University of Toronto Mississauga
An Introduction to Probability and Modelling
STA107H5, Winter 2024

Solutions Key

Term Test 2
Duration: 75 Minutes

Last Name / Surname (please print): ....

First Name (please print): ....

Student ID Number: ....

INSTRUCTIONS and POLICIES:

- Please DO NOT rip off any page from this booklet.
- For all questions, complete solutions are required. Show your work to earn full marks and then circle the final answer where appropriate.
- You are allowed to use a non-programmable calculator.

Question	**1**	**2**	**3**	**4**	**TOTAL**
Value	10	10	10	10	**40**
Points					

Question 1 (10 points = 5+5). From 8 married couples who have attended a party, we want to select a group of 4 people and award them prizes for the best outfit of the night.

a) How many choices are there if the group does not contain a married couple?

b) How many choices are there if the group does not contain a married couple and further it consists of 2 men and 2 women?

Part (a).

Solution 1: We first select 4 out of the 8 married couples in $\binom{8}{4}$ different ways since each individual in the group of 4 must come from a different married couple. Then, for each of the 4 chosen couples, we select one individual (either the husband or the wife, but not both) in 2^4 ways as there are 2 choices (husband or wife) for each couple. So, the total number of ways is

$$\binom{8}{4} \times 2^4 = \frac{8!}{4!\,4!} \times 2^4 = \frac{8 \times 7 \times 6 \times 5}{4!} \times 2^4 = 1120$$

Solution 2: We select the first person in 16 different ways. For the second person, however, we must avoid selecting the spouse of the first person, which is possible in 14 different ways remaining for the second person. Similarly, there will be 12 and 10 choices remaining for the third and fourth persons. But, a group of 4 is the same regardless of the order in which they are selected, so we must divide by 4!. Thus, the total number of ways is

$$\frac{16 \times 14 \times 12 \times 10}{4!} = 1120$$

Remark Notice the following relation between solutions 1 and 2:

$$\frac{16 \times 14 \times 12 \times 10}{4!} = \frac{(2 \times 8) \times (2 \times 7) \times (2 \times 6) \times (2 \times 5)}{4!} = \frac{8 \times 7 \times 6 \times 5}{4!} \times 2^4$$

Part (b).

We first select 2 out of the 8 men in $\binom{8}{2}$ different ways. Then, excluding the spouses of the selected men, 2 women are selected in $\binom{6}{2}$ different ways. So, the total number of ways is

$$\binom{8}{2} \times \binom{6}{2} = \frac{8 \times 7}{2} \times \frac{6 \times 5}{2} = 420$$

Question 2 (10 points). A player throws darts at a target. On each trial, independently of the other trials, the player hits the target with a probability of 0.1 . How many times should this player throw darts to achieve a probability of at least 0.8 of hitting the target at least once? You can assume the experiment gets terminated once the target is hit.

Let E_i represent the event of hitting the target on the ith trial of n independent throws.

Then, $E_1 \cup E_2 \cup \ldots \cup E_n$ is the event corresponding to hitting the target at least once and $E_1' \cap E_2' \cap \ldots \cap E_n'$ that of not hitting the target at all. Thus,

$$\begin{aligned} \mathrm{P}(E_1 \cup E_2 \cup \ldots\ E_n) &= 1 - \mathrm{P}(E_1' \cap E_2' \cap \ldots \cap E_n') \\ &= 1 - (1 - 0.1)^n \\ &= 1 - 0.9^n \end{aligned}$$

We then set this equal (or greater) to 0.8 and solve for n:

$$\begin{aligned} &1 - 0.9^n = 0.8 \implies \\ &0.9^n = 0.2 \implies \\ &\ln 0.9^n = \ln 0.2 \implies \\ &n \ln 0.9 = \ln 0.2 \implies \\ &n = \frac{\ln 0.2}{\ln 0.9} \approx 15.275 \end{aligned}$$

However, the number of throws is an integer value, so $n = 16$.

Question 3 (10 points). A large department store wants to study the purchasing habits of its customers for three specific products, I, II and III. The data given in the following table come from a market research that the store conducted and show the proportions of customers who purchase one or more among the three products during a specific period

Product	Percentage of buyers
I	0.25
II	0.40
III	0.12
I & II	0.17
I & III	0.10
II & III	0.07
I, II & III	0.05

Assuming that these proportions reflect the probability for any customer to buy these product(s), calculate the probability that a customer buys product III given (s)he has bought at least one of products I and II.

Let E_1, E_2 and E_3 denote the events that the customer buys products I, II and III, respectively. Then,

$$\begin{aligned}
\mathrm{P}(E_3 \mid E_1 \cup E_2) &= \frac{\mathrm{P}(E_3 \cap (E_1 \cup E_2))}{\mathrm{P}(E_1 \cup E_2)} \\
&= \frac{\mathrm{P}((E_1 \cap E_3) \cup (E_2 \cap E_3))}{\mathrm{P}(E_1 \cup E_2)} \\
&= \frac{\mathrm{P}(E_1 \cap E_3) + \mathrm{P}(E_2 \cap E_3) - \mathrm{P}(E_1 \cap E_2 \cap E_3)}{\mathrm{P}(E_1) + \mathrm{P}(E_2) - \mathrm{P}(E_1 \cap E_2)} \\
&= \frac{0.1 + 0.07 - 0.05}{0.25 + 0.4 - 0.17} \\
&= 0.25
\end{aligned}$$

Question 4 (10 points). A PCR test is 90% effective in detecting COVD-19 when a person suffers from the disease (test result is positive if a person has the disease). However, the test also yields a "false positive" result for 2% of the healthy persons tested (if a healthy person is tested, with a probability of 0.02 , the test result will be positive, implying that he or she has COVID-19). If 0.4% of the population actually has the disease, what is the probability that a person has the disease, given that the test result is positive?

HINT: Use the Bayes' Theorem.

Let us define the following events:

$$D := \text{event that the person has the disease}$$

$$T^+ := \text{event that the person tests positive}$$

Then,

$$\begin{aligned}
\mathrm{P}(D \mid T^+) &= \frac{\mathrm{P}(T^+ \mid D)\,\mathrm{P}(D)}{\mathrm{P}(T^+ \mid D)\,\mathrm{P}(D) + \mathrm{P}(T^+ \mid D')\,\mathrm{P}(D')} \\
\mathrm{P}(D \mid T^+) &= \frac{\mathrm{P}(T^+ \mid D)\,\mathrm{P}(D)}{\mathrm{P}(T^+ \mid D)\,\mathrm{P}(D) + \mathrm{P}(T^+ \mid D')(1 - \mathrm{P}(D))} \\
&= \frac{0.9 \times 0.004}{0.9 \times 0.004 + 0.02 \times 0.996} \\
&\approx 0.153
\end{aligned}$$