Assignment 4

1. **Outline.**

- General Science Topics:
  i. Age determination of stars giving key insight into evolution
  ii. Understanding the mass-age relationship for stars
  iii. How star clocks help determine useful data in other fields
  iv. Peek into the infant universe/formation of baryonic matter

- Star Cluster Research Outcomes (Age Dating Techniques):
  1. Eclipsing Binaries
     i. Observation of eclipsing binaries in crowded fields
     ii. Measuring radial velocities and mass
     iii. 2. Gyrochronology
     i. Mass-rotation-age intimately related
     ii. Determination of star mass (binaries) and rotation rate
     iii. Bigger stars, faster rotation $\Rightarrow$ shorter lifespan
  3. White Dwarf Cooling Ages
     i. Obtain a measurement of the WD's luminosity
     ii. $t_{cool} \propto L^{-\frac{5}{7}}$ for WD
     iii. Comparison of observed ages with theoretical ages

- Scientific Application:
  1. Precise measurement of stellar ages
  2. Obtain an insight to how stars evolve
  3 .Pindown ages of observed clusters
  4 .Insight to formation of the early universe

2. Opacity.

Starting with the Rosseland mean opacity equation,

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T}\ d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\ d\nu} \tag{1}$$

we assume the two situations for $\kappa_{\nu,L} = \kappa_L$ and $\kappa_{\nu,H} = \kappa_H$ for a fraction f having opacity κ_L:

$$\frac{1}{\kappa_{\nu,L}} = f \cdot \frac{1}{\kappa_L} = \frac{f}{\kappa_H}$$

$$\frac{1}{\kappa_{\nu,H}} = (1-f) \cdot \frac{1}{\kappa_L} = \frac{1-f}{\kappa_H}$$

The total opacity is then $\frac{1}{\kappa_\nu} = \frac{1}{\kappa_{\nu,L}} + \frac{1}{\kappa_{\nu,H}}$:

$$\frac{1}{\kappa_\nu} = \frac{f}{\kappa_L} + \frac{1-f}{\kappa_H} \tag{2}$$

Plugging this into eqn. (1),

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \left(\frac{f}{\kappa_L} + \frac{1-f}{\kappa_H}\right) \frac{\partial B_\nu}{\partial T}\ d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\ d\nu} = \left(\frac{f}{\kappa_L} + \frac{1-f}{\kappa_H}\right) \frac{\int_0^\infty \frac{\partial B_\nu}{\partial T}\ d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\ d\nu}$$

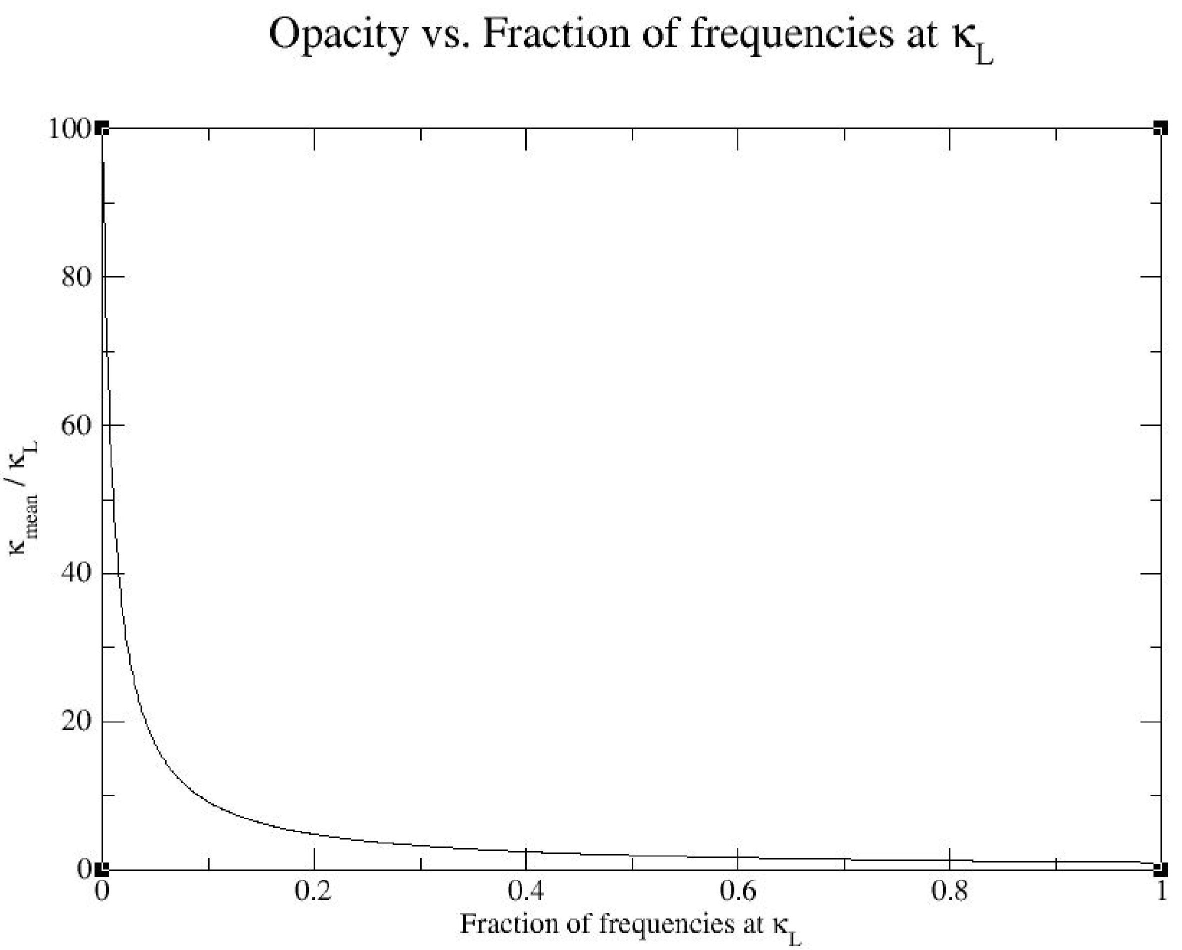
$$\frac{1}{\bar{\kappa}} = \frac{f}{\kappa_L} + \frac{1-f}{\kappa_H} \tag{3}$$

Taking the reciprocal of eqn. (3) and multiplying both sides by $\frac{1}{\kappa_L}$, the expression becomes,

$$\bar{\kappa} = \frac{1}{\frac{f}{\kappa_L} + \frac{1-f}{\kappa_H}} \Rightarrow \frac{\bar{\kappa}}{\kappa_L} = \frac{1}{f + (1-f)\frac{\kappa_L}{\kappa_H}}$$

Letting $\kappa_H = 100\kappa_L$, the following simple code was used to produce the graph below to compare $\frac{\bar{\kappa}}{\kappa_L}$:

```
PROGRAM opacity

  ! ------ Variables and Constants ------

  IMPLICIT NONE
  INTEGER :: i
  REAL :: k_over_kL , f , df

  ! ------ Input/Output Files ------

  OPEN ( UNIT=10 , FILE='opacity.dat', STATUS='UNKNOWN' )

  ! ------ Code -------

  ! Initializing opacity and fraction

  k_over_kL = 0.
  f = 0. ; df = 0.001

  ! DO WHILE loop to compute opacities

  DO WHILE ( f <= 1. )

     k_over_kL = 1. / ( f + (1.-f)*0.01 )
     write (10,*) f , k_over_kL
     f = f + df

  END DO

  STOP

END PROGRAM opacity
```



From the graph, at fractions approaching 1, $\bar{\kappa}$ will approximately be equivalent to κ_L, where continuum opacity sources dominate, allowing most frequencies of light to pass through relatively easy in the outer layers of the star producing a continuous spectra. High opacity represents $\approx 10\%$ of the total opacity graphed, inferring that $\bar{\kappa}$ does not sensitivity depend on fluctuations of high opacity.

3. **Radiation in the Outer Layers.**

a) Assuming surface conditions for a star, $\ell_r \approx L$ and $r \approx R$, the net flux can be directly related to the photosphere temperature T_{eff} as well as the luminosity at the surface, assuming no energy generation,

$$F_{net} \approx \sigma T_{\text{eff}}^4 = \frac{L}{4\pi R^2} \tag{1}$$

The key here being that the net flux through two spherical layers within the photosphere is constant, which allows for the use of the Stefan-Boltzmann Law.

b) Starting from the radiative transport equation with surface conditions for a star,

$$\frac{dT}{dr} = -\frac{3}{16\pi ac} \cdot \frac{\kappa\rho L}{R^2 T^3} \tag{2}$$

Eqn. (1) can be arranged so that

$$\frac{L}{R^2} = 4\pi\sigma T_{\text{eff}}^4$$

and this can be substituted into Eqn. (2) to obtain,

$$\frac{dT}{dr} = -\frac{3}{16\pi ac} \cdot \frac{\kappa\rho \cdot 4\pi\sigma T_{\text{eff}}^4}{T^3} = -\frac{3}{4ac} \cdot \frac{\kappa\rho\sigma T_{\text{eff}}^4}{T^3}$$

Again, rearranging this expression and substituting $a = \frac{4\sigma}{c}$, we get

$$\frac{dT}{dr} = -\frac{3}{16} \cdot \kappa\rho \cdot \frac{T_{\text{eff}}^4}{T^3} \tag{3}$$

Using the definition for optical depth and letting $ds = -dr$ be the photon distance traveled between outer layers in the star, with the negative sign accounting for the optical depth *increasing* towards the center, with the path length of the traveling photon radially outward from the star's center. Eqn. (3) can then be rearranged, $\frac{dr}{d\tau}$

$$d\tau = -\kappa\rho \, dr \Rightarrow \frac{dr}{d\tau} = -\frac{1}{\kappa\rho}$$

Multiplying this expression through by both sides of Eqn. (3) and making mathematicians cringe, we obtain

$$\frac{dr}{d\tau} \cdot \frac{dT}{dr} = -\frac{3}{16} \cdot -\frac{\kappa\rho}{\kappa\rho} \cdot \frac{T_{\mathrm{eff}}^4}{T^3} \Rightarrow \boxed{\frac{dT}{d\tau} = \frac{3}{16}\frac{T_{\mathrm{eff}}^4}{T^3}}$$

c) Rearranging this differential equation and integrating to obtain a general solution for the function $T(\tau)$,

$$\int T^3 dT = \frac{3}{16} T_{\mathrm{eff}}^4 \int d\tau \Rightarrow \frac{T^4}{4} = \frac{3}{16} T_{\mathrm{eff}}^4 \, \tau + C$$

where C is an arbitrary constant. Solving for T,

$$T^4 = \frac{3}{4} T_{\mathrm{eff}}^4 \, \tau + K \tag{4}$$

Where K is, again, another constant. Applying the boundary condition, $T^4(\tau = \frac{2}{3}) = T_{\mathrm{eff}}^4$ for this general and solving for the unknown constant,

$$T_{\mathrm{eff}}^4 = \frac{3}{4} T_{\mathrm{eff}}^4 \cdot \frac{2}{3} + K \Rightarrow K = \frac{1}{2} T_{\mathrm{eff}}^4$$

Substituting this into equation (4) and factoring out a factor of $\frac{3}{4} T_{\mathrm{eff}}^4$,

$$T^4 = \frac{3}{4} T_{\mathrm{eff}}^4 \, \tau + \frac{1}{2} T_{\mathrm{eff}}^4$$

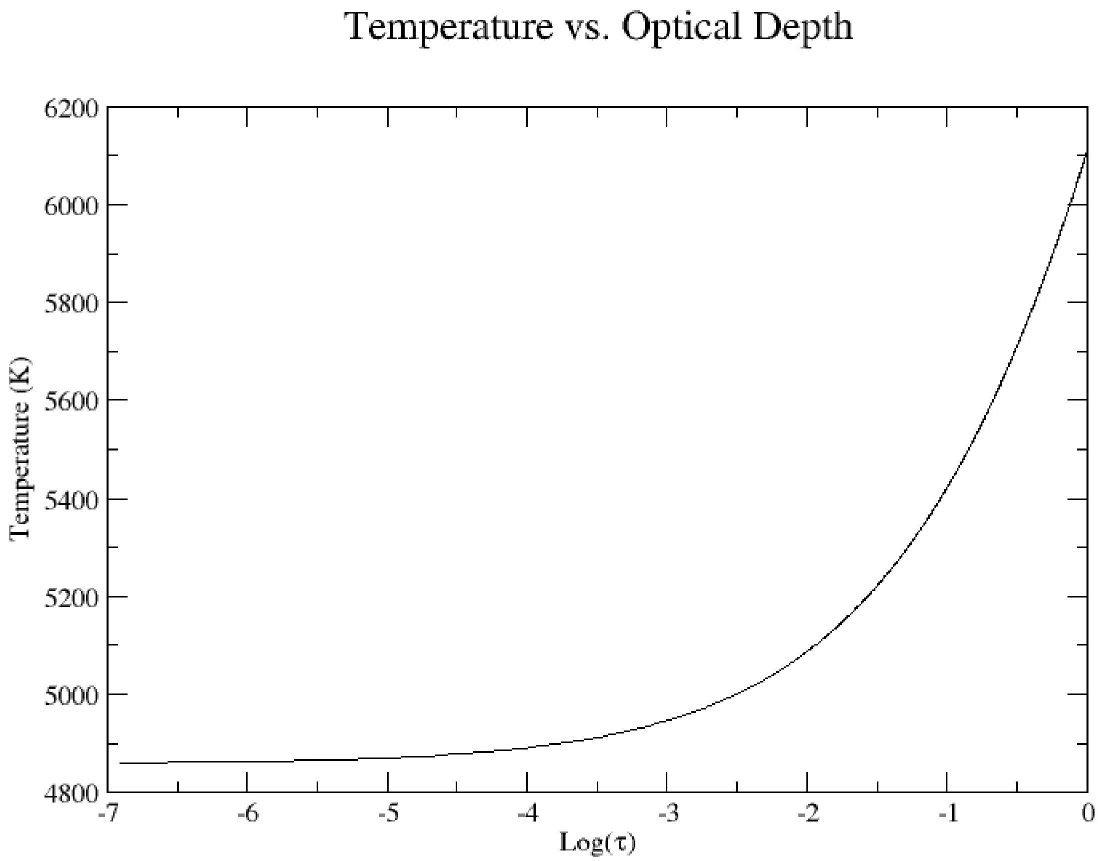
We arrive at the exact solution:

$$\boxed{T^4 = \frac{3}{4}T_{\text{eff}}^4\left(\tau + \frac{2}{3}\right)} \tag{5}$$

d) This relation holds true in the photosphere also in the presence of convective outer layers due to τ only depending on the mean free path, ℓ_m, of photons. For the top of the photosphere where $\tau \approx 0$, $\boxed{T = \frac{1}{2}T_{\text{eff}}}$.

e) The following code was used to obtain the graph of T vs. $\log\tau$:

```
PROGRAM optical_depth

  ! ------ Variables and Constants ------

  IMPLICIT NONE
  REAL :: T, tau, dtau
  REAL, PARAMETER :: T_sun = 5777.

  ! ------ Input/Output Files ------

  OPEN ( UNIT=10, FILE='T_vs_logtau.dat', STATUS='UNKNOWN' )

  ! ------ Code ------

  tau = 0. ; dtau = 0.001

  DO WHILE ( tau <= 1. )

     tau = tau + dtau
     T = T_sun * (3./4.)**(1./4.) * ( tau + 2./3. )**(1./4.)
     write (10,*) log(tau), T

  END DO

  STOP
END PROGRAM optical_depth
```


Temperature vs. Optical Depth
Temperature (K)
6200
6000
5800
5600
5400
5200
5000
4800
-7
-6
-5
-4
-3
-2
-1
0
Log(τ)

4. **Stellar Model Calculations.**

a) Parameters for each star:

Table 1: STATSTAR Star Comparison

Mass:	ρ_c (g/cm^3)	P_c (dynes/cm^2)	T_c (K)	$L/L_\odot$	$R/R_\odot$
1.0 $M_\odot$	77.2529	1.46284×10^{17}	1.41421×10^{7}	0.86071	1.020998
2.5 $M_\odot$	51.2330	1.61620×10^{17}	2.35600×10^{7}	56.805	1.431432

The $2.5M_\odot$ hold both the higher radius and luminosity, as well as the lower density, agreeing with observations of high mass stars being higher luminosity and radius than lower stars yet surprisingly being less dense than lower mass stars.

Roughly estimating the values at which m_r/M and $\ell_r/L = 50\%$ and 99% by looking at the `starmodl.dat` and data files produced from my code for each star (in terms of r/R):

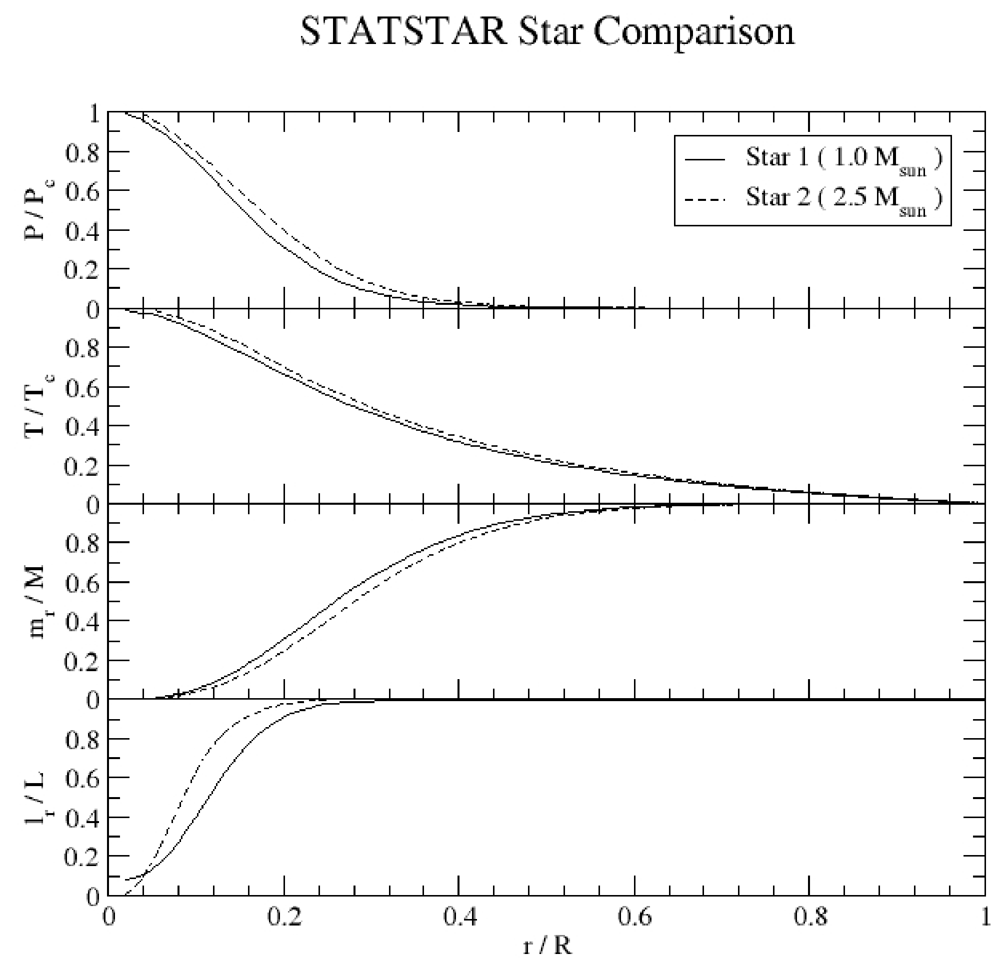
Table 2: STATSTAR Star Comparison of m_r/M

Mass:	$m_r/M = 50\%$	$m_r/M = 99\%$
1.0 $M_\odot$	0.259	0.640
2.5 $M_\odot$	0.278	0.659

Table 3: STATSTAR Star Comparison of $\ell r/L$

Mass:	$\ell r/L = 50\%$	$\ell r/L = 99\%$
1.0 $M_\odot$	$r/R \approx 0.119$ $m_r/M \approx 0.0850$	$r/R \approx 0.288$ $m_r/M \approx 0.594$
2.5 $M_\odot$	$r/R \approx 0.0880$ $m_r/M \approx 0.0270$	$r/R \approx 0.288$ $m_r/M \approx 0.525$

b)



c) Convection zones for $1.0M_{\odot}$:

$$\Rightarrow 0 < m_r/M < 0.027$$

Convection zones for $2.5M_{\odot}$:

$$\Rightarrow 0 < m_r/M < 0.161$$

```
Joey@Joey ~/astr440/statstar
$ ./Statstar
  Enter the mass of the star (in solar units):
2.5
  Enter the luminosity of the star (in solar units):
56.805
  Enter the effective temperature of the star (in K):
13240
  Enter the mass fraction of hydrogen (X):
0.7
  Enter the mass fraction of metals (Z):
0.008



               CONGRATULATIONS, I THINK YOU FOUND IT!
         However, be sure to look at your model carefully.



 The surface conditions are:          The central conditions are:

 Mtot =      2.500000 Msun            Mc/Mtot     =  2.48887E-04
 Rtot =      1.431432 Rsun            Rc/Rtot     =  1.80000E-02
 Ltot =     56.805000 Lsun            Lc/Ltot     =  7.69581E-02
 Teff =  13240.000000 K               Density     =  5.12330E+01 g/cm**
 X    =      0.700000                 Temperature =  2.35600E+07 K
 Y    =      0.292000                 Pressure    =  1.61620E+17 dynes/
 Z    =      0.008000                 epsilon     =  1.35148E+04 ergs/s
                                      dlnP/dlnT   =  2.48781E+00


           ***** The integration has been completed *****
             The model has been stored in starmodl.dat
```