Gauss Integration

Let $f(x)$, whose Taylor expansion around 0 is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

or, $f(x) = a + bx + cx^2 + dx^3 + ex^4 + \cdots +$ a, b, c, d, e constants

We want to compute the integral of

$f(x) = a + bx$; to the first approximation

then $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} a\,dx + \int_{-1}^{1} bx\,dx$

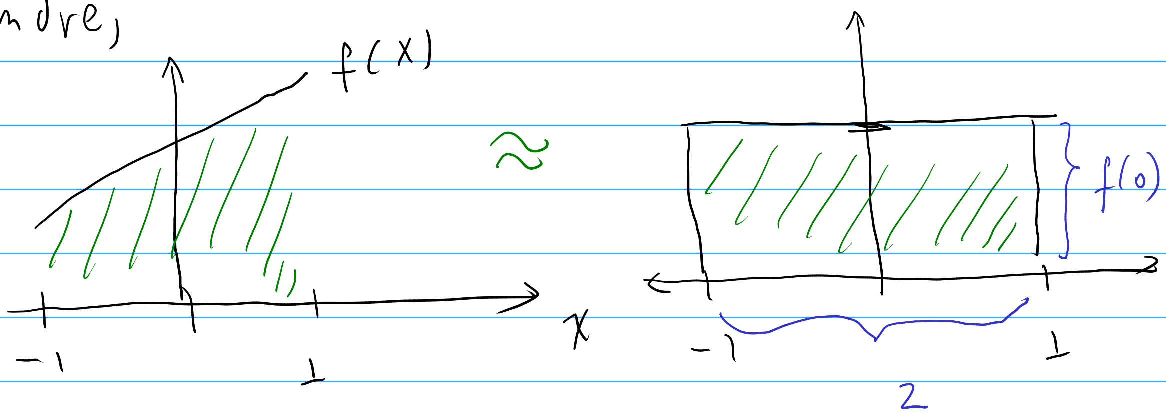
but bx, is an odd function, hence $\int_{-1}^{1} bx\,dx = 0$

$$\int_{-1}^{1} f(x)\,dx = \int a\,dx = ax\Big|_{-1}^{1} = a - [-a] = 2a$$

And, $a = f(0)$, hence $\int_{-1}^{1} f(x)\,dx = 2f(0)$

Note that, $W_0 = [2]$ $x_0 = [0]$, for the first coefficient of Gauss Legendre,

Geometrically,



Now, Let's compute, $\int_{-1}^{1} f(x)dx$ with $f(x) = a + bx + cx^2$

Given that bx is an odd function. $\int_{-1}^{1} bx = 0$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} (a + cx^2)dx = \left[ax + \frac{cx^3}{3}\right]_{-1}^{1} = 2\left[a + \frac{c}{3}\right]$$

Now, note that

$$f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) = 2\left[a + \frac{c}{3}\right]$$

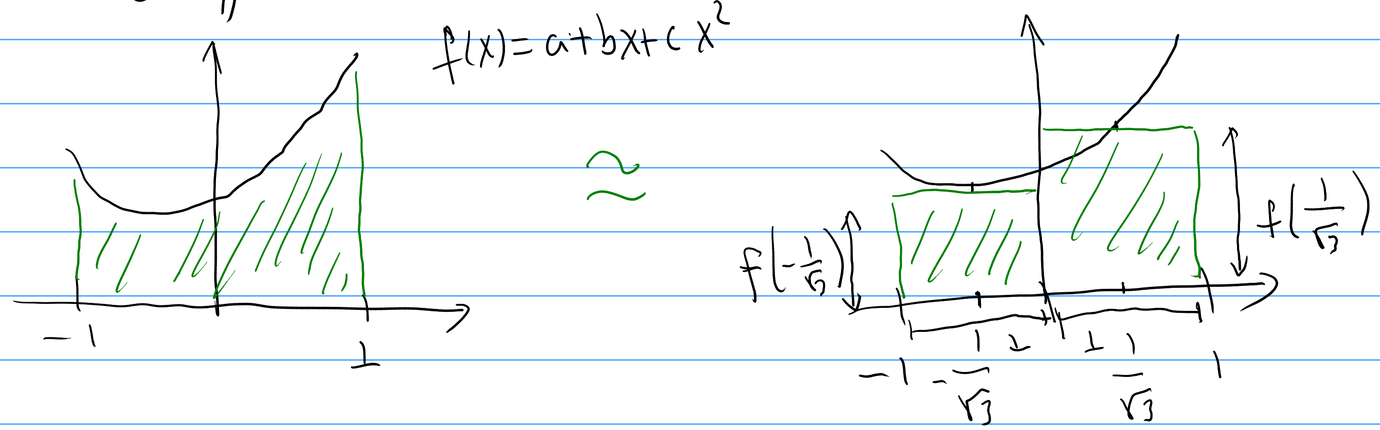
Hence,

$$\int_{-1}^{1} f(x)dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} f\left(-\frac{1}{\sqrt{3}}\right) \\ f\left(\frac{1}{\sqrt{3}}\right) \end{bmatrix}$$

Hence $w_1 = [1, 1]$ and $x_1 = \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$

are the values of gauss,

Geometrically,



Now, Let's compute, $\int_{-1}^{1} f(x)dx$ with $f(x) = a + bx + cx^2 + dx^3$

Given that, bx and dx^3 are odd functions

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} (a + cx^2)dx = 2\left[a + \frac{c}{3}\right],$$

Hence, the values w_0, and x_1, are sufficient to integrate a function of third degree.

Now, Let's compute $\int_{-1}^{1} f(x)\,dx$ with $f(x)=a+bx+cx^2+dx^3+ex^4$

then, as before, bx, dx^3 are odd, hence

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (a+cx^2+ex^4)\,dx = \left[ax+\frac{cx^3}{3}+\frac{ex^5}{5}\right]_{-1}^{1}$$

$$= 2\left[a+\frac{c}{3}+\frac{e}{5}\right]$$

Now, note that,

$$f(0) = a \qquad f\left(-\sqrt{\tfrac{3}{5}}\right) = a + c\frac{3}{5} + e\frac{3^2}{5^2} = f\left(\sqrt{\tfrac{3}{5}}\right)$$

And, $\frac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right) = \frac{10}{9} f\left(\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0)$

$$= \frac{10}{9}\left[a + \frac{3c}{5} + \frac{9e}{5^2}\right] + \frac{8}{9}a$$

$$= \frac{10}{9}a + \frac{2c}{3} + \frac{2e}{5} + \frac{8}{9}a = \frac{18}{9}a + \frac{2c}{3} + \frac{2e}{5}$$

$$= 2\left[a + \frac{c}{3} + \frac{e}{5}\right]$$

Then, $\int_{-1}^{1} f(x)\,dx = \begin{bmatrix} \frac{5}{9} & \frac{8}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} f\left(-\sqrt{\tfrac{3}{5}}\right) \\ f(0) \\ f\left(\sqrt{\tfrac{3}{5}}\right) \end{bmatrix} = \vec{w}_2 \cdot f(\vec{x}_2)$

Geometrically,