UNIVERSITY OF COPENHAGEN

FACULTY OF SCIENCE

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Assignment

Advisors: Jonatan Siegrist & Johan Emil Westergaard

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Chapter 1

Black-Scholes

The outset

Consider the price evolution of the S&P 500 and Russell 2000 indices over the last 20 years.

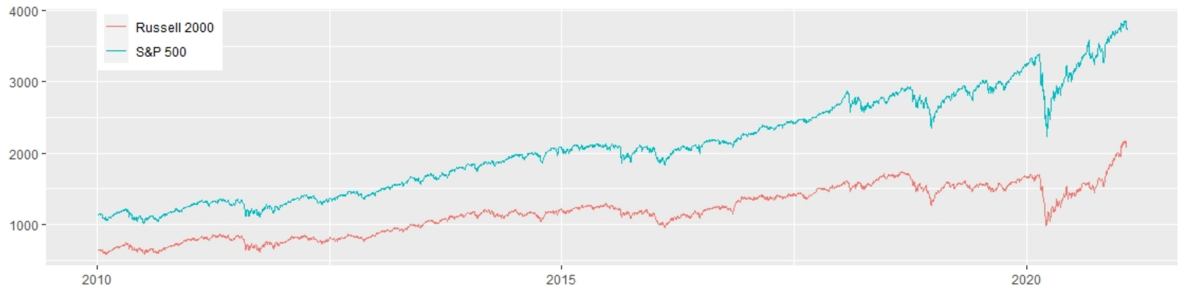

Figure 1.1: The S&P 500 and the Russell 2000 indices from the 1st of January 2010 to the 1st of February 2021

Evidently, the stock indices have increased steadily albeit with some term random fluctuations. In an effort to describe the behaviour of a stock[1] in a more formal setting, we propose the following model[2]:

Consider a financial market in which the uncertainty is governed by the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ equipped with the Wiener process $(W_t)_{t\geq 0}$. Here, Ω represents the state space, $\mathcal{F}$ is the σ-algebra representing measurable events, $(\mathcal{F}_t)_{t\geq 0}$ is the filtration, and $\mathbb{P}$ the probability measure. We will think of $\mathbb{P}$ as the real-world or physical measure. The only stochasticity derives from the Wiener process. Consequently, the filtration is considered to be

[1]Or a stock index

[2]Also known as the Black-Scholes model

the natural filtration of the Wiener process, namely

$$\mathcal{F}_t = \mathcal{F}_t^W \quad t \geq 0 \tag{1.1}$$

where

$$\mathcal{F}_t^W = \sigma(W_s \mid 0 \leq s \leq t) \tag{1.2}$$

Concentrating on the stock once more, the idea is to model the stock process under the physical measure $\mathbb{P}$ as a stochastic process $(S_t)_{t\geq 0}$ with the dynamics

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ S_0 &= s \end{aligned} \tag{1.3}$$

where $\mu \in \mathbb{R}$ and $s, \sigma \in \mathbb{R}^+$ are constant. A stochastic process with this dynamics is called a geometric Brownian motion. One can show, that the solution to the stochastic differential equation is given by

$$S(t) = S_0 \cdot \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right) \tag{1.4}$$

In addition to the stock, we also define a risk-free asset B which is governed by the dynamics

$$\begin{aligned} dB_t &= rB_t dt \\ B_0 &= 1 \end{aligned} \tag{1.5}$$

Contrary to what one might think, financial assets are not priced under the real-world measure, $\mathbb{P}$, but under the so-called risk neutral measure, $\mathbb{Q}$: Assuming that the market is arbitrage-free and complete there exists a unique equivalent martingale measure denoted by $\mathbb{Q}$. Under $\mathbb{Q}$, the stock has the dynamics

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^Q \\ S_0 &= s \end{aligned} \tag{1.6}$$

Here, W^Q is a Wiener process under $\mathbb{Q}$. Thus for a financial asset which pays $g(S_T)$ at some fixed maturity T one arrives at the time t arbtrage-free price, $\Pi(t)$, given by

$$\Pi(t) = E^Q\left[e^{-\int_t^T r du} g(S(T)) \mid \mathcal{F}_t\right] = e^{-r(T-t)} E^Q[g(S(T)) \mid S(t) = s] \tag{1.7}$$

the above is often referred to as $risk - neutral$ pricing.

1.1 Model assumptions

The aim of this section is to provide some intuition about the model in the above and its underlying assumptions. Moreover, a brief discussion on the use of model assumptions in mathematical modelling might also appear.

a) Examine eq. 1.1, 1.2, and 1.3. How are they to be understood intuitively?

b) Contemplate the model assumptions presented in the text:

i) What are the model assumptions under both $\mathbb{P}$ and $\mathbb{Q}$ and what do they imply about the behaviour of the stock as well as the market participants?

ii) Do the model assumptions seem realistic and do they in any case need to be?

c) A prevailing paradigm in finance is risk-neutral pricing. Explain in your own words why this is a clever concept[3].

1.2 Geometric Brownian motion

In this section, we will investigate the implications of the assumption that the stock price process follows a geometric Brownian motion.

a) Derive eq.1.4 and solve the differential equation 1.5. Hint: Consider the process $X_t = \log(S_t)$ and apply Itô's formula

b) Derive the stock price distribution under $\mathbb{P}$.

c) Derive the distribution of $\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$ under $\mathbb{P}$. We assume that the time points are equidistant meaning that $t_i - t_{i-1} = \Delta t$.

d) Download price data on S&P 500. This can be done with the package `quantmod` in R.

e) Use statistical analysis to investigate the model assumptions regarding the stock. Do you find that data supports the model assumptions?

[3]For inspiration, see Björk section 7.4 and 15.6

f) Derive closed-form maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ. Hint: Recall the distributional result from c).

g) Compute $\hat{\mu}$ and $\hat{\sigma}$ based on the observed price data. Do this for i) All data, ii) Data from 2020 only. What are the differences? Can you explain these differences?

h) Derive closed-form expressions for the variance of $\hat{\mu}$ and $\hat{\sigma}$. What are the properties of the variance of $\hat{\mu}$ and how do they affect the pricing of options in Black-Scholes? Hint: Use the Delta method to derive the covariance matrix for $(\hat{\mu}, \hat{\sigma})$.

1.3 Monte Carlo - Simulation

Monte Carlo simulation is a powerful tool to estimate the expectation of a random variable X or of a stochastic process at a specific time T, X_T. The idea of Monte Carlo is to exploit the law of large numbers. Given a stock price process, $(S_t)_{t\geq 0}$, imagine that we are interested in the expected stock price at time T, i.e. $E(S_T)$. With Monte Carlo in mind, we therefore simulate n iid realisations of S_T, $S_{T_1}, ..., S_{T_n}$, potentially by simulating n independent trajectories/paths of $(S_t)_{t\geq 0}$ (if we weren't gifted with the relation 1.4). It then follows from the law of large numbers that

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} S_{T_i} \overset{a.s.}{\to} E(S_T) = \theta \tag{1.8}$$

where we coin θ the Monte Carlo estimator. Monte Carlo simulation therefore relies on the ability to simulate realisations on S_T and not on the skills to analytically (if possible) compute $E(S_T)$.

a) Explain why Monte Carlo is useful in mathematical finance. Furthermore, discuss the pros and cons of the method. Include a discussion of the sampling error and discretization error in your answer.

b) Argue that the variance and standard error of the Monte Carlo estimator is approximately proportional to n^{-1} and $n^{-1/2}$ respectively. Hint: standard error is defined as the standard deviation of $\hat{\theta} - \theta$

c) Write a simulation scheme of the stock price process using Euler discretization. Hint: Seydel page 46

d) Assume $S(0) = 10$, $\mu = 0.07$ and $\sigma = 0.2$. Write code that simulates and plots 10 paths of S under $\mathbb{P}$ for different choices of Δt. Is is possible for the stock price to become negative? Explain why, and if it is coherent with the model assumptions.

e) Compute $E(S_T)$ using Monte Carlo and compare with the analytical solution (i.e. you also have to compute $E(S_T)$ analytically).

f) From a) it is evident, that we can reduce the variance of the Monte Carlo estimator, by simulating more paths. Discuss why it would be preferable to reduce the variance of the Monte Carlo estimator without simulating additional paths. In this discussion, describe how the two variance reduction techniques *antithetic variates* and *control variates* works.

g) Rewrite your simulation scheme using antithetic variates.

Chapter 2

Option Pricing

2.1 European options

Let $S(t)$ denote the stock price at time t and $S = (S(t))_{t\geq 0}$ the corresponding stock price process.
A European call option gives the investor the right but not the obligation to buy the underlying asset at a strike price K at the expiry date T. The payoff at expiry can thus be described by the payoff function below

$$\max(0, S(T) - K) = (S(T) - K)^+ \tag{2.1}$$

A European put option gives the investor the right but not the obligation to sell the underlying asset at a strike price K at the expiry date T. Consequently, the payoff function is given by

$$\max(0, K - S(T)) = (K - S(T))^+ \tag{2.2}$$

For the purpose of the an inspired discussion in later questions, American options are also briefly introduced. With an American option, the investor has the right to exercise the option at some time between time t and expiry, T. Provided τ is the optimal stopping time, the payoff function of such an option is

$$\begin{aligned} &\max(S(\tau) - K, 0) \quad (\text{ call }) \\ &\max(K - S(\tau), 0) \quad (\text{ put }) \end{aligned} \tag{2.3}$$

The financial model presented in the previous chapter implies the existence of an analytical pricing formula for such options. The Black-Scholes pricing formula states that the price of a European call option is given by

$$\begin{aligned} C(t, S(t), K, r, \sigma) &= E^Q\left[e^{-\int_t^T r du} \max(S(T) - K, 0) \mid \mathcal{F}_t\right] \\ &= S(t)\phi\left(d_1\right) - e^{-r(T-t)} K\phi\left(d_2\right) \end{aligned} \tag{2.4}$$

where

$$
\begin{aligned}
d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right) \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned} \tag{2.5}
$$

and r, σ denote the short rate and the volatility respectively.
Moreover, the relationship between the price of European call and put options can be described by the identity known as the put-call parity:

$$
P(t, S(t), K, r, \sigma) = Ke^{-r(T-t)} + C(t, S(t), K, r, \sigma) - S(t) \tag{2.6}
$$

a) Why are we interested in financial options i.e. when are they useful?

b) Provide a proof of 2.6. Hint: Replicate a European put using the risk-free asset, the stock, and the European call.

c) Compute prices of European options using both Monte Carlo simulation and eq. 2.4 for different values of S, T, r, and σ. Explain the intuition behind the results and demonstrate the validity of the chosen Monte Carlo method.

d) Retrieve prices of European(SPX) and American(SPY) options on the S&P 500. Use eq. 2.4 to compute the equivalent theoretical prices and compare them to the data.

e) Explain the relationship between American and European options. Hint: Considering the concepts *intrinsic value* and *time value* might be good a idea.

f) Take the European option data and compute implied volatilities. The implied volatility is the value of model volatility, σ^*, such that the theoretical option price matches the one observed in the market.

g) Plot the results of e) against the log-moneyness $\left(\log(\frac{S}{K})\right)$ and interpret the result.

h) Recall the empirical volatility estimate from Chapter 1 and compare it to the implied volatilities. Two relevant questions to consider in this are:

i) How are the the two types of volatility to be understood in relation to the $\mathbb{P}$ and $\mathbb{Q}$ measure

ii) What is the expected relationship between the volatilities provided the model assumptions from Chapter 1 hold true? What do the findings show?