Finkont0

- Introduktion -

Københavns Universitet

Last updated: 9. februar 2021

Praktisk information

- Kontaktoplysninger
  - Johan: johanemilw@gmail.com
  - Jonatan: jonatansiegrist@hotmail.com
- Vejledning
  - Tirsdag kl. 16.30 hver/hver anden uge
  - Relevante dokumenter findes på: `https://github.com/JohanEmilW/FinKont0_2021`
- Forventninger:
  - I forbereder jer før møderne
  - Send gerne spørgsmål løbende på mail, så samler vi op til møderne
  - Hvis I sender kode, skal det være selvindeholdt!
  - Hvis I føler, tempoet er for højt/lavt, så sig til

Gode råd

- Tag back-ups
- Brug af versionsstyring(GitHub, GitLab, etc.)
- Tal med hinanden
- Skriv modulær kode

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
A random variable X is a mapping

$$X : \Omega \to \mathbb{R} \tag{1}$$

such that X is $\mathscr{F}$-measurable.
A stochastic process, $(X_t)_{t \geq 0}$ is mapping

$$X : \mathbb{R}^+ \times \Omega \to \mathbb{R} \tag{2}$$

such that for each $t \in \mathbb{R}^+$ the mapping

$$X(t, \cdot) : \Omega \to \mathbb{R} \tag{3}$$

is $\mathscr{F}$-measurable
For fixed t, it is a random variable: $\omega \to X(t, \omega)$
For fixed ω, it is a function of time: $t \to X(t, \omega)$
We call this function a sample path/realization/trajectory

Wiener process - Definition 4.1

The stochastic process $(W_t)_{t \geq 0}$ is a Wiener process if

1. $W_0 = 0$ a.s
2. The process W has independent increments, i.e. if $r < s \leq t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent stochastic variables
3. For $s < t$ the stochastic variable $W_t - W_s$ is Gausssian with mean 0 and variance $t - s$
4. W has continuous trajectories

Useful properties

- $W_t - W_s \perp\!\!\!\perp W_s$ for $t > s$
- $W_{t-s} = W_{t-s} - W_0 \sim \mathcal{N}(0, t-s) \Rightarrow W_{t-s} \overset{D}{=} W_t - W_s$

Let X be a stochastic process. Suppose that there exists two adapted processes μ and σ such that the following holds for all $t \geq 0$:

$$X(t) = a + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s) \tag{4}$$

where $a \in \mathbb{R}$.
X can then be written as

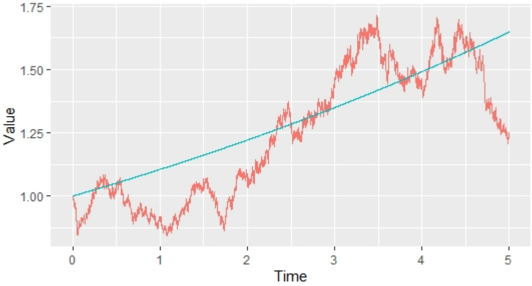
$$dX(t) = \mu(t)dt + \sigma(t)dW(t) \tag{5}$$

$$X(0) = a \tag{6}$$

Here, (5) is the stochastic differential equation, whereas (6) is the initial condition.

In continuous time finance, models are typically stated in terms of stochastic differential equations. When you become accustomed to it, infinitesimal dynamics tend to provide some intuition:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ S_0 &= 1 \end{aligned} \tag{7}$$



Theorem 4.10

Assume that the process X has a stochastic differential given by

$$dX(t) = \mu(t)dt + \sigma(t)dW(t), \tag{8}$$

where μ and σ are adapted processes, and let f be a $C^{1,2}$-function. Define the process Z by $Z(t) = f(t, X(t))$. Then Z has a stochastic differential given by

$$df(t, X(t)) = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma \frac{\partial f}{\partial x} dW(t) \tag{9}$$

Proposition 4.11

With assumptions as in 4.10, df is given by

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2 \tag{10}$$

where we use the following formal multiplication table

$$(dt)^2 = 0, \quad dt \cdot dW = 0, \quad (dW)^2 = dt$$

Itôs formel - eksempel

Let $(S_t)_{t\geq 0}$ be governed by (7) and consider $Z_t = f(t, S_t) = \exp(S_t)$. We find that

$$\frac{\partial t}{\partial t} = 0 \ ; \ \frac{\partial f}{\partial S} = \frac{\partial^2 f}{\partial S^2} = \exp(S_t) \tag{11}$$

Itô's formula now yields the dynamics of Z

$$\begin{aligned} dZ_t &= \exp(S_t)dS_t + \frac{1}{2}\exp(S_t)(dS_t)^2 \\ &= \exp(S_t)(dS_t + \underbrace{\frac{1}{2}\mu S_t(dt)^2 + \mu\sigma S_t^2(dt\cdot dW)}_{=0} + \frac{1}{2}\sigma^2 S_t^2(dW_t)^2) \\ &= \exp(S_t)\left((\mu + \frac{1}{2}\sigma^2 S_t)S_t dt + \sigma S_t dW_t\right) \end{aligned} \tag{12}$$