Exercise 1 *30%*
Recall the (Minimum) Vertex Cover and (Maximum) Independent Set problems. For the Vertex Cover problem, give a graph $G = (V, E)$, the goal is to find a smallest subset $S \subseteq V$ such that for any edge $(u, v) \in E$, either $u \in S$ or $v \in S$. For the Independent Set problem, given a graph $G = (V, E)$, the goal is to find a largest subset $I \subseteq V$ such that there is no edge between any two vertices in I.
A simple observation is: if S^* is the *minimum* vertex cover in G, then $V \setminus S^*$ is *maximum* independent set in G.
Suppose you have given a 2-approximation algorithm for Vertex Cover. Consider the following approximation algorithm for Independent Set problem: Given a graph $G = (V, E)$, use the 2-approximation algorithm for Vertex Cover on G to get a vertex cover S and output $V \setminus S$.

- Give an upper bound on the approximation ratio of the above algorithm for the Independent Set problem. (The smaller your bound is, the better.)

Solution for Exercise 1:

First of all, we know that:

$$|S^*| + |IS^*| = |V|.$$

notation:

$$\begin{cases} |S^*|: \text{the size of Minimum Vertex Cover.} \\ |IS^*|: \text{the size of Maximum Independent Set.} \\ |V|: \text{the size of set of the whole vertexs.} \end{cases}$$

and Besides, According to this algorithm.

$|S| + |IS| = |V|$ also holds,

where $|S|$ and $|IS|$ are the size of its outputted Vertex Cover and Independent set, correspondingly.

Let us denote the approximation-ratio of this Algorithm, for Max Independent Set problem is "r". $(0<r<1)$

Apparently,

$$|IS| \geq r \cdot |IS^*|$$, always hold.

(IS^* is the orcale optimal solution for MIS, while IS is this algorithm's output).

$$\Rightarrow \quad r \leq \frac{|IS|}{|IS^*|}$$, always hold.

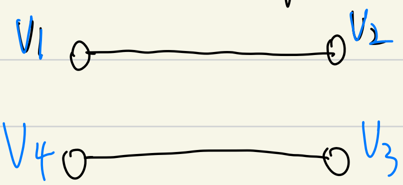
$$\Rightarrow \quad r \leq \frac{|IS|}{|IS^*|} = \frac{|V|-|S|}{|V|-|S^*|}$$, always hold. ①

meanwhile, according to " This Algorithm is a 2-approximation algorithm for Minimum Vertex Cover"

$$\Rightarrow \quad |S| \leq 2 \cdot |S^*|$$, always, hold.

Take the Algorithm on the slides as an example, (" GREEDY-VC" in slides 8)

as for the below example:



the greedy VC will output $S = \{v_1, v_2, v_3, v_4\}$. $= V$.

$\therefore$ according to ①: $r \leq \frac{|V|-|S|}{|V|-|S^*|} = \frac{0}{2} = 0$.

$\Rightarrow$ the upperbound of approx. ration for MIS problem. of such a algorithm $= 0$.

Exercise 2 *30%*
Consider the LCR algorithm for the leader election problem on the ring network.

- Give a UID assignment for which $\Omega(n^2)$ messages are sent.
- Give a UID assignment for which only $O(n)$ messages are sent.

Solution for Exercise2:

(1): Give a UID assignment for which $\Omega(n^2)$ messages are sent.

Solution for (1):

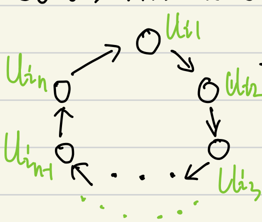
Suppose there are n different UIDs:

$$u_1, u_2, u_3, \cdots, u_n.$$

Resort these above n UIDs according to their value:

$$u_{i_1} > u_{i_2} > \cdots > u_{i_n}$$

Let us Assume that the message will be passed clock-wisely



Then, assign $u_{i_1}, u_{i_2}, \cdots, u_{i_n}$ one-by-one clockwisely, will make $\Omega(n^2)$ messages be sent in total during the execution of LCR algorithm.

Proof for (1):

According to the description of Algorithm LCR, in our case, message containing UID: u_{i_1} will be passed (sent) n times,

message containing UID: u_{i_2} will be sent $(n-1)$ times.

$\vdots$

message containing UID: u_{i_k} $(1 \le k \le n)$ will be sent $(n+1-k)$ times

To conclude, This assignment will make $\sum_{i=1}^{n} (n+1-i) = \frac{n(n+1)}{2} = \Omega(n^2)$ messages be sent in total.

(2) Give a UID assignment for which only $O(n)$ messages are sent.

Solution for (2):

Suppose that we have n different UIDs:

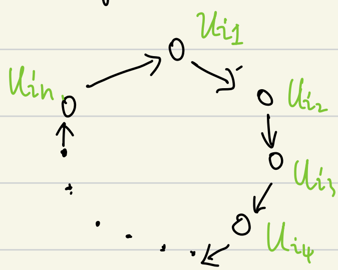
$u_1, u_2, \cdots, u_n$

Resort them increasingly: $\boxed{u_{i_1} < u_{i_2} < \cdots < u_{i_n}.}$

Let us still Assume that the message's will be passed clockwisely.

Then assign $u_{i_1}, u_{i_2}, \cdots, u_{i_n}$ one-by-one clockwisely, will make $O(n)$ messages be passed during the execution.

Of LCR algorithm in Our Case.



Proof of (2): According to the description of Algorithm LCR:

message containing u_{i1} will be sent only once.

message containing u_{i2} will be sent only once.

⋮

message containing u_{ik} will be sent only once ($1 \le k \le n$)

To conclude, There will be $\sum_{i=1}^{n} 1 = n = O(n)$ messages being sent.

$(1-\frac{1}{n}) \times m$ $\leq \frac{1}{n^2} \times k$

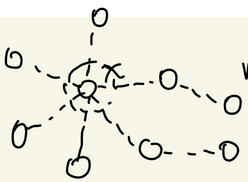
$(1-\frac{1}{n^2})$ $(1-\frac{k}{n^2})$

$(1-\frac{}{n^2}$

Exercise 3 *40%*

Consider a regularized variant of Luby's MIS algorithm as follows: The algorithm consists of $\log \Delta + 1$ phases, each made of $100 \cdot \log n$ consecutive rounds. Here Δ denotes the maximum degree in the graph. Let S be a set initialized to be empty. In each round of the i-th phase, each remaining node is marked with probability $\frac{2^i}{10\Delta}$. Different nodes are marked independently. Then marked nodes who do not have any marked neighbor are added to S, and removed from the graph along with their neighbors. If at any time, a node v becomes isolated and none of its neighbors remain, then v is also added to S and is removed from the graph.

- Argue that the set S at the end of the algorithm is an independent set.
- Prove that with high probability (i.e., with probability at least $1 - \frac{1}{n^C}$, for some constant $C > 1$), by the end of the i-th phase, in the remaining graph each node has degree at most $\frac{\Delta}{2^i}$.
- Conclude that the set S at the end of the algorithm is a maximal independent set, with high probability.



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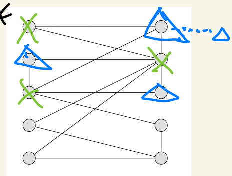

Solution for Problem 3:

(1): Argue that the set S at the end of the algorithm is an I.S.

Let us prove this by contradiction,

First, Assume that the set S is not an independent set. Then, we further induce that there exist at least 2 nodes (Let's denote them as 'V_1' and 'V_2', correspondingly), so that V_1 and V_2 are connected in the original graph.

①: if V_1 is added to S, prior to V_2, there will be a contradiction, for after V_1 is added to S, all of the neighbors (containing V_2) are supposed to be removed from the current graph., thus making it impossible for V_2 to appear in the S.

②: if V_2 is added to S, prior to V_1, we can similarly induce a contradiction.

$\Longrightarrow$ there will always have a contradiction.

$\Rightarrow$ Our initial assumption does not exist!

$\Rightarrow$ There are no 2 nodes in the S, so that they are directly connected in the original graph (neighbors).

$\Rightarrow$ S is an independent set.

Solution 2

We attempt to prove it by Mathematical Induction. that after the i-th phase, the max-degree of the remaining graph is at most. $\frac{\Delta}{2^i}$.

1°. Considering that $i=0$, Then the max-degree $\le \Delta = \frac{\Delta}{2^i}$.

$\Rightarrow$ The above conclusion holds when $i=0$

2°. Supposing that when $i=k$, the above conclusion holds, Which is that: the max-degree $\le \frac{\Delta}{2^k}$, with high $(1-\frac{1}{n^2})$ prob.

Then, we will induce that: the max-degree of the remaining graph after the $(k+1)$-phase $\le \frac{\Delta}{2^{k+1}}$, with hi Probability. $(1-\frac{1}{n^2})$

Firstly, Let us assume that after $(k+1)$-phase, there exist nodes whose degree $\ge \frac{\Delta}{2^{k+1}}+1$, denote one node in them as V, denote the set of "v" as M.

Then, we know: during each round in the $(k+1)$-phase,

$$\therefore \Pr[\{\text{one of } v\text{'s neighbors got marked}\}]$$

$$\geq 1 - \left(1 - \frac{2^{k+1}}{10\Delta}\right)^{\text{degree}(v)}$$

$$\geq 1 - \left(1 - \frac{2^{k+1}}{10\Delta}\right)^{\frac{\Delta}{2^{k+1}}+1} \geq 1 - \left(1 - \frac{2^{k+1}}{10\Delta}\right)^{\frac{\Delta}{2^{k+1}}}$$

$$\underset{\text{def: } \frac{\Delta \cdot 10}{2^{k+1}} = t}{=\!=\!=} 1 - \left(1 - \frac{1}{t}\right)^{\frac{t}{10}} = 1 - \left(\frac{1}{1+\frac{1}{t-1}}\right)^{\frac{t}{10}} = 1 - \left[\frac{1}{\left(1+\frac{1}{t-1}\right)^{t}}\right]^{\frac{1}{10}} \quad (*)$$

for, $e \leq \left(1+\frac{1}{t-1}\right)^{t} \leq 4$, we know that:

$$(*) \geq 1 - e^{-\frac{1}{10}} \geq 1 - 2^{-\frac{1}{10}} \quad ①$$

Then, if one node u is marked,

we need to consider the probability that none of

its neighbor got marked is:

$$\Pr[\text{none of } u\text{'s neighbor got marked}]$$

$$\geq \left(1 - \frac{2^{k+1}}{10\Delta}\right)^{\text{upper-bound}(\text{degree}(u))}$$

$$\geq \left(1 - \frac{2^{k+1}}{10\Delta}\right)^{\frac{\Delta}{2^k}} = \left(1 - \frac{2^k}{5\Delta}\right)^{\frac{\Delta}{2^k}}$$

according to the induction-assumptation: $\text{degree}(u) \leq \frac{\Delta}{2^k}$

$$\underset{\text{def: } \frac{5\Delta}{2^k} = t}{=\!=\!=} \left(1 - \frac{1}{t}\right)^{\frac{t}{5}} = \left[\left(\frac{1}{\frac{1}{t-1}+1}\right)^{t}\right]^{\frac{1}{5}}$$

$$\geq \left(\frac{1}{4}\right)^{\frac{1}{5}} = 4^{-\frac{1}{5}} \quad ②$$

Conjucting ① and ②, We can induce that, in each round,

$$\Pr[v \text{ is removed}] \geq ① \cdot ② = \left(1 - 2^{-\frac{1}{10}}\right) \cdot 4^{-\frac{1}{5}} \geq \frac{1}{25}.$$

$$Pr[v \text{ is not removed}] = 1 - Pr[v \text{ is removed}] \le 1 - \frac{1}{25}$$

in each phase, there are $[100 \cdot \log n]$ round in total.

$\Rightarrow$ $Pr[v \text{ is not removed, in } (k+1)\text{-th phase}]$

$$= \left(Pr[v \text{ is not removed}]\right)^{100 \cdot \log n}.$$

$$\le \left(1 - \frac{1}{25}\right)^{100 \log n} \quad \left(1 - \frac{1}{25}\right)^{25 \cdot 4 \log n}.$$

$$= \left[\left(1 - \frac{1}{25}\right)^{25}\right]^{4 \log n} \le 2^{-4 \cdot \log n} = \frac{1}{n^4} \quad ③$$

Considering all of the nodes in set M.

$Pr.[\text{any node in } M \text{ is not removed, in } (k+1)\text{-th phase}]$

$\le \frac{1}{n^4} \times n = \frac{1}{n^3}$. $\Rightarrow$ with at least $1 - \frac{1}{n^3}$, the conclusion holds.

$\Rightarrow$ the new "high probability"

$$\ge \left(1 - \frac{1}{n^2}\right) \cdot \left(1 - \frac{1}{n^3}\right) = 1 - \frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^5} \approx 1 - \frac{1}{n^2}$$

3°. Conclude that S is a maximal-independent-set with high probability.

Solution:

① in the (1)-part: we have proved that

what we finally get is an independent set.

② in the (2)-part, we can further induce that. after $\log(\Delta)+1$ phases, the max-degree of nodes in the remaining graph $\leq \frac{\Delta}{2^i} = \frac{\Delta}{2^{1+\log\Delta}} = \frac{1}{2} < 1$

$\Rightarrow$ there is no edges in the remaining graph 2.1°

Besides, if $v \notin S$, we can know that $\exists u$. s.t. v and u were directly connected in the original graph. 2.2°

Take all of the ①, 2.1°, 2.2° into consideration. We can know that. S is a maximal independent set. (with. a probability of at least $1-\frac{1}{n^2}$).