School of Computer Science and Technology
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Exercise Sheet 2 for
Design and Analysis of Algorithms
Autumn 2022

Due 18 Oct 2022 at 23:59

Exercise 1

Suppose each CPU can execute at most one process at any time. Consider the following experiment, which proceeds in a sequence of rounds. For the first round, we have n processes, which are assigned independently and uniformly at random to n CPUs. For any $i \geq 1$, after round i, we first find all the processes p such that p has been assigned to a CPU C *by itself*, i.e., p is the *unique* process that has been assigned to C; then we remove all such processes (as they would be executed) in round i. The remaining processes are retained for round $i+1$, in which they are assigned independently and uniformly at random to the n CPUs.

(a) If there are b processes at the start of a round, what is the expected number of processes at the start of the next round?

(b) Suppose that every round the number of removed processes was exactly the expected number of removed processes. Show that all the processes will be removed in $O(\log\log n)$ rounds.

Hint: If x_j is the expected number of processes left after j rounds, show and use that $x_{j+1} \leq x_j^2/n$. You can use the fact that $1 - kx \leq (1-x)^k$ for $0 < x < 1$ and $k \leq \frac{1}{x}$.

Solution 1:
(a):
Define random variables $X_i (1 \leq i \leq n)$ as following:

$$X_i = \begin{cases} 1, & \textit{exactly a unique process is assigned to } CPU_i \\ 0, & \textit{otherwise} \end{cases} \tag{1}$$

Given that we have b processes at the start of this round, we have the following calculation:
$P(X_i = 1) = \binom{b}{1} \cdot \frac{1}{n} \cdot (1 - \frac{1}{n})^{b-1} = \frac{b}{n} \cdot (1 - \frac{1}{n})^{b-1}$, $P(X_i = 0) = 1 - P(X_i = 1) = 1 - \frac{b}{n} \cdot (1 - \frac{1}{n})^{b-1}$.
So we have that the expectation $E[X_i] = 1 \times P(X_i = 1) + 0 \times P(X_i = 0) = \frac{b}{n} \cdot (1 - \frac{1}{n})^{b-1}$.
Further, We consider the expectation of total number(X) of the used GPUs at this round:
$X = \Sigma_{i=1}^{n} X_i$;
$E[X] = E[\Sigma_{i=1}^{n} X_i] = \Sigma_{i=1}^{n} E[X_i] = b \cdot (1 - \frac{1}{n})^{b-1}$.
And random variable X also represents the number of processes which are removed after the end of this round, so **the expected number of processes at the start of next round** is:
$b - E[X] = b - b \cdot (1 - \frac{1}{n})^{b-1}$.
(b):
Firstly, we need to prove the Lemma :
Note that $x_j$is a the expected number of processes left after j rounds, then we have $x_{j+1} \leq x_j^2/n$
Prove: according to **(a)**'s conclusion, we have that:
let $x_j = b$, then $x_{j+1} = b - b \cdot (1 - \frac{1}{n})^{b-1}$;
We only need to prove that $b - b \cdot (1 - \frac{1}{n})^{b-1} \leq \frac{b^2}{n}$.
And this is equivalent to the inequation: $1 - \frac{b}{n} \leq (1 - \frac{1}{n})^{b-1}$,
According to the fact that $1 - kx \leq (1-x)^k$ for $0 < x < 1$ and $k \leq \frac{1}{x}$, we have that $1 - \frac{b}{n} \leq (1 - \frac{1}{n})^b \leq (1 - \frac{1}{n})^{b-1}$
and this is the end of prove of Lemma.
Then, we are going to prove that **all the processes will be removed in O($loglogn$) rounds**.

Proof: We have that for 0-th round and 1-th round $x_0 = n$; $x_1 = n - n \cdot (1 - \frac{1}{n})^{n-1}$
Note that $x_1 = n - n \cdot (1 - \frac{1}{n})^{n-1} = n \cdot (1 - \frac{1}{(1+\frac{1}{n-1})^{n-1}})$.
Given the fact that $(1 + \frac{1}{n-1})^{n-1} \le e$, we have that $x_1 = n \cdot (1 - \frac{1}{(1+\frac{1}{n-1})^{n-1}}) \le n\dot(1 - \frac{1}{e})$.
So, for $\forall\ k \ge 2$, according to the previously proved lemma,
We have that $x_k \le \frac{x_{k-1}^2}{n} \le ... \le \frac{x_1^{2^{k-1}}}{n^{2^{k-1}-1}} \le \frac{(n\dot(1-\frac{1}{e}))^{2^{k-1}}}{n^{2^{k-1}-1}} = n \cdot (1 - \frac{1}{e})^{2^{k-1}}$,
If all of the n processes are removed exactly after k+1 rounds, we know that $x_k = O(1)$ according to the conclusion of (a).
So this results in that the upperbound of x_k, which is $\mathbf{n \cdot (1 - \frac{1}{e})^{2^{k-1}}} > O(1)$.
The above inequation is equivalent to guaranteeing $(\frac{e}{e-1})^{2^{k-1}} \le O(n)$, resulting that $k = O(loglogn)$. And this is the end of the prove.

Exercise 2

Suppose you are given a biased coin that has Pr[HEADS] $= p \ge a$, for some fixed a, without being given any other information about p.

(a) Devise a procedure that outputs a value $\tilde{p}$ such that you can guarantee that $\Pr[|p - \tilde{p}| \ge \varepsilon p] \le \delta$, for any choice of the constants $0 < a, \varepsilon, \delta < 1$. (The value $\tilde{p}$ is often called the estimate of p.)

(b) Let N be the number of times you need to flip the biased coin to obtain the estimate. What is the smallest value of N for which you can still give the above guarantee?

Hint: flip the coin a few times and consider the fraction of times seeing HEADS.
Solution 2:
(a):
The algorithm **Estimate-p** devised by myself could be described as follows:

Algorithm 1: Estimate-p

Require: a biased coin **C**, flip times **N**.
Ensure: an estimate value $\tilde{\mathbf{p}}$ for the Pr[HEADS]=p.

```
count = 0; headcount = 0
for count < N do
    flip the coin C, and observe the which side is upwards.
    if head side is upwards then
        headcount = headcount + 1;
    end if
    count = count + 1;
end for
p̃ = headcount/N
return p̃
```

In the following solution (in solution of (b)), We will further prove that **for any choice of the constants** $0 < a, \varepsilon, \delta < 1$**, we can guarantee that** $\Pr[|p - \tilde{p}| \ge \varepsilon p] \le \delta$ **holds with a big enough N**.
(b):
Define random variables $X_i (1 \le i \le N)$ as following:

$$X_i = \begin{cases} 1, & the\ head\ side\ was\ upwards\ after\ the\ i-th\ flipping \\ 0, & otherwise \end{cases} \tag{2}$$

Define random variable $X = \Sigma_{i=1}^{N} X_i$, and this random variable represents the total number of the head sides appearing upward.
Apparently, the expectation of X, $E[X] = \Sigma_{i=1}^{N} E[X_i] = N \times p$.
Recalling the procedure devised in the solution of (a), we have that:

$\Pr[|p-\tilde{p}| \geq \varepsilon p] = \Pr[|\frac{X}{N} - p| \geq \varepsilon p] = \Pr[|X - N \times p| \geq N \times \varepsilon p] = \Pr[|X - E[X]| \geq \varepsilon E[X]]$,
According to **Chernoff's Bound**, we have that:
$\Pr[|\mathbf{p} - \tilde{\mathbf{p}}| \geq \varepsilon \mathbf{p}] = \Pr[|X - E[X]| \geq \varepsilon E[X]] \leq 2exp(\frac{-E[X]\varepsilon^2}{3}) = \mathbf{exp}(\frac{-\mathbf{Np}\varepsilon^{\mathbf{2}}}{\mathbf{3}})$;
Let $\mathbf{exp}(\frac{-\mathbf{Np}\varepsilon^{\mathbf{2}}}{\mathbf{3}}) \leq \delta$, we can solve that $N \geq \frac{\mathbf{3}}{\mathbf{p}} \cdot \frac{\mathbf{1}}{\varepsilon^{\mathbf{2}}} \cdot \mathbf{ln}(\frac{\mathbf{2}}{\delta})$.
Given that $p \geq a$, we can know that $\mathbf{N_{min}} = \lceil \frac{\mathbf{3}}{\mathbf{a}} \cdot \frac{\mathbf{1}}{\varepsilon^{\mathbf{2}}} \cdot \mathbf{ln}(\frac{\mathbf{2}}{\delta}) \rceil$.

Exercise 3

Let X and Y be finite sets and let Y^X denote the set of all functions from X to Y. We will think of these functions as "hash" functions. A family $\mathcal{H} \subseteq Y^X$ is said to be strongly 2-universal if the following property holds, with $h \in \mathcal{H}$ picked uniformly at random:

$$\forall x, x' \in X \ \forall y, y' \in Y \left(x \neq x' \Rightarrow \Pr_h[h(x) = y \wedge h(x') = y'] = \frac{1}{|Y|^2} \right).$$

We are give a a stream $\mathcal{S}$ of elements of X, and suppose that $\mathcal{S}$ contains at most s distinct elements. Let $\mathcal{H} \subseteq Y^X$ be a strongly 2-universal hash family with $|Y| = cs^2$ for some constant $c > 0$. Suppose we use a random function $h \in \mathcal{H}$ to hash.
Prove that the probability of a collision (i.e., the event that two distinct elements of $\mathcal{S}$ hash to the same location) is at most $1/(2c)$.
Solution 3:
Prove:
We first give a formal definition of collision for a certain hash function $\mathbf{h}$ as follows:
for $\forall x, x' \in \mathbf{X}, \exists y \in \mathbf{Y}, s.t.\ \mathbf{h}(x) = \mathbf{h}(x') = y$;
Given that all of the hash functions in this exercise have the strongly 2-universal quality.
So, for certain x, x', y, the collision probability is :
$\Pr_h[h(x) = y \wedge h(x') = y] = \frac{1}{|Y|^2} = \frac{1}{c^2 \cdot s^4}$
Annotate the collision event "$\mathbf{h}(\mathbf{x}) = \mathbf{y} \wedge \mathbf{h}(\mathbf{x'}) = \mathbf{y}$" as $C(x, x', y)$;
$\Pr[C(x, x', y)] = \Pr_h[h(x) = y \wedge h(x') = y] = \frac{1}{c^2 \cdot s^4}$
Considering Traverse all of the (x,x',y), we have that:
$\Pr[\textbf{collision happens}] = \Pr[\cup_{x,x' \in \mathbf{X}, y \in \mathbf{Y}, x \neq x'} C(x, x', y)] \leq \Sigma_{x,x' \in \mathbf{X}, y \in \mathbf{Y}, x \neq x'} \Pr[C(x, x', y)]$
$= \binom{|\mathbf{Y}|}{1} \cdot \binom{|\mathbf{X}|}{2} \cdot \frac{1}{c^2 \cdot s^4} = \frac{s(s-1)}{2} \cdot cs^2 \cdot \frac{1}{c^2 \cdot s^4} = \frac{s-1}{2cs} \leq \frac{1}{2c}$.
This is the end of the prove.
