Finkont0

- Hedging with Greeks -

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Greeks

Financial quants are in general not especially interested in the valuation of transactions. They are interested in risk management i.e. hedging of transactions. As a result, market sensitivities are a point of interest. Sensitivities are no more than the partial derivatives of the pricing function. Theses derivatives are called the Greeks.

$$\Delta = \frac{\partial P}{\partial s}, \tag{1}$$

$$\Gamma = \frac{\partial^2 P}{\partial s^2}, \tag{2}$$

$$\rho = \frac{\partial P}{\partial r}, \tag{3}$$

$$\Theta = \frac{\partial P}{\partial t}, \tag{4}$$

$$\mathscr{V} = \frac{\partial P}{\partial \sigma} \tag{5}$$

Can you hedge the gamma risk of the portfolio by buying/selling the underlying stock?

Δ

Sketch of how to derive Δ:
Remember the pricing function of a European call.

$$C(t,s) = s\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2) \tag{6}$$

Now we have that

$$\Delta = \frac{\partial C}{\partial s} \tag{7}$$

$$= \Phi(d_1) + s\phi(d_1)\frac{\partial}{\partial s}(d_1(t,s)) - e^{-r(T-t)}K\phi(d_2)\frac{\partial}{\partial s}(d_2(t,s)) \tag{8}$$

Note that

$$\frac{\partial}{\partial s}(d_1(t,s)) = \frac{\partial}{\partial s}(d_2(t,s)) = ? \tag{9}$$

and that

$$\phi(d_2) = \phi(d_1 - \sigma\sqrt{T-t}) \tag{10}$$

$$= \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}(d_1 - \sigma\sqrt{T-t})^2} \tag{11}$$

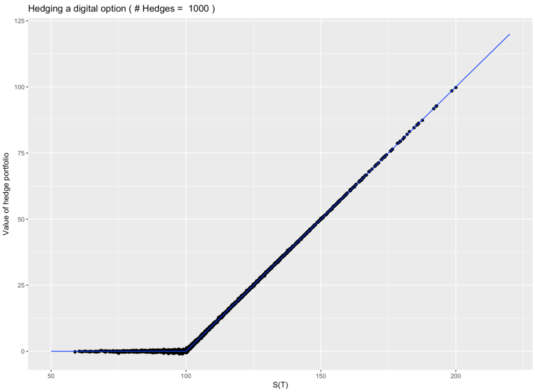
$$= \ldots = \phi(d_1)\cdot ? \tag{12}$$

Discrete hedge experiment

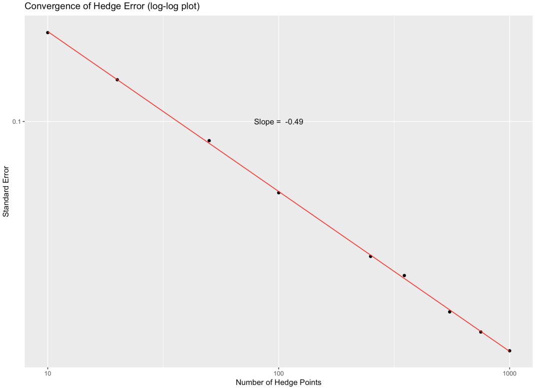
1. Choose number of hedge point from time 0 until expiry T. Denote the hedge points t_i
2. Simulate stock paths at the time points t_i
3. Suppose you sell a call / someone gives you the dollar amount equal the price of a European option. Use this money to buy $\frac{\partial C}{\partial S}(S(0),0)$ units of the stock potentially financed with a loan in the bank
4. At time t_i you rebalance your portfolio such that you hold $\frac{\partial C}{\partial S}(S(t_i),t_i)$ units in the stock. If you at any time need/have extra funds you loan/deposit these. Remember that the loan/deposit accrues interest.
5. Do this until time T where you liquidate the portfolio. At all time keep track of the value of your portfolio. Compare the value of the portfolio with the true Black-Scholes call price. Call the difference the hedge error.

Note that this procedure (in the continuously rebalanced case) will replicate the value of the derivative (Proposition 9.7).

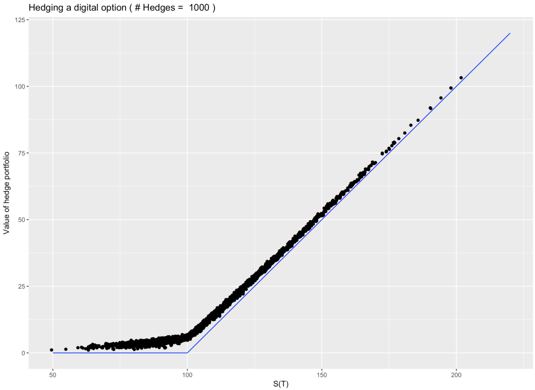
The hedge should work (if you hedge often enough) regardless of the evolution of the stock... and it does:



The convergence of the standard deviation of the hedge error seems to be of order 0.5



If we hedge with the wrong volatility (i.e. imagine that you as a risk manager have misspecified/miscalibrated the model volatility), then the hedge error doesn't vanish in the limit.



Amazingly, we can be very precise about how wrong we are when we hedge with a "wrong"volatility. This is called The Fundamental Theorem of Derivatives Trading.

The Fundamental Theorem of Derivatives Trading(FTODT)

Context: A trader buys a European call option at time 0 at the applied volatility σ_0^i. To mitigate the risk, the trader Δ-hedges her position with hedging volatility, σ^H, using the underlying stock and the money account. This is sometimes known as gamma trading.

The *profit-and-loss*(PnL) of the hedged position can be described by what has been dubbed *the fundamental theorem of derivatives trading*.

Assuming the Black-Scholes model, the **present value** of the PnL of the strategy during time $[0,T]$ is given by:

$$PnL_T = C\left(0, S_0; \sigma_0^h\right) - C\left(0, S_0; \sigma_0^i\right) + \int_0^T e^{-rt} \frac{1}{2} \left(\sigma_t^2 - \left(\sigma_t^h\right)^2\right) S_t^2 \Gamma\left(t, S_t; \sigma_t^h\right) dt \tag{13}$$

Before we start the proof: Useful concepts

Björk Thm. 7.7(Black-Scholes Equation)
Assume the Black-Scholes model and suppose that we want to price a contingent claim, $\mathscr{X} = g(S_T)$, with pricing function $\Pi(t) = F(t, S_t)$. Then the only pricing function, F, consistent with the absence of arbitrage is the solution to the following boundary value problem in $[0, T] \times \mathbb{R}_+$:

$$F_t(t,s) + rsF_s(t,s) + \frac{1}{2}s^2\sigma^2(t,s)F_{ss}(t,s) - rF(t,s) = 0 \tag{14}$$

$$F(T,s) = g(s) \tag{15}$$

(14) is a PDE, NOT an SDE!

Self-financing portfolio - Björk ch. 6,8, and 9

A self-financing portfolio, is a portfolio with no exogenous infusion or withdrawal of money. Thus, the buying an asset must be financed by selling another.

Björk definition 6.2

A portfolio with value process

$$V^h(t) = h_0(t)B(t) + \sum_{i=1}^{N} h_i(t)S_i(t) \tag{16}$$

is self-financing if the value process is governed by the dynamics

$$dV^h(t) = h_0(t)dB(t) + \sum_{i=1}^{N} h_i(t)dS_i(t) \tag{17}$$

Self-financing portfolio - Björk ch. 6,8, and 9

A self-financing portfolio, is a portfolio with no exogenous infusion or withdrawal of money. Thus, the buying an asset must be financed by selling another.

Björk definition 8.1 A claim, $X = g(S_t)$, can be replicated/hedged if there exist a self-replicating portfolio, h, such that

$$V^h(T) = g(S_T) \quad \mathbb{P} - a.s \tag{18}$$

h is called a hedge against X, or the replicating/hedging portfolio.

Björk Thm. 8.5

Consider the claim $X = g(S_t)$ and a pricing function, F, satisfying the boundary value problem. The corresponding hedging portfolio is

$$h^{Bank}(t) = \frac{F(t, S(t) - S(t)F_s(t, S(t)))}{B(t)} \tag{19}$$

$$h^{Stock}(t) = F_s(t, S(t)) \tag{20}$$

with value process $V^h(t) = F(t, S(t))$

General idea: Buy the option and sell the hedging portfolio. The combined portfolio, $V^h(t)$, is self-financing(why?) so the dynamics of the value process should look like this:

$$dV_t^h = dC\left(t, S_t; \sigma_t^i\right) + h_t^0 dB_t - C_s\left(t, S_t; \sigma_t^h\right) dS_t \tag{21}$$

$$= C_s\left(t, S_t; \sigma_t^h\right)\left(rS_t dt - dS_t\right) - rC\left(t, S_t; \sigma_t^i\right) dt + dC\left(t, S_t; \sigma_t^i\right) \tag{22}$$

Now, apply îto to obtain the dynamics of $dC\left(t, S_t; \sigma_t^h\right)$

$$dC\left(t, S_t; \sigma_t^h\right) = C_t\left(t, S_t; \sigma_t^h\right) dt + C_s\left(t, S_t; \sigma_t^h\right) dS_t + \frac{1}{2} C_{ss}\left(t, S_t; \sigma_t^h\right) \sigma_t^2 S_t^2 dt \tag{23}$$

Why are there two different volatilities present in (23)?

Recall, that Thm. 7.7 yields

$$C_t\left(t,S_t;\sigma_t^h\right)=rC\left(t,S_t;\sigma_t^h\right)-rS_tC_s\left(t,S_t;\sigma_t^h\right)-\frac{1}{2}\left(\sigma_t^h\right)^2S_t^2C_{ss}\left(t,S_t;\sigma_t^h\right) \quad (24)$$

Now, plug (24) into (23) and rearrange such that

$$0=-dC\left(t,S_t;\sigma_t^h\right)+C_s\left(t,S_t;\sigma_t^h\right)dS_t \quad (25)$$
$$+\left(rC\left(t,S_t;\sigma_t^h\right)-rS_tC_s\left(t,S_t;\sigma_t^h\right)+\frac{1}{2}\left(\sigma_t^2-\left(\sigma_t^h\right)^2\right)S_t^2C_{ss}\left(t,S_t;\sigma_t^h\right)\right)dt$$

Add (25) to (22) to obtain

$$dV_t^h = -dC\left(t, S_t; \sigma_t^h\right) + rC\left(t, S_t; \sigma_t^i\right) dt + dC\left(t, S_t; \sigma_t^i\right) \tag{26}$$

$$+ rC(t, S_t; \sigma_t^h) dt + \frac{1}{2}\left(\sigma_t^2 - \left(\sigma_t^h\right)^2\right) S_t^2 C_{ss}\left(t, S_t; \sigma_t^h\right) dt$$

$$= dC\left(t, S_t; \sigma_t^i\right) - dC\left(t, S_t; S\sigma_t^h\right) - r\left(C\left(t, s_t; \sigma_t^i\right) - c\left(t, s_t; \sigma_t^h\right)\right) dt \tag{27}$$

$$+ \frac{1}{2}\left(\sigma_t^2 - (\sigma_t^h)^2\right) S_t^2 C_{ss}\left(t, S_t; \sigma_t^h\right) dt$$

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With a clever application of Itô, we get

$$dV_t^h = e^{rt} d\left(e^{-rt}\left(C\left(t,S_t;\sigma_t^i\right) - C\left(t,S_t;\sigma_t^h\right)\right)\right) \\ + \frac{1}{2}\left(\sigma_t^2 - (\sigma_t^h)^2\right) S_t^2 \Gamma\left(t,S_t;\sigma_t^h\right) dt \tag{28}$$

The PnL of the portfolio at time T is given by $PnL := \int_0^T e^{-rt} dV_t^h$, hence

$$PnL = C\left(0,S_0;\sigma_0^h\right) - C\left(0,S_0;\sigma_0^i\right) + \frac{1}{2}\int_0^T e^{-rt}\left((\sigma_t)^2 - (\sigma_t^h)^2\right) S_t^2 \Gamma\left(t,S_t;\sigma_t^h\right) dt \tag{29}$$

Wilmott's hedge experiment:

1. Assume the Black-Scholes model holds and that market implied volatilities are constant, i.e. $\sigma = \sigma_t$; $\sigma_t^i = \sigma^i \ \forall t$.
2. Let $\sigma^i < \sigma$ implying that the option is underpriced in the market according to model
3. As budding financial professionals, we wish to collect the premium
4. Idea: The option is cheap, so we can buy it and subsequently hedge it in the market to reduce the risk
5. Pursuant to the FTODT, our PnL at expiry is given by

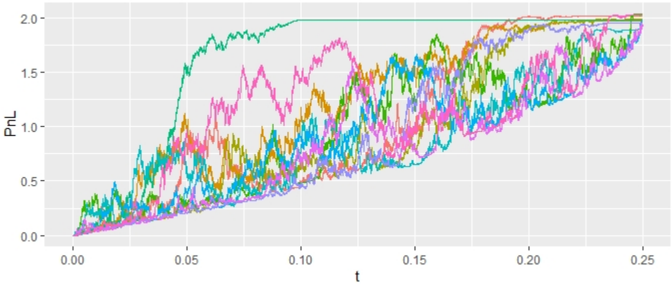
$$PnL_T = C(0, S_0, \sigma) - C(0, S_0, \sigma^i) > 0 \tag{30}$$

   if we hedge using the model volatility.
6. If we hedge using the implied volatility instead, the PnL will be

$$\int_0^T e^{-rt} \frac{1}{2} \left(\sigma_t^2 - \left(\sigma_t^i \right)^2 \right) S_t^2 \Gamma \left(t, S_t; \sigma_t^i \right) dt \tag{31}$$

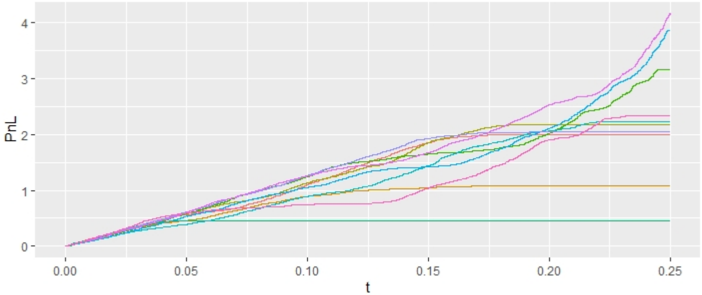
   which is positive and stochastic.

Case I - hedging w. model volatility



Parameters: $S_0 = 1, \mu = 0.1, \sigma = 0.2, \sigma^i = 0.1, r = 0.02, T = 1/4$, and $K = 100$. Note, that $e^{rT}(C(0, S_0, \sigma) - C(0, S_0, \sigma^i)) = 1.99$

Case II - hedging w. implied volatility



The PnL appears much smoother, but highly path dependant!