightarrow (\text{indMax}^\infty t_1) \leq (\text{indMax}^\infty t_2)$

However, the most important property we want from indMax^∞ is that indMax is idempotent with respect to it. The first step to showing this is realizing that we can take the maximum of t and $\text{indMax}^\infty t$ and we have a tree that is no larger than $\text{indMax}^\infty t$: because it is already an infinite chain of applications, adding one more makes no difference.

$\text{indMax-}\infty\text{lt1} : \forall t \rightarrow \text{indMax } (\text{indMax}^\infty t) t \leq \text{indMax}^\infty t$
 $\text{indMax-}\infty\text{lt1 } t = \leq\text{-limiting } _ \lambda k \rightarrow \text{helper } (\text{Iso.fun } \text{CNIso } k)$

where

$\text{helper} : \forall n \rightarrow \text{indMax } (\text{nindMax } t n) t \leq \text{indMax}^\infty t$

$\text{helper } n =$

$\leq\text{-cocone } _ (\text{Iso.inv } \text{CNIso } (\mathbb{N}.\text{suc } n))$

$(\text{subst } (\lambda sn \rightarrow \text{nindMax } t (\mathbb{N}.\text{suc } n) \leq \text{nindMax } t sn)$

$(\text{sym } (\text{Iso.rightInv } \text{CNIso } (\text{suc } n)))$

$(\leq\text{-refl } _))$

If adding one more $\text{indMax } t$ has no effect, then adding n more will also have no effect:

$\text{indMax-}\infty\text{ltN} : \forall n t$
 $\rightarrow \text{indMax } (\text{indMax}^\infty t) (\text{nindMax } t n) \leq \text{indMax}^\infty t$
 $\text{indMax-}\infty\text{ltN } \mathbb{N}.\text{zero } t = \text{indMax-}\leq Z (\text{indMax}^\infty t)$
 $\text{indMax-}\infty\text{ltN } (\mathbb{N}.\text{suc } n) t =$
 $\text{indMax-monoR } (\text{indMax-commut } (\text{nindMax } t n) t)$
 $\leq \S \text{indMax-assocL } (\text{indMax}^\infty t) t (\text{nindMax } t n)$
 $\leq \S \text{indMax-monoL } (\text{indMax-}\infty\text{lt1 } t)$
 $\leq \S \text{indMax-}\infty\text{ltN } n t$

By our inductive definition of indMax , we have that

$\text{indMax } (\text{indMax}^\infty t) (\text{indMax}^\infty t)$

is equal to

$\mathbb{N}\text{Lim } (\lambda n \rightarrow \text{indMax } (\text{nindMax } n t) (\text{indMax}^\infty t))$

Our previous lemma gives that, for any n , $\text{indMax}^\infty t$ is an upper bound for $\text{indMax} (\text{nIndMax } n \ t) (\text{indMax}^\infty t)$. So $\leq\text{-limiting}$ gives that the limit over all n is also bounded by $\text{indMax}^\infty t$, i.e. Lim constructs the least of all upper bounds. This gives us our key result: up to \leq , indMax is idempotent on values constructed with indMax^∞ .

```
indMax∞-idem : ∀ t
  → indMax (indMax∞ t) (indMax∞ t) ≤ indMax∞ t
indMax∞-idem t =
  ≤-limiting _ λ k →
    (indMax-commut (nindMax t (Iso.fun CNIso k)) (indMax∞ t))
  ≤ § indMax∞lt (Iso.fun CNIso k) t
```

There is one last property to prove that will be useful in the next section: $\text{indMax}^\infty t$ is a lower bound on t , and hence equivalent to it, whenever indMax is idempotent on t . If taking indMax of t with itself does not increase its size, doing so n times will not increase its size, so again the result follows from Lim being the least upper bound.

```
indMax∞-≤ : ∀ {t} → indMax t t ≤ t → indMax∞ t ≤ t
indMax∞-≤ lt
  = ≤-limiting _
    λ k → nindMax-≤ (Iso.fun CNIso k) lt
where
  nindMax-≤ : ∀ {t} n → indMax t t ≤ t → nindMax t n ≤ t
  nindMax-≤ N.zero lt = ≤-Z
  nindMax-≤ {t = t} (N.suc n) lt
    = indMax-monoL (nindMax-≤ n lt)
    ≤ § lt
```

An immediate corollary of this is that $\text{indMax}^\infty (\text{indMax}^\infty t)$ is equivalent to $\text{indMax}^\infty t$.

4.2 Strictly Monotone Brouwer Trees

Now that we have identified a substantial class of well behaved Brouwer trees, we want to define a new type containing only those trees. These are SMB-trees: strictly monotone Brouwer trees. In this section, we will define them, and show how they can be given a similar interface to Brouwer trees.

To begin, we declare a new Agda module, with the same parameters we have been working with thus far: a type of codes, interpretations of those codes into types, and a code whose interpretation is isomorphic to \mathbb{N} .

```
module SMBTree {ℓ}
  (C : Set ℓ)
  (El : C → Set ℓ)
  (CN : C) (CNIso : Iso (El CN) N) where
```

We import all of our definitions so far, using the “Brouwer” prefix to distinguish them from the trees and ordering we are about to define. Critically, we do not instantiate these with the same interpretation function. Instead, we interpret each code wrapped in Maybe . This ensures that we always

take Brouwer limits over non-empty sets, an assumption that was critical for the definitions of Section 3.2. However, we place no such restriction on SMB-trees.

```
import Brouwer
C
(λ c → Maybe (El c))
CN (maybeNatIso CNIso) as Brouwer
```

4.2.1 Refining Brouwer Trees. We define SMB-trees as a dependent record, containing an underlying Brouwer tree, and a proof that indMax is idempotent on this tree.

```
record SMBTree : Set ℓ where
  constructor MkTree
  field
    rawTree : Brouwer.Tree
    isIdem : (indMax rawTree rawTree) Brouwer.≤ rawTree
open SMBTree
```

We can then define so-called “smart-constructors” corresponding to each of the constructors for Brouwer-trees: zero, successor, and limit. Zero and successor directly correspond to the Brouwer tree zero and successor. Their proofs of idempotence are trivial from the properties of Brouwer \leq .

```
opaque
  unfolding indMax
  Z : SMBTree
  Z = MkTree Brouwer.Z Brouwer.≤-Z
  ↑ : SMBTree → SMBTree
  ↑ (MkTree t pf) = MkTree (Brouwer.↑ t) (Brouwer.≤-sucMono pf)
```

However, constructing the limit of a sequence of SMB-trees is not so easy. Since we instantiated El to wrap its result in Maybe , we need to handle nothing for each limit, but we can use Z as a default value, since adding it to any sequence does not change the least upper bound. More challenging is how, as we saw in Section 3.2, Brouwer trees do not have $\text{indMax} (\text{Lim } c \ f) (\text{Lim } c \ f) \leq \text{Lim } c \ f$, so we cannot directly produce a proof of idempotence.

Our key insight is to define limits of SMB-trees using indMax^∞ on the underlying trees: for any function producing SMB-trees, we take the limit of the underlying trees, then indMax that result with itself an infinite number of times. The idempotence proof is then the property of indMax^∞ that we proved in Section 4.1.

```
Lim : ∀ (c : C) → (f : El c → SMBTree) → SMBTree
Lim c f =
  MkTree
    (indMax∞ (Brouwer.Lim c (maybe' (λ x → rawTree (f x)) Brouwer.Z)))
    (indMax∞-idem _)
```

4.2.2 Ordering SMB-trees. SMB-trees are ordered by the order on their underlying Brouwer trees:

```
record _≤_ (t1 t2 : SMBTree) : Set ℓ where
  constructor mk≤
  inductive
  field
  get≤ : (rawTree t1) Brouwer.≤ (rawTree t2)
open _≤_
```

Having a successor function allows us to define a strict ordering on SMB-trees.

```
_<_ : SMBTree → SMBTree → Set ℓ
_<_ t1 t2 = (↑ t1) ≤ t2
```

The next step is to prove that our SMB-tree constructors satisfy the same inequalities as Brouwer trees. Since SMB-trees are ordered by their underlying Brouwer trees, most properties can be directly lifted from Brouwer trees to SMB-trees.

```
opaque
  unfolding Z ↑
  ≤↑ : ∀ t → t ≤ ↑ t
  ≤↑ t = mk≤ (Brouwer.≤↑ t _)

  _≤_ : ∀ {t1 t2 t3} → t1 ≤ t2 → t2 ≤ t3 → t1 ≤ t3
  _≤_ (mk≤ lt1) (mk≤ lt2) = mk≤ (Brouwer.≤-trans lt1 lt2)

  ≤-refl : ∀ {t} → t ≤ t
  ≤-refl = mk≤ (Brouwer.≤-refl _)
```

The constructors for \leq each have a counterpart for SMB-trees. For zero and successor, these are trivially lifted.

```
≤-Z : ∀ {t} → Z ≤ t
≤-Z = mk≤ Brouwer.≤-Z

≤-sucMono : ∀ {t1 t2} → t1 ≤ t2 → ↑ t1 ≤ ↑ t2
≤-sucMono (mk≤ lt) = mk≤ (Brouwer.≤-sucMono lt)
```

The constructors for ordering limits require more attention. To show that an SMB-tree limit is an upper-bound, we use the fact that the underlying limit was an upper bound, and the fact that indMax is an upper bound, since the SMB-tree Lim wraps its result in indMax . Note that, since we already have transitivity for our new \leq , we can simply show that $f\ k$ is less than the limit of f , avoiding the more complicated form of $\leq\text{-cocone}$.

```
≤-limUpperBound : ∀ {c : C} → {f : El c → SMBTree}
  → ∀ k → f k ≤ Lim c f
≤-limUpperBound {c = c} {f = f} k
  = mk≤ (Brouwer.≤-cocone _ (just k) (Brouwer.≤-refl _))
  Brouwer.≤ ; indMax∞-self (Brouwer.Lim c _)
```

Finally, we need to show that the SMT-tree limit is less than all other upper bounds. Suppose $t : \text{SMBTree}$ is an upper bound for f , and t_u is the underlying tree for t , and f_u computes the underlying trees for f . Then $\leq\text{-limiting}$ gives that the underlying tree for t is an upper bound for the trees underlying the image of f . However, the SMB-tree limit wraps its result in indMax . The monotonicity of indMax then gives that $\text{indMax}(\text{Lim } c\ f_u)$ is less than $\text{indMax } t'$. In Section 4.1, we showed that indMax had no effect on Brouwer trees that indMax was idempotent on. This is exactly what the isIdem field of SMB-trees contains! So we have $\text{indMax } t' \leq t'$, and transitivity gives our result.

```
≤-limLeast : ∀ {c : C} → {f : El c → SMBTree}
  → {t : SMBTree}
  → (∀ k → f k ≤ t) → Lim c f ≤ t
≤-limLeast {f = f} {t = MkTree t idem} lt
  = mk≤ (
    indMax∞-mono
    (Brouwer.≤-limiting _
     (maybe (λ k → get≤ (lt k)) Brouwer.≤-Z))
    Brouwer.≤ ; (indMax∞-self idem) )
```

4.2.3 The Join for SMB-trees. Our whole reason for defining SMB-trees was to define a well-behaved maximum operator, and we finally have the tools to do so. We can define the join in terms of indMax on the underlying trees. The proof that the indMax is idempotent on the result follows from associativity, commutativity, and monotonicity of indMax .

```
opaque
  unfolding indMax Z ↑ indMaxView
  max : SMBTree → SMBTree → SMBTree
  max t1 t2 =
    MkTree
    (indMax (rawTree t1) (rawTree t2))
    (indMax-swap4
     Brouwer.≤ ; indMax-mono (isIdem t1) (isIdem t2))
```

For Brouwer trees, indMax had all the properties we wanted except for idempotence. All of these can be lifted directly to SMB-trees:

```
max-≤L : ∀ {t1 t2} → t1 ≤ max t1 t2
max-≤R : ∀ {t1 t2} → t2 ≤ max t1 t2
max-mono : ∀ {t1 t1' t2 t2'} → t1 ≤ t1' → t2 ≤ t2' →
  max t1 t2 ≤ max t1' t2'
max-idem≤ : ∀ {t} → t ≤ max t t
max-commut : ∀ t1 t2 → max t1 t2 ≤ max t2 t1
max-assocL : ∀ t1 t2 t3 → max t1 (max t2 t3) ≤ max (max t1 t2) t3
max-assocR : ∀ t1 t2 t3 → max (max t1 t2) t3 ≤ max t1 (max t2 t3)
```


In particular, `max` is strictly monotone, and distributes over the successor:

`max-strictMono` : $\forall \{t_1 \ t'_1 \ t_2 \ t'_2 : \text{SMBTree}\}$
 $\rightarrow t_1 < t'_1 \rightarrow t_2 < t'_2 \rightarrow \text{max } t_1 \ t_2 < \text{max } t'_1 \ t'_2$

`max-sucMono` : $\forall \{t_1 \ t_2 \ t'_1 \ t'_2 : \text{SMBTree}\}$
 $\rightarrow \text{max } t_1 \ t_2 \leq \text{max } t'_1 \ t'_2 \rightarrow \text{max } t_1 \ t_2 < \text{max } (\uparrow t'_1) (\uparrow t'_2)$

However, because we restricted SMB-trees to only contain Brouwer trees that `indMax` is idempotent on, we can prove that `Max` is idempotent for SMB-trees:

`max-idem` : $\forall \{t : \text{SMBTree}\} \rightarrow \text{max } t \ t \leq t$
`max-idem` $\{t = \text{MkTree } t \ pf\} = \text{mk} \leq pf$

These together are enough to prove that our maximum is the least of all upper bounds.

`max-LUB` : $\forall \{t_1 \ t_2 \ t\} \rightarrow t_1 \leq t \rightarrow t_2 \leq t \rightarrow \text{max } t_1 \ t_2 \leq t$
`max-LUB` $lt1 \ lt2 = \text{max-mono } lt1 \ lt2 \leq ; \text{max-idem}$

Perhaps surprisingly, this means that an SMB-tree version of `limMax` is equivalent to `max`, since they are both the least upper bound:

`INLim` : $(\mathbb{N} \rightarrow \text{SMBTree}) \rightarrow \text{SMBTree}$
`INLim` $f = \text{Lim } \text{CN } (\lambda \ cn \rightarrow f \ (\text{Iso.fun } \text{CNIso } cn))$

`max'` : $\text{SMBTree} \rightarrow \text{SMBTree} \rightarrow \text{SMBTree}$
`max'` $t_1 \ t_2 = \text{INLim } (\lambda \ n \rightarrow \text{if0 } n \ t_1 \ t_2)$

`max'-sL` : $\forall \{t_1 \ t_2\} \rightarrow t_1 \leq \text{max}' \ t_1 \ t_2$

`max'-sR` : $\forall \{t_1 \ t_2\} \rightarrow t_2 \leq \text{max}' \ t_1 \ t_2$

`max'-LUB` : $\forall \{t_1 \ t_2 \ t\} \rightarrow t_1 \leq t \rightarrow t_2 \leq t \rightarrow \text{max}' \ t_1 \ t_2 \leq t$

`max≤max'` : $\forall \{t_1 \ t_2\} \rightarrow \text{max } t_1 \ t_2 \leq \text{max}' \ t_1 \ t_2$
`max≤max'` = `max-LUB` `max'-sL` `max'-sR`

`max'≤max` : $\forall \{t_1 \ t_2\} \rightarrow \text{max}' \ t_1 \ t_2 \leq \text{max } t_1 \ t_2$
`max'≤max` = `max'-LUB` `max-sL` `max-sR`

4.2.4 Well Founded Ordering on SMB-trees. Our motivation for defining SMB-trees was defining well founded recursion, so the final piece of our definition is a proof that the strict ordering of SMB-trees is well founded. Intuitively this should hold: if there are no infinite descending chains of Brouwer trees, and there are fewer SMB-trees than Brouwer trees, then there can be no infinite descending chains of SMB-trees. The key lemma is that an SMB-tree is accessible if its underlying Brouwer tree is.

`sizeWF` : `WellFounded` `_<_`
`sizeWF` $t = \text{sizeAcc } (\text{Brouwer.ordWF } (\text{rawTree } t))$
 where
`sizeAcc` : $\forall \{t\}$
 $\rightarrow \text{Acc } \text{Brouwer.}_<_ (\text{rawTree } t)$
 $\rightarrow \text{Acc } _<_ t$

`sizeAcc` $\{t\} (\text{acc } x)$
 $= \text{acc } ((\lambda \ y \ lt \rightarrow \text{sizeAcc } (x \ (\text{rawTree } y) \ (\text{get} \leq lt))))$

Thus, we have an ordinal type with limits, a strictly monotone join, and well founded recursion.

5 An Algebraic Perspective

As a final contribution, we give an algebraic viewpoint of SMBTrees, in terms of equivalences rather than orderings. There are no new results in this section, but the equational view highlights the ways in which a strictly monotone join is useful in reasoning.

SMB-trees cannot be completely characterized using first order equations, since `Lim` is an infinitary operator. Nevertheless, we anticipate that many of the equations here could be useful in developing automated rewriting tools or tactics for reasoning about SMBTrees. The constraint of $t_1 < t_2$ can be translated into the equation $\uparrow t_1 \vee t_2 \approx t_2$, which can then be mechanically simplified according to the equations in the following sections.

5.1 Semilattices and Setoids

Unfortunately, SMB-trees are only a preorder, not a partial order. Because we are working in vanilla Agda, we have no function extensionality, so applying `Lim` to definitionally distinct but extensionally equal functions produces trees that are equivalent but not equal. Postulating that equivalent terms are equal would be inconsistent: inductive and limit-based joins are equivalent for SMB-trees, so $\uparrow (t_1 \ t_2)$ is equivalent to `limMax` $(\uparrow t_1) (\uparrow t_2)$, even though their heads are distinct datatype constructors. Likewise, `Lim` $c \ f$ is equivalent to `Z` any time $El \ c$ is uninhabited.

As such, we present our equations in the setoid style i.e. up to an equivalence relation, but the results in this section could be adapted to quotient types in a system like Cubical Agda [Vezzosi et al. 2019]. First, we establish that SMB-trees are a bounded join-semilattice, up to the symmetric closure of the \leq relation:

`TreeSemiLat` : `BoundedJoinSemilattice` $\ell \ \ell \ \ell$
`Carrier` `TreeSemiLat` = `SMBTree`
`_≈_` `TreeSemiLat` $t_1 \ t_2 = t_1 \ \text{SMBTree.}_\leq \ t_2 \times t_2 \ \text{SMBTree.}_\leq \ t_1$
`_≤_` `TreeSemiLat` = `SMBTree.}_≤_`
`_∨_` `TreeSemiLat` = `SMBTree.max`
`⊥` `TreeSemiLat` = `SMBTree.Z`
`-- ...`

Orderings between trees can then be expressed equationally using the join: t_1 is smaller than t_2 iff their join is t_2 .

`ord→equiv` : $\forall \{t_1 \ t_2\} \rightarrow t_1 \leq t_2 \rightarrow t_1 \vee t_2 \approx t_2$
`equiv→ord` : $\forall \{t_1 \ t_2\} \rightarrow t_1 \vee t_2 \approx t_2 \rightarrow t_1 \leq t_2$

The semilattice properties can be presented algebraically: a semilattice is a commutative, idempotent semigroup.

991 $\text{assoc} : \forall \{t_1 \ t_2 \ t_3\} \rightarrow t_1 \vee (t_2 \vee t_3) \approx (t_1 \vee t_2) \vee t_3$
 992 $\text{commut} : \forall \{t_1 \ t_2\} \rightarrow t_1 \vee t_2 \approx t_2 \vee t_1$
 993 $\text{idem} : \forall \{t\} \rightarrow t \vee t \approx t$

995 5.2 Successor: The Inflationary Endomorphism

996 The algebraic version of strict monotonicity is that the suc-
 997 cesor function is what Bezem and Coquand [2022] call an
 998 *inflationary endomorphism*, i.e. a unary operator whose in-
 999 teractions with the join behave like the successor on natural
 1000 numbers. To our knowledge, SMB-trees are the first ordinal
 1001 notation in type theory for which the successor is inflation-
 1002 ary.

1003 There are two laws to inflationary endomorphisms. First,
 1004 the maximum of $\uparrow t$ and t must be $\uparrow t$, which captures the
 1005 idea that t is less than $\uparrow t$.

1006 $\uparrow \text{absorb} : \forall \{t\} \rightarrow t \vee (\uparrow t) \approx \uparrow t$
 1007 $\uparrow \text{absorb} =$
 1008 $\text{max-mono } (\leq \uparrow _) \leq \text{refl} \leq \text{max-idem}$
 1009 $, \text{max-}\leq R$

1010 Second, the successor must distribute over the join. Recall
 1011 that this was precisely the condition we used to establish
 1012 strict monotonicity.

1013 $\uparrow \text{dist} : \forall \{t_1 \ t_2\} \rightarrow \uparrow (t_1 \vee t_2) \approx \uparrow t_1 \vee \uparrow t_2$
 1014 $\uparrow \text{dist} \{t_1\} \{t_2\} =$
 1015 $\text{max-sucMono } \leq \text{refl}$
 1016 $, \text{max-LUB } (\leq \text{sucMono max-}\leq L) (\leq \text{sucMono max-}\leq R)$

1020 5.3 Characterizing Limits

1021 Finally, we present some equations regarding joins and lim-
 1022 its. Since limits are essentially (possibly-)infinitary joins, we
 1023 write them using \bigvee .

1024 $\bigvee : \forall \{c\} \rightarrow (El \ c \rightarrow \text{SMBTree}) \rightarrow \text{SMBTree}$
 1025 $\bigvee f = \text{Lim } _ f$

1026 Since limits are an upper bound, joining any element from
 1027 a sequence with the limit of that sequence has no effect:

1028 $\bigvee\text{-Bound} : \forall \{c : C\} \{f : El \ c \rightarrow \text{SMBTree}\} \{k\}$
 1029 $\rightarrow f \ k \vee (\bigvee f) \approx \bigvee f$
 1030 $\bigvee\text{-Bound} = \text{ord} \rightarrow \text{equiv } (\leq \text{limUpperBound } _)$

1031 Moreover, limits are an actual supremum: the limit of a func-
 1032 tion is absorbed by any upper bound of the function's image.

1033 $\bigvee\text{-Supremum} : \forall \{c : C\} \{f : El \ c \rightarrow \text{SMBTree}\} \{t\}$
 1034 $\rightarrow (\forall k \rightarrow f \ k \vee t \approx t) \rightarrow (\bigvee f) \vee t \approx t$
 1035 $\bigvee\text{-Supremum} \{f = f\} \text{lt}$
 1036 $= \text{ord} \rightarrow \text{equiv } (\leq \text{limLeast } (\lambda k \rightarrow \text{equiv} \rightarrow \text{ord } (\text{lt } k)))$

1037 The join of a constant, non-empty sequence is the singu-
 1038 lar element of that sequence:

1039 $\bigvee\text{-const} : \forall \{c \ t\}$
 1040 $\rightarrow El \ c$

1041 $\rightarrow \text{Lim } c (\lambda _ \rightarrow t) \approx t$

1042 $\bigvee\text{-const } k = \leq \text{limLeast } (\lambda _ \rightarrow \leq \text{refl}), \leq \text{limUpperBound } k$

1043 The join of an empty sequence is zero:

1044 $\bigvee\text{-empty} : \forall \{c \ f\}$
 1045 $\rightarrow \neg (El \ c)$
 1046 $\rightarrow \text{Lim } c \ f \approx Z$

1047 $\bigvee\text{-empty } \text{empty}$
 1048 $= (\leq \text{limLeast } (\lambda k \rightarrow \text{contradiction } k \ \text{empty})), \leq Z$

1049 More interesting is the interaction between limits and
 1050 joins. For two limits over the same index set, the join of
 1051 those two limits is the same as the limit of joining the se-
 1052 quences together.

1053 $\bigvee\text{-distHomo} : \forall \{c : C\} \{f \ g : El \ c \rightarrow \text{SMBTree}\}$
 1054 $\rightarrow (\text{Lim } c \ f) \vee (\text{Lim } c \ g) \approx \text{Lim } c (\lambda x \rightarrow f \ x \vee g \ x)$

1055 $\bigvee\text{-distHomo}$

1056 $= \text{max-LUB}$
 1057 $(\leq \text{extLim } (\lambda _ \rightarrow \text{max-}\leq L))$
 1058 $(\leq \text{extLim } (\lambda _ \rightarrow \text{max-}\leq R))$

1059 $, \leq \text{limLeast } (\lambda k \rightarrow \text{max-mono } (\leq \text{limUpperBound } _) (\leq \text{limUpperBound } _))$

1060 We can obtain a more general result, distributing a limit
 1061 over a join, so long as the limit is over a non-empty sequence.
 1062 The join of a non-zero tree with an empty limit will be non-
 1063 zero, but pushing the join under a limit will produce zero,
 1064 so the following result only applies to non-empty limits.

1065 $\bigvee\text{-distHet} : \forall \{c : C\} \{f : El \ c \rightarrow \text{SMBTree}\} \{t\}$

1066 $\rightarrow El \ c$
 1067 $\rightarrow (\bigvee f) \vee t \approx \bigvee (\lambda x \rightarrow f \ x \vee t)$

1068 $\bigvee\text{-distHet } k$

1069 $= \text{max-LUB}$
 1070 $(\leq \text{extLim } (\lambda _ \rightarrow \text{max-}\leq L))$
 1071 $(\leq \text{limUpperBound } k \leq \text{refl } (\leq \text{extLim } \lambda _ \rightarrow \text{max-}\leq R))$
 1072 $, \leq \text{limLeast } (\lambda k \rightarrow \text{max-monoL } (\leq \text{limUpperBound } _))$

1080 6 Discussion

1085 6.1 Comparison to Other Ordinal Systems

1086 Kraus et al. [2023] give a comprehensive overview of ordinal
 1087 notation systems in type theory, with a detailed comparison
 1088 of their comparative properties. They define three different
 1089 systems: Cantor normal forms that represent ordinals as bi-
 1090 nary trees, restricted Brouwer trees that represent ordinals
 1091 as infinitely branching trees, and well founded types that
 1092 represent ordinals as types with a certain sort of relation on
 1093 their elements.

1094 The definitions Kraus et al. give are more restrictive than
 1095 ours. For example, for Brouwer trees they require that **Lim**
 1096 only operate on functions that are strictly increasing, pre-
 1097 venting the definition of **limMax**. These restrictions make
 1098 their ordinals very well behaved, so they can examine their
 1099 mathematical properties. SMB-trees have less rich theory,

but the properties they do satisfy are specifically tailored to proving termination of higher-order programs.

6.1.1 Transitivity, Extensionality and Well Foundedness. Kraus et al. show three properties for each system they present: transitivity of the ordering, well foundedness (as in Section 2.2), and *extensionality*, the property that two ordinals are equal iff their sets of smaller terms are equal. They also show a strict version of extensionality for each system: to ordinals are equal iff their sets of strictly smaller terms are equal.

SMB-trees satisfy each of the above properties: the transitivity of \leq is inherited from Brouwer trees, and we show well foundedness in Section 2.2. Extensionality for \leq is trivially true for our setoid version of equivalence. For propositional equality, extensionality cannot be proved without some form of quotient type. We conjecture that the strict order $<$ is not extensional for SMB-trees, since it does not hold for Brouwer trees without quotient types.

Well founded types lack a basic transitivity property for the strict order: without additional axioms, one cannot conclude $x < z$ from $x \leq y$ and $y < z$. So, though well founded types have binary and infinitary suprema like SMB-trees, they lack the basic principle for reasoning about strict orders, making them ill suited for defining recursive procedures.

6.1.2 Classifiability. Classifiability is the property that each ordinal is either zero, a successor, or a limit, and that exactly one of those properties holds. Restricted Brouwer trees and Cantor normal forms both satisfy classifiability, but SMB-trees do not. Even our version of Brouwer trees do not have this property: since we allow non-increasing sequences, the limit of the constant-zero sequence is equivalent to zero.

Not having classifiability does negatively affect the decidability properties of SMB-trees. For example, for restricted Brouwer trees, it is decidable whether a tree is infinite or not, but this is not the case for SMB-trees, since some limits are actually finite. However, since SMB-trees are defined specifically around well founded recursion, losing decidability properties is an acceptable compromise.

6.1.3 Joins and Suprema. The main novelty of SMB-trees is the existence of both binary suprema (joins) and infinitary suprema (limits) that interact well with the strict ordering. Cantor normal forms have binary joins and strict monotonicity (as a by-product of decidable ordering), but lack infinitary joins. Well founded types have binary and infinitary suprema, but without additional axioms even their successor function is not monotone, so strict monotonicity is out of the question. For restricted Brouwer trees, binary joins cannot exist without further axioms. This is an artifact of allowing *Lim* only on strictly increasing sequences, since it

disallows *limMax* or other similar constructs. So even without strict monotonicity, the capability of SMB-trees exceeds that of restricted Brouwer trees. The cost of this is SMB-trees fulfill fewer nice properties with respect to propositional equality. Since setoid reasoning is sufficient for well founded recursion, we find this tradeoff acceptable.

JE ▶ [TODO table](#) ◀

6.2 Conclusion

Designing an ordinal library is an exercise in compromise, balancing the desired properties with the limitations of decidability and constructive reasoning. With SMB-trees, we have identified a point in the design space well suited to proving termination. The algebraic framework of SMB-trees lays the groundwork for future developments on reasoning mechanically about ordinals. Beyond of our specific use-case, the development of SMB-trees shows that sometimes careful design with dependent types can avoid the need for additional axioms or language features.

References

- Guillaume Allais, Edwin Brady, Nathan Corbyn, Ohad Kammar, and Jeremy Yallop. 2023. Frex: dependently-typed algebraic simplification. arXiv:2306.15375 [cs.PL]
- Yves Bertot and Pierre Castéran. 2004. Interactive Theorem Proving and Program Development. Springer-Verlag.
- Marc Bezem and Thierry Coquand. 2022. Loop-checking and the uniform word problem for join-semilattices with an inflationary endomorphism. Theoretical Computer Science 913 (2022), 1–7. <https://doi.org/10.1016/j.tcs.2022.01.017>
- Edwin C. Brady. 2021. Idris 2: Quantitative Type Theory in Practice. CoRR abs/2104.00480 (2021). arXiv:2104.00480 <https://arxiv.org/abs/2104.00480>
- Jonathan H.W. Chan. 2022. Sized dependent types via extensional type theory. Master's thesis. University of British Columbia. <https://doi.org/10.14288/1.0416401>
- Nathan Corbyn. 2021. Proof Synthesis with Free Extensions in Intensional Type Theory. Technical Report. University of Cambridge. MEng Dissertation.
- Leonardo de Moura, Soonho Kong, Jeremy Avigad, Floris van Doorn, and Jakob von Raumer. 2015. The Lean Theorem Prover (System Description). In Automated Deduction - CADE-25, Amy P. Felty and Aart Middeldorp (Eds.). Springer International Publishing, Cham, 378–388.
- Agda Developers. 2017. Github Issue: Equality is incompatible with sized types. <https://github.com/agda/agda/issues/2820>.
- Joseph S. Eremondi. 2023. On the design of a gradual dependently typed language for programming. Ph.D. Dissertation. University of British Columbia. <https://doi.org/10.14288/1.0428823>
- John Hughes, Lars Pareto, and Amr Sabry. 1996. Proving the Correctness of Reactive Systems Using Sized Types. In Proceedings of the 23rd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (St. Petersburg Beach, Florida, USA) (POPL '96). Association for Computing Machinery, New York, NY, USA, 410–423. <https://doi.org/10.1145/237721.240882>
- Nicolai Kraus, Fredrik Nordvall Forsberg, and Chuangjie Xu. 2023. Type-theoretic approaches to ordinals. Theoretical Computer Science 957 (2023), 113843. <https://doi.org/10.1016/j.tcs.2023.113843>
- CONOR MCBRIDE and JAMES MCKINNA. 2004. The view from the left. Journal of Functional Programming 14, 1 (2004), 69–111. <https://doi.org/10.1017/S0956796803004829>

Ulf Norell. 2009. Dependently Typed Programming in Agda. In Proceedings of the 4th International Workshop on Types in Language Design and Implementation (Savannah, GA, USA) (TLDI '09). ACM, New York, NY, USA, 1–2. <https://doi.org/10.1145/1481861.1481862>

The Univalent Foundations Program. 2013. Homotopy Type Theory: Univalent Foundations of Mathematics. <https://homotopytypetheory.org/>

book, Institute for Advanced Study.

Andrea Vezzosi, Anders Mörtberg, and Andreas Abel. 2019. Cubical Agda: A Dependently Typed Programming Language with Univalence and Higher Inductive Types. Proc. ACM Program. Lang. 3, ICFP, Article 87 (jul 2019), 29 pages. <https://doi.org/10.1145/3341691>