Reducing Redundancy in Cut-Elimination by Resolution ⋆

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Abstract. `CERes` is a method of cut-elimination that uses resolution proof search to avoid some kinds of redundancies that affect reductive cut-elimination methods. This paper shows that, unfortunately, there are also cases where `CERes` can produce proofs that are more redundant and even exponentially larger than the proofs produced by reductive cut-elimination methods. The paper then describes a few novel variants of `CERes` that are much less susceptible to these redundancies.

1 Introduction

The cut-elimination method `CERes` invented by A. Leitsch and M. Baaz [3–5] uses resolution proof search to avoid certain kinds of redundancies that affect reductive cut- elimination methods: there are proofs for which reductive cut-elimination methods may require non-elementary many reduction steps and produce non- elementarily large intermediary proofs, while `CERes`'s more global and search-based approach produces a proof in atomic cut normal form without performing such expensive intermediary steps [6].

However, as explained in Section 3 of this paper, there are also cases where reductive cut-elimination methods can produce short proofs while the proofs generated by `CERes` are exponentially larger. Thanks to a simplified description of `CERes` in Section 2, it becomes evident that the source of redundancy is the naive transformation to clause form that is implicitly used by `CERes`.

Sections 4 and 5 develop two techniques to tame the redundancy. The first takes inference permutability into account when performing the clause form transformation, thus avoiding the duplication of literals when disjunctions are distributed over conjunctions. The second proposes the use of structural clause form transformation, which is known not to cause a worst-case exponential blow-up in the formula size. These two techniques can be combined, resulting in the least redundant `CERes` variant described in Section 6.

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2 Cut-Elimination by Resolution

The `CERes` method consists of four steps. Firstly, the proof with cuts ψ is skolemized into a proof ψ'. Secondly, a *characteristic formula* $\mathcal{S}_\psi$ is extracted from ψ'. Thirdly, as this formula is always unsatisfiable, it can be converted to a *characteristic clause set* and refuted by resolution. The resulting resolution refutation δ (whose existence is guaranteed by the completeness of the resolution calculus [10]) is made of resolution and factoring inferences. In the last step, a sequent calculus proof $CERes(\psi, \delta,)$ having the same end-sequent as ψ' and whose cuts are atomic can be obtained by converting resolution and factoring inferences from δ to, respectively, atomic cuts and contractions and replacing the leaves from δ by cut-free parts of ψ' known as *projections*.

Definition 1 (Characteristic Formula). *The* characteristic formula $\mathcal{S}_\varphi$ *of a proof φ is the characteristic formula $\mathcal{S}_{\rho^*}$ where ρ^* is its lowermost inference and $\mathcal{S}_\rho$ is defined for each inference ρ as follows:*

- *If ρ is an axiom with sequent $A \vdash A$, four cases are distinguished:*
  - *Only the succedent's A is a cut-ancestor: $\mathcal{S}_\rho = A$*
  - *Only the antecedent's A is a cut-ancestor: $\mathcal{S}_\rho = \neg A$*
  - *None is a cut-ancestor: $\mathcal{S}_\rho = \bot$*
  - *Both are cut-ancestors: $\mathcal{S}_\rho = \top$*
- *If ρ is an n-ary inference and $\rho_1, \ldots, \rho_n$ are the inferences deriving the premises of ρ, two cases are distinguished:*
  - *ρ operates on cut-ancestors:*

$$\mathcal{S}_\rho = \mathcal{S}_{\rho_1} \wedge \ldots \wedge \mathcal{S}_{\rho_n}$$

  - *ρ does not operate on cut-ancestors::*

$$\mathcal{S}_\rho = \mathcal{S}_{\rho_1} \vee \ldots \vee \mathcal{S}_{\rho_n}$$

Example 1. Let φ be the proof below:

$$\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}\wedge_r^1}{A \wedge B \vdash A \wedge B}\wedge_l^6 \quad \cfrac{\cfrac{B \vdash B \quad A \vdash A}{A, B \vdash B \wedge A}\wedge_r^2}{A \wedge B \vdash B \wedge A}\wedge_l^7}{A \wedge B \vdash B \wedge A}cut^3 \quad \cfrac{C \vdash C \quad C \vdash C}{C \vdash C}cut^4}{(A \wedge B) \vee C \vdash B \wedge A, C}\vee_l^5$$

Its characteristic formula is: $\mathcal{S}_\varphi \equiv ((A \wedge^1 B) \wedge^3 (\neg B \vee^2 \neg A)) \vee^5 (C \wedge^4 \neg C)$ $\square$

Theorem 1. $\forall \overline{\alpha_\varphi}.\mathcal{S}_\varphi$, *where $\overline{\alpha_\varphi}$ are the eigenvariables of φ, is unsatisfiable.*

Proof. Recursively transform each subproof ψ of φ having end-sequent $\Gamma, \Gamma^* \vdash \Delta, \Delta^*$, where $\Gamma^* \vdash \Delta^*$ are cut-ancestors, into a proof ψ' of $\forall \overline{\alpha_\psi}.\mathcal{S}_\rho, \Gamma^* \vdash \Delta^*$. By doing so, φ itself is transformed into a proof φ' with end-sequent $\forall \overline{\alpha_\varphi}.\mathcal{S}_\varphi \vdash$. $\square$

Definition 2 (Simple Transformation to Conjunctive Normal Form). *A formula in negative normal form can be transformed into conjunctive normal form by rewriting it according to the following rule:*

$$S \vee (S_1 \wedge \ldots \wedge S_n) \rightsquigarrow (S \vee S_1) \wedge \ldots \wedge (S \vee S_n)$$

Definition 3 (Sequent Notation). *A formula in conjunctive normal form*

$$\bigwedge_{i \in I} \Big(\bigvee_{1 \leq j' \leq j_i} \neg A_{ij'} \vee \bigvee_{1 \leq h' \leq h_i} B_{ih'} \Big)$$

can be written in sequent notation as the set $\{A_{i1}, \ldots, A_{ij_i} \vdash B_{i1}, \ldots, B_{ih_i} | i \in I\}$.

Definition 4 (Clause Set). *The* clause set $\mathcal{C}_\varphi$ *of a proof* φ *is the conjunctive normal form of* $\mathcal{S}_\varphi$ *written in sequent notation.*

Example 2. Let φ be the proof shown in Example 1. Its characteristic formula $\mathcal{S}_\varphi$ normalizes as:

$$\mathcal{S}_\varphi \rightsquigarrow^* (A \vee C) \wedge (A \vee \neg C) \wedge (B \vee C) \wedge (B \vee \neg C) \wedge (\neg B \vee \neg A \vee C) \wedge (\neg B \vee \neg A \vee \neg C)$$

Hence, the clause set of φ is:

$$\mathcal{C}_\varphi \equiv \{\vdash A, C \quad ; \quad C \vdash A \quad ; \quad \vdash B, C \quad ; \quad C \vdash B \quad ; \quad B, A \vdash C \quad ; \quad B, A, C \vdash\}$$

and it can be refuted by the following resolution refutation δ:

$$\dfrac{\dfrac{\dfrac{\vdash A, C \qquad C \vdash A}{\vdash A, A}\,r}{\vdash A}\,f_r \qquad \dfrac{\dfrac{\dfrac{\vdash B, C \qquad C \vdash B}{\vdash B, B}\,r}{\vdash B}\,f_r \qquad \dfrac{\dfrac{\dfrac{B, A \vdash C \qquad C, B, A \vdash}{B, A, B, A \vdash}\,r}{B, A, B \vdash}\,f_r}{A, B \vdash}\,f_r}{A \vdash}\,r}{\vdash}\,r$$

□

Resolution and factoring inferences are essentially atomic cuts and contractions with unification. Therefore, to obtain the `CERes`-normal-form (an **LK**-proof, with only atomic cuts, of the skolemized end-sequent of the original proof with cuts), one can replace each leaf of the refutation by a *projection* with an appropriate end-sequent, apply all the unifiers and convert resolution and factoring inferences to atomic cuts and contractions. Since a projection's purpose is to replace a leaf in a refutation of a clause set, its end-sequent must contain the leaf's clause as a subsequent. Moreover, if its end-sequent contains any other formula, then this formula must appear in the end-sequent of the original proof with cuts, because this formula is propagated downward after the replacement and thus necessarily appears in the end-sequent of the `CERes`-normal-form. Otherwise, if the formula were not in the end-sequent of the original proof, the `CERes`-normal-form's end-sequent would be necessarily different from that of the skolemized proof with cuts. Finally, a projection must, of course, be cut-free, otherwise the `CERes`-normal-form would contain more cuts in addition to the inessential atomic cuts originating from the refutation. These three conditions are formally expressed in Definition 5.

Definition 5 (Projection). *Let φ be a proof with end-sequent $\Gamma \vdash \Delta$ and $\Gamma_c \vdash \Delta_c \in \mathcal{C}_\varphi$. Any cut-free proof of $\Gamma', \Gamma_c \vdash \Delta', \Delta_c$, where $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, is a* projection *of φ with respect to $\Gamma_c \vdash \Delta_c$.*

Projections can be easily constructed by extracting cut-free parts of the original proof with cuts. The original method [6, 2, 4, 3, 14] generates projections where $\Gamma' = \Gamma$ and $\Delta' = \Delta$. Here a slightly more optimized method that constructs less redundant projections, where $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, is described.

Definition 6 (Algorithm for Constructing Projections). *Let φ be a proof and c be a clause from $\mathcal{C}_\varphi$. Let A be the set of axioms of φ that contain formulas that contribute to c. Then, $\lfloor \varphi \rfloor_c$ is constructed by taking from φ only the inferences that operate on formulas that are both descendents of axioms in A and end-sequent ancestors, and adding weakening inferences when necessary.*

Example 3. For φ of Example 1, the projections $\lfloor \varphi \rfloor_{\vdash A, C}$ and $\lfloor \varphi \rfloor_{\vdash B, C}$ are:

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,w_l}{A \wedge B \vdash A}\,\wedge_l^6 \qquad C \vdash C}{(A \wedge B) \vee C \vdash A, C}\,\vee_l^5 \qquad\qquad \dfrac{\dfrac{\dfrac{B \vdash B}{A, B \vdash B}\,w_l}{A \wedge B \vdash B}\,\wedge_l^6 \qquad C \vdash C}{(A \wedge B) \vee C \vdash B, C}\,\vee_l^5$$

The projections $\lfloor \varphi \rfloor_{C \vdash A}$ and $\lfloor \varphi \rfloor_{C \vdash B}$ are:

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,w_l}{A \wedge B \vdash A}\,\wedge_l^6 \qquad \dfrac{C \vdash C}{C, C \vdash C}\,w_l}{(A \wedge B) \vee C, C \vdash A, C}\,\vee_l^5 \qquad\qquad \dfrac{\dfrac{\dfrac{B \vdash B}{A, B \vdash B}\,w_l}{A \wedge B \vdash B}\,\wedge_l^6 \qquad \dfrac{C \vdash C}{C, C \vdash C}\,w_l}{(A \wedge B) \vee C, C \vdash B, C}\,\vee_l^5$$

The projections $\lfloor \varphi \rfloor_{B, A \vdash C}$ and $\lfloor \varphi \rfloor_{B, A, C \vdash}$ are:

$$\dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A, B \vdash B \wedge A}\,\wedge_r^2}{A \wedge B, A, B \vdash B \wedge A}\,w_l \qquad C \vdash C}{(A \wedge B) \vee C, A, B \vdash B \wedge A, C}\,\vee_l^5 \qquad\qquad \dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A, B \vdash B \wedge A}\,\wedge_r^2}{A \wedge B, A, B \vdash B \wedge A}\,w_l \qquad \dfrac{C \vdash C}{C, C \vdash C}\,w_l}{(A \wedge B) \vee C, A, B \vdash B \wedge A, C}\,\vee_l^5$$

$\square$

Theorem 2. *Let ψ be a proof and $\Gamma_c \vdash \Delta_c$ be a clause in its characteristic clause set. Then $\lfloor\psi\rfloor_{\Gamma_c \vdash \Delta_c}$ is a projection of ψ with respect to the clause $\Gamma_c \vdash \Delta_c$.*

Proof. A detailed proof is available in [15]. Here only a sketch is provided. Let A be a set of axioms from ψ that contain literals that contribute to $\Gamma_c \vdash \Delta_c$. Note that, since the literals from $\Gamma_c \vdash \Delta_c$ occur as cut-ancestors in the axioms in A, and inferences operating on cut-ancestors are not performed in the construction of $\lfloor\psi\rfloor_{\Gamma_c \vdash \Delta_c}$, these literals are simply propagated down to the end-sequent of $\lfloor\psi\rfloor_{\Gamma_c \vdash \Delta_c}$. Therefore, the end-sequent of $\lfloor\psi\rfloor_{\Gamma_c \vdash \Delta_c}$ is a supersequent of $\Gamma_c \vdash \Delta_c$. Among the formulas in the end-sequent of ψ, let $\Gamma' \vdash \Delta'$ be the sequent containing all and only those formulas that contain descendents of axioms in A. Since $\lfloor\psi\rfloor_{\Gamma_c \vdash \Delta_c}$ contains all the inferences that operate on formulas that are both end-sequent ancestors and descendents from axioms in A, the end-sequent of $\lfloor\psi\rfloor_{\Gamma_c \vdash \Delta_c}$ is a supersequent of $\Gamma' \vdash \Delta'$. Finally, $\lfloor\psi\rfloor_{\Gamma_c \vdash \Delta_c}$ is cut-free, because it contains no inference operating on cut ancestors and thus contains no cut. $\square$

The construction of the characteristic clause set in original descriptions of the `CERes` method [3–5] differs slightly from the construction presented in this paper. There, instead of a characteristic formula, one consructs a *characteristic clause term*, that has $\oplus$ instead of $\wedge$ (for binary inferences operating on cut-ancestors), $\otimes$ instead of $\vee$ (for binary inferences operating on end-sequent-ancestors), and singleton clause sets instead of formulas (for axioms). The operators $\oplus$ and $\otimes$ are then interpreted respectively as a set union and as a clause set merge operation in order to generate the characteristic clause set. Both approaches are clearly equivalent, but the approach described here is simpler as it does not require the invention of new operators and relies on the standard technique of clause form transformation for the generation of the characteristic clause set. This is crucial for understanding why proofs generated by `CERes` may contain redundancies.

Definition 7 (`CERes`-normal-form). *The* `CERes`-normal-form *of the proof φ w.r.t. the resolution refutation δ of its clause set $\mathcal{C}_\varphi$ is denoted $CERes(\varphi, \delta)$ and obtained by:*

1. *converting resolution and factoring inferences from δ to, respectively, atomic cuts and contractions, using a substitution σ obtained by the composition of all unifiers.*
2. *replacing each axiom clause c in δ by its corresponding projection $\lfloor\psi\rfloor_c\sigma$.*
3. *adding contractions in the bottom of the proof, if necessary.*

Example 4. Let φ be the proof shown in Example 1 and δ be the refutation of its characteristic clause set, as shown in Example 2. Then, $CERes(\varphi, \delta)$, obtained by replacing the leaves of δ by the respective projections shown in Example 3, converting resolution and factoring inferences respectively to cuts and contractions and adding contractions at the bottom, is:

$$
\begin{prooftree}
\AxiomC{$\lfloor\varphi\rfloor_{\vdash A,C}$}
\noLine
\UnaryInfC{$(A \wedge B) \vee C \vdash A, C$}
\AxiomC{$\lfloor\varphi\rfloor_{C\vdash A}$}
\noLine
\UnaryInfC{$(A \wedge B) \vee C, C \vdash A, C$}
\RightLabel{cut}
\BinaryInfC{$(A \wedge B) \vee C, (A \wedge B) \vee C \vdash A, A, C$}
\RightLabel{c_r}
\UnaryInfC{$(A \wedge B) \vee C, (A \wedge B) \vee C \vdash A, C$}
\AxiomC{ψ}
\noLine
\UnaryInfC{$A, (A \wedge B) \vee C, (A \wedge B) \vee C, (A \wedge B) \vee C, (A \wedge B) \vee C \vdash B \wedge A, B \wedge A, C$}
\RightLabel{cut}
\BinaryInfC{$(A \wedge B) \vee C, (A \wedge B) \vee C, (A \wedge B) \vee C, (A \wedge B) \vee C, (A \wedge B) \vee C, (A \wedge B) \vee C \vdash B \wedge A, B \wedge A, C, C$}
\doubleLine
\RightLabel{c^*}
\UnaryInfC{$(A \wedge B) \vee C \vdash B \wedge A, C$}
\end{prooftree}
$$

Where ψ is:

$$
\begin{prooftree}
\AxiomC{$\lfloor\varphi\rfloor_{\vdash B,C}$}
\noLine
\UnaryInfC{$(A \wedge B) \vee C \vdash B, C$}
\AxiomC{$\lfloor\varphi\rfloor_{C\vdash B}$}
\noLine
\UnaryInfC{$(A \wedge B) \vee C, C \vdash B, C$}
\RightLabel{cut}
\BinaryInfC{$(A \wedge B) \vee C, (A \wedge B) \vee C \vdash B, B, C$}
\RightLabel{c_r}
\UnaryInfC{$(A \wedge B) \vee C, (A \wedge B) \vee C \vdash B, C$}
\AxiomC{$\lfloor\varphi\rfloor_{B,A\vdash C}$}
\noLine
\UnaryInfC{$(A \wedge B) \vee C, A, B \vdash B \wedge A, C$}
\AxiomC{$\lfloor\varphi\rfloor_{B,A,C\vdash}$}
\noLine
\UnaryInfC{$C, (A \wedge B) \vee C, A, B \vdash B \wedge A, C$}
\RightLabel{cut}
\BinaryInfC{$A, B, A, B, (A \wedge B) \vee C, (A \wedge B) \vee C \vdash B \wedge A, B \wedge A, C$}
\RightLabel{c_r}
\UnaryInfC{$A, B, B, (A \wedge B) \vee C, (A \wedge B) \vee C \vdash B \wedge A, B \wedge A, C$}
\RightLabel{c_r}
\UnaryInfC{$A, B, (A \wedge B) \vee C, (A \wedge B) \vee C \vdash B \wedge A, B \wedge A, C$}
\RightLabel{cut}
\BinaryInfC{$A, (A \wedge B) \vee C, (A \wedge B) \vee C, (A \wedge B) \vee C, (A \wedge B) \vee C \vdash B \wedge A, B \wedge A, C$}
\end{prooftree}
$$

$\square$

3 Redundancy

A closer inspection of the projections in example 3 reveals that they are quite redundant. Inference $\wedge_l^6$, for example, repeatedly appears in four of the six projections; $\wedge_r^2$ appears in two projections; and $\vee_l^5$ appears in all six projections. Furthermore, all six projections derive the formula $(A \wedge B) \vee C$ in their end-sequents, and then all occurrences of this formula must be contracted in the bottom of $CERes(\varphi, \delta)$ shown in example 4.

The redundancy is mainly a consequence of the simple transformation to conjunctive normal form used by `CERes`. When disjunction is distributed over conjunction, it is often the case that literals must be duplicated. Hence, each projection w.r.t. a clause that contains a copy of a duplicated literal will have to contain a copy of every inference that operates on descendants of this literal.

In the worst case, by Theorem 4, this redundancy can make `CERes` normal forms exponentially larger than normal forms produced by reductive cut-elimination methods. Theorem 4 is a corollary of Theorem 3, which shows that exponential blow-up already occurs in the size of the characteristic clause set.

Theorem 3 (Size of Characteristic Clause Set). *There exist positive constants k and k' and a sequence of proofs $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ such that $|\varphi_n| \leq k4^n$ and $|\mathcal{C}_{\varphi_n}| \geq k'2^{2^n}$.*

Proof. Let $\psi_1(s)$ be the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{A_1(s) \vdash A_1(s) \qquad B_1(s) \vdash B_1(s)}{A_1(s), B_1(s) \vdash A_1(s) \wedge B_1(s)}\wedge_r}{A_1(s) \wedge B_1(s) \vdash A_1(s) \wedge B_1(s)}\wedge_l \qquad \dfrac{\dfrac{C_1(s) \vdash C_1(s) \qquad D_1(s) \vdash D_1(s)}{C_1(s), D_1(s) \vdash C_1(s) \wedge D_1(s)}\wedge_r}{C_1(s) \wedge D_1(s) \vdash C_1(s) \wedge D_1(s)}\wedge_l}{(A_1(s) \wedge B_1(s)) \vee (C_1(s) \wedge D_1(s)) \vdash A_1(s) \wedge B_1(s), C_1(s) \wedge D_1(s)}\vee_l}{(A_1(s) \wedge B_1(s)) \vee (C_1(s) \wedge D_1(s)) \vdash (A_1(s) \wedge B_1(s)) \vee (C_1(s) \wedge D_1(s))}\vee_r$$

And let $\psi_n(s)$ $(n > 1)$ be:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\psi_{n-1}(a.s)\\ A_n(s) \vdash A_n(s)\end{array} \qquad \begin{array}{c}\psi_{n-1}(b.s)\\ B_n(s) \vdash B_n(s)\end{array}}{A_n(s), B_n(s) \vdash A_n(s) \wedge B_n(s)}\wedge_r}{A_n(s) \wedge B_n(s) \vdash A_n(s) \wedge B_n(s)}\wedge_l \qquad \dfrac{\dfrac{\begin{array}{c}\psi_{n-1}(c.s)\\ C_n(s) \vdash C_n(s)\end{array} \qquad \begin{array}{c}\psi_{n-1}(d.s)\\ D_n(s) \vdash D_n(s)\end{array}}{C_n(s), D_n(s) \vdash C_n(s) \wedge D_n(s)}\wedge_r}{C_n(s) \wedge D_n(s) \vdash C_n(s) \wedge D_n(s)}\wedge_l}{(A_n(s) \wedge B_n(s)) \vee (C_n(s) \wedge D_n(s)) \vdash A_n(s) \wedge B_n(s), C_n(s) \wedge D_n(s)}\vee_l}{(A_n(s) \wedge B_n(s)) \vee (C_n(s) \wedge D_n(s)) \vdash (A_n(s) \wedge B_n(s)) \vee (C_n(s) \wedge D_n(s))}\vee_r$$

where $x.s$ denotes the the result of prepending x in the list s and (for $n > 1$):

$$\begin{aligned}
A_n(s) &= (A_{n-1}(a.s) \wedge B_{n-1}(a.s)) \vee (C_{n-1}(a.s) \wedge D_{n-1}(a.s))\\
B_n(s) &= (A_{n-1}(b.s) \wedge B_{n-1}(b.s)) \vee (C_{n-1}(b.s) \wedge D_{n-1}(b.s))\\
C_n(s) &= (A_{n-1}(c.s) \wedge B_{n-1}(c.s)) \vee (C_{n-1}(c.s) \wedge D_{n-1}(c.s))\\
D_n(s) &= (A_{n-1}(d.s) \wedge B_{n-1}(d.s)) \vee (C_{n-1}(d.s) \wedge D_{n-1}(d.s))
\end{aligned}$$

Let φ_n be the proof below:

$$\frac{\psi_n([]) \qquad \psi_n([])}{(A_n([]) \wedge B_n([])) \vee (C_n([]) \wedge D_n([])) \vdash (A_n([]) \wedge B_n([])) \vee (C_n([]) \wedge D_n([]))}\ cut$$

Let $S^l_{\psi_k(s)}$ be the subformula of $\mathcal{S}_{\varphi_n}$ corresponding to the root inference of the subproof $\psi_k(s)$ in the left side of φ_n. Analogously, let $S^r_{\psi_k(s)}$ be the subformula of $\mathcal{S}_{\varphi_n}$ at the root inference of the subproof $\psi_k(s)$ in the right side of φ_n. Then:

$$\mathcal{S}_{\varphi_n} = S^l_{\psi_n([])} \wedge S^r_{\psi_n([])}$$

where:

$$S^l_{\psi_j(s)} = \begin{cases} (A_1(s) \wedge B_1(s)) \vee (C_1(s) \wedge D_1(s) & \text{, if } j = 1 \\ (S^l_{\psi_{j-1}(a.s)} \wedge S^l_{\psi_{j-1}(b.s)}) \vee (S^l_{\psi_{j-1}(c.s)} \wedge S^l_{\psi_{j-1}(d.s)}) & \text{, otherwise} \end{cases}$$

$$S^r_{\psi_j(s)} = \begin{cases} (\neg A_1(s) \vee \neg B_1(s)) \wedge (\neg C_1(s) \vee \neg D_1(s)) & \text{, if } j = 1 \\ (S^r_{\psi_{j-1}(a.s)} \vee S^r_{\psi_{j-1}(b.s)}) \wedge (S^r_{\psi_{j-1}(c.s)} \vee S^r_{\psi_{j-1}(d.s)}) & \text{, otherwise} \end{cases}$$

Let $f_l(n)$ be the number of clauses in $\mathcal{C}_{\varphi_n}$ stemming from the left branch of the cut. Analogously, let $f_r(n)$ be the number of clauses stemming from the right branch of the cut. Clearly, $|\mathcal{C}_{\varphi_n}| = f_l(n) + f_r(n)$. By analyzing the structure of the subformulas of $\mathcal{S}_{\varphi_n}$, it is possible to see that:

$$f_l(n) = \begin{cases} 4 & \text{, if } n = 1 \\ 4(f_l(n-1))^2 & \text{, otherwise} \end{cases}$$

$$f_r(n) = \begin{cases} 2 & \text{, if } n = 1 \\ 2(f_r(n-1))^2 & \text{, otherwise} \end{cases}$$

It can be easily proved by induction that $f_l(n) = 4^{(2^n-1)}$ and $f_r(n) = 2^{(2^n-1)}$. Therefore:

$$|\mathcal{C}_{\varphi_n}| = 4^{(2^n-1)} + 2^{(2^n-1)} \geq 2^{(2^n)}$$

□

Theorem 4 (Size of CERes-Normal-Form). *There exist positive constants k and k'' and a sequence of proofs $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ such that:*

- $|\varphi_n| \leq k4^n$.
- $|CERes(\varphi_n, \delta_n)| \geq k''2^{2^n}$, *for any refutation δ_n.*
- $|\varphi_n^*| \leq k4^n$, *for any φ_n^* obtained from φ_n by reductive cut-elimination.*

Proof. Consider the proofs φ_n defined in the proof of the Theorem 3. As proved there, $|\mathcal{C}_{\varphi_n}| \geq k'2^{2^n}$, for some positive rational constant k'. Moreover, in any refutation δ_n of $\mathcal{C}_{\varphi_n}$, every clause of $\mathcal{C}_{\varphi_n}$ has to be used at least once. Therefore, $|\delta_n| \geq k''2^{2^n}$, for some $k'' > k'$. Since $|CERes(\varphi_n, \delta_n)| \geq |\delta_n|$, $|CERes(\varphi_n, \delta_n)| \geq k''2^{2^n}$ as well. As φ_n contains neither implicit nor explicit contractions, the size strictly decreases with every reductive cut-elimination step. Hence, for any proof φ_n^* obtained from φ_n by reductive cut-elimination, $|\varphi_n^*| < |\varphi_n| < k4^n$. □

While the sequence of proofs used to prove Theorems 3 and 4 is rather artificial, it is important to note that redundancy can be expected to occur often in practice as well. Whenever the input proof has a structure with alternations of inferences operating on cut-ancestors and end-sequent-ancestors, the characteristic formula has alternations of disjunctions and conjunctions, and its literals are duplicated during the clause form transformation of the characteristic formula. Therefore, this is an issue that must be addressed for `CERes` to be efficiently applicable to proofs with complex structure.

4 Taking Inference Permutability into Account

Redundancy originates in the distribution of disjunction over conjunction during the clause form transformation of the characteristic formula. As conjunctions and disjunctions in the characteristic formula correspond to binary inferences operating, respectively, on cut-ancestors and end-sequent-ancestors in the proof, the amount of distributions can be reduced by first pre-processing the proof and permuting inferences operating on cut-ancestors downward. The rules for inference permutation are shown in Appendix B and in [15].

Example 5. Downward permutation of all inferences operating on cut ancestors transforms proof φ from Example 1 into the proof ψ shown below:

$$\dfrac{\dfrac{\dfrac{\psi'}{(A \wedge B) \vee C \vdash A \wedge B, C} \qquad C \vdash C}{(A \wedge B) \vee C \vdash A \wedge B, C}\, cut_2 \qquad \dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A, B \vdash B \wedge A}\, \wedge_r}{A \wedge B \vdash B \wedge A}\, \wedge_l}{}}{(A \wedge B) \vee C \vdash B \wedge A, C}\, cut_1$$

where ψ' is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\, w_l}{A \wedge B \vdash A}\, \wedge_l \qquad C \vdash C}{(A \wedge B) \vee C, \vdash A, C}\, \vee_l \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{A, B \vdash B}\, w_l}{A \wedge B \vdash B}\, \wedge_l \qquad C \vdash C}{(A \wedge B) \vee C \vdash B, C}\, \vee_l}{(A \wedge B) \vee C, (A \wedge B) \vee C \vdash A \wedge B, C, C}\, \wedge_r}{(A \wedge B) \vee C \vdash A \wedge B, C, C}\, c_l}{(A \wedge B) \vee C \vdash A \wedge B, C}\, c_r$$

Its characteristic formula is:

$$\mathcal{S}_\psi \equiv (((A \vee C) \wedge (B \vee C)) \wedge \neg C) \wedge (\neg B \vee \neg A)$$

And its characteristic clause set is:

$$\mathcal{C}_\psi \equiv \{\vdash A, C \quad ; \quad \vdash B, C \quad ; \quad C \vdash \quad ; \quad B, A \vdash \quad \}$$

Thanks to inference permutations, fewer duplications occur in $\mathcal{C}_\psi$ than in $\mathcal{C}_\varphi$. □

However, local proof rewriting such as inference permutation are inefficient. It is typical of reductive cut-elimination methods and it is something that cut-elimination by resolution strives to avoid and improve. Therefore, a method for transforming the characteristic formula into clause form that takes the possibility of inference permutation into account without actually having to perform inference permutations is desirable. Such a method ($\overset{S}{\rightsquigarrow}$) is defined in this section, and it is shown in Lemma 1 that every rewriting of the characteristic formula according to $\overset{S}{\rightsquigarrow}$ corresponds to a sequence of inference permutation steps according to the rules in Appendix B. Thus, while $\rightsquigarrow$ does full distribution of disjunction over conjunction, as if inferences operating on cut-ancestors were always indirectly dependent on the inferences operating on end-sequent ancestors above them, $\overset{S}{\rightsquigarrow}$ does partial distribution when the corresponding inferences are independent and permutable without the need for duplications. For the partial distribution to be possible, the characteristic formula must contain extra information to allow the retrieval of the dependencies between the branching inferences.

Definition 8. *Let ρ be an inference in a proof φ. Then $\Omega_\rho(\varphi)$ denotes the set of descendents of formulas that occur in axiom sequents containing ancestors of auxiliary formulas of ρ.*

Example 6. In the proof φ below, the formulas belonging to $\Omega_{\vee_l}(\varphi)$ have been highlighted in blue:

$$\dfrac{\dfrac{\dfrac{\dfrac{{\color{blue}A \vdash A} \quad {\color{blue}B \vdash B}}{{\color{blue}A, B \vdash A \wedge B}}\wedge_r}{{\color{blue}A \wedge B \vdash A \wedge B}}\wedge_l \quad \dfrac{\dfrac{B \vdash B \quad A \vdash A}{A, B \vdash B \wedge A}\wedge_r}{A \wedge B \vdash B \wedge A}\wedge_l}{{\color{blue}A \wedge B} \vdash B \wedge A}cut_1 \quad \dfrac{{\color{blue}C \vdash C} \quad C \vdash C}{{\color{blue}C} \vdash C}cut_2}{({\color{blue}A} \wedge {\color{blue}B}) \vee {\color{blue}C} \vdash B \wedge A, C}\vee_l$$

And below, the formulas belonging to $\Omega_{cut_1}(\varphi)$ have been highlighted in blue:

$$\dfrac{\dfrac{\dfrac{\dfrac{{\color{blue}A \vdash A} \quad {\color{blue}B \vdash B}}{{\color{blue}A, B \vdash A \wedge B}}\wedge_r}{{\color{blue}A \wedge B \vdash A \wedge B}}\wedge_l \quad \dfrac{\dfrac{{\color{blue}B \vdash B} \quad {\color{blue}A \vdash A}}{{\color{blue}A, B \vdash B \wedge A}}\wedge_r}{{\color{blue}A \wedge B \vdash B \wedge A}}\wedge_l}{{\color{blue}A \wedge B \vdash B \wedge A}}cut_1 \quad \dfrac{C \vdash C \quad C \vdash C}{C \vdash C}cut_2}{({\color{blue}A} \wedge {\color{blue}B}) \vee C \vdash {\color{blue}B} \wedge {\color{blue}A}, C}\vee_l$$

Definition 9 ($\overset{S}{\rightsquigarrow}$). *In the rewriting rules below, let ρ be the inference in φ corresponding to $\vee_\rho$. For the rewriting rules to be applicable, $S_{n+1}, \ldots, S_{n+m}$ and S must contain at least one formula from $\Omega_\rho(\varphi)$ each (i.e. there must be an atomic subformula A of S_{n+k} such that $A \in \Omega_\rho(\varphi)$), and $S_1, \ldots, S_n$ and S' and S'' should not contain any formula from $\Omega_\rho(\varphi)$. Moreover, an innermost rewriting strategy is enforced.*

$$S \vee_\rho (S_1 \wedge \ldots \wedge S_n \wedge S_{n+1} \wedge \ldots \wedge S_{n+m}) \overset{S}{\rightsquigarrow} S_1 \wedge \ldots \wedge S_n \wedge (S \vee_\rho S_{n+1}) \wedge \ldots \wedge (S \vee_\rho S_{n+m})$$

$$S \wedge_\rho S' \overset{S}{\rightsquigarrow} S' \qquad\qquad S' \vee_\rho S'' \overset{S}{\rightsquigarrow} S' \qquad\qquad S' \wedge_\rho S'' \overset{S}{\rightsquigarrow} S'$$

Definition 10 (Degenerate Inferences). *An inference ρ in a proof φ is* degenerate *when all its auxiliary formulas are descendants of main formulas of weakening inferences. When only some auxiliary (sub)formulas of ρ are descendants of main formulas of weakening inferences, ρ is* partially degenerate.

Remark 1. The last three rules in Definition 9 handle connectives that correspond to (partially) *degenerate inferences*, whose auxiliary formulas are introduced by weakening. These rules are related to downward permutation of weakening inferences, as shown in Lemma 1. Because of the last two rules, $\overset{S}{\rightsquigarrow}$ is not confluent.

Definition 11 (Swapped Clause Set). *A* swapped clause set $\mathcal{C}^S_{\varphi|S}$ *of a proof φ is the $\overset{S}{\rightsquigarrow}$-normal-form S of $\mathcal{S}_\varphi$ written in sequent notation.*

Remark 2. In cases where $\mathcal{C}^S_{\varphi|S_1} = \mathcal{C}^S_{\varphi|S_1}$ for any $\overset{S}{\rightsquigarrow}$-normal-forms S_1 and S_2 of $\mathcal{S}_\varphi$, the unique swapped clause set is denoted simply as $\mathcal{C}^S_\varphi$.

Remark 3. Swapped clause sets are very similar to *profiles*, which have been defined in [7]. In fact, the concept of swapped clause set evolved from attempts to find a simpler explanation for profiles. They differ only on proofs that contain degenerate inferences. In such cases, swapped clause sets are always smaller [15].

Definition 12 (`SCERes`-normal-form). *$SCERes(\varphi, \delta)$ denotes* `SCERes` normal form *of the proof φ w.r.t. the resolution refutation δ of any swapped clause set $\mathcal{C}^S_{\varphi|S}$. It is obtained in the same way as a `CERes`-normal-form, but using a swapped clause set $\mathcal{C}^S_{\varphi|S}$ instead of the clause set $\mathcal{C}_\varphi$.*

Example 7. Let φ be the proof below:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\wedge^1_r}{A \wedge B \vdash A \wedge B}\wedge_l \qquad \dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A, B \vdash B \wedge A}\wedge^2_r}{A \wedge B \vdash B \wedge A}\wedge_l}{A \wedge B \vdash B \wedge A}cut^3 \qquad \dfrac{C \vdash C \qquad C \vdash C}{C \vdash C}cut^4}{(A \wedge B) \vee C \vdash B \wedge A, C}\vee^5_l$$

Its characteristic formula is:

$$\mathcal{S}_\varphi \equiv ((A \wedge^1 B) \wedge^3 (\neg B \vee^2 \neg A)) \vee^5 (C \wedge^4 \neg C)$$

Considering that $\{A, B, C\} \subset \Omega_{\vee^5_l}(\varphi)$ and $\{A, B, C\} \cap \Omega_{\vee^5_l}(\varphi) = \emptyset$, the characteristic formula $\mathcal{S}_\varphi$ can be normalized in the two ways shown below:

$$\begin{aligned}
\mathcal{S}_\varphi &\equiv ((A \wedge^1 B) \wedge^3 (\neg B \vee^2 \neg A)) \vee^5 (C \wedge^4 \neg C) \\
&\overset{S}{\rightsquigarrow} ((A \wedge^1 B) \vee^5 (C \wedge^4 \neg C)) \wedge^3 (\neg B \vee^2 \neg A) \\
&\overset{S}{\rightsquigarrow} ((A \vee^5 (C \wedge^4 \neg C)) \wedge^1 (B \vee^5 (C \wedge^4 \neg C))) \wedge^3 (\neg B \vee^2 \neg A) \\
&\overset{S}{\rightsquigarrow} (((A \vee^5 C) \wedge^4 \neg C) \wedge^1 ((B \vee^5 C) \wedge^4 \neg C)) \wedge^3 (\neg B \vee^2 \neg A) \\
&\equiv (A \vee^5 C) \wedge \neg C \wedge (B \vee^5 C) \wedge \neg C \wedge (\neg B \vee^2 \neg A) \\
&\equiv S_1
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_\varphi &\equiv ((A \wedge^1 B) \wedge^3 (\neg B \vee^2 \neg A)) \vee^5 (C \wedge^4 \neg C) \\
&\overset{S}{\rightsquigarrow} ((A \wedge^1 B) \vee^5 (C \wedge^4 \neg C)) \wedge^3 (\neg B \vee^2 \neg A) \\
&\overset{S}{\rightsquigarrow} ((((A \wedge^1 B) \vee^5 C) \wedge^4 \neg C)) \wedge^3 (\neg B \vee^2 \neg A) \\
&\overset{S}{\rightsquigarrow} ((((A \vee^5 C) \wedge^1 (B \vee^5 C)) \wedge^4 \neg C)) \wedge^3 (\neg B \vee^2 \neg A) \\
&\equiv (A \vee^5 C) \wedge (B \vee^5 C) \wedge \neg C \wedge (\neg B \vee^2 \neg A) \\
&\equiv S_2
\end{aligned}$$

The swapped clause sets are:

$$\mathcal{C}^S_{\varphi|S_1} = \{\ \vdash A, C \ ;\ \vdash B, C \ ;\ C \vdash \ ;\ C \vdash \ ;\ B, A \vdash\ \}$$

$$\mathcal{C}^S_{\varphi|S_2} = \{\ \vdash A, C \ ;\ \vdash B, C \ ;\ C \vdash \ ;\ B, A \vdash\ \}$$

It is interesting to note that $\mathcal{C}^S_{\varphi|S_1} = \mathcal{C}^S_{\varphi|S_2}$ (because they are sets). This is not a coincidence. It always occurs when the non-confluence is due to non-degenerated applications of the first rewriting rule. A refutation δ of $\mathcal{C}^S_\varphi$ is shown below:

$$\dfrac{\dfrac{\dfrac{\vdash A, C \qquad \dfrac{\vdash B, C \qquad B, A \vdash}{A \vdash C}\,r}{\vdash C, C}\,r}{\vdash C}\,f_r \qquad C \vdash}{\vdash}\,r$$

$\mathit{SCERes}(\varphi, \delta)$ is the `CERes`-normal-form obtained with the swapped clause set:

$$\dfrac{\dfrac{\begin{array}{c}\varphi_0 \\ (A \wedge B) \vee C, (A \wedge B) \vee C \vdash B \wedge A, C\end{array} \qquad C \vdash C}{(A \wedge B) \vee C, (A \wedge B) \vee C \vdash B \wedge A, C}\,cut}{(A \wedge B) \vee C \vdash B \wedge A, C}\,c_l$$

where φ_0 is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,w_l}{A \wedge B \vdash A}\,\wedge_l \qquad C \vdash C}{(A \wedge B) \vee C \vdash A, C}\,\vee_l \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{A, B \vdash B}\,w_l}{A \wedge B \vdash B}\,\wedge_l \qquad C \vdash C}{(A \wedge B) \vee C \vdash B, C}\,\vee_l \qquad \dfrac{B \vdash B \qquad A \vdash A}{A, B \vdash B \wedge A}\,\wedge_r}{(A \wedge B) \vee C, A \vdash B \wedge A, C}\,cut}{(A \wedge B) \vee C, (A \wedge B) \vee C \vdash B \wedge A, C, C}\,cut}{(A \wedge B) \vee C, (A \wedge B) \vee C \vdash B \wedge A, C}\,c_r$$

As expected, less redundant clause sets and projections result in a $\mathit{SCERes}(\varphi, \delta)$ significantly smaller than $\mathit{CERes}(\varphi, \delta)$ shown in Example 4. □

Lemma 1 (Correspondence between $\overset{S}{\rightsquigarrow}$ and $\gg$). *If φ is skolemized and $\mathcal{S}_\varphi \overset{S}{\rightsquigarrow} S$, then there exists a proof ψ such that $\varphi \gg^* \psi$ and $\mathcal{S}_\psi = S$.*

Proof. In Appendix C.

Theorem 5 (Unsatisfiability of the Swapped Clause Set). *For any skolemized proof φ and $\overset{S}{\rightsquigarrow}$-normal-form S of $\mathcal{S}_\varphi$, $\mathcal{C}^S_{\varphi|S}$ is unsatisfiable.*

Proof. By Lemma 1, there is ψ such that $\mathcal{S}_\psi = S$. Clearly, $\mathcal{C}^S_{\varphi|S} = \mathcal{C}^S_\psi$. But $\mathcal{C}^S_\psi = \mathcal{C}_\psi$, since S is also a $\rightsquigarrow$-normal-form. Therefore, $\mathcal{C}^S_{\varphi|S} = \mathcal{C}_\psi$ and, by Theorem 1 and the fact that $\mathcal{C}_\psi$ is equisatisfiable with $\mathcal{S}_\psi$, $\mathcal{C}^S_{\varphi|S}$ is unsatisfiable. □

Remark 4. It is not possible to prove Theorem 5 analogously to the proof of the unsatisfiability of the profile shown in [7, 8], which is essentially based on the fact that the profile of φ subsumes $\mathcal{C}_\varphi$. Unfortunately, $\mathcal{C}^S_{\varphi|S}$ does not subsume $\mathcal{C}_\varphi$ in general. In particular, the subsumption fails when φ contains degenerate inferences, in which case $\overset{S}{\rightsquigarrow}$ prunes too much of the characteristic formula and then some clauses of $\mathcal{C}_\varphi$ are not subsumed by any clause of $\mathcal{C}^S_{\varphi|S}$.

Although `SCERes`-normal-forms are usually much less redundant than `CERes`-normal-forms. In the worst case, there is still an exponential blow-up when the characteristic formula is converted to a swapped clause set.

Theorem 6 (Size of Swapped Clause Set). *There exist positive constants k and k' and a sequence of proofs $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ such that $|\varphi_n| \leq k4^n$ and $|\mathcal{C}^S_{\varphi_n}| \geq k'2^{2^n}$.*

Theorem 7 (Size of `SCERes`-Normal-Form). *There exist positive constants k and k'' and a sequence of proofs $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ such that:*

- $|\varphi_n| \leq k4^n$.
- $|SCERes(\varphi_n, \delta_n)| \geq k''2^{2^n}$, *for any refutation* δ_n.

Proof. For both theorems above, the same sequence of proofs used in Theorem 3 can be used, because $\mathcal{C}_{\varphi_k} = \mathcal{C}^S_{\varphi_k}$ for any proof φ_k in that sequence. □

5 Using Structural Clause Form Transformation

In order to avoid the exponential blow-up in the size of the clause set altogether, this section investigates the use of *structural clause form transformation* [1], which avoids the distribution of disjunction over conjunction by introducing new defined predicate symbols for each subformula of the formula to be normalized. The number of new predicate symbols is, therefore, linear w.r.t. to the size of the formula. Moreover, each literal is duplicated only a constant number of times.

Definition 13 ($\overset{D}{\rightsquigarrow}$). *For every conjunctive or disjunctive formula S with free variables $x_1, \ldots, x_m$, a new predicate symbol $N_S(x_1, \ldots, x_m)$ can be created together with its corresponding defining formula:*

$$D_S = N_{S_1 \wedge \ldots \wedge S_n}(x_1, \ldots, x_m) \leftrightarrow n(S_1) \wedge \ldots \wedge n(S_n)$$

$$D_S = N_{S_1 \vee \ldots \vee S_n}(x_1, \ldots, x_m) \leftrightarrow n(S_1) \vee \ldots \vee n(S_n)$$

where:

$$n(S_k) = \begin{cases} S_k & \text{, if } S_k \text{ is a literal} \\ N_{S_k}(y_1, \ldots, y_j) & \text{, if } S_k \text{ is a non-literal with free variables } y_1, \ldots, y_j \end{cases}$$

Then:
$$S \overset{D}{\rightsquigarrow} n(S) \wedge \bigwedge_{S' \in sub(S)} D_{S'}$$
where $sub(S)$ is the set of non-literal subformulas of S.

Remark 5. The connective $\leftrightarrow$ is considered to be just an abbreviation:

$$A \leftrightarrow B_1 \vee \ldots \vee B_n \quad = \quad (\overline{A} \vee B_1 \vee \ldots \vee B_n) \wedge (\overline{B_1} \vee A) \wedge \ldots \wedge (\overline{B_n} \vee A)$$

$$A \leftrightarrow B_1 \wedge \ldots \wedge B_n \quad = \quad (\overline{B_1} \vee \ldots \vee \overline{B_n} \vee A) \wedge (\overline{A} \vee B_1) \wedge \ldots \wedge (\overline{A} \vee B_n)$$

where $\overline{C}$ is $\neg D$, if $C = D$, and D, if $C = \neg D$.

Definition 14 (Definitional Clause Set). *The* definitional clause set $\mathcal{C}^D_\varphi$ *of a proof φ is the $\overset{D}{\rightsquigarrow}$-normal-form of $\mathcal{S}_\varphi$ written in sequent notation. Clauses in $\mathcal{C}^D_\varphi$ are either the* proper clause $\vdash n(\mathcal{S}_\varphi)$ *or* definitional clauses *originating from the defining formulas $D_{S'}$ for each subformula S' of $\mathcal{S}_\varphi$.*

Example 8. Let φ be the proof shown in Example 1, whose characteristic formulas is:
$$\mathcal{S}_\varphi \equiv ((A \wedge B) \wedge (\neg B \vee \neg A)) \vee (C \wedge \neg C)$$

New predicate symbols can be created and defined by the following formulas:

$D \leftrightarrow C \wedge \neg C \qquad E \leftrightarrow \neg B \vee \neg A \qquad F \leftrightarrow A \wedge B \qquad G \leftrightarrow F \wedge E \qquad H \leftrightarrow G \vee D$

The $\overset{D}{\rightsquigarrow}$-normal-form of $\mathcal{S}_\varphi$ is:

$$\begin{array}{l}
H \wedge \\
(\neg D \vee C) \wedge (\neg D \vee \neg C) \wedge (\neg C \vee D \vee C) \wedge \\
(\neg E \vee \neg B \vee \neg A) \wedge (E \vee B) \wedge (E \vee A) \wedge \\
(\neg F \vee A) \wedge (\neg F \vee B) \wedge (\neg A \vee \neg B \vee F) \wedge \\
(\neg G \vee F) \wedge (\neg G \vee E) \wedge (\neg E \vee \neg F \vee G) \wedge \\
(\neg H \vee G \vee D) \wedge (\neg G \vee H) \wedge (\neg D \vee H)
\end{array}$$

The definitional clause set $\mathcal{C}^D_\varphi$ consists of the following clauses. The proper clause is $\vdash H$. All other clauses are definitional clauses.

$$\begin{array}{lll}
\vdash H & & \\
D \vdash C & D, C \vdash & C \vdash D, C \\
E, B, A \vdash & \vdash E, B & \vdash E, A \\
F \vdash A & F \vdash B & A, B \vdash F \\
G \vdash F & G \vdash E & E, F \vdash G \\
H \vdash G, D & G \vdash H & D \vdash H
\end{array}$$

$\square$

The construction of projections requires special care when definitional clause sets are used. The reason is that the clauses now contain many new predicate symbols which do not occur in the input proof. Fortunately, for all definitional clauses of a definitional clause set, projections can be constructed easily and independently of the input proof, by using definition inference rules in the sequent calculus.

Definition 15 (Definitional Projection). *The* definitional projection $\lfloor\varphi\rfloor_c^D$ *of a proof φ w.r.t. a definitional clause c from $\mathcal{C}_\varphi^D$ is constructed according to one of the following templates:*

$$\dfrac{\dfrac{n(S_1)\vdash n(S_1) \qquad \dots \qquad n(S_n)\vdash n(S_n)}{n(S_1)\lor\dots\lor n(S_n)\vdash n(S_1),\dots,n(S_n)}\,\lor_l^*}{N_{S_1\lor\dots\lor S_n}(x_1,\dots,x_m)\vdash n(S_1),\dots,n(S_n)}\,d_l
\qquad
\dfrac{\dfrac{\dfrac{n(S_k)\vdash n(S_k)}{n(S_k)\vdash n(S_1),\dots,n(S_n)}\,w_r^*}{n(S_k)\vdash n(S_1)\lor\dots\lor n(S_n)}\,\lor_r^*}{n(S_k)\vdash N_{S_1\lor\dots\lor S_n}(x_1,\dots,x_m)}\,d_r$$

$$\dfrac{\dfrac{n(S_1)\vdash n(S_1) \qquad \dots \qquad n(S_n)\vdash n(S_n)}{n(S_1),\dots,n(S_n)\vdash n(S_1)\land\dots\land n(S_n)}\,\land_r^*}{n(S_1),\dots,n(S_n)\vdash N_{S_1\land\dots\land S_n}(x_1,\dots,x_m)}\,d_r
\qquad
\dfrac{\dfrac{\dfrac{n(S_k)\vdash n(S_k)}{n(S_1),\dots,n(S_n)\vdash n(S_k)}\,w_l^*}{n(S_1)\land\dots\land n(S_n)\vdash n(S_k)}\,\land_l^*}{N_{S_1\land\dots\land S_n}(x_1,\dots,x_m)\vdash n(S_k)}\,d_l$$

If S_k is a negative literal, negation inferences must be added in the templates.

Example 9. Definitional projections w.r.t. the clauses shown in Example 8 are:

$\lfloor\varphi\rfloor_{D\vdash C}^D$:

$$\dfrac{\dfrac{\dfrac{C\vdash C}{C,\neg C\vdash C}\,w_l}{C\land\neg C\vdash C}\,\land_l}{D\vdash C}\,d_l$$

$\lfloor\varphi\rfloor_{D,C\vdash}^D$:

$$\dfrac{\dfrac{\dfrac{\dfrac{C\vdash C}{C,C\vdash C}\,w_l}{C,\neg C,C\vdash}\,\neg_l}{C\land\neg C,C\vdash}\,\land_l}{D,C\vdash}\,d_l$$

$\lfloor\varphi\rfloor_{C\vdash D,C}^D$:

$$\dfrac{\dfrac{C\vdash C \qquad \dfrac{C\vdash C}{\vdash C,\neg C}\,\neg_r}{C\vdash C,C\land\neg C}\,\land_r}{C\vdash C,D}\,d_r$$

$\lfloor\varphi\rfloor_{\vdash E,A}^D$:

$$\dfrac{\dfrac{\dfrac{\dfrac{A\vdash A}{\vdash\neg A,A}\,\neg_r}{\vdash\neg B,\neg A,A}\,w_r}{\vdash\neg B\lor\neg A,A}\,\lor_r}{\vdash E,A}\,d_r$$

$\lfloor\varphi\rfloor_{\vdash E,B}^D$:

$$\dfrac{\dfrac{\dfrac{\dfrac{B\vdash B}{\vdash\neg B,B}\,\neg_r}{\vdash\neg B,\neg A,B}\,w_r}{\vdash\neg B\lor\neg A,B}\,\lor_r}{\vdash E,B}\,d_r$$

$\lfloor\varphi\rfloor_{E,A,B\vdash}^D$:

$$\dfrac{\dfrac{\dfrac{B\vdash B}{\neg B,B\vdash}\,\neg_l \qquad \dfrac{A\vdash A}{\neg A,A\vdash}\,\neg_l}{\neg B\lor\neg A,B,A\vdash}\,\lor_l}{E,B,A\vdash}\,d_l$$

$\lfloor\varphi\rfloor_{F\vdash A}^D$:

$$\dfrac{\dfrac{\dfrac{A\vdash A}{A,B\vdash A}\,w_l}{A\land B\vdash A}\,\land_l}{F\vdash A}\,d_l$$

$\lfloor\varphi\rfloor_{F\vdash B}^D$:

$$\dfrac{\dfrac{\dfrac{B\vdash B}{A,B\vdash B}\,w_l}{A\land B\vdash B}\,\land_l}{F\vdash B}\,d_l$$

$\lfloor\varphi\rfloor_{A,B\vdash F}^D$:

$$\dfrac{\dfrac{A\vdash A \qquad B\vdash B}{A,B\vdash A\land B}\,\land_r}{A,B\vdash F}\,d_r$$

$\lfloor\varphi\rfloor_{G\vdash F}^D$:

$$\dfrac{\dfrac{\dfrac{F\vdash F}{F,E\vdash F}\,w_l}{F\land E\vdash F}\,\land_l}{G\vdash F}\,d_l$$

$\lfloor\varphi\rfloor_{G\vdash E}^D$:

$$\dfrac{\dfrac{\dfrac{E\vdash E}{F,E\vdash E}\,w_l}{F\land E\vdash E}\,\land_l}{G\vdash E}\,d_l$$

$\lfloor\varphi\rfloor_{F,E\vdash G}^D$:

$$\dfrac{\dfrac{F\vdash F \qquad E\vdash E}{F,E\vdash F\land E}\,\land_r}{F,E\vdash G}\,d_r$$

$\lfloor\varphi\rfloor^{D}_{D\vdash H}$:

$$\cfrac{\cfrac{\cfrac{D \vdash D}{D \vdash G, D}\,w_r}{D \vdash G \vee D}\,\vee_r}{D \vdash H}\,d_r$$

$\lfloor\varphi\rfloor^{D}_{G\vdash H}$:

$$\cfrac{\cfrac{\cfrac{G \vdash G}{G \vdash G, D}\,w_r}{G \vdash G \vee D}\,\vee_r}{G \vdash H}\,d_r$$

$\lfloor\varphi\rfloor^{D}_{H\vdash G,D}$:

$$\cfrac{\cfrac{G \vdash G \qquad D \vdash D}{G \vee D \vdash G, D}\,\vee_l}{H \vdash G, D}\,d_l$$

While projections w.r.t. definitional clauses can always be easily constructed using the four templates given in Definition 15, the proper clause requires a more sophisticated inductively constructed projection that actually depends on the input proof.

Definition 16 (Proper Projection). *The* proper projection $\lfloor\varphi\rfloor^{P}_{\vdash n(\mathcal{S}_\varphi)}$ *of a proof φ w.r.t. its proper clause $\vdash n(\mathcal{S}_\varphi)$ is constructed inductively. Let φ' be a subproof of φ and ρ be its last inference. Let S' be the subformula of $\mathcal{S}_\varphi$ at φ'. The following cases can be distinguished:*

– *ρ is an axiom inference: Then φ' is of the form:*

$$\cfrac{}{A \vdash A}\,\rho$$

- *If both occurrences of A are cut-ancestors, then let φ'' be:*

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\,\rho}{\vdash \neg A, A}\,\neg_r}{\vdash \neg A \vee A}\,\vee_r}{\vdash n(S')}\,d_r$$

- *If only the antecedent's A is a cut-ancestor, then let φ'' be:*

$$\cfrac{\cfrac{}{A \vdash A}\,\rho}{\vdash \neg A, A}\,\neg_r$$

- *Otherwise, let $\varphi'' = \varphi'$*

– *ρ is a n-ary inference (with $n \geq 2$): Then φ' is of the form:*

$$\cfrac{\begin{matrix}\psi'_1\\ \Gamma'_1 \vdash \Delta'_1\end{matrix} \qquad \ldots \qquad \begin{matrix}\psi'_n\\ \Gamma'_n \vdash \Delta'_n\end{matrix}}{\Gamma' \vdash \Delta'}\,\rho$$

By induction, ψ''_k is of the form:

$$\begin{matrix}\psi''_k\\ \Gamma''_1 \vdash \Delta''_1, n(S'_{\psi'_k})\end{matrix}$$

where $S'_{\psi'_k}$ is the substruct of S' corresponding to ψ'_k.

- *ρ operates on end-sequent-ancestors: Then let φ'' be:*

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\psi_1'' \\ \Gamma_1'' \vdash \Delta_1'', n(S'_{\psi_1'})\end{matrix} \qquad \dots \qquad \begin{matrix}\psi_n'' \\ \Gamma_n'' \vdash \Delta_n'', n(S'_{\psi_n'})\end{matrix}}{\Gamma'' \vdash \Delta'', n(S'_{\psi_1'}), \dots, n(S'_{\psi_n'})}\,\rho}{\Gamma'' \vdash \Delta'', n(S'_{\psi_1'}) \vee \dots \vee n(S'_{\psi_n'})}\,\vee_r}{\Gamma'' \vdash \Delta'', n(S')}\,d_r$$

- *ρ operates on cut-ancestors: Then let φ'' be:*

$$\dfrac{\dfrac{\begin{matrix}\psi_1'' \\ \Gamma_1'' \vdash \Delta_1'', n(S'_{\psi_1'})\end{matrix} \qquad \dots \qquad \begin{matrix}\psi_n'' \\ \Gamma_n'' \vdash \Delta_n'', n(S'_{\psi_n'})\end{matrix}}{\Gamma'' \vdash \Delta'', n(S'_{\psi_1'}) \wedge \dots \wedge n(S'_{\psi_n'})}\,\wedge_r}{\Gamma'' \vdash \Delta'', n(S')}\,d_r$$

– *ρ is a unary inference: Then φ' is of the form:*

$$\dfrac{\psi'}{\Gamma' \vdash \Delta'}\,\rho$$

- *ρ operates on cut-ancestors: then ρ is simply skipped and thus $\varphi'' = \psi''$.*
- *ρ operates on end-sequent-ancestors: then ρ is performed on the transformed premise and thus φ'' is of the form:*

$$\dfrac{\psi''}{\Gamma'' \vdash \Delta''}\,\rho$$

The proper projection $\lfloor \varphi \rfloor^{P}_{\vdash n(\mathcal{S}_\varphi)}$ *is the final result of this inductive construction. $\lfloor \varphi \rfloor^{P}_{\vdash n(\mathcal{S}_\varphi)} = \varphi''$ when $\varphi' = \varphi$.*

Example 10. The proper projection $\lfloor \varphi \rfloor^{P}_{\vdash H}$ for the proof φ from the previous example is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}\,\wedge_r}{A, B \vdash F}\,d_r}{A \wedge B \vdash F}\,\wedge_l \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{\vdash \neg B, B}\,\neg_r \quad \dfrac{A \vdash A}{\vdash \neg A, A}\,\neg_r}{\vdash \neg B, \neg A, B \wedge A}\,\wedge_r}{\vdash \neg B \vee \neg A, B \wedge A}\,\vee_r}{\vdash E, B \wedge A}\,d_r}{A \wedge B \vdash F \wedge E, B \wedge A}\,\wedge_r}{A \wedge B \vdash G, B \wedge A}\,d_r \qquad \dfrac{\dfrac{C \vdash C \quad \dfrac{C \vdash C}{\vdash \neg C, C}\,\neg_r}{C \vdash C \wedge \neg C, C}\,\wedge_r}{C \vdash D, C}\,d_r}{(A \wedge B) \vee C \vdash G, D, B \wedge A, C}\,\vee_l}{(A \wedge B) \vee C \vdash G \vee D, B \wedge A, C}\,\vee_r}{(A \wedge B) \vee C \vdash H, B \wedge A, C}\,d_r$$

A step-by-step construction of this projection can be found in [15]. □

Theorem 8 (Size of Definitional Clause Set). *There exists a positive constant k such that $|\mathcal{C}^{D}_{\varphi}| \leq k|\varphi|$, for any proof φ.*

Proof. The number of subformulas in $\mathcal{S}_\varphi$ is linearly proportional to the size of $\mathcal{S}_\varphi$, and structural clause form transformation adds three definitional clauses for each subformula. The definitional clauses have at most $k_a + 1$ literals each, where k_a is the maximum arity of disjunctions and conjunctions in $\mathcal{S}_\varphi$. Therefore, $|\mathcal{C}^{D}_{\varphi}| \leq k|\mathcal{S}_\varphi| \leq k|\varphi|$, for some positive k. □

Theorem 9 (Size of Proper and Definitional Projections). *There exist positive constants k and k' such that, for any proof φ:*

- $|\lfloor\varphi\rfloor_{c}^{P}| \leq k|\varphi_n|$
- $|\lfloor\varphi\rfloor_{c'}^{D}| \leq k'$

Proof. Definitional projections have a size that is bounded by a constant proportional to the maximum arity of disjunctions and conjunctions. The proper projection $\lfloor\varphi\rfloor_{c}^{P}$ is just φ with inferences operating on cut-ancestors replaced by a bounded number of $\neg_r$, $\vee_r$, $\wedge_r$ and d_r inferences. $\square$

Definition 17 **(`DCERes`-normal-form).** *$DCERes(\varphi, \delta)$ denotes the* `DCERes`-normal-form *of the proof φ w.r.t. a resolution refutation δ of the definitional clause set $\mathcal{C}^D_\varphi$. It is obtained in the same way as a `CERes`-normal-form, but using the definitional clause set $\mathcal{C}^D_\varphi$ instead of $\mathcal{C}_\varphi$ and using definitional and proper projections instead of the usual projections.*

Example 11 (`DCERes`-Normal-Form). By combining the proper projection shown in Example 10 and the definitional projections shown in Example 9 with a resolution refutation δ of $\mathcal{C}^D_\varphi$, the resulting normal form $DCERes(\varphi, \delta)$ is:

$$
\dfrac{\overset{\lfloor\varphi\rfloor^P_{\vdash H}}{(A \land B) \lor C \vdash H, B \land A, C} \qquad \dfrac{\dfrac{\lfloor\varphi\rfloor^D_{H\vdash G,D} \qquad \dfrac{\dfrac{\lfloor\varphi\rfloor^D_{G\vdash F} \qquad \dfrac{\lfloor\varphi\rfloor^D_{G\vdash E} \qquad \dfrac{\dfrac{\lfloor\varphi\rfloor^D_{F\vdash A} \qquad \dfrac{\lfloor\varphi\rfloor^D_{F\vdash B} \qquad \lfloor\varphi\rfloor^D_{E,B,A\vdash}}{F, E, A \vdash}}{F, F, E \vdash}\,cut}{F, E \vdash}\,c_l}{G, F \vdash}\,cut}{G, G \vdash}\,cut}{G \vdash}\,c_l}{H \vdash D}\,cut \qquad \dfrac{\dfrac{\lfloor\varphi\rfloor^D_{D\vdash C} \qquad \lfloor\varphi\rfloor^D_{D,C\vdash}}{D, D \vdash}\,cut}{D \vdash}\,c_l}{H \vdash}\,cut}{(A \land B) \lor C \vdash B \land A, C}\,cut
$$

$\square$

Although in a `DCERes`-normal-form the cuts appear to be atomic, the atomic cut-formulas are actually defined predicate symbols that stand for complex propositional formulas. Therefore, it can be argued that `DCERes` does not eliminate cuts as much as `CERes` and `SCERes` do. Although this is a reasonable objection to `DCERes`, it should be noted that as long as the cuts are quantifier-free, as is the case for `DCERes`-normal-forms, summarized mathematical information in the form of Herbrand sequents can still be obtained from the proof [12, 13, 9].

6 A Combined Approach

Although the number of defined symbols introduced by the construction of definitional clause sets is bounded linearly with respect to the size of the characteristic formula, it still creates a new symbol for every subformula of the characteristic formula. The number of new symbols can be reduced with a technique that combines ideas from swapped and definitional clause sets. The idea is to use $\overset{S}{\rightsquigarrow}$ as long as no duplications occur and then use $\overset{D}{\rightsquigarrow}$ only for the subformulas that cannot be normalized with $\overset{S}{\rightsquigarrow}$ without duplications.

Definition 18 **($\overset{DS}{\rightsquigarrow}$).** *$\overset{S'}{\rightsquigarrow}$ denotes a restricted form of $\overset{S}{\rightsquigarrow}$ where the distribution of disjunction over conjunction cannot lead to duplications. The first rewriting rule from Definition 9 is replaced by the rewriting rule below, where S is distributed to at most one conjunct S_k:*

$$S \vee (S_1 \wedge \ldots \wedge S_k \wedge \ldots \wedge S_n) \overset{S'}{\rightsquigarrow} S_1 \wedge \ldots \wedge (S \vee S_k) \wedge \ldots \wedge S_n$$

$\overset{D'}{\rightsquigarrow}$ denotes a restricted form of $\overset{D}{\rightsquigarrow}$, defined by the following rewriting rule, which can be applied only if $S \vee (S_1 \wedge \ldots \wedge S_n)$ is already in $\overset{DS}{\rightsquigarrow}$-normal-form:

$$C[S \vee (S_1 \wedge \ldots \wedge S_n)] \overset{D'}{\rightsquigarrow} C[S \vee N(x_1, \ldots, x_m)] \wedge (N(x_1, \ldots, x_m) \leftrightarrow S_1 \wedge \ldots \wedge S_n)$$

where N is a new symbol and $x_1, \ldots, x_m$ are free-variables of $(S_1 \wedge \ldots \wedge S_n)$.

The relation $\overset{DS}{\rightsquigarrow}$ is the union of $\overset{S'}{\rightsquigarrow}$ and $\overset{D'}{\rightsquigarrow}$.

Definition 19 **(Definitional Swapped Clause Set).** *A* definitional swapped clause set $\mathcal{C}^{DS}_{\varphi|S}$ *of a proof φ w.r.t. to a $\overset{DS}{\rightsquigarrow}$-normal-form S of $\mathcal{S}_\varphi$ is S written in sequent notation. Clauses originating from defining equations introduced by $\overset{D'}{\rightsquigarrow}$ are* definitional clauses*. Non-definitional clauses not containing new symbols are* pure clauses*. All other clauses are* mixed clauses*.*

Remark 6. In cases where $\mathcal{C}^{DS}_{\varphi|S_1} = \mathcal{C}^{DS}_{\varphi|S_2}$ for any S_1 and S_2, the unique definitional swapped clause set is denoted simply as $\mathcal{C}^{DS}_\varphi$.

Example 12. Let φ be the proof shown in Example 7. Its characteristic formula can be normalized as follows:

$$\begin{aligned}
\mathcal{S}_\varphi &\equiv ((A \wedge^1 B) \wedge^3 (\neg B \vee^2 \neg A)) \vee^5 (C \wedge^4 \neg C) \\
&\overset{S'}{\rightsquigarrow} ((A \wedge^1 B) \vee^5 (C \wedge^4 \neg C)) \wedge^3 (\neg B \vee^2 \neg A) \\
&\overset{S'}{\rightsquigarrow} ((((A \wedge^1 B) \vee^5 C) \wedge^4 \neg C)) \wedge^3 (\neg B \vee^2 \neg A) \\
&\overset{D'}{\rightsquigarrow} (((D_{A \wedge B} \vee^5 C) \wedge^4 \neg C)) \wedge^3 (\neg B \vee^2 \neg A) \wedge (D_{A \wedge B} \leftrightarrow (A \wedge^1 B)) \\
&\equiv (D_{A \wedge B} \vee^5 C) \wedge^4 \neg C \wedge^3 (\neg B \vee^2 \neg A) \wedge \\
&\quad\; (\neg D_{A \wedge B} \vee A) \wedge (\neg D_{A \wedge B} \vee B) \wedge (D_{A \wedge B} \vee (\neg A \vee \neg B))
\end{aligned}$$

And the corresponding definitional swapped clause set is:

$$\mathcal{C}_{\varphi}^{DS} \equiv \left\{ \begin{array}{l} \vdash D_{A \wedge B}, C \quad ; \\ C \vdash \quad ; \\ B, A \vdash \quad ; \\ D_{A \wedge B} \vdash A \quad ; \\ D_{A \wedge B} \vdash B \quad ; \\ A, B \vdash D_{A \wedge B} \end{array} \right\}$$

$D_{A\wedge B} \vdash A$, $D_{A\wedge B} \vdash B$ and $A, B \vdash D_{A\wedge B}$ are definitional clauses. $C \vdash$ and $B, A \vdash$ are pure clauses. And $\vdash D_{A\wedge B}, C$ is a mixed clause. □

While construction of definitional swapped clause sets is reasonably straightforward, the construction of projections presents some difficulties. As in the case of definitional clause sets, some clauses in definitional swapped clause sets are definitional, and their projections can be easily constructed according to Definition 15. Other clauses are pure in the sense that they do not contain any defined predicate symbol, and hence their projections can be constructed with the usual method explained in Definition 6. However, for mixed clauses, which contain a mix of defined and undefined predicate symbols, it is necessary to construct a *mixed projection*, which combines the usual algorithm with the construction method for proper projections described in Definition 16.

Definition 20 (Encapsulated Formulas). *Let S be a characteristic formula and S' be a subformula of S corresponding to a new predicate symbol $N_{S'}$ created during the $\overset{DS}{\rightsquigarrow}$-normalization of S. Then, the* encapsulated formulas *of $N_{S'}$ are all the atomic formulas of S'.*

Example 13. The formulas encapsulated by the new predicate symbol $D_{A\wedge B}$ of the $\overset{DS}{\rightsquigarrow}$-normal-form of the struct $\mathcal{S}_\varphi$ shown in Example 12 are: A and B. □

Definition 21 (Mixed Projection). *Let φ be a proof and c a mixed clause in $\mathcal{C}_{\varphi|S}^{DS}$. The* mixed projection *$\lfloor \varphi \rfloor_c^M$ of φ w.r.t. c is constructed in two steps:*

1. *Apply the usual algorithm from Definition 6 to φ, letting A be the set of axioms that contain formulas that contribute to c or that are encapsulated by a new predicate symbol in c.*
2. *Replace inferences operating on descendants of encapsulated formulas by $\neg_r$, $\wedge_r$, $\vee_r$ and d_r, analogously to what is done in the construction of proper projections (Definition 16).*

Example 14. Let φ be the proof from previous examples. Noting that the axiom sequent $C \vdash C$ contains a formula that directly contributes to $\vdash D_{A\wedge B}, C$, while the axiom sequents $A \vdash A$ and $B \vdash B$ contain formulas that are encapsulated by $D_{A\wedge B}$, the first step in the construction of the mixed projection $\lfloor \varphi \rfloor^M_{\vdash D_{A\wedge B}, C}$ generates the following proof:

$$
\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\,\wedge_r^1}{A \wedge B \vdash A \wedge B}\,\wedge_l \qquad C \vdash C}{(A \wedge B) \vee C \vdash A \wedge B, C}\,\vee_l^5
$$

The second step introduces definition inferences to reencapsulate the formulas:

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\,\wedge_r}{A, B \vdash D_{A \wedge B}}\,d_r}{A \wedge B \vdash D_{A \wedge B}}\,\wedge_l \qquad C \vdash C}{(A \wedge B) \vee C \vdash D_{A \wedge B}, C}\,\vee_l^5
$$

□

Definition 22 (DSCERes-normal-form). *$DSCERes(\varphi, \delta)$ denotes the* `DSCERes` normal-form *of the proof φ w.r.t. the resolution refutation δ of any definitional swapped clause set $\mathcal{C}^{DS}_{\varphi|S}$. It is obtained in the same way as a* `CERes`*-normal-form, but using a definitional swapped clause set $\mathcal{C}^{DS}_{\varphi|S}$ instead of $\mathcal{C}_\varphi$ and using definitional and mixed projections when necessary.*

Example 15. The shortest refutation δ of $\mathcal{C}^{DS}_{\varphi}$ is shown below:

$$
\dfrac{\dfrac{\vdash D_{A \wedge B}, C \qquad C \vdash}{\vdash D_{A \wedge B}}\,r \qquad \dfrac{\dfrac{D_{A \wedge B} \vdash A \qquad \dfrac{D_{A \wedge B} \vdash B \qquad B, A \vdash}{D_{A \wedge B}, A \vdash}\,r}{D_{A \wedge B}, D_{A \wedge B} \vdash}\,r}{D_{A \wedge B} \vdash}\,f_l}{\vdash}\,r
$$

Using the mixed projection shown in Example 14, pure projections shown in Example 3 and definitional projections shown in Example 9, $DSCERes(\varphi, \delta)$ is:

$$
\dfrac{\dfrac{\lfloor \varphi \rfloor^M_{\vdash D_{A \wedge B}, C} \qquad C \vdash C}{(A \wedge B) \vee C \vdash D_{A \wedge B}, C}\,cut \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,w_l}{A \wedge B \vdash A}\,\wedge_l}{D_{A \wedge B} \vdash A}\,d_l \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{A, B \vdash B}\,w_l}{A \wedge B \vdash B}\,\wedge_l}{D_{A \wedge B} \vdash B}\,d_l \qquad \dfrac{B \vdash B \qquad A \vdash A}{B, A \vdash B \wedge A}\,\wedge_r}{D_{A \wedge B}, A \vdash B \wedge A}\,cut}{D_{A \wedge B}, D_{A \wedge B} \vdash B \wedge A}\,cut}{D_{A \wedge B} \vdash B \wedge A}\,c_l}{(A \wedge B) \vee C \vdash B \wedge A, C}\,cut
$$

□

7 Ignoring Quantifier-Free Cuts

When all that is desired is the possibility to summarize the proof by means of Herbrand sequents [12, 13, 9], it is only necessary to remove cuts that contain quantifiers. This can be achieved with an easy modification of `CERes`: in the construction of the characteristic formula and of the projections, ancestors of quantifier-free cuts should be treated in the same way as end-sequent ancestors.

Any of the previously defined variants of `CERes` can be modified in this manner. When this is done, more axioms are mapped to $\perp$ in the characteristic formula, and some inferences that were previously mapped to conjunctions are now mapped to disjunctions. Consequently, fewer distributions, redundant duplications and new predicate symbols are necessary. This leads to smaller and more easily refutable clause sets.

8 Conclusion

In this paper, a source of redundancy in cut-elimination with the `CERes` method was identified and three variants (`SCERes`, `DCERes` and `DSCERes`) were developed to succesfully tackle this problem. By using `DCERes` or `DSCERes`, it is possible to avoid an exponential increase in the size of the clause set extracted from the proof with cuts. This improves the efficiency of cut-elimination by resolution and allows it to generate smaller essentially cut-free proofs.

The redundancy uncovered here is closely linked to the fact that the classical `CERes` method relies heavily on the use of contraction and weakening. This brings difficulties when applying `CERes` for logics that treat these structural inference rules more restrictively, such as substructural and intuitionistic logics [11]. Therefore, investigating redundancy sources in `CERes` not only is beneficial for improving the efficiency of `CERes` and for reducing the complexity of proofs it produces, but might also lead to insights on how to broaden the set of logics for which `CERes` is applicable.

A Sequent Calculus

The sequent calculus used in this paper is shown below. Γ,Δ,Γ_1,Δ_1,Γ_2,Δ_2 are multisets of formulas called *contexts*. For each rule, the active formula below the line, colored in red, is its *main* formula, while the active formulas in the premises, colored in blue, are its *auxiliary* formulas.

$$\frac{}{A \vdash A}\ axiom \qquad \frac{\Gamma_1 \vdash \Delta_1, F \quad F, \Gamma_2 \vdash \Gamma_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

$$\frac{\Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}\ w_l \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, F}\ w_r \qquad \frac{F, F, \Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}\ c_l \qquad \frac{\Gamma \vdash \Delta, F, F}{\Gamma \vdash \Delta, F}\ c_r$$

$$\frac{F_1, F_2, \Gamma \vdash \Delta}{F_1 \wedge F_2, \Gamma \vdash \Delta}\ \wedge_l \qquad \frac{\Gamma_1 \vdash \Delta_1, F_1 \quad \Gamma_2 \vdash \Delta_2, F_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, F_1 \wedge F_2}\ \wedge_r \qquad \frac{\Gamma \vdash \Delta, F}{\neg F, \Gamma \vdash \Delta}\ \neg_l$$

$$\frac{F_1, \Gamma_1 \vdash \Delta_1 \quad F_2, \Gamma_2 \vdash \Delta_2}{F_1 \vee F_2, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \vee_l \qquad \frac{\Gamma \vdash \Delta, F_1, F_2}{\Gamma \vdash \Delta, F_1 \vee F_2}\ \vee_r \qquad \frac{F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg F}\ \neg_r$$

$$\frac{\Gamma_1 \vdash \Delta_1, F_1 \quad F_2, \Gamma_2 \vdash \Delta_2}{F_1 \rightarrow F_2, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \rightarrow_l \qquad \frac{F_1, \Gamma \vdash \Delta, F_2}{\Gamma \vdash \Delta, F_1 \rightarrow F_2}\ \rightarrow_r$$

$$\frac{F\{x/t\}, \Gamma \vdash \Delta}{(\forall x)F, \Gamma \vdash \Delta}\ \forall_l \qquad \frac{\Gamma \vdash \Delta, F\{x/\alpha\}}{\Gamma \vdash \Delta, (\forall x)F}\ \forall_r \qquad \frac{F\{x/\alpha\}, \Gamma \vdash \Delta}{(\exists x)F, \Gamma \vdash \Delta}\ \exists_l \qquad \frac{\Gamma \vdash \Delta, F\{x/t\}}{\Gamma \vdash \Delta, (\exists x)F}\ \exists_r$$

α should occur neither in Γ nor in Δ nor in F. t must not contain a variable that is bound in F.

$$\frac{F[x_1,\ldots,x_n],\Gamma\vdash\Delta}{P(x_1,\ldots,x_n),\Gamma\vdash\Delta}\ d_l \qquad \frac{\Gamma\vdash\Delta,F[x_1,\ldots,x_n]}{\Gamma\vdash\Delta,P(x_1,\ldots,x_n)}\ d_r$$

where $x_1,\ldots,x_n$ are the free variables of F and P is defined by: $P(x_1,\ldots,x_n)\leftrightarrow F[x_1,\ldots,x_n]$.

B Inference Permutation

This section describes a proof rewriting system (Definition 31) for permutting inferences. The rules are subdivided according to the kind of dependence (Definition 23) between the inferences that they permute. If the lower inference is independent of the upper inference, then they can easily be permuted (Definition 24), with no increase of proof size. However, if the lower inference is indirectly dependent on the upper inference, then the permutation requires a duplication of the lower inference, as well as the introduction of weakening and contraction inferences (Definition 25). The case of eigen-variable dependence can be avoided by considering skolemized proofs only. Even though two inferences cannot generally be permuted if there is direct dependence between them, this is possible in the particular case when the upper inference is contraction (Definition 27) or weakening (Definition 27).

Remark 7. Many symmetric and analogous cases have been ommitted in the definitions of inference permutation rules in this section. Full versions of the definitions, showing all cases, are available at [15].

Definition 23 (Dependences between Inferences). *An inference ρ_2 is* directly dependent *on another inference ρ_1, denoted $\rho_2 \prec_D \rho_1$, if and only if a main occurrence of ρ_1 is an ancestor of an auxiliary occurrence of ρ_2. A strong quantifier inference ρ_2 is* eigenvariable-dependent *on weak quantifier inference ρ_1 occurring above ρ_2, denoted $\rho_2 \prec_Q \rho_1$, if and only if the term used by ρ_1 to substitute the weakly quantified variable contains an occurrence of the eigenvariable of ρ_2. An inference ρ_2 is* indirectly dependent *on another inference ρ_1 occurring above ρ_2, denoted $\rho_2 \prec_I \rho_1$, if and only if it is not directly dependent on ρ_1 and the auxiliary occurrences of ρ_2 have ancestors in more than one premise sequent of ρ_1. An inference ρ_2 is* independent *of another inference ρ_1 if and only if ρ_2 is neither directly dependent nor eigenvariable-dependent nor indirectly dependent on ρ_1.*

Definition 24 ($\gg_I$). *For the permutation of independent inferences, the following rules can be used.*

When both inferences are unary:

$$\cfrac{\cfrac{\begin{array}{c}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1\vdash\Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{array}}{\Gamma^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1\vdash\Delta^{\rho_1},\Delta_1^{\rho_2},\Delta_1}\ \rho_1}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1\vdash\Delta^{\rho_1},\Delta^{\rho_2},\Delta_1}\ \rho_2$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1}}{\Gamma_1^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta^{\rho_2}, \Delta_1}\,\rho_2}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta_1}\,\rho_1$$

When the lower inference is unary and the upper inference is binary:

$$\dfrac{\dfrac{\begin{matrix}\varphi_1 \\ \Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1\end{matrix} \qquad \begin{matrix}\varphi_2 \\ \Gamma_2^{\rho_1}, \Gamma_2 \vdash \Delta_2^{\rho_1}, \Delta_2\end{matrix}}{\Gamma^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1, \Delta_2}\,\rho_1}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta_1, \Delta_2}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\dfrac{\begin{matrix}\varphi_1 \\ \Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1\end{matrix}}{\Gamma_1^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta^{\rho_2}, \Delta_1}\,\rho_2 \qquad \begin{matrix}\varphi_2 \\ \Gamma_2^{\rho_1}, \Gamma_2 \vdash \Delta_2^{\rho_1}, \Delta_2\end{matrix}}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta_1, \Delta_2}\,\rho_1$$

When the upper inference is unary and the lower inference is binary:

$$\dfrac{\dfrac{\begin{matrix}\varphi_1 \\ \Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1\end{matrix}}{\Gamma^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1}\,\rho_1 \qquad \begin{matrix}\varphi_2 \\ \Gamma_2^{\rho_2}, \Gamma_2 \vdash \Delta_2^{\rho_2}, \Delta_2\end{matrix}}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta_1, \Delta_2}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\dfrac{\begin{matrix}\varphi_1 \\ \Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1\end{matrix} \qquad \begin{matrix}\varphi_2 \\ \Gamma_2^{\rho_2}, \Gamma_2 \vdash \Delta_2^{\rho_2}, \Delta_2\end{matrix}}{\Gamma_1^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta_1^{\rho_1}, \Delta^{\rho_2}, \Delta_1, \Delta_2}\,\rho_2}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta_1, \Delta_2}\,\rho_1$$

When both inferences are binary:

$$\dfrac{\dfrac{\begin{matrix}\varphi_1 \\ \Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1\end{matrix} \qquad \begin{matrix}\varphi_2 \\ \Gamma_2^{\rho_1}, \Gamma_2 \vdash \Delta_2^{\rho_1}, \Delta_2\end{matrix}}{\Gamma^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1, \Delta_2}\,\rho_1 \qquad \begin{matrix}\varphi_3 \\ \Gamma_3^{\rho_2}, \Gamma_3 \vdash \Delta_3^{\rho_2}, \Delta_3\end{matrix}}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta_1, \Delta_2, \Delta_3}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\dfrac{\begin{matrix}\varphi_1 \\ \Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1\end{matrix} \qquad \begin{matrix}\varphi_3 \\ \Gamma_3^{\rho_2}, \Gamma_3 \vdash \Delta_3^{\rho_2}, \Delta_3\end{matrix}}{\Gamma_1^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_3 \vdash \Delta_1^{\rho_1}, \Delta^{\rho_2}, \Delta_1, \Delta_3}\,\rho_2 \qquad \begin{matrix}\varphi_2 \\ \Gamma_2^{\rho_1}, \Gamma_2 \vdash \Delta_2^{\rho_1}, \Delta_2\end{matrix}}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta_1, \Delta_2, \Delta_3}\,\rho_1$$

Definition 25 **($\gg_{ID}$).** *For the permutation of indirectly dependent inferences, the following rule can be used:*

$$\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1} \qquad \stackrel{\varphi_2}{\Gamma_2^{\rho_1}, \Gamma_2^{\rho_2}, \Gamma_2 \vdash \Delta_2^{\rho_1}, \Delta_2^{\rho_2}, \Delta_2}}{\Gamma^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_2^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta^{\rho_1}, \Delta_1^{\rho_2}, \Gamma_2^{\rho_2}, \Delta_1, \Delta_2}\,\rho_1}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta_1, \Delta_2}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1}}{\Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_2^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_2^{\rho_2}, \Delta_1}\,w^*}{\Gamma_1^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta^{\rho_2}, \Delta_1}\,\rho_2 \qquad \dfrac{\dfrac{\stackrel{\varphi_2}{\Gamma_2^{\rho_1}, \Gamma_2^{\rho_2}, \Gamma_2 \vdash \Delta_2^{\rho_1}, \Delta_2^{\rho_2}, \Delta_2}}{\Gamma_2^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_2^{\rho_2}, \Gamma_2 \vdash \Delta_2^{\rho_1}, \Delta_1^{\rho_2}, \Delta_2^{\rho_2}, \Delta_2}\,w^*}{\Gamma_2^{\rho_1}, \Gamma^{\rho_2}, \Gamma_2 \vdash \Delta_2^{\rho_1}, \Delta^{\rho_2}, \Delta_2}\,\rho_2}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta^{\rho_2}, \Delta_1, \Delta_2}\,\rho_1}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1, \Gamma_2 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta_1, \Delta_2}\,c^*$$

Definition 26 **($\gg_{DC}$).** *In cases when the dependent inference is a contraction and the inference ρ on which it depends occurs twice above the contraction, they can be swapped in a smarter way according to the rules below:*

$$\dfrac{\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma_1, \Gamma_\rho, \Gamma_\rho \vdash \Delta_1, \Delta_\rho, \Delta_\rho}}{\Gamma_1, \Gamma_\rho, \Pi_\rho \vdash \Delta_1, \Delta_\rho, \Lambda_\rho}\,\rho}{\Gamma_1, \Pi_\rho, \Pi_\rho \vdash \Delta_1, \Lambda_\rho, \Lambda_\rho}\,\rho}{\Gamma_1, \Pi_\rho \vdash \Delta_1, \Lambda_\rho}\,c^*$$

$$\Downarrow$$

$$\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma_1, \Gamma_\rho, \Gamma_\rho \vdash \Delta_1, \Delta_\rho, \Delta_\rho}}{\Gamma_1, \Gamma_\rho \vdash \Delta_1, \Delta_\rho}\,c^*}{\Gamma_1, \Pi_\rho \vdash \Delta_1, \Lambda_\rho}\,\rho$$

$$\dfrac{\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma_1, \Gamma_1^\rho, \Gamma_1^\rho \vdash \Delta_1, \Delta_\rho, \Delta_\rho} \qquad \stackrel{\varphi_2}{\Gamma_2, \Gamma_2^\rho \vdash \Delta_2, \Delta_2^\rho}}{\Gamma_1, \Gamma_\rho, \Pi_\rho \vdash \Delta_1, \Delta_\rho, \Lambda_\rho}\,\rho \qquad \stackrel{\varphi_2}{\Gamma_2, \Gamma_2^\rho \vdash \Delta_2, \Delta_2^\rho}}{\Gamma_1, \Pi_\rho, \Pi_\rho \vdash \Delta_1, \Lambda_\rho, \Lambda_\rho}\,\rho}{\Gamma_1, \Pi_\rho \vdash \Delta_1, \Lambda_\rho}\,c^*$$

$$\Downarrow$$

$$\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma_1, \Gamma_1^\rho, \Gamma_1^\rho \vdash \Delta_1, \Delta_1^\rho, \Delta_1^\rho}}{\Gamma_1, \Gamma_\rho \vdash \Delta_1, \Delta_\rho}\,c^* \qquad \stackrel{\varphi_2}{\Gamma_2, \Gamma_2^\rho \vdash \Delta_2, \Delta_2^\rho}}{\Gamma_1, \Pi_\rho \vdash \Delta_1, \Lambda_\rho}\,\rho$$

Definition 27 **($\gg_C$).** *The permutation of an inference over contractions causes duplication of the inference, as shown in the rules below:*

$$
\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1, \Gamma_\rho, \Gamma'_\rho \vdash \Delta_1, \Delta_\rho, \Delta'_\rho\end{array}}{\Gamma_1, \Gamma_\rho \vdash \Delta_1, \Delta_\rho}\, c^*}{\Gamma_1, \Pi_\rho \vdash \Delta_1, \Lambda_\rho}\, \rho
$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1, \Gamma_\rho, \Gamma'_\rho \vdash \Delta_1, \Delta_\rho, \Delta'_\rho\end{array}}{\Gamma_1, \Gamma_\rho, \Gamma_\rho \vdash \Delta_1, \Delta_\rho, \Delta_\rho}\, w^*}{\Gamma_1, \Gamma_\rho, \Pi_\rho \vdash \Delta_1, \Delta_\rho, \Lambda_\rho}\, \rho}{\Gamma_1, \Pi_\rho, \Pi_\rho \vdash \Delta_1, \Lambda_\rho, \Lambda_\rho}\, \rho}{\Gamma_1, \Pi_\rho \vdash \Delta_1, \Lambda_\rho}\, c^*
$$

$$
\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1, \Gamma_1^\rho, \Gamma_1\rho' \vdash \Delta_1, \Delta_1^\rho, \Delta_1^{\rho'}\end{array}}{\Gamma_1, \Gamma_\rho \vdash \Delta_1, \Delta_\rho}\, c^* \qquad \begin{array}{c}\varphi_2\\ \Gamma_2, \Gamma_2^\rho \vdash \Delta_2, \Delta_2^\rho\end{array}}{\Gamma_1, \Pi_\rho \vdash \Delta_1, \Lambda_\rho}\, \rho
$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1, \Gamma_1^\rho, \Gamma_1^{\rho'} \vdash \Delta_1, \Delta_1^\rho, \Delta_1^{\rho'}\end{array}}{\Gamma_1, \Gamma_1^\rho, \Gamma_1^\rho \vdash \Delta_1, \Delta_\rho, \Delta_\rho}\, w^* \qquad \begin{array}{c}\varphi_2\\ \Gamma_2, \Gamma_2^\rho \vdash \Delta_2, \Delta_2^\rho\end{array}}{\Gamma_1, \Gamma_\rho, \Pi_\rho \vdash \Delta_1, \Delta_\rho, \Lambda_\rho}\, \rho \qquad \begin{array}{c}\varphi_2\\ \Gamma_2, \Gamma_2^\rho \vdash \Delta_2, \Delta_2^\rho\end{array}}{\Gamma_1, \Pi_\rho, \Pi_\rho \vdash \Delta_1, \Lambda_\rho, \Lambda_\rho}\, \rho}{\Gamma_1, \Pi_\rho \vdash \Delta_1, \Lambda_\rho}\, c^*
$$

Definition 28 **($\gg_{WI}$).** *An independent inference can be permuted with a weakening on which it does not depend. This is done according to the rules below, which are just special cases of the rules in Definition 24:*

$$
\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1^\rho, \Gamma \vdash \Delta_1^\rho, \Delta\end{array}}{F, \Gamma_1^\rho, \Gamma \vdash \Delta_1^\rho, \Delta}\, w_l}{F, \Gamma^\rho, \Gamma \vdash \Delta^\rho, \Delta}\, \rho
$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1^\rho, \Gamma \vdash \Delta_1^\rho, \Delta\end{array}}{\Gamma^\rho, \Gamma \vdash \Delta^\rho, \Delta}\, \rho}{F, \Gamma^\rho, \Gamma \vdash \Delta^\rho, \Delta}\, w_l
$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1}}{F, \Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1}\ w_l \qquad \overset{\varphi_2}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2}}{F, \Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\ \rho$$

$$\Downarrow$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1} \qquad \overset{\varphi_2}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2}}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\ \rho}{F, \Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\ w_l$$

Definition 29 ($\gg_{WD}$)**.** *When an inference ρ is directly dependent on a weakening, permuting the weakening down according ot the rules below leads to the disappearance of ρ.*

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta}}{\Gamma_1^\rho, \Gamma \vdash \Delta_1^\rho, \Delta}\ w^*}{\Gamma^\rho, \Gamma \vdash \Delta^\rho, \Delta}\ \rho$$

$$\Downarrow$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta}}{\Gamma^\rho, \Gamma \vdash \Delta^\rho, \Delta}\ w^*$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1}}{\Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1}\ w^* \qquad \overset{\varphi_2}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2}}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\ \rho$$

$$\Downarrow$$

$$\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1}}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\ w^*$$

Definition 30 ($\gg_W$)**.** *The proof rewriting relation $\gg_W$ for* downward permutation of weakening *is the union of $\gg_{WI}$ and $\gg_{WD}$.*

Definition 31 ($\gg$)**.** *The proof rewriting relation $\gg$ is:*

$$\gg_I \cup \gg_{ID} \cup \gg_{DC} \cup \gg_C \cup \gg_W$$

C Correspondence between $\overset{S}{\rightsquigarrow}$ and $\gg$

Lemma 2. *If φ is skolemized and $\mathcal{S}_\varphi \overset{S}{\rightsquigarrow} S$, then there exists a proof ψ such that $\varphi \gg^* \psi$ and $\mathcal{S}_\psi = S$.*

Proof. The proof can be subdivided according to all possible cases of rewriting according to $\overset{S}{\rightsquigarrow}$. Here only three cases are shown, but every other case is either symmetric or analogous to one of these three cases.

Case of rewriting with no duplication, when the redex has the form $S\vee_{\rho_2}(S' \wedge_{\rho_1} S'')$ and is rewritten to $S' \wedge_{\rho_1} (S\vee_{\rho_2}S''))$: Since ρ_1 operates on cut ancestors and ρ_2 operates on end-sequent ancestors, ρ_2 is not directly dependent on ρ_1. Moreover, φ is skolemized and thus ρ_2 is also not eigen-variable dependent on ρ_1. Since S is distributed only to S'', only S'' contains formulas from $\Omega_{\rho_2}(\varphi)$. Therefore, $\Omega_{\rho_2}(\varphi)$ has formulas in at most one premise of ρ_1, and hence ancestors of auxiliary formulas of ρ_2 occur in at most one premise of ρ_1. Therefore ρ_2 is independent of ρ_1. Moreover, any inference ρ_i on the path between ρ_2 and ρ_1 and on which ρ_2 directly depends is also independent of ρ_1. Consequently, there exists a proof ψ with $\varphi \gg_I^* \psi$ where ρ_2 and all inferences ρ_i on which it depends have been swapped above ρ_1, so that $\mathcal{S}_\psi$ is equal to $\mathcal{S}_\varphi$ with $S\vee(S' \wedge S'')$ rewritten to $S' \wedge (S\vee S'')$.

Case of rewriting with duplication, when the redex has the form $(S' \wedge_{\rho_1} S'')\vee_{\rho_2}S$ and is rewritten to $(S'\vee_{\rho_2}S) \wedge_{\rho_1} (S''\vee_{\rho_2}S)$: Since ρ_1 operates on cut ancestors and ρ_2 operates on end-sequent ancestors, ρ_2 is not directly dependent on ρ_1. Moreover, φ is skolemized and thus ρ_2 is also not eigen-variable dependent on ρ_1. However, as both S' and S'' contain formulas from $\Omega_{\rho_2}(\varphi)$, it must be the case that $\rho_2 \prec_I \rho_1$. In the sequent calculus **LK**, this can only happen if there exists a sequence of unary[1] inferences $\rho_D^* \equiv (\rho_{D_1}, \ldots, \rho_{D_n})$ on the path between ρ_2 and ρ_1 such that ρ_{D_i} is indirectly dependent on ρ_1 and ρ_2 depends on ρ_{D_i}, for any i such that $1 \leq i \leq n$. Moreover, any inference ρ_{D_i} (on the path between ρ_2 and ρ_1) on which ρ_2 directly depends is also independent of ρ_1.

Let φ' be the subproof of φ having the conclusion sequent of ρ_2 as its end-sequent. Then, there exists a proof φ'' with $\varphi' \gg^* \varphi''$ (permuting ρ_D^* and ρ_2 above all inferences on which they do not depend) such that φ'' has the following form (or a form symmetric to it):

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma_1, \Gamma_1^{\rho_2}, \Gamma_1^{\rho_1} \vdash \Delta_1, \Delta_1^{\rho_2}, \Delta_1^{\rho_1}} \qquad \dfrac{\varphi_2}{\Gamma_2, \Gamma_2^{\rho_2}, \Gamma_2^{\rho_1} \vdash \Delta_2, \Delta_2^{\rho_2}, \Delta_2^{\rho_1}}}{\Gamma_1, \Gamma_2, \Gamma_1^{\rho_2}, \Gamma_2^{\rho_2}, \Gamma^{\rho_1} \vdash \Delta_1, \Delta_2, \Delta_1^{\rho_2}, \Delta_2^{\rho_2}, \Delta^{\rho_1}}\ \rho_1}{\Gamma_1, \Gamma_2, \Gamma^{\rho_1} \vdash \Delta_1, \Delta_2, F_{12}^{\rho_2}, \Delta^{\rho_1}}\ \rho_D^* \qquad \dfrac{\varphi_3}{\Gamma_3 \vdash \Delta_3, F_3^{\rho_2}}}{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma^{\rho_1} \vdash \Delta_1, \Delta_2, \Delta_3, F^{\rho_2}, \Delta^{\rho_1}}\ \rho_2$$

[1] If the inferences were not unary, the redex would not be of the form $(S' \wedge_{\rho_1} S'')\vee_{\rho_2}S$.

Then there exists a proof φ''' with $\varphi'' \gg^* \varphi'''$ (permuting ρ_D^* above ρ_1) such that φ''' has the following form:

$$
\begin{prooftree}
\AxiomC{φ_1}
\noLine
\UnaryInfC{$\Gamma_1, \Gamma_1^{\rho 2}, \Gamma_1^{\rho 1} \vdash \Delta_1, \Delta_1^{\rho 2}, \Delta_1^{\rho 1}$}
\doubleLine
\RightLabel{w^*}
\UnaryInfC{$\Gamma_1, \Gamma_1^{\rho 2}, \Gamma_2^{\rho 2}, \Gamma_1^{\rho 1} \vdash \Delta_1, \Delta_1^{\rho 2}, \Delta_2^{\rho 2}, \Delta_1^{\rho 1}$}
\doubleLine
\RightLabel{ρ_D^*}
\UnaryInfC{$\Gamma_1, \Gamma_1^{\rho 1} \vdash \Delta_1, F_{12}^{\rho 2}, \Delta_1^{\rho 1}$}
\AxiomC{φ_2}
\noLine
\UnaryInfC{$\Gamma_2, \Gamma_2^{\rho 2}, \Gamma_2^{\rho 1} \vdash \Delta_2, \Delta_2^{\rho 2}, \Delta_2^{\rho 1}$}
\doubleLine
\RightLabel{w^*}
\UnaryInfC{$\Gamma_2, \Gamma_1^{\rho 2}, \Gamma_2^{\rho 2}, \Gamma_2^{\rho 1} \vdash \Delta_2, \Delta_1^{\rho 2}, \Delta_2^{\rho 2}, \Delta_2^{\rho 1}$}
\doubleLine
\RightLabel{ρ_D^*}
\UnaryInfC{$\Gamma_2, \Gamma_2^{\rho 1} \vdash \Delta_2, F_{12}^{\rho 2}, \Delta_2^{\rho 1}$}
\RightLabel{ρ_1}
\BinaryInfC{$\Gamma_1, \Gamma_2, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, F_{12}^{\rho 2}, F_{12}^{\rho 2}, \Delta^{\rho 1}$}
\RightLabel{c_r}
\UnaryInfC{$\Gamma_1, \Gamma_2, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, F_{12}^{\rho 2}, \Delta^{\rho 1}$}
\AxiomC{φ_3}
\noLine
\UnaryInfC{$\Gamma_3 \vdash \Delta_3, F_3^{\rho 2}$}
\RightLabel{ρ_2}
\BinaryInfC{$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, \Delta_3, F^{\rho 2}, \Delta^{\rho 1}$}
\end{prooftree}
$$

Then there exists a proof $\varphi^{(4)}$ with $\varphi''' \gg_C \varphi^{(4)}$ (permuting ρ_2 above the contraction) such that $\varphi^{(4)}$ has the following form:

$$
\begin{prooftree}
\AxiomC{φ_1}
\noLine
\UnaryInfC{$\Gamma_1, \Gamma_1^{\rho 2}, \Gamma_1^{\rho 1} \vdash \Delta_1, \Delta_1^{\rho 2}, \Delta_1^{\rho 1}$}
\doubleLine
\RightLabel{w^*}
\UnaryInfC{$\Gamma_1, \Gamma_1^{\rho 2}, \Gamma_2^{\rho 2}, \Gamma_1^{\rho 1} \vdash \Delta_1, \Delta_1^{\rho 2}, \Delta_2^{\rho 2}, \Delta_1^{\rho 1}$}
\doubleLine
\RightLabel{ρ_D^*}
\UnaryInfC{$\Gamma_1, \Gamma_1^{\rho 1} \vdash \Delta_1, F_{12}^{\rho 2}, \Delta_1^{\rho 1}$}
\AxiomC{φ_2}
\noLine
\UnaryInfC{$\Gamma_2, \Gamma_2^{\rho 2}, \Gamma_2^{\rho 1} \vdash \Delta_2, \Delta_2^{\rho 2}, \Delta_2^{\rho 1}$}
\doubleLine
\RightLabel{w^*}
\UnaryInfC{$\Gamma_2, \Gamma_1^{\rho 2}, \Gamma_2^{\rho 2}, \Gamma_2^{\rho 1} \vdash \Delta_2, \Delta_1^{\rho 2}, \Delta_2^{\rho 2}, \Delta_2^{\rho 1}$}
\doubleLine
\RightLabel{ρ_D^*}
\UnaryInfC{$\Gamma_2, \Gamma_2^{\rho 1} \vdash \Delta_2, F_{12}^{\rho 2}, \Delta_2^{\rho 1}$}
\RightLabel{ρ_1}
\BinaryInfC{$\Gamma_1, \Gamma_2, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, F_{12}^{\rho 2}, F_{12}^{\rho 2}, \Delta^{\rho 1}$}
\AxiomC{φ_3}
\noLine
\UnaryInfC{$\Gamma_3 \vdash \Delta_3, F_3^{\rho 2}$}
\RightLabel{ρ_2}
\BinaryInfC{$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, \Delta_3, F_{12}^{\rho 2}, F^{\rho 2}, \Delta^{\rho 1}$}
\AxiomC{φ_3}
\noLine
\UnaryInfC{$\Gamma_3 \vdash \Delta_3, F_3^{\rho 2}$}
\RightLabel{ρ_2}
\BinaryInfC{$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_3, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_3, F^{\rho 2}, F^{\rho 2}, \Delta^{\rho 1}$}
\RightLabel{c_r}
\UnaryInfC{$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_3, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_3, F^{\rho 2}, \Delta^{\rho 1}$}
\doubleLine
\RightLabel{c^*}
\UnaryInfC{$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, \Delta_3, F^{\rho 2}, \Delta^{\rho 1}$}
\end{prooftree}
$$

Finally, there exists a proof ψ' with $\varphi^{(4)} \gg_I^* \psi'$ (permuting each copy of ρ_2 above ρ_1) such that ψ' has the following form:

$$
\begin{prooftree}
\AxiomC{φ_1}
\noLine
\UnaryInfC{$\Gamma_1, \Gamma_1^{\rho 2}, \Gamma_1^{\rho 1} \vdash \Delta_1, \Delta_1^{\rho 2}, \Delta_1^{\rho 1}$}
\doubleLine
\RightLabel{w^*}
\UnaryInfC{$\Gamma_1, \Gamma_1^{\rho 2}, \Gamma_2^{\rho 2}, \Gamma_1^{\rho 1} \vdash \Delta_1, \Delta_1^{\rho 2}, \Delta_2^{\rho 2}, \Delta_1^{\rho 1}$}
\doubleLine
\RightLabel{ρ_D^*}
\UnaryInfC{$\Gamma_1, \Gamma_1^{\rho 1} \vdash \Delta_1, F_{12}^{\rho 2}, \Delta_1^{\rho 1}$}
\AxiomC{φ_3}
\noLine
\UnaryInfC{$\Gamma_3 \vdash \Delta_3, F_3^{\rho 2}$}
\RightLabel{ρ_2}
\BinaryInfC{$\Gamma_1, \Gamma_3, \Gamma_1^{\rho 1} \vdash \Delta_1, \Delta_3, F^{\rho 2}, \Delta_1^{\rho 1}$}
\AxiomC{φ_2}
\noLine
\UnaryInfC{$\Gamma_2, \Gamma_2^{\rho 2}, \Gamma_2^{\rho 1} \vdash \Delta_2, \Delta_2^{\rho 2}, \Delta_2^{\rho 1}$}
\doubleLine
\RightLabel{w^*}
\UnaryInfC{$\Gamma_2, \Gamma_1^{\rho 2}, \Gamma_2^{\rho 2}, \Gamma_2^{\rho 1} \vdash \Delta_2, \Delta_1^{\rho 2}, \Delta_2^{\rho 2}, \Delta_2^{\rho 1}$}
\doubleLine
\RightLabel{ρ_D^*}
\UnaryInfC{$\Gamma_2, \Gamma_2^{\rho 1} \vdash \Delta_2, F_{12}^{\rho 2}, \Delta_2^{\rho 1}$}
\AxiomC{φ_3}
\noLine
\UnaryInfC{$\Gamma_3 \vdash \Delta_3, F_3^{\rho 2}$}
\RightLabel{ρ_2}
\BinaryInfC{$\Gamma_2, \Gamma_3, \Gamma_2^{\rho 1} \vdash \Delta_2, \Delta_3, F^{\rho 2}, \Delta_2^{\rho 1}$}
\RightLabel{ρ_1}
\BinaryInfC{$\Gamma_1, \Gamma_2, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, F_{12}^{\rho 2}, F_{12}^{\rho 2}, \Delta^{\rho 1}$}
\RightLabel{c_r}
\UnaryInfC{$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_3, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, \Delta_3, \Delta_3, F^{\rho 2}, \Delta^{\rho 1}$}
\doubleLine
\RightLabel{c^*}
\UnaryInfC{$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma^{\rho 1} \vdash \Delta_1, \Delta_2, \Delta_3, F^{\rho 2}, \Delta^{\rho 1}$}
\end{prooftree}
$$

Consequently, there exists a proof ψ with $\varphi \gg^* \psi$ (namely, the proof obtained from φ by rewriting its subproof φ' to ψ' as shown above) where ρ_2 and all unary inferences ρ_{D_i} on which it depends have been permuted above ρ_1, so that $\mathcal{S}_\psi$ is equal to $\mathcal{S}_\varphi$ with $(S' \wedge S'')\vee S$ rewritten to $(S' \vee S) \wedge (S'' \vee S)$.

Degenerate case, when the redex has the form $S \vee_\rho (S_1 \wedge \ldots \wedge S_n)$ and is rewritten to $S_1 \wedge \ldots \wedge S_n$: Let φ' be the subproof ending with ρ, φ'_1 be its left subproof (corresponding to S) and φ'_2 be its right subproof (corresponding to $(S_1 \wedge \ldots \wedge S_n)$). Since φ'_2 contains no formula from $\Omega_\rho(\varphi)$, it must be the case that all auxiliary formulas of ρ occurring in its right premise are descendants of main formulas of weakening inferences. ρ is a (partially) degenerate inference. Moreover, since the innermost rewriting strategy guarantees that $S \vee (S_1 \wedge \ldots \wedge S_n)$ is a minimal redex, the auxiliary formulas in the right premise of ρ are not ancestors of any binary inference operating (for if they were, there would be a redex in $(S_1 \wedge \ldots \wedge S_n)$). Therefore, in the sequence rewriting φ'_2 into its $\gg_W$-normal-form φ''_2, none of the rewriting rules of $\gg_{WD}$ that delete binary inferences is used. Consequently, the characteristic formula remains unchanged when φ'_2 is rewritten into φ''_2. Let φ'' be the result of replacing φ'_2 by φ''_2 in φ'.φ'' is of the following form:

$$
\cfrac{\begin{array}{c}\varphi'_1 \\ \Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1\end{array} \qquad \begin{array}{c}\varphi'''_2 \\ \Gamma_2 \vdash \Delta_2 \\ \hline\hline \Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2\end{array}\ w^*}{\Gamma^\rho, \Gamma_2, \Gamma_1 \vdash \Delta^\rho, \Delta_2, \Delta_1}\ \rho
$$

with φ''_2 being:

$$
\begin{array}{c}\varphi'''_2 \\ \Gamma_2 \vdash \Delta_2 \\ \hline\hline \Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2\end{array}\ w^*
$$

By using one of the rewriting rules of $\gg_{WD}$, φ'' can be rewritten to ψ' below:

$$
\begin{array}{c}\varphi'''_2 \\ \Gamma_2 \vdash \Delta_2 \\ \hline\hline \Gamma^\rho, \Gamma_2, \Gamma_1 \vdash \Delta^\rho, \Delta_2, \Delta_1\end{array}\ w^*
$$

Therefore, there exists a proof ψ (namely, the proof obtainable from φ by replacing its subproof φ' by ψ') such that $\varphi \gg_W \psi$ and $\mathcal{S}_\psi$ is $\mathcal{S}_\varphi$ with $S \vee (S_1 \wedge \ldots \wedge S_n)$ rewritten to $S_1 \wedge \ldots \wedge S_n$. □

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