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Fundamental Views

This time we take a look at some consequences and caveats of a central result in quantitative finance.

Assume the interest rate is 0 and consider a dividend-free asset with risk-neutral price dynamics

$$dS_t = \sigma_t S_t dW_t,$$

where $\{\sigma_t\}$ is some random process. Suppose we buy an option with expiry T at implied volatility σ_H; by this we mean that the option's market (or over-the-counter) price is such that if we valued it in a Black–Scholes model with volatility σ_H, then model and market prices would match. We now Black–Scholes Δ-hedge with hedge volatility σ_H; for instance, for a call this means that at any time $t \leq T$ we hold $N\left(\frac{\ln(S_t/K)+\frac{1}{2}\sigma_H^2(T-t)}{\sigma_H\sqrt{T-t}}\right)$. The Fundamental Theorem of Derivative Trading[1] states that the profit-and-loss (PnL) at expiry T of this trading operation is

$$PnL(T) = \frac{1}{2}\int_0^T \Gamma_t S_t^2(\sigma_t^2 - \sigma_H^2)dt, \tag{1}$$

where Γ_t is the Black–Scholes Gamma of the option. Proofs in the literature are essentially the same, but styles vary considerably. In its simplest form, the proof is an unproblematic exam question.[2] I think my favorite way is by Mark Davis,[3] whereas Ellersgaard, Jönsson, and Poulsen (2017) have all the bells and whistles.

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Different views

Equation (1) has been viewed in the literature both as a negative and as a positive result. In El Karoui, Jeanblanc-Picque, and Shreve (1998), which to my knowledge is the first published, peer-review paper with the result clearly (if not prominently) stated, it is viewed primarily as negative: unless we can truly bound volatility, we cannot bound our PnL. Subsequently, the result came to be seen as a positive one; it's an explict representation and in practice the right-hand side is small if we/the market can predict volatility well. Also, the right-hand side has no dW-integral, only a dt-integral. This has led to the interpretation that "model risk causes bleeding not blow-up." However, for the representation in equation (1) to be valid, it is important that we look at the PnL at expiry T. At times prior to expiry, there would be a mark-to-market term that we cannot control without making further assumptions. Ahmad and Wilmott (2001) show that *if* implied volatility is constant *and* we use that as hedge volatility, then PnL-paths are smooth; otherwise they will not be, even if we hedge with implied volatility. Although in my opinion it is fair to add the disclaimer "strictly speaking" or "in theory" to the last part of the previous sentence; an opinion built on, among other things, the empirical analysis in Ellersgaard, Jönsson, and Poulsen (2017). But there is a third way to view equation (1), namely as an explanatory model of the profit-and-loss on trading operations. The reason for this is that for any trading operation, the left-hand side and the right-hand side of equation (1) are, if not directly observable, then at least objectively measurable. This is analogous to one of the most ubiquitious results in finance: the Capital Asset Pricing Model (CAPM). Viewed mathematically, the CAPM equation is nothing but a rewriting of the first-order conditions for mean–variance optimality of a given portfolio. However, easy applicability and testable implications of the CAPM equation means that it takes on a life of its own. Rather than being a consequence, it becomes the model in itself. Is that good or bad? Well, it's interesting.

Volatility exposure

Suppose $\frac{1}{T}\int_0^T \sigma_t^2 dt = \sigma_H^2$ (i.e. hedge volatility = implied volatility = realized volatility, all the pieces fit) But I still lose money on the hedge. How is that possible? To see why, note the Γ_t factor in equation (1). It creates a path-dependence; an error when Γ_t is high (think short time-to-expiry, option at-the-money) can be very costly. So a Δ-hedged position on a call does not give us a straight bet on realized volatility. But equation (1) also tells us how to remedy that; Δ-hedge a contract for which $\Gamma = 1/S^2$. That contract is (up to an affine term that plays no role here) the log-contract, for which $\Delta = 1/S$. And since the log-contract can be replicated statically by puts and calls (dK/K^2 units of each out-of-the-money option by the spanning formula from Carr and Madan (2001)) we get model-independent valuation of variance swaps and the story behind the volatility index VIX.

Local volatility

Suppose we look at a call. When we hedge with implied volatility, notationally explicitly writing $\sigma_{imp}^2(K,T)$, the initial investment is 0, and thus $E^Q(PnL(T)) = 0$. Now assume we have a local volatility model, $\sigma_t = \sigma_{loc}(S_t, t)$, in which we use $(y,u) \mapsto \phi(y,u)$to denote the density of S_u. Taking expectation in equation (1) and rearranging gives us what is called Dupire's zero-sigma formula:

$$\sigma_{imp}^2(T,K) = \frac{\int_0^T \int_0^\infty \phi(S,t)\Gamma(S,t)S^2\sigma_{loc}^2(S,t)dSdt}{\int_0^T \int_0^\infty \phi(S,t)\Gamma(S,t)S^2 dSdt}.$$

This means that we can view (squared) implied volatility as a weighted average (in time and space) of (squared) local volatility, the weights being $\phi(S,t)\Gamma(S,t)$ S^2dSdt up to normalization. In general, $\phi(\cdot,0)$ is concentrated at S_0, and for the call $\Gamma(\cdot,T)$ is concentrated at K, so for short times-to-expiry in a time-homogenous model, we have the approximation

$$\sigma_{imp}(K,T) \approx \sqrt{\frac{\sigma_{loc}^2(S_0) + \sigma_{loc}^2(K)}{2}}.$$

This tells us that (i) short-expiry at-the-money implied volatility is approximately equal to local volatility (fairly obvious one might think), (ii) the at-the-money skew in implied volatility is approximately half the slope local volatility (less obvious).

Stochastic volatility

A casual reading of equation (1) leads one to say: if I just Black–Scholes Δ-hedge with σ_t every day, my PnL will be identically 0. Suppose now that such hedging is possible; either the person in question is very good at estimating volatility or God calls him every day to tell him what volatility is. This person will still find that his PnL is not identically 0. The point that he has missed is that the argument for equation (1) critically uses the constant volatility assumption; there needs to be a partial differential equation linking hedge volatility and Greeks. In the case where $\{\sigma_t\}$ is a diffusion process (a stochastic volatility model) extensions of equation (1) can be formulated, see for instance (Savine, 2017, pp. 42–57); these involve Volga (the second sigma-derivative) and Vanna (the sigma-spot-derivative).

Jumps

The final word of warning is about jumps, of which there are assumed to be none when deriving equation (1). The result can be generalized to include jumps; see Nielsen, Jönsson, and Poulsen (2017) for compound Poisson type and Davis (2010) for general Levy type. The resulting formulas look quite complicated, but the important message is not: if you are Δ-hedging a short position in an option with a positive Gamma, then jumps will hurt you – irrespective of whether they are up or down; you are picking up pennies in front of a steam train.

About the Author

Rolf Poulsen is in the Department of Mathematical Sciences at the University of Copenhagen. His main research interest is quantitative methods for pricing and hedging of derivatives. He will talk about exchange rate markets at length to all who will listen – and some who won't.

ENDNOTES

1. This is a term that I have been trying to push ever since I stole it from Jesper Andreasen, http://www.math.ku.dk/ rolf/jandreasen.pdf.
2. See http://web.math.ku.dk/ rolf/teaching/mfe04/mfe04.html.
3. See https://tinyurl.com/yammlqqk.

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