Solutions key

University of Toronto Mississauga

An Introduction to Probability and Modelling
STA107H5 S LEC0101
Summer 2023

Midterm
Duration: 90 Minutes
Date: July 21, 2023

Last Name / Surname (please print): ....

First Name (please print): ....

Student ID Number: ....

INSTRUCTIONS and POLICIES:

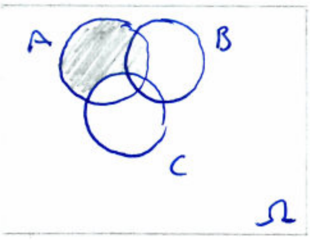
- Please DO NOT rip off any page from this booklet.
- For all questions, complete solutions are required. Show your work to earn full marks and then circle the final answer where appropriate.
- You are allowed to use a non-programmable calculator.
- Simplify final answers and round to 3 decimal places where appropriate.

Question	**1**	**2**	**3**	**4**	**5**	**TOTAL**
Value	10	10	5	10	15	**50**
Points						

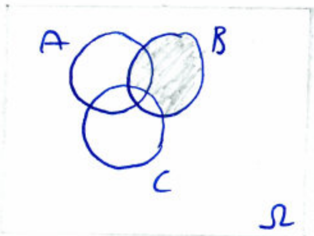
Question 1 (10 points). Let A, B, and C be three events on a sample space Ω. Use the Venn diagrams to examine whether the following relation is always true:

$$(A - B) - C = (A - C) - (B - C).$$

RHS:

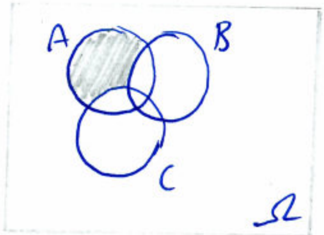


$A - C$



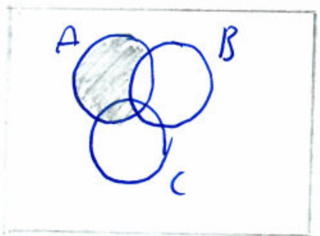
$B - C$

$\Rightarrow$



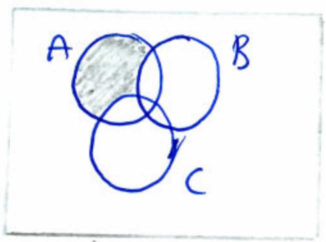
$(A - C) - (B - C)$

LHS:



$A - B$

$\Rightarrow$



$(A - B) - C$

$\therefore$ The given relation holds true. In other words, it is an identity.

Question 1 (cont'd).

Question 2 (10 points). Simplify the following expression using the properties of operations among events:

$$(A \cup B \cup C)(A \cup BC').$$

Hint: *You should* ***justify*** *each step of your solution by referring to at least one of the properties $SP1$-$SP11$, included within the appendix page of this exam, or any other property such as the De Morgan's Laws.*

Question 2 (cont'd).

$$(A \cup B \cup C)(A \cup BC') \overset{SP6}{=} \left[(A \cup B \cup C)A\right] \cup \left[(A \cup B \cup C)BC'\right]$$

$$\overset{SP11}{=} A \cup \left[(A \cup B \cup C)BC'\right]$$

$$\overset{SP6}{=} A \cup \left[ABC' \cup BBC' \cup CBC'\right]$$

$$\overset{SP1}{=} A \cup \left[ABC' \cup BC' \cup CBC'\right]$$

$$\overset{SP4}{=} A \cup \left[ABC' \cup BC' \cup BCC'\right]$$

$$\overset{SP7,\, SP2}{=} A \cup \left[ABC' \cup BC' \cup \emptyset\right]$$

$$\overset{SP2}{=} A \cup \left[ABC' \cup BC'\right]$$

$$\overset{SP5}{=} \left[A \cup ABC'\right] \cup BC'$$

$$\overset{SP11}{=} (A \cup A) \cup BC'$$

$$\overset{SP1}{=} A \cup BC' = \boxed{A \cup (B - C)}$$

Question 3 (5 points). Let A, B, and C be three events in a sample space Ω, such that

$$P(A) = 2P(B), \quad P(B) = 2P(C) \quad \text{and} \quad A = (B \cup C)'.$$

Assuming that the events B and C are mutually exclusive, compute $P(A)$.

First, note that

$$P(A) = 2P(B) \implies P(B) = \frac{P(A)}{2}$$

and

$$P(B) = 2P(C) \implies P(C) = \frac{P(B)}{2} = \frac{1}{2}\left(\frac{P(A)}{2}\right) = \frac{P(A)}{4}$$

Now,

$$A = (B \cup C)' \implies P(A) = P((B \cup C)')$$

$$\implies P(A) = 1 - P(B \cup C)$$

$$\implies P(A) = 1 - \left[P(B) + P(C) - P(BC)\right]$$

$$\implies P(A) = 1 - \left[\frac{P(A)}{2} + \frac{P(A)}{4} + P(\emptyset)\right]$$

$$\implies P(A) + \frac{P(A)}{2} + \frac{P(A)}{4} = 1$$

$$\implies \frac{7P(A)}{4} = 1 \implies \boxed{P(A) = \frac{4}{7}}$$

Question 4 (10 points). How many distinct 4-digit numbers can be formed using the digits $\{1, 2, \ldots, 9\}$, which are strictly greater than 4721?

We need to consider 4 different cases:

$\boxed{\text{Case I}}$ Numbers whose thousands are greater than 4:

$$\underset{1}{\underline{5}} \times \underset{2}{\underline{9}} \times \underset{3}{\underline{8}} \times \underset{4}{\underline{7}} = 2520$$

$\boxed{\text{Case II}}$ Numbers whose thousands are 4 and their hundreds are greater than 7:

$$\underset{1}{\underline{1}} \times \underset{2}{\underline{2}} \times \underset{3}{\underline{8}} \times \underset{4}{\underline{7}} = 112$$

$\boxed{\text{Case III}}$ Numbers whose thousands and hundreds are 4 and 7, respectively, but their tens are greater than 2:

$$\underset{1}{\underline{1}} \times \underset{2}{\underline{1}} \times \underset{3}{\underline{5}} \times \underset{4}{\underline{7}} = 35$$

$\boxed{\text{Case IV}}$ Numbers whose thousands, hundreds and tens are 4, 7 and 2, respectively, but their ones are greater

Question 4 (cont'd).

than $\underline{\underline{1}}$:

$$\frac{1}{1} \times \frac{1}{2} \times \frac{1}{3} \times \frac{5}{4} = 5$$

Thus, using the Addition Principal, we get :

$$2520 + 112 + 35 + 5 = \boxed{2672}$$

Question 5 (15 points = 5+5+5). A high-school class is attended by 10 boys $B_1, B_2, \ldots, B_{10}$ and 5 girls $G_1, G_2, \ldots, G_5$. After their final exams, all students are graded and ranked from 1 to 15.

a) How many different rankings are possible?

b) What is the probability that all 10 boys have consecutive ranks (e.g. from rank 2 to 11)?

c) What is the probability that at least one girl is ranked immediately after another girl?

a) $15!$

b) We will consider the group of boys as a single object and find the permutations as follows:

$$G_1, G_2, \boxed{B_1, B_2, \ldots, B_9, B_{10}}, G_3, G_4, G_5$$

Let A be the event that all boys are ranked consecutively. Then,

Question 5 (cont'd).

$$|A| = \underbrace{(6!)}_{\text{permutations of girls with the group of boys}} \underbrace{(10!)}_{\text{permutations of boys within the box}}$$

So,

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6! \cdot 10!}{15!}$$

$$= \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 10!}{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10!}$$

$$= \frac{4}{(14)(13)(11)}$$

$$= \frac{2}{1001} \approx \boxed{1.998 \times 10^{-3}}$$

Question 5 (cont'd).

c) Note that the complement of having at least one girl ranked immediately after/before another girl is when a girl is never ranked immediately after/before another girl. So, let's first fix the boys and compute their permutations and then use the empty spots in between to order girls.

$$\square B_1 \square B_2 \square \cdots \square B_{10} \square$$

Let E be the event that at least a girl is ranke[d] immediately after another girl.

$|E'| = (10!)\,(11)_5 \Rightarrow |E| = 15! - (11)_5 \cdot 10! \Rightarrow P(E) = \frac{11}{13}$ [...]

≈ 0.8[...]

APPENDIX

Proposition 1.1 *The following properties hold true for the operations among events:*

$SP1.$ $A \cup A = A \qquad AA = A$

$SP2.$ $A \cup \emptyset = A \qquad A\emptyset = \emptyset$

$SP3.$ $A \cup \Omega = \Omega \qquad A\Omega = A$

$SP4.$ $A \cup B = B \cup A \qquad AB = BA$

$SP5.$ $A \cup (B \cup C) = (A \cup B) \cup C \qquad A(BC) = (AB)C$

$SP6.$ $A \cup (BC) = (A \cup B)(A \cup C) \qquad A(B \cup C) = (AB) \cup (AC)$

$SP7.$ $A \cup A' = \Omega \qquad AA' = \emptyset$

$SP8.$ $(A')' = A$

$SP9.$ *If* $A \subseteq B$ *and* $B \subseteq C$, *then* $A \subseteq C$

$SP10.$ *If* $A \subseteq B$, *then* $B' \subseteq A'$ *and vice versa*

$SP11.$ *If* $A \subseteq B$, *then* $AB = A$ *and* $A \cup B = B$.