UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

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Assignment

Advisors: Jonatan Siegrist & Johan Emil Westergaard

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Chapter 1

Black-Scholes

The outset

Consider the price evolution of the S&P 500 and Russell 2000 indices over the last 20 years.

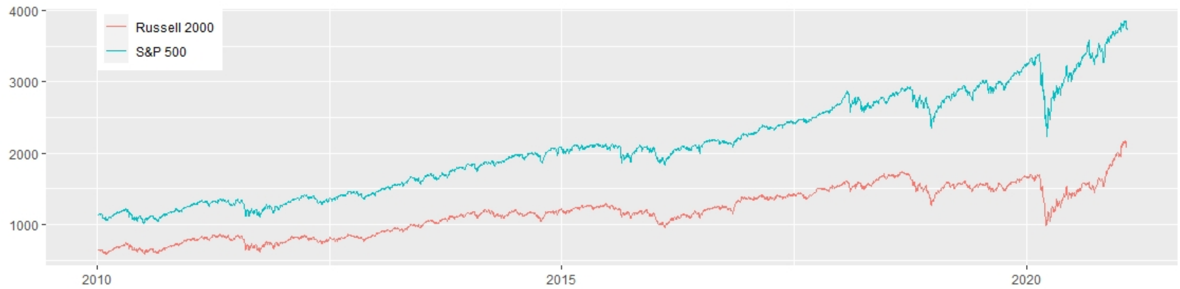

Figure 1.1: The S&P 500 and the Russell 2000 indices from the 1st of January 2010 to the 1st of February 2021

Evidently, the stock indices have increased steadily albeit with some term random fluctuations. In an effort to describe the behaviour of a stock[1] in a more formal setting, we propose the following model[2]:

Consider a financial market in which the uncertainty is governed by the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ equipped with the Wiener process $(W_t)_{t\geq 0}$. Here, Ω represents the state space, $\mathcal{F}$ is the σ-algebra representing measurable events, $(\mathcal{F}_t)_{t\geq 0}$ is the filtration, and $\mathbb{P}$ the probability measure. We will think of $\mathbb{P}$ as the real-world or physical measure. The only stochasticity derives from the Wiener process. Consequently, the filtration is considered to be

[1] Or a stock index

[2] Also known as the Black-Scholes model

the natural filtration of the Wiener process, namely

$$\mathcal{F}_t = \mathcal{F}_t^W \quad t \geq 0 \tag{1.1}$$

where

$$\mathcal{F}_t^W = \sigma(W_s \mid 0 \leq s \leq t) \tag{1.2}$$

Concentrating on the stock once more, the idea is to model the stock process under the physical measure $\mathbb{P}$ as a stochastic process $(S_t)_{t\geq 0}$ with the dynamics

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ S_0 &= s \end{aligned} \tag{1.3}$$

where $\mu \in \mathbb{R}$ and $s, \sigma \in \mathbb{R}^+$ are constant. A stochastic process with this dynamics is called a geometric Brownian motion. One can show, that the solution to the stochastic differential equation is given by

$$S(t) = S_0 \cdot \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right) \tag{1.4}$$

In addition to the stock, we also define a risk-free asset B which is governed by the dynamics

$$\begin{aligned} dB_t &= rB_t dt \\ B_0 &= 1 \end{aligned} \tag{1.5}$$

Contrary to what one might think, financial assets are not priced under the real-world measure, $\mathbb{P}$, but under the so-called risk neutral measure, $\mathbb{Q}$: Assuming that the market is arbitrage-free and complete there exists a unique equivalent martingale measure denoted by $\mathbb{Q}$. Under $\mathbb{Q}$, the stock has the dynamics

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^Q \\ S_0 &= s \end{aligned} \tag{1.6}$$

Here, W^Q is a Wiener process under $\mathbb{Q}$. Thus for a financial asset which pays $g(S_T)$ at some fixed maturity T one arrives at the time t arbtrage-free price, $\Pi(t)$, given by

$$\Pi(t) = E^Q\left[e^{-\int_t^T r du} g(S(T)) \mid \mathcal{F}_t\right] = e^{-r(T-t)} E^Q[g(S(T)) \mid S(t) = s] \tag{1.7}$$

the above is often referred to as $risk - neutral$ pricing.

1.1 Model assumptions

The aim of this section is to provide some intuition about the model in the above and its underlying assumptions. Moreover, a brief discussion on the use of model assumptions in mathematical modelling might also appear.

a) Examine eq. 1.1, 1.2, and 1.3. How are they to be understood intuitively?

b) Contemplate the model assumptions presented in the text:

i) What are the model assumptions under both $\mathbb{P}$ and $\mathbb{Q}$ and what do they imply about the behaviour of the stock as well as the market participants?

ii) Do the model assumptions seem realistic and do they in any case need to be?

c) A prevailing paradigm in finance is risk-neutral pricing. Explain in your own words why this is a clever concept[3]. If possible, show analytically why risk-neutral pricing yields the "true" price of a financial derivative.

1.2 Geometric Brownian motion

In this section, we will investigate the implications of the assumption that the stock price process follows a geometric Brownian motion.

a) Derive eq.1.4 and solve the differential equation 1.5. Hint: Consider the process $X_t = \log(S_t)$ and apply Itô's formula

b) Derive the stock price distribution under $\mathbb{P}$.

c) Assume equidistant time points. Derive the distribution of $\frac{\log(S_n)}{\log(S_{n-1})}$ under $\mathbb{P}$.

d) Download price data on S&P 500. This can be done with the package `quantmod` in R.

e) Use statistical analysis to investigate the model assumptions regarding the stock. Do you find that data supports the model assumptions?

[3]For inspiration, see Björk section 7.4 and 15.6

f) Derive closed-form maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ. Hint: Recall the distributional result from c).

g) Compute $\hat{\mu}$ and $\hat{\sigma}$ based on the observed price data. Do this for i) All data, ii) Data from 2020 only. What are the differences? Can you explain these differences?

h) Derive closed-form expressions for the variance of $\hat{\mu}$ and $\hat{\sigma}$. What are the properties of the variance of $\hat{\mu}$ and how do they affect the pricing of options in Black-Scholes? Hint: Use the Delta method to derive the covariance matrix for $(\hat{\mu}, \hat{\sigma})$.

1.3 Monte Carlo - TBA