The “Polarisation Parallelogram” for the Visualisation and Statistical Comparison of Ordinal Scale Group Polarisation

Here I introduce a new mathematical and graphical framework for analysis of group cleavages. It allows for standardised and interpretable quantitative measures of polarisation and opinion extremity between two groups (such as political parties, religions or ideological identities) that considers the full distribution of answers meaningfully and an intuitive visual representation thereof. As I will demonstrate, it has academic, business analytic and data journalistic value. It is intuitive and visually illustrative, but still lends itself well to statistical analysis and has desirable mathematical properties.

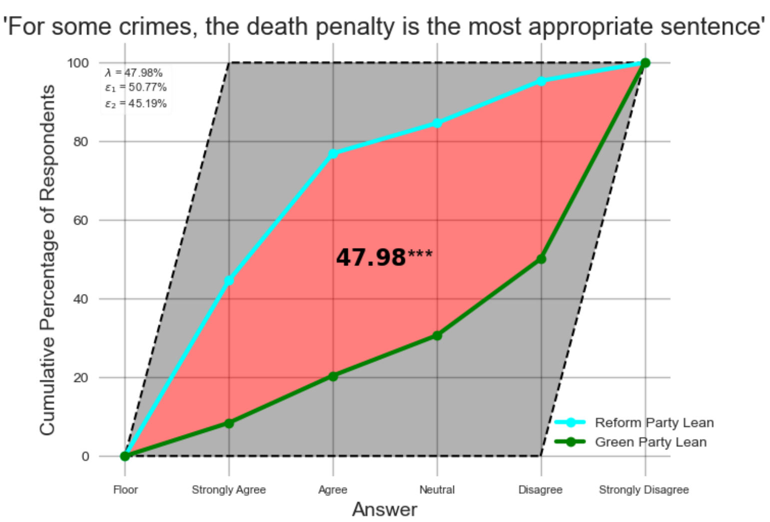


Figure 1: Green vs Reform Party leaners on the death penalty

Let us take an issue where two groups are expected to have differing distributions. I have chosen the death penalty as an example of an issue of this type, and the two groups as UK citizens who lean towards the **Green Party** - a left-leaning party - and citizens that lean towards the **Reform Party** - a right-wing party. I am using data from the British Social Attitudes Survey of 2021. The x axis contains an ordinal opinion scale with n (here, 5) categories on a given attitude or set of attitudes (for example, strongly agree to strongly disagree). The y axis is the cumulative percentage of respondents within a group who have answered along that scale (or have that score) up to that point.The two lines therefore show the cumulative distribution of opinions of two groups G_1 and G_2. The area between the two graphs, A, as a percentage of the total area of the parallelogram, is our polarisation λ where $\lambda \in [0, 1]$.

The maximum value is 1, or 100% this would be the case if the wedge area covered the entire parallelogram; in other words, if groups sorted entirely into respective extremes. The minimum value is 0; this would be the case if the area =0 (i.e. group distributions are identical). Let differences between categories along the x axis equal 1. Here are two ways of formulating the polarisation λ; one in terms of cumulative ‘percentage answered’ differences between groups in a given category $(y_{i1} - y_{i2})$, or D_i (i.e. vertical distances on the graph) and one in terms of differences *within* an individual category d_i:

$$1) \quad \lambda = \frac{\sum_{i=1}^{n-1}(y_{i1} - y_{i2})}{n-1} = \frac{\sum_{i=1}^{n-1} D_i}{n-1}$$

$$2) \quad \lambda = \frac{\sum_{i=1}^{n-1}\sum_{j=1}^{i} d_i}{n-1}$$

Proof of λ Formula

The area A is equal to the difference between the areas under G_1 and G_2. Using the trapezoidal method to calculate the areas under G_1 and G_2:

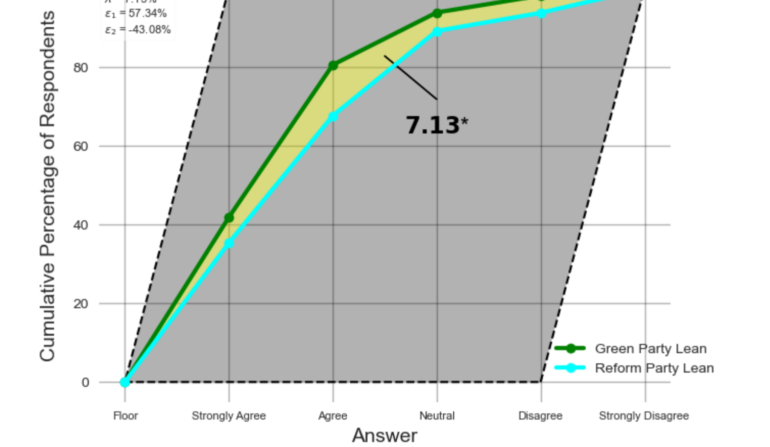
$$
\begin{aligned}
A = &\left(\frac{1}{2}\left(\frac{S_n - 0}{n}\right)(0 + 1 + 2(y_{11} + y_{21} + \ldots + y_{(n-1)1})) - \frac{1}{2}\cdot 1 \cdot 1\right) - \\
&\left(\frac{1}{2}\left(\frac{S_n - 0}{n}\right)(0 + 1 + 2(y_{12} + y_{22} + \ldots + y_{(n-1)2})) - \frac{1}{2}\cdot 1 \cdot 1\right) \\
&= \frac{1}{2}\left(2[(y_{11} - y_{12}) + (y_{21} - y_{22}) + \ldots + (y_{(n-1)1} - y_{(n-1)2})]\right) \\
&= \sum_{i=1}^{n-1}(y_{i1} - y_{i2})
\end{aligned}
$$

The area of the parallelogram is equal to $(n-1) * 1 = n-1$. Thus, Area A as a proportion of the parallelogram $= \frac{A}{A+B_1+B_2} = \frac{A}{n-1}$

$$
\begin{aligned}
&= \frac{\sum_{i=1}^{n-1}(y_{i1} - y_{i2})}{n-1} \\
&= \lambda \in [0, 1] = \textbf{Polarisation}
\end{aligned}
$$

In other words, the polarisation measure can be computed entirely by summing the cumulative distances from scale positions 1 to n-1, where n is the number of categories. This can also be thought of as the 'mean cumulative difference' across all categories with a non-trivial deviation (as deviation will by definition be zero once the final category is reached). My proofs of all the formulae I have derived can be found in the appendix.

In our first example, the figure is around 0.48, or just under 48%. The area is just under half the total parallelogram, or the 'theoretical extreme' where the Reform Party unanimously selects 'Strongly Agree' and the Green Party selects 'Strongly Disagree'. Generally speaking, we would expect a scenario anywhere close to this type to be extremely rare, especially in scales where strength of agreement or disagreement is possible. The asterisks next to the lambda figure indicate statistical significance.



Let us now take a second example. The question is on whether big businesses 'benefit owners at the expense of workers'. Here, the distributions between those that lean Green Party and those that lean Reform Party are much more similar. Lambda is just 0.0713. One can also notice that the Green Party is now 'on the top'. This makes no difference to Lambda, but illustrates that Green Party voters are now more likely to agree with the statement. This shows the value of the parallelogram in comparing polarisation *between* issues, especially when scales are identical. The first question is more about social values and tradition, whereas the second is more about an attitude to economic inequality. Both parties being somewhat populist, the distributions of answers are more similar.

Now let us move to our measures of extremity ϵ_1 and ϵ_2, derived from the areas B_1 and B_2. If $\frac{B_j}{n-1} = 0$, then Group G_j's opinions are 100% 'extreme'. $\frac{B_j}{n-1} = 0.5$ would include, for example, a uniform distribution of answers or a situation in which all respondents chose the middle option. For a natural interpretation of $\epsilon = 0$ for central distributions and $\epsilon = \pm 1$ for maximum extremity:

$$\epsilon_j = 1 - \frac{2B_j}{n-1}$$

It can be shown that:

$$\epsilon_1 = \frac{2\sum_{i=1}^{n-1} y_{i1}}{n-1} - 1, \quad \epsilon_2 = 1 - \frac{2\sum_{i=1}^{n-1} y_{i2}}{n-1}$$

An ϵ value less than zero indicates that both groups skew in the same direction on an issue. We will see later that attitudes towards race-based affirmative action are an example. My suggestion is that λ, ϵ_1 and ϵ_2 are all reported together along with the visualisation (as this paper will do). Contextualising λ with ϵ allows for the reader to distinguish between, for example, low values of λ drawn from overall consensus on one side of a debate (this would mean ϵs have opposite signs) from low values of λ drawn from a consensus of neutrality (this would mean ϵs have values close to 0). Additionally, λ and ϵ are linked mathematically by the elegant relationship:

$$\lambda = \frac{\epsilon_1 + \epsilon_2}{2}$$

Proof of ϵ Formulae

Let $\epsilon_j = 1 - \dfrac{2B_j}{n-1}$

For our purposes it is easier to start with the derivation of ϵ_2.

$$\begin{aligned}
B_2 &= \frac{1}{2}\left(\frac{S_n - 0}{n}\right)(0 + 1 + 2(y_{12} + y_{22} + \ldots + y_{(n-1)2})) - \frac{1}{2} \\
&= \frac{1}{2}(1 + 2(y_{12} + y_{22} + \ldots + y_{(n-1)2})) - \frac{1}{2} \\
&= \sum_{i=1}^{n-1} y_{i2}
\end{aligned}$$

Thus, $\epsilon_2 = 1 - \dfrac{2\sum_{i=1}^{n-1} y_{i2}}{n-1}$

ϵ_1 can now be derived using our formula forB_2 :

$$\begin{aligned}
B_1 &= (n-1) - A - B_2 \\
&= (n-1) - \sum_{i=1}^{n-1}(y_{i1} - y_{i2}) - \sum_{i=1}^{n-1} y_{i2} \\
&= (n-1) - \sum_{i=1}^{n-1} y_{i1} + \sum_{i=1}^{n-1} y_{i2} - \sum_{i=1}^{n-1} y_{i2} \\
&= (n-1) - \sum_{i=1}^{n-1} y_{i1}
\end{aligned}$$

Substituting into the formula for ϵ_1 :

$$\begin{aligned}
\epsilon_1 &= 1 - \frac{2(n - 1 - \sum_{i=1}^{n-1} y_{i1})}{n-1} \\
&= 1 - \frac{2(n-1)}{n-1} + \frac{2\sum_{i=1}^{n-1} y_{i1}}{n-1} \\
&= \frac{2\sum_{i=1}^{n-1} y_{i1}}{n-1} - 1
\end{aligned}$$

Hence,

$$\begin{aligned}
\epsilon_1 &= \frac{2\sum_{i=1}^{n-1} y_{i1}}{n-1} - 1 \\
\epsilon_2 &= 1 - \frac{2\sum_{i=1}^{n-1} y_{i2}}{n-1}
\end{aligned}$$

Proof of λ and ϵ Relationship

Recall that $\epsilon_j = 1 - \frac{2B_j}{n-1}$

By rearranging, $\frac{B_j}{n-1} = \frac{1-\epsilon_j}{2}$

Recall that the parallelogram area is equal to $A + B_1 + B_2 = n - 1$

Thus, $\frac{A}{n-1} + \frac{B_1}{n-1} + \frac{B_2}{n-1} = 1$

$\lambda + \frac{1-\epsilon_1}{2} + \frac{1-\epsilon_2}{2} = 1$

$\lambda + \frac{1}{2} + \frac{1}{2} - \frac{\epsilon_1 + \epsilon_2}{2} = 1$

$\lambda = \frac{\epsilon_1 + \epsilon_2}{2}$

Hence, polarisation λ is equal to the midpoint of the two extremity measures ϵ_1 and ϵ_2.

Intuitively, as respective extremity values move closer to 1, polarisation strictly increases and if $\epsilon_1 = -\epsilon_2$, the distributions will be identical and polarisation will be zero.

To return to our two examples - in the death penalty case, both *epsilon* values are positive. This means that both groups diverge in opposite directions from neutrality. $epsilon_1$ is higher (0.57, or 57%) than $epsilon_2$ (0.45, or 45%), meaning that the Reform Party leaners are slightly more in favour of the death penalty than Green Party leaners are against, but both distributions are reasonably strongly in the expected direction.

However, in our second example, both Green Party and Reform Party leaners are more likely to answer on the side of agreement. This means that the 'lower' group (Reform) cross neutrality into 'negative' territory (remember that the possible range of *epsilon* values is -1 to 1.)

It can also show that λ is equivalent to half the absolute value of the difference between the recentered group means were the scale quantified on a [-1,1] scale.

Proof of Equivalence to Recentered Means

For this result it is once again easier to start with ϵ_2.

Recall that $B_2 = \sum_{i=1}^{n-1} y_{i2}$

Each cumulative percentage is itself the sum of all previous percentages.

Thus, if P_i is the proportion of respondents in the ith category in group 2:

$$B_2 = (n-1)P_1 + (n-2)P_2 + \ldots + P_{n-1}$$

$$\epsilon_2 = 1 - \frac{2[(n-1)P_1 + (n-2)P_2 + \ldots + P_{n-1}]}{n-1}$$

For an integer ordinal scale, the mean score across all respondents can be computed as:

$$\mu = P_1 + 2P_2 + \ldots + nP_n$$

$$P_n = 1 - (P_1 + P_2 + \ldots + P_{n-1})$$

Thus,

$$\mu = n(1 - (P_1 + P_2 + \ldots + P_{n-1})) + P_1 + 2P_2 + \ldots + P_{n-1}$$

$$= n + (1-n)P_1 + (2-n)P_2 + \ldots - P_{n-1}$$

We want our scale to move from [1,n] to [-1,1]. If x $\in [1, n]$ is a point on the old scale it can be rescaled with: $\frac{2}{n-1} \cdot (x - \frac{n+1}{2})$

By the linearity of expected values, this also applies to our mean:

$$\begin{aligned}
\tilde{\mu} &= \frac{2}{n-1} \cdot (\mu - \frac{n+1}{2}) \\
&= \frac{2}{n-1} \cdot \mu - \frac{n+1}{2} \cdot \frac{2}{n-1} \\
&= \frac{2[(n + (1-n)P_1 + (2-n)P_2 - P_{n-1})]}{n-1} - \frac{n+1}{n-1} \\
&= \frac{2n}{n-1} - \frac{n+1}{n-1} + \frac{2[(1-n)P_1 + (2-n)P_2 - P_{n-1}]}{n-1} \\
&= 1 + \frac{2[(1-n)P_1 + (2-n)P_2 - P_{n-1}]}{n-1} \\
&= 1 - \frac{2[(n-1)P_1 + (n-2)P_2 + P_{n-1}]}{n-1} \\
&= \epsilon_2
\end{aligned}$$

Thus, our extremity value ϵ_2 is equal to the mean score if the ordinal scale were centered at 0 with equally spaced values ranging from -1 to 1.

Almost the same applies to ϵ_1; however, in this case it is the negative recentered mean.

If we multiply our ϵ_1 formula by -1 we get

$-\epsilon_1 = 1 - \frac{2\sum_{i=1}^{n-1} y_{i1}}{n-1}$ and then the proof is identical to that of ϵ_2.

As $\lambda = \frac{\epsilon_1 + \epsilon_2}{2}$, by substituting $-\tilde{\mu_1}$ for ϵ_1 and $\tilde{\mu_2}$ for ϵ_2, λ can be written as:

$$\lambda = \frac{\tilde{\mu_2} - \tilde{\mu_1}}{2}$$

This means that the statistical significance of lambda is also equivalent to a difference in means test. The statistical significance of this test is indicated by the asterisks next to the lambda figure in the diagrams.

The visual interpretation (i.e. in terms of areas) is subject to the assumption of no crossing between group cumulative distributions. However, the mathematics holds no matter what and for groups where opinion polarisation is expected, crossing is unlikely to occur. Mathematically speaking, if the distributions cross, the area is equal to the *absolute value* of the sum of cumulative differences. In this scenario, it would no longer be equal to the difference in means and so the statistical interpretation would be different.

For the study of relative ideological polarisation and group extremity in two theoretically meaningful groups within and between issues, the parallelogram provides a concise, illuminative, reliable and understandable method of extracting and understanding these multiple relevant characteristics. Like the Esteban model, it involves splitting into defined groups. However, it is designed for a measure of polarisation more similar to that of Draca (2018) where the focus of analysis is between predefined groups and the distribution of survey scores. It strikes an elegant balance between interpretability and rigour, not only having the advantage of relaxing any unidimensionality assumption, but being a) comparable across scales with differing numbers of options, b) comparable across groups of any relative size, c) representable in an intuitive visual way and, e) directly related to equally interpretable measures of individual group extremity, both of which are also computable from the parallelogram.

Of course, while the numerical figure has an intuitive interpretation geometrically (i.e. percentage of the area were there theoretical maximum polarisation) it doesn't exactly mean too much on its own. So what's the best way to use this parallelogram? Well, I find that it has most use as a *comparative* measure. The two main types of comparison I find it to be fruitful for are *within-groups, between issues* and *within-issue, different groups.*

The former is essentially what we did above. We found that, on an agree-disagree scale, those who leaned to the Green Party were *very* polarised compared to those who learned towards Reform on the death penalty, but both seemed to basically agree that big businesses will try to benefit themselves at the expense of workers. This is pretty interesting. The analysis would then be up to whoever generated the graph. Is is that both parties are essentially 'populist' or have their own interpretation of what it means to 'help working people'? Generating graphs for other 'traditional values' questions versus economics questions seems to suggest that the trend of 'disagreement on moral issues, agreement on economic issues' basically holds true in general for these two parties (try it!)

a) Green Party and all other parties

b) Reform Party and all other parties

c) Other parties with each other

What are the patterns? Once you know, it's up to you how you interpret or dive deeper!