School of Computer Science and Technology
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Exercise Sheet 1 for
Design and Analysis of Algorithms
Autumn 2022
Due 1 Oct 2022 at 23:59

Exercise 1

Suppose you have an unfair dice such that if you roll it, the probability that each odd number (i.e., $1, 3, 5$) appears on the top is $\frac{1}{12}$, and the probability that each even number (i.e., $2, 4, 6$) appears on the top is $\frac{1}{4}$.
(1) Suppose that you roll this dice exactly once.

- What is the expected value of the number X on top of the dice?
- What is the variance of X?

(2) Suppose that you roll this dice n times. Let Y denote the sum of the numbers that are on the top of the dice throughout all the n rolls.

- What is the expected value of Y?
- What is the variance of Y?
- Give an upper bound of $\Pr[Y > 4n]$ by using Markov inequality.
- Give an upper bound of $\Pr[Y > 4n]$ by using Chebyshev's inequality.
- Give an upper bound of $\Pr[Y > 4n]$ by using Chernoff Bound.

Solution 1:
(1): $E[X] = \frac{1}{12} \times (1 + 3 + 5) + \frac{1}{4} \times (2 + 4 + 6) = 3.75$

(2): $Var[X] = E[X^2] - (E[X])^2 = \frac{1}{12} \times (1^2 + 3^2 + 5^2) + \frac{1}{4} \times (2^2 + 4^2 + 6^2) - 3.75^2 = 2.854$

(3):$E[Y] = E[\Sigma_{i=1}^{n} X_i] = \Sigma_{i=1}^{n} E[X_i] = nE[X] = 3.75n$

(4):$Var[Y] = Var[\Sigma_{i=1}^{n} X_i] = \Sigma_{i=1}^{n} Var[X_i] = nVar[X] = 2.854n$

(5):$According\ to\ Markov\ inequality:$
$Pr[Y > 4n] \leq \frac{E[Y]}{4n} = \frac{3.75n}{4n} = 0.9375$

(6):$According\ to\ Chebyshev' s inequality:$
$Pr[Y > 4n] = Pr[Y - 3.75n > 0.25n] = Pr[Y - E[Y] > 0.25n] \leq Pr[|Y - E[Y]| > 0.25n] \leq \frac{2.854n}{(0.25n)^2} = \frac{45.664}{n}$

(7):First, Let $x_i = 0.1X_i(for \forall i \in [1, n]), y = 0.1Y$, then $x_i \in [0, 1](for \forall i \in [1, n]), y = \Sigma_{i=1}^{n} x_i$.
$Pr[Y > 4n] = Pr[y > 0.4n]$
$According\ to\ Chernoff\ Bound:$
$Pr[Y > 4n] = Pr[y > 0.4n] = Pr[y - 0.375n > 0.025n] \leq Pr[|y - 0.375n| > 0.025n] \leq 2exp(\frac{-0.375n \times (\frac{25}{375})^2}{3})$
$\approx \frac{2}{exp(5.56 \times 10^{-4} \times n^2)}$

Exercise 2 *(An application of Markov's inequality: turning a Las Vegas algorithm to a Monte Carlo algorithm)*

Let $\mathcal{A}$ be a randomized algorithm for some decision problem P (e.g., to decide if a graph is connected or not). Suppose that for any given instance of P of size n, $\mathcal{A}$ runs in *expected* $T(n)$ time and always outputs the correct answer.
Use $\mathcal{A}$ to give a new randomized algorithm NEWALG for the problem P such that NEWALG *always* runs in $100T(n)$ time and

- if the input is a 'Yes' instance[1], then it will be accepted with probability at least $\frac{5}{6}$;
- if the input is a 'No' instance, then it will always be rejected.

Describe your algorithm and justify its correctness and running time.

Hint: You may use Markov's inequality.

Solution 2:
the algorithm **NEWALG** is described as follows

Algorithm 1 NEWALG

Input: Original Algorithm **A**, Original Expected Running Time $\mathbf{T(n)}$, Problem **P**.
Output: A decision('**Accept**' or '**Reject**')

run algorithm **A** on the problem **P**
for t **in** $range(0, 100\mathbf{T(n)}, \mathbf{T(n)})$ **do**
  check whether the algorithm **A** successfully outputs an answer or not.
  if algorithm **A** output an answer **then**
    return the answer outputted by **A**
  end if
end for
return Reject

Now we are going to prove that :

- if the input is a 'Yes' instance[2], then it will be accepted with probability at least $\frac{5}{6}$;
- if the input is a 'No' instance, then it will always be rejected. Prove: (1) if the input is a 'No' instance, obviously **NEWALG** will output a '**Reject**'. (2) if the input is a 'Yes' instance, Suppose that the algorithm **A** will run t time to output a correct answer. According to Markov inequality:
  $Pr[output\ a\ \textbf{'Reject'}] = Pr[\mathbf{A} does\ not\ output\ an\ answer\ in\ 100T(n)\ time] = Pr[t \geq 100T(n)]$
  $\leq \frac{E(t)}{100T(n)} = \frac{1}{100}$
  $Pr[output\ a\ \textbf{'Accept'}] = 1 - Pr[output\ a\ \textbf{'Reject'}] \geq \frac{99}{100} \geq \frac{5}{6}$

Exercise 3
Take R to be the IQ of a random person you pull off the street. What is the probability that some's IQ is at least 200 given that the average IQ is 100 and the variance of R is 100?
Solution 3:
According to Chebyshev's inequality:
$Pr[R \geq 200] = Pr[R - 100 \geq 100] \leq Pr[|R - 100| \geq 100] = Pr[|R - E[R]| \geq 10 \times \sqrt{var[R]}] \leq \frac{1}{10^2} = 0.01$
So, the probability that someone's IQ is at least 200 is less than (or equal to) 0.01.

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[1] An instance is a 'Yes'-instance if the correct answer to the Problem is 'Yes'. Otherwise (the correct answer is 'No'), it is a 'No'-instance.
[2] An instance is a 'Yes'-instance if the correct answer to the Problem is 'Yes'. Otherwise (the correct answer is 'No'), it is a 'No'-instance.