Problem Set #4;

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Problem 1

Let $M_1 = (Q_1, \Sigma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma_2, \delta_2, q_2, F_2)$. Such that $L(M_1) = A$ and $L(M_2) = B$. Build $M = (Q, \Sigma, \delta, q_0, F)$ such that $L(M) = f(A, B)$.

$$
\begin{aligned}
Q &= Q_1 \times Q_2 \times \{1, 2\} \\
\delta(q, t) &= \delta_1(q, t) \rightarrow q \times t \times 1 && \forall q \in Q_1, \forall t \in \Sigma \\
\delta(q, t) &= \delta_2(q, t) \rightarrow q \times t \times 2 && \forall q \in Q_2, \forall t \in \Sigma \\
q_0 &= q_1 \\
F &= F_1 \times F_2
\end{aligned}
$$

This works for any state in $f(A, B)$ because for any state in Q_2 a Σ symbol can be applied specifically to $q \in Q_1$ to progress the sub DFA A and the inverse is true for the DFA for B.

Problem 2

Let $M_1 = (Q_1, \Sigma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma_2, \delta_2, q_2, F_2)$. Such that $L(M_1) = A$ and $L(M_2) = B$. Build $M = (Q, \Sigma, \delta, q_0, F)$ such that $L(M) = f(A, B)$.

$$
\begin{aligned}
Q &= Q_1 \times Q_2 \times \{1, 2\} \\
\delta(q, t) &= \delta_1(q, t) \rightarrow q \times t \times 1 && \forall q \in Q_1, \forall t \in \Sigma \\
\delta(q, t) &= \delta_2(q, t) \rightarrow q \times t \times 2 && \forall q \in Q_2, \forall t \in \Sigma \\
\delta(q, \epsilon) &= q \\
q_0 &: initialstate \\
F &= F_1 \times F_2
\end{aligned}
$$

This works for any state in $f(A, B)$ because for any state in Q_2 a Σ symbol can be applied specifically to $q \in Q_1$ to progress the sub DFA A and the inverse is true for the DFA for B. Also for any ϵ symbol the sub DFAs don't progress.

Problem 3

Let $M_1 = (Q_1, \Sigma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma_2, \delta_2, q_2, F_2)$. Such that $L(M_1) = A$ and $L(M_2) = B$. Build $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ such that $L(M) = f(A, B)$.

$$
\begin{aligned}
Q &= Q_1 \times Q_2 \times \{1, 2\} \\
\Gamma &= \{X\} \\
\delta(q_0, \epsilon, \$) &= \{(q_1, \$)\} \\
\delta(q, t, \epsilon) &= q \times t \times 1 \to X && \forall q \in Q_1, t \in \Sigma^* \\
\delta(q, t, X) &= q \times t \times 2 \to \epsilon && \forall q \in Q_2, t \in \Sigma^* \\
q_0 &: initialstate \\
F &= F_1 \times F_2
\end{aligned}
$$

This will keep track of the length of each input string by placing a symbol on the stack and popping it off as the two sub routines run to their final states.

Problem 4

To show that $B \setminus A$ is decidable with A being a CFL and B being a regular language we make a decider D in the following form:

$$
\begin{aligned}
D =&\text{"}Given\ input < A, B, w > . \\
&1.Convert\ A\ to\ CNF. \\
&2.Simulate\ A\ on\ w\ for\ 2n-1\ steps,\ n\ being\ the\ amount\ of\ rules\ in\ A. \\
&3.Simulate\ B\ on\ w. \\
&4.Accept\ if\ B\ accepts\ and\ A\ rejects,\ otherwise\ reject.\text{"}
\end{aligned}
$$

Problem 5

The language is defined in the following manner:
$A = \{< M > \mid$ given a representation of a program M accept when M crashes$\}$
$A_{TM} \leqslant_m A$ using the mapping function $f(< M, w >) \mapsto < M >$. Assuming that A is RE then A_{TM} would also be RE which has been proven otherwise creating a contradiction. Meaning A is not RE.

Problem 6

By definition of ALL_{TM} any input TM is seen to accept Σ^* meaning that ALL_{TM} is not decidable because for some input TMs will loop while running. Let A be a TM that runs ALL_{TM} to prove that it is RE. A is defined in the following manner:

$$
\begin{aligned}
&A = \text{"}On\ input\ < M > .\\
&\quad 1.Run\ EQ_{TM}\ < M, C >\ with\ C\ defined\ as\ so:\\
&\quad\quad C\ ='\ Accept\ all\ strings.'\\
&\quad 2.If\ EQ_{TM}\ accepts,\ accept,\ otherwise\ reject.\text{"}
\end{aligned}
$$

We have proven that EQ_{TM} is RE then by having a TM for ALL_{TM} the compliment of ALL_{TM} is not RE because for the compliment to be RE the TM would have to be a decider which it is not.

Problem 7

Assume for contradiction that there a procedure P that on input $< G >$ that will return $< G_{normal} >$ such taht if G and G' are CFG's such that $L(G) = L(G')$then $G_{normal} = G'_{normal}$ and if $L(G) \neq L(G')$ then $G_{normal} \neq G'_{normal}$. Let $A = \{a^n b^m | n, m \geqslant 0\}$ .Let $A' = A$ applying P to A and A' will yelid us with G_{normal} and G'_{normal} respectively and to show that the results are not unique:

$$
\begin{aligned}
G_{normal} = &S \rightarrow aSbT|a\\
&T \rightarrow bT|b
\end{aligned}
$$

$$
\begin{aligned}
G'_{normal} = &S \rightarrow aTbS|b\\
&T \rightarrow aT|a
\end{aligned}
$$

Both of these CFGs produce A and with no smaller possible CFG we show the contradiction with $G_{normal} \neq G'_{normal}$ with $G = G'$.