Proof

(a) $\nabla_x (x^T A) = \nabla_x (a^T x) = a$

given :

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

let $Q = x^T a = [x_1 \; x_2 \cdots x_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$\Rightarrow Q = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$

Now. $\nabla_x Q = \begin{bmatrix} \frac{\partial Q}{\partial x_1} \\ \frac{\partial Q}{\partial x_2} \\ \vdots \\ \frac{\partial Q}{\partial x_n} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a$

Also, $P = a^T x = [a_1 a_2 \cdots a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$P = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

Can also be written as

$$P = x_1a_1 + x_2a_2 + \cdots x_na_n$$

$$\Rightarrow P = Q$$

$$\Rightarrow \nabla_x Q = \nabla_x P = a$$

$$\Rightarrow \nabla_x(x^Ta) = \nabla_x(a^Tx) = a$$

(b) $\nabla_x(x^TAx) = (A + A^T)\,x$

given: $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ A_{n1} & \cdots & & A_{nn} \end{bmatrix}$

let $Q = x^T A x$

$$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \\ A_{n1} & & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} x_i A_{i1} & \sum_{i=1}^{n} x_{i2} & \cdots & \sum_{i=1}^{n} x_i A_{in} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Q = \sum_{i=1}^{n} x_i A_{i1} x_1 + \sum_{i=1}^{n} x_i A_{i2} x_2 + \cdots + \sum_{i=1}^{n} x_i A_{in} x_n$$

Now,

$$\nabla_x Q = \begin{bmatrix} \frac{\partial Q}{\partial x_1} \\ \frac{\partial Q}{\partial x_2} \\ \vdots \\ \frac{\partial Q}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1}\left(x_1 A_{11} x_1 + x_2 A_{21} x_1 + x_1 A_{12} x_2 + \cdots + x_n A_{n1} x_1 + x_n A_{nn} x_n\right) \\ \frac{\partial}{\partial x_2}\left(x_1 A_{11} x_1 + \cdots + x_n A_{nn} x_n\right) \\ \vdots \\ \frac{\partial}{\partial x_n}\left(x_1 A_{11} x + \cdots + x_n A_{nn} x_n\right) \end{bmatrix}$$

$$Q = \begin{bmatrix} 2A_{11}x_1 + (A_{12}+A_{21})x_2 + \cdots + (A_{1n}+A_{n1})x_n \\ (A_{12}+A_{21})x_1 + 2A_{22}x_2 + \cdots + (A_{1n}+A_{n1})x_n \\ \vdots \\ (A_{n1}+A_{1n})x_1 + (A_{n2}+A_{2n})x_2 + \cdots + 2(A_{nn})x_n \end{bmatrix}$$

Now, let $P = (A+A^T)x$

$$\Rightarrow P = \left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \ddots & & \vdots \\ A_{n1} & \cdots & & A_{nn} \end{bmatrix} + \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ \vdots & \ddots & & \\ A_{1n} & & & A_{nn} \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} 2A_{11} & (A_{12}+A_{21}) & \cdots & (A_{1n}+A_{n1}) \\ (A_{21}+A_{12}) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ (A_{n1}+A_{1n}) & \cdots & \cdots & 2A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\Rightarrow P = Q$

$\therefore$ proved.

(c) Given A is a symmetric matrix, which implies, $A = A^T$ — ①

$$\text{L.H.S.} = \nabla_x (x^T A x)$$

$$= (A + A^T) x \quad (\text{from (b)})$$

$$= (A + A) x \quad (\text{from ①})$$

$$= 2Ax = \text{R.H.S.}$$

$$\therefore \nabla_x (x^T A x) = 2Ax$$

(d) $\nabla_x \left[(Ax+b)^T (Ax+b) \right] = 2A^T(Ax+b)$

Proof: L.H.S. $= \nabla_x \left[(Ax+b)^T (Ax+b) \right]$

$= \nabla_x \left[\left((Ax)^T + b^T \right) (Ax+b) \right]$

$= \nabla_x \left[\left(x^T A^T + b^T \right) (Ax+b) \right]$

$= \nabla_x \left(x^T A^T A x + x^T A^T b + b^T A x + b^T b \right)$

$= \nabla_x x^T A^T A x + \nabla_x x^T A^T b + \nabla_x b^T A x + \nabla_x b^T b$

$= \nabla_x x^T (A^T A) x + \nabla_x x^T (A^T b) + \nabla_x (A^T b)^T x + 0$

$= \left(A^T A + (A^T A)^T \right) x + A^T b + A^T b$

$= \left(A^T A + A^T A \right) x + 2A^T b$

$= 2A^T A x + 2A^T b$

$= 2A^T (Ax+b) \quad = \text{R.H.S.}$

$$\therefore \nabla_x \left[(Ax+b)^T (Ax+b) \right] = 2A^T (Ax+b)$$